Set theory INC$^{\#}$ based on intuitionistic logic with restricted modus ponens rule.

Jaykov Foukzon

[1] *Israel Institute of Technology, Haifa, Israel,*
jaykovfoukzon@gmail.com

Received: XX XX
Accepted: XX XX

Jaykov Foukzon

Israel Institute of Technology
jaykovfoukzon@gmail.com

Abstract:

In this article Russell's paradox and Cantors paradox resolved successfully using intuitionistic logic with restricted modus ponens rule.

1.Introduction

Considering only pure sets, the naive set comprehension principle says, for any condition, that there is a set containing all and only the sets satisfying this condition. In first-order logic, this can be formulated as the following schematic principle, where ϕ may be any formula in whichy does not occur freely:

$$\exists y \forall x (x \in y \leftrightarrow \phi). \tag{1.1}$$

Russell's paradox shows that the instance obtained by letting ϕ be $x \notin x$ is inconsistent in classical logic. One response to the paradox is to restrict naive set comprehension by ruling out this and other problematic instances: only for each of some special conditions is it claimed there is a set containing all and only the sets satisfying the condition. Many well known set theories can be understood as instances of this generic response,differing in how they understand special.For example,the axiom schema of separation (1.1) in Zermelo-Fraenkel set theory (ZF) restricts set comprehension to conditions which contain, as a conjunct, the condition of being a member of some given set:

$$\exists y \forall x (x \in y \leftrightarrow \phi \wedge x \in z). \tag{1.2}$$

Similarly, in Quine's New Foundations (NF) set comprehension is restricted to conditions which are stratified, where ϕ is stratified just in case there is a mapping f from individual variables to natural numbers such that for each subformula of ϕ of the form $x \in y, f(y) = f(x) + 1$ and for each subformula of ϕ of the form $x = y, f(x) = f(y)$. Both of these responses block Russell's paradox by ruling out the condition $x \notin x$. Must every restriction of naive comprehension take the form of simply ruling out certain instances? In this article, I have suggest and explore a different approach. As we have seen,standard set comprehension axioms restrict attention to some special conditions: for each of these special conditions, they provide for the existence of a set containing all and only the sets which satisfy it.

Instead of restricting the conditions one is allowed to consider, we propose restricting the way in which the sets in question satisfy a given condition: for every condition, our comprehension axiom will assert the existence of a set containing all and only the sets satisfying that condition in a special way using intuitionistic first-order logic with restricted modus ponens rule.

2.Russell's paradox resolution using intuitionistic first-order logic with restricted modus ponens rule.

2.1.Russell's paradox

The comprehension principle (1.1) for the condition $x \notin x$ gives

$$\exists \Re \forall x (x \in \Re \Leftrightarrow x \notin x). \tag{2.1}$$

Thus $\Re$ is the set whose members are exactly those sets that are not members of themselves. It follows from (2.1)

$$\Re \in \Re \Leftrightarrow \Re \notin \Re. \tag{2.2}$$

Is $\Re$ a member of itself? If it is,i.e.$\Re \in \Re$ then it must satisfy the condition of not being a member of itself and so it is not,i.e.$\Re \notin \Re$. If it is not, then it must not satisfy the condition of not being a member of itself, and so it must be a member of itself. Since by classical logic only one case or the other one must hold – either $\Re$ is a member of itself or it is not – it follows that the theory implies a contradiction.

Remark 2.1.Remind classical logic mandates that any contradiction trivializes a theory by making every sentence of the theory provable. This is because, in classical logic, the following is a theorem:

$$\text{Ex Falso Quadlibet : } A \Rightarrow (\neg A \Rightarrow B). \tag{2.3}$$

Remark 2.2.Now, virtually the only way to avoid EFQ is to give up disjunctive syllogism

also known as disjunction elimination :

$$\frac{P \vee \neg Q}{Q} \tag{2.4}$$

that is, given the usual definitions of the connectives, modus ponens! So altering basic sentential logic in this way is radical indeed – but possible.

Remark 2.3.Unfortunately, even giving up EFQ is not enough to retain a semblance of naive Cantor set theory (NC). One also has to give up the following additional theorem of basic sentential logic:

$$\text{Contraction: } (A \supset (A \supset B)) \supset (A \supset B). \tag{2.5}$$

It can then be argued that NC leads directly, not merely to an isolated contradiction, but to triviality. (For the argument that this is so,see the entry on Curry's paradox [1].

Thus it seems that the woes of NC are not confined to Russell's paradox but also include a negation-free paradox due to Curry.

Remark 2.4. Another suggestion might be to conclude that the paradox depends upon an instance of the principle of Excluded Middle, that either $\Re$ is a member of $\Re$ or it is not. This is a principle that is rejected by some non-classical approaches to logic, including intuitionism [2],[3].

Remind that in classical logic, we often discuss the truth values that a formula can take. The values are usually chosen as the members of a Boolean algebra. The meet and join operations in the Boolean algebra are identified with the $\wedge$ and $\vee$ logical connectives, so that the value of a formula of the form $A \wedge B$ is the meet of the value of A and the value of B in the Boolean algebra. Then we have the useful theorem that a formula is a valid proposition of classical logic if and only if its value is 1 for every valuation-that is, for any assignment of values to its variables. A corresponding theorem is true for intuitionistic logic, but instead of assigning each formula a value from a Boolean algebra, one uses values from an Heyting algebra, of which Boolean algebras are a special case. A formula is valid (or holds) in intuitionistic logic if and only if it receives the value of the top element for any valuation on any Heyting algebra. It can be shown that to recognize valid formulas, it is sufficient to consider a single Heyting algebra whose elements are the open subsets of the real line $\mathbb{R}$ [2]. In this algebra we have: (1) $\mathbf{Value}[\bot] = \varnothing$, (2) $\mathbf{Value}[\top] = \mathbb{R}$, (3) $\mathbf{Value}[A \wedge B] = \mathbf{Value}[A] \cap \mathbf{Value}[B]$, (4) $\mathbf{Value}[A \vee B] = \mathbf{Value}[A] \cup \mathbf{Value}[B]$,

(5) $\mathbf{Value}[A \Rightarrow B] = \mathbf{Int}\left(\mathbf{Value}[A]^{\complement} \cup \mathbf{Value}[B]\right)$, (6) $\mathbf{Value}[\neg A] = \mathbf{Int}\left(\mathbf{Value}[A]^{\complement}\right)$, where $\mathbf{Int}(X)$ is the interior of X and $X^{\complement}$ its complement.

Remark 2.5. With these assignments (1)-(6), intuitionistically valid formulas are precisely those that are assigned the value of the entire line [2]. For example, the formula $\neg(A \wedge \neg A)$ is valid,since $\mathbf{Value}[\neg(A \wedge \neg A)] = \mathbb{R}$. So the valuation of this formula is true, and indeed the formula is valid. But the law of the excluded middle, $A \vee \neg A$, can be easily shown to be invalid by using a specific value of the set of positive real numbers for A : $\mathbf{Value}[A] = \{x | x > 0\} = \mathbb{R}_{+}$. For such A one obtains $\mathbf{Value}[\neg(A \wedge \neg A)] \neq \mathbb{R}$.

We do now as follows:

Case I.Assume now that: (a) $\Re \in \Re$ holds,i.e.$\mathbf{Value}[\Re \in \Re] = \mathbb{R}$ and therefore $\Re \notin \Re$ is not holds,since $\mathbf{Value}[\Re \notin \Re] = \varnothing$.

From (2.2) it follows that (b) $\Re \in \Re \Rightarrow \Re \notin \Re$. From (a) and (b) by modus ponens rule it follows that

$$\Re \in \Re, \Re \in \Re \Rightarrow \Re \notin \Re \vdash \Re \notin \Re. \tag{2.6}$$

From (2.6) and (a) one obtains the following formula $\Re \in \Re \wedge \Re \notin \Re$. But by the Law of Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$. Thus we obtain a contradiction and therefore $\Re \in \Re$ is not holds.

Case II.Assume now that:

(a) $\Re \notin \Re$ holds,i.e.$\mathbf{Value}[\Re \notin \Re] = \mathbb{R}$ and therefore $\Re \in \Re$ is
not holds,since $\mathbf{Value}[\Re \in \Re] = \varnothing$.

From (2.2) it follows that (b) $\Re \notin \Re \Rightarrow \Re \in \Re$. From (a) and (b) by modus ponens rule it
follows that

$$\Re \notin \Re, \Re \notin \Re \Rightarrow \Re \in \Re \vdash \Re \in \Re. \quad (2.7)$$

From (2.7) and (b) one obtains the following formula $\Re \in \Re \wedge \Re \notin \Re$. But by the Law of
Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$. Thus we obtain a contradiction and
therefore $\Re \notin \Re$ is not holds. Thus bouth $\Re \in \Re$ and $\Re \notin \Re$ is not holds, but by absent
the Excluded Middle but by absent the law Excluded Middle this does not pose any
problems.

Remark 2.6. However it well known that it is possible to derive the contradiction only from
the statement (2.2) i.e., $\Re \in \Re \Leftrightarrow \Re \notin \Re$. We do so as follows:

Assume now that: $\Re \in \Re \Leftrightarrow \Re \notin \Re$ holds and therefore $\Re \in \Re \Rightarrow \Re \notin \Re$. But
we also know that $\Re \in \Re \Rightarrow \Re \in \Re$. So $\Re \in \Re \Rightarrow \Re \in \Re \wedge \Re \notin \Re$.

But by the Law of Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$

So by modus tollens we conclude that $\Re \notin \Re$.

At the same time we also know that $\Re \notin \Re \Rightarrow \Re \in \Re$, and thus by modus ponens
we conclude that $\Re \in \Re$.

So we can deduce both $\Re \in \Re$ and its negation $\Re \notin \Re$ using only intuitionistically
acceptable methods.

Remark 2.7. Another suggestion might be to conclude that the paradox depends upon
an instance of the Law of Non-contradiction, that $\neg(\Re \in \Re \wedge \Re \notin \Re)$. This is a principle
that is rejected by some non-classical approaches to logic,including paraconsistent logic
[3].Nevertheless even paraconsistent logic can not safe NC from a triviality [3].

3.Russell's paradox resolution using intuitionistic first-order logic with restricted modus ponens rule.

3.1. The intuitionistic propositional calculus $\mathbf{Pp}^{\#}$ with restricted modus ponens rule.

The first step in the metamathematical study of any part of logic or mathematics is to specify a formal language $\mathcal{L}$. For propositional or sentential logic, the standard language has denumerably many distinct proposition letters $P_0, P_1, P_2, \ldots$ and symbols $\&, \vee, \rightarrow, \neg, \perp$ for the propositional connectives "and," "or," "if ...then," and "not" respectively, with left and right parentheses (,) (sometimes written "[,]" for ease of reading). Classical logic actually needs only two connectives (since classical $\vee$ and $\rightarrow$ can be defined in terms of $\&$ and $\neg$), but the four intuitionistic connectives are independent. The classical language is thus properly contained in the intuitionistic, which is more expressive. The most important tool of metamathematics is generalized induction, a method Brouwer endorsed. The class of wff's (well-formed formulas) of the language $\mathcal{L}(\mathbf{Pp}^{\#})$ of $\mathbf{Pp}^{\#}$ is defined inductively by the rules:

(i) Each proposition letter is a (prime) formula.

(ii) If A,B are formulas so are $(A\&B),(A \vee B),(A \rightarrow B)$ and $(\neg A)$.

(iii) Nothing is a formula except as required by (i) and (ii).

(iv)The class of wff's of the language $\mathcal{L}(\mathbf{Pp}^{\#})$ we will denoted by $\mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$.

As in classical logic, $(A \leftrightarrow B)$ abbreviates $((A \rightarrow B)\&(B \rightarrow A))$.

The axioms are all formulas of the following forms:

$\mathbf{Pp}^{\#}$ 1. $A \rightarrow (B \rightarrow A)$.

$\mathbf{Pp}^{\#}$ 2. $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \ \rightarrow (A \rightarrow C))$.

$\mathbf{Pp}^{\#}$ 3. $A \rightarrow (B \rightarrow A\&B)$.

$\mathbf{Pp}^{\#}$ 4. $A\&B \rightarrow A$.

$\mathbf{Pp}^{\#}$ 5.$A\&B \rightarrow B$.

$\mathbf{Pp}^{\#}$ 6. $A \rightarrow A \vee B$.

$\mathbf{Pp}^{\#}$ 7. $B \rightarrow A \vee B$.

$\mathbf{Pp}^{\#}$ 8. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$.

$\mathbf{Pp}^{\#}$ 9. $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$.

$\mathbf{Pp}^{\#}$10. $\neg A \rightarrow (A \rightarrow B)$.

$\mathbf{Pp}^{\#}$11. $\perp \rightarrow A$.

Remark 3.1.The system of classical logic is obtained by adding any one of the following axioms: 1.$\phi \vee \neg \phi$ (Law of the excluded middle. May also be formulated as

$(\phi \rightarrow \chi) \rightarrow ((\neg \phi \rightarrow \chi) \rightarrow \chi))$

2.$\neg \neg \phi \rightarrow \phi$ (Double negation elimination)

$((\phi \rightarrow \chi) \rightarrow \phi) \rightarrow \phi$ (Peirce's law)

$(\neg \phi \rightarrow \neg \chi) \rightarrow (\chi \rightarrow \phi)$ (Law of contraposition)

The rule of inference of $\mathbf{Pp}^{\#}$ is

$\mathbf{R}^{\#}$1.RMP (Restricted Modus Ponens).

From A and $A \rightarrow B$, conclude B iff $A \notin \Delta_1$and $(A \rightarrow B) \notin \Delta_2$,where $\Delta_1,\Delta_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$

We abbraviate by $A,A \rightarrow B \vdash_{RMP} B$.

$\mathbf{R}^{\#}$2.MT (Restricted Modus Tollens)

$P \rightarrow Q,\neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta'_1$and $(P \rightarrow Q) \notin \Delta'_2$,where $\Delta'_1,\Delta'_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$

If Γ is any collection of formulas and $E_1,\ldots,E_k$ any finite sequence of formulas each of which is a member of Γ, an axiom, or an immediate consequence by RMP of two preceding formulas, then $E_1,\ldots,E_k$ is a derivation in $\mathbf{Pp}^{\#}$ of its last formula E_k from the assumptions Γ. We write $\Gamma \vdash$Pp E to denote that such a derivation exists with $E_k = E$. The following theorem is proved by induction over the definition of a derivation; its converse follows from $\mathbf{R}^{\#}$1.

Deduction Theorem. If Γ is any collection of formulas and A,B are any formulas such that $\Gamma \cup \{A\} \vdash_{RMP} B$, then also $\Gamma \vdash_{RMP} (A \rightarrow B)$.

3.2. The intuitionistic first-order predicate calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule.

The pure firstorder language $\mathcal{L}(\mathbf{Pd}^{\#})$ has individual variables $a_1,a_2,a_3,\ldots,$ and countably infinitely many distinct predicate letters $P_1(\ldots),P_2(\ldots),P_3(\ldots),\ldots$ of arity n for each $n = 0,1,2,\ldots,$ including the 0-ary proposition letters. There are two new logical symbols ∀ ("for all") and ∃ ("there exists"). The terms of the language $\mathcal{L}(\mathbf{Pd}^{\#})$ of $\mathbf{Pd}^{\#}$ are the individual variables. The well formed formulas are defined by by the rules:

(i) If $P(\ldots)$ is an n-ary predicate letter and $t_1,\ldots,t_n$ are terms then $P(t_1,\ldots,t_n)$ is a (prime) formula.

(ii) If A, B are formulas so are $(A\&B), (A \vee B), \quad (A \to B)$ and $(\neg A)$.

(iii) If A is a formula and x an individual variable, then (∀xA) and (∃xA) are formulas.

(iv) Nothing else is a formula.

(v) The class of wff's of the language $\mathcal{L}(\mathbf{Pd}^{\#})$ we will denoted by $\mathcal{F}_{\mathbf{wff}}(\mathbf{Pd}^{\#})$

We use $x,y,z,w,x_1,y_1,\ldots$ and $A,B,C,\ldots,A(x),A(x,y),\ldots$ as metavariables for variables and formulas, respectively. Anticipating applications (e.g. to arithmetic), $s,t,s_1,t_1,\ldots$ vary over terms. In omitting parentheses, $\forall x$ and $\exists x$ are treated like $\neg$. The scope of a quantifier, and free and bound occurrences of a variable in a formula, are defined as usual. A formula in which every variable is bound is a sentence or closed formula.

If x is a variable, t a term, and $A(x)$ a formula which may or may not contain x free, then $A(t)$ denotes the result of substituting an occurrence of t for each free occurrence of x in $A(x)$. The substitution is free if no free occurrence in t of any variable becomes bound in $A(t)$; in this case we say t is free for x in $A(x)$.

In addition to $\mathbf{Pp}1$ - $\mathbf{Pp}11$, $\mathbf{Pd}^{\#}$ has two new axiom schemas, where $A(x)$ may be any formula and t any term free for x in $A(x)$:

$\mathbf{Pd}^{\#}12$. $\forall xA(x) \to A(t)$.

$\mathbf{Pd}^{\#}13$. $A(t) \to \exists xA(x)$.

The rules of inference are:

$\mathbf{R}^{\#}1$.RMP (Restricted Modus Ponens).

From A and $A \to B$, conclude B iff $A \notin \Delta_1$and $(A \to B) \notin \Delta_2$,where $\Delta_1,\Delta_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pd}^{\#})$

We abbraviate $\mathbf{R}^{\#}1$ by $A,A \to B \vdash_{RMP} B$.

$\mathbf{R}^{\#}2$.MT (Restricted Modus Tollens)

$P \to Q,\neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \to Q) \notin \Delta_2'$,where $\Delta_1',\Delta_2' \subsetneqq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$

$\mathbf{R}^{\#}3$. From $C \to A(x)$ where x does not occur free in C, conclude $C \to \forall xA(x)$.

$\mathbf{R}^{\#}4$. From $A(x) \to C$ where x does not occur free in C, conclude $\exists xA(x) \to C$.

A deduction (or derivation) in $\mathbf{Pd}^{\#}$ of a formula E from a collection Γ of assumption formulas is a finite sequence of formulas, each of which is an axiom by $\mathbf{Pd}^{\#}1$ - $\mathbf{Pd}^{\#}13$, or a member of Γ, or follows immediately by $\mathbf{R}^{\#}1$, $\mathbf{R}^{\#}2$ or $\mathbf{R}^{\#}3$ from one or two formulas occurring earlier in the sequence. A proof is a deduction from no assumptions. If Γ is a collection of sentences and E a formula, the notation $\Gamma \vdash_{RMP} E$ means that a deduction of E from Γ exists. If Γ is a collection of formulas, we write $\Gamma \vdash_{RMP} E$ only if there is a deduction of E from Γ in which neither $\mathbf{R}^{\#}2$ nor $\mathbf{R}^{\#}3$ is used with respect to any variable free in Γ. With this restriction, the deduction theorem extends to $\mathbf{Pd}^{\#}$: If $\Gamma \cup \{A\} \vdash_{RMP}$ B then $\Gamma \vdash_{RMP} (A \to B)$.such that $\Gamma \cup \{A\} \vdash_{RMP} B$, then also $\Gamma \vdash_{RMP} (A \to B)$.

3.2.Russell's paradox resolution using first-order predicate calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule.

Assume now that: $\Re \in \Re \Leftrightarrow \Re \notin \Re$ holds and therefore $\Re \in \Re \Rightarrow \Re \notin \Re$.

Remark 3.2.We set now $(\Re \notin \Re) \in \Delta_1$and $(\Re \notin \Re \Rightarrow \Re \in \Re) \in \Delta_2$.

We also know that $\Re \in \Re \Rightarrow \Re \in \Re$.So $\Re \in \Re \Rightarrow \Re \in \Re \wedge \Re \notin \Re$.

But by the Law of Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$

So by canonical (unrestricted) modus tollens we conclude that $\Re \notin \Re$.

At the same time we also know that $\Re \notin \Re \Rightarrow \Re \in \Re$, and thus by restricted modus ponens we can not conclude that $\Re \in \Re$.

From $\Re \in \Re \Leftrightarrow \Re \notin \Re$ we obtain $\Re \notin \Re \Rightarrow \Re \in \Re$. We also know that $\Re \notin \Re \Rightarrow \Re \notin \Re$. So $\Re \notin \Re \Rightarrow \Re \in \Re \wedge \Re \notin \Re$. But by the Law of Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$,

So by unrestricted modus tollens we conclude that $\neg(\Re \notin \Re)$ and therefore we obtain that $\neg(\Re \notin \Re)$ & $\Re \notin \Re$. We set now $(\Re \notin \Re) \in \Delta_1'$and $(\Re \notin \Re \Rightarrow \Re \in \Re) \in \Delta_2'$, and thus by restricted modus tollens we can not conclude that $\neg(\Re \notin \Re)$.

Thus by using calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule and restricted modus tollens Russell's paradox dissipears.

4.Intuitionistic Set Theory INC$^{\#}$ based on first-order predicate calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule.

4.1.AXIOMS AND BASIC DEFINITIONS

Intuitionistic set theory $\mathbf{INC}^{\#}$ is formulated as a system of axioms in the same first order language as its classical counterpart, only based on intuitionistic logic with restricted modus ponens rule.. The language of set theory is a first-order language $\mathcal{L}^{\#}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg\,(x = y)$ and $x \notin y$ for $\neg(x\neg y)$. Individual variables $x,y,z,\ldots$ of $\mathcal{L}^{\#}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x), \exists! x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x)\ \&\ \forall y(\varphi(y) \Rightarrow x = y)]$. $\mathcal{L}^{\#}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called nonclassical sets; we shall use upper case letters $A,B,\ldots$ for such sets. For each nonclassical set $A = \{x|\varphi(x)\}$ the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x,A)]$is called the defining axioms for the nonclassical set A.

Remark 4.1.Note that (1) the formula $\forall x[x \in A \Leftrightarrow \varphi(x)]$ is not always asserts that

Remark 4.2.(1) Two nonclassical sets A,B are defined to be equal and we write $A = B$ if

$\forall x[x \in A \Leftrightarrow x \in B]$. (2) A is a subset of B, and we write $A \subseteqq B$, if $\forall x[x \in A \Rightarrow x \in B]$. (3) We also write $\mathbf{Cl.Set}(A)$ for the formula $\exists u \forall x[x \in A \Leftrightarrow x \in u]$. (4) We also write $\mathbf{NCl.Set}(A)$ for the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x,A)]$.

Remark 4.3.$\mathbf{Cl.Set}(A)$) asserts that the set A is a classical set. For any classical set u, it follows from the defining axiom for the classical set $\{x|x \in u\}$ that

$\mathbf{Cl.Set}(\{x : x \in u\})$.

We shall identify $\{x|x \in u\}$ with u, so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as $u \subset A, u \subseteqq A, u = A$, etc.

If A is a nonclassical set, we write $\exists x \in A\ \varphi(x,A)$ for $\exists x[x \in A \wedge \varphi(x,A)]$ and $\forall x \in A\varphi(x,A)$ for $\forall x[x \in A \Rightarrow \varphi(x,A)]$.

We define now the following sets:

1.$\{u_1,u_2,\ldots,u_n\} = \{x|x = u_1 \vee x = u_2 \vee \ldots \vee x = u_n\}$. 2. $\{A_1,A_2,\ldots,A_n\} =$
$= \{x|x = A_1 \vee x = A_2 \vee \ldots \vee x = A_n\}$. 3.$\cup A = \{x|\exists y[y \in A \wedge x \in y]\}$.
4.$\cap A = \{x|\forall y[y \in A \Rightarrow x \in y]\}$. 5.$A \cup B = \{x|x \in A \vee x \in B\}$.
5.$A \cap B = \{x|x \in A \wedge x \in B\}$. 6.$A - B = \{x|x \in A \wedge x \notin B\}$. 7.$u^+ = u \cup \{u\}$.
8.$\mathbf{P}(A) = \{x|x \subseteq A\}$. 9.$\{x \in A|\varphi(x,A)\} = \{x|x \in A \wedge \varphi(x,A)\}$. 10.$\mathbf{V} = \{x|: x = x\}$.

11.$\varnothing = \{x|x \neq x\}$.

The system $\mathbf{INC}^{\#}$ of set theory is based on the following axioms:

Extensionality1: $\forall u \forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$
Extensionality2: $\forall A \forall B[\forall x(x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$
Universal Set: $\mathbf{NCl.Set}(\mathbf{V})$
Empty Set: $\mathbf{Cl.Set}(\varnothing)$
Pairing1: $\forall u \forall v\ \mathbf{Cl.Set}(\{u,v\})$
Pairing2: $\forall A \forall B\ \mathbf{NCl.Set}(\{A,B\})$
Union1: $\forall u\ \mathbf{Cl.Set}(\cup u)$
Union2: $\forall A\ \mathbf{NCl.Set}(\cup A)$
Powerset1: $\forall u\ \mathbf{Cl.Set}(\mathbf{P}(u))$
Powerset2: $\forall A\ \mathbf{NCl.Set}(\mathbf{P}(A))$
Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^{+} \in a)]$
Separation1$\forall u_1 \forall u_2, \ldots \forall u_n \mathbf{Cl.Set}(\{x \in a|\varphi(x,u_1,u_2,\ldots,u_n)\})$
Separation2$\forall u_1 \forall u_2, \ldots \forall u_n \mathbf{NCl.Cl.Set}(\{x \in A|\varphi(x,A;u_1,u_2,\ldots,u_n)\})$

4.2.Cantor paradox resolution

Theorem1. (On diagonalization) If the domain $\mathbf{T}$ of function F is contained in a set A and if the values of F are subsets of A, then the set $Z = \{t \in T : t \notin F\{t)\}$ is not a value of the function F.

Proof. We have to show that for every $t \in T, F(t) \neq Z.$ From the definition of the set Z it follows that if $t \in T$, then $t \in Z \Leftrightarrow t \notin F\{t).$ Thus if $F(t) = Z$ we obtain the contradiction: $t \in Z \Leftrightarrow t \notin Z.$

Theorem 2. The set $\mathbf{P}(A)$ is not equipollent to A nor to any subset of A.

For otherwise there would exist a one-to-one function whose domain is a subset of A and whose range is the family of all subsets of A. But this contradicts Theorem 1.

Theorem 3. No two of the sets $A, \mathbf{P}(A), \mathbf{P}(\mathbf{P}(A)),$ etc. are equipollent, i.e. $\overline{\overline{A}} < \overline{\overline{\mathbf{P}(A)}} < \overline{\overline{\mathbf{P}(\mathbf{P}(A))}},$ etc.

Tus we obtain $\overline{\overline{\mathbf{V}}} < \overline{\overline{\mathbf{P}(\mathbf{V})}}$ and $\overline{\overline{\mathbf{P}(\mathbf{V})}} \leq \overline{\overline{\mathbf{V}}},$ since $\mathbf{P}(\mathbf{V}) \subset \mathbf{V},$ but this is a contradiction.

We assume now that$\exists \bar{t}[F(\bar{t}) = Z]$,i.e.there exists $\bar{t}$ such that the following statement holds

$$\bar{t} \in Z \Leftrightarrow \bar{t} \notin Z. \tag{4.1}$$

We set now (i) the following statement holds $\bar{t} \in Z,$(ii) $(\bar{t} \notin Z) \in \Delta_1$and $(\bar{t} \notin Z \Rightarrow \bar{t} \in Z) \in \Delta_2.$From (4.1) we know that $\bar{t} \in Z \Rightarrow \bar{t} \in Z.$So $\bar{t} \in Z \Rightarrow \bar{t} \in Z \wedge \bar{t} \notin Z.$

But by the Law of Non-contradiction we know that $\neg(\bar{t} \in Z \wedge \bar{t} \notin Z).$So by canonical (unrestricted) modus tollens we conclude that $\bar{t} \notin Z.$At the same time we also know that $\bar{t} \notin Z \Rightarrow \bar{t} \in Z$, and thus by restricted modus ponens we can not conclude that $\bar{t} \in Z.$

From $\bar{t} \in Z \Leftrightarrow \bar{t} \notin Z$ we obtain $\bar{t} \notin Z \Rightarrow \bar{t} \in Z.$We also know that $\bar{t} \notin Z \Rightarrow \bar{t} \notin Z.$So $\bar{t} \notin Z \Rightarrow \bar{t} \in Z \wedge \bar{t} \notin Z.$But by the Law of Non-contradiction

we know that $\neg(\bar{t} \in Z \wedge \bar{t} \notin Z)$.
So by unrestricted modus tollens we conclude that $\neg(\bar{t} \notin Z)$ and therefore we obtain that $\neg(\bar{t} \notin Z)$ & $\bar{t} \notin Z \equiv \bar{t} \in Z \wedge \bar{t} \notin Z$, but this is a contradiction. In order to avoid the contradiction, we set now $(\bar{t} \notin Z) \in \Delta_1'$ and $(\bar{t} \notin Z \Rightarrow \bar{t} \in Z) \in \Delta_2'$,
and thus by restricted modus tollens we can not conclude that $\neg(\bar{t} \notin Z) \equiv \bar{t} \in Z$.
Thus by using calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule and restricted modus tollens Cantor paradox dissipears.

REFERENCES

[1] J. Foukzon. Pure and Applied Mathematics Journal. Special Issue:Modern Combinatorial Set Theory and Large Cardinal Properties. Vol. 4, No. 1-1, 2015, pp.6-12 DOI: 10.11648/j.pamj.s.2015040101.12 arXiv:0804.4818v2 [math.LO].

[2] Sørensen, Morten Heine B; Paweł Urzyczyn (2006). Lectures on the Curry-Howard Isomorphism. Studies in Logic and the Foundations of Mathematics. Elsevier. p. 42. ISBN 978-0-444-52077-7

[3] A.I. Arruda,Remarks In Da Costa’s Paraconsistent Set Theories,Revista colombiana de matematicas.Volume 19 / 1985 / Article.