The *Fog* Covering Cantor's *Paradise*[1]: Some Paradoxes on Infinity and Continuum

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Abstract. We challenge Georg Cantor's theory about infinity. By attacking the concept of "countable/uncountable" and diagonal argument, we reveal the uncertainty, which is obscured by the lack of clarity. The problem arises from the basic understandings of infinity and continuum. We perform many thought experiments to refute current standard views. The results support the opinion that no potential infinity leads to an actual infinity, nor is there any continuum composed of indivisibles statically, nor is Cantor's theory consistent in itself.

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Opening Words: Some Flowers in the *Fog*

Is this paper accessible and interesting? Let's observe the set of natural numbers and a sequence of variations on it: $\{1, 2, 3, \dots\}$, $\{2, 3, 4, \dots\}$, $\{3, 4, 5, \dots\}$, $\dots$. Each variation set is a little bit different from its predecessor, and each is thought to be predictable whilst the general trend obvious. Well, what set is the target aimed at by this evolution? Simple as the question may be, the answer deserves careful thought. Our illustration is in **Thought Experiment 5.3**. Next, consider another perhaps simple question. If infinitely many glasses of salt water with the same concentration add together, they form an infinitely large salt lake. What is the concentration of the lake water? Why is that a question? A shock is waiting for you in **Thought Experiment 5.2**. Now, turn to one more such amazing question. Every point in $[-1, 1]$ has a position. Which one takes the mean position (or, is of the arithmetic mean value)? Certainly point 0 does. Well, think about $(-1, 1]$. Which point this time? We face such an issue in **Thought Experiment 4.2**. Hopefully, by now, we have answered the question at the beginning of this section — there is a simple, familiar and yet strange world ahead for all to rediscover and enjoy.

1. Introduction

Following Georg Cantor's key ideas in set theory[1, Ch.41 sec.7–8], we meet the puzzle of logic uncertainty while approaching actual infinity. Surprisingly though, the problem has long been being avoided, or at least insufficiently noticed, and nowadays it's just too easy (but no use) to blame human intuition for all this mess. We design a series of concise thought experiments first to highlight the logic disaster, then to expose the cause — a misconception of potential infinity, a half-baked myth of actual infinity and, quite related to the former two, a misperception of continuum.

[1] We borrow the words "*fog*" and "*paradise*" from Hermann Weyl and David Hilbert respectively. Weyl regards Cantor's hierarchy of transfinite cardinals as "*a fog on a fog*"[1, p.1003]. Hilbert comments[1, p.1003], "*No one shall expel us from the paradise which Cantor created for us.*"

Outline of the paper. In **Section 2**, we give a counterexample against Cantor's diagonal argument, provided all rational numbers are countable as in Cantor's theory. Next, in **Section 3**, to push the chaos to a new high, we present a plausible method for putting all real numbers to a list. Then, to explore the cause of the paradoxes we turn to some basic and primitive issues. In **Section 4** and **5** we reexamine the prevailing opinion on continuum and infinity. After that, in **Section 6** we discuss at the basic logic level the origin of all the confusion.

2. Facing the *Fog*: Countability and Diagonal Argument

Cantor's idea of countability/uncountability is the starting point of his theory of transfinite numbers. Correspondingly, his diagonal argument is the most important method in set theory and mathematical logic. But we do not think his reasoning is clear. We run a thought experiment with the rational numbers in (0, 1), all of which, in Cantor's view, can be listed as an infinite sequence (with no duplicates; the same hereinafter). Given such a sequence $\langle q_1, q_2, q_3, \ldots \rangle$, we rewrite each of its elements as a repeating decimal and denote the result by $\langle q_{01}, q_{02}, q_{03}, \ldots \rangle$.

Thought Experiment 2.1 A "Lost" Rational Number

Let q equal (and be with the same appearance as) an arbitrarily given element of the sequence. Go through all decimal digits of q in natural order. For the nth decimal digit of q, swap positions of the nth element of the current sequence and the nearest following element that differs from q at the nth decimal digit. (For each n, the swap is executed exactly one time.) As explained in **Appendix A**, we can always find an element for each two-position swap. We exhibit the rule in the form of step-by-step operations:

Step 1: In $\langle q_{01}, q_{02}, q_{03}, \ldots \rangle$, search from q_{02} successively until find an element differing from q at the 1st decimal digit. Swap positions of the search result and q_{01}; and denote the rearranged sequence by $\langle q_{11}, q_{12}, q_{13}, \ldots \rangle$.

Step 2: In $\langle q_{11}, q_{12}, q_{13}, \ldots \rangle$, search from q_{13} until find an element differing from q at the 2nd decimal digit. Swap positions of the search result and q_{12}; and denote the rearranged sequence by $\langle q_{21}, q_{22}, q_{23}, \ldots \rangle$.

Step 3: $\ldots$, search from q_{24} $\ldots$ 3rd $\ldots$. Swap $\ldots$ and q_{23}; and $\ldots$.

Go on throughout the decimal part of q.

The stage result sequences are distinct from one another, for the results of nth and $(n+k)$th steps (n, $k = 1, 2, 3, \ldots$) differ at least at the $(n+1)$st element.

Write all the stage result sequences as *list B*:

$\langle \boldsymbol{q_{11}}, q_{12}, q_{13}, \ldots \rangle$,

$\langle q_{21}, \boldsymbol{q_{22}}, q_{23}, \ldots \rangle$,

$\langle q_{31}, q_{32}, \boldsymbol{q_{33}}, \ldots \rangle$,

$\ldots$

Apply the diagonal argument to *list B* on two levels separately, beginning with the outer level. The diagonal of the list yields a sequence, $\langle q_{11}, q_{22}, q_{33}, \ldots \rangle$. Is this a rearranged sequence of $\langle q_{01}, q_{02}, q_{03}, \ldots \rangle$? If the answer is yes, $\{q_{11}, q_{22}, q_{33}, \ldots\}$ contains q. But, observing the list of $q_{11}, q_{22}, q_{33}, \ldots$ on the level of decimal digits and applying the diagonal argument again shows q is not in the sequence, for q_{nn} differs from q at the nth decimal digit. If the answer is no, $\{q_{11}, q_{22}, q_{33}, \ldots\}$ is a proper subset of $\{q_{01}, q_{02}, q_{03}, \ldots\}$. That means the so called diagonal, $q_{11}, q_{22}, q_{33}, \ldots$, covers *list B* vertically rather than both horizontally and vertically. Since the number of the lines of *list B* is exactly that of all the decimal digits of q, and the number of the columns is exactly that of all the rational numbers in (0, 1), a clear problem presents itself: How can Cantor's diagonal exhausts the proposed list of real numbers in his proof both horizontally and vertically (with the columns of the list there matching the lines of our *list B*, and the lines of his list containing but not limited to all the content of our columns)?

Aside: *The sequences* $\langle q_{01}, q_{02}, q_{03}, \ldots\rangle$, $\langle q_{11}, q_{12}, q_{13}, \ldots\rangle$, $\langle q_{21}, q_{22}, q_{23}, \ldots\rangle, \ldots$ *are based on the same set and each two of them differ at finitely many positions. Although they are mutually rearranged sequences, each of them, if exists, could stand on its own (in other words, as for existence, none of them is prior to another).*

While dealing with finite many items, Cantor's diagonal argument is clear. As for infinitely many items, his argument is also hoped to be rigorous, but that is not the case before us. And once logical uncertainty has crept in, everyone may "prove" whatever at will.

3. Under Cover of the *Fog*: Listing all Real Numbers

According to Cantor's theory, all real numbers or even the reals in a bounded interval cannot be listed as a sequence. We are going to list them to reveal more confusion.

We just create a sequence for all the reals in $[0, C)$, $C \in \mathbb{R}^+$, and leave the rest job to some well-known solutions. As there is a mapping between real numbers and points on straight line, we may mix numbers with points. And it is also the same for intervals and line segments (including open or half-open segments; the same hereinafter).

Preparation: Bend $[0, C)$ into a circle (henceforth, we straighten or bend it on demand without declaring; and to ensure clarity, we do not use any chord). Denote point 0 by P_0. Suppose that there is a light beam emitted from P_0, aimed at and reflected by another point, which is denoted by P_1, on the circumference. If the line segments $[0, P_1)$ and $[0, C)$ are incommensurable, and P_0 is the only point that has no reflective feature, then the light beam produces a series of reflection points, $P_1, P_2, P_3, \ldots$, and does not make a repeat or return to P_0 within finite reflection steps. We refer to P_0 and all the reflection points as the *bright points*.

Lemma 3.1 *Bright points* are dense on the circumference.

(That is to say, for any two points on the circumference, however close they may be to each other, there is a *bright point* in between. For the sake of self-containedness of this paper, we include a proof in **Appendix B**.)

Note: *We use the character of ideal light beam for offering a logic clue to thread all the relevant points. However, from another view angle, whether or not a point is a* bright point *is determined by itself (its specific position) once C, P_0 and P_1 given. A point $x \in [0, C)$ is a* bright point *if it makes an indefinite equation ($x + mC = nP_1$, where m, n are unknown non-negative integers) solvable. (And for a positive n, point x turns out to be the nth reflection point.)*

A natural question is: Are all the points on the circumference *bright*?

Thought Experiment 3.1 A Sequence of All the Real Numbers in $[0,\ C)$

If the answer to the above question is yes, the sequence $P_0,\ P_1,\ P_2,\ P_3, \ldots$ contains all the real numbers in $[0,\ C)$. If the answer is no, we may classify all the points into two sets — set B for all the *bright points* and set D for the rest, which we call the *dark points*. Naturally, we have a sequence of all *bright points*; and we aim to add all *dark points* to it.

For visualization, attach $[C,\ 2C]$ to $[0,\ C)$ as a "handle". Hold the "handle", cut $[0,\ 2C]$ at all *dark points* simultaneously while keeping each of the *dark points* as the right-hand endpoint of its own fragment. Then throw away the "handle" together with the perhaps existing remainder of $[0,\ C)$.

Note: *The course of disintegration can be easily described in formal logic. As mentioned above, set* B *is* $\{P_0,\ P_1,\ P_2,\ P_3, \ldots\}$, *and set* D *is* $\{x : 0 \leqslant x < C\} \setminus B$. *We denote* $\{x : 0 \leqslant x \leqslant 2C\}$ *by* A, *and* $D \bigcup \{2C\}$ *by* D'. *The elements of* D' *determine a partition of* A. *For* $d_j \in D'$, *the equivalence class* $[d_j]$ *is* $\{x \in [0,\ d_j] : \forall d_i \in D(d_i < d_j \rightarrow d_i < x)\}$. *Then ignore the equivalence class [2C], just consider the others ....*

Now, we focus on the fragments, each of which contains exact one *dark point*. According to **Lemma 3.1** , any two *dark points* are separated from each other by some *bright points*. Considering also that the leftmost point of $[0,\ C)$ is a *bright* one, (thus, intuitively, within the whole segment each *dark point* may look to the right or left like a human being, and the scene coming into its sight is the same as if it were located somewhere in the middle of an otherwise wholly *bright* segment,) none of the fragments can be free of *bright point* (since, otherwise, if the unique point of an exceptional fragment looks to the left while located in the whole segment, the scene coming into sight would be the same as if it looked to the left while located in the middle of a wholly *dark* segment. Why such an awkward interpretation? We come back to this issue later in **Thought Experiment 4.4**). In this sense, the number of the fragments, which equals the number of *dark points*, is not greater than that of *bright points*. Now that all the *bright points* are in $\{P_0,\ P_1,\ P_2,\ P_3, \ldots\}$, according to the *well-ordering principle* (which states that *every nonempty subset of the natural numbers has a least member*), in each fragment there is a *bright point* with the least index number. Pair up each *dark point* with such a *bright* one of the same fragment. Then execute a series of insertions based on $\langle P_0,\ P_1,\ P_2,\ P_3, \ldots\rangle$ — for each *bright point*, if it is an unpaired one just skip it, else insert right after it its *dark* partner. The target sequence appears.

Up to this point everything is seemingly normal. But the assumption of the existence of the *dark point* still leads to confusion. If there is a *dark point*, then we can find an endless chain of *dark points* by reasoning backwards repeatedly — tracing the *dark point* back iteratively along the circumference with the same step-length as the reflection of the light beam (but in the reverse direction) to other *dark points*. Consequently the number of the *dark points* could not be smaller than that of the *bright* ones, and in this case the structure of each fragment is unimaginable. (Moreover, an even annoying question is: How many isolated *dark* chains are there? We choose to walk around this muddy place.) Needless to say, the *dark points*, if exist, are also dense on the circumference. Therefore, in each fragment the *dark point* would be the only point, or another *dark point* would thereupon exist in the same fragment. This means no *bright point* could exist, but it is absurd. Therefore, the assumption of the existence of *dark point* is false.

A much more direct way to show the confliction led to by *dark points* is adopting another cutting rule — cutting $[0, C)$ at each *bright* point while keeping the *bright point* as the left-hand endpoint of its fragment. Then similar reasoning as above indicates that there is no room for any *dark point.*

We are familiar with the saying that a number system, for example the rational numbers, can be dense on real line without completely filling the line. That means "dense" does not need to be "without any gap". But is that clear in logic?

Thought Experiment 3.2 The Puzzling "Dense"

The property of "dense" implies that between any two members, however close they may be, there is always a third one. The rational number system has this property. Let's observe an infinite sequence of nested rational pairs: $\langle -1, 1\rangle$, $\langle -1/2, 1/2\rangle$, $\langle -1/4, 1/4\rangle$, $\langle -1/8, 1/8\rangle, \ldots$. If a third number is embraced by an inner pair, it is also embraced by every outer one. Considering the property of "dense", we naturally expect a common rational number that is between the two partners of each pair; and sure enough, 0 is such a number. However, a sticky fact is that 0 is the only one. Does that mean if it is taken away, "dense" would no longer hold? If the answer is yes, then the existence of 0 as a rational is a must for the denseness property. But what if the thoroughly wrapped number is not a rational (actually, there are such examples for some other sequences of rational pairs)? And if the answer is no, then to ensure that "dense" still holds, should there be another rational that is wrapped as deeply as 0 is? Neither of the answers is doubtless.

We are bothered by the question: Do the rational numbers (or, every "dense" collection of numbers with no bound) fill up the whole real line? For example, we may design a strictly increasing rational sequence $p_1, p_2, p_3, \ldots$ and a strictly decreasing rational sequence $q_1, q_2, q_3, \ldots$, and they have the same limit, say $\sqrt{2}$. (See a numerical example in the **Appendix C**). Now we have an infinite series of nested pairs: $\langle p_1, q_1\rangle$, $\langle p_2, q_2\rangle$, $\langle p_3, q_3\rangle, \ldots$; apparently $\sqrt{2}$ is the only number to penetrate into all the pairs. If we take away $\sqrt{2}$ (for it is an irrational number, which should have no effect on the denseness property of the rational system), then, does the property of "dense" still hold for the rational system? If the answer is yes, should we take on the hopeless task of finding a rational that is located in the same depth as $\sqrt{2}$ is? If the answer is no, should we accept $\sqrt{2}$ as a rational number?

4. Is a Continuum Composed of Indivisibles Statically?

In comparing "dense" with "without any gap", we get curious about the relationship between line segment (as continuum) and point (as indivisible). Now we turn to thinking about the prevailing opinion — a continuum is a collection of (stationary) indivisibles. Is that clear in logic? Why have there always been some people standing on the opposite side since ancient times?

In mathematics a point is thought to have no extension in any direction. As questioned by many, how can zero-magnitude points accumulate to a positive-magnitude segment?

An intuitive example adding to the doubt is: The segment [0, 1] can cover [0, 1), but the latter cannot cover up the former. On the ground of this difference, the size of one point does matter and cannot be zero. On the other hand, if someone tries to assign a nonzero value to the size, which value could work? Why not, say, half the value? Well, to suspend the debate over the detail value, let us just say one point is a point-measured entity.

We devise a group of thought experiments to illustrate that a fixed aggregation of points can never reach the status of a continuum, in other words, the static-indivisible-composed model of a continuum, which we simply refer to as the *indivisible-model* (of a continuum) hereafter, is untenable. What we aim to argue against is not the existence of continuum or indivisible, but the reasonability of the *indivisible-model*.

First of all, what is a continuum? It is said to be a continuous entity; but if only "continuous" were not equally in need of clarifying. Here we just mention some understanding about continuum: A straight line is a simple example (with real line being its *indivisible-model*). Informally, it is so perfect with regard to uniformity as to have no structure or detail (so to speak) — while looking at it, no matter which part or what a scale to focus on, one would never find more than the first glance tells. The perfect uniformity requires the *indivisible-model* of a straight line to be either free of gap or full of gap — strictly unified throughout. If there is no gap in the *indivisible-model*, as expected, each possible magnitude value would have its point representation on a real line, and each reasonably deduced point would be found in the *model*. It follows that the *model* has reflectional and translational symmetries.

The notation [0, 1) refers to the set of all real numbers between 0 and 1, including 0 but excluding 1. However, what we want to illustrate below is it is impossible for such a stationary aggregation to be "without any missing number". So, in our discussion, "without gap" is not a presupposed or known property. From now on, we use a modified notation, $\underset{\cdots}{[0,\,1)}$ — the dotted underline stands for the uncertainty about its completeness property. And, for convenience, we usually omit the underline if without causing any confusion.

Thought Experiment 4.1 The *Enclosed Point*

Between any two distinct real points there is always another one, which we call an *enclosed point* between the two given points. If $0 < X_1 < X_2 < \cdots < X_n$, and Y is an *enclosed point* between 0 and X_1, then Y is also an *enclosed point* between 0 and X_i for $i = 2, 3, \ldots, n$. Accordingly, we call Y an *enclosed point* between 0 and $\{X_1, X_2, \ldots, X_n\}$. Further onwards, 1 is an *enclosed point* between 0 and (2, 3].

Is there an *enclosed point* between 0 and $[x,\,1]$ for each x $(0 < x < 1)$? Everyone would answer yes without hesitation, since $(0 + x)/2$ is an example. Then, is there an *enclosed point* between 0 and $(x,\,1]$ for each x $(0 \leqslant x < 1)$? Is the answer just the same as the former? Or, can "$(x,\,1]$ *with* $0 \leqslant x < 1$" stretch even closer to 0 and make a difference? As far as can be told from the concept of "open/closed intervals", it has advantage in approaching to 0 — both "$[x,\,1]$ *with* $0 < x < 1$" and "$(x,\,1]$ *with* $0 \leqslant x < 1$" may be regarded as two sets of *indivisible-model* of segments that are each right-ended by point 1, and (0, 1] is the only element that can cover any other element of the two sets. (But is that clear? We continue the discussion in **Thought Experiment 4.4**.) However, considering that $\bigcup\{[x,\,1] : 0 < x < 1\}$ and $\bigcup\{(x,\,1] : 0 \leqslant x < 1\}$ are the same set of real numbers (points) — both of them are $\{x : 0 < x \leqslant 1\}$, the aforementioned view needs careful thinking. The identity of the two point-level sets suggests that the two original sets cover exactly the same range, and the two answers for the relevant questions, on this account, should be the same.

Actually, what we want to ask in the second question is, whether there is an *enclosed point* Y between 0 and $\underset{\cdots}{(0, 1]}$? Now, take $\underset{\cdots}{(0, 1]}$ as an immutable aggregation of points and face the question directly. If the answer is yes, $\underset{\cdots}{(0, 1]}$ is not as complete as expected, for missing at least Y. If the answer is no, there is no gap between the point 0 and the segment of (0, 1] (in other words, they touch each other). In this case, if a segment is composed exclusively of the static points, the consequent question is: Which point of (0, 1] are so close to the point 0 that they bear no gap in between (or, which point of the half-open segment has direct contact with the point 0)?

If a segment could be explained as a set of points with each point having a position value, then there is an arithmetic mean for all the values. And the position of the centroid is of this very value.

Thought Experiment 4.2 The Absent Centroid

Observe the segment (–1, 1), obviously 0 is the arithmetic mean. But what if the object is (–1, 1)$\bigcup$$\{1\}$? Does the segment (–1, 1] have a centroid? Of course, just like every object has its center of mass (and the remaining question is whether the center is within the body. For some objects, say, a doughnut, it is not). And certainly, the "center of mass" for each bounded connected part of a straight line should be within the "body" and identical with the centroid. Now, where is the centroid for (–1, 1]? (For those who think that one more point is not enough to make any difference, we have some questions: Whether a one-point-figure has a centroid? What is the centroid of $\{0\}$? And then what is the centroid of $\{0\}\bigcup\{1\}$?) Intuitively the newly joined point 1 would "drag" it to the right a tiny little bit. Another simple fact is that, for every bounded geometric figure, any part (but not whole) that is centrally symmetric about the centroid may be omitted without affecting the position of the centroid. So we can easily examine any supposed centroid of (–1, 1]. Unfortunately, no known point can stand the test. Is it an *enclosed point* between 0 and (0, 1] discussed in **Thought Experiment 4.1**?

How about switching back to the concept of the arithmetic mean? Well, because the segments (–1, 1) is just a left part of (–1, 1], the two mean positions cannot coincide. For those interested in calculating the precise mean position for all the points of the latter, the results tend to favor $0 +$ (*half a point-measured length*). Sure enough, no known point takes this position.

If, as we are taught, there are no adjacent points on a real line, it is impossible for any bounded entity to make an exact point-measured displacement along it. But on second thought, if a point is an individual entity, and taking away one point from a real line would result in a point-measured gap, we can certainly expect the existence of a point-measured distance and a motion of such a distance.

Thought Experiment 4.3 Moving a Point-measured Distance

Observe [0, 2) and (0, 2]. From the view angle of linear motion, they may be taken as each other's displacement result — a motion of a point-measured distance comes to light.

By the way, the two mean positions, referring to **Thought Experiment 4.2**, would be 1− (*half a point-measured length*) and 1+ (*half a point-measured length*), which differ by one point in position.

The points of a real line somewhat resemble the sheets of paper of a book, which can be divided into two groups by a bookmark without disrupting the order. (Next, if something recalls a classical thought to mind, just forget the famous idea for a moment.) Suppose that a virtual bookmark partitions a real line into $(-\infty,\ b)$ and $[b,\ +\infty)$, and we notate the "bookmark" together with its position by ϒ**b**. And a fellow notation, **b**ϒ, is for the "bookmark" that is "one sheet of paper" after ϒ**b**. If there is no gap between, say, $(-\infty,\ b)$ and $[b,\ +\infty)$, the "thickness" of a "bookmark" is 0 and it is possible to make a virtual insertion immediately before or after each real point. That being the case, the virtual bookmark can move a point-measured distance — the distance between ϒ**b** and **b**ϒ is exactly a point-measured long. Moreover, if there is a point immediately before the "bookmark", according to the property of symmetry, there should be another point located after the "bookmark" symmetrically — a pair of adjacent points comes to light. In fact, the trouble might spring out even earlier — before the above discussion we should ask in advance whether an extra point can be put at the very position of the "bookmark". This reminds us of **Thought Experiment 4.1**.

***Aside**: All the above examples seem to describe an abrupt "movement". Indeed, our goal is to refute the* indivisible-model *of continuum by showing the existence of some embarrassing tiny distance. For those who care much more about the continuity of a movement, we provide an example: When point x moves continuously from position 2 to 4, the arithmetic mean position for all the points of $(0,\ 2)\bigcup\{x\}$ moves continuously from 1+ (half a point-measured length) to 1+ (half a point-measured length)+ (one point-measured length).*

Remember "why such an awkward interpretation" in **Thought Experiment 3.1**? And the question arises in **Thought Experiment 4.1**: How can "$[x,\ 1]$ *with* $0 < x < 1$" and "$(x,\ 1]$ *with* $0 \leqslant x < 1$" cover exactly the same range, while the super element $(0,\ 1]$ existing in the latter but not in the former? For a definitive answer, we come to another question: Is an open interval really open?

In standard opinion, $[0,\ 3]$ could be "cut off" into $[0,\ 2)$ and $[2,\ 3]$. But why $[0,\ 2)$ has no right endpoint (whereas $[2,\ 3]$ has its left endpoint)? If it has one, that one would be the left adjacent of point 2. We are taught that no adjacent points exist on a real line. So, there is nothing strange. However, such an explanation can never eliminate our doubt. From the viewpoint of uniformity of straight line, each point of real line has nothing special except for its unique position. That means if any of them can act as an endpoint, so can others. As concerns the edge of a bounded connected part, adding or removing a point would cause nothing more than a shift of position to the related boundary — there would be no change in its form or style.

Thought Experiment 4.4 The Suspicious Open Interval

Suppose that, along a real line orbit, an antimatter object $[-1+t,\ 1+t]$ is sliding towards a normal object $[2,\ 4]$, and its position varies directly with time, t. A special collision would be inevitable and the two would cancel each other out during the process, for at any moment they match each other symmetrically.

But what would happen if there are two antimatter points $-1+t$ and $1+t$, instead of the object $[-1+t,\ 1+t]$, moving towards [2, 4]? A stage result would be that one of the moving points has disappeared together with the left endpoint of [2, 4], and the other point is sliding towards (2, 4] from left side. What would happen next? Which point of (2, 4] would vanish?

Now turn to another case. Suppose that a point x moves from 1 to 3 continuously along a real line. If [1, 3] has no gap, all points of the interval would be exactly all the possible positions of x. Therefore, "*taking the position of each of the points in orderly accordance with time*" is "*moving continuously through the interval*". Then observe a growing segment $[0,\ x]$ with x running through all the points of [1, 3]. The growing segment would apparently run through all the connected parts (of [0, 3]) that contains [0, 1], for the growth is a continuous one. Thus there exists a point X_i $(1 \leqslant X_i \leqslant 3)$ such that $[0,\ X_i]$ is exactly [0, 2), which is surely a connected part of [0, 3] and contains [0, 1]. That means [0, 2) is right-ended by X_i. Denying the existence of X_i means a gap in the *indivisible-model*. But accepting its existence also violates the standard opinion about adjacent points.

There is a motion-independent variant of the experiment: Now that real line is assumed to be without gap, it is naturally the most accurate ruler with each of its point working as a scale mark. The "most accurate" means, while making a measurement, there are always plenty of marks such that the concept of tolerance or estimation can be dismissed — every finite length value can be read off directly from some point on the "ruler" and even a difference of one point would be reflected in the scale readings without additional description. If not, the accuracy can be improved simply by adding (to the original model as new point, new scale mark) every known exceptional length value (that cannot be directly read off yet) and hence the original model cannot be "without gap". From another aspect, every bounded connected part of a straight line is a definite length value in itself, no matter there is a ruler (that can directly measure it) or not.

5. The "Final Result" of a Potentially Infinite Process

The *indivisible-model* of continuum cannot stand further thought. Now we are faced with the sequential question (a classical one): What is the final result of iterative bisection of a line segment?

To discuss this question, we begin with clarifying what does "final result" mean. While describing the consequence of a potentially infinite process we use "after all steps", "final result", and the like. What is the intuitive origin of the underlying supposition? Here are two examples:

Example 1: For a potentially infinite collecting task, the initial status is an empty set; and the stage results are $\{1\}$, $\{1,\ 2\}$, $\{1,\ 2,\ 3\}$, and so on; and the "final result" is thought to be $\{1,\ 2,\ 3, \ldots\}$.

Example 2: For the geometric series related to the first one of Zeno's paradoxes[2, p.349], the initial status is an empty record; and the stage results are $\frac{1}{2}$, $\frac{1}{2}+\frac{1}{4}$, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, and so on; and the "final result" is thought to be $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$.

For a potentially infinite process, we denote the first n steps by $S_1, S_2, \ldots, S_n$, the initial status by C_0, the nth stage result by C_n and the “final status” C. Now we execute a collecting task — collect the notations of executed steps successively. If the main process can reach completion, our collecting task can thereupon reach completion and we can get an actual infinite set $\{S_1, S_2, S_3, \ldots\}$, that is to say C_0 can become C after experiencing $\langle S_1, S_2, S_3, \ldots\rangle$. From the viewpoint of transformation function, the first n step(s) as a group determines a function T_n such that $C_n = T_n(C_0)$, whilst all the steps as a group determine a function T such that $C = T(C_0)$.

Nevertheless, we believe that all the so-called final results of the potentially infinite processes are just imaginations of human beings, or more precisely, some preexisting or supposed existing objects are assigned to act as the nonexistent “final results”. Hence, no wonder, sometimes people are puzzled by different versions of “final result” of one potentially infinite process. Usually it is difficult to take sides between (or among) the different versions, as shown in the case of the Ross–Littlewood paradox[3](which has offered a good chance to rethink infinity).

Thought Experiment 5.1 The “Final Results”?

Take the sequence of all natural numbers as a row of road lamps settled at regular intervals. We begin with all the lamps lit, and denote the status by $\langle \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\rangle$, a sequence of period 1.

Step 1: For all the lit ones, from left to right, switch off the 1st, skip the 2nd, switch off the 3rd, skip the 4th, and so on. Then denote the result by $\langle \cancel{1}, \mathbf{2}, \cancel{3}, \mathbf{4}, \cancel{5}, \mathbf{6}, \ldots\rangle$, a sequence of period 2. (Those have the form $2k{+}1$ for non-negative integer k are switched off, and all the multiples of 2 remain.)

Step 2: For all the lit ones, execute the same operation as above. Denote the result by $\langle \cancel{1}, \cancel{2}, \cancel{3}, \mathbf{4}, \cancel{5}, \cancel{6}, \cancel{7}, \mathbf{8}, \cancel{9}, \cancel{10}, \cancel{11}, \mathbf{12}, \ldots\rangle$, a sequence of period 4. (Those have the form $4k{+}2$ are switched off this time, and all the multiples of 4 remain.)

Step 3: .... (Those have the form $8k{+}4$ are switched off this time, and all the multiples of 8 remain.)

Go on iteratively ad infinitum.

We denote the “final result” by $\langle \cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}, \cancel{8}, \cancel{9}, \cancel{10}, \cancel{11}, \cancel{12}, \ldots\rangle$, which shows none of the lamps is on. But the strange thing is that the period length turns out to be 1, whereas, to tell from its monotone increasing trend in the process, it cannot be a finite number.

However, if we simulate the above process following another rule — “switch off the 2nd and every other one thereafter”, the stage results would be $\langle \mathbf{1}, \cancel{2}, \mathbf{3}, \cancel{4}, \mathbf{5}, \cancel{6}, \ldots\rangle$, $\langle \mathbf{1}, \cancel{2}, \cancel{3}, \cancel{4}, \mathbf{5}, \cancel{6}, \cancel{7}, \cancel{8}, \mathbf{9}, \cancel{10}, \cancel{11}, \cancel{12}, \ldots\rangle, \ldots$ with the same period lengths as above, respectively. (Some more detail is: only those have the form $2k{+}1$ remain to the end of the first step; only those have the form $4k{+}1$ remain to the end of the second step; only those have the form $8k{+}1$ remain to the end of the third step; and so on.) We would have $\langle \mathbf{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}, \cancel{8}, \cancel{9}, \cancel{10}, \cancel{11}, \cancel{12}, \ldots\rangle$ as the “final result”, which, in accord with the process, has no finite period length.

Now introduce a concept, *relative lighting rate*, which is a ratio like approval rate or occupancy rate. The *relative lighting rate* of $\langle \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots\rangle$ is 1, and that of $\langle \cancel{1}, \cancel{2}, \cancel{3}, \ldots\rangle$ is 0. So, we may describe the evolution of the *relative lighting rate* in the former process as 1, $1-\frac{1}{2}$, $1-\frac{1}{2}-\frac{1}{4}$, etc., and the “final result” as

$(1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\cdots)$. For the latter process, we may describe the course as 1, $1-\frac{1}{2}$, $1-\frac{1}{2}-\frac{1}{4}$, etc., and the "final result" as $(1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\cdots)$. In this sense, the two processes coincide with each other all the way. But, strangely enough, the two "final situations" are different by one lamp. That might be the reason why there are always some people rejecting the opinion that 0.999... is exactly 1 — there is a lamp remaining lit somewhere at the back of their minds.

Once again return to the initial condition. This time we think of the lamps as a row of dots arranged at regular intervals on a rubber band that has been pre-stretched sufficiently. Take the position of the 1st lamp as the reference point. Next we are concerned only with the lit lamps.

Step I: For all the lit lamps, switch off the 2nd and every other one thereafter. Then let the rubber band contract around the reference point until the current 2nd lit lamp occupies the initial spatial position of the original 2nd lit one. (As a consequence, the current 3rd lit lamp, in turn, occupies the initial position of the original 3rd lit one. And so forth.)

Repeat ad infinitum.

What is the "final result" this time? To judge from the form of numbers, only 1 would remain, since for a natural number n, to survive the first k steps means n–1 is divisible by 2^k and vice versa. Nevertheless, it is easy to see, from the duplicate images presented by the steps, that there is no trend towards such a "final result" and there is no hope to escape from the initial status. This gives rise to a growing feeling that all the "final results" do not really exist.

All the potentially infinite processes, when looked at their steps (with no detail in any step being taken into account), share the same endless sequential structure, which we call the *N-structure* because the natural numbers as endless ordinals are abstracted from various examples of it — each example has a unique starting element, which has one and only one successor, which, in the same way, has its own successor, and so forth (there is nothing else — no branch, no loop, and each following element is a finite "distance" away from the starter). A conclusion at this step-sequence level (thus has nothing to do with the details inside any step), no matter which specific process it is drawn from, relates only to the macrostructure and holds equally for all examples of the *N-structure*. Therefore the falsity of any "final result" means none of the examples of the *N-structure* (the processes of potentially infinite steps) can ever reach a completion.

Why there are so many people accept the so-called final results as a given? This is due to that from the beginning their minds are fed with some specially selected cases, each of which suggests an apparent "final result", which is too natural and perfect to be questioned. But, are they really so perfect?

Thought Experiment 5.2 Fresh or Salty?

Take each of the natural numbers as a glass of salt water. We denote the initial status by $\langle \underline{\underline{\mathbf{1}}}, \underline{\underline{\mathbf{2}}}, \underline{\underline{\mathbf{3}}}, \ldots \rangle$. All of them are of the same concentration, and, if added together, can make an infinitely large salt lake of the same salinity.

Step 1: Replace the 1st glass of salt water with a glass of fresh water, which is free of salt. Denote the result by $\langle 1, \underline{\underline{\mathbf{2}}}, \underline{\underline{\mathbf{3}}}, \ldots \rangle$.

Step 2: Replace the next glass of salt water with a glass of fresh water. Denote the result by $\langle 1,\, 2,\, \underline{\underline{\mathbf{3}}}, \ldots\rangle$.

Go on ad infinitum.

After all steps, so to speak, we denote the "final result" by $\langle 1,\, 2,\, 3, \ldots\rangle$. Obviously all the glasses of water, with none of them containing any salt, can be added together to form an infinitely large lake of fresh water. As to the perfect completion, there seems to be no doubt. Yet, the story does not end here.

On the other hand, we can deduce the concentration of the "final" lake step by step. We denote the original concentration by C_0. Suppose that the result lake of Step 1 has the concentration C_1 (after the uniform state has been reached). Taking into account the effect of the first replacement, we have $C1 > C_0 - C_0/4 = (1-1/4)C_0$. The result lake of Step 2 has the concentration C_2. Taking into account also the second replacement, we have $C_2 > C_1 - C_0/8 > (1-1/4-1/8)C_0$. Accordingly, we have $C_3 > C_2 - C_0/16 > (1-1/4-1/8-1/16)C_0$, and so on. Consequently, the "final" lake has the concentration C, which satisfies $C > [1-(1/4+1/8+1/16+\cdots)]C_0$. Conceivably, by choosing different coefficients on the right side of the inequalities, we can even conclude that C is higher than any given concentration that is below C_0.

We also provide a variant of the experiment from a static viewpoint. There is an infinitely large salt lake that consists of infinitely many disjoint units of salt water indexed by natural numbers. All the concentrations are the same, which we denote by C_0. But, maybe an earnest person would bother to think the relationship between the concentration of the lake and that of each unit of salt water. Evidently, the salinity of the lake is backed up by the salt of all the units; and the contribution of the salt of the nth unit is less than $C_0/2^{n+1}$. As a consequence, the concentration of the lake, which is given as C_0, may be less than $(1/4+1/8+1/16+\cdots)C_0$. (And besides, from some angle, one can even conclude that the concentration is higher than C_0.) A contradiction.

A truth may be observed from every angle — there is no contradiction between any two views. An illusion of human beings is another thing. All the confusion we have exposed above reveals that all the "final results" are the work of man.

Thought Experiment 5.3 Poor or Rich?

There are infinitely many piggy banks (indexed by natural numbers) with the 1st one containing 1 coin, the 2nd containing 2 coins, the 3rd containing 3 coins, and so on. We denote the initial status by $\langle 1,\, 2,\, 3, \ldots\rangle$.

Step 1: Take away the first piggy bank, which contains 1 coin. Denote the result by $\langle 2,\, 3,\, 4, \ldots\rangle$.

Step 2: Take away the next piggy bank, which contains 2 coins. Denote the result by $\langle 3,\, 4,\, 5, \ldots\rangle$.

Go on ad infinitum.

After all steps, so to speak, there would be nothing left over.

Now, return to the initial status and begin a contrastive process.

Step I: Add one coin to each of the piggy banks. Denote the result by $\langle 2,\, 3,\, 4, \ldots\rangle$.

Step II: Add one coin to each of the piggy banks again. Denote the result by $\langle 3, 4, 5, \ldots \rangle$.

Go on ad infinitum.

After all steps, so to speak, we denote the "final result" by $\langle a_1, a_2, a_3, \ldots \rangle$. According to Cantor's theory, each of the piggy banks contains $\aleph_0$ coins. However, here we are only interested in whether the result is null or not.

Make a comparison of the stage results between the two processes. In set theory, they are the same respectively. Thus an unavoidable question is: How can the same path (with the same starting point) lead up to two opposite destinations?

No wonder there are always some people accepting only potential infinity and reluctant to go any further — lack of clarity in both logic and intuitive sense proves the critical hurdle in the way to active infinity.

The "final result" and "total finish" are two sides of the same coin. On this account a contradiction about the "final result" denies the existence of either.

6. A Look Back at the *Fog*

After Cantor's time, in the field of the foundations of mathematics, most mathematicians readily yield to a kind of asymmetrical reasoning — their logic standard differs depending on whether a normally deduced result is for or against the dominating opinion. For example, "*x is different from each element of an infinite set*" is thought to be sufficient to prove "*x is not an element of the set*", whilst "*point 1 is nonzero away from each point of [0, 1)*" is not regarded as adequate to conclude that "*point 1 separates from [0, 1) (there is a gap in between)*", though both assertions are based on the same principle — *a set is determined by its members* (hence a proposition about a set can be verified by testing all its members literally).

However, a mathematical theory can never rest mainly on asymmetrical reasoning for long. Underlying the prosperity and wonderfulness of the theory of transfinite numbers there must be some more or less plausible pillars.

Cantor asserts[4], "*Each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite.*" As Morris Kline explains[2, p.200], "*He argued that the potentially infinite in fact depends upon a logically prior actually infinite.*"

Cantor's words show his acceptance of actual infinity and infinite set. We call this belief the *Belief A*. The actual infinities at this point are still lacking vitality, they cannot promise many exciting stories in such status. Nevertheless, what makes them vivid and substantial, as in many controversial proofs with regard to infinity, is a related but distinct belief: *Each potential infinity, if rigorously applicable, leads up to an actual infinity.* This belief is a bridge across the impossible gulf between finite and infinite. We call it the *Belie B*. With it in mind, everyone may draw a conclusion based on "finishing" a potentially infinite process, and figure out an actual infinity's properties from an associated potential infinity. Like many others, we think of the *Belief B* as a super troublemaker. As Hermann Weyl puts it[1, p.1200], "*... the sequence of numbers which grows beyond any stage already reached ... is a manifold of possibilities opening to infinity; it remains forever in the status of creation, but is not a closed realm of things existing in themselves. That we blindly converted one into the other is the true source of our difficulties, ...*"

If a potential infinity leads to an actual infinity, then a potentially infinite process could be taken as a function mapping initial status inputs to final status outputs. But many pieces of evidence suggest otherwise. Strictly following the *Belief B* results in the confusion of **Thought Experiment 5.2**, and combining the *Belief B* with the *axiom of extensionality* causes the contradiction in **Thought Experiment 5.3**. (In axiomatic set theory, the *axiom of extensionality* conveys the idea that, as in naïve set theory, *a set is determined by its members rather than by any particular way of describing the set.*) Although the axiom is clear and safe for finite sets, our experiment indicates that it is not reliable while combined with the *Belief B*.

Well, would everything be all right if it were not for the affection of the *Belief B*? No. Current situation is really worrisome. The *Belief A* has been artfully used for presenting a stirring magic show launched by Cantor. And in order to sustain the epic performance, the standard of plausibility and clarity has been distorted; some ingenious arguments only in partial agreement with logic have occupied center stage in the foundations of mathematics ever since. Yet it must be said in fairness to the mathematicians that no one intentionally resorts to trickery or deception.

One-to-one correspondence, the basic principle of set theory, is simple and clear for finite sets. But, when applied to infinite set, it brings about a big shock to humans' intuitive mind. For Cantor "*a set is infinite if it can be put into one-to-one correspondence with part of itself*"[1, p.995]. However, such a supposed basic property of infinite sets violates *the fifth common notion of Euclid* (which states *the whole is greater than the part*), thereby kicking off the chaos and leading to paradoxes such as Hilbert's Hotel[5].

Let's rethink, from another perspective, the problem mentioned in **Thought Experiment 5.2**. Suppose that there are two infinitely large salt lakes, with one of them having the concentration C ($C > 0$) and the other $C/2$. If the salt and the water are separated, each lake would become an infinite amount of salt and an infinite amount of fresh water, and the two lakes would yield, according to Cantor's theory, just the same amounts of water and the same amounts of salt respectively. That means, reversely, if given an infinite amount of salt and an infinite amount of water, one can obtain a salt lake of concentration C or $C/2$ or some other value. Thus, *the law of contradiction* (a fundamental principle of logic, which states *something cannot be true and not true at the same time*) is violated.

Note: *The unusual concentration is not so abrupt because the paradox of Hilbert's Hotel has already shown a similar ratio, the occupancy rate, which is also without certainty. Given the infinite constant of persons and rooms in Hilbert's Hotel, the occupancy rate can be adjusted as one pleases (with each room containing at most one person).*

How about *the law of excluded middle* (another fundamental principle of logic, which states *any proposition is true or its negation is true*)? We arbitrarily use the law in **Thought Experiment 3.1**, but does it hold for infinite sets? No. To explain our view, we provide a thought experiment adapted from the relevant ideas of **Thought Experiment 3.1**.

Thought Experiment 6.1 Containing or not?

Consider all rational numbers between (and inclusive of) 0 and 1 in the form of reduced fraction (in particular, 0 is taken as 0/1, and 1 as 1/1). We classify the numbers into two sets — set B for those have an even number as its numerator or denominator, and set D for all others. Let set $A = B \bigcup D$. We notice that between any two elements of A there are both an element of B and an element of D.

Note: *The fact may be easily seen. Just observe 2/(2n+1), 4/(2n+1), . . . , 2n/(2n+1), when written in reduced form, each of them is an element of B.*

> *They spread on* [0, 1] *with a fixed interval that is smaller than 1/n. Similarly, observe 1/(2n+1), 3/(2n+1), . . . , (2n-1)/(2n+1), . . . of D. They . . . .*
>
> The elements of D determine a partition of A in the way that, for $d_j \in D$, the equivalence class $[d_j]$ is $\{$*rational number* $x \in [0, d_j] : \forall d_i \in D\,(d_i < d_j \rightarrow d_i < x)\}$. Of course, each equivalence class contains exactly one element of D. Our question is: Does $[d_j]$ contain any element of B? There is a similar question in **Thought Experiment 3.1** (where we include a more intuitive operation — cutting the segment at infinitely many points). The answer cannot be yes, otherwise $[d_j]$ contains another element of D. On the other hand, the answer cannot be no, otherwise, for the same sake, each of the other equivalence classes does not contain any element of B, but that implies $D = A$.

Hilbert has a famous comment on Cantor's work[1, p.1003]: "*No one shall expel us from the paradise which Cantor created for us.*" However, without basic logic, who can tell a mirage from reality?

7. Some More Words

Once the series of thought experiments reach the public, our mission is fulfilled; we may sit back and see whether the *paradise* will be gone with all the *fog*. Before diverse voices refocus on the "long-settled" controversy respecting Cantor's work, we would like to say some things in addition.

How to understand infinity? A closely associated question is, How to understand continuum? These questions have been puzzling human beings for thousands of years, and there have been plentiful doctrines formed on unclear arguments since at least Aristotle's time. Our goal is to present some simple and convincing evidence against the current overwhelming opinion, which we have been trying but failing to understand.

Note: *Aristotle writes[6, Book III, Ch.6], "There will not be an actual infinite." And in his view ". . . nothing that is continuous can be composed of 'indivisibles': e.g. a line cannot be composed of points, the line being continuous and the point indivisible."[6, Book VI, Ch.1]*

Our demonstration supports such a view:

(i) No potential infinity leads to an actual infinity. The existence of actual infinities depends upon axiom, and Cantor's theory, though influential, is not reliable. (ii) Connectedly, indivisibles can never statically form a continuum. The existence of continuum depends upon axiom, and its static-indivisible-composed model, though useful, is not a logical one.

Aside: *What does* "exist" *or* "existence" *mean in our discussion? In mathematics, informally, anything that can find its room in a consistent system is said to exist in that system. A thing may exist in one system, while not in another. Although two conflicting things cannot coexist in one system (otherwise the system is inconsistent), they may exist separately in two systems, say, Euclidean geometry and non-Euclidean geometry. And many existing things (e.g., irrational numbers) were once unknown or thought to be impossible. Some others (e.g., infinitesimal) have their ups and downs, though whether something exists or not is objective.*

Gauss states[1, p.993], "*I protest against the use of an infinite quantity as an actual entity; this is never allowed in mathematics. The infinite is only a manner of speaking, in which one properly speaks of limits . . . .*" His words indicate he believes that the concept of actual infinity cannot be derived from others and there is no reasonable way of introducing it. However, Cantor does not think so. He points out[4], "*In order for there to be a variable quantity in some mathematical study, the domain of its variability must strictly speaking be known beforehand .... Thus this domain is a definite, actually infinite set of values.*"

In Cantor's judgment, it is a must to accept actual infinity. Normally, next step is to come up with several fresh concepts around this idea and vitalize them by mining some new connections and relationships. The key is to keep a full consistency and clarity, as the lack of which may shake everything. Cantor has done a lot but, disappointingly, left behind a *fog*-wrapped "*paradise*". Worse than that, what threatens his theory is not a leaky roof, but a shaky foundation.

In reality, Cantor's thought has ridiculously won a miraculous victory, but still there are some who hold the opposing perspective. Is there a middle ground between the views of both sides? Perhaps there is. Abraham Robinson's position is based on the two main points or principles[7, p.507]: "*(i) Infinite totalities do not exist in any sense .... (ii) Nevertheless, we should continue the business of Mathematics 'as usual,' i.e., we should act as if infinite totalities really existed.*" His flexible attitude is helpful for accommodating and accomplishing both of the two sides' goals before actual infinity is recognized just as it is. However, the point is that a tolerance limit should be set for "*as usual*" or "*as if*", otherwise mathematics will always be surrounded by such things as "*a fog on a fog*" ("*Hermann Weyl spoke of Cantor's hierarchy of alephs as a fog on a fog*"[1, p.1003]), as it is today. (The outline of the limit is not our concern for now.)

At this point, the fate of infinity attracts most attention. Just like Cantor's "*paradise*" is not the final destination, its collapse does not mark the point of terminal decline for infinity. As far as existence is concerned, actual infinity should not be more abrupt than imaginary number (which contributes a lot to the development of mathematics). It is not a matter of proving or disproving; the difficulty for infinity lies in taking on more positive significance than sustaining and entertaining itself. It seems that, in the fading *fog*, actual infinity may walk out of a mythical dream and sip on some fresh air.

Usually, there is more than one belief surrounding the same fact. And those with various opinions are always ready to fight for the truth and continue what is thought to be the right way. Anyway, pushing and hiding the contradictions into the depth of *fog*, instead of eliminating them at the root, is always an option, but never a solution.

Aside*: What do we mean by "*... the depth of fog*"? We just give a simple example. Let's slip back into Cantor's theory once more. In* ***Thought Experiment 2.1****, if taking into account the concept of order type, the paradox may be discussed further. Some may argue that infinite may times of position swaps would change the order type. Others do not agree. They think, in the sequence, the relationship between a position and its element is just like the relation between a fixed seat and the person sitting in it — the seat would remain still whilst the person may not. If there is neither an empty seat nor a standing person, the order type of the persons is just the same as that of the seats, which are stationary all the way. Nevertheless, some may think from another aspect. For the sequence of all natural numbers:* $1, 2, 3, \ldots,$ *repeatedly swap positions of the element 1 and its immediate right neighbor. For those who insist on there being a change of the order type, the "final result" would be* $2, 3, 4, \ldots, 1$ *with the order type* $\omega+1$ *in Cantor's theory, hence the paradox of* ***Thought Experiment 2.1*** *can be explained. It seems to be reasonable. But what if go a step further along the same lines? Next, repeatedly swap positions of the element 2 and its direct right neighbor. The question is: Whether* $3, 4, 5, \ldots, 2, 1$ *would be a stage result? Confusion is knocking at the door. Some may go on to ask: So what if the element*

2 turns out to be in trouble? Does that relate to the realization of the aforementioned 2, 3, 4, . . . , 1*? Well, just compare beginning with* 1, 2, 3, . . . *and with* 1, 2, 3, . . . , 0.

Like it or not, the concept of infinity does raise a paradise, which is for imagination. However, even a reasonable imagination, which may be scientifically helpful in virtue of its reasonable aspect(s), still has unreasonable aspect(s), (or it is a fact, a truth). The key is never to indulge in the *fog* while making clever use of imagination. It is not the imagination that hurts, but the mistaking of it for truth.

Aside*: As the public knows, in physics, a rigid object is commonly treated as if all its mass were concentrated in a single point called the center of mass, though, in fact, the mass is everywhere in the object. This imagination simplifies the solution of problems of force and motion. But it does not always work, for it is not equal to the truth, otherwise there would be no need for the concept of center of gravity. The same is true for the imaginations about infinities and continua in mathematics.*

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Appendix

A. Explanation for "we can always find . . . " in Experiment 2.1

For base-10 system, every rational number has a terminating or repeating decimal representation, and every terminating decimal can be converted to a repeating decimal. The same is true for binary system. Moreover, the expression capacity of a base-n numeral system does not depend upon which n distinct symbols to employ, so we can use 0 and 1 to establish a binary system, or 5 and 6 to do the same. In each of the binary systems, every rational number in (0, 1) has its repeating expansion, which shares the same visual appearance with some base-10 rational number in (0, 1). Therefore, each search task for the two-position swap cannot fail — at least there are infinite choices that look like some type of binary rational numbers and free of any single digit to be avoided.

B. Proof for Lemma 3.1

(The term arc means specifically inferior arc below.)

1. The light beam does not return to any *bright point* again within finite many reflections because the line segments $[0, P_1)$ and $[0, C)$ are incommensurable.

2. It suffices to show that for any given arc of the circle, no matter how short it is or where it locates, there is a *bright point* on it.

We denote the length of the given arc by $\varepsilon\,(0 < \varepsilon \leqslant C/2)$. Let k equal the greatest integer not exceeding C/ε, so $(k + 1) > C/\varepsilon$. The first $k{+}1$ *bright points* divide the circumference into $k{+}1$ disjoint arcs. As $C/(k + 1) < \varepsilon$, at least one of the $k{+}1$ arc lengths is shorter than ε. For such a short one, we denote its arc length by $S\,(0 < S < \varepsilon)$. For the two *bright points* that determine this short arc, we denote the difference between their indexes by $i\,(i \in \mathbb{N},\, 0 < i \leqslant k)$.

Denote the biggest integer not exceeding C/S by m. As the arc distance between any two *bright points* is determined only by the difference between their indexes, the $m+1$ points P_0, P_i, $P_{2i}, \ldots, P_{mi}$ (ignoring all others) divide the circumference into $m+1$ disjoint arcs with m of them having the length $S\,(0 < S < \varepsilon)$ and the remaining one being even shorter. That means it is impossible for any ε-lengthed arc on the circumference to avoid all the $m+1$ points. □

C. A Numerical Example for Thought Experiment 3.2

A strictly increasing rational sequence that has limit $\sqrt{2}$: $7/5$, $41/29$, $239/169, \ldots$ with general term $\sqrt{2b_n^2 - 1}/b_n$, where $b_1 = 5$, $b_{n+1} = 3b_n + 2\sqrt{2b_n^2 - 1}$ for $n = 1, 2, 3, \ldots$. And a strictly decreasing rational sequence that has limit $\sqrt{2}$: $3/2$, $17/12$, $99/70, \ldots$ with general term $\sqrt{2c_n^2 + 1}/c_n$, where $c_1 = 2$, $c_{n+1} = 3c_n + 2\sqrt{2c_n^2 + 1}$ for $n = 1, 2, 3, \ldots$.

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