Some Inconsistence in Logic

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Abstract

This paper brings up some important points about logic, e.g., mathematical logic, and also an inconsistence in logic as per Godel's incompleteness theorems which state that there are mathematical truths that are not decidable or provable. These incompleteness theorems have shaken the solid foundation of mathematics where innumerable proofs and theorems have pride of place. The great mathematician David Hilbert had been much disturbed by them. There are much long unsolved famous conjectures in mathematics, e.g., the twin primes conjecture, the Goldbach conjecture, the Riemann hypothesis, et al. Perhaps, by Godel's incompleteness theorems the proofs for these famous conjectures will not be possible and the numerous mathematicians attempting to find solutions for these conjectures are simply banging their heads against the metaphorical wall. Besides mathematics, Godel's incompleteness theorems will have ramifications in other areas involving logic. The paper looks at the ramifications of the incompleteness theorems, which pose the serious problem of inconsistency, and offers a solution to this dilemma. The paper also looks into the apparent inconsistence of the axiomatic method in mathematics.

Keywords: Logical deductions; Godel's incompleteness theorems; not provable; validity; truths; theorems; axioms; conjectures.

1 Making Logical Statements

How should we think or act in a logical manner? We should exercise pain, caution and care when making statements, e.g., do our research first and get our facts or premises right, make a careful choice of words, terms or expressions to be used, aim at clarity and at being understood, listen to, consider and accept or adopt others' ideas or points of view if they are relevant, win the support of or acceptance by others for our logical propositions or ideas, et al. A logical statement could be simple and short or it could be complex, detailed and lengthy. The facts or premises contained in the statement should be true facts, facts whose truthfulness could be verified. It is important that the statement could be verified or proved to be true, e.g., confirmed by an experiment or experiments, or, some other kind of test. For those statements whose truths are not certain or verifiable, we could make them with some qualification or caveat - we could make such statements with a probabilistic, but reasoned, approach, e.g., we could state that something is probably true, most probably true, unlikely to be true, has little likelihood of being true, in all probability true (or untrue or false), true under certain circumstances, or, false under certain

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circumstances, et al., depending on our intuition and how strongly we felt about the probability of its being true, or, false, though we were not certain of or able to verify its truth or falseness. It is of course important to have clarity and alertness of mind while making statements.

The following is a listing of the kinds of statement we may encounter from our fellow-beings or counterparts:-

(1) The statement is entirely true. (The truth is verifiable.)
(2) The statement is entirely false. (The falseness is verifiable.)
(3) The statement is partially true and partially false. (The partial truth and partial falseness are verifiable.)
(4) The statement is partially true and partially false, while the rest of the statement is not verifiable as true, or, false but may be considered probably, or, most probably true, or, false. (The partial truth and partial falseness are verifiable.)
(5) The statement is partially true, while the rest of the statement is not verifiable as true, or, false but may be considered probably, or, most probably true, or, false. (The partial truth is verifiable.)
(6) The statement is partially false, while the rest of the statement is not verifiable as true, or, false but may be considered probably, or, most probably true, or, false. (The partial falseness is verifiable.)
(7) The statement is neither true nor false, but may be considered probably, or, most probably true, or, false. (The statement could not be verified to be true, or, false.)
(8) The statement is entirely true under certain circumstances. (This is verifiable.)
(9) The statement is entirely false under certain circumstances. (This is verifiable.)
(10) The statement is partially true and partially false under certain circumstances. (These partial truth and partial falseness are verifiable.)
(11) The statement is partially true and partially false under certain circumstances, while the rest of the statement is not verifiable as true, or, false but may be considered probably, or, most probably true, or, false. (These partial truth and partial falseness are verifiable.)
(12) The statement is partially true under certain circumstances, while the rest of the statement is not verifiable as true, or, false but may be considered probably, or, most probably true, or, false. (This partial truth is verifiable.)
(13) The statement is partially false under certain circumstances, while the rest of the statement is not verifiable as true, or, false but may be considered probably, or, most probably true, or, false. (This partial falseness is verifiable.)
(14) The statement is neither true nor false under certain circumstances, but may be considered probably, or, most probably true, or, false. (This statement could not be verified to be true, or, false.)
(15) The statement is neither true nor false under certain circumstances, and there is little or hardly any probability that it is true, or, false. (This statement could not be verified to be true, or, false.)
(16) The statement is none of the above, i.e., it is neither true nor false under any circumstances, nor probably, or, most probably true, or, false - we cannot make anything or form any conclusion about the statement at all. (This statement could not be verified to be true, or, false.)

The above listing of 16 kinds of statement which we may encounter may not be an exhaustive listing. The listing may be further classified, e.g., there may be a statement which is three-quarter true and one-quarter false, a statement which is two-third true and one-third probably, or, most probably true, or, false, a statement which is one-third true, one-third false and one-third probably, or, most probably true, or, false, a statement which is one-fifth true, two-fifth false and two-fifth probably, or, most probably true, or, false, et al. There could be lots of fine, subtle distinctions in the logical statements or propositions, which we should be keenly aware of. We should be alert to all the possible implications.

2 Inconsistence in Logic as per Godel's Incompleteness Theorems

Recall that by Godel's incompleteness theorems there are true statements whose truth is not provable and false statements whose falseness is not provable. The question is if we could not prove or verify the truth or falseness of a statement how could we be certain or know that the statement is true, or, false? (Is it just a hunch, feeling or intuition?) Wouldn't it be a contradiction (of mathematical reasoning wherein rigorous or solid proof is demanded) or absurd to state thus "This statement is true (or, false) but its trueness (or, falseness) cannot be proved"? The most we could say about such a statement is that it is probably, or, most probably true, or, false. This appears to have been an erroneous conception, for it is practically not possible to know whether a statement or proposition is true, or, false, if its truth, or, falseness is not verifiable or provable, i.e., a statement could only be true, or, false, if it could be proved to be so - otherwise, it would be just a conjecture, e.g., a mathematical statement which has not been proved is a conjecture while a mathematical statement which has been proved is a theorem. In mathematics, a statement is true only if it is proved to be so. Could an "undecidable" or "unprovable" mathematical statement be accepted as true (or, false) as per Godel's incompleteness theorems (which is actually a contradiction of the important mathematical principle of the need for proofs)? In other words, could we say "By Godel's incompleteness theorems, this mathematical statement is true (or, false) but its proof is an impossibility", or, "I know this mathematical statement is true (or, false) though its proof is an impossibility"? Which could invite a counter-argument "If you cannot prove that this mathematical statement is true (or, false), how do you know that this mathematical statement is true (or, false)?", which would make it all look rather absurd.

3 Further Inconsistence in Logic

It should also be noted that Godel had asserted that a non-provable true, or, false statement could only be possibly proved through utilizing a more powerful set of axioms from outside the system. An axiom, by the way, is an obviously or evidently true statement which does not require a proof for ascertaining its validity. Axioms are used in mathematical reasoning. But there appears to be some intrinsic arbitrariness in axioms for what is obviously or evidently true to one mathematician might not be so to another mathematician, which would depend on the mathematicians' mental capacity or capacity for abstract thought, e.g., what mathematical truth is obvious to a mathematical genius might not be obvious to an ordinary mathematician. Let us look at the following curious example: In *Principia Mathematica*, the magnum opus of two well-known philosophers and mathematicians Bertrand Russell and Alfred North Whitehead, which attempts to reduce mathematics to logic, more than 100 pages of dense mathematical reasoning are used to prove the simple statement one plus one equals two. This simple statement one plus one equals two would be evidently true and an axiom to practically everyone, a statement which a primary school or even kindergarten kid would instinctively understand to be true. Wouldn't it imply that the two well-known philosophers and mathematicians are mentally dull for apparently doubting the simple statement one plus one equals two and requiring more than 100 pages of dense mathematical and symbolic reasoning to be assured of its truth or validity? This is indeed a curiousity. It could be categorically stated that the person who really doubts that one plus one equals two has yet to be found, at least by the author.

The moot point is how one decides what statement is an axiom, e.g., what criteria is the decision based on, and, what statement is not an axiom and requires a proof for ascertaining its validity, i.e., how this arbitrariness in axioms could be dealt with. Could some of the great, long unsolved mathematical conjectures, e.g., the twin primes conjecture and the Goldbach conjecture, be regarded as axioms, e.g., by a vastly superior intellect, perhaps that belonging to an alien race, wherein the truth or validity of the conjectures is obvious or certain to him as one plus one equals two is obvious to us ordinary mortals?

This is apparently another inconsistence, viz., inconsistence in the axiomatic method in mathematics.

4 Conclusion

To be consistent, all true, or, false statements should be provable to be so. As intelligent beings, we demand proofs for everything, e.g., proof of innocence/guilt in a court of law, proof that a mathematical theorem is valid, proof that a statement is logical or valid, et al. By Godel's incompleteness theorems, anyone could proclaim that something (even if it is evidently absurd, even if it is really a lie) is valid but whose proof of validity is an impossibility (and who is to know better or be wiser for it). For example, and interestingly, could someone who has committed a crime state "I am innocent but by Godel's incompleteness theorems it is not possible to prove that I am innocent" or something like that? Also, could a mathematician, e.g., state that a famous outstanding conjecture, for instance the twin primes conjecture, is true but by Godel's incompleteness theorems this is not provable? (Mathematics is an abstract art whose underpinnings are proofs which are formulated by its practitioners, the mathematicians, whose principal responsibility is to produce the proofs for theorems. Godel's incompleteness theorems by negating the possibility of proofs might theoretically at least put mathematics (where proofs are bread and butter issues) and mathematicians out of business.) Couldn't someone just simply invoke Godel's incompleteness theorems to defend the truth (or lie) of any statement by stating that by these theorems the statement is true but not provable? Finished. Full stop. Everyone has to accept the "true" statement. Godel's incompleteness theorems could thus become the de facto defense for falsehood.

It is a great inconsistence in logic when a logician or mathematician states that as per Godel's incompleteness theorems a conclusion is true but is not decidable or provable, because in logic or mathematics reasons, proofs and explanations are strictly needed for confirming the validity of the conclusion. (A proof is an explanation which renders the truth of a statement clear, evident, obvious, and indisputable. An axiom is an evidently or obviously true statement which does not require any proof or explanation.) The solution to this dilemma evidently is to regard all statements not firmly supported by proofs or reasons as conjectures, at most strong conjectures, and not truths or theorems. The exception would be axioms, the evidently or obviously true statements which do not require any proof or explanation (though what might be an axiom or obviously true statement to some might not be so to others, depending on the person's level of intelligence, which is explained below).

With regards to proof or explanation, an important point should be noted, viz., that a person of higher intelligence would likely understand and accept the truth (or, falseness) of a statement without much explanation needed or possibly with no explanation needed at all, while a person of lower intelligence would likely need a more detailed and less complex explanation to understand the statement and might still have difficulty understanding the statement despite the more detailed and less complex explanation. To the person of extremely high intelligence the true (or, false) statement could be so evidently or obviously true (or, false) that it appears an axiom. Thus, depending on the individual's level of intelligence, a true (or, false) statement could be right away accepted as obviously true (or, false), accepted as true (or, false) after some deliberation, "undecidable", i.e., with no idea whether it is true or false, or accepted as false/true due to faulty reasoning and/or misunderstanding or misinterpretation. In short, the more intelligent person would likely grasp the truth, or, falseness of a statement more easily than the less intelligent person, similar to the case of the more intelligent students likely comprehending a difficult subject more easily while the less intelligent ones are likely to have a harder time doing so despite the extra effort of the teacher in explaining it to them.

Hence, logic, proofs and axioms are not without problems as is indicated above.

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