MULTIVARIATE EXPANSIVITY THEORY

T. AGAMA

ABSTRACT. In this paper we launch an extension program for single variable expansivity theory. We study this notion under tuples of polynomials belonging to the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. As an application we show that

$$\min\{\max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s} + 1\}_{i=1}^{l} < \frac{1}{l}\sum_{i=1}^{l}\max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s} + 2 + \mathcal{J}$$

where $\mathcal{J} := \mathcal{J}(l) \geq 0$ and $\mathrm{Ind}_{f_k}(x_j)$ is the largest power of x_j $(1 \leq j \leq n)$ in the polynomial $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$.

1. Introduction

The notion of the **single variable** expansivity theory had been developed quite extensively by the author [1]. This notion turns out to be an important tool in studying Sendov's conjecture. This theory also has wide range of applications in determining the insolubility of certain systems of differential equations. In the current paper we launch an extension program where the study is carried out for polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ with real number base field $\mathbb{R}$. It turns out that various basic notion studied under the single variable theory carry over to this setting.

Throughout this paper, we keep the usual standard notion $\mathcal{S}$ for all tuples whose entries belong to the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Occasionally we might choose to index these tuples by $\mathcal{S}_j$ over the natural numbers $\mathbb{N}$ if we have two or more and we want to keep them distinct from each other. The tuples $\mathcal{S}_0 = (0, 0, \ldots, 0)$ and $\mathcal{S}_e = (1, 1, \ldots, 1)$ are still reserved for the null and the unit tuple respectively. Further to the above requirements any tuple of polynomial will be assumed to contain exactly n entries and two tuples under the operation of addition or subtraction will be assumed to contain the same number of entries.

2. Expansion in mixed and specified directions

In this section we introduce the notion of an expansion in a **mixed** and **specific** directions. We launch the following extension program

Definition 2.1. Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. Then by an expansion on $\mathcal{S} \in \mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ in the direction x_i

Date: January 19, 2021.

2010 *Mathematics Subject Classification.* Primary 11Pxx, 11Bxx; Secondary 11Axx, 11Gxx.

Key words and phrases. mixed, polynomial, diagonalization, dropler effect, destabilization, ring.

for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \longrightarrow \mathcal{F}$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

with

$$\nabla_{[x_i]}(\mathcal{S}) = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i}\right).$$

The value of the l th expansion at a given value a of x_i is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^l_{[x_i](a)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^l_{[x_i](a)}(\mathcal{S})$ is a tuple of polynomials in $\mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Similarly by an expansion in the mixed direction $\otimes_{i=1}^l [x_{\sigma(i)}]$ we mean

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^l [x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

for any permutation $\sigma : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, l\}$. The value of this expansion on a given value a_i of $x_{\sigma(i)}$ for all $i \in [\sigma(1), \sigma(l)]$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}](a_i)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^l [x_{\sigma(i)}](a_i)}(\mathcal{S})$ is tuple of real numbers $\mathbb{R}$.

Proposition 2.2. *A multivariate expansion is linear.*

Proof. It suffices to show that each of the operators $\nabla_{[x_i]} : \{\mathcal{S}_i\}_{i=1}^\infty \longrightarrow \{\mathcal{S}_i\}_{i=1}^\infty$ for a fixed direction $[x_i]$, $\gamma : \{\mathcal{S}_i\}_{i=1}^\infty \longrightarrow \{\mathcal{S}_i\}_{i=1}^\infty$ and $\beta \circ \gamma : \{\mathcal{S}_i\}_{i=1}^\infty \longrightarrow \{\mathcal{S}_i\}_{i=1}^\infty$ is linear, since the map $\gamma : \{\mathcal{S}_i\}_{i=1}^\infty \longrightarrow \{\mathcal{S}_i\}_{i=1}^\infty$ is bijective. Let $\mathcal{S}_a = (f_1, f_2, \ldots, f_n), \mathcal{S}_b = (g_1, g_2, \ldots, g_n) \in \mathcal{F} = \{\mathcal{S}_i\}_{i=1}^\infty$ and let $\lambda, \mu \in \mathbb{R}$, then it follows that

$$\begin{aligned}
\nabla_{[x_i]}(\lambda \mathcal{S}_a + \mu \mathcal{S}_b) &= \nabla(\lambda(f_1, f_2, \ldots, f_n) + \mu(g_1, g_2, \ldots, g_n)) \\
&= \nabla_{[x_i]}((\lambda f_1, \lambda f_2, \ldots, \lambda f_n) + (\mu g_1, \mu g_2, \ldots, \mu g_n)) \\
&= \nabla_{[x_i]}((\lambda f_1 + \mu g_1, \lambda f_2 + \mu g_2, \ldots \lambda f_n + \mu g_n)) \\
&= (\frac{\partial(\lambda f_1 + \mu g_1)}{\partial x_i}, \frac{\partial(\lambda f_2 + \mu g_2)}{\partial x_i}, \ldots, \frac{\partial(\lambda f_n + \mu g_n)}{\partial x_i}) \\
&= (\lambda \frac{\partial f_1}{\partial x_i} + \mu \frac{\partial g_1}{\partial x_i}, \lambda \frac{\partial f_2}{\partial x_i} + \mu \frac{\partial g_2}{\partial x_i}, \ldots, \lambda \frac{\partial f_n}{\partial x_i} + \mu \frac{\partial g_n}{\partial x_i}) \\
&= (\lambda \frac{\partial f_1}{\partial x_i}, \lambda \frac{\partial f_2}{\partial x_i}, \ldots, \lambda \frac{\partial f_n}{\partial x_i}) + (\mu \frac{\partial g_1}{\partial x_i}, \mu \frac{\partial g_2}{\partial x_i}, \ldots, \mu \frac{\partial g_n}{\partial x_i}) \\
&= \lambda(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i}) + \mu(\frac{\partial g_1}{\partial x_i}, \frac{\partial g_2}{\partial x_i}, \ldots, \frac{\partial g_n}{\partial x_i}) \\
&= \lambda \nabla_{[x_i]}(\mathcal{S}_a) + \mu \nabla_{[x_i]}(\mathcal{S}_b).
\end{aligned}$$

Similarly,

$$\begin{aligned}\gamma(\lambda\mathcal{S}_a+\mu\mathcal{S}_b) &= \begin{pmatrix}\lambda f_1+\mu g_1\\ \lambda f_2+\mu g_2\\ \vdots\\ \lambda f_n+\mu g_n\end{pmatrix}\\ &= \begin{pmatrix}\lambda f_1\\ \lambda f_2\\ \vdots\\ \lambda f_n\end{pmatrix}+\begin{pmatrix}\mu g_1\\ \mu g_2\\ \vdots\\ \mu g_n\end{pmatrix}\\ &= \lambda\gamma(\mathcal{S}_a)+\mu\gamma(\mathcal{S}_b).\end{aligned}$$

Similarly

$$\begin{aligned}\beta\circ\gamma(\lambda\mathcal{S}_a+\mu\mathcal{S}_b) &= \begin{pmatrix}0 & 1 & \cdots 1\\ 1 & 0 & 1\cdots 1\\ \vdots & \vdots & \cdots\vdots\\ 1 & 1 & \cdots 0\end{pmatrix}\begin{pmatrix}\lambda f_1+\mu g_1\\ \lambda f_2+\mu g_2\\ \vdots\\ \lambda f_n+\mu g_n\end{pmatrix}\\ &= \begin{pmatrix}0 & 1 & \cdots 1\\ 1 & 0 & 1\cdots 1\\ \vdots & \vdots & \cdots\vdots\\ 1 & 1 & \cdots 0\end{pmatrix}\left\{\begin{pmatrix}\lambda f_1\\ \lambda f_2\\ \vdots\\ \lambda f_n\end{pmatrix}+\begin{pmatrix}\mu g_1\\ \mu g_2\\ \vdots\\ \mu g_n\end{pmatrix}\right\}\\ &= \lambda\begin{pmatrix}0 & 1 & \cdots 1\\ 1 & 0 & 1\cdots 1\\ \vdots & \vdots & \cdots\vdots\\ 1 & 1 & \cdots 0\end{pmatrix}\begin{pmatrix}f_1\\ f_2\\ \vdots\\ f_n\end{pmatrix}+\mu\begin{pmatrix}0 & 1 & \cdots 1\\ 1 & 0 & 1\cdots 1\\ \vdots & \vdots & \cdots\vdots\\ 1 & 1 & \cdots 0\end{pmatrix}\begin{pmatrix}g_1\\ g_2\\ \vdots\\ g_n\end{pmatrix}\\ &= \lambda(\beta\circ\gamma)(\mathcal{S}_a)+\mu(\beta\circ\gamma)(\mathcal{S}_b).\end{aligned}$$

This proves the linearity of expansion. □

Remark 2.3. Next we prove a fundamental result which shows that an expansion is commutative. This reinforces the very notion that there is no need to give precedence to the direction of an expansion. In essence, it gives some flexibility to the way and manner an expansion could be carried out.

Proposition 2.4. *An expansion is commutative.*

Proof. Consider $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ the collection of tuples in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. It suffices to show that for any $\mathcal{S}\in\mathcal{F}$ then

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]\otimes[x_j]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]\otimes[x_i]}(\mathcal{S}).$$

First we can write

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}) = \left(\left(\sum_{\substack{t\in[1,n]\\ t\neq 1}}\sum_{k=t}\frac{\partial f_k}{\partial x_i}\right),\ldots,\left(\sum_{\substack{t\in[1,n]\\ t\neq n}}\sum_{k=t}\frac{\partial f_k}{\partial x_i}\right)\right)$$

and make the assignment

$$\begin{aligned}\mathcal{S}_{g_k} &= (g_{k1}, g_{k2}, \ldots, g_{kn})\\ &= \bigg(\bigg(\sum_{\substack{t\in[1,n]\\ t\neq 1}}\sum_{k=t}\frac{\partial f_k}{\partial x_i}\bigg), \ldots, \bigg(\sum_{\substack{t\in[1,n]\\ t\neq n}}\sum_{k=t}\frac{\partial f_k}{\partial x_i}\bigg)\bigg)\end{aligned}$$

for $g_{ki} \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. Next we carry out the second expansion on $\mathcal{S}_{g_k}$ and we get

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{g_k}) = \bigg(\sum_{\substack{s\in[1,n]\\ s\neq 1}}\sum_{k=s}\frac{\partial g_{ks}}{\partial x_j}, \ldots, \sum_{\substack{s\in[1,n]\\ s\neq n}}\sum_{k=s}\frac{\partial g_{ks}}{\partial x_j}\bigg)$$

so that by combining the two expansions in both directions, we have

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]\otimes[x_j]}(\mathcal{S}) &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{g_k})\\ &= \bigg(\bigg(\sum_{\substack{s\in[1,n]\\ s\neq 1}}\sum_{k=s}\frac{\partial g_{ks}}{\partial x_j}\bigg), \ldots, \bigg(\sum_{\substack{s\in[1,n]\\ s\neq n}}\sum_{k=s}\frac{\partial g_{ks}}{\partial x_j}\bigg)\bigg)\\ &= \bigg(\bigg(\sum_{\substack{s\in[1,n]\\ s\neq 1}}\sum_{k=s}\sum_{\substack{t\in[1,n]\\ t\neq 1}}\sum_{k=t}\frac{\partial^2 f_k}{\partial x_j \partial x_i}\bigg),\\ &\quad \ldots, \bigg(\sum_{\substack{s\in[1,n]\\ s\neq n}}\sum_{k=s}\sum_{\substack{t\in[1,n]\\ t\neq n}}\sum_{k=t}\frac{\partial^2 f_k}{\partial x_j \partial x_i}\bigg)\bigg)\end{aligned}$$

by appealing to the linearity of the operator $\frac{\partial}{\partial x_i}$. By carrying out the expansion in the opposite direction and appealing to the linearity of the operator

$$\frac{\partial}{\partial x_i}$$

we have

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]\otimes[x_i]}(\mathcal{S}) &= \bigg(\bigg(\sum_{\substack{s\in[1,n]\\ s\neq 1}}\sum_{k=s}\sum_{\substack{t\in[1,n]\\ t\neq 1}}\sum_{k=t}\frac{\partial^2 f_k}{\partial x_j \partial x_i}\bigg),\\ &\quad \ldots, \bigg(\sum_{\substack{s\in[1,n]\\ s\neq n}}\sum_{k=s}\sum_{\substack{t\in[1,n]\\ t\neq n}}\sum_{k=t}\frac{\partial^2 f_k}{\partial x_j \partial x_i}\bigg)\bigg)\end{aligned}$$

by exploiting the condition

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}$$

for each polynomial $g_i, f_i \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. By comparing the result of both expansions in reverse directions, the claim follows immediately. $\square$

3. The totient and residue of an expansion

In this section we introduce the notion of the **residue** and the **totient** of an expansion. These two notions are analogous to the notion of the rank and the degree of an expansion under the single variable theory. We launch more formally the following languages.

Definition 3.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. Then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ is **free** with **totient** k, denoted $\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]$, if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S})=\mathcal{S}_0$$

where $k>0$ is the smallest such number. We call the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k-1}_{[x_i]}(\mathcal{S})$ the **residue** of the expansion, denoted by $\Theta[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k-1}_{[x_i]}(\mathcal{S})]$. Similarly by the totient of the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$, we mean the smallest value of k such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0.$$

We denote the totient of the mixed expansion with

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})].$$

Proposition 3.2. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. If the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_k)$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_l)$ are free with totients s and t, respectively. Then the expansion*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_k+\mathcal{S}_l)$$

is also free with totient $\max\{s,t\}$.

Proof. Suppose the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_k)$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_l)$ are free with totients s and t, respectively. Then it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_i]}(\mathcal{S}_k)=\mathcal{S}_0$$

with $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s-m}_{[x_i]}(\mathcal{S}_k)\neq\mathcal{S}_0$ for all $m\leq s$ and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t}_{[x_i]}(\mathcal{S}_l)=\mathcal{S}_0$$

with $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t-m}_{[x_i]}(\mathcal{S}_l)\neq\mathcal{S}_0$ for all $m\leq t$. Now let us apply $\max\{s,t\}$ copies of the expansion maps to the tuple $\mathcal{S}_k+\mathcal{S}_l$ so that we have by appealing to the linearity of an expansion map we have

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{s,t\}}_{[x_i]}(\mathcal{S}_k+\mathcal{S}_l)&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{s,t\}}_{[x_i]}(\mathcal{S}_k)\\&+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{s,t\}}_{[x_i]}(\mathcal{S}_l)\\&=\mathcal{S}_0\end{aligned}$$

since $s,t\leq\max\{s,t\}$. Next we see that for any $1\leq r\leq\max\{s,t\}$ then by appealing to the linearity of the expansion map

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{s,t\}-r}_{[x_i]}(\mathcal{S}_k+\mathcal{S}_l)&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{s,t\}-r}_{[x_i]}(\mathcal{S}_k)\\&+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\max\{s,t\}-r}_{[x_i]}(\mathcal{S}_l)\\&\neq\mathcal{S}_0\end{aligned}$$

since at least one of the inequality $\max\{s,t\} - r < s$ or $\max\{s,t\} - r < t$ must hold. Thus $\max\{s,t\}$ is the totient of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_k+\mathcal{S}_l)$. This completes the proof of the proposition. $\square$

Remark 3.3. Next we expose an important relationship that exists between the totient of the mixed expansion and the underlying expansion in specific directions. One could view this result as a sub-additivity property of the totient of an expansion.

Theorem 3.4. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. Then we have the inequality*

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})] \leq \frac{1}{l}\sum_{i=1}^{l}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]+\mathcal{K}$$

where $\mathcal{K}(l)=\mathcal{K}>0$.

It is easily noticeable that the inequality allows us to control the totient of a mixed expansion by the average of the totient of expansions in specific directions involved in the mixed expansion. We relegate the proof of this to latter sections, where we develop the required tools needed. It is fair to say that this inequality is crude; However, we will obtained a much stronger version in the sequel that gives much information.

4. The dropler effect induced by an expansion

In this section we introduce the notion of the **dropler** effect induced by an expansion. This phenomena is mostly induced by expansion on several other expansions in a specific direction.

Definition 4.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. Then the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is said to induce a **dropler** effect with **intensity** k, denoted $\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]=k$, on the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{k}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

is free with $k<\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]$ and k is the smallest such number. In other words, we say the expansion admits a dropler effect from the source $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ with intensity k. The energy saved $\mathbb{E}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}](\mathcal{S})$ by the expansion under the dropler effect is given by

$$\mathbb{E}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}](\mathcal{S})=\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})].$$

We call this equation the energy-dropler effect intensity equation.

Proposition 4.2. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. If the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ each admits a dropler effect from the same source with intensities k_1 and k_2, respectively, then the expansion*

$$\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}\right](\mathcal{S})$$

also admits a dropler effect from the same source with intensity $\max\{k_1,k_2\}$.

Proof. Suppose the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ each admits a dropler effect from the same source with intensities k_1 and k_2, respectively. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ be their source, then it follows by virtue of Definition 4.1

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_1}_{[x_s]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0 \tag{4.1}$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_2}_{[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0. \tag{4.2}$$

Let us consider the expansion map $\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}\right]$ and apply $\max\{k_1,k_2\}$ copies to the source $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$. It follows by the linearity of an expansion and further appealing to (4.1) and (4.2)

$$\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}\right]^{\max\{k_1,k_2\}}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0.$$

It is easy to observe that

$$\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}+(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}\right]^{\max\{k_1,k_2\}-r}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})\neq\mathcal{S}_0 \tag{4.3}$$

for any $r\geq 1$, by appealing to the linearity of a multivariate expansion and exploiting the fact that at least one of the inequality must hold

$$\max\{k_1,k_2\}-r\leq k_1 \quad \text{or} \quad \max\{k_1,k_2\}-r\leq k_2.$$

Thus $\max\{k_1,k_2\}$ is the intensity of the dropler effect induced on the concatenations of the expansions under the same source. □

Remark 4.3. One could ask whether an analogue of this result exists for expansions with concatenated directions. While a general answer to this question may seem very baffling, we can somehow obtain a variant by imposing some conditions that ensure expansion in one direction does not wear off and interfere with the direction of the other. We make this assertion more precise in the following proposition.

Proposition 4.4. *Let* $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ *be a collection of tuples of polynomials in the ring* $\mathbb{R}[x_1,x_2,\ldots,x_n]$. *Let the expansions* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ *and* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ *each admits a dropler effect with intensities* k_1 *and* k_2*, respectively, from the source* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$. *If the expansions* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ *and* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ *admits no dropler effect from the sources*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

respectively, then the mixed expansion

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]\otimes[x_t]}(\mathcal{S})$$

also admits a dropler effect from the same source with intensity $\min\{k_1,k_2\}$.

Proof. Suppose the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ each admits a dropler effect from the same source with intensities k_1 and k_2, respectively. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ be their source, then it follows by virtue of Definition 4.1

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_1}_{[x_s]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0 \tag{4.4}$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_2}_{[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0. \tag{4.5}$$

Let us apply $\min\{k_1,k_2\}$ copies of the mixed expansion operator $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]\otimes[x_t]}$ to the source $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ then we see that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\min\{k_1,k_2\}}_{[x_s]\otimes[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=\mathcal{S}_0$$

by appealing to the commutative property of the expansion operator and (4.4) and (4.5). Again by appealing to the commutative property of an expansion operator the relation holds

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\min\{k_1,k_2\}-r}_{[x_s]\otimes[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})\neq\mathcal{S}_0$$

for any $\min\{k_1,k_2\}\geq r\geq 1$, since $\min\{k_1,k_2\}-r<k_1$ and $\min\{k_1,k_2\}-r<k_2$ and k_1,k_2 are the intensities of the dropler effects and the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ admit no dropler effect from the sources

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

respectively. This proves that $\min\{k_1,k_2\}=\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]\otimes[x_t]}(\mathcal{S})]$, the intensity of the dropler effect induced on the mixed expansion. $\square$

Proposition 4.5. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ be expansions with totients k_1 and k_2, respectively. If the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ admits no dropler effect from the source*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_t]}(\mathcal{S})$$

for $u<k_2$ and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_s]}(\mathcal{S})$$

for $v<k_1$, respectively, then

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]\otimes[x_s]}(\mathcal{S})]=\min\{k_1,k_2\}.$$

Proof. Suppose $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ are expansions with totients k_1 and k_2, respectively. Then it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_1}_{[x_s]}(\mathcal{S})=\mathcal{S}_0 \tag{4.6}$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k_2}_{[x_t]}(\mathcal{S})=\mathcal{S}_0 \tag{4.7}$$

where k_1, k_2 are the smallest such number. Appealing to the commutative property of the expansion operator, we can write by virtue of (4.6) and (4.7)

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\min\{k_1,k_2\}}_{[x_t]\otimes[x_s]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\min\{k_1,k_2\}}_{[x_s]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\min\{k_1,k_2\}}_{[x_t]}(\mathcal{S})$$
$$= \mathcal{S}_0.$$

Under the assumption that the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})$ admits no dropler effect from the source

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_t]}(\mathcal{S})$$

for $v < k_2$ and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_s]}(\mathcal{S})$$

for $u < k_1$, respectively, then it certainly follows that $\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})]$ and $\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})]$ are the smallest numbers, respectively, such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_s]}(\mathcal{S})]}_{[x_s]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_t]}(\mathcal{S}) = \mathcal{S}_0$$

and

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]}(\mathcal{S})]}_{[x_t]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_s]}(\mathcal{S}) = \mathcal{S}_0$$

so that for any $\min\{k_1,k_2\} \geq c \geq 1$ then

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\min\{k_1,k_2\}-c}_{[x_t]\otimes[x_s]}(\mathcal{S}) \neq \mathcal{S}_0$$

by exploiting the linearity of an expansion operator. This proves that $\min\{k_1,k_2\} = \Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_t]\otimes[x_s]}(\mathcal{S})]$, the totient of the mixed expansion. $\square$

5. **Destabilization of an expansion**

In this section we introduce the notion of destabilization induced by an expansion. This notion will form an essential toolbox in proving some result in this sequel. We launch more formally the following languages.

Definition 5.1. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. We say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ is said to undergo natural **destabilization** if $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{0}_{[x_i](0)}(\mathcal{S}) \neq \mathcal{S}_0$. We say it undergoes **destabilization** at stage $k \geq 1$ if $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{j}_{[x_i](0)}(\mathcal{S}) = \mathcal{S}_0$ for all $1 \leq j \leq k-1$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i](0)}(\mathcal{S}) \neq \mathcal{S}_0$. In other words, we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ admits a destabilization at stage $k \geq 1$. We say it is **strongly** destabilized if the vector

$$\overrightarrow{O(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i](0)}(\mathcal{S})}$$

has no zero entry.

Remark 5.2. Next we prove a result that tells us that destabilization should by necessity happen in an expansion. The following result confines this stage to a certain range.

Proposition 5.3. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Then the stage of destabilization k of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ satisfies the inequality*

$$0 \leq k < \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})].$$

Proof. If the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ admits a natural destabilization then the stage $k = 0$. Thus we may assume the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S})$ do not admit a natural destabilization. Let us suppose to the contrary that the stage of destabilization of some expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S}_m)$ satisfies $k \geq \Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S}_m)]$ so that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i](0)}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_m)]-1}(\mathcal{S}_m) = \mathcal{S}_0.$$

This is a contradiction, since the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_m)]-1}(\mathcal{S}_m)$$

is the residue of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S}_m)$ and thus has no direction of form $[x_i]$. □

Theorem 5.4. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Then for all directions $[x_j]$ with $1 \leq j \leq n$ every expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})$ is strongly destabilized at the stage $\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-1$*

Proof. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})$ be any expansion in an arbitrary direction $[x_j]$. Then by virtue of Definition 3.1 the expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-1}(\mathcal{S})$$

is the residue of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})$. Let us suppose to the contrary the vector

$$\overrightarrow{O(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j](0)}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-1}(\mathcal{S})}$$

has at least a zero entry. Then it follows that the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-1}(\mathcal{S})$ contains the direction $[x_j]$ and hence

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]}(\mathcal{S}) \neq \mathcal{S}_0$$

which contradicts the fact that $\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})]$ is the totient of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})$. This completes the proof. □

Remark 5.5. Next we relate the notion of the dropler effect induced by a mixed expansion on expansions in a specific direction to the notion of destabilization. We show that these two notions are somewhat related.

6. Diagonalization and sub-expansion of an expansion

In this section we introduce the notion of **diagonalization** of an expansion and sub-expansion of an expansion. This notion is mostly applied to expansions in **mixed** directions. We launch the following languages to ease our work.

Definition 6.1. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. We say the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is **diagonalizable** in the direction $[x_j]$ $(1 \leq j \leq n)$ at the spot $\mathcal{S}_r \in \mathcal{F}$ with order k with $\mathcal{S} - \mathcal{S}_r$ not a tuple of $\mathbb{R}$ if

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_r).$$

We call the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_r)$ the **diagonal** of the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ of **order** $k \geq 1$. We denote with $\mathcal{O}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_r)]$ the order of the diagonal.

Proposition 6.2. *Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Let the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_t)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_r)$ be both diagonalizable in the fixed direction $[x_i]$ at the spot $\mathcal{S}_a$ with order u and $\mathcal{S}_k$ with order v, respectively. If $u > v$ (resp. $v > u$) then*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_t + \mathcal{S}_r)$$

is also diagonalizable at the spot $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{u-v}_{[x_i]}(\mathcal{S}_a) + \mathcal{S}_k$ with order v, respectively

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{v-u}_{[x_i]}(\mathcal{S}_k) + \mathcal{S}_a$$

with order u.

Proof. Suppose the expansions $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_t)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_r)$ be both diagonalizable in the fixed direction $[x_i]$ at the spots $\mathcal{S}_a$ with order u and $\mathcal{S}_k$ with order v, respectively. Then it follows by virtue of Definition 6.1

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{u}_{[x_i]}(\mathcal{S}_a)$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{v}_{[x_i]}(\mathcal{S}_k).$$

Then by concatenating the two mixed expansion and appealing to the linearity of an expansion with $u > v$, we have

$$\begin{aligned}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_t + \mathcal{S}_r) &= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_t) + (\gamma^{-1} \circ \beta \circ \\ &\quad \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_r) \\ &= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{u}_{[x_i]}(\mathcal{S}_a) + (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{v}_{[x_i]}(\mathcal{S}_k).\blacksquare\end{aligned}$$

Under the assumption $u > v$ and appealing to the linearity of an expansion operator, we deduce

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_t + \mathcal{S}_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{v}_{[x_i]}\bigg((\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{u-v}_{[x_i]}(\mathcal{S}_a) + \mathcal{S}_k\bigg).\blacksquare$$

The claim follows by choosing the spot

$$\mathcal{S}_f = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{u-v}_{[x_i]}(\mathcal{S}_a) + \mathcal{S}_k.$$

□

Remark 6.3. Next we launch the notion of the sub-expansion of an expansion. The same notion under the **single variable theory** still carries over to this setting.

Definition 6.4. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. We say the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z)$ is a sub-expansion of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_j]}(\mathcal{S}_t)$, denoted $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_j]}(\mathcal{S}_t)$ if there exist some $0 \leq m$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+m}_{[x_j]}(\mathcal{S}_t).$$

We say the sub-expansion is proper if $m + k = l$. We denote this proper sub-expansion by $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_j]}(\mathcal{S}_t)$. On the other hand, we say the sub-expansion is **ancient** if $m + k > l$.

Remark 6.5. Next we relate the notion of the **sub-expansion** of an expansion to the notion of **Diagonalization** of a mixed expansion. We expose this profound relationship in the following proposition

Proposition 6.6. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. If the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is* ***diagonalizable*** *in the direction $[x_j]$ $(1 \leq j \leq n)$ at the spots $\mathcal{S}_t, \mathcal{S}_r \in \mathcal{F}$ such that $\mathcal{S}_t - \mathcal{S}_r$ is not a tuple of $\mathbb{R}$ with orders k_t and k_r, respectively and $k_r > k_t$. Then*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_t}_{[x_j]}(\mathcal{S}_t) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_r}_{[x_j]}(\mathcal{S}_r).$$

Proof. Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$ and let the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ be diagonalizable in the direction $[x_j]$ $(1 \leq j \leq n)$ at the spots $\mathcal{S}_t, \mathcal{S}_r \in \mathcal{F}$ such that $\mathcal{S}_t - \mathcal{S}_r$ is not a tuple of $\mathbb{R}$ with orders k_t and k_r, so that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_t}_{[x_j]}(\mathcal{S}_t) \tag{6.1}$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_r}_{[x_j]}(\mathcal{S}_r). \tag{6.2}$$

It follows by combining (6.1) and (6.2) the relation

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_t}_{[x_j]}(\mathcal{S}_t) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_r}_{[x_j]}(\mathcal{S}_r)$$

since $\mathcal{S}_t - \mathcal{S}_r$ is not a tuple of $\mathbb{R}$. Since $k_r > k_t$, it follows that there exist some $m \geq 1$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_t+m}_{[x_j]}(\mathcal{S}_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_t}_{[x_j]}(\mathcal{S}_t)$$

so that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_t}_{[x_j]}(\mathcal{S}_t) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k_r}_{[x_j]}(\mathcal{S}_r).$$

□

Remark 6.7. The converse of Proposition 6.6 may not necessarily hold because the sub-expansion may be ancient. But we can be certain the converse will hold if we allow the sub-expansion to be a proper sub-expansion. This relation is espoused in the following result as a weaker converse of the above result.

Proposition 6.8. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. If the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S}_t)$ is a diagonal with order k of the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ and*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i]}(\mathcal{S}_r) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S}_t)$$

then the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S}_r)$ is also a diagonal with order l of the same mixed expansion.

Proof. Let us suppose the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S}_t)$ is the diagonal with order k of the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$. Then it follows by virtue of Definition 6.1

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S}_t).$$

Since

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i]}(\mathcal{S}_r) < (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S}_t)$$

it follows by appealing to Definition 6.4

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i]}(\mathcal{S}_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l+m}_{[x_i]}(\mathcal{S}_t).$$

for some $0 \leq m$ with $l + m = k$ so that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i]}(\mathcal{S}_r) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_i]}(\mathcal{S}_t).$$

The result follows from this relation, since $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}(\mathcal{S}_t)$ is a diagonal with order k of the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$. $\square$

Remark 6.9. The notion of the totient, the droppler effect and the diagonalization of an expansion may seem to be quite seperate disparate notion of the theory but the following Proposition indicates a subtle connection among these three.

Proposition 6.10. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. If the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ induces a dropler effect with intensity k on the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})$ and is diagonalizable in the direction $[x_j]$ at the spot $\mathcal{S}_t$ with order s, then the expansion*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t)$$

is free with totient

$$\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t)] = k + s.$$

Proof. First let us suppose the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ induces a dropler effect with intensity k on the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})$. Then it follows by virtue of Definition 4.1

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = \mathcal{S}_0$$

with $k < \Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]$ and k is the smallest such number. Under the assumption the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})$ and is diagonalizable in the direction $[x_j]$ at the spot $\mathcal{S}_t$ with order s, it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_j]}(\mathcal{S}_t)$$

so that we have

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k+s}_{[x_j]}(\mathcal{S}_t) = \mathcal{S}_0.$$

Now let us suppose there exist some $r \leq k+s$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k+s-r}_{[x_j]}(\mathcal{S}_t) = \mathcal{S}_0$$

then it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k-r}_{[x_j]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = \mathcal{S}_0.$$

This is a contradiction, since $k = \mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]$ is the intensity of the dropler effect and is the smallest such number. It follows that

$$\begin{aligned}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t)] &= \mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})] + \mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t)]\\ &= k+s\end{aligned}$$

and the claim follows immediately. □

Lemma 6.11. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Then the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})$ for all $1 \leq i \leq l$ admits dropler effect from the source*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{i}[x_{\sigma(l)}]}(\mathcal{S}).$$

Proof. Let us consider an arbitrary expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}(\mathcal{S})$ for all $1 \leq j \leq l$. Then by appealing to the commutative property of an expansion, we can rewrite

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}&\\ \circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{\substack{i=1\\ i\neq j}}^{l}[x_{\sigma(i)}]}(\mathcal{S}).&\end{aligned}$$

It follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}(\mathcal{S})]-1}_{[x_{\sigma(j)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = \mathcal{S}_0.$$

It follows that there exists some smallest number $k \leq \Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}(\mathcal{S})] - 1 < \Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}(\mathcal{S})]$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_{\sigma(j)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = \mathcal{S}_0.$$

This proves the claim that each expansion of the form $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(j)}]}(\mathcal{S})$ for all $1 \leq j \leq l$ admits a dropler effect from the source

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}).$$

□

Remark 6.12. Next we show that the notion of diagonalization exist for mixed expansion in each direction involved in the mixed expansion. The proof is quite iterative in nature and will be employed in the sequel.

Proposition 6.13. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials belonging to the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Then the mixed expansion*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

is diagonalizable in each direction $[x_{\sigma(i)}]$ for $1 \leq i \leq l$.

Proof. Let us consider the mixed expansion

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

and let $[x_{\sigma(j)}]$ for $1 \leq j \leq l$ be our targeted direction, then by appealing to the commutative property of an expansion we have

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(j)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{\substack{i=1 \\ i \neq j}}^{l}[x_{\sigma(i)}]}(\mathcal{S}).$$

Next let us consider the residual mixed expansion

$$\begin{aligned}(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{\substack{i=1 \\ i \neq j}}^{l}[x_{\sigma(i)}]}(\mathcal{S}) &= (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{\substack{i=1 \\ i \neq j}}^{l}[x_{\sigma(i)}]} \\ &\quad \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S}).\end{aligned}$$

If there exist some tuple $\mathcal{S}_a \in \mathcal{F}$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(j)}]}(\mathcal{S}_a)$$

then we make a substitution and obtain two copies of the expansion operator $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(j)}]}$ by virtue of the commutative property of an expansion. Otherwise we choose

$$\mathcal{S}_b = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

and apply the remaining operators on it. By repeating the iteration in this manner, we will obtain

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_{\sigma(j)}]}(\mathcal{S}_t)$$

for $k \geq 1$ and for some $\mathcal{S}_t \in \mathcal{F}$. This completes the proof of the proposition. □

Remark 6.14. We are now ready to prove the inequality announced at the outset of the paper. We bring together the tools developed in the foregone section to obtain a stronger version of the inequality.

Theorem 6.15. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Then we have the inequality*

$$\begin{aligned}\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})] &< \frac{1}{l}\sum_{i=1}^{l}\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(\mathcal{S})] \\ &\quad + \frac{1}{l}\sum_{\substack{1 \leq i \leq l \\ \mathcal{S}_r \in \mathrm{Diag}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})] \\ \mathcal{S}_r = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)}} \mathcal{O}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)]\end{aligned}$$

where $\mathrm{Diag}[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]$ *is the set of all diagonals of the expansion*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}).$$

Proof. Let us consider the mixed expansion

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}).$$

Then by appealing to Proposition 6.13 then for each direction $[x_{\sigma(i)}]$ for $1\leq i\leq l$ there exist some spot $\mathcal{S}_t$ and a number $k\geq 1$ such that we can write

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_{\sigma(i)}]}(\mathcal{S}_t).$$

Again appealing to Lemma 6.11 each of the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_{\sigma(i)}]}(\mathcal{S})$ admits a dropler effect from the source

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}).$$

The upshot is that we can write for each direction $[x_{\sigma(i)}]$ the relation

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_t)]=\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]+\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_t)].$$

By appealing to Definition 4.1, we obtain further the inequality

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_t)]<\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]+\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_t)].$$

Again we see that the inequality is valid

$$\begin{aligned}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]&=\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S}_t)]\\&\leq\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_t)]\end{aligned}$$

so that we have the refined inequality

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]<\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]+\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_t)].$$

Since there are l directions under consideration, we add l such chains of the inequality and obtain

$$\begin{aligned}l\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]&<\sum_{i=1}^{l}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]\\&+\sum_{\substack{1\leq i\leq l\\ \mathcal{S}_r\in\mathrm{Diag}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]\\ \mathcal{S}_r=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)}}\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)].\end{aligned}$$

This completes the proof of the theorem. □

Corollary 6.16. *Let $\mathcal{F}:=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. If $\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)]=1$ for each*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)\in\mathrm{Diag}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]$$

then

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]<\frac{1}{l}\sum_{i=1}^{l}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]+1.$$

Proof. This is a consequence of the inequality in Theorem 6.15 by taking
$$\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)]=1$$
for each
$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)\in \mathrm{Diag}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})].$$
□

Appealing further to the energy dropler effect-intensity equation in Definition 4.1
$$\mathbb{E}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}](\mathcal{S})=\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]-\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S})]$$
and Theorem 6.15, we obtain a refined inequality
$$\begin{aligned}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})] &< \frac{1}{l}\sum_{i=1}^{l}\mathbb{E}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]\\ &+\frac{1}{l}\sum_{i=1}^{l}\mathcal{I}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]\\ &+\frac{1}{l}\sum_{\substack{1\leq i\leq l\\ \mathcal{S}_r\in \mathrm{Diag}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]\\ \mathcal{S}_r=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)}}\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)].\end{aligned}$$
We call this inequality the totient, energy, dropler effect intensity inequality.

7. **Hybrid expansions**

In this section we introduce and study the notion of **hybrid** expansions and explore some connections.

Definition 7.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. We say the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}^{k}(\mathcal{S}_a)$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{t}(\mathcal{S}_b)$ with $i\neq j$ are **hybrid** if
$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}^{k}(\mathcal{S}_a)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{t}(\mathcal{S}_b).$$
We denote this relationship with
$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}^{k}(\mathcal{S}_a)\bowtie(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{t}(\mathcal{S}_b).$$

Proposition 7.2. *Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. If the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is diagonalizable at the spot $\mathcal{S}_a$ with order k in the direction $[x_i]$ and*
$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}^{k}(\mathcal{S}_a)\bowtie(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{t}(\mathcal{S}_b)$$
then the mixed expansion is also diagonalizable at the spot $\mathcal{S}_b$ with order t in the direction $[x_j]$.

Proof. Suppose the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is diagonalizable at the spot $\mathcal{S}_a$ with order k in the direction $[x_i]$, then by appealing to Definition 6.1 we have

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S}_a).$$

Under the assumption the expansions are hybrid, it follows by appealing to Definition 7.2

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t}_{[x_j]}(\mathcal{S}_b)$$

and the claim follows immediately. □

Proposition 7.3. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomial in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_a)$ be a diagonal of the mixed expansion*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

with order $k \geq 1$. If

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S}_a) \bowtie (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t}_{[x_j]}(\mathcal{S}_b)$$

then

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t}_{[x_j]}(\mathcal{S}_b)] < \max\{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]\}_{i=1}^{l} + \max\{\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)]\}^{l}_{\substack{i=1\\ \mathcal{S}_t\in \mathrm{Diag}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]}}.$$

Proof. Let us suppose the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S}_a)$ is a diagonal of the mixed expansion

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$$

with order $k \geq 1$. Then it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S}_a).$$

Since

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S}_a) \bowtie (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t}_{[x_j]}(\mathcal{S}_b)$$

it follows that we can write

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t}_{[x_j]}(\mathcal{S}_b)$$

so that by appealing to Theorem 6.15, we obtain the inequality

$$\begin{aligned}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{t}_{[x_j]}(\mathcal{S}_b)] &= \Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]\\ &< \frac{1}{l}\sum_{i=1}^{l}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]\\ &+ \frac{1}{l}\sum_{\substack{1\leq i\leq l\\ \mathcal{S}_t\in \mathrm{Diag}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})]}}\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S}_t)]\end{aligned}$$

and the claim follows by further controlling the two sums on the right hand-side of the inequality. □

Remark 7.4. Next we express the relationship between hybrid expansion and the notion of diagonalization of a mixed expansion.

Proposition 7.5. *Let $\mathcal{F} = \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1, x_2, \ldots, x_n]$. If*

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k}(\mathcal{S}_a) \bowtie (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{t}(\mathcal{S}_b)$$

then the mixed expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{t}(\mathcal{S}_b)$ respectively $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k}(\mathcal{S}_a)$ is diagonalizable at the spots $\mathcal{S}_a$ with order $k+1$ respectively $\mathcal{S}_b$ with order $t+1$.

Proof. Let us suppose

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k}(\mathcal{S}_a) \bowtie (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{t}(\mathcal{S}_b)$$

then it follows that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k}(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{t}(\mathcal{S}_b)$$

so that by applying a copy of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}$ respectively $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}$ on both sides, we have the following relations

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{t}(\mathcal{S}_b) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k+1}(\mathcal{S}_a)$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]}^{k}(\mathcal{S}_a) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}^{t+1}(\mathcal{S}_b)$$

and the claim follows immediately from these two relations. □

8. Applications of the totient inequality

In this section we explore some applications of the theory. We obtain an inequality that will be useful for the study of the Pierce-Birkhoff conjecture. We first make the following terminologies more precise.

Definition 8.1. Let $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ be a polynomial. By the index of x_i for $1 \leq i \leq n$ relative to f_k, denoted $\mathrm{Ind}_{f_k}(x_i)$, we mean the largest power of x_i in the polynomial f_k.

Lemma 8.2. *Let $\mathcal{S} = (f_1, f_2, \ldots, f_s)$ be a tuple of polynomials such that $f_i \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ for $1 \leq i \leq s$. Then for any $1 \leq j \leq n$*

$$\Phi[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S})] = \max\{\mathrm{Ind}_{f_i}(x_j)\}_{i=1}^{s} + 1.$$

Proposition 8.3. *Let $f_1, f_2, \ldots, f_s \in \mathbb{R}[x_1, x_2, \ldots . x_n]$ be polynomials. Then there exist some $\mathcal{J} := \mathcal{J}(l) \geq 0$ such that*

$$\min\{\max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s} + 1\}_{i=1}^{l} < \frac{1}{l}\sum_{i=1}^{l} \max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s} + 2 + \mathcal{J}.$$

Proof. First let us consider the tuple $\mathcal{S}=(f_1,f_2,\ldots,f_s)$. Next, we break the proof into two special cases: The case were each of the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})$ for $1\leq i\leq l$ does not admit and the case at least one admits a dropler effect from the source
$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}).$$
In the case each of the expansions admit no dropler effect from the underlying source then by appealing to Proposition 4.4 and Lemma 8.2, we can write
$$\begin{aligned}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})] &= \min\{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]\}_{i=1}^{l}\\ &= \min\{\max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s}+1\}_{i=1}^{l}.\end{aligned}$$
Again by appealing to Lemma 8.2 we can write
$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})]=\max\{\mathrm{Ind}_{f_i}(x_j)\}_{i=1}^{s}+1$$
so that by appealing to Corollary 6.16, we can write
$$\min\{\max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s}+1\}_{i=1}^{l}<\frac{1}{l}\sum_{i=1}^{l}\max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s}+2.$$
We now turn to the case where at least one of the expansions $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})$ for $1\leq i\leq l$ admits a dropler effect. In this case we would have by appealing to Proposition 4.4
$$\begin{aligned}\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})] &= \min\{(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(i)}]}(\mathcal{S})\}_{i=1}^{l}-\mathcal{J}\\ &= \min\{\max\{\mathrm{Ind}_{f_k}(x_{\sigma(i)})\}_{k=1}^{s}+1\}_{i=1}^{l}-\mathcal{J}\end{aligned}$$
for some $\mathcal{J}:=\mathcal{J}(l)>0$. The right hand side expression is not impacted in this case. By combining the inequalities in both cases, the claim inequality follows as a consequence. □

References

1. Agama, Theophilus *A proof of Sendov's conjecture*, arXiv preprint arXiv:1907.12825, 2019.

Department of Mathematics, African Institute for mathematical sciences, Ghana.
E-mail address: Theophilus@aims.edu.gh/emperordagama@yahoo.com