The link between Algebraic and Rational Homogeneous Linear Recurrences

Deepak Ponvel Chermakani

Email: deepakc@pmail.ntu.edu.sg

Abstract: - We show that the nth term of an algebraic homogeneous linear-recurrence, can be expressed as a weighted sum of the nth terms of a finite number of rational homogeneous linear recurrences. The weights in this weighted sum belong to a finite set of algebraic constants, no two of which are rational multiples of each other.

1. Introduction

The topic of Homogenous Linear Recurrences (HLRs) has long intrigued mathematicians for more than a century. The construction of HLRs is simple indeed, that the next term in the sequence is a linear function of the preceding L terms. Yet HLR Problems such as the Skolem Problem (on whether some term in the HLR can be zero) and the Positivity Problem (on whether all terms of the HLR will always be positive) continue to remain unsolved.

Recent research [1] has found efficient algorithms for computing the nth term of the Rational HLR (RHLR), where all starting points and coefficients are rational constants. But research has also acknowledged the difficulty of finding a similar efficient algorithm for the Algebraic HLR (AHLR). By "computing" the nth term of an AHLR, we shall mean expressing the nth term as a rationally weighted sum of algebraic constants, such that no two of these algebraic constants are rational multiples of each other.

In this paper, we express the problem of computing the nth term of an ALHR, as a problem of computing the nth terms of a finite set of RHLRs. Though the complexity of our approach can grow super exponentially to the problem size, the importance of this link between ALHRs and RHLRs is discussed.

<u>Notations</u>

We use the following well-known notations that:

1. A number x is said to be algebraic, if x is a root of some univariate polynomial with integer coefficients. That is, x is said to be algebraic if there exist integers $\{c_0, c_1, c_2, \ldots, c_d, d\}$ where $d > 0$, such that $c_d x^d + c_{d-1} x^{d-1} + c_{d-2} x^{d-2} + \ldots + c_0 = 0$.
2. A number x is said to be rational if $x=p/q$, where both p and q are integers and $q\neq 0$.
3. (x y) denotes the product of x and y.
4. x^y and x^y both denote x to the power of y. We use both notations to our convenience. Here, x is called the base, while y is called the exponent.
5. If A and B are two boolean statements, then
   $A\rightarrow B$ denotes "If A is true, then B is true"
   $A\leftrightarrow B$ denotes "A is true, if and only if, B is true"
6. LCM denotes Least Common Multiple. For example, LCM(2, 5, 15) = 30.
7. A Homogeneous Linear-Recurrence (HLR) whose nth term, f_n, is defined by the following conditions:
   $f_n = 0$, for all integers $n < 0$
   $f_n = a_n$, for all integers n in [0, (L-1)]
   $f_n = b_1 f_{n-1} + b_2 f_{n-2} + b_3 f_{n-3} + \ldots + b_L f_{i-L}$, for all integers $n \geq L$
8. In the definition above, if each of $\{a_0, a_1, a_2, \ldots, a_{L-1}, b_1, b_2, b_3, \ldots, b_L\}$ is a given constant algebraic number, then the HLR is called an Algebraic HLR or AHLR. In the rest of this paper, "algebraic" shall imply "real algebraic", even though all the results of our paper are easily extensible to all types of algebraic starting points and coefficients of the HLR. If each of $\{a_0, a_1, a_2, \ldots, a_{L-1}, b_1, b_2, b_3, \ldots, b_L\}$ is a given constant rational number, then the HLR is called a Rational HLR or RHLR.
9. L is called the degree of the HLR.

2. The Approach

2.1. Definitions

We further make the following definitions:

1. We say that an algebraic number is in Standard Algebraic Format (SAF), if it is expressed as num/den, where each exponent in num and den, is a positive rational ≤ 1. For example, if we are given the algebraic constant
   $b_1 = 0.1(2)^{-2/5} + 1.5(2 + (2/7)^{-2/3})^{-1/2}$,
   then we can express b_1 in SAF as follows:
   $b_1 = 0.1(2)^{-2/5} + 1.5(2 + (2/7)^{-2/3})^{-1/2}$
   $= 0.1/2^{2/5} + 1.5/(2 + (7/2)^{2/3})^{-1/2})$
   $= ((2 + (7/2)^{2/3})^{1/2} + 15\ (2^{2/5}))/(10(2^{2/5})(2 + (7/2)^{2/3})^{1/2})$
   $= ((2(2)^{2/3} + (7)^{2/3})^{1/2} / (2^{1/3}) + 15\ (2^{2/5}))/(10(2^{2/5})(2 + (7/2)^{2/3})^{1/2})$
   $= ((2(2)^{2/3} + (7)^{2/3})^{1/2} + 15\ (2^{2/3})(2^{2/5}))/(10(2^{2/5})(2(2^{2/3}) + 7^{2/3})^{1/2})$, which is finally in SAF.
   In this example, b_1 = num/den. Here, num = $((2(2)^{2/3} + (7)^{2/3})^{1/2} + 15\ (2^{2/3})(2^{2/5}))$, where the set of exponents are {1, 2/3, 1/2, 2/5}. Also, den = $(10(2^{2/5})(2(2^{2/3}) + 7^{2/3})^{1/2})$, where the set of exponents are {1, 2/3, 1/2, 2/5}.
2. In an ALHR, we assume, without loss of generality, that each element of the set of given algebraic constants $\{a_0, a_1, a_2, \ldots, a_{L-1}, b_1, b_2, b_3, \ldots, b_L\}$ is expressed in SAF. This is because any algebraic number can be expressed in SAF. We accordingly denote the set S = {num_a_0 , den_a_0 , num_a_1 , den_a_1 , num_a_2 , den_a_2 , ... , num_a_{L-1} , den_a_{L-1} , num_b_1 , den_b_1 , num_b_2 , den_b_2 , num_b_3 , den_b_3 , ... , num_b_L , den_b_L}.
3. From all the elements of S, we identify the set P of all possible unique factor algebraic constants, where the exponent of each algebraic constant is the inverse of a positive integer. That is, P = {p_1^(1/s_1) , p_2^(1/s_2) , p_3^(1/s_3) , ... , p_u^(1/s_u)}, where u is the number of elements in set P, where each s_i is a positive integer > 1, and where $p_i \neq p_j$ for $i \neq j$. If S contains two or more candidate factors p^(1/s_{i1}) , p^(1/s_{i2}) , p^(1/s_{i3}) , ... , p^(1/s_{im}) then we exclude p^(1/s_{i1}) , p^(1/s_{i2}) , p^(1/s_{i3}) , ... , p^(1/s_{im}) from P, and instead include p^(1/LCM(s_{i1} , s_{i2} , s_{i3} , ... , s_{im})) into P. In our previously used example of b_1 in SAF where $b_1 = ((2(2)^{2/3} + (7)^{2/3})^{1/2} + 15\ (2^{2/3})(2^{2/5}))/(10(2^{2/5})(2(2^{2/3}) + 7^{2/3})^{1/2})$, the candidate elements to be inserted into P are $\{2^{1/LCM(3,5)}, 7^{1/3}, (2(2)^{2/3} + (7)^{2/3})^{1/2}\}$, that is $\{2^{1/15}, 7^{1/3}, (2(2)^{2/3} + (7)^{2/3})^{1/2}\}$,
4. Using P, we define the initial version of set Q to contain all possible $(s_1 s_2 s_3 \ldots s_u)$ combinations of the following product of algebraic numbers:
   $(p_1^{i1/s1}\ p_2^{i2/s2}\ p_3^{i3/s3} \ldots p_u^{iu/su})$,
   for all integers i_1 in $[0, s_1 - 1]$
   for all integers i_2 in $[0, s_2 - 1]$
   …
   for all integers i_u in $[0, s_u - 1]$.
   If P = $\{3^{1/2}, 2^{1/2}, 2^{1/3}\}$, then we make P = $\{3^{1/2}, 2^{1/LCM(2,3)}\}$, which is $\{3^{1/2}, 2^{1/6}\}$, for which, Q = $\{(3^0\ 2^0), (3^0\ 2^{1/6}), (3^0\ 2^{2/6}), (3^0\ 2^{3/6}), (3^0\ 2^{4/6}), (3^0\ 2^{5/6}), (3^{1/2}\ 2^0), (3^{1/2}\ 2^{1/6}), (3^{1/2}\ 2^{2/6}), (3^{1/2}\ 2^{3/6}), (3^{1/2}\ 2^{4/6}), (3^{1/2}\ 2^{5/6})\}$.
5. We then iteratively prune set Q to remove elements that are rational multiples of each other. That is, we repeatedly check if Q has two elements q_i and q_j such that $i \neq j$ and $q_i = rq_j$ where r is a rational, then q_i is included in Q while q_j is excluded from Q. We denote the final set Q = $\{q_1, q_2, q_3, \ldots, q_v\}$, that has no two elements q_i and q_j such that $i \neq j$ and $q_i = rq_j$ where r is rational. One reason why this pruning is needed, is that the product of two irrational algebraic numbers can be a rational. For example, $((37^{0.5} + 1)(37^{0.5} - 1))^{0.5} = 6$. Another example is $(2^{0.5}\ 2^{0.5}) = 2$.
6. DEN_PRODUCT is the product of all the denominators of the given algebraic constants in SAF, that is, DEN_PRODUCT = (den_a_0 den_a_1 den_a_2 ... den_a_{L-1} den_b_1 den_b_2 den_b_3 ... den_b_L).

We now state the main result of this paper in Theorem 1.

Theorem 1: In an ALHR, for every n, f_n = SUM (($k_{n,k}\ q_k$), for all integers k in [1, v]), where:
1. **each $k_{n,k}$ follows a RHLR of finite degree**
2. **$\{q_1, q_2, q_3, \ldots, q_v\}$ is a finite set of algebraic constants, no two of which are rational multiples of each other**

Proof: We prove this via two Lemmas. In Lemma 1.1, we show that f_n can be expressed as a rationally weighted sum of algebraic constants from a fixed set of algebraic constants. In Lemma 1.2, we show that each of these rational weights follows a RHLR, with respect to n.

Lemma 1.1: f_n = SUM ($k_{n,k}\ q_k$, over all integers k in [1, v])

Proof: Consider the recursive AHLR definition in Section 1, which can be rewritten as below:

for integers $n < 0$,
$f_n = 0$.
for integers n in [0, (L-1)],
$f_n = (num\_a_n/den\_a_n)$.
for integers $n \geq L$,
$f_n = (num\_b_1/den\_b_1)\ f_{n-1} + (num\_b_2/den\_b_2)\ f_{n-2} + (num\_b_3/den\_b_3)\ f_{n-3} + \ldots + (num\_b_L/den\_b_L)\ f_{i-L}$.

For each integer n in [0, (L-1)], multiply both sides of the equation for f_n with den_a_n. For each integer $n \geq L$, multiply both sides of the equation for f_n with DEN_PRODUCT. We get the following:

for integers $n < 0$,
$f_n = 0$.
for integers n in [0, (L-1)],
$den\_a_n\ f_n = num\_a_n$
for integers $n \geq L$,
$DEN\_PRODUCT\ f_n =$
$(DEN\_PRODUCT\ num\_b_1/den\_b_1)\ f_{n-1} +$
$(DEN\_PRODUCT\ num\_b_2/den\_b_2)\ f_{n-2} +$
$(DEN\_PRODUCT\ num\_b_3/den\_b_3)\ f_{n-3} +$
$\ldots +$
$(DEN\_PRODUCT\ num\_b_L/den\_b_L)\ f_{n-L}$.

We shall refer to the above formulation as the *recursive formulation* of f_n. The above recursive formulation ensures that there is no algebraic constant with a negative power, on both sides of the equations. As f_n is evaluated in sequence for $n \geq 0$, the various algebraic constants are multiplied and summed with each other, and the resulting algebraic constants produced in f_n continue to be rational multiples of the elements of Q. So each f_n can continue to be described as a rationally weighted sum of the algebraic constants in Q. That is, $f_n = k_{n,1}\ q_1 + k_{n,2}\ q_2 + k_{n,3}\ q_3 + \ldots + k_{n,v}\ q_v$, for $n \geq 0$, which we refer to as the *absolute formulation* of f_n.

Hence Proved Lemma 1.1

Lemma 1.2: Each $k_{n,k}$ is a rational that follows some RHLR definition with respect to n.

Proof: Substitute the absolute formulations for f_n , f_{n-1} , f_{n-2} , ... , f_{n-L} into the recursive definition of f_n , and we get:

$DEN\_PRODUCT\ (k_{n,1}\ q_1 + k_{n,2}\ q_2 + k_{n,3}\ q_3 + \ldots + k_{n,v}\ q_v) =$
$(DEN\_PRODUCT\ num\_b_1/den\_b_1)\ (k_{n-1,1}\ q_1 + k_{n-1,2}\ q_2 + k_{n-1,3}\ q_3 + \ldots + k_{n-1,v}\ q_v) +$
$(DEN\_PRODUCT\ num\_b_2/den\_b_2)\ (k_{n-2,1}\ q_1 + k_{n-2,2}\ q_2 + k_{n-2,3}\ q_3 + \ldots + k_{n-2,v}\ q_v) +$
$(DEN\_PRODUCT\ num\_b_3/den\_b_3)\ (k_{n-3,1}\ q_1 + k_{n-3,2}\ q_2 + k_{n-3,3}\ q_3 + \ldots + k_{n-3,v}\ q_v) +$
$\ldots +$
$(DEN\_PRODUCT\ num\_b_L/den\_b_L)\ (k_{n-L,1}\ q_1 + k_{n-L,2}\ q_2 + k_{n-L,3}\ q_3 + \ldots + k_{n-L,v}\ q_v)$.

The resulting Left Hand Side (LHS) and Right Hand Side (RHS) will contain rational multiples of algebraic constants, where all such algebraic constants are elements of Q.

For each integer i in [1, v], we then compare coefficients of q_i , on the LHS and RHS, to obtain the following equation:

$w_{i,0,1}\ k_{n,1} + w_{i,0,2}\ k_{n,2} + \ldots + w_{i,0,v}\ k_{n,v}$
$=$
$w_{i,1,1}\ k_{n-1,1} + w_{i,1,2}\ k_{n-1,2} + \ldots + w_{i,1,v}\ k_{n-1,v}$
$+$
$w_{i,2,1}\ k_{n-2,1} + w_{i,2,2}\ k_{n-2,2} + \ldots + w_{i,2,v}\ k_{n-2,v}$
$+$
$\ldots$
$+$
$w_{i,L,1}\ k_{n-L,1} + w_{i,L,2}\ k_{n-L,2} + \ldots + w_{i,L,v}\ k_{n-L,v}$.

In the above equation, $w_{i,j,k}$ is a rational constant, for each integer i in [1, v], for each integer j in [0, v], for each integer k in [1, v].

We get v equations from above, and so for each integer i in [1,v], we are able to express $k_{n,i}$ by the following equation:

$k_{n,i}$
$=$
$t_{i,1,1}\ k_{n-1,1} + t_{i,1,2}\ k_{n-1,2} + \ldots + t_{i,1,v}\ k_{n-1,v}$
$+$

$t_{i,2,1}\, k_{n-2,1} + t_{i,2,2}\, k_{n-2,2} + \ldots + t_{i,2,v}\, k_{n-2,v}$
+
...
+
$t_{i,L,1}\, k_{n-L,1} + t_{i,L,2}\, k_{n-L,2} + \ldots + t_{i,L,v}\, k_{n-L,v}$.

In the above equation, $t_{i,j,k}$ is a rational constant, for each integer i in [1, v], for each integer j in [1, v], for each integer k in [1, v].

Though the above seems to suggest that $k_{n,i}$ follows a Rational Non-Homogeneous Linear Recurrence (RNHLR), we can show that it can also be expressed as a RHLR. To do this, we use the concept of Generating Functions, where we define $G_i(x)$ = SUM($k_{n,i}\, x^n$, over all integers n in [0, infinity]). We know that $k_{n,i} = 0$ for each integer i in [1, v], for n<0. So we multiply the LHS and the RHS of the above equation for $k_{n,i}$ by x^n, and sum up the LHS and RHS separately from n=0 to infinity, to get the following expression of $G_i(x)$, for each integer i in [1, v] :

$G_i(x)$
=
$t_{i,1,1}\, x^1\, G_1(x) + t_{i,1,2}\, x^1\, G_2(x) + \ldots + t_{i,1,v}\, x^1\, G_v(x)$
+
$t_{i,2,1}\, x^2\, G_1(x) + t_{i,2,2}\, x^2\, G_2(x) + \ldots + t_{i,2,v}\, x^2\, G_v(x)$
+
...
+
$t_{i,L,1}\, x^L\, G_1(x) + t_{i,L,2}\, x^L\, G_2(x) + \ldots + t_{i,L,v}\, x^L\, G_v(x)$
+
$P_i(x)$.

In the above equation, $P_i(x)$ is a polynomial with constant rational coefficients whose values depend on the initial given algebraic constants $\{a_0, a_1, a_2, \ldots, a_{L-1}, b_1, b_2, b_3, \ldots, b_L\}$.

The above equation of $G_i(x)$ may be rewritten in a neater form as follows, for each integer i in [1, v] :
$H_{i,1}(x)G_1(x) + H_{i,2}(x)G_2(x) + \ldots + H_{i,L}(x)G_L(x) = P_i(x)$, where each of $\{P_i(x), H_{i,1}(x), H_{i,2}(x), \ldots, H_{i,L}(x), G_i(x)\}$ is a polynomial with constant rational coefficients.

Solving for $G_i(x)$ for each integer i in [1, v], yields a rational polynomial with constant rational coefficients, that is $G_i(x) = NUM_i(x)/DEN_i(x)$, where each of $NUM_i(x)$ and $DEN_i(x)$ is a polynomial in x with constant integer coefficients. That is, each of $NUM_i(x)$ and $DEN_i(x)$ is a polynomial of form $c_d\, x^d + c_{d-1}\, x^{d-1} + c_{d-2}\, x^{d-2} + \ldots + c_0$, where each of $\{c_0, c_1, c_2, \ldots, c_d, d\}$ is an integer constant and d>0.

Next, we know that for a generic RHLR, defined in Section 1 (Introduction), the generating function [2] is given by the following rational polynomial with constant rational coefficients:
$(a_0 - x(a_0b_1 - a_1) - x^2(a_0b_2 + a_1b_1 - a_2) - x^3(a_0b_3 + a_1b_2 + a_2b_1 - a_3) - \ldots - x^{L-1}(a_0b_{L-1} + a_1b_{L-2} + a_2b_{L-3} + \ldots + a_{L-2}b_1 - a_{L-1})) / (1 - xb_1 - x^2b_2 - x^3b_3 - \ldots - x^{L-1}b_{L-1})$.

So, for each integer i in [1, v], we express $G_i(x) = NUM_i(x)/DEN_i(x)$, such that the coefficient of x^0 in $DEN_i(x)$ is equal to 1, and then the coefficients of the other powers of x in each of $NUM_i(x)$ and $DEN_i(x)$ can become rational constants. We are thus able to find the starting values and the coefficients, and thus define the RHLR for $k_{n,i}$ for each integer i in [1, v], for each integer $n \geq 0$.

<u>Hence Proved Lemma 1.2</u>
Hence Proved Theorem 1

Theorem 2: The nth term f_n of an ALHR can be expressed as a weighted sum of the nth terms of RHLRs, where the algebraic weights are from a finite set of algebraic constants.
Proof: This follows directly from Theorem 1, where Q is the finite set of algebraic constants
Hence Proved Theorem 2

Theorem 3: In an ALHR, the following two statements are true:
1. **$(f_n = 0) \leftrightarrow (k_{n,1} = k_{n,2} = k_{n,3} = \ldots = k_{n,v} = 0)$**
2. **(f_n is periodic) ↔ (each of $\{k_{n,1}, k_{n,2}, \ldots, k_{n,v}\}$ is periodic).**

Proof: Since no two elements of Q are rational multiples of each other, there is no non-trivial rationally weighted average of $\{q_1, q_2, q_3, \ldots, q_v\}$ that can be equal to 0, with the exception of the trivial case where all rational weights are equal to 0. Hence, $(f_n = 0) \rightarrow (k_{n,1} = k_{n,2} = k_{n,3} = \ldots = k_{n,v} = 0)$. Also, $(k_{n,1} = k_{n,2} = k_{n,3} = \ldots = k_{n,v} = 0) \rightarrow (f_n = 0)$, is obvious.

Again, since no two elements of Q are rational multiples of each other, we can say that ($f_t = f_{t+mT}$ for two integers {t, T} and all non-negative integers m) → (each of {$k_{t,1} = k_{t+mT,1}$, $k_{t,2} = k_{t+mT,2}$, ... , $k_{t,v} = k_{t+mT,v}$ }). The converse is obvious, that is, (each of {$k_{t,1} = k_{t+mT,1}$, $k_{t,2} = k_{t+mT,2}$, ... , $k_{t,v} = k_{t+mT,v}$ }) →($f_t = f_{t+mT}$).
Hence Proved Theorem 3

2.2. The Algorithm

We now give the following algorithm, to express the ALHR as the weighted average of a finite number of RHLRs, where the weights are algebraic constants from a finite set:

1. In the given ALHR, express each of the given starting points {a_0 , a_1 , a_2 , ... , a_{L-1}} and each of the given coefficients {b_1 , b_2 , b_3 , ... , b_L} in SAF.
2. Generate set S.
3. Generate set P from S.
4. Generate the initial version of set Q_0 from P, and the final pruned version and denote it as set Q having v elements.
5. Write $f_n = k_{n,1}\, q_1 + k_{n,2}\, q_2 + k_{n,3}\, q_3 + ... + k_{n,v}\, q_v$, for $n \geq 0$, which is the absolute formulation for f_n. Substitute this absolute formulation into the recursive formulation, and then equate the coefficients of each of the elements of Q on both sides of the resulting equation. As explained in Theorem 1, for each integer i in [1, v], for each integer j in [1, v], for each integer k in [1, v], find the values of the rational constants $w_{i,j,k}$ and $t_{i,j,k}$. This allows us to express the rational nth term $k_{n,i}$ as a RNHLR, for each integer i in [1, v].
6. As explained in Theorem 1, derive the expression of G_i , the generating function of $k_{n,i}$, as $G_i(x) = NUM_i(x)/DEN_i(x)$, where each of $NUM_i(x)$ and $DEN_i(x)$ is a polynomial in x with rational coefficients, and the coefficient of x^0 in $DEN_i(x)$ is equal to 1.
7. Compare $G_i(x)$ with the well-known standard form of the generating function of a HLR, and express $k_{n,i}$ as a RHLR for every integer i in [1, v].
8. Finally, express $f_n = k_{n,1}\, q_1 + k_{n,2}\, q_2 + k_{n,3}\, q_3 + ... + k_{n,v}\, q_v$, for $n \geq 0$.

2.3. An example

Consider the following ALHR f_n defined by the following conditions:

$f_n = 0$, for integers $n < 0$
$f_n = a_n$, for integers n in [0, 1]
$f_n = b_1\, f_{n-1} + b_2\, f_{n-2}$, for integers $n \geq 2$

where:

$a_0 = 1$
$a_1 = (1 + (2 + 7^{2/3})^{1/2})/10$
$b_1 = (1 + (2 + 7^{2/3})^{1/2})/5$
$b_2 = -1$

The above AHLR is the same one used for generating the values of cos(nx) = 2cos(x)cos((n-1)x) - cos((n-2)x), where for our example, we take $\cos(x) = (1 + (2 + 7^{2/3})^{1/2})/10$. Here, our set P = {$7^{1/3}$, $(2 + 7^{2/3})^{1/2}$}. Our initial set Q_0 = {1 , ($7^{1/3}$), ($7^{2/3}$), ($(2 + 7^{2/3})^{1/2}$) , ($(2 + 7^{2/3})^{1/2}\, 7^{1/3}$) , ($(2 + 7^{2/3})^{1/2}\, 7^{2/3}$)}. In this example, we are given the starting points and coefficients, directly in SAF, and there are also no denominators, making our example easier to manually compute. Since there is nothing to prune, set Q_0 is the same as the final version of set Q having v=6 elements. So our absolute formulation is: $f_n = k_{n,1} + k_{n,2}\,(7^{1/3}) + k_{n,3}\,(7^{2/3}) + k_{n,4}\,((2 + 7^{2/3})^{1/2}) + k_{n,5}\,((2 + 7^{2/3})^{1/2}\, 7^{1/3}) + k_{n,6}\,((2 + 7^{2/3})^{1/2}\, 7^{2/3})$, for $n \geq 0$. Substituting this absolute formulation into the recursive formulation yields:

$k_{n,1} + k_{n,2}\,(7^{1/3}) + k_{n,3}\,(7^{2/3}) + k_{n,4}\,((2 + 7^{2/3})^{1/2}) + k_{n,5}\,((2 + 7^{2/3})^{1/2}\, 7^{1/3}) + k_{n,6}\,((2 + 7^{2/3})^{1/2}\, 7^{2/3})$
=
$((1 + (2 + 7^{2/3})^{1/2})/5)(k_{n-1,1} + k_{n-1,2}\,(7^{1/3}) + k_{n-1,3}\,(7^{2/3}) + k_{n-1,4}\,((2 + 7^{2/3})^{1/2}) + k_{n-1,5}\,((2 + 7^{2/3})^{1/2}\, 7^{1/3}) + k_{n-1,6}\,((2 + 7^{2/3})^{1/2}\, 7^{2/3}))$
-
$(k_{n-2,1} + k_{n-2,2}\,(7^{1/3}) + k_{n-2,3}\,(7^{2/3}) + k_{n-2,4}\,((2 + 7^{2/3})^{1/2}) + k_{n-2,5}\,((2 + 7^{2/3})^{1/2}\, 7^{1/3}) + k_{n-2,6}\,((2 + 7^{2/3})^{1/2}\, 7^{2/3}))$.

Now we equate the coefficients of q_i on the LHS and RHS, for each integer i in [1,6].

Equating the coefficient of q_1=1 gives, for each integer $n \geq 2$:

$k_{n,1} = (k_{n-1,1}/5) + (2k_{n-1,4}/5) + (7k_{n-1,5}/5) - k_{n-2,1}$.

Equating the coefficient of q_2=($7^{1/3}$) gives, for each integer $n \geq 2$:

$k_{n,2} = (k_{n-1,2}/5) + (2k_{n-1,5}/5) + (7k_{n-1,6}/5) - k_{n-2,2}$.

Equating the coefficient of q_3=($7^{2/3}$) gives, for each integer $n \geq 2$:

$k_{n,3} = (k_{n-1,3}/5) + (k_{n-1,4}/5) + (2k_{n-1,6}/5) - k_{n-2,3}$.

Equating the coefficient of $q_4=(2 + 7^{2/3})^{1/2}$ gives, for each integer $n\geq 2$:

$k_{n, 4} = (k_{n-1, 1}/5) + (k_{n-1, 4}/5) - k_{n-2, 4}$.

Equating the coefficient of $q_5=((2 + 7^{2/3})^{1/2} (7^{1/3}))$ gives, for each integer $n\geq 2$:

$k_{n, 5} = (k_{n-1, 2}/5) + (k_{n-1, 5}/5) - k_{n-2, 5}$.

Equating the coefficient of $q_6=((2 + 7^{2/3})^{1/2} (7^{2/3}))$ gives, for each integer $n\geq 2$:

$k_{n, 6} = (k_{n-1, 3}/5) + (k_{n-1, 6}/5) - k_{n-2, 6}$.

We also know from the starting points a_0 and a_1, that:

$k_{0, 1} = 1$.

$k_{0, 2} = k_{0, 3} = k_{0, 4} = k_{0, 5} = k_{0, 6} = 0$.

$k_{1, 1} = k_{1, 4} = 1/10$.

$k_{1, 2} = k_{1, 3} = k_{1, 5} = k_{1, 6} = 0$.

And finally, we know that for each integer i in [1, 6], and for each integer $n < 0$:

$k_{n, i} = 0$.

We have expressed $k_{n, i}$ as a RNHLR of degree 2. Our aim is now to express $k_{n, i}$ as a RHLR, for each integer i in [1, 6]. For this, we derive the generating function G_i of $k_{n, i}$ for each integer i in [1, 6]. Using the method of generating functions on each of the RNHLR equations obtained (multiplying the LHS and RHS of each equation by x^n and summing for all $n\geq 2$, we obtain:

From the RNHLR for $k_{n, 1}$:

$(G_1 - k_{0, 1} - xk_{1, 1}) = (x(G_1 - k_{0, 1})/5) + (2x(G_4 - k_{0, 4})/5) + (7x(G_5 - k_{0, 5}))/5) - (x^2G_1)$.

From the RNHLR for $k_{n, 2}$:

$(G_2 - k_{0, 2} - xk_{1, 2}) = (x(G_2 - k_{0, 2})/5) + (2x(G_5 - k_{0, 5})/5) + (7x(G_6 - k_{0, 6}))/5) - (x^2G_2)$.

From the RNHLR for $k_{n, 3}$:

$(G_3 - k_{0, 3} - xk_{1, 3}) = (x(G_3 - k_{0, 3})/5) + (x(G_4 - k_{0, 4})/5) + (2x(G_6 - k_{0, 6}))/5) - (x^2G_3)$.

From the RNHLR for $k_{n, 4}$:

$(G_4 - k_{0, 4} - xk_{1, 4}) = (x(G_1 - k_{0, 1})/5) + (x(G_4 - k_{0, 4})/5) - (x^2G_4)$.

From the RNHLR for $k_{n, 5}$:

$(G_5 - k_{0, 5} - xk_{1, 5}) = (x(G_2 - k_{0, 2})/5) + (x(G_5 - k_{0, 5})/5) - (x^2G_5)$.

From the RNHLR for $k_{n, 6}$:

$(G_6 - k_{0, 6} - xk_{1, 6}) = (x(G_3 - k_{0, 3})/5) + (x(G_6 - k_{0, 6})/5) - (x^2G_6)$.

The above 6 equations may be respectively rewritten in a concise form as follows:

$G_1(x^2 - x/5 + 1) + G_4(-2x/5) + G_5(-7x/5) = 1 - (x/10)$.

$G_2(x^2 - x/5 + 1) + G_5(-2x/5) + G_6(-7x/5) = 0$.

$G_3(x^2 - x/5 + 1) + G_4(-x/5) + G_6(-2x/5) = 0$.

$G_4(x^2 - x/5 + 1) + G_1(-x/5) = -(x/10)$.

$G_5(x^2 - x/5 + 1) + G_2(-x/5) = 0$.

$G_6(x^2 - x/5 + 1) + G_3(-x/5) = 0$.

For each integer i in [1,6], solving for G_i gives a solution of the form $(NUM_i(x)/DEN_i(x))$, where each of $NUM_i(x)$ and $DEN_i(x)$ is a polynomial in x whose coefficients are rational constants, and the coefficient of x^0 in $DEN_i(x)$ is equal to 1. For example, for some i, if $G_i = ((x^2 - 3x + 5)/(x^3 - 3x^2 + 7x - 13))$, then write $G_i = (-1/13)((x^2 - 3x + 5)/(-(x^3/13) + (3x^2/13) - (7x/13) + 1))$

Following the final step 7 of our algorithm, compare the expression of $G_i = (NUM_i(x)/DEN_i(x))$ with the standard form of the generating function of a generic RHLR, to find the starting points and coefficients and thus express each of $\{k_{n, 1} , k_{n, 2} , k_{n, 3} , k_{n, 4} , k_{n, 5} , k_{n, 6}\}$ as a RHLR.

3. Conclusion and Future Work

Our paper showed the link between the ALHR and RHLRs. Specifically, it showed that within a finite set of steps (i.e. using a deterministic algorithm), the nth term of an ALHR can be expressed as an weighted sum of the nth terms of a finite number of RHLRs, where the weights belong to a finite set of algebraic constants derived from the set of starting points and coefficients of the ALHR. Some applications where our results could be useful are:

1. Finding when the nth terms of multiple RHLRs could be simultaneously zero for the same n. This problem could be converted into the problem of whether or not the nth term of equivalent AHLR (representing the RHLRs) is equal to

0 for some n.

2. Evaluation of the nth term of an AHLR. Since the nth term of a RHLR can be efficiently evaluated, we would need to find the values of the nth terms of multiple RHLRs that arise from the AHLR. The main drawback of our approach would be that the number of such RHLRs can potentially grow super-exponentially to the data size.
3. Determination of whether or not an angle is an integer multiple of PI. If we are given the algebraic value of the sine or cosine of an angle x, then we would be dealing with the AHLR similar to what we covered in our example in Section 2.3, where the conclusion would be that x is an integer multiple of PI, if and only if, cos(nx) = 1 for some n>0.
4. An interesting observation is that the degree of the numerator or denominator of the generating functions of the RHLRs, can be > L. In other words, if an ALHR of degree L is expressible as a set of RHLRs of degree > L, the reverse is also possible (i.e. it should be possible to "express" an RHLR of degree L as an AHLR of degree lower than L). We know that the Skolem problem, the question of whether or not there exists some n>0 such that $f_n = 0$, is open for RHLRs of degree > 4. As the degree is reduced, expressing RHLRs as the equivalent AHLR, this might prove to be a useful approach in solving the open Skolem problem.

References

[1] Dmitry I. Khomovsky, “Efficient computation of terms of linear recurrence sequences of any order”, Integers, 18, 2018.
[2] Herbert Wilf, "Generatingfunctionology", 1990.

About the author

I, Deepak Ponvel Chermakani, wrote this paper out of my own interest and initiative, during my spare time. In Aug 2015, I completed a fulltime two-year Master of Science Degree in Electrical Engineering, from University of Hawaii at Manoa USA (www.hawaii.edu). In Sep 2010, I completed a fulltime one-year Master of Science Degree in Operations Research with Computational Optimization, from University of Edinburgh UK (www.ed.ac.uk). In Jul 2003, I completed a fulltime four-year Bachelor of Engineering Degree in Electrical and Electronic Engineering, from Nanyang Technological University Singapore (www.ntu.edu.sg). In Jul 1999, I completed fulltime high schooling from National Public School in Bangalore in India. I am most grateful to my parents (especially my mother Mrs. Kanaga Rathinam Chermakani) for their sacrifices in educating me and bringing me up.