# HOW HARD IS THE TENSOR RANK?

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Abstract. We build a combinatorial technique to solve several long standing problems in linear algebra with a particular focus on algorithmic complexity of matrix completion and tensor decomposition problems. For all appropriate integral domains $\mathcal{R}$, we show the polynomial time equivalence of the problem of the solvability of a system of polynomial equations over $\mathcal{R}$ to

- the *minimum rank matrix completion* problem (in particular, we answer a question asked by Buss, Frandsen, Shallit in 1999),
- the determination of *matrix rigidity* (we answer a question posed by Mahajan, Sarma in 2010 by showing the undecidability over $\mathbb{Z}$, and we solve recent problems of Ramya corresponding to $\mathbb{Q}$ and $\mathbb{R}$),
- the computation of *tensor rank* (we answer a question asked by Gonzalez, Ja'Ja' in 1980 on the undecidability over $\mathbb{Z}$, and, additionally, the special case with $\mathcal{R} = \mathbb{Q}$ solves a problem posed by Bläser in 2014),
- the computation of the *symmetric rank* of a symmetric tensor, whose algorithimic complexity remained open despite an extensive discussion in several foundational papers. In particular, we prove the NP-hardness conjecture proposed by Hillar, Lim in 2013.

In addition, we solve two problems on *fractional minimal ranks* of incomplete matrices recently raised by Grossmann, Woerdeman, and we answer, in a strong form, a recent question of Babai, Kivva on the dependence of the solution to the matrix rigidity problem on the choice of the target field.

## 1. Introduction

One particular motivation of this study is to reach the full understanding of the algorithmic complexity of *tensor rank*, which is the function sending a tensor $T \in U \otimes V \otimes W$ to the smallest integer $r$ for which one can write

$$T = \lambda_1 T_1 + \ldots + \lambda_r T_r$$

with $T_i = u_i \otimes v_i \otimes w_i$ and $\lambda_i \in \mathcal{F}$. Here, one usually takes $\mathcal{F}$ to be a field, but it is convenient for us to allow it to be an arbitrary commutative ring. As one can see from this definition, we restrict our attention to three-way tensor products because, as we show in this paper, this setting is already sufficient to understand the algorithmic complexity of rank decompositions. The rank of a three-way tensor corresponds to the invariant known as the *multiplicative complexity* for bilinear programs [32, 35, 80], which appears in the famous problem on the complexity of matrix multiplication [22, 48, 79]. The general problem of tensor rank decompositions was introduced eighty years ago [37] and, apart from the above mentioned application in computational complexity theory, it appears as a fundamental tool

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in statistics [68], signal processing [49], psychology [19], linguistics [77], chemometrics [21] and many other contexts, see a more detailed survey in [45].

The first step towards understanding the computational complexity of tensor rank was made by Håstad [35], who showed that this problem is NP-hard over $\mathbb{Q}$ and NP-complete over finite fields. A recent paper of Hillar and Lim [36] shows that Håstad's approach works well enough to prove the NP-hardness over $\mathbb{R}$ and $\mathbb{C}$ as well. In this paper, we show that, for any integral domain $\mathcal{R}$, the computation of the tensor rank over $\mathcal{R}$ is polynomial time equivalent to the general problem of the solvability of a system of polynomial equations over $\mathcal{R}$. We note a recent paper of Schaefer and Štefankovič [66], which proves the same result but restricted to the case when $\mathcal{R}$ is a field. Our result is more general, and, in particular, it shows that the tensor rank over $\mathbb{Z}$ is undecidable, which answers the question asked by Gonzalez and Ja'Ja' in 1980. Moreover, we note that our result is original even if $\mathcal{R}$ is a field because, although the work [66] is already peer reviewed, the first version of the current paper [70] appeared on arXiv earlier than the first version of [66]. In the perspective of applied mathematics, one is usually focused with the real number version of tensor rank, and our result implies that the corresponding problem is $\exists\mathbb{R}$-complete, that is, it is polynomial time equivalent to the *existential theory of the reals* [2, 7, 23, 53, 55, 61, 64, 65, 71, 72]. Also, Bläser [9] asked a question on the complexity of the rational number version of tensor rank, and we get an answer as the special case $\mathcal{R} = \mathbb{Q}$ in our descrption.

The *symmetric tensor rank* appears if we assume $U = V = W$ and $u_i = v_i = w_i$ in the above definition of the rank, and this invariant is also relevant in pure mathematics and engineering [20, 43, 56, 76]. The practical applications motivate a search of algorithms computing this invariant as well as the study of its computational complexity, and, indeed, many authors went on to discuss these issues. In particular, the potential hardness of the computation of the symmetric rank was discussed in the foundational paper of Comon, Golub, Lim, Mourrain [20] and in subsequent works [6, 11, 56, 62] focused on the algorithmic computation of symmetric decompositions. However, although the authors of these papers wrote that the problem should be hard, no particular hardness result was known, and Hillar, Lim posed its NP-hardness as a conjecture in a further notable paper [36]. This question was reiterated in subsequent studies [31] and remained open before the publication of our work. We prove this conjecture, and, moreover, we show that the symmetric rank admits the same description of the complexity as the one given for tensor rank in the above paragraph, whenever $\mathcal{R}$ is a field with $|\mathcal{R}| \geqslant 4$.

The starting point of the technical part of this paper is an appropriate reduction of the problem of the solvability of a family of polynomial equations to another problem known as the *minimum rank matrix completion*. In this problem, one is given a positive integer $r$ and a matrix $A$ with entries in $\mathcal{R} \cup \{*\}$, where $*$ is a placeholder symbol for an unknown entry, and the task is to fill in the placeholders in $A$ with elements of $\mathcal{R}$ so that the rank of the resulting matrix is at most $r$. As it turns out, this simply looking problem takes one of the central places in modern applied mathematics [1, 13, 15, 14, 16, 17, 39, 59, 84], and the current paper is the first one that gives a full description of its algorithmic complexity. In particular, it turns out that the minimum rank matrix completion is $\exists\mathbb{R}$-complete in the most relevant case of the real numbers, and hence we get a solution to the problem discussed by Buss, Frandsen, Shallit [12] and Laurent [47]. More generally, we show that the minimum rank matrix completion is polynomial time equivalent to the solvability of a system of polynomial equations over any commutative ring $\mathcal{R}$, and this conclusion remains valid even if we require that the desired minimum rank equals three. In addition to these results, our approach leads to a resolution of two recent problems of Grossmann, Woerdeman on *fractional minimal rank* [34].

Another application of our technique comes from the problem of *matrix rigidity*, which was introduced many decades ago [33, 82] and became one of the central topics in arithmetic complexity [3, 4, 30]. In matrix rigidity, we are given two positive integers $k$, $r$ and a matrix $A$ with entries in $\mathcal{R}$, and we need to express $A$ as the sum $L+S$, where $L$ has rank at most $r$, and $S$ has at most $k$ nonzero entries. This problem is NP-hard over any field [27], but until now, beyond the NP-hardness, there has been no clear understanding of its complexity in the general case, and the dependence of its answer on the choice of $\mathcal{R}$ remained unclear [4, 29, 51, 60, 63]. For instance, Fomin, Lokshtanov, Meesum, Saurabh, Zehavi [29] and Mahajan, Sarma [51, 63] discuss the question of the decidability of the integer version of matrix rigidity. In fact, Mahajan, Sarma [51, 63] state that *'an upper bound of NP is not obvious'* over $\mathbb{Z}$, so it might have been considered plausible that the integer version of matrix rigidity is in NP. However, as we will see, this problem is algorithmically undecidable! Also, a recent paper of Babai, Kivva [4] raises the question of the dependence of the answer to the matrix rigidity problem on the choice of the ground field $\mathcal{R}$, and the current paper answers this question in a strong form. More generally, we prove the polynomial time equivalence of matrix rigidity to the solvability of polynomial equations for all relevant choices of $\mathcal{R}$, which solves several additional problems posed by Ramya [60].

Our paper is structured as follows. In the forthcoming Sections 2, 3, 4, we collect precise definitions of notions discussed in this paper, we formulate our main results, and we recall several basic techniques. In particular, Section 2 is devoted to tensor decompositions, Section 3 points out their relation to the substitution method and matrix completion problems, and Section 4 introduces the notion of matrix rigidity and a related algorithmic question. The remaining technical Sections 5–9 collect the proofs of our results, and their content is outlined in Section 5.

We conclude this introduction with a notational convention that is necessary because we discuss the issues of algorithmic complexity.

*Remark* 1.1. Throughout this paper, the letter $\mathcal{R}$ denotes a ring whose elements are encoded by strings in some finite alphabet so that the addition and multiplication in $\mathcal{R}$ are performed by polynomial time algorithms.

## 2. Tensor decompositions

We consider tensors of order three over a commutative ring $\mathcal{R}$. From the combinatorial point of view, a *tensor* is a three-dimensional array $T$ with elements $T(i|j|k)$ in $\mathcal{R}$, where $i, j, k$ run over corresponding *indexing sets* $I, J, K$. We write

$$T \in \mathcal{R}^{I\times J\times K}$$

and say that $T$ is an $I\times J\times K$ tensor over $\mathcal{R}$. The *size* of $T$ is defined as $|I|\times|J|\times|K|$. A tensor $T$ is called *symmetric* if $I = J = K$ and $T(i|j|k) = T(i'|j'|k')$ whenever $(i, j, k)$ is a permutation of $(i', j', k')$. Given three vectors

$$a \in \mathcal{R}^I,\ b \in \mathcal{R}^J,\ c \in \mathcal{R}^K,$$

we define the $I \times J \times K$ tensor $a \otimes b \otimes c$ by setting its $(i,j,k)$th entry to be $a_i b_j c_k$. Tensors arising in this way are called *decomposable* or *simple* with respect to $\mathcal{R}$. We note that, if we allow the vectors $a, b, c$ to contain elements not from $\mathcal{R}$ but rather from some extension $\mathcal{S}$, we may possibly get a different set of simple tensors.

**Definition 2.1.** Let $\mathcal{R} \subseteq \mathcal{S}$ be commutative rings, and let $T$ be a tensor over $\mathcal{R}$. The *rank* of $T$ *with respect to* $\mathcal{S}$ is the smallest integer $r$ such that $T$ can be written as a sum of $r$ tensors decomposable over $\mathcal{S}$. This quantity is denoted by $\operatorname{rk}_{\mathcal{S}} T$.

It is well known that the rank of a tensor with entries in $\mathcal{R}$ may depend on $\mathcal{S}$ even if $\mathcal{R}$ is a field [5, 24]. In the setting of integral domains, the rank may depend on the extension even for matrices, which we think of as $m \times n \times 1$ tensors.

**Example 2.2.** (Example 17 in [69].) *The rank of the matrix*

$$\begin{pmatrix} x & -z & 0 \\ 0 & y & x \\ y & 0 & z \end{pmatrix}$$

*is three over the ring* $\mathbb{R}[x,y,z]$ *and two over the field* $\mathbb{R}(x,y,z)$.

Now we are ready to formulate one of the main results. We recall that, as in Remark 1.1, the elements of the ring $\mathcal{R}$ are labeled by strings in a fixed finite alphabet, and the arithmetic operations on $\mathcal{R}$ are represented by polynomial time algorithms taking the corresponding strings as inputs. We note that this assumption may not transfer to the extension $\mathcal{S}$ as in Definition 2.1 and similar occasions, including Theorem 2.4 below, so $\mathcal{S}$ can be an arbitrary ring containing $\mathcal{R}$.

**Example 2.3.** In particular, Theorem 2.4 below holds for the real ranks of rational tensors, which corresponds to the special case $\mathcal{R} = \mathbb{Q}$, $\mathcal{S} = \mathbb{R}$.

**Theorem 2.4.** *Let $\mathcal{R} \subseteq \mathcal{S}$ be integral domains, and let $f_1, \ldots, f_p$ be polynomials with coefficients in $\mathcal{R}$. There is a polynomial time algorithm that constructs an order-three tensor $\mathcal{T}$ over $\mathcal{R}$ and an integer $r$ such that the following are equivalent:*
*(1) the equations $f_1 = 0, \ldots, f_p = 0$ have a simultaneous solution in $\mathcal{S}$;*
*(2) the rank of $\mathcal{T}$ with respect to $\mathcal{S}$ does not exceed $r$.*

*Moreover, these $\mathcal{T}$ and $r$ do not depend on the choice of $\mathcal{S}$.*

On the other hand, a straightforward formulation of Definition 2.1 gives a system of polynomial equations with coefficients in $\mathcal{R}$, and the inequality $\operatorname{rk}_{\mathcal{S}} T \leqslant r$ holds if and only if this system has a solution in $\mathcal{S}$. Therefore, Theorem 2.4 gives a complete description of the algorithmic complexity of the tensor rank.

**Theorem 2.5.** *Let $\mathcal{R} \subseteq \mathcal{S}$ be integral domains. Given a tensor $T$ over $\mathcal{R}$ and $r \in \mathbb{Z}$, checking the inequality $\operatorname{rk}_{\mathcal{S}} T \leqslant r$ is polynomial time equivalent to deciding if a given system of polynomial equations with coefficients in $\mathcal{R}$ has a solution in $\mathcal{S}$.*

A particularly important special case $\mathcal{R} = \mathbb{Q}$, $\mathcal{S} = \mathbb{R}$ shows that the real tensor rank is what is called an $\exists\mathbb{R}$-*complete* problem [53]. In other words, the real tensor rank is polynomial time equivalent to many classical problems in geometry, which include oriented matroids [55], polytope realizability [61], Nash equilibria [23], graph drawings [7], art galleries [2], linkages [64]. A similar characterization is known for other problems on rank decompositions in linear algebra, including the nonnegative rank [71] and positive semidefinite rank of matrices [72].

Theorem 2.5 with $\mathcal{R} = \mathcal{S} = \mathbb{Q}$ shows that the rational tensor rank is polynomial time equivalent to deciding if a given *Diophantine equation* has a rational solution, which answers the question of Bläser, see Open Problem 2 on page 119 in [9]. We recall that the rational Diophantine solvability is a famous problem that is believed to be undecidable, but its complexity status remains open despite extensive research [41, 42, 52, 54, 58]. Therefore, Theorem 2.4 can be seen as a conditional proof of the undecidability of the rational tensor rank, which would confirm Conjecture 13.3 in the paper [36] by Hillar and Lim. We remark that the solvability of Diophantine equations over $\mathbb{Z}$ was the content of *Hilbert's tenth problem*, and this question was proved to be undecidable a half century ago [52]. Using Theorem 2.4 together with this undecidability result, we get the following.

**Corollary 2.6.** *Tensor rank over $\mathbb{Z}$ is undecidable.*

Corollary 2.6 answers the question by Gonzalez and Ja'Ja' dating back to 1980, see page 77 of [32]. Finally, we note that the solvability of Diophantine equations is NP-hard over any integral domain [41], so we have another corollary of Theorem 2.4, generalizing the results of Håstad [35] and Hillar and Lim [36], who stated the NP-hardness of the tensor rank over $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and over finite fields.

**Corollary 2.7.** *Tensor rank is NP-hard over any integral domain.*

Now we switch to the symmetric case. As explained in the introduction, the *symmetric rank* of a symmetric tensor $T$ *with respect to* a field $\mathcal{S}$ is the smallest number of simple symmetric tensors over $\mathcal{S}$ whose linear span contains $T$. We can prove the symmetric counterpart of Theorem 2.4.

**Theorem 2.8.** *Let $\mathcal{F} \subseteq \mathcal{K}$ be fields with $|\mathcal{K}| \geqslant 4$. The problem of checking if*

$$\operatorname{srk}_{\mathcal{K}} T \leqslant r$$

*for a symmetric tensor $T$ over $\mathcal{F}$ and $r \in \mathbb{Z}$ is polynomial time equivalent to deciding if a given family of polynomials with coefficients in $\mathcal{F}$ have a common zero over $\mathcal{K}$.*

In particular, the computation of the symmetric rank is NP-hard over any field with at least four elements, which proves the conjecture of Hillar and Lim [36] discussed in the introduction. The most well studied cases are $\mathbb{R}$ and $\mathbb{C}$, see [20, 36, 56], so our additional assumption on the cardinality seems to be quite mild. In particular, the symmetric ranks are often studied in terms of the Waring rank function of homogeneous polynomials, which is not well defined over fields of characteristic 2 or 3, see [10]. Our technique could cover the case of cardinality three as well, but, as said above, this case is not especially relevant and leads to significant technical difficulties, so we decided to omit it. Also, we did not work with the case $|\mathcal{K}| = 2$ because it is quite pathological, which can be seen, in particular, from the fact that the symmetric tensor

$$e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1$$

does not admit any symmetric decomposition over $\mathbb{Z}/2\mathbb{Z}$.

## 3. Substitutions and matrix completions

In comparison to many recent studies that approach the general and symmetric tensor decomposition problems from the point of view of algebraic geometry [6, 18, 45], our methods involve a more combinatorial background. In particular, one of the ideas of our approach is to combine the standard *substitution method* of tensor rank computation with a classical problem of linear algebra, known as the *minimal rank matrix completion*, which we discuss later in this section.

**Definition 3.1.** If $T$ is an $I \times J \times K$ tensor, then we define the $k$th 3-*slice* of $T$ as an $I \times J$ matrix whose $(i, j)$ entry equals $T(i|j|k)$. For all $i \in I$, $j \in J$, we can define the $i$th 1-slice of $T$ and the $j$th 2-slice of $T$ similarly.

The substitution method rests on the following easy lemma. We refer the reader to [46] for a recent account on this method and to [73, 74] for further developments. The paper [38] gives an earlier appearance of a related result.

**Lemma 3.2.** *Let $\mathcal{F}$ be a field, and let $T$ be a tensor in $\mathcal{F}^{I\times J\times K}$ with $K = \{1, \ldots, k\} \cup \{1', \ldots, \tau'\}$. Let $S_i$ be the $i$th 3-slice of $T$ and assume that $S_{1'}, \ldots, S_{\tau'}$ are linearly independent and rank-one. Then $\operatorname{rank}_{\mathcal{F}} T$ is equal to*

$$\tau' + \min \operatorname{rank}_{\mathcal{F}} \mathcal{T}(V_1, \ldots, V_k),$$

*where $\mathcal{T}(V_1, \ldots, V_k)$ is the tensor formed by the slices $S_1 - V_1, \ldots, S_k - V_k$, and the matrices $V_1, \ldots, V_k$ are taken in the $\mathcal{F}$-linear span of $S_{1'}, \ldots, S_{\tau'}$.*

Another tool important for our paper is the *minimal rank matrix completion* problem. If $*$ is a new placeholder symbol, we say that a matrix $M$ with entries in $\mathcal{R} \cup \{*\}$ is an *incomplete matrix* over $\mathcal{R}$; any matrix obtained by replacing the $*$'s with elements in $\mathcal{S}$ is called a *completion* of $M$ over $\mathcal{S}$. What is the smallest value that the rank of a completion of a given incomplete matrix may take? The reduction of the problem of computing the rank of a tensor over a field to the minimal rank matrix completion is straightforward by Lemma 3.2, see Section 7 for details. The opposite reduction was given by Derksen in [26], so he showed that the minimal rank completion and tensor rank are polynomial time equivalent problems in the case of fields. However, the algorithmic complexity of both problems remained open, so we need to prove the following result on the way to Theorem 2.5.

**Theorem 3.3.** *Let $\mathcal{R} \subseteq \mathcal{S}$ be commutative rings. The problem of deciding if a given incomplete matrix with entries in $\mathcal{R} \cup \{*\}$ has a completion of rank three with respect to $\mathcal{S}$ is polynomial time equivalent to the problem of deciding if a given system of polynomial equations with coefficients in $\mathcal{R}$ has a solution over $\mathcal{S}$.*

The author believes that this result is new even in the case of fields. Numerous related problems are shown to be NP-hard, and, as noted by Derksen in [25], the NP-hardness of the problem being discussed in the case of fields follows from the earlier paper by Peeters [57]. There are several related problems whose complexity is described completely, which include a result similar to Theorem 3.3 but for the version of the minimal rank problem in which some of the $*$ entries may be required to take the same value [12]. However, the complexity of our version was discussed by Buss, Frandsen, Shallit without any progress on lower bounds, see page 575 in [12]. Also, Laurent writes that the minimal rank completion seems to be a difficult task, but again she does not mention any particular result on the complexity of this problem, see page 1972 in [47]. As said above, our Theorem 3.3 does not only prove the NP-hardness, but fully determines the complexity of the problem over any commutative ring. Other related completion problems whose complexity has been known are the *Euclidean distance completion* [47], *minimal rank sign pattern completion* [8], and the problem of minimizing the rank of matrices over a finite field that *fit* a given graph [57]. We note that the approximate version of the minimal rank completion problem naturally arises in applied mathematics [15, 40], and a version with random positions of the $*$'s is particularly important [16].

## 4. On the complexity of matrix rigidity

As explained in the introduction, the rigidity of matrices is a well known topic useful in arithmetic complexity theory [3, 4, 30, 33, 82]. Although the algorithmic complexity of this concept was discussed in many research papers [29, 51, 60, 63], no general lower bound on its complexity was known except the NP-hardness result of Deshpande [27]. In particular, Fomin, Lokshtanov, Meesum, Saurabh, Zehavi [29] give an extensive study of parametrized versions of matrix rigidity over $\mathbb{R}$, but if the restrictions on the corresponding parameters are omitted, the time complexity of their algorithm becomes exponential. Mahajan, Sarma [51, 63] use the connection of matrix rigidity to matrix completion and prove that, if the former problem belongs to NP, then the latter problem is in NP as well. Also, as explained by Mahajan, Sarma [51, 63], Ramya [60], the real version of the problem is in PSPACE.

Moreover, before the recent work of Babai, Kivva [4], there has been no progress reported on a related question of the dependence of the answer to the matrix rigidity problem on the choice of the field of the output. Another motivation to raise this issue comes, as explained in [4], from the recent paper of Dvir, Edelman [28] in which a certain family of matrices, previously expected to be rigid by some experts in the topic, were shown to be non-rigid with respect to some extension of the initial field of their definition as at the same time their rigidity with respect to the initial field remained open. Our approach to the computational complexity of matrix rigidity covers the situation when the output matrices are taken in a larger field, so we formulate this issue in the decision form as follows.

*Remark* 4.1. As explained above, the practical applications are focused on the conventional definition of matrix rank over a field. However, we can take a slightly more general perspective because, as in Definition 2.1, we define the rank of a matrix $A$ with respect to an arbitrary commutative ring $\mathcal{S}$ as the smallest integer $k$ such that $A$ represents as the product of an $m \times k$ and $k \times n$ matrices over $\mathcal{S}$. Since this definition coincides with the conventional rank over the fraction field of $\mathcal{S}$ whenever $\mathcal{S}$ is a principal ideal domain, by the *Smith normal form* [78], we can assume that the ring $\mathcal{S}$ in Problem 4.2 below is a principal ideal domain.

**Problem 4.2** (*rigidity* of a matrix over $\mathcal{R}$ *with respect to* $\mathcal{S}$)**.**
Given: A matrix $A$ with entries in $\mathcal{R}$, integers $k$, $r$.
Question: Do there exist matrices $L$, $S$ with entries in $\mathcal{S}$ such that $A = L + S$, the rank of $L$ is at most $r$, and $S$ has at most $k$ nonzero entries?

In our considerations on the complexity of matrix rigidity, we sometimes need the following mild technical assumption, which is obviously true in all relevant cases.

**Definition 4.3.** Let $\mathcal{R}$ be an infinite ring as in Remark 1.1. An *injective sequence oracle* on $\mathcal{R}$ is an algorithm that

- takes a positive integer $n$ as input,
- halts in time polynomial in $n$,
- returns a sequence $(r_1, \ldots, r_n)$ of pairwise different elements of $\mathcal{R}$.

**Example 4.4.** For instance, the function $n \to (1, 2, \ldots, n)$ represents an injective sequence oracle whenever $\mathcal{R}$ is a field of characteristic 0.

Now we formulate our main result on the complexity of matrix rigidity.

**Theorem 4.5.** *Let $\mathcal{S}$ be a principal ideal domain containing $\mathcal{R}$. Then,*

(1) *if $\mathcal{R}$ is endowed with an injective sequence oracle, then Problem 4.2 allows a polynomial reduction from the problem to decide if a given system of polynomial equations with coefficients in $\mathcal{R}$ has a solution in $\mathcal{S}$,*
(2) *if $\mathcal{S}$ is a field, then there is a polynomial reduction in the opposite direction.*

**Corollary 4.6.** *If $\mathcal{S}$ is a field, and $\mathcal{R} \subseteq \mathcal{S}$ is endowed with an injective sequence oracle, then Problem 4.2 is polynomial time equivalent to the problem to decide if a given system of polynomial equations with coefficients in $\mathcal{R}$ has a solution in $\mathcal{S}$.*

*Remark* 4.7. If $\mathcal{S}$ is finite, then both problems in Corollary 4.6 are NP-complete [27], so the conclusion remains valid. Therefore, it may only fail if $\mathcal{S}$ is infinite but the subring $\mathcal{R}$ has no injective sequence oracle, and a generalization of our reduction to this setting may require more technical efforts. However, it may not be relevant because Theorem 4.5 is sufficient for all cases considered in current work.

Now we look at some earlier work related to Theorem 4.5. Mahajan, Sarma [51, 63] study the complexities of several versions of matrix rigidity restricted to the class NP by declaring, for instance, that the entries of the corresponding matrix $S$ are taken in some finite subset of $\mathcal{S}$ fixed in advance. As said in the introduction, they state that *'an upper bound of NP is not obvious'* over $\mathbb{Z}$, so it might have been considered plausible that the integer version of matrix rigidity is in NP.

**Question 4.8** ([51, 63])**.** If $\mathcal{R} = \mathcal{S} = \mathbb{Z}$, do we have Problem 4.2 $\in$ NP?

This question was reiterated in a weaker form by the authors of [29], who said that the corresponding problem is not known to be decidable. As we see from Theorem 4.5, the setting $\mathcal{R} = \mathcal{S} = \mathbb{Z}$ makes Problem 4.2 undecidable, so we get a negative answer to Question 4.8 and resolve the above mentioned weaker problem in [29]. In other words, the integer version of matrix rigidity is undecidable, and the rational number analogue of this question was also discussed in literature.

**Question 4.9** ([29, 60])**.** If $\mathcal{R} = \mathcal{S} = \mathbb{Q}$, what is the complexity of Problem 4.2?

Corollary 4.6 gives a full answer to this question in the way similar to the discussion after Theorem 2.5 above. Namely, the rational number vesion of matrix rigidity turns out to be polynomial time equivalent to the famous decision problem of whether a given *Diophantine equation* has a rational solution or not [41, 42, 52, 54, 58]. Let us also comment on the real number analogue of this question.

**Question 4.10** ([60])**.** If $\mathcal{R} = \mathbb{Q}$ and $\mathcal{S} = \mathbb{R}$, what is the complexity of Problem 4.2?

Ramya [60] noticed that this problem belongs to PSPACE (by using the membership of the minimum rank matrix completion in $\exists\mathbb{R}$ and an exhaustive search over all possible zero-nonzero patterns of the matrices $S$ in Problem 4.2). From Corollary 4.6, we immediately see that Problem 4.2 is $\exists\mathbb{R}$-complete in the setting $\mathcal{R} = \mathbb{Q}$, $\mathcal{S} = \mathbb{R}$, so we get a resolution of Question 4.10 as well. We proceed with a related question of the dependence of the answer to the matrix rigidity problem on the choice of $\mathcal{S}$, which was raised in a recent paper of Babai, Kivva [4].

**Question 4.11** ([4])**.** Suppose $\mathcal{S} \subset \mathcal{S}'$ are different fields containing the entries of a matrix $A$ in Problem 4.2. Can the answer be different for $\mathcal{S}$ and $\mathcal{S}'$?

Babai, Kivva [4] exhibit a field $\mathbb{F}$ and a matrix whose rigidity with respect to $\mathbb{F}$ is different from the one over $\mathcal{S} = \mathbb{F}[\sqrt{2}]$, so they obtain an affirmative answer to Question 4.11. Apart from the algorithmic complexity results in Theorem 4.5, our technique gives an answer to a much stronger form of this question.

**Theorem 4.12.** *If $\mathbb{F}$ is an infinite field, and $\mathbb{F}'$ is a proper algebraic extension of $\mathbb{F}$, then there are a matrix $A$ with entries in $\mathbb{F}$ and integers $k$, $r$ such that*

- *$(A, k, r)$ is a no-instance of Problem 4.2 with $\mathbb{F}$ in the role of both $\mathcal{R}$, $\mathcal{S}$,*
- *$(A, k, r)$ is a yes-instance of Problem 4.2 with $(\mathbb{F}, \mathbb{F}')$ in the role of $(\mathcal{R}, \mathcal{S})$.*

As in the main results of Sections 2 and 3, the proofs of Theorems 4.5 and 4.12 are to be given in due course in the forthcoming technical part of our paper.

## 5. Organization of the paper

The following Section 6 is devoted to the proof of Theorem 3.3, which can be seen as a variation of a recent investigation of the complexity of the *positive semidefinite rank* of a matrix [72]. As a byproduct of our approach, we get solutions of two problems on *fractional minimal rank* posed by Grossmann and Woerdeman [34]. In Section 7, we employ the construction recently used by Derksen [25] and deduce Theorem 2.4 from Theorem 3.3. As said above, an immediate consequence of this construction is that Theorem 3.3 implies Theorem 2.4 over a field, and we adapt this method to any integral domain. In Section 8, we switch to symmetric tensor decompositions and complete the proof of Theorem 2.8. The remaining Section 9 is devoted to matrix rigidity and contains proofs of Theorems 4.5 and 4.12.

## 6. The complexity of minimal rank matrix completion

The main goal of this section is to prove Theorem 3.3 and hence determine the algorithmic complexity of the minimal rank matrix completion problem. To this end, it would be sufficient to assume that $\mathcal{R}$ and $\mathcal{S}$ are commmutative rings satisfying $\mathcal{R} \subseteq \mathcal{S}$, but the above mentioned application to the problems of Grossmann and Woerdeman [34] requires us to consider a slightly more general setting.

*Remark* 6.1. In this section, we assume that $\mathcal{R}$ and $\mathcal{S}$ are rings satisfying $\mathcal{R} \subseteq \mathcal{S}$, and, additionally, we assume that the implication

$$(C'C = I) \to (CC' = I) \tag{6.1}$$

holds for the $3 \times 3$ matrices $C$, $C'$ over $\mathcal{S}$. In particular, this happens when $\mathcal{S}$ is commutative or when $\mathcal{S}$ itself is a matrix ring over a field.

We proceed with several notational conventions needed to reflect the algorithmic context of the problem. In this section, we represent polynomials with variables $x_1, \ldots, x_n$ and coefficients in $\mathcal{R}$ as elements of the free ring $\mathbb{Z}\langle \mathcal{R}, x_1, \ldots, x_n \rangle$, which are non-commutative polynomials with integer coefficients, where apart from the variables $x_1, \ldots, x_n$ we have one additional variable for every element in $\mathcal{R}$. In other words, the elements of $\mathbb{Z}\langle \mathcal{R}, x_1, \ldots, x_n \rangle$ are the sums

$$p_1 + \ldots + p_s \tag{6.2}$$

in which every $p_i$ is a monomial with either $p_i = w_i$ or $p_i = -w_i$, where

$$w_i = \pi_{i1}\pi_{i2}\dots\pi_{iq_i} \tag{6.3}$$

is a word in the alphabet $\mathcal{R} \cup \{x_1, \dots, x_n\}$. The product of monomials is defined as the concatenation of the corresponding words, and the sign of the product is the product of the signs of the multipliers. Two elements of the form (6.2) are equal if they can be brought to the same form by appropriate permutations of the monomials and cancellations of the pairs of identical words appearing with opposite signs. Every element $f$ in $\mathbb{Z}\langle\mathcal{R}, x_1, \dots, x_n\rangle$ naturally defines a function from $\mathcal{S}^n$ to $\mathcal{S}$, where the value $f(\xi_1, \dots, \xi_n)$ is computed by replacing every occurrence of $x_i$ with $\xi_i$ in $f$ with a subsequent evaluation of the obtained expression over $\mathcal{S}$.

*Remark* 6.2. We write $\mathbb{O}$ to denote the empty sum as in (6.2), and we define $\mathbb{I}$ as the empty word of the form (6.3) taken with the positive sign. The elements $\mathbb{O}$ and $\mathbb{I}$ are the zero and one of the ring $\mathbb{Z}\langle\mathcal{R}, x_1, \dots, x_n\rangle$, respectively.

**Example 6.3.** For $a \in \mathcal{R}$, the polynomial $xy - yx + a$ belongs to $\mathbb{Z}\langle\mathcal{R}, x, y\rangle$. If $\mathcal{S}$ is commutative, it represents the constant function always equal to $a$.

We proceed with a reduction needed for the proof of Theorem 3.3. For any monomial $p_i = +w_i$ or $p_i = -w_i$ with $w_i$ as in (6.3), we define

$$\sigma(p_i) = \{\pm\mathbb{I},\, \pm\pi_{i1},\, \pm\pi_{i1}\pi_{i2},\, \pm\pi_{i1}\pi_{i2}\pi_{i3},\, \dots,\, \pm p_i\} \cup \{\pm\pi_{i1},\, \pm\pi_{i2},\, \dots,\, \pm\pi_{iq_i}\}\,.$$

For a general input polynomial $f$ represented as $p_1 + \ldots + p_s$, we set

$$\sigma(f) = \sigma(p_1) \cup \ldots \cup \sigma(p_s) \cup \{\mathbb{O}, \pm p_1, \pm(p_1 + p_2), \dots, \pm f\},$$

and for a finite set $F = \{f_1, \dots, f_t\}$, we take $\sigma(F) = \sigma(f_1) \cup \ldots \cup \sigma(f_t)$. Clearly, the construction of the set $\sigma(F)$ can be done in polynomial time.

**Example 6.4.** If $F = \{xy - yx + a\}$ with $a \in \mathcal{R}$, then

$$\sigma(F) = \{\mathbb{O},\, \pm\mathbb{I},\, \pm x,\, \pm y,\, \pm xy,\, \pm yx,\, \pm a,\, \pm(xy - yx),\, \pm(xy - yx + a)\}. \tag{6.4}$$

Now we fix a finite set $F$ of input polynomials, each of which is represented as a sum of monomials in $\mathbb{Z}\langle\mathcal{R}, x_1, \dots, x_n\rangle$. In the rest of this section, we simply write $\sigma$ instead of $\sigma(F)$, and $\sigma^3$ stands for the set of all triples of elements in $\sigma$.

**Definition 6.5.** We denote by $\mathcal{H} = \mathcal{H}(F)$ the set of those vectors in $\sigma^3$ that have one of the coordinates equal to $\mathbb{I}$ or $-\mathbb{I}$. We define the matrix $\mathcal{U} = \mathcal{U}(x_1, \dots, x_n)$ whose columns are vectors in $\mathcal{H}$, and we define $\mathcal{W}(x_1, \dots, x_n) = \mathcal{U}^\top\mathcal{U}$.

**Example 6.6.** If $F = \{xy - yx + a\}$ with $a \in \mathcal{R}$, then $\mathcal{H}$ is the set of all triples of the elements in (6.4) which have at least one of the coordinates equal to either $\mathbb{I}$ or $-\mathbb{I}$. Since there are 15 elements in (6.4) different from $\pm\mathbb{I}$, we have a total of

$$2^3 + 3 \cdot 2^2 \cdot 15 + 3 \cdot 2 \cdot 15^2$$

or 1538 elements in $\mathcal{H}$ in this case.

In what follows, we label the columns of $\mathcal{U}$ by the corresponding elements of $\mathcal{H}$. In particular, the $(u, v)$ entry of $\mathcal{W}$ with $u, v \in \mathcal{H}$ equals the dot product $u \cdot v$.

**Definition 6.7.** For all $u, v \in \mathcal{H}$, the polynomial $\delta(u, v) \in \mathbb{Z}\langle\mathcal{R}, x_1, \dots, x_n\rangle$ is defined as $u \cdot v$, or, equivalently, by the formula $\delta(u, v) = \mathcal{W}(u|v)$. We define the $\mathcal{H} \times \mathcal{H}$ matrix $\mathcal{B} = \mathcal{B}(F)$ with entries in $\mathcal{R} \cup \{*\}$ as follows:

(B1) if $\delta(u, v) = \rho$ with $\rho \in \mathbb{Z}\langle\mathcal{R}\rangle$, then we define $\mathcal{B}(u|v) \in \mathcal{R}$ as the value of $\rho$,

(B2) if $\delta(u, v) = f$ with $f \in F$, then we take $\mathcal{B}(u|v) = 0$,
(B3) in the remaining cases, we set $\mathcal{B}(u|v) = *$.

**Example 6.8.** We note that $\mathcal{B}(xy - yx + a)$ is an incomplete matrix with known entries in $\mathbb{Z}[a] \subseteq \mathcal{R}$. According to Example 6.6, its size is $1538 \times 1538$.

It is clear that the matrix $\mathcal{W}(\xi_1, \ldots, \xi_n)$ is a rank-three completion of $\mathcal{B}$ provided that $(\xi_1, \ldots, \xi_n) \in \mathcal{S}^n$ is a simultaneous solution of the equations $f_1 = 0, \ldots, f_t = 0$. We are going to show that all rank-three completions arise in this way up to the natural action of the group of invertible $3 \times 3$ matrices.

**Lemma 6.9.** *Let $P$ be an $\mathcal{H} \times 3$ matrix over $\mathcal{S}$, and let $L$ be a $3 \times \mathcal{H}$ matrix over $\mathcal{S}$ such that the product $PL$ is a completion of $\mathcal{B}$. Let $C$ be the matrix obtained by taking the columns of $L$ with indexes in $E = \{(\mathbb{I}, \mathbb{O}, \mathbb{O}), (\mathbb{O}, \mathbb{I}, \mathbb{O}), (\mathbb{O}, \mathbb{O}, \mathbb{I})\}$. Then*

$$PC = \mathcal{U}(\xi_1, \ldots, \xi_n)^\top \quad \text{and} \quad C^{-1}L = \mathcal{U}(\xi_1, \ldots, \xi_n), \tag{6.5}$$

*where $(\xi_1, \ldots, \xi_n)$ is a simultaneous solution of the equations $f_1 = 0, \ldots, f_t = 0$.*

*Proof. Step 1.* Since the $3 \times 3$ submatrix of $\mathcal{B}$ with row and column indexes in $E$ is the unity matrix, we get that the submatrix $C'$ of $P$ formed by the rows with indexes in $E$ satisfies $C'C = I$, where $I$ is the $3 \times 3$ identity matrix. Since the condition (6.1) in Remark 6.1 applies, we also have $CC' = I$, and the transformation $(P, L) \to (PC, C'L)$ cannot change the property of $PL$ to be a completion of $\mathcal{B}$. Therefore, we can assume without loss of generality that $C$ is the unity matrix, which means that the rows of $P$ with indexes in $E$ and the columns of $L$ with indexes in $E$ already satisfy the desired conclusion as in the equalities (6.5).

*Step 2.* For any $u \in \mathcal{H}$, we denote the $u$th row of $P$ by $p(u)$, and we denote the $u$th column of $L$ by $l(u)$. The assumption of the lemma states that the product $PL$ is a completion of $\mathcal{B}$, which means that

$$p(u) \cdot l(v) = \mathcal{B}(u|v) \quad \text{whenever } \mathcal{B}(u|v) \neq *. \tag{6.6}$$

Using this language, we can rewrite the result of Step 1 as

$$p(\mathbb{I}, \mathbb{O}, \mathbb{O}) = (1, 0, 0), \;\; p(\mathbb{O}, \mathbb{I}, \mathbb{O}) = (0, 1, 0), \;\; p(\mathbb{O}, \mathbb{O}, \mathbb{I}) = (0, 0, 1), \tag{6.7}$$

$$l(\mathbb{I}, \mathbb{O}, \mathbb{O}) = (1, 0, 0), \;\; l(\mathbb{O}, \mathbb{I}, \mathbb{O}) = (0, 1, 0), \;\; l(\mathbb{O}, \mathbb{O}, \mathbb{I}) = (0, 0, 1). \tag{6.8}$$

Now let $u \in \mathcal{H}$ be a vector whose $j$-th coordinate belongs to $\mathbb{Z}\langle\mathcal{R}\rangle$, that is, this coordinate does not depend on $x_1, \ldots, x_n$, so, as a function, it identically equals to some $u_j \in \mathcal{R}$. Straightforwardly, if we now write $e_j$ for the length three vector with $\mathbb{I}$ at the $j$-th position and with $\mathbb{O}$'s everywhere else, the product $e_j \cdot u$ equals the $j$-th coordinate of $u$, and we get

$$\mathcal{B}(e_j|u) = u_j \tag{6.9}$$

by the item (B1) of Definition 6.7, and hence

$$L(j|u) = p(e_j) \cdot l(u) = u_j, \tag{6.10}$$

where the first equality comes from (6.7), and the second equality is deduced from (6.9) by the condition (6.6). From (6.10) we get that $L(j|u)$ equals the value of $\mathcal{U}(j|u)$ whenever $\mathcal{U}(j|u) \in \mathbb{Z}\langle\mathcal{R}\rangle$, and, either by the symmetry of the construction or with a similar argument using (6.8) instead of (6.7), we get that $P(u|j)$ equals the result of the evaluation of $\mathcal{U}(j|u)$, again provided that $\mathcal{U}(j|u) \in \mathbb{Z}\langle\mathcal{R}\rangle$.

*Step 3.* Using Step 2, we get that, for any variable $x_i$, there is $y_i \in \mathcal{S}$ such that

$$l(\mathbb{I}, \mathbb{O}, x_i) = (1, 0, y_i). \tag{6.11}$$

In what follows, we denote by $y_i$ the element of $\mathcal{S}$ such that the formula (6.11) is satisfied. We also write $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

*Step 4.* We say that the label $u = (a, b, c)$ is *p-good* if

$$p(u) = (a(y), b(y), c(y)),$$

where $a(y)$ is the result of the evaluation of the polynomial $a \in \mathbb{Z}\langle \mathcal{R}, x_1, \ldots, x_n\rangle$ at the point $y = (y_1, \ldots, y_n)$. Similarly, we say that $u$ is *l-good* if

$$l(u) = (a(y), b(y), c(y)).$$

By Step 2, the labels consisting of elements in $\mathbb{Z}\langle \mathcal{R}\rangle$ are both necessarily $p$-good and $l$-good. In order to complete the proof, we need to check that

(i) every label in $\mathcal{H}$ is both $p$-good and $l$-good, and

(ii) $f(y) = 0$ for all $f \in F$.

*Step 5.* Now let us see what happens if a vector $(g, \mathbb{O}, h)$ is $l$-good.

*Step 5.1.* Since either $g = \pm\mathbb{I}$ or $h = \pm\mathbb{I}$ by Definition 6.5, we have $gh = hg$ and

$$(-h, g, g) \cdot (g, \mathbb{O}, h) = (-h, g, g) \cdot (\mathbb{O}, -\mathbb{I}, \mathbb{I}) = \mathbb{O}, \tag{6.12}$$

so we can use the item (B1) of Definition 6.7 to get

$$\mathcal{B}(-h, g, g|g, \mathbb{O}, h) = \mathcal{B}(-h, g, g|\mathbb{O}, -\mathbb{I}, \mathbb{I}) = 0, \tag{6.13}$$

and then the application of (6.6) to (6.13) gives

$$p(-h, g, g) \cdot l(g, \mathbb{O}, h) = p(-h, g, g) \cdot l(\mathbb{O}, -\mathbb{I}, \mathbb{I}) = 0. \tag{6.14}$$

Since the vector $(g, \mathbb{O}, h)$ is $l$-good by the assumption of Step 5, and the vector $(\mathbb{O}, -\mathbb{I}, \mathbb{I})$ is $l$-good by the result of Step 2, the equalities (6.14) imply

$$p(-h, g, g) \cdot (g(y), 0, h(y)) = p(-h, g, g) \cdot (0, -1, 1) = 0$$

or that $p(-h, g, g) = (\pi_1, \pi_2, \pi_3) \in \mathcal{S}^3$ with

$$\pi_1 g(y) + \pi_3 h(y) = 0 \text{ and } \pi_2 = \pi_3. \tag{6.15}$$

Again, since either $g = \pm\mathbb{I}$ or $h = \pm\mathbb{I}$, the equalities (6.15) can be used to check that $(-h, g, g)$ is a $p$-good vector[1]. A similar consideration starting from the equalities

$$(-h, g, g) \cdot (g, h, \mathbb{O}) = (\mathbb{O}, \mathbb{O}, \mathbb{I}) \cdot (g, h, \mathbb{O}) = \mathbb{O},$$

taken instead of (6.12), shows that the vector $(g, h, \mathbb{O})$ is $l$-good.

*Step 5.2.* Still assuming that $(g, \mathbb{O}, h)$ is $l$-good, we deduce that $(-h, \mathbb{O}, g)$ is $p$-good by the argument as in Step 5.1 starting from

$$(-h, \mathbb{O}, g) \cdot (g, \mathbb{O}, h) = (-h, \mathbb{O}, g) \cdot (\mathbb{O}, \mathbb{I}, \mathbb{O}) = \mathbb{O}.$$

Similarly, the condition that $(-h, \mathbb{O}, g)$ is $p$-good in turn implies that $(g, \mathbb{O}, h)$ is $l$-good, again by the same argument but starting from

$$(-h, \mathbb{O}, g) \cdot (g, \mathbb{O}, h) = (\mathbb{O}, \mathbb{I}, \mathbb{O}) \cdot (g, \mathbb{O}, h) = \mathbb{O}.$$

*Step 5.3.* Now we can get the main conclusion of Step 5, using the symmetry and results of Steps 5.1 and 5.2. Namely, a vector $(g, \mathbb{O}, h)$ is $l$-good if and only

[1]For instance, if $g = \mathbb{I}$, then $\pi_2 = \pi_3 = 1$ by the result of Step 2, and we get $\pi_1 = -h(y)$ from the first equality in (6.15). The cases when $g = -\mathbb{I}$ or $h = \pm\mathbb{I}$ can be treated in a similar fashion.

if any permutation of $(g, \mathbb{O}, h)$ is $l$-good, which in turn happens if and only if any permutation of $(-h, \mathbb{O}, g)$ is $p$-good.

*Step 6.* Now we assume that $(\mathbb{I}, \mathbb{O}, \alpha)$, $(\mathbb{I}, \mathbb{O}, \beta)$ are $l$-good vectors.

*Step 6.1.* Further, we assume that $\alpha + \beta$ is in the set $\sigma$ from Definition 6.5. Using the argument outlined in Step 5.1 but with

$$(\mathbb{I}, \mathbb{I}, \mathbb{O}) \cdot (-\mathbb{I}, \mathbb{I}, \alpha) = (\mathbb{O}, -\alpha, \mathbb{I}) \cdot (-\mathbb{I}, \mathbb{I}, \alpha) = \mathbb{O}$$

instead of (6.12), we conclude that $(-\mathbb{I}, \mathbb{I}, \alpha)$ is an $l$-good vector[2]. Similarly, the vector $(-\beta, -\alpha - \beta, \mathbb{I})$ can be shown to be $p$-good starting from the equalities

$$(-\beta, -\alpha - \beta, \mathbb{I}) \cdot (-\mathbb{I}, \mathbb{I}, \alpha) = (-\beta, -\alpha - \beta, \mathbb{I}) \cdot (\mathbb{I}, \mathbb{O}, \beta) = \mathbb{O}$$

involving the $l$-good vectors $(-\mathbb{I}, \mathbb{I}, \alpha)$ and $(\mathbb{I}, \mathbb{O}, \beta)$. Finally, the fact that the vector $(\mathbb{O}, \mathbb{I}, \alpha + \beta)$ is $l$-good can be shown from the equalities

$$(\mathbb{I}, \mathbb{O}, \mathbb{O}) \cdot (\mathbb{O}, \mathbb{I}, \alpha + \beta) = (-\beta, -\alpha - \beta, \mathbb{I}) \cdot (\mathbb{O}, \mathbb{I}, \alpha + \beta) = \mathbb{O}$$

involving the $p$-good vectors $(-\beta, -\alpha - \beta, \mathbb{I})$ and $(\mathbb{I}, \mathbb{O}, \mathbb{O})$. Therefore, we can conclude that the vector $(\mathbb{I}, \mathbb{O}, \alpha + \beta)$ is $l$-good by the result of Step 5.3.

*Step 6.2.* Now we switch to the case when $\beta\alpha$ belongs to $\sigma$. Then, the vector $(\beta\alpha, \mathbb{I}, \alpha)$ is $l$-good by the argument similar to that in Step 5.1 but starting from

$$(\mathbb{O}, -\alpha, \mathbb{I}) \cdot (\beta\alpha, \mathbb{I}, \alpha) = (\mathbb{I}, \mathbb{O}, -\beta) \cdot (\beta\alpha, \mathbb{I}, \alpha) = \mathbb{O}$$

instead of (6.12), where the vectors $(\mathbb{O}, -\alpha, \mathbb{I})$ and $(\mathbb{I}, \mathbb{O}, -\beta)$ are $p$-good by Step 5. Similarly, the vector $(\mathbb{I}, -\beta\alpha, \mathbb{O})$ is $p$-good by the same argument starting with

$$(\mathbb{I}, -\beta\alpha, \mathbb{O}) \cdot (\mathbb{O}, \mathbb{O}, \mathbb{I}) = (\mathbb{I}, -\beta\alpha, \mathbb{O}) \cdot (\beta\alpha, \mathbb{I}, \alpha) = \mathbb{O}.$$

Therefore, the vector $(\mathbb{I}, \mathbb{O}, \beta\alpha)$ is $l$-good by Step 5.3.

*Step 7.* The results of Step 6 show that the vector $(\mathbb{I}, \mathbb{O}, s)$ is $l$-good for all $s \in \sigma$.

*Step 8.* In order to check the condition (i) as in Step 4, because of the symmetry, it is sufficient to check that the vector $(\mathbb{I}, u, v)$ is $l$-good whenever $u, v$ are taken in $\sigma$. Again, this follows from the argument similar to Step 5.1 applied to the equalities

$$(-u, \mathbb{I}, \mathbb{O}) \cdot (\mathbb{I}, u, v) = (-v, \mathbb{O}, \mathbb{I}) \cdot (\mathbb{I}, u, v) = \mathbb{O}$$

involving the vectors $(-u, \mathbb{I}, \mathbb{O})$ and $(-v, \mathbb{O}, \mathbb{I})$ that are $p$-good by Steps 5 and 7.

*Step 9.* In order to check the condition (ii) as in Step 4, we consider an arbitrary polynomial $f$ in $F$. Since we have

$$(\mathbb{O}, \mathbb{O}, \mathbb{I}) \cdot (\mathbb{I}, \mathbb{O}, f) = f,$$

we get $\mathcal{B}(\mathbb{O}, \mathbb{O}, \mathbb{I} | \mathbb{I}, \mathbb{O}, f) = 0$ by the item (B2) of Definition 6.7. This implies

$$p(\mathbb{O}, \mathbb{O}, \mathbb{I}) \cdot l(\mathbb{I}, \mathbb{O}, f) = 0$$

by the condition (6.6), and hence

$$(0, 0, 1) \cdot (1, 0, f(y)) = 0 \tag{6.16}$$

because, according to Step 8, the vectors $(\mathbb{O}, \mathbb{O}, \mathbb{I})$, $(\mathbb{I}, \mathbb{O}, f)$ are both $p$-good and $l$-good. Since the equality (6.16) means that $f(y) = 0$, the proof is complete. □

The following corollary is immediate from Lemma 6.9.

**Corollary 6.10.** *The matrix $\mathcal{B}(F)$ admits a completion of rank three with respect to $\mathcal{S}$ if and only if the equations $f_1 = 0, \ldots, f_t = 0$ have a common solution in $\mathcal{S}$.*

[2]This is possible because $(\mathbb{I}, \mathbb{I}, \mathbb{O})$ is $p$-good by Step 2 and $(\mathbb{O}, -\alpha, \mathbb{I})$ is $p$-good by Step 5.

*Proof.* As said above, the matrix $\mathcal{W}(\xi_1, \ldots, \xi_n)$ is a rank-three completion of $\mathcal{B}$, provided that $(\xi_1, \ldots, \xi_n)$ is a simultaneous solution of the polynomial equations $f_1 = 0, \ldots, f_t = 0$. Conversely, if there is no such a solution over $\mathcal{S}$, then by Lemma 6.9 the matrix $\mathcal{B}$ admits no completion of rank three with respect to $\mathcal{S}$. □

Since the reduction $F \to \mathcal{B}(F)$ is polynomial time, we get Theorem 3.3 from Corollary 6.10. Now we proceed with the application of our technique to the problems of Grossmann and Woerdeman [34]. Namely, for a positive integer $b$ and an incomplete $m \times n$ matrix $A$ with known entries in a field $\mathcal{F}$, they define $A \otimes I_b$ to be the $mb \times nb$ incomplete matrix seen as the $m \times n$ block matrix in which

(1) if $A(i|j) = *$, then the $(i, j)$ block is the $b \times b$ block of $*$'s,

(2) if $A(i|j) \neq *$, then the $(i, j)$ block is the scalar matrix $A(i|j)I_b$.

The authors of [34] define the *fractional minimal rank* of $A$ as

$$\operatorname{fmr}(A) = \inf_b \frac{\min \operatorname{rk} A \otimes I_b}{b} \tag{6.17}$$

with the numerator being the minimal rank of any completion of $A \otimes I_b$ over $\mathcal{F}$. Question 4 in Section 5 of [34] asked, is the infimum in (6.17) necessarily attained? We give a negative resolution of this question. We decided to work with $\mathcal{F} = \mathbb{Q}$, but the following construction is vaild over any field of characteristic zero.

**Example 6.11.** Let $\mathcal{R} = \mathcal{F} = \mathbb{Q}$ and consider the matrix $B = \mathcal{B}(xy - yx + 1)$ as in Example 6.8. Then $\operatorname{fmr}(B) = 3$ and $\min \operatorname{rk} B \otimes I_b \geqslant 3b + 1$ for all $b$.

*Proof.* For $b > 1$, we can think of $B \otimes I_b$ as the matrix $\mathcal{B}(xy - yx + 1)$ but constructed with respect to the ring $\mathcal{R} = \operatorname{Mat}_b(\mathbb{Q})$ instead of $\mathcal{R} = \mathbb{Q}$. We note that the polynomial $xy - yx + 1$ cannot be vanished by matrices over a field of characteristic zero because the traces of $xy$ and $yx$ are equal. We apply Lemma 6.9, which is valid in this setting because of Remark 6.1, and we conclude that no completion of $B \otimes I_b$ can be represented as the product $PL$ of rational matrices unless $P$ has more than $3b$ columns. In other words, we have $\min \operatorname{rk} B \otimes I_b \geqslant 3b + 1$ for all $b$.

Now let $D_b$ be the $b \times b$ matrix with numbers $1, \ldots, 1, 1 - b$ on the main diagonal and zeros everywhere else. A standard result of matrix theory [75] shows that every trace-zero matrix over $\mathbb{Q}$ is a commutator, so, if we considered the matrix $B'_b$ defined as $\mathcal{B}(xy - yx + D_b)$ with respect to the ring $\mathcal{S} = \operatorname{Mat}_b(\mathbb{Q})$, we would have

$$\min \operatorname{rk} B'_b \leqslant 3b.$$

As we see from the discussion in Example 6.8, the matrix $B \otimes I_b$ can be obtained from $B'_b$ by altering a fixed number of its entries, which implies

$$|\min \operatorname{rk} B \otimes I_b - \min \operatorname{rk} B'_b| = O(1) \text{ as } b \to \infty$$

and thus $\operatorname{fmr}(B) = 3$. □

Our technique allows one to solve another problem in [34]. The formulation of this problem requires the notion of the *triangular minimum rank* tmr of a partial matrix, but, since the corresponding definition is relatively complicated, we decided not to reproduce it here. The relevant properties are that

(T1) $\operatorname{tmr}(A) \leqslant \operatorname{fmr}(A) \leqslant \min \operatorname{rk} A$ for any incomplete matrix $A$, and

(T2) $\operatorname{tmr}(A) \geqslant \operatorname{rk} A'$ for any complete submatrix $A'$ of $A$.

**Conjecture 6.12** (Problem 3 in Section 5 of [34])**.** For all incomplete matrices $A$, we have either $\operatorname{tmr}(A) < \operatorname{fmr}(A) < \min \operatorname{rk} A$ or $\operatorname{tmr}(A) = \operatorname{fmr}(A) = \min \operatorname{rk} A$.

Using the matrix $B$ in Example 6.11, we can disprove this conjecture.

**Example 6.13.** We have $\operatorname{tmr}(B) = \operatorname{fmr}(B) = 3$ and $\min \operatorname{rk} B > 3$.

*Proof.* The equality $\operatorname{fmr}(B) = 3$ is immediate from Example 6.11. Further, we have $\min \operatorname{rk} B > 3$ by Corollary 6.10 because the equation $xy - yx + 1 = 0$ has no solutions over $\mathbb{Q}$. Also, we have $\operatorname{tmr}(B) \geqslant 3$ from the condition (T2) because $B$ contains a unit $3 \times 3$ submatrix (namely, this is the submatrix with the row and column indexes in the set $E$ as in Lemma 6.9). Finally, we have $\operatorname{tmr}(B) \leqslant 3$ from the condition (T1) because we already know that $\operatorname{fmr}(B) = 3$. □

## 7. A proof of Theorem 2.4

We switch to the setting of Theorem 2.4, so many results of this section require that the rings $\mathcal{R} \subseteq \mathcal{S}$ are integral domains, that is, they are commutative and have no zero divisors. In several lemmas, we will need to refer to the construction of the matrix $\mathcal{B}(F)$ and the corresponding family of polynomials $F = \{f_1, \ldots, f_t\}$ as in the previous section. The matrix $\mathcal{B}(F)$ is simply denoted as $\mathcal{B}$.

**Lemma 7.1.** *Let $\mathcal{F}$ be a field containing an integral domain $\mathcal{S}$. Assume that $W_1, W_2, W_3$ are rank-one matrices over $\mathcal{F}$ such that $W_1 + W_2 + W_3$ is a completion of $\mathcal{B}$. Assume that, for some $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{F}$, some rank-one matrix $W_0$ coincides with $\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3$ everywhere except possibly several entries that are $*$'s in $\mathcal{B}$. Then there is an element $\mu \in \mathcal{F}$ such that $W_0$ is one of $\mu W_1, \mu W_2, \mu W_3$.*

*Proof.* We define the *$E$-submatrix* of $\mathcal{B}$ as the one formed by the rows and columns with indexes in $E = \{(\mathbb{I}, \mathbb{O}, \mathbb{O}), (\mathbb{O}, \mathbb{I}, \mathbb{O}), (\mathbb{O}, \mathbb{O}, \mathbb{I})\}$. We note that this submatrix is the unity matrix and, in particular, it does not contain the $*$'s. The corresponding $E$-submatrices of $W_1, W_2, W_3$ are rank-one and sum to a rank-three matrix, so the rank of the $E$-submatrix of $\lambda_1 W_1 + \lambda_2 W_2 + \lambda_3 W_3$ equals the number of those $\lambda_i$'s that are nonzero. Therefore, it suffices to consider the case when $W_0$ coincides with $W_3$ everywhere except possibly several entries that are $*$'s in $\mathcal{B}$. We are going to show that $W_0 = W_3$; for $j = 0, 1, 2, 3$, we write

$$W_j = a_j b_j^\top \text{ with } a_j, b_j \in \mathcal{F}^{\mathcal{H}}.$$

Using an appropriate scaling, we assume that the coordinates of $a_0$ and $b_0$ with indexes in $E$ are equal to the corresponding coordinates of $a_3$ and $b_3$.

We define $P$ as the matrix formed by the columns $a_1, a_2, a_3$, and we take $L$ to be the matrix formed by the rows $b_1^\top, b_2^\top, b_3^\top$. The matrix $PL = W_1 + W_2 + W_3$ is a completion of $\mathcal{B}$, and Lemma 6.9 implies that

$$C^{-1} L = \mathcal{U}(\xi_1, \ldots, \xi_n),$$

where $(\xi_1, \ldots, \xi_n)$ is a simultaneous solution of $f_1 = 0, \ldots, f_t = 0$, and $C$ is the $3 \times 3$ matrix formed by the columns of $L$ with indexes in $E$. Similarly, we define $Q$ as the matrix formed by the rows $b_1^\top, b_2^\top, b_0^\top$, and we get that

$$C^{-1} Q = \mathcal{U}(\psi_1, \ldots, \psi_n),$$

where $(\psi_1, \ldots, \psi_n)$ is another simultaneous solution of $f_1 = 0, \ldots, f_t = 0$. The entries of $Q - L$ are all zero except possibly those in the third row, so the matrix

$$\mathcal{U}(\psi_1, \ldots, \psi_n) - \mathcal{U}(\xi_1, \ldots, \xi_n) = C^{-1}(Q - L) \tag{7.1}$$

has rank at most one. By its definition, the set $\mathcal{H}$ consists of vectors one of whose coordinates is constant, so that the matrix (7.1) has a zero in every column. Since the rank of (7.1) is at most one, it has a zero row, which suffices to conclude that $(\psi_1, \ldots, \psi_n) = (\xi_1, \ldots, \xi_n)$. Therefore, the matrix (7.1) is zero, which means that $Q = L$ or $b_0 = b_3$. Using the symmetry, we get $a_0 = a_3$ and $W_0 = W_3$. □

We are going to construct a reduction from the matrix completion problem to tensor rank, and we use the construction that previously appeared in [25].

**Definition 7.2.** Let $\mathcal{B}$ be the matrix as in Section 6; we recall that the rows and columns of $\mathcal{B}$ have indexes in the set $\mathcal{H}$. We enumerate by

$$k_1 = (i_1, j_1),\ \ldots,\ k_\tau = (i_\tau, j_\tau)$$

the entries which correspond to the $*$'s in $\mathcal{B}$, so $\tau$ is the number of such entries. We set $K = \{0, 1, \ldots, \tau\}$, and we define the $\mathcal{H} \times \mathcal{H} \times K$ tensor $\mathcal{A} = \mathcal{A}(\mathcal{B})$ as follows:

(1) $\mathcal{A}(u|v|t) = \mathcal{B}(u|v)$ if $t = 0$ and $\mathcal{B}(u|v) \neq *$,
(2) $\mathcal{A}(u|v|t) = 1$ if $k_t = (u, v)$,
(3) $\mathcal{A}(u|v|t) = 0$ in the remaining cases.

In other words, we begin by taking the matrix $\mathcal{B}$ as in Section 6, and we substitute its $*$'s with zeros to obtain the matrix which we further call $A$. Then we get $\mathcal{A}$ by the addition to $A$ of the $\tau$ new 3-slices equal to the matrix units corresponding to the positions of the $*$ entries of $\mathcal{B}$. Derksen [25] showed that

$$\operatorname{rk}_{\mathcal{S}} \mathcal{A}(\mathcal{B}) = \tau + \min \operatorname{rk}_{\mathcal{S}} \mathcal{B} \tag{7.2}$$

in the case when $\mathcal{S}$ is a field; this result comes from Lemma 3.2 as well. We are going to adapt the substitution technique and prove an appropriate analogue of (7.2) for any integral domain $\mathcal{S}$. More precisely, we prove that the inequality $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \leqslant \tau + 3$ is valid if and only if $\mathcal{B}$ admits a completion of rank three with respect to $\mathcal{S}$.

**Lemma 7.3.** *If $\mathcal{S}$ is an integral domain, then $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \geqslant \tau + 3$.*

*Proof.* Since $\mathcal{S}$ is an integral domain, there is a field containing $\mathcal{S}$, and the assertion follows from Lemma 3.2 or from the above mentioned result by Derksen. □

**Lemma 7.4.** *Let $\mathcal{S}$ be an integral domain. If $\mathcal{B}$ admits a completion of rank three with respect to $\mathcal{S}$, then $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \leqslant \tau + 3$.*

*Proof.* If $B$ is such a completion, then we can get a tensor $B_0$ of rank three over $\mathcal{S}$ by setting $B_0(u|v|t) = B(u|v)$ if $t = 0$ and $B_0(u|v|t) = 0$ otherwise. Further, we define a rank-one tensor $S_t$ whose entries are all zeros except $S_t(i_t|j_t|0) = -B(i_t|j_t)$ and $S_t(i_t|j_t|t) = 1$. We get $\mathcal{A} = B_0 + S_1 + \ldots + S_\tau$, so the result follows. □

**Lemma 7.5.** *If $\mathcal{S}$ is an integral domain and $\operatorname{rk}_{\mathcal{S}} \mathcal{A} \leqslant \tau + 3$, then $\mathcal{B}$ admits a completion of rank three with respect to $\mathcal{S}$.*

*Proof.* Let $\mathcal{F}$ be a field containing $\mathcal{S}$, and let

$$\mathcal{A} = S_1 + \ldots + S_{\tau+3}$$

be a decomposition of $\mathcal{A}$ into the sum of tensors that are rank-one with respect to $\mathcal{S}$. Let $V$ be the $\mathcal{F}$-linear space spanned by the zeroth 3-slices of $S_1, \ldots, S_{\tau+3}$ with the coordinates $k_1, \ldots, k_\tau$ removed. Since the 3-slices of $\mathcal{A}$ with indexes $1, \ldots, \tau$ are linearly independent and have zeros outside $k_1, \ldots, k_\tau$, we get $\dim V \leqslant 3$. We say that a 3-slice is *non-trivial* if it has a non-zero element somewhere except $k_1, \ldots, k_\tau$.

Therefore, if there were at least four $S_i$'s with non-trivial zeroth 3-slices, then these 3-slices would become linearly dependent after the removal of the $*$ positions.

Using Lemma 7.1, we would get that there are two $S_i$'s whose zeroth 3-slices are non-zero and coincide up to scalings by nonzero elements of $\mathcal{F}$. The sum of these two $S_i$'s would still be a simple tensor with respect to $\mathcal{F}$, which would imply $\operatorname{rk}_{\mathcal{F}} \mathcal{A} \leqslant \tau+2$ and contradict to Lemma 7.3. Therefore, there are at most three $S_i$'s with non-trivial zeroth 3-slices, and the sum of these 3-slices is a desired completion of $\mathcal{B}$. □

Lemmas 7.4 and 7.5 prove that $\operatorname{rk}_{\mathcal{S}} \mathcal{A}(\mathcal{B}) \leqslant \tau + 3$ if and only if $\mathcal{B}$ admits a completion of rank three with respect to an integral domain $\mathcal{S}$. By Corollary 6.10, such a completion exists if and only if the polynomials in the family $F$ as in Section 6 have a common zero over $\mathcal{S}$. Since the reduction $F \to \mathcal{A}(\mathcal{B}(F))$ can be computed in polynomial time, we complete the proof of Theorem 2.4.

## 8. Symmetric tensors

The goal of this section is to prove Theorem 2.8 and, more generally, the analogue of Theorem 2.4 for the symmetric rank of tensors over a field. Our argument is a reduction of the standard tensor rank problem to the symmetric version.

**Definition 8.1.** We say that a tensor $T_0$ is obtained from an $I \times J \times K$ tensor $T$ by *adjoining* an $I \times J$ matrix $A$ as a 3-slice if the 3-slices of $T_0$ are precisely those of $T$ and $A$. We use similar definitions for adjoining 1-slices and 2-slices.

**Definition 8.2.** For all $p, q \in I \cap J$, we define the *$(p, q)$-unit* as the $I \times J$ matrix $M$ such that $M(i|j) = 1$ if $i, j \in \{p, q\}$ and $M(i|j) = 0$ otherwise. In particular, such a matrix becomes a conventional matrix unit whenever $p = q$.

Now we are ready to present the main tool of this section.

**Definition 8.3.** Let $I$, $J$, $K$ be disjoint indexing sets, and let $T$ be an $I \times J \times K$ tensor over a field $\mathcal{F}$. We define the new indexing set $H = I \cup J \cup K$ and the $H \times H \times H$ tensor $S = S(T)$ as follows. For all $\alpha, \beta, \gamma \in H$, we take

(S1) $S(\alpha|\beta|\gamma) = T(i|j|k)$ if $(\alpha, \beta, \gamma)$ is a permutation of $(i, j, k) \in I \times J \times K$,

(S2) $S(\alpha|\beta|\gamma) = 0$ otherwise.

**Definition 8.4.** Let $I, J, K, H$ be the indexing sets as in Definition 8.3, and let $S$ be an $H \times H \times H$ tensor over $\mathcal{F}$. We define $I^2$ as the set of all $\{p, q\}$ with $p, q$ in $I$. The sets $J^2, K^2$ are defined similarly, and we denote $\mathcal{H} = H \cup I^2 \cup J^2 \cup K^2$. We define the $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ tensor $\mathcal{T} = \mathcal{T}(S)$ by adjoining of the $\pi$-unit 1-slices to $S$, and then the subsequent adjoining of the $\pi$-unit 2-slices and the $\pi$-unit 3-slices to the resulting tensors. Here, an index $\pi$ runs over the set $I^2 \cup J^2 \cup K^2$.

*Remark* 8.5. In Definition 8.4 and in what follows, we use $\pi \in I^2 \cup J^2 \cup K^2$ as the label of the slice of $\mathcal{T}(S(T))$ corresponding to the adjoined $\pi$-unit matrix.

*Remark* 8.6. The resulting tensor $\mathcal{T}(S(T))$ is symmetric.

In what follows, we assume $|I| = |J| = |K| = n$, and we are going to prove that

$$\operatorname{srk}_{\mathcal{F}} \mathcal{T}(S(T)) = \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2 + n) \tag{8.1}$$

whenever $|\mathcal{F}| \geqslant 4$. This would show that $T \to \mathcal{T}(S(T))$ is a polynomial time many-one reduction from the standard rank problem to the symmetric one; we note that the assumption $|I| = |J| = |K|$ does not cause a loss of generality because the rank of a tensor remains unchanged if one adjoins several zero slices to it. In particular, the equality (8.1) implies Theorem 2.8, which is the main goal of this section. Therefore, the rest of our paper is devoted to the confirmation of (8.1).

**Lemma 8.7.** *Let $S(T)$ be the tensor as in Definition 8.3, and $\mathcal{T}(S(T))$ be the tensor as in Definition 8.4. Then* $\operatorname{rk}_{\mathcal{F}} \mathcal{T}(S(T)) \geqslant \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2+n)$.

*Proof.* Let $M_3$ be a linear combination of the 3-slices of $\mathcal{T}$ with indexes in $I^2 \cup J^2 \cup K^2$. By Definition 8.4, all non-zero entries of these slices belong to the blocks $(I|I)$, $(J|J)$, $(K|K)$, and the same conclusion holds for $M_3$. Therefore, the addition of $M_3$ to any slice of $\mathcal{T}$ does not change its $(I|J|K)$ block. Similarly, the addition of any linear combination of the 2-slices (and 1-slices, afterwards) with indexes in $I^2 \cup J^2 \cup K^2$ to any 2-slice (or 1-slice, respectively) of the resulting tensor does not affect its $(I|J|K)$ block.

We apply Lemma 3.2 to the 1-slices with indexes in $I^2 \cup J^2 \cup K^2$, then to the 2-slices with these indexes, and then to the 3-slices. We get the desired inequality because the total number of adjoined linearly independent slices is

$$3(|I^2| + |J^2| + |K^2|) = 4.5(n^2+n)$$

and because the $(I|J|K)$ block of $\mathcal{T}$ is $T$. □

Since the rank cannot exceed the symmetric rank, Lemma 8.7 implies

$$\operatorname{srk}_{\mathcal{F}} \mathcal{T}(S(T)) \geqslant \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2+n),$$

which gives one direction of the equality (8.1). Our proof of the opposite direction is more technical, and we need some more notation.

**Definition 8.8.** Let $T$ be an $I \times I \times I$ symmetric tensor over a field $\mathcal{F}$. Let $\rho$ be a permutation of $I$, and let $(f_i)$ be a family of non-zero elements of $\mathcal{F}$, where the index $i$ runs over $I$. We say that the tensor whose $(i|j|k)$ entry equals

$$f_i\, f_j\, f_k\, T(\rho_i|\rho_j|\rho_k)$$

is obtained from $T$ by a *monomial transformation.* Indexes $i, \hat{\imath} \in I$ are called *twins* for $T$ if the $\hat{\imath}$th 1-, 2-, and 3-slices are equal to the corresponding $i$th slices. The *removal* of an index $i$ is the operation of restricting $T$ to the indexing set $I \setminus \{i\}$.

**Observation 8.9.** *A monomial transformation and the removal of a twin do not change the symmetric ranks of a given tensor.*

**Lemma 8.10.** *Let $\mathcal{F}$ be a field with $|\mathcal{F}| \geqslant 4$, and let $x$ be a scalar in $\mathcal{F}$. We define the $2 \times 2 \times 2$ symmetric tensor $A$ such that*

$$A(1|1|1) = x,\ A(1|1|2) = 1,\ A(1|2|2) = A(2|2|2) = 0.$$

*Then* $\operatorname{srank}_{\mathcal{F}} A \leqslant 3$.

*Proof.* We define the values

$$q = \frac{p}{px-1},\ s_1 = \frac{1}{p(2-px)},\ s_2 = \frac{(px-1)^2}{p(px-2)},\ s_3 = \frac{p^2}{px-1}$$

depending on a parameter $p$, and we check that

$$A = s_1(1,p)^{\otimes 3} + s_2(1,q)^{\otimes 3} + s_3(0,1)^{\otimes 3}$$

provided that we can choose a value of $p$ satisfying $0 \notin \{p, 1-px, 2-px\}$. This is possible unless we have both $x = 0$ and $2 = 0$ at the same time, but then

$$A = \sigma_1(1,1)^{\otimes 3} + \sigma_2(1,q)^{\otimes 3} + \sigma_3 \left(1, \frac{q}{q+1}\right)^{\otimes 3}$$

with

$$\sigma_1 = \frac{q^2}{q+1},\ \sigma_2 = \frac{1}{q^3+q^2},\ \sigma_3 = \frac{(q+1)^3}{q^2}$$

and $q \notin \{0,1\}$. $\square$

Further, let us define an $(i,j)$th 3-*transversal* of a tensor $T$ as the set of entries in which the first two coordinates are equal to $i$ and $j$, respectively. The notions of 1- and 2-transversals are defined in a similar way.

**Lemma 8.11.** *Let $I, J, K, H$ be the indexing sets as in Definition 8.3, and let $U$ be a symmetric $H \times H \times H$ tensor over a field $\mathcal{F}$ with $|\mathcal{F}| \geqslant 4$. If $U(i|j|k) = 0$ whenever $i \in I$, $j \in J$, $k \in K$, then*

$$\operatorname{srk}_{\mathcal{F}} \mathcal{T}(U) \leqslant 4.5(n^2+n),$$

*where $\mathcal{T}(U)$ is the tensor as in Definition 8.4.*

*Proof.* We take an arbitrary total order $\succeq$ on $I \cup J \cup K$, and we write $p \succ q$ if $p \succeq q$ and $p \neq q$. For any $X \in \{I, J, K\}$ and $\pi = \{p, q\} \in X^2$ with $p \succ q$, we define the $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ tensor $\mathcal{L}_\pi$ as follows. For all $r, s \in \{p, q\}$, $h \in H$, $x, y, z \in \mathcal{H}$, we set

(L1) $\mathcal{L}_\pi(r|s|h) = U(p|q|h)$ if either $h \notin X$ or $h \succ p$,

(L2) $\mathcal{L}_\pi(r|s|\pi) = 1$,

(L3) $\mathcal{L}_\pi(z|x|y) = \mathcal{L}_\pi(y|z|x) = \mathcal{L}_\pi(x|y|z)$ if at least one of these is already defined,

(L4) the entries which are not yet defined are zero.

Every $\mathcal{L}_\pi$ can be reduced to the tensor as in Lemma 8.10 by the transformations as in Observation 8.9. We have

$$\operatorname{srk}_{\mathcal{F}} \mathcal{L}_\pi \leqslant 3,$$

so the result of the lemma would follow if we check that the tensor

$$\Phi = \mathcal{T}(U) - \sum \mathcal{L}_\pi$$

has symmetric rank at most $9n$ with respect to $\mathcal{F}$, where the summation goes over all possibilities of $\pi$ for which we defined $\mathcal{L}_\pi$ above. We can check that all the non-zero entries of $\Phi$ are covered by the union of the $(u,u)$-th 1-, 2-, 3-transversals over all $u \in H$. We get

$$\Phi = \sum_{u \in H} \mathcal{M}_u,$$

where $\mathcal{M}_u$ is defined as, for all $p, q, r \in \mathcal{H}$,

(M1) $\mathcal{M}_u(p|q|r) = \Phi(p|q|r)$ if $u$ appears at least twice among $p, q, r$,

(M2) $\mathcal{M}_u(p|q|r) = 0$ otherwise.

This implies $\operatorname{srk}_{\mathcal{F}} \Phi \leqslant 9n$ because each of the $3n$ tensors $\mathcal{M}_u$ has symmetric rank at most three again by Observation 8.9 and Lemma 8.10. $\square$

**Lemma 8.12.** *Let $\mathcal{T}(S(T))$ be the tensor as in Definition 8.4. If $|\mathcal{F}| \geqslant 4$, then*

$$\operatorname{srk}_{\mathcal{F}} \mathcal{T}(S(T)) \leqslant \operatorname{rk}_{\mathcal{F}} T + 4.5(n^2+n).$$

*Proof.* Assuming that $\operatorname{rk}_{\mathcal{F}} T = r$, we consider an appropriate decomposition

$$T = \sum_{t=1}^{r} a_t \otimes b_t \otimes c_t,$$

where each of $(a_t)$, $(b_t)$, $(c_t)$ is a family of $r$ vectors with indexing sets $I$, $J$, $K$, respectively. We construct the family $(w_t)$ of vectors indexed with $\mathcal{H}$ by setting the $I$ part of $w_t$ equal to $a_t$, the $J$ part equal to $b_t$, the $K$ part equal to $c_t$, and setting all the other entries equal to zero, which means that the $I^2 \cup J^2 \cup K^2$ part of $w_t$ is zero. Now we see that the tensor

$$\mathcal{T}(S(T)) - \sum_{t=1}^{r} w_t \otimes w_t \otimes w_t$$

satisfies the assumptions imposed on the tensor $\mathcal{T}(U)$ as in Lemma 8.11, and the application of this lemma completes the proof. □

Lemmas 8.7 and 8.12 prove the equality (8.1) for any field $\mathcal{F}$ of cardinality at least four. This shows that $T \to \mathcal{T}(S(T))$ is a desired reduction of the tensor rank problem to the symmetric rank, which completes the proof of Theorem 2.8.

## 9. A reduction of matrix completion to matrix rigidity

In order to prove Theorem 4.5, we need to construct a polynomial reduction to Problem 4.2 from the solvability of systems of polynomial equations, and, in case when $\mathcal{S}$ is a field, we need a polynomial reduction in the opposite direction. Both reductions are non-trivial, and we begin with the one that is somewhat easier.

**Lemma 9.1.** *Let $m$, $n$, $s$ be positive integers such that $s \leqslant n$. Let $\Phi$ be a family of polynomial equations with variables separated into two families $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$, and assume that the coefficients of $\Phi$ are taken in a subring $\mathcal{R}$ of a field $\mathcal{S}$. We define the new family $\Phi'$ of polynomials whose variables are the initial families $x$, $y$ and new families $(z_t)$, $(\alpha_t)$ indexed with $t \in \{1, \ldots, n\}$, and, additionally, the family $(w_{ij})$ indexed with integers $i$, $j$ such that $0 \leqslant i < j \leqslant n$, and two single variables $\alpha_0$ and $\tau$. The family $\Phi'$ itself consists of all equations*

($\Phi 0$) *in $\Phi$,*
($\Phi 1$) $(\tau^i - \tau^j) w_{ij} = 1$ *for all $i$, $j$ as above,*
($\Phi 2$) $x_t z_t = 0$, $z_t(z_t - \tau + 1) = 0$ *for all $t$,*
($\Phi 3$) $\alpha_t = (z_t + 1)\alpha_{t-1}$ *for all $t$,*
($\Phi 4$) $\alpha_n = \tau^{n-s}$ *and $\alpha_0 = 1$.*

*Then the following are equivalent:*

- *the polynomials in $\Phi'$ have a simultaneous solution in $\mathcal{S}$,*
- *the polynomials in $\Phi$ have a simultaneous solution in $\mathcal{S}$ in which not more than $s$ of the variables $(x_1, \ldots, x_n)$ take nonzero values.*

*Proof.* After a straightforward elimination of $(w_{ij})$, the equations ($\Phi 1$) tell that the powers $(\tau^0, \tau^1, \ldots, \tau^n)$ are pairwise distinct, and, since $\mathcal{S}$ is infinite, an appropriate assignment of $\tau$ always exists. Similarly, the equations ($\Phi 2$) say that

(Z1) $z_t = 0$ if $x_t \neq 0$, and
(Z2) either $z_t = 0$ or $z_t = \tau - 1$ if $x_t = 0$,

and hence the equations ($\Phi 3$) tell that

($\alpha 1$) $\alpha_t = \alpha_{t-1}$ if $x_t \neq 0$, and

($\alpha 2$) either $\alpha_t = \alpha_{t-1}$ or $\alpha_t = \tau\alpha_{t-1}$ if $x_t = 0$,

for all appropriate $t$, so we see that the variables $(z_t)$ got eliminated as well. Therefore, for any solution $(x_1, \ldots, x_n)$ to the equation ($\Phi 0$), the corresponding equations ($\Phi 4$) can satisfy if and only we had at least $n - s$ steps of the form ($\alpha 2$), which means that at least $n - s$ variables in $(x_1, \ldots, x_n)$ are equal to zero. □

Therefore, we can justify the second statement in Theorem 4.5.

**Corollary 9.2.** *If $\mathcal{R}$ is a subring of a field $\mathcal{S}$, then there is a polynomial reduction from Problem 4.2 to the problem to decide whether a given system of polynomial equations with coefficients in $\mathcal{R}$ has a solution in $\mathcal{S}$ or not.*

*Proof.* If $\mathcal{S}$ is finite, then the result is immediate because both problems are NP-complete [27]. If $\mathcal{S}$ is infinite, then we take an instance $(A, k, r)$ of Problem 4.2, and, for every row index $\rho$ of $A$, for every column index $\sigma$ of $A$, we replace the corresponding entry $A(\rho|\sigma)$ of the matrix $A$ with $x_{\rho\sigma} + A(\rho|\sigma)$, where every $x_{\rho\sigma}$ is a new variable. The resulting matrix $A(x)$ is of rank at most $r$ if and only if there exist a $p \times r$ matrix $Y_1$ and an $r \times q$ matrix $Y_2$ such that

$$Y_1 Y_2 = A(x). \tag{9.1}$$

Moreover, the matrix $A(x)$ certifies a positive answer to Problem 4.2 if and only if the equation (9.1) admits a solution in $\mathcal{S}$ in which at most $k$ of the variables $(x_{\rho\sigma})$ are nonzero. According to Lemma 9.1, this question in turn reduces to the conventional problem for systems of polynomial equations with coefficients in $\mathcal{R}$ on their solvability over $\mathcal{S}$ (without constraints on the number of zero variables). □

Now we can focus on a proof of the first statement of Theorem 4.5, and our approach is based on a classical result on a question of Zarankiewicz [83].

**Definition 9.3** (Zarankiewicz problem [44, 83, 85])**.** Let $m$, $n$, $s$, $t$ be integers such that $2 \leqslant \max\{s, t\} \leqslant \min\{m, n\}$. One defines

$$z(m, n, s, t)$$

as the maximum number $z$ for which there exists an $m \times n$ matrix $M$ such that

- exactly $z$ entries of $M$ are zeros,
- $M$ contains no $s \times t$ submatrix of all zeros.

**Theorem 9.4** (Kővári, Sós, Turán [44])**.** *One has $z(n, n, t, t) < (t-1)^t n^{2-1/t} + tn$.*

Our reduction also requires a standard result on *Cauchy matrices.*

**Definition 9.5.** Let $G$ be a matrix with entries in a field $\mathbb{F}$. For purposes of this paper, we say that $G$ is *generic* if every square submatrix of $G$ is non-singular.

**Theorem 9.6** (Cauchy matrices [50, 67])**.** *Let $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ be a sequence of $2n$ pairwise distinct elements in a field $\mathbb{F}$. Then the $n \times n$ matrix*

$$G(i|j) = \frac{1}{a_i - b_j}$$

*is generic.*

We proceed with the remaining reduction, in which we employ the *rank three matrix completion* as an intermediate problem. As in the above considerations, the symbol $*$ denotes the unspecified entries of an incomplete matrix, and, of course, we assume that $*$ is not used as the name of any element in $\mathcal{R}$ or $\mathcal{S}$.

**Definition 9.7.** Let $A$ be a $k \times k$ matrix with entries in $\mathcal{R} \cup \{*\}$, and let $G$ be an $n \times n$ matrix with entries in $\mathcal{R}$. We assume that the rows and columns of $A$ are indexed with a set $I$, and the rows and columns of $G$ are indexed with $J$. We define the $kn \times kn$ matrix $\mathcal{R}_{A\,G}$ with rows and columns labeled by $I \times J$ with the formula

$$\mathcal{M}_{A\,G}(i_1, j_1|i_2, j_2) = \begin{cases} A(i_1|i_2), & \text{if } A(i_1|i_2) \in \mathcal{R}, \\ G(j_1|j_2), & \text{if } A(i_1|i_2) = *. \end{cases}$$

**Theorem 9.8.** *Let $\mathcal{R}$ be a subring of a principal ideal domain $\mathcal{S}$. Let $A$ be a $k \times k$ matrix over $\mathcal{R} \cup \{*\}$ with exactly $q$ entries equal to $*$. For $n = (10k)^8$ we take a generic $n \times n$ matrix $G$ over $\mathcal{R}$. Then*

*(i) $A$ admits a completion of rank at most three over $\mathcal{S}$*

*if and only if*

*(ii) $(\mathcal{M}_{A\,G}, qn^2, 3)$ is a yes-instance of Problem 4.2.*

*Proof.* Similarly to Definition 9.7, we declare that $I$ is the indexing set for the rows and columns of $A$, and $J$ is the indexing set for the rows and columns of $G$.

If the condition (i) applies, then we can put an appropriate element $a_{i\hat{\imath}} \in \mathcal{S}$ at every $(i|\hat{\imath})$ entry of $A$ with $A(i|\hat{\imath}) = *$ so that the resulting matrix $A'$ has rank at most three. In this case, for all $i$, $\hat{\imath}$ as above, and for arbitrary $j, \hat{\jmath} \in J$, we replace the $(i, j|\hat{\imath}, \hat{\jmath})$ entry of $\mathcal{M}_{A\,G}$ with $a_{i\hat{\imath}}$. Therefore, after having at most $qn^2$ entries of $\mathcal{M}_{A\,G}$ changed, we get the matrix $R'$ partitioned into the $k \times k$ blocks in which every block equals $A'$. This implies $\operatorname{rank}(R') \leqslant 3$, which confirms the condition (ii).

Conversely, we assume the condition (ii). Then we have $\mathcal{M}_{A\,G} = L + S$, where

$$\text{the matrix } S \text{ has at most } qn^2 \text{ nonzero entries,} \tag{9.2}$$

$\operatorname{rank}(L) \leqslant 3$, and both matrices $L$, $S$ have all entries in $\mathcal{S}$. For all fixed $i, \hat{\imath} \in I$, we consider the $n \times n$ submatrix $S_{i\hat{\imath}}$ obtained from $S$ by taking the rows with indexes $(i, j)$ and columns with indexes $(\hat{\imath}, \hat{\jmath})$, for arbitrary $j, \hat{\jmath} \in J$. If $A(i|\hat{\imath}) = *$, we get $G$ as the submatrix of $\mathcal{M}_{A\,G}$ obtained by taking the same rows and columns as in $S_{i\hat{\imath}}$, and, since $G$ is generic, the matrix $S_{i\hat{\imath}}$ cannot have a $4 \times 4$ submatrix of all zeros. By Theorem 9.4, the matrix $S_{i\hat{\imath}}$ has at most

$$3^4\, n^{1.75} + 4n < 85\, n^{1.75}$$

zero entries, or, in other words,

$$\text{the matrix } S_{i\hat{\imath}} \text{ has at least } n^2 - 85\, n^{1.75} \text{ nonzero entries.} \tag{9.3}$$

Since there are exactly $q$ possible choices of $(i, \hat{\imath})$ with $A(i|\hat{\imath}) = *$, an application of the condition (9.3) shows that $S$ has at least

$$q(n^2 - 85\, n^{1.75})$$

nonzero entries in *tangible* positions, that is, those $(i, j|\hat{\imath}, \hat{\jmath})$ with $A(i|\hat{\imath}) = *$. In view of the condition (9.2), there are at most

$$q \cdot 85\, n^{1.75} \leqslant k^2 \cdot 85\, n^{1.75} = \sqrt[4]{n/10^8} \cdot 85\, n^{1.75} = 0.85\, n^2 \tag{9.4}$$

nonzero entries of $S$ that are not tangible. Since the quantities (9.4) are less than $n^2$, there exist $j, \hat{\jmath} \in J$ such that no non-tangible nonzero entry gets into the $k \times k$ submatrix $\overline{S}$ obtained from $S$ by taking the rows with indexes $(i', j)$ and columns with indexes $(\hat{\imath}', \hat{\jmath})$, for arbitrary $i, \hat{\imath}' \in I$. Now let $\overline{M}$, $\overline{L}$ be the submatrices obtained from $\mathcal{M}_{A\,G}$, $L$, respectively, by taking the same rows and columns as in $\overline{S}$. Then

$\overline{M} = \overline{L} + \overline{S}$, and since $\overline{S}$ has all its nonzero entries tangible, the matrix $\overline{M}$ agrees with $\overline{L}$ everywhere except possibly several tangible entries. Therefore, $\overline{M}$ is a rank-three completion of $A$, and hence the condition (i) is satisfied. □

In the first statement of Theorem 4.5, the ring $\mathcal{R}$ is endowed with an injective sequence oracle, so a generic matrix $G$ can be constructed in polynomial time by Theorem 9.6. Therefore, Theorem 9.8 gives a polynomial reduction to Problem 4.2 from the rank three completion problem of matrices with specified entries in $\mathcal{R}$ and replaced entries in $\mathcal{S}$. According to Theorem 3.3, this rank three completion problem is polynomial time equivalent to deciding whether a given system of polynomial equations with coefficients in $\mathcal{R}$ has a solution in $\mathcal{S}$, and hence this latter problems admits a polynomial reduction to Problem 4.2 as well. This confirms the first statement of Theorem 4.5, and, since the second statement is already verified in Corollary 9.2, the proof of Theorem 4.5 is complete.

Now we switch to Theorem 4.12 and take appropriate fields $\mathbb{F}$ and $\mathbb{F}'$ as in its formuation. In view of Corollary 6.10, we can construct a matrix with entries in $\mathbb{F}$ which admits a rank three completion over $\mathbb{F}'$ and no rank three completion over $\mathbb{F}$, and hence the conclusion of Theorem 4.12 follows by the construction in Theorem 9.8. However, since the example used in Corollary 6.10 is large, and since the proof of this corollary is quite complicated, it makes sense to explain how to give a simple solution to the above mentioned problem of Babai, Kivva [4] on the dependence of the solution to Problem 4.2 on the choice of $\mathcal{S}$.

*Remark* 9.9. For many natural choices of $\mathcal{S}$, an affirmative answer to Question 4.11 can be obtained from Theorem 9.8 alone. For instance, the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 1 \\ 0 & 2 & * & 1 & 0 \\ 0 & 1 & 0 & * & 1 \\ 0 & 1 & 1 & 1 & * \end{pmatrix}$$

admits a rank three completion over $\mathbb{Q}[\sqrt{2}]$ but no rank three completion over $\mathbb{Q}$, so we can apply the construction in Theorem 9.8 to the matrix $\mathcal{A}$ and get an instance of the matrix rigidity problem with different answers over $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}]$.

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