A Group of Order 1575 with a Normal Sylow 3-subgroup is Abelian

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Abstract

In this paper we lay out the proof of this result in group theory using an old-fashioned approach.

If H is a subgroup of G, then $C(H) = \{x \in G | xh = hx \text{ for all } h \in H\}$

Theorem 1

Let G be a group with a normal subgroup H and a subgroup K. If N is a subgroup of G such that $N \subset C(H)$ and $N \subset C(K)$, then $N \subset C(HK)$.

Theorem 2

Let A and B be normal subgroups of G such that $A \cap B = (e)$. Then $A \subset C(B)$.

Theorem 3

Let G be a group of order 1575 with a normal Sylow 3-subgroup. Then G is abelian.

Proof.

By the hypothesis, G has a normal subgroup H of order 3^2. By Sylow's theorem, G has a subgroup K of order 5^2. Since, by Sylow's theorem, the number of subgroups of G/H of order 5^2 is 1, it follows that HK/H is the only subgroup of G/H of order 5^2 and hence HK/H is normal in G/H. Consequently HK is normal in G. Let φ be an automorphism of HK. Since K is a subgroup of HK of order 5^2, $\varphi(K)$ is a subgroup of HK of order 5^2. By Sylow's theorem, the number of subgroups of HK of order 5^2 is 1. Thus $\varphi(K) = K$. Since HK is normal in G and K is a characteristic subgroup of HK, K must be normal in G. By Sylow's theorem, G has a subgroup N of order 7. Since, by Sylow's theorem, the number of subgroups of G/H of order 7 is 1, it follows that HN/H is the only subgroup of G/H of order 7 and hence HN/H is normal in G/H. Consequently HN is normal in G. Let φ be an automorphism of HN. Since N is a subgroup of HN of order 7, $\varphi(N)$ is a subgroup of HN of order 7. By Sylow's theorem, the number of subgroups of HN of order 7 is 1. Thus $\varphi(N) = N$. Since HN is normal in G and N is a characteristic subgroup of HN, N must be normal in G. Since H is a group of order 3^2 where 3 is a prime, H is abelian and hence $H \subset C(H)$. Moreover since H and K are normal in G such that $H \cap K = (e)$, $H \subset C(K)$ by Theorem 2. So G has a normal subgroup H and a subgroup K such that $H \subset C(H)$ and $H \subset C(K)$. Thus $H \subset C(HK)$ by Theorem 1. Since H and N are normal in G such that $H \cap N = (e)$, $H \subset C(N)$ by Theorem 2. So G has a normal subgroup HK and a subgroup N such that $H \subset C(HK)$ and $H \subset C(N)$. Thus

$$H \subset C((HK)N) \tag{1}$$

by Theorem 1. Since K and H are normal in G such that $K \cap H = (e)$, $K \subset C(H)$ by Theorem 2. Moreover since K is a group of order 5^2 where 5 is a prime, K is abelian and hence $K \subset C(K)$. So G has a normal subgroup H and a subgroup K such that $K \subset C(H)$ and $K \subset C(K)$. Thus $K \subset C(HK)$ by Theorem 1. Since K and N are normal in G such that $K \cap N = (e)$, $K \subset C(N)$ by Theorem 2. So G has a normal subgroup HK and a subgroup N such that $K \subset C(HK)$ and $K \subset C(N)$. Thus

$$K \subset C((HK)N) \tag{2}$$

by Theorem 1. By (1) and (2),

$$HK \subset C((HK)N). \tag{3}$$

Since N and H are normal in G such that $N \cap H = (e)$, $N \subset C(H)$ by Theorem 2. Moreover since N and K are normal in G such that $N \cap K = (e)$, $N \subset C(K)$ by Theorem 2. So G has a normal subgroup H and a subgroup K such that $N \subset C(H)$ and $N \subset C(K)$. Thus $N \subset C(HK)$ by Theorem 1. Since N is a group of order 7 where 7 is a prime, N is cyclic and hence abelian. Thus $N \subset C(N)$. So G has a normal subgroup HK and a subgroup N such that $N \subset C(HK)$ and $N \subset C(N)$. Thus by Theorem 1

$$N \subset C((HK)N). \tag{4}$$

By (3) and (4),

$$(HK)N \subset C((HK)N).$$

Note that $|(HK)N| = 1575$. Since $(HK)N \subset G$ and $|(HK)N| = |G|$, it follows that $(HK)N = G$. Since $G \subset (HK)N$ and $(HK)N \subset C((HK)N)$, it follows that $G \subset C((HK)N)$. Since $G \subset (HK)N$, it follows that $C((HK)N) \subset C(G)$. Since $G \subset C((HK)N)$ and $C((HK)N) \subset C(G)$, it follows that $G \subset C(G)$. Since $C(G) \subset G$ and $G \subset C(G)$, it follows that $C(G) = G$. To conclude G is abelian.

References

[1] I. N. Herstein, *Abstract Algebra*, Macmillan Publishing Company, New York, 1990.