Cramer's rule for overdetermined systems of linear equations.

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Abstract:

In this article, with the help of introduced cross product in n-dimensional Euclidean space, the Jacobi identity and Cramer's rule for overdetermined system of linear equations are displayed.

1 Cross product in n-dimensional Euclidean space.

Let we have a matrix:

$$\Delta = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{1}$$

as $a_{ij} \in$ R - set of real numbers and $m \leq n$.
Let's look on determinants of size m:

$$\Delta_k = \sum (-1)^{[i_1 i_2 \dots i_m]} a_{1i_1}.a_{2i_2}\dots a_{mi_m} \tag{2}$$

where $1+2+3+\ldots+m+i_1+i_2+\ldots+i_m = k, 1 \leq i_1 \neq i_2 \neq \ldots \neq i_m \leq n$
we will call (2) deteminants of matrix (1)

Theorem: Let we have a matrix:

$$\Delta = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n+1} \\ a_{21} & a_{22} & \dots & a_{2n+1} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn+1} \end{pmatrix} \tag{3}$$

Then all determinants of Δ are:

$$\Delta_{n(n+1)}, \Delta_{n(n+1)+1}, \dots, \Delta_{n(n+2)}$$

and if:

$$\vec{c}((-1)^{n(n+2)}\Delta_{n(n+2)}, \dots, -\Delta_{n(n+1)+1}, \Delta_{n(n+1)}),$$

then

$$\begin{aligned} &\vec{c}.\vec{e}_1 = 0 \\ &\vec{c}.\vec{e}_2 = 0 \\ &\dots\dots\dots \\ &\vec{c}.\vec{e}_n = 0, \end{aligned} \tag{4}$$

where $\vec{e}_1, \dots, \vec{e}_n$ are vector-rows in Δ

Proof:

We will show that:

$$a_{11}(-1)^{n(n+2)}C_1 + a_{12}(-1)^{n(n+2)-1}C_2 + \dots - a_{1n}C_n + a_{1n}C_{n+1} = 0, \tag{5}$$

where C_i is a determinant formed by Δ when removing i column from (3). Let's view the cases for n are even and odd, then (5) is an obvious, because it represents a expansion along the first row on:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1n+1} \\ a_{11} & a_{12} & \dots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2n+1} \\ \dots & \dots & \dots & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{nn+1} \end{vmatrix}$$

and also:

$$C_1 = \Delta_{n(n+2)}, ..., C_{n+1} = \Delta_{n(n+1)}$$

Definition:
Cross product n - number n + 1-dimensional vectors :

$$\vec{e}_1(a_{11}, ..., a_{1n+1}), ..., \vec{e}_n(a_{1n}, ..., a_{nn+1}),$$

we call vector:

$$\vec{c}((-1)^{n(n+2)}\Delta_{n(n+2)}, ..., -\Delta_{n(n+1)+1}, \Delta_{n(n+1)})$$

we will denote it with:

$$\vec{c} = \vec{e}_1 \wedge \vec{e}_2 \wedge ...\vec{e}_n$$

2 Cross product – base properties.

Property 1

$$\vec{e}_1 \wedge \vec{e}_2 \wedge ...\vec{e}_n = (-1)^{[i_1 i_2 ... i_n]}\vec{e}_{i_1} \wedge \vec{e}_{i_2} \wedge ...\vec{e}_{i_n}, \tag{1}$$

where:

$i_1, i_2, ..., i_n$ – permutations on the numbers from 1 to n,
$[i_1, i_2, ..., i_n]$ – number of inversions of the respective permutation.

Proof:

The proof follows from the fact that in even inversion $[i_1 i_2 ... i_n] = 2k, k \in N$ in the coordinates on $\vec{e}_1 \wedge \vec{e}_2 \wedge ...\vec{e}_n$, which are determinants, we change even number times the rows in coordinates, i.e. they retain their value.

Property 2

$$(\lambda_1\vec{e}_1) \wedge (\lambda_2\vec{e}_2) \wedge ... \wedge (\lambda_n\vec{e}_n) = \lambda_1\lambda_2...\lambda_n(\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n), \text{ where } \lambda_1, \lambda_2, ..., \lambda_n \in R$$

Proof:

Let we have: $\lambda_1\vec{e_1}, \lambda_2\vec{e_2}, ..., \lambda_n\vec{e_n}$, then:

$$(\lambda_1\vec{e_1}) \wedge (\lambda_2\vec{e_2}) \wedge ..., \wedge(\lambda_n\vec{e_n}) = ((-1)^{n(n+2)}\Delta_{n(n+2)}, ..., \Delta_{n(n+1)}) =$$

$$((-1)^{n(n+2)}\begin{vmatrix} \lambda_1 a_{12} & \dots & \lambda_1 a_{1n+1} \\ \dots & \dots & \dots \\ \lambda_n a_{n2} & \dots & \lambda_n a_{nn+1} \end{vmatrix}, ..., \begin{vmatrix} \lambda_1 a_{11} & \dots & \lambda_1 a_{1n} \\ \dots & \dots & \dots \\ \lambda_n a_{n1} & \dots & \lambda_n a_{nn} \end{vmatrix}) =$$

$$\lambda_1\lambda_2...\lambda_n(\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n)$$

Property 3

$$\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_k \wedge (\vec{b}_1 + \vec{b}_2 + ... + \vec{b}_m) \wedge \vec{e}_{k+2} \wedge ... \wedge \vec{e}_n =$$
$$\vec{e}_1 \wedge ...\vec{e}_k \wedge \vec{b}_1 \wedge \vec{e}_{k+2} \wedge ... \wedge \vec{e}_n + ... + \vec{e}_1 \wedge ...\vec{e}_k \wedge \vec{b}_m \wedge \vec{e}_{k+2} \wedge ... \wedge \vec{e}_n$$

where:

$$\vec{e}_1, ..., \vec{e}_n \in R^{n+1}, \vec{b}_1, ..., \vec{b}_m \in R^{n+1}$$

Proof:

Again it follows from the properties of the determinants.

Property 4

Let:

$$\vec{c} = \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n, \text{ then } |\vec{c}| = 0 \Leftrightarrow \text{ when } \vec{e}_1, \vec{e}_2, ...\vec{e}_n \text{are linearly dependent.}$$

Proof:

Let:

$$\vec{e_1}(a_{11}, ..., a_{1n+1}), ..., \vec{e_n}(a_{n1}, ..., a_{nn+1}) \Rightarrow$$

$$\vec{c}((-1)^{n(n+2)}\Delta_{n(n+2)}, ..., \Delta_{n(n+1)}) \Rightarrow$$

$$|\vec{c}| = ((\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n).(\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n))^{\frac{1}{2}}$$

we consider the expression under the root:

$$\Delta_{n(n+2)} \begin{vmatrix} a_{12} & \dots & a_{1n+1} \\ \dots & \dots & \dots \\ a_{n2} & \dots & a_{nn+1} \end{vmatrix} + ... + \Delta_{n(n+1)} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & \dots & a_{1n} & a_{1n+1} \\ \dots & \dots & \dots & \\ a_{n1} & \dots & a_{nn} & a_{nn+1} \\ (-1)^{n(n+2)}\Delta_{n(n+2)} & \dots & -\Delta_{n(n+1)+1} & \Delta_{n(n+1)} \end{vmatrix} = D$$

Now, if $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n$ are linearly dependent, then D=0 $\Rightarrow |\vec{c}| = 0$. In the other direction we can start from D.

3 Scalar triple product.

Definition:

Let we have a vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n, \vec{e}_{n+1} \in R^{n+1}$. Scalar triple product on vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n, \vec{e}_{n+1}$ in this order, we call scalar multiplication on vector:

$$\vec{e} = \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n$$

with vector $\vec{e}_{n+1}$ i.e. number:

$$k = (\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n)\vec{e}_{n+1}$$

Property 5

$$(\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n)\vec{e}_{n+1} = \vec{e}_1(\vec{e}_2 \wedge ... \wedge \vec{e}_n \wedge \vec{e}_{n+1})$$

Proof:

Let:

$\vec{e}_1(a_{11}, a_{12}, ..., a_{1n+1})$
$\vec{e}_2(a_{21}, a_{22}, ..., a_{2n+1})$
....
$\vec{e}_n(a_{n1}, a_{n2}, ..., a_{nn+1})$
$\vec{e}_{n+1}(a_{n+11}, a_{n+22}, ..., a_{n+1n+1}) \Rightarrow$ $(\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n)\vec{e}_{n+1} =$

$$((-1)^{n(n+2)} \begin{vmatrix} a_{12} & \dots & a_{1n+1} \\ \dots & \dots & \dots \\ a_{n2} & \dots & a_{nn+1} \end{vmatrix} a_{n+11}, ..., \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} a_{n+1n+1}) =$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2n+1} \\ \dots & \dots & \dots & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{nn+1} \\ a_{n+11} & a_{n+22} & \dots & a_{n+1n} & a_{n+1n+1} \end{vmatrix} = \vec{e}_1(\vec{e}_2 \wedge ... \wedge \vec{e}_n \wedge \vec{e}_{n+1})$$

From *Property 5* $\Rightarrow$:

Property 6

$$(\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n)\vec{e}_{n+1} = (-1)^{[i_1 i_2 ... i_{n+1}]}(\vec{e}_{i_1} \wedge \vec{e}_{i_2} \wedge ... \wedge \vec{e}_{i_n})\vec{e}_{i_{n+1}}$$

4 Hyper-parallelepiped volume.

Property 7

$$(|\vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n|)^2 = det\, G$$

where G is Gramian matrix for: $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$

Lemma: Let we have a system of linear equations:

$$1) \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ .... \\ a_{i_1 1}x_1 + a_{i_2 2}x_2 + ... + a_{i_n}x_n = \mu \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = 0 \end{cases}$$

Then if:

$$\mu = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

system 1) is the system is compatible.

Proof:

From Theorem in "Cross product in n-dimensional Euclidean space $\Rightarrow$ that the one solution of 2):

$$2) \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ .... \\ a_{i-1\,1}x_1 + a_{i-1\,2}x_2 + ... + a_{i-1\,n}x_n = 0 \\ a_{i+1\,1}x_1 + a_{i+1\,2}x_2 + ... + a_{i+1\,n}x_n = 0 \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = 0 \end{cases}$$

is:

$$3) \begin{matrix} x_1 = (-1)^{n(n+2)}\Delta_{n(n+2)} \\ , \dots, \\ x_{n-1} = -\Delta_{n(n+1)+1}, \\ x_n = \Delta_{n(n+1)} \end{matrix}$$

$\Rightarrow$

$$x_1^{'} = (-1)^{i+1}(-1)^{n(n+1)}\Delta_{n(n+1)},$$
$$...$$
$$x_{n-1}^{'} = -(-1)^{i+n+1}\Delta_{n(n+1)+1},$$
$$x_n^{'} = (-1)^{i+n}\Delta_{n(n+1)}$$

also are solutions on 2).

Let replace the above solutions in i equation $\Rightarrow$:

$$\mu = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Proof on Property 7:

Let see the vector $\vec{k} = \vec{p} \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1}$, $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, \vec{e}_n \in R^{n+1}$, and $\vec{p} = \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1} \wedge \vec{e}_n$. $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, \vec{e}_n$ are linearly independent (id they are linearly dependent , then $|\vec{k}| = 0$), then $\vec{p} \perp \vec{e}_1, ..., \vec{p} \perp \vec{e}_{n-1}, \vec{p} \perp \vec{e}_n$, but $\vec{p} \perp \vec{k} \Rightarrow \vec{k}, \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ are linearly dependent $\Rightarrow$

$$\vec{k} = (\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1} \wedge \vec{e}_n) \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1} = \lambda_1 \vec{e}_1 + \lambda_2 \vec{e}_2 + ... + \lambda_n \vec{e}_n,$$

multiply the above equality sequentially scalar by $\vec{e}_1, ..., \vec{e}_n$, then from Properties 5 and 4, $\Rightarrow$:

$$\begin{cases} \lambda_1 \vec{e}_1^2 + \lambda_2 \vec{e}_1 \vec{e}_2 + \dots + \lambda_n \vec{e}_1 \vec{e}_n = 0 \\ \lambda_1 \vec{e}_2 \vec{e}_1 + \lambda_2 \vec{e}_2^2 + \dots + \lambda_n \vec{e}_2 \vec{e}_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots.. \\ \lambda_1 \vec{e}_{n-1} \vec{e}_1 + \lambda_2 \vec{e}_{n-1} \vec{e}_2 + \dots + \lambda_n \vec{e}_{n-1} \vec{e}_n = 0 \\ \lambda_1 \vec{e}_n \vec{e}_1 + \lambda_2 \vec{e}_n \vec{e}_2 + \dots + \lambda_n \vec{e}_n^2 = \mu \end{cases}$$

From the Lemma $\Rightarrow$

$$\mu = \begin{vmatrix} \vec{e}_1^2 & \vec{e}_1\vec{e}_2 & \dots & \vec{e}_1\vec{e}_n \\ \vec{e}_2\vec{e}_1 & \vec{e}_2^2 & \dots & \vec{e}_2\vec{e}_n \\ \dots & \dots & \dots & \dots \\ \vec{e}_n\vec{e}_1 & \vec{e}_n\vec{e}_2 & \dots & \vec{e}_n^2 \end{vmatrix}$$

But:

$$\mu = \lambda_1 \vec{e}_n \vec{e}_1 + \lambda_2 \vec{e}_n \vec{e}_2 + \dots + \lambda_n \vec{e}_n^2 = ((\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1} \wedge \vec{e}_n) \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1})\vec{e}_n$$

and from Property 5 $\Rightarrow$

$$((\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1} \wedge \vec{e}_n) \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1})\vec{e}_n = (\vec{e}_1 \wedge ... \wedge \vec{e}_n)(\vec{e}_1 \wedge ... \wedge \vec{e}_n) = (\vec{e}_1 \wedge ... \wedge \vec{e}_n)^2 \Rightarrow$$

$$(\vec{e}_1 \wedge ... \wedge \vec{e}_n)^2 = \begin{vmatrix} \vec{e}_1^2 & \vec{e}_1\vec{e}_2 & \dots & \vec{e}_1\vec{e}_n \\ \vec{e}_2\vec{e}_1 & \vec{e}_2^2 & \dots & \vec{e}_2\vec{e}_n \\ \dots & \dots & \dots & \dots \\ \vec{e}_n\vec{e}_1 & \vec{e}_n\vec{e}_2 & \dots & \vec{e}_n^2 \end{vmatrix}$$

5 Product of scalar triple products.

Property 8 Let we have vectors:

$$\vec{e}_1(a_{11}, a_{12}, \dots, a_{1n+1})$$
$$\vec{e}_2(a_{21}, a_{22}, \dots, a_{2n+1})$$
$$\dots\dots\dots\dots$$
$$\vec{e}_n(a_{n1}, a_{n2}, \dots, a_{nn+1})$$
$$\vec{e}_{n+1}(a_{n+11}, a_{n+22}, \dots, a_{n+1n+1})$$

и

$$\vec{g}_1(g_{11}, g_{12}, \dots, g_{1n+1})$$
$$\vec{g}_2(g_{21}, g_{22}, \dots, g_{2n+1})$$
$$\dots\dots\dots\dots$$
$$\vec{g}_n(g_{n1}, g_{n2}, \dots, g_{nn+1})$$
$$\vec{g}_{n+1}(g_{n+11}, g_{n+22}, \dots, g_{n+1n+1})$$

then:

$$((\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \vec{e}_n)\vec{e}_{n+1})((\vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \vec{g}_n)\vec{g}_{n+1}) = \begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_{n+1} \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_{n+1} \\ \vec{e}_{n+1}\vec{g}_1 & \vec{e}_{n+1}\vec{g}_2 & \ldots & \vec{e}_{n+1}\vec{g}_{n+1} \end{vmatrix}$$

Proof:

In proof on Property 5 we have a determinant presentation on scalar triple products:

$$(\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n)\vec{e}_{n+1} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \ldots & a_{2n} & a_{2n+1} \\ \ldots & \ldots & \ldots & \ldots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} & a_{nn+1} \\ a_{n+11} & a_{n+22} & \ldots & a_{n+1n} & a_{n+1n+1} \end{vmatrix}$$

analogous:

$$(\vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \wedge \vec{g}_n)\vec{g}_{n+1} = \begin{vmatrix} g_{11} & g_{12} & \ldots & g_{1n} & g_{1n+1} \\ g_{21} & g_{22} & \ldots & g_{2n} & g_{2n+1} \\ \ldots & \ldots & \ldots & \ldots & \\ g_{n1} & g_{n2} & \ldots & g_{nn} & g_{nn+1} \\ g_{n+11} & g_{n+22} & \ldots & g_{n+1n} & g_{n+1n+1} \end{vmatrix}$$

then:

$$((\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \vec{e}_n)\vec{e}_{n+1})((\vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \vec{g}_n)\vec{g}_{n+1}) =$$

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \ldots & a_{2n} & a_{2n+1} \\ \ldots & \ldots & \ldots & \ldots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} & a_{nn+1} \\ a_{n+11} & a_{n+22} & \ldots & a_{n+1n} & a_{n+1n+1} \end{vmatrix} \begin{vmatrix} g_{11} & g_{12} & \ldots & g_{1n} & g_{1n+1} \\ g_{21} & g_{22} & \ldots & g_{2n} & g_{2n+1} \\ \ldots & \ldots & \ldots & \ldots & \\ g_{n1} & g_{n2} & \ldots & g_{nn} & g_{nn+1} \\ g_{n+11} & g_{n+22} & \ldots & g_{n+1n} & g_{n+1n+1} \end{vmatrix} =$$

$$\begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_{n+1} \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_{n+1} \\ \vec{e}_{n+1}\vec{g}_1 & \vec{e}_{n+1}\vec{g}_2 & \ldots & \vec{e}_{n+1}\vec{g}_{n+1} \end{vmatrix}$$

6 Dot product on two cross products.

Property 9

$$(\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \vec{e}_n)(\vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \vec{g}_n) = \begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_n \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_n \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_n \end{vmatrix}$$

Proof:

Let:

$$p = \vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \vec{e}_n$$
$$q = \vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \vec{g}_n$$

then:

$$pq = qp$$
$$pq = (\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \vec{e}_n)q$$
$$qp = (\vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \vec{g}_n)p$$
$$\Rightarrow$$
$$(pq)^2 = ((\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \vec{e}_n)q)((\vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \vec{g}_n)p) =$$
$$\begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_n & \vec{e}_1 p \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_n & \vec{e}_2 p \\ \ldots & \ldots & \ldots & \ldots & \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_n & \vec{e}_n p \\ q\vec{g}_1 & q\vec{g}_2 & \ldots & q\vec{g}_n & qp \end{vmatrix} =$$
$$\begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_n & 0 \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_n & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_n & 0 \\ 0 & 0 & \ldots & 0 & qp \end{vmatrix} =$$
$$qp \begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_n \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_n \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_n \end{vmatrix}$$
$$\Rightarrow$$
$$pq = \begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_n \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_n \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_n \end{vmatrix}$$

Property 10

$$(\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n) \wedge \vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \wedge \vec{g}_{n-1} =$$
$$\begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_{n-1} & \vec{e}_1 \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_{n-1} & \vec{e}_2 \\ \ldots & \ldots & \ldots & \ldots & \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_{n-1} & \vec{e}_n \end{vmatrix}$$

Proof:

Let $\vec{k} = (\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n) \wedge \vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \wedge \vec{g}_{n-1}$ and $\vec{p} = \vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n$,
but $\vec{p} \perp \vec{e_1}, \vec{e_2}, ..., \vec{e_n}$ and $\vec{k} \perp \vec{p} \Rightarrow$
$\vec{k}, \vec{e_1}, \vec{e_2}, ..., \vec{e_n}$ are linearly dependent.
Then $\vec{k} = \lambda_1 \vec{e_1} + \lambda_2 \vec{e_2} + ... + \lambda_n \vec{e_n}$

We multiply scalarly both sides of the above equation by any non-zero vector $\vec{v} \Rightarrow$:

$$((\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n) \wedge \vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \wedge \vec{g}_{n-1})\vec{v} = (\vec{e}_1 \wedge \vec{e}_2 \wedge \ldots \wedge \vec{e}_n)(\vec{g}_1 \wedge \vec{g}_2 \wedge \ldots \wedge \vec{g}_{n-1} \wedge \vec{v}) =$$
$$\begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_{n-1} & \vec{e}_1\vec{v} \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_{n-1} & \vec{e}_2\vec{v} \\ \ldots & \ldots & \ldots & \ldots & \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_{n-1} & \vec{e}_n\vec{v} \end{vmatrix} = \lambda_1 \vec{e_1}\vec{v} + \lambda_2 \vec{e_2}\vec{v} + ... + \lambda_n \vec{e_n}\vec{v}$$

In the above equation we develop the determinant on the last column and we get:

$$(-1)^{n+1}\vec{e_1}\vec{v} \begin{vmatrix} \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_{n-1} \\ \vec{e}_3\vec{g}_1 & \vec{e}_3\vec{g}_2 & \ldots & \vec{e}_3\vec{g}_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \ldots & \vec{e}_n\vec{g}_{n-1} \end{vmatrix} + ... + \vec{e_n}\vec{v} \begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \ldots & \vec{e}_1\vec{g}_{n-1} \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \ldots & \vec{e}_2\vec{g}_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ \vec{e}_{n-1}\vec{g}_1 & \vec{e}_{n-1}\vec{g}_2 & \ldots & \vec{e}_{n-1}\vec{g}_{n-1} \end{vmatrix} = \lambda_1 \vec{e_1}\vec{v} + \lambda_2 \vec{e_2}\vec{v} + ... + \lambda_n \vec{e_n}\vec{v}$$

$\vec{v}$ is an arbitrary vector, from here we can equate both sides of the above equality and we get :

$$\lambda_1 = (-1)^{n-1}\begin{vmatrix} \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \dots & \vec{e}_2\vec{g}_{n-1} \\ \vec{e}_3\vec{g}_1 & \vec{e}_3\vec{g}_2 & \dots & \vec{e}_3\vec{g}_{n-1} \\ \dots & \dots & \dots & \dots \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \dots & \vec{e}_n\vec{g}_{n-1} \end{vmatrix}, ..., \lambda_n = \begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \dots & \vec{e}_1\vec{g}_{n-1} \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \dots & \vec{e}_2\vec{g}_{n-1} \\ \dots & \dots & \dots & \dots \\ \vec{e}_{n-1}\vec{g}_1 & \vec{e}_{n-1}\vec{g}_2 & \dots & \vec{e}_{n-1}\vec{g}_{n-1} \end{vmatrix}$$

We substitute $\lambda_1, \lambda_2, ..., \lambda_n$ in $\lambda_1\vec{e_1} + \lambda_2\vec{e_2} + ... + \lambda_n\vec{e_n} \Rightarrow$

$$(\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n) \wedge \vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge \vec{g}_{n-1} = \begin{vmatrix} \vec{e}_1\vec{g}_1 & \vec{e}_1\vec{g}_2 & \dots & \vec{e}_1\vec{g}_{n-1} & \vec{e}_1 \\ \vec{e}_2\vec{g}_1 & \vec{e}_2\vec{g}_2 & \dots & \vec{e}_2\vec{g}_{n-1} & \vec{e}_2 \\ \dots & \dots & \dots & \dots & \\ \vec{e}_n\vec{g}_1 & \vec{e}_n\vec{g}_2 & \dots & \vec{e}_n\vec{g}_{n-1} & \vec{e}_n \end{vmatrix}$$

7 Jacobi identity

$$\begin{aligned} &(\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n) \wedge \vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge \vec{g}_{n-1} = \\ &(\vec{e}_1 \wedge \vec{g}_1 \wedge \dots \wedge \vec{g}_{n-1}) \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge \dots \wedge \vec{e}_n + ...+ \\ &+ (\vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge e_i \wedge \dots \wedge \vec{g}_{n-1}) \wedge \vec{e}_1 \wedge \dots \wedge \vec{e}_{i-1} \wedge \vec{e}_{i+1} \wedge ... \wedge \vec{e}_n + \dots + \\ &+ (\vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge \vec{g}_{n-1} \wedge \vec{e}_n) \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \vec{e}_{n-1} \end{aligned} \tag{1}$$

as $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n \in R_n,\ \vec{g}_1, \vec{g}_2, ..., \vec{g}_{n-1} \in R_{n-1}$

Proof:

From *Property 9* and *Property 10* $\Rightarrow$:

$$\begin{aligned} &(\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n) \wedge \vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge \vec{g}_{n-1} = \\ &(-1)^{1+n}\vec{e_1}(\vec{e}_2 \wedge \vec{e}_3 \wedge \dots \wedge \vec{e}_n)(\vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge \vec{g}_{n-1}) + ...+ \\ &(-1)^{i+n}\vec{e_i}(\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e_{i-1}} \wedge \vec{e_{i+1}} \wedge +... + \wedge\vec{e}_n)(\vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge \vec{g}_{n-1}) + ...+ \\ &(-1)^{1+n}\vec{e_n}(\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_{n-1})(\vec{g}_1 \wedge \vec{g}_2 \wedge \dots \wedge \vec{g}_{n-1}) \end{aligned} \tag{2}$$

We will use induction . Let n=2, then:

$$(\vec{e}_1 \wedge \vec{e}_2) \wedge \vec{g}_1 = (\vec{e}_1 \wedge \vec{g}_1) \wedge \vec{e}_2 + (\vec{g}_1 \wedge \vec{e}_2) \wedge \vec{e}_1$$

and this is Jacobi identity.
Let n=3.
We must proof that:

$$(\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3) \wedge \vec{g}_1 \wedge \vec{g}_2 = (\vec{e}_1 \wedge \vec{g}_1 \wedge \vec{g}_2) \wedge \vec{e}_2 \wedge \vec{e}_3 + (\vec{g}_1 \wedge \vec{e}_2 \wedge \vec{g}_2) \wedge \vec{e}_1 \wedge \vec{e}_3 + (\vec{g}_1 \wedge \vec{g}_2 \wedge \vec{e}_3) \wedge \vec{e}_1 \wedge \vec{e}_2 \tag{3}$$

From (2) $\Rightarrow$:

$$\begin{aligned} &\vec{e}_1(\vec{e}_2 \wedge \vec{e}_3)(\vec{g}_1 \wedge \vec{g}_2) - \vec{e}_2(\vec{e}_1 \wedge \vec{e}_3)(\vec{g}_1 \wedge \vec{g}_2) + \vec{e}_3(\vec{e}_1 \wedge \vec{e}_2)(\vec{g}_1 \wedge \vec{g}_2) = \\ &= \vec{e}_1(\vec{g}_1 \wedge \vec{g}_2)(\vec{e}_2 \wedge \vec{e}_3) - \vec{g}_1(\vec{e}_1 \wedge \vec{g}_2)(\vec{e}_2 \wedge \vec{e}_3) + \vec{g}_2(\vec{e}_1 \wedge \vec{g}_1)(\vec{e}_2 \wedge \vec{e}_3) + \\ &+ \vec{g}_1(\vec{e}_2 \wedge \vec{g}_2)(\vec{e}_1 \wedge \vec{e}_3) - \vec{e}_2(\vec{g}_1 \wedge \vec{g}_2)(\vec{e}_1 \wedge \vec{e}_3) + \vec{g}_2(\vec{g}_1 \wedge \vec{e}_2)(\vec{e}_1 \wedge \vec{e}_3) + \\ &+ \vec{g}_1(\vec{g}_2 \wedge \vec{e}_3)(\vec{e}_1 \wedge \vec{e}_2) - \vec{g}_2(\vec{g}_1 \wedge \vec{e}_3)(\vec{e}_1 \wedge \vec{e}_2) + \vec{e}_3(\vec{g}_1 \wedge \vec{g}_2)(\vec{e}_1 \wedge \vec{e}_2) \end{aligned}$$

We group the coefficients before $\vec{g}_1$ and $\vec{g}_2 \Rightarrow$:
For $\vec{g}_1$:

$$\begin{aligned} &(\vec{e}_2 \wedge \vec{g}_2)(\vec{e}_1 \wedge \vec{e}_3) - (\vec{e}_1 \wedge \vec{g}_2)(\vec{e}_2 \wedge \vec{e}_3) + (\vec{e}_1 \wedge \vec{e}_2)(\vec{g}_2 \wedge \vec{e}_3) = \\ &= ((\vec{e}_2 \wedge \vec{g}_2) \wedge \vec{e}_1 - (\vec{e}_1 \wedge \vec{g}_2) \wedge \vec{e}_2 + (\vec{e}_1 \wedge \vec{e}_2) \wedge \vec{g}_2)\vec{e}_3 = \vec{o}\vec{e}_3 = 0 \end{aligned}$$

Analogous for $\vec{g}_2$ and $\Rightarrow$(3).

Let (1) is true for n-1. We reveal (1) using (2) and in left part on (1).
Let $\vec{g}_i, 1 \leq i \leq n-1 \Rightarrow$

$(-1)^{n+i-1}\vec{g}_i(\vec{e}_1 \wedge \vec{g}_1 \wedge ... \wedge \vec{g}_{i-1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1})(\vec{e}_2 \wedge ... \wedge \vec{e}_n) + ...+$
$(-1)^{n+i-1}\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{e}_i \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1})(\vec{e}_1 \wedge ... \wedge \vec{e}_{i-1} \wedge \vec{e}_{i-1} \wedge ... \wedge \vec{e}_n)+$
$(-1)^{n+i}\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{e}_{i+1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1})(\vec{e}_1 \wedge ... \wedge \vec{e}_i \wedge \vec{e}_{i+2} \wedge ... \wedge \vec{e}_n) + ...+$
$(-1)^{n+i}\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{g}_{i-1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{e}_n)(\vec{e}_1 \wedge ... \wedge \vec{e}_{n-2}) = 0 \Leftrightarrow$

$(-1)^{n+i-1}\vec{g}_i(\vec{e}_1 \wedge \vec{g}_1 \wedge ... \wedge \vec{g}_{i-1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1}) \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n + ...+$
$(-1)^{n+i-1}\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{e}_i \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1}) \wedge \vec{e}_1 \wedge ... \wedge \vec{e}_{i-1} \wedge \vec{e}_{i-1} \wedge ... \wedge \vec{e}_n+$
$(-1)^{n+i}\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{e}_{i+1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1}) \wedge \vec{e}_1 \wedge ... \wedge \vec{e}_i \wedge \vec{e}_{i+2} \wedge ... \wedge \vec{e}_n + ...+$
$(-1)^{n+i}\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{g}_{i-1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{e}_n) \wedge \vec{e}_1 \wedge ... \wedge \vec{e}_{n-2} = 0 \Leftrightarrow$

$\vec{g}_i(\vec{e}_1 \wedge \vec{g}_1 \wedge ... \wedge \vec{g}_{i-1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1}) \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n + ...+$
$\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{e}_i \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1}) \wedge \vec{e}_1 \wedge ... \wedge \vec{e}_{i-1} \wedge \vec{e}_{i-1} \wedge ... \wedge \vec{e}_n+$
$\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{e}_{i+1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{g}_{n-1}) \wedge \vec{e}_1 \wedge ... \wedge \vec{e}_i \wedge \vec{e}_{i+2} \wedge ... \wedge \vec{e}_n + ...+$
$\vec{g}_i(\vec{g}_1 \wedge ... \wedge \vec{g}_{i-1} \wedge \vec{g}_{i+1} \wedge ... \wedge \vec{e}_n) \wedge \vec{e}_1 \wedge ... \wedge \vec{e}_{n-2} = \vec{g}_i\vec{o} = \vec{o}$

8 Cramer's rule for overdetermined system of linear equations.

Let we have a linearly independent vectors: $\vec{e}, \vec{e}_1, ..., \vec{e}_n \in R_{n+1} \Rightarrow \exists \lambda_1, ..., \lambda_n \in R$:

$$\vec{e} = \lambda_1\vec{e}_1 + \lambda_2\vec{e}_2 + ... + \lambda_n\vec{e}_n \tag{1}$$

We multiply 1) right sequentially with:

$$\begin{array}{l} \wedge\, \vec{e}_2 \wedge \vec{e}_3 \wedge ... \wedge \vec{e}_n \\ \wedge\, \vec{e}_1 \wedge \vec{e}_3 \wedge ... \wedge \vec{e}_n \\ \wedge\, \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_n \\ \cdots \\ \wedge\, \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_{n-2} \wedge \vec{e}_n \\ \wedge\, \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_{n-1} \end{array}$$

$\Rightarrow$:

$$\left\{ \begin{array}{l} \vec{e} \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge ... \wedge \vec{e}_n = \lambda_1\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge ... \wedge \vec{e}_n \\ \vec{e} \wedge \vec{e}_1 \wedge \vec{e}_3 \wedge ... \wedge \vec{e}_n = \lambda_2\vec{e}_2 \wedge \vec{e}_1 \wedge \vec{e}_3 \wedge ... \wedge \vec{e}_n \\ \cdots \\ \vec{e} \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_{n-2} \wedge \vec{e}_n = \lambda_{n-1}\vec{e}_{n-1} \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_{n-2} \wedge \vec{e}_n \\ \vec{e} \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_{n-1} = \lambda_n\vec{e}_n \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge ... \wedge \vec{e}_{n-1} \end{array} \right. \tag{2}$$

We multiply scalar (2): the first equation with vector from on his right and so on. and if we denote:

$$\begin{array}{l} \vec{p}_i = \vec{e} \wedge \vec{e}_1 \wedge \cdots \wedge \vec{e}_{i-1} \wedge \vec{e}_{i+1} \wedge \cdots \wedge \vec{e}_n \\ \vec{q}_i = \vec{e}_i \wedge \vec{e}_1 \wedge \cdots \wedge \vec{e}_{i-1} \wedge \vec{e}_{i+1} \wedge \cdots \wedge \vec{e}_n \\ 1 \le i \le n \end{array} \tag{3}$$

from Property 7 $\Rightarrow$:

$$\begin{array}{l} \lambda_1 = sgn(\vec{p}_1\vec{q}_1)\left|\dfrac{\vec{e} \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n}{\vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n}\right| = sgn(\vec{p}_1\vec{q}_1)\sqrt{\dfrac{detG(\vec{p}_1)}{detG(\vec{q}_1)}} \\ \lambda_2 = sgn(\vec{p}_2\vec{q}_2)\left|\dfrac{\vec{e} \wedge \vec{e}_1 \wedge \cdots \wedge \vec{e}_n}{\vec{e}_2 \wedge \vec{e}_1 \wedge \vec{e}_3 \wedge \cdots \wedge \vec{e}_n}\right| = sgn(\vec{p}_2\vec{q}_2)\sqrt{\dfrac{detG(\vec{p}_2)}{detG(\vec{q}_2)}} \\ \cdots \\ \lambda_{n-1} = sgn(\vec{p}_{n-1}\vec{q}_{n-1})\left|\dfrac{\vec{e} \wedge \vec{e}_1 \wedge \cdots \wedge \vec{e}_{n-2} \wedge \vec{e}_n}{\vec{e}_{n-1} \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_{n-2} \wedge \vec{e}_n}\right| = sgn(\vec{p}_{n-1}\vec{q}_{n-1})\sqrt{\dfrac{detG(\vec{p}_{n-1})}{detG(\vec{q}_{n-1})}} \\ \lambda_n = sgn(\vec{p}_n\vec{q}_n)\left|\dfrac{\vec{e} \wedge \vec{e}_1 \wedge \cdots \wedge \vec{e}_{n-1}}{\vec{e}_n \wedge \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_{n-1}}\right| = sgn(\vec{p}_n\vec{q}_n)\sqrt{\dfrac{detG(\vec{p}_n)}{detG(\vec{q}_n)}} \end{array} \tag{4}$$

References

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