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Granule Description based on Compound Concepts *

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Concise granule descriptions for describable granules and approaching description methods for indescribable granules are challenging and important issues in granular computing. The concept with only common attributes has been frequently studied. To investigate the granules with some special needs, we propose two new types of compound concepts in this paper: bipolar concept and common-and-necessary concept. Based on the definitions of concept-forming operations, the logical formulas are derived for each of the following types of concepts: formal concept, three-way concept, object oriented concept, bipolar concept and common-and-necessary concept. Furthermore, by utilizing the logical relationship among various concepts, we have derived concise and unified equivalent conditions for describable granules and approaching description methods for indescribable granules for all five kinds of concepts.

Keywords: Granular computing; Granule description; Approaching description; Bipolar concept; Common-and-necessary concept

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1 Introduction

A group of objects with some common attributes is called a granule [21, 22]. Concept-based granule representation is a popular topic, and concept lattice is a key tool for information processing and analysis. The mathematical basis of concept lattice is lattice theory, the visualization tool is the Hasse graph, and the related research methods are abstract algebra, discrete mathematics, data structure and algorithm analysis, fuzzy set [20], rough set [12], granular computing [21, 22], etc. So far, formal concept analysis has been frequently used in information retrieval [10], knowledge discovery [11], association analysis [16], recommendation system [28] and other fields [3, 17].

When we study a certain kind of concepts, we need to first consider how to find out all the concepts from the given data. This problem is called concept lattice construction

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[1, 2, 13]. Second, in order to better analyze data and save storage space, it is necessary to reduce concept lattice [4, 15, 18, 23]. Furthermore, the nodes of concept lattice can infer from each other, and on this basis, one can extract rules [10].

Zhi and Qi in [27] observed that the existing types of concepts cannot simultaneously investigate the common attributes and possible attributes of granules. Thus they proposed the common-possible concepts, where all attributes are from the same collection, and explored the relationships among the common-possible concepts, formal concepts and object oriented concepts.

In this paper, we still consider a scenario that in an international travel agency, one tour guide can speak several frequently used languages, or can speak infrequently used languages. We first introduce the compound context (U, A, I, B, J), where A is the set of frequently used attributes, and B is the set of infrequently used attributes, on which common-and-necessary concept can be defined.

At school, teachers usually pay more attention to two types of students. One kind of students have good test scores, and the other kind of students have poor test scores. Considering this kind of scenario, we need to define the bipolar concept.

Li and Liu in [9] proposed the concepts of covering element and inserting element of a granule by which equivalent conditions of describable granules of formal concept and three-way concept were obtained.

Usually, the definitions of concept operators are given in a description language. However, sometimes the logical formula can reflect the essence of the problem. For example, the essence of formal concept is in the logical formula: $X = a_1 \wedge a_2 \wedge \cdots \wedge a_k$, and the essence of object oriented concept is in the logical formula: $X = a_1 \vee a_2 \vee \cdots \vee a_k$, where (U, A, I) is a formal context, $X \subseteq U$, and $a_1, a_2, \cdots a_k \in A$. It is easy to decipher that there is a kind of equivalent relationship between the two. In this paper, the logical formulas are given for all five kinds of concepts. Thus, by utilizing the logical relationship among various concepts, and based on the definitions of concept-forming operations, we can derive much concise and unified equivalent conditions of describable granules and approaching description method of indescribable granules for all five kinds of concepts.

The main contributions in this paper can be summarized as follows.

1. We propose two new types of compound concepts: bipolar concept and common-and-necessary concept.

2. The logical formulas have been given for all five kinds of concepts.

3. Utilizing the logical relationship among various concepts, we have derived much concise and unified equivalent conditions of describable granules and approaching description method of indescribable granules for all five kinds of concepts.

The structure of this paper is organized as follows. In Section 2, some frequently used definitions and lemmas related to this paper are reviewed. In Section 3, we present concise and unified equivalent conditions of describable granules for five kinds of concepts. In Section 4, some explicit approaching description methods for indescribable granules are presented for four kinds of concepts. Finally, in Section 5, the paper is concluded with a summary and an outlook for future work.

2 Preliminary notions and properties

For the convenience of discussion, we first review some frequently used notions and properties related to this paper, such as formal concept, concept-forming operations, logical language, describable granules, and the description of granule.

2.1 The formal context and its operations

We first present the definitions of formal context and its operations as follows.

Definition 1. [6] We say that a triplet (U, A, I) is a formal context, if $U = \{x_1, x_2, \ldots, x_m\}$, $A = \{a_1, a_2, \ldots, a_n\}$, and $I \subseteq U \times A$ is a binary relation. Here we call each $x_i (i \leq m)$ as an object, and each $a_j (j \leq n)$ as an attribute. xIa or $(x, a) \in I$ indicates that an object $x \in U$ has the attribute $a \in A$.

In the rest of this paper, (U, A, I) is always used to represent a formal context. Some operations can be defined as below.

Definition 2. [6] Given a (U, A, I). For any $X \subseteq U$ and $B \subseteq A$, two concept-forming operations are defined respectively:

$f : P(U) \to P(A)$, $f(X) = \{m \in A | \forall x \in X, (x, m) \in I\}$

$g : P(A) \to P(U)$, $g(B) = \{x \in U | \forall m \in B, (x, m) \in I\}$

The following are the definitions of formal concepts and concept lattices.

Definition 3. [6] Given a (U, A, I). For any $X \subseteq U$ and $B \subseteq A$, if $f(X) = B$ and $g(B) = X$, then we define (X, B) as a formal concept, where X and B are said to be the extension and the intension of (X, B), respectively. For concepts $(X_1, B_1), (X_2, B_2)$, where $X_1, X_2 \subseteq U$, $B_1, B_2 \subseteq A$, we can define the partial order as follows:

$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow B_2 \subseteq B_1$

Furthermore, we have the following definitions:

$(X_1, B_1) \bigwedge (X_2, B_2) = (X_1 \cap X_2, fg(B_1 \cup B_2))$ or $(X_1 \cap X_2, f(X_1 \cap X_2))$

$(X_1, B_1) \bigvee (X_2, B_2) = (gf(X_1 \cup X_2), B_1 \cap B_2)$ or $(g(B_1 \cap B_2), B_1 \cap B_2)$

Thus, one can observe that all formal concepts from (U, A, I) would form a complete lattice, which is defined as a concept lattice, and denoted by $L(U, A, I)$.

Lemma 1. [6] For any $X_1, X_2, X \subseteq U$, $B_1, B_2, B \subseteq A$, here (U, A, I) is a context, it is easy to show that the following statements hold:

(1) $X_1 \subseteq X_2 \Rightarrow f(X_2) \subseteq f(X_1)$, $B_1 \subseteq B_2 \Rightarrow g(B_2) \subseteq g(B_1)$;

(2) $X \subseteq gf(X)$, $B \subseteq fg(B)$;

(3) $f(X) = fgf(X)$, $g(B) = gfg(B)$;

(4) $X \subseteq g(B) \;\Leftrightarrow\; B \subseteq f(X)$;

(5) $f(X_1 \cup X_2) = f(X_1) \cap f(X_2)$, $g(B_1 \cup B_2) = g(B_1) \cap g(B_2)$;

(6) $f(X_1 \cap X_2) \supseteq f(X_1) \cup f(X_2)$, $g(B_1 \cap B_2) \supseteq g(B_1) \cup g(B_2)$.

A (U, A, I) is typically represented by a table of 0 and 1, with 1s meaning binary relations between objects (rows) and attributes (columns). A simple example of a (U, A, I) is presented as follows:

Table 1: (U, A, I)

U	a_1	a_2	a_3	a_4	a_5
1	0	1	1	0	0
2	1	1	0	0	0
3	1	0	0	0	0
4	0	0	0	0	1
5	0	0	0	1	1
6	0	0	1	1	1
7	1	1	1	0	0

The formal concepts in Table 1 can be calculated as given in the following Table 2:

Table 2: Formal concepts in Table 1

		$C_0 = (\{1,2,3,4,5,6,7\}, \emptyset)$	
$C_4 = (\{2,3,7\}, \{a_1\})$	$C_3 = (\{1,2,7\}, \{a_2\})$	$C_2 = (\{1,6,7\}, \{a_3\})$	$C_1 = (\{4,5,6\}, \{a_5\})$
	$C_5 = (\{2,7\}, \{a_1, a_2\})$	$C_6 = (\{1,7\}, \{a_2, a_3\})$	$C_7 = (\{5,6\}, \{a_4, a_5\})$
	$C_9 = (\{7\}, \{a_1, a_2, a_3\})$		$C_8 = (\{6\}, \{a_3, a_4, a_5\})$
		$C_{10} = (\emptyset, \{a_1, a_2, a_3, a_4, a_5\})$	

One can visualise a formal concept in a table of 0 and 1 as a closed rectangle of 1s, where the rows and columns are not necessarily contiguous [1]. We define the cell of the ith row and jth column as (i, j). Thus in Table 1, $(5, 4)$, $(5, 5)$, $(6, 4)$ and $(6, 5)$ form the concept C_7, and C_7 is a rectangle of height 2 and width 2. Similarly $(6, 3)$, $(6, 4)$ and $(6, 5)$ form the concept C_8, and C_8 is a rectangle of height 1 and width 3. $(1, 3)$, $(6, 3)$ and $(7, 3)$ form the concept C_2, and C_2 is a rectangle of height 3 and width 1, here $(1, 3)$ and $(6, 3)$ are not contiguous. Next, we first investigate the describable granule.

2.2 Describable granule and its description

Given a (U, A, I). We present a logical language used for describing a granule $X \subseteq U$. For any $b \in B \subseteq A$, we call b as an atomic formula. By joining all the atomic formulas in B together with the connective $\wedge$, we can obtain a compound formula $\wedge B = \bigwedge_{b \in B} b$.

Furthermore, if the object $x \in U$ has the attribute b, then we say that x fulfil the atomic formula b, denoted by $x \mapsto b$. Obviously, if $x \mapsto b$ for any $b \in B$, then x is said to fulfil the compound formula $\wedge B$, denoted by $x \mapsto \wedge B$. Thus the semantics of $\wedge B$ is defined as the set of the objects fulfilling $\wedge B$ as given below,

$$m(\wedge B) = \{x \in U | \forall b \in B, x \mapsto b\}$$

Proposition 1. [25] Given a (U, A, I) and $B \subseteq A, B \neq \emptyset$. Then we have $m(\wedge B) = g(B)$.

With the semantics operator m, we now give the definition of a $\wedge$-describable granule.

Definition 4. [25] Given a (U, A, I) and $X \subseteq U$. If there exists $B \subseteq A, B \neq \emptyset$, such that $g(B) = X$, then we say that X is $\wedge$-describable and $\wedge B$ is a description of X, denoted by $d(X) = \wedge B$.

The following proposition is an immediate result from Proposition 1.

Proposition 2. [25] Given a (U, A, I), $Y \subseteq U$ and $f(Y) \neq \emptyset$. If Y is the extension of a formal concept, then the granule Y must be $\wedge$-describable and $d(Y) = \wedge f(Y)$.

For example in Table 1, we can rewrite $a_1 = (0, 1, 1, 0, 0, 0, 1)$, $a_2 = (1, 1, 0, 0, 0, 0, 1)$ as column vectors, then $a_1 \wedge a_2 = (0, 1, 0, 0, 0, 0, 1)$, where $a_1, a_2 \in A$. At the same time, $x_2 = (1, 1, 0, 0, 0)$, $x_7 = (1, 1, 1, 0, 0)$, then $x_2 \wedge x_7 = (1, 1, 0, 0, 0)$, where $x_2, x_7 \in U$. Thus $X = \{x_2, x_7\} = a_1 \wedge a_2$, and X is $\wedge$-describable, $d(X) = a_1 \wedge a_2$, $\{x_2, x_7\} = m(a_1 \wedge a_2)$, also $C_5 = (\{x_2, x_7\}, \{a_1, a_2\})$ is a concept.

In essence, $X \subseteq U$ is $\wedge$-describable $\Longleftrightarrow$ there exist $a_1, a_2, \cdots a_k \in A$, such that $X = a_1 \wedge a_2 \wedge \cdots \wedge a_k$. It is worth noticing that $\{x_6\} = a_3 \wedge a_4$, also $\{x_6\} = a_3 \wedge a_4 \wedge a_5$. So $d(\{x_6\}) = a_3 \wedge a_4$ and $d(\{x_6\}) = a_3 \wedge a_4 \wedge a_5$. One can see that the description of $X \subseteq U$, maybe not unique.

3 Equivalent conditions of describable granules

Based on the definitions of concept-forming operations, we present some unified equivalent conditions of describable granules for each of the following types of concepts: formal concept, three-way concept, object oriented concept, bipolar concept and common-and-necessary concept.

3.1 $\wedge$-describable granules

In [9], the notion of covering element was proposed, and the following proposition was obtained.

Proposition 3. [9] Given a (U, A, I), $X \subseteq U$ and $f(X) \neq \emptyset$. Then X is $\wedge$-describable $\Longleftrightarrow$ X does not have any covering element $y \notin X$, such that $f(X) \subseteq f(y)$.

Based on concept-forming operations of Definition 2, we give the following simple theorem.

Theorem 1. Given a (U, A, I), $X \subset U$ and $f(X) \neq \emptyset$. Then X is $\wedge$-describable $\Longleftrightarrow$ $X = gf(X)$.

Proof. $\Longrightarrow$ By Definition 4, there exists $B \subseteq A, B \neq \emptyset$, such that $m(\wedge B) = X \Longleftrightarrow g(B) = X$. By Lemma 1.(3), $X = g(B) = gfg(B) = gf(X)$.

$\Longleftarrow$ Let $B = f(X)$. Then $g(B) = gf(X) = X$. So $m(\wedge B) = X$, thus X is $\wedge$-describable. □

By Lemma 1.(2), $X \subseteq gf(X)$. If $X \subset gf(X)$, then there exists $y \in gf(X)$ and $y \notin X$, such that $f(y) \supseteq f(X)$. Thus y must be a covering element. Therefore, the conditions of Theorem 1 and Proposition 3 are equivalent.

If X is $\wedge$-describable, $B = f(X) = \{a_1, a_2, \cdots a_k\} \Longrightarrow X = g(B) \Longrightarrow X = a_1 \wedge a_2 \wedge \cdots \wedge a_k$. If Y is $\wedge$-describable, $C = f(Y) = \{b_1, b_2, \cdots b_j\} \Longrightarrow Y = g(C) \Longrightarrow Y =$

$b_1 \wedge b_2 \wedge \cdots \wedge b_j$. Thus

$$X \bigcap Y = a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge b_1 \wedge b_2 \wedge \cdots \wedge b_j$$

$$g(f(X) \bigcup f(Y)) = g(\{a_1, a_2, \cdots a_k, b_1, b_2, \cdots b_j\}) = X \bigcap Y$$

So the following Proposition 4 is obvious.

Proposition 4. [9] Given a (U, A, I), $X, Y \subset U$ and $f(X) \neq \emptyset$, $f(Y) \neq \emptyset$. If X and Y are $\wedge$-describable, then granule $X \bigcap Y$ must be $\wedge$-describable, and $d(X \bigcap Y) = \wedge(f(X) \bigcup f(Y))$.

However, as $X \bigcup Y = a_1 \wedge a_2 \wedge \cdots \wedge a_k \vee b_1 \wedge b_2 \wedge \cdots \wedge b_j$, $X \bigcup Y$ maybe not $\wedge$-describable.

3.2 $(\wedge, \wedge, \neg)$-describable granules

Based on Definition 3 in [9], from a (U, A, I), we form a compound context (U, A, I, B, J), where $a \in A \Leftrightarrow b \in B$, $(x, a) \in I \Leftrightarrow (x, b) \notin J$.

In the rest of this paper, (U, A, I, B, J) is always used to represent a compound context.

For any $X \subseteq U$ and $C \subseteq A \bigcup B$, we further extend concept-forming operations f and g_T as follows:

$$f : P(U) \to P(A \bigcup B), f(X) = \{m \in A \bigcup B | \forall x \in X, (x, m) \in I \bigcup J\}$$

$$g_T : P(A \bigcup B) \to P(U),$$

$$g_T(C) = \{u \in U | \forall m \in A \bigcap C, (u, m) \in I\} \bigcap \{u \in U | \forall m \in B \bigcap C, (u, m) \in J\}$$

Evidently, f is the combination of f^+ and f^- in [9], and g_T is the combination of g^+ and g^- in [9].

Given a (U, A, I, B, J). For any $X \subseteq U$ and $C \subseteq A \bigcup B$, if $f(X) = C$ and $g_T(C) = X$, then we say that (X, C) is a three-way concept in [14]. As involving both the logic connectives $\wedge$ and $\neg$, we call it as a compound concept.

Based on Definition 4 in [9], we give a new notion of $(\wedge, \wedge, \neg)$-describable.

Definition 5. Given a (U, A, I, B, J) and $X \subseteq U$, if there exist $C \neq \emptyset$, $D \neq \emptyset$, $C \subseteq A$ and $D \subseteq B$, such that $g_T(C \bigcup D) = X$, then we say that the granule X is $(\wedge, \wedge, \neg)$-describable, and $d(X) = \wedge(C \bigcup D)$.

Similar to $\wedge$-describable, we can obtain the following concise result about $(\wedge, \wedge, \neg)$-describable.

Theorem 2. Given a (U, A, I, B, J), $X \subset U$ and $f(X) \bigcap A \neq \emptyset$, $f(X) \bigcap B \neq \emptyset$. Then X is $(\wedge, \wedge, \neg)$-describable $\Longleftrightarrow X = g_T f(X)$.

Proof. Necessity. By Definition 5, there exist $C \neq \emptyset$, $D \neq \emptyset$, $C \subseteq A$ and $D \subseteq B$, such that $g_T(C \bigcup D) = X \Longrightarrow X = g(C) \bigcap g(D)$, where g is defined in Definition 2.

By Lemma 1.(5), $X = g(C) \bigcap g(D) = g(C \bigcup D)$.

By Lemma 1.(3), $X = g(C \bigcup D) = gfg(C \bigcup D) = gf(X) = g_T f(X)$.

Sufficiency. Let $C \bigcup D = f(X)$, where $f(X) \bigcap A = C$, $f(X) \bigcap B = D$. Then $g_T(C \bigcup D) = g_T f(X) = X$. Thus X is $(\wedge, \wedge, \neg)$-describable. $\Box$

One can observe that Theorem 2 in [9] also gave a equivalent condition of $(\wedge, \wedge, \neg)$-describable granule only for the case of $X \subseteq U$, X is $\wedge$-indescribable in (U, A, I). In comparison with the above result, Theorem 2 in this paper is broader than Theorem 2 in [9]

Furthermore, if X is $(\wedge, \wedge, \neg)$-describable, $f(X) = \{a_1, a_2, \cdots a_k\} \Longrightarrow X = g_T f(X)$ $\Longrightarrow$

$$X = a_1 \wedge a_2 \wedge \cdots \wedge a_k$$

where $a_1, a_2, \cdots a_k \in A \bigcup B$, $\{a_1, a_2, \cdots a_k\} \bigcap A \neq \emptyset$ and $\{a_1, a_2, \cdots a_k\} \bigcap B \neq \emptyset$.

Also, if Y is $(\wedge, \wedge, \neg)$-describable, $f(Y) = \{b_1, b_2, \cdots b_j\} \Longrightarrow Y = g_T f(Y) \Longrightarrow$

$$Y = b_1 \wedge b_2 \wedge \cdots \wedge b_j$$

where $b_1, b_2, \cdots b_j \in A \bigcup B$, $\{b_1, b_2, \cdots b_j\} \bigcap A \neq \emptyset$ and $\{b_1, b_2, \cdots b_j\} \bigcap B \neq \emptyset$.

Therefore,

$$X \bigcap Y = a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge b_1 \wedge b_2 \wedge \cdots \wedge b_j$$

$$g_T(f(X) \bigcup f(Y)) = g(\{a_1, a_2, \cdots a_k, b_1, b_2, \cdots b_j\}) = X \bigcap Y$$

By combining the above results, one can see the following Proposition 5 is obvious.

Proposition 5. Given a (U, A, I, B, J), $X, Y \subset U$ and $f(X) \neq \emptyset$, $f(Y) \neq \emptyset$. If X and Y are $(\wedge, \wedge, \neg)$-describable, then the granule $X \bigcap Y$ must be $(\wedge, \wedge, \neg)$-describable with $d(X \bigcap Y) = \wedge(f(X) \bigcup f(Y))$.

However, as $X \bigcup Y = a_1 \wedge a_2 \wedge \cdots \wedge a_k \vee b_1 \wedge b_2 \wedge \cdots \wedge b_j$, $X \bigcup Y$ may be not $(\wedge, \wedge, \neg)$-describable.

As Proposition 11 in [9] only discuss the case of $X \subseteq U$, X is $\wedge$-indescribable in (U, A, I), which indicates that Proposition 5 here is broader than Proposition 11 in [9].

For example, from a (U, A, I) in Table 1, we can create a (U, A, I, B, J) described as below in Table 3, where $a \in A \Leftrightarrow b \in B$, and $(x, a) \in I \Leftrightarrow (x, b) \notin J$.

Table 3: (U, A, I, B, J)

U	a_1	a_2	a_3	a_4	a_5	b_1	b_2	b_3	b_4	b_5
1	0	1	1	0	0	1	0	0	1	1
2	1	1	0	0	0	0	0	1	1	1
3	1	0	0	0	0	0	1	1	1	1
4	0	0	0	0	1	1	1	1	1	0
5	0	0	0	1	1	1	1	1	0	0
6	0	0	1	1	1	1	1	0	0	0
7	1	1	1	0	0	0	0	0	1	1

From Table 2, $C_5 = (\{2,7\}, \{a_1, a_2\})$ is a concept over (U, A, I). However, $(\{2,7\}, \{a_1, a_2, b_4, b_5\})$ is a concept over (U, A, I, B, J) and $\{2,7\} = a_1 \wedge a_2 \wedge b_4 \wedge b_5$, which is not covered by Theorem 2 in [9].

Note that $\{2,3\} = a_1 \wedge b_3 \wedge b_4 \wedge b_5$, by Proposition 5, $\{2,3\} \bigcap \{2,7\} = \{2\} = a_1 \wedge a_2 \wedge b_3 \wedge b_4 \wedge b_5$, which is still $(\wedge, \wedge, \neg)$-describable. Next, we will describe important $\vee$-describable granules.

3.3 $\vee$-describable granules

To investigate the other aspects of granules, we next present the possibility operator $(\cdot)^{\diamond}$ and necessity operator $(\cdot)^{\Box}$. For convenience, $(\cdot)^{\Box}$ is denoted as $f_{\vee}$, and $(\cdot)^{\diamond}$ is denoted as $g_{\vee}$ in the remaining part of this paper.

Definition 6. [5, 26] Given a (U, A, I). For any $X \subseteq U$ and $B \subseteq A$, two concept-forming operations are defined respectively:

$$f_{\vee} : P(U) \to P(A), f_{\vee}(X) = \{m \in A | \forall g \in U, (g, m) \in I \Rightarrow g \in X\}$$

$$g_{\vee} : P(A) \to P(U), g_{\vee}(B) = \{g \in U | \exists m \in B, (g, m) \in I\}$$

Based on the above two concept-forming operations, one can present the object oriented concept as proposed in [5, 26]. For any $X \subseteq U$ and $B \subseteq A$, if $f_{\vee}(X) = B$ and $g_{\vee}(B) = X$, then (X, B) is called an object oriented concept.

Definition 7. [26] Given a (U, A, I) and $X \subseteq U$. If there exists $B \subseteq A, B \neq \emptyset$, such that $g_{\vee}(B) = X$, then we say that X is $\vee$-describable and $d(X) = \vee B$.

In essence, $X \subseteq U$ is $\vee$-describable $\Longleftrightarrow$ there exist $a_1, a_2, \cdots a_k \in A$, such that

$$X = a_1 \vee a_2 \vee \cdots \vee a_k$$

Furthermore, from a (U, A, I), we can create a context (U, B, J), where $a \in A \Leftrightarrow b \in B$, and $(x, a) \in I \Leftrightarrow (x, b) \notin J$. In this case, we have

$$X = a_1 \vee a_2 \vee \cdots \vee a_k \Longleftrightarrow U \backslash X = b_1 \wedge b_2 \wedge \cdots \wedge b_k$$

Therefore, X is $\vee$-describable over $(U, A, I) \Longleftrightarrow U \backslash X$ is $\wedge$-describable over (U, B, J). The following Theorem 3 is immediate from Theorem 1.

Theorem 3. Given a (U, A, I) and $X \subset U$. Then X is $\vee$-describable $\Longleftrightarrow U \backslash X = gf(U \backslash X)$ over formal context (U, B, J), where $a \in A \Leftrightarrow b \in B$, and $(x, a) \in I \Leftrightarrow (x, b) \notin J$.

If X is $\vee$-describable, one will have $f(U \backslash X) = \{a_1, a_2, \cdots a_k\} \Longrightarrow U \backslash X = gf(U \backslash X) \Longrightarrow U \backslash X = a_1 \wedge a_2 \wedge \cdots \wedge a_k$.

Similarly, if Y is $\vee$-describable, $f(U \backslash Y) = \{b_1, b_2, \cdots b_j\} \Longrightarrow U \backslash Y = gf(U \backslash Y) \Longrightarrow U \backslash Y = b_1 \wedge b_2 \wedge \cdots \wedge b_j$. By combining the above two results, one will have

$$(U \backslash X) \bigcap (U \backslash Y) = a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge b_1 \wedge b_2 \wedge \cdots \wedge b_j$$

$$g(f(X) \bigcup f(Y)) = g(\{a_1, a_2, \cdots a_k, b_1, b_2, \cdots b_j\}) = (U \backslash X) \bigcap (U \backslash Y)$$

So the following Proposition 6 is obvious.

Proposition 6. Given a (U, A, I) and $X, Y \subset U$. If X and Y are $\vee$-describable, then $U \backslash (U \backslash X) \bigcap (U \backslash Y)$ is $\wedge$-describable, and $d((U \backslash X) \bigcap (U \backslash Y)) \quad = \wedge(f(U \backslash X) \bigcup f(U \backslash Y))$.

As $(U \backslash X) \bigcup (U \backslash Y) = a_1 \wedge a_2 \wedge \cdots \wedge a_k \vee b_1 \wedge b_2 \wedge \cdots \wedge b_j$, $(U \backslash X) \bigcup (U \backslash Y)$ may be not the $\wedge$-describable.

In fact, similar to Theorem 1, it is easy to obtain the following Proposition 7, which is equivalent to Theorem 3.

Proposition 7. Given a (U, A, I), $X \subset U$ and $f_\vee(X) \neq \emptyset$. Then X is $\vee$-describable $\Longleftrightarrow X = g_\vee f_\vee(X)$.

For example, from a (U, A, I) in Table 1, we can create a formal context (U, B, J), given below in Table 4, where $a \in A \Leftrightarrow b \in B$, and $(x, a) \in I \Leftrightarrow (x, b) \notin J$.

Table 4: Formal context (U, B, J)

U	b_1	b_2	b_3	b_4	b_5
1	1	0	0	1	1
2	0	0	1	1	1
3	0	1	1	1	1
4	1	1	1	1	0
5	1	1	1	0	0
6	1	1	0	0	0
7	0	0	0	1	1

On one hand, $(\{1, 4, 5, 6, 7\}, \{a_3, a_4, a_5\})$ is an object oriented concept over (U, A, I) in Table 1, and $\{1, 4, 5, 6, 7\} = a_3 \vee a_4 \vee a_5$. On the other hand, $(\{2, 3\}, \{b_3, b_4, b_5\})$ is a formal concept over formal context (U, B, J) in Table 4, and $U \backslash \{1, 4, 5, 6, 7\} = \{2, 3\} = b_3 \wedge b_4 \wedge b_5$.

Similarly, $(\{1, 2, 5, 6, 7\}, \{a_2, a_3, a_4\})$ is an object oriented concept over (U, A, I) in Table 1, and $\{1, 2, 5, 6, 7\} = a_2 \vee a_3 \vee a_4$. At the same time, $(\{3, 4\}, \{b_2, b_3, b_4\})$ is a formal concept over formal context (U, B, J) in Table 4, and $U \backslash \{1, 2, 5, 6, 7\} = \{3, 4\} = b_2 \wedge b_3 \wedge b_4$.

As $\{2, 3\} \cap \{3, 4\} = \{3\} = b_2 \wedge b_3 \wedge b_4 \wedge b_5$, from Proposition 6, $\{1, 2, 4, 5, 6, 7\} = a_2 \vee a_3 \vee a_4 \vee a_5$.

As there are many fast algorithms for computing formal concepts [1, 2], so these algorithms can be used for computing the object oriented concepts and attribute oriented concepts [19].

3.4 $(\wedge, \neg)$-describable granules

In Subsection 3.1, we usually extract the common features of a specific set to explore the embedded useful patterns for decision making, which is usually called as common attribute analysis [27]. In this subsection, we will consider the following scenario. At school, teachers usually pay more attention to two types of students. One kind of students have good test scores, and the other kind of students have poor test scores.

For any $X \subseteq U$ and $C \subseteq A\bigcup B$, we further extend concept-forming operations f and g as follows:

$$f : P(U) \to P(A\bigcup B), f(X) = \{m \in A\bigcup B|\forall x \in X, (x, m) \in I\bigcup J\}$$

$$g : P(A\bigcup B) \to P(U), g(C) = \{u \in U|\forall m \in C, (u, m) \in I\bigcup J\}$$

Evidently, f is the combination of f^+ and f^- in [9], and g is the combination of g^+ and g^- in [9]. It is easy to show that Lemma 1 still holds for f, g in the compound context (U, A, I, B, J).

Given a (U, A, I, B, J). For any $X \subseteq U$ and $C \subseteq A\bigcup B$, if $f(X) = C$ and $g(C) = X$, then we say that (X, C) is called a **bipolar concept**. As involving both the logic connectives $\wedge$ and $\neg$, we call it as a compound concept.

We now give a new notion called $(\wedge, \neg)$-describable.

Definition 8. Given a (U, A, I, B, J) and $X \subseteq U$, if there exist $C\bigcup D \neq \emptyset$, $C \subseteq A$ and $D \subseteq B$, such that $g(C\bigcup D) = X$, then we say that the granule X is $(\wedge, \neg)$-describable, and $d(X) = \wedge(C\bigcup D)$.

Similar to $\wedge$-describable, we can obtain the following concise result about $(\wedge, \neg)$-describable.

Theorem 4. Given a (U, A, I, B, J), $X \subset U$ and $f(X)\bigcap(A\bigcup B) \neq \emptyset$. Then X is $(\wedge, \neg)$-describable $\Longleftrightarrow X = gf(X)$.

Proof. $\Longrightarrow$ By Definition 5, there exist $C \neq \emptyset$, $D \neq \emptyset$, $C \subseteq A$ and $D \subseteq B$, such that $g(C\bigcup D) = X$. By Lemma 1.(3), $X = g(C\bigcup D) = gfg(C\bigcup D) = gf(X)$.

$\Longleftarrow$ Let $C\bigcup D = f(X)$, where $f(X)\bigcap A = C$, $f(X)\bigcap B = D$. Then $g(C\bigcup D) = gf(X) = X$. Thus X is $(\wedge, \neg)$-describable. □

Furthermore, if X is $(\wedge, \neg)$-describable, $f(X) = \{a_1, a_2, \cdots a_k\} \Longrightarrow X = gf(X) \Longrightarrow$

$$X = a_1 \wedge a_2 \wedge \cdots \wedge a_k$$

where $a_1, a_2, \cdots a_k \in A\bigcup B$. Also, if Y is $(\wedge, \neg)$-describable, $f(Y) = \{b_1, b_2, \cdots b_j\} \Longrightarrow Y = gf(Y) \Longrightarrow$

$$Y = b_1 \wedge b_2 \wedge \cdots \wedge b_j$$

where $b_1, b_2, \cdots b_j \in A\bigcup B$. Therefore,

$$X\bigcap Y = a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge b_1 \wedge b_2 \wedge \cdots \wedge b_j$$

$$g(f(X)\bigcup f(Y)) = g(\{a_1, a_2, \cdots a_k, b_1, b_2, \cdots b_j\}) = X\bigcap Y$$

By combining the above results, one can see the following Proposition 5 is obvious.

Proposition 8. Given a (U, A, I, B, J), $X, Y \subset U$ and $f(X) \neq \emptyset$, $f(Y) \neq \emptyset$. If X and Y are $(\wedge, \neg)$-describable, then the granule $X\bigcap Y$ must be $(\wedge, \neg)$-describable with $d(X\bigcap Y) = \wedge(f(X)\bigcup f(Y))$.

However, as $X\bigcup Y = a_1 \wedge a_2 \wedge \cdots \wedge a_k \vee b_1 \wedge b_2 \wedge \cdots \wedge b_j$, $X\bigcup Y$ may be not $(\wedge,\neg)$-describable.

By Definition 5 and Definition 8, one can observe that $(\wedge,\neg)$-describable covers a wider range than $(\wedge,\wedge,\neg)$-describable. Specifically, $(\wedge,\wedge,\neg)$-describable only covers the case of $f(X)\bigcap A \neq \emptyset$ and $f(X)\bigcap B \neq \emptyset$. However, $(\wedge,\neg)$-describable also covers the case of $f(X)\bigcap A \neq \emptyset$, $f(X)\bigcap B = \emptyset$ and the case of $f(X)\bigcap A = \emptyset$, $f(X)\bigcap B \neq \emptyset$.

For example, from Table 3, $\{2,3,4\} = b_3 \wedge b_4$, so $\{2,3,4\}$ is $(\wedge,\neg)$-describable. As $A\bigcap\{b_3,b_4\} = \emptyset$, thus $\{2,3,4\}$ is not $(\wedge,\wedge,\neg)$-describable.

Similarly, $\{1,6,7\} = a_3$, so $\{1,6,7\}$ is $(\wedge,\neg)$-describable. As $B\bigcap\{a_3\} = \emptyset$, thus $\{1,6,7\}$ is not $(\wedge,\wedge,\neg)$-describable.

3.5 $(\wedge,\wedge,\vee)$-describable granules

In [27], the common-possible concept was proposed for concurrently investigating the common attributes and possible attributes of granules, where all attributes are from the same collection. In this subsection, we still consider the scenario that in an international travel agency, one guide can speak several frequently used languages, or can speak infrequently used languages.

We introduce a (U,A,I,B,J), where A is the set of frequently used attributes, and B is the set of infrequently used attributes. For any $X \subseteq U$ and $E \subseteq A\bigcup B$, we further introduce the concept-forming operations f_{CN} and g_{CN} as follows:

$$f_{CN} : P(U) \rightarrow P(A\bigcup B),$$

$$f_{CN}(X) = \max_{a_1\wedge a_2\wedge\cdots\wedge a_k \supseteq X}\{a_1,a_2,\cdots,a_k \in A\}\bigcup \min_{b_1\vee b_2\vee\cdots\vee b_j \supseteq X}\{b_1,b_2,\cdots,b_j \in B\}$$

$$g_{CN} : P(A\bigcup B) \rightarrow P(U),$$

$$g_{CN}(E) = \{u \in U|\forall m \in A\bigcap E, (u,m) \in I\}\bigcap(B\bigcap E)^{\diamond}$$

where $(B\bigcap E)^{\diamond} = \{u \in U|\exists m \in B\bigcap E, (u,m) \in J\}$.

Given a (U,A,I,B,J), for any $X \subseteq U$ and $E \subseteq A\bigcup B$, if $f_{CN}(X) = E$ and $g_{CN}(E) = X$, then the pair (X,E) is called a **common-and-necessary concept**. As this concept also includes two logic connectives $\wedge$ and $\vee$, we call it as a compound concept.

Furthermore, we give the definition of $(\wedge,\wedge,\vee)$-describable.

Definition 9. Given a (U,A,I,B,J) and $X \subseteq U$, if there exist $C \neq \emptyset$, $D \neq \emptyset$, $C \subseteq A$ and $D \subseteq B$, such that $g_{CN}(C\bigcup D) = X$, then we say that the granule X is $(\wedge,\wedge,\vee)$-describable, and $d(X) = \wedge(C) \wedge \vee(D)$.

Based on the definitions of f_{CN} and g_{CN}, we prove the following theorem.

Theorem 5. Given a (U,A,I,B,J), $X \subset U$ and $f_{CN}(X)\bigcap A \neq \emptyset$, $f_{CN}(X)\bigcap B \neq \emptyset$. Then X is $(\wedge,\wedge,\vee)$-describable $\Longleftrightarrow X = g_{CN}f_{CN}(X)$.

Proof. Necessity. By Definition 9, there exist $C \neq \emptyset$, $D \neq \emptyset$, $C \subseteq A$ and $D \subseteq B$, such that $g_{CN}(C\bigcup D) = X$. By the definition of g in Definition 2, let $X_1 = g(C)$, $X_2 = D^{\diamond}$. Then $X_1\bigcap X_2 = X$.

By the definition of f_{CN}, and note that $A \bigcap B = \emptyset$, $f_{CN}(X) \bigcap A$ is the maximum over A and $f_{CN}(X) \bigcap B$ is the minimum over B. Then,
$X_1 \bigcap X_2 = X \Longrightarrow C \subseteq (f_{CN}(X) \bigcap A)$ and $D \supseteq (f_{CN}(X) \bigcap B)$

$$\Longrightarrow g_{CN}((f_{CN}(X) \bigcap A) \bigcup (f_{CN}(X) \bigcap B)) \subseteq g(C) \bigcap D^{\diamond} = X_1 \bigcap X_2 = X$$

On the other hand,

$$g(f_{CN}(X) \bigcap A) \supseteq X$$

and

$$(f_{CN}(X) \bigcap B)^{\diamond} \supseteq X$$

$$\Longrightarrow g_{CN}((f_{CN}(X) \bigcap A) \bigcup (f_{CN}(X) \bigcap B)) \supseteq X$$

Therefore, $g_{CN} f_{CN}(X) = X$.

Sufficiency. Let $C \bigcup D = f_{CN}(X)$, where $f_{CN}(X) \bigcap A = C$, $f_{CN}(X) \bigcap B = D$. Then $g_{CN}(C \bigcup D) = g_{CN} f_{CN}(X) = X$. Thus X is $(\wedge, \wedge, \vee)$-describable. □

In fact, X is $(\wedge, \wedge, \vee)$-describable $\Longleftrightarrow X = a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge (b_1 \vee b_2 \vee \cdots \vee b_j)$, where $a_1, a_2, \cdots, a_k \in A$ and $b_1, b_2, \cdots, b_j \in B$.

Table 5: (U, A, I, B, J)

U	a_1	a_2	a_3	a_4	a_5	b_1	b_2	b_3	b_4
1	0	1	1	0	0	1	0	0	0
2	1	1	0	0	0	0	0	1	1
3	1	0	0	0	0	0	0	1	1
4	0	0	0	0	1	0	0	1	0
5	0	0	0	1	1	0	0	1	0
6	0	0	1	1	1	1	0	0	0
7	1	1	1	0	0	1	1	0	1

For example, in Table 5, $\{2,3,7\} = a_1$, $\{2,3,7\} = b_2 \vee b_4$, thus $\{2,3,7\} = a_1 \wedge (b_2 \vee b_4)$, where $f_{CN}(\{2,3,7\}) = \{a_1, b_2, b_4\}$, and $g_{CN} f_{CN}(\{2,3,7\}) = g_{CN}(\{a_1, b_2, b_4\}) = \{2,3,7\}$. Thus $X = \{2,3,7\}$ is $(\wedge, \wedge, \vee)$-describable. However, we also have $\{2,3,7\} = a_1 \wedge b_4$, thus $d(\{2,3,7\})$ is not unique.

Similarly, $g_{CN} f_{CN}(\{1,6,7\}) = g_{CN}(\{a_3, b_1, b_2\}) = \{1,6,7\}$. Thus $X = \{1,6,7\}$ is $(\wedge, \wedge, \vee)$-describable.

Also, $g_{CN} f_{CN}(\{2,3\}) = g_{CN}(\{a_1, b_3\}) = \{2,3\}$. Thus $X = \{2,3\}$ is $(\wedge, \wedge, \vee)$-describable. Note that here $X \subset a_1 = \{2,3,7\}$ and $X \subset b_3 = \{2,3,4,5\}$.

Next, we will describe the indescribable granules.

4 The approaching descriptions of indescribable granules

Based on the definitions of concept-forming operations, we will present unified approaching description methods of indescribable granules for each of the following types of concepts: formal concept, three-way concept, object oriented concept, bipolar concept and common-and-necessary concept.

4.1 The approaching descriptions of $\wedge$-indescribable granules

Suppose that the target granule X is $\wedge$-indescribable, if we can find $\wedge$-describable granules X_l and X_u, such that $X_l \subset X \subset X_u$, then we say that X_l and X_u are the approaching descriptions of $\wedge$-indescribable granule X [9].

By Lemma 1.(3), $gf(gf(X)) = gf(X)$, thus $gf(X)$ is $\wedge$-describable from Theorem 1.

By Lemma 1.(2), $X \subseteq gf(X)$. Furthermore, if granule Y is $\wedge$-describable and $X \subset Y$, then $gf(X) \subseteq gf(Y) = Y$. Thus we get the following.

Theorem 6. Given a (U, A, I), $\wedge$-indescribable $X \subset U$ and $f(X) \neq \emptyset$. Then $gf(X)$ must be the smallest $\wedge$-describable granule containing X.

It is easy to show that here Theorem 6 is equivalent to Theorem 4 in [9].

Now consider the $\wedge$-describable granules X_l, such that $X_l \subset X$.

From a (U, A, I), we can create a context (U, B, J), where $a \in A \Leftrightarrow b \in B$, and $(x, a) \in I \Leftrightarrow (x, b) \notin J$. Then

$$X_l = a_1 \wedge a_2 \wedge \cdots \wedge a_k \Longleftrightarrow U \backslash X_l = b_1 \vee b_2 \vee \cdots \vee b_k$$

From $X_l \subset X$, we have

$$b_1 \vee b_2 \vee \cdots \vee b_k \supseteq U \backslash X$$

So, to find the largest granule $X_l \subset X$

$$\Longleftrightarrow \min_{b_1 \vee b_2 \vee \cdots \vee b_k \supseteq U \backslash X \text{ and } (U \backslash X) \bigcap b_i \neq \emptyset, 1 \leq i \leq k} \{b_1, b_2, \cdots, b_k \in B\}$$

Table 6: (U, A, I) in [9]

U	a_1	a_2	a_3	a_4	a_5
1	1	0	0	0	0
2	1	1	0	1	0
3	1	0	1	1	1
4	1	1	1	0	1
5	1	1	1	0	0
6	0	1	0	0	1

Table 7: Formal context (U, B, J) from (U, A, I) in [9]

U	b_1	b_2	b_3	b_4	b_5
1	0	1	1	1	1
2	0	0	1	0	1
3	0	1	0	0	0
4	0	0	0	1	0
5	0	0	0	1	1
6	1	0	1	1	0

For example, in Table 6, let us take $X = \{4, 5, 6\}$, thus we consider $U \backslash X = \{1, 2, 3\}$ in Table 7.

It is easy to see that $\{1, 2, 3\} = b_2 \vee b_3$ or $\{1, 2, 3\} = b_2 \vee b_5$. Therefore, $X_l = a_2 \wedge a_3 = \{4, 5\}$ or $X_l = a_2 \wedge a_5 = \{4, 6\}$.

4.2 The approaching descriptions of $(\wedge, \wedge, \neg)$-indescribable granules

Similar to Subsection 3.2, from a (U, A, I), we establish a (U, A, I, B, J), where $a \in A \Leftrightarrow b \in B$, $(x, a) \in I \Leftrightarrow (x, b) \notin J$.

Similar to Subsection 4.1, we can get the following result.

Theorem 7. Given a (U, A, I, B, J), $(\wedge, \wedge, \neg)$-indescribable $X \subset U$, $f(X) \bigcap A \neq \emptyset$ and $f(X) \bigcap B \neq \emptyset$. Then $g_T f(X)$ must be the smallest $(\wedge, \wedge, \neg)$-describable granule and $g_T f(X) \supseteq X$, where f and g_T are defined in Subsection 3.2.

It is easy to show that here Theorem 7 is equivalent to Theorem 6 in [9].

Now consider the $(\wedge, \wedge, \neg)$-describable granules X_l, such that $X_l \subset X$. From a compound context (U, A, I, B, J), we construct a context (U, C, K), where $a \in A$ or $a \in B \Leftrightarrow c \in C$, and $(x, a) \in I$ or $J \Leftrightarrow (x, c) \notin K$. Then

$$X_l = a_1 \wedge a_2 \wedge \cdots \wedge a_k \Longleftrightarrow U \backslash X_l = c_1 \vee c_2 \vee \cdots \vee c_k$$

From $X_l \subset X$, we have

$$c_1 \vee c_2 \vee \cdots \vee c_k \supseteq U \backslash X$$

So, to find the largest granule $X_l \subset X$

$$\Longleftrightarrow \min_{c_1 \vee c_2 \vee \cdots \vee c_k \supseteq U \backslash X \text{ and } (U \backslash X) \bigcap c_i \neq \emptyset, 1 \leq i \leq k} \{c_1, c_2, \cdots, c_k \in C\}$$

For $(\wedge, \neg)$-indescribable granules, we can obtain the similar approximation results.

4.3 The approaching descriptions of $\vee$-indescribable granules

Similar to Subsection 3.3, from a (U, A, I), we build a context (U, B, J), where $a \in A \Leftrightarrow b \in B$, $(x, a) \in I \Leftrightarrow (x, b) \notin J$. Then

$$X = a_1 \vee a_2 \vee \cdots \vee a_k \Longleftrightarrow U \backslash X = b_1 \wedge b_2 \wedge \cdots \wedge b_k$$

Similar to Subsection 4.1, the following new result can be obtained.

Theorem 8. Given a context (U, B, J), $\vee$-indescribable $X \subset U$ and $f(X) \neq \emptyset$. Then $U \backslash gf(U \backslash X)$ must be the largest $\vee$-describable granule contained in X.

Now consider the $\vee$-describable granules X_u, such that $X_u \supset X$. Then

$$X_u = a_1 \vee a_2 \vee \cdots \vee a_k \Longrightarrow a_1 \vee a_2 \vee \cdots \vee a_k \supset X$$

So, to find the smallest granule $X_u \supset X$

$$\Longleftrightarrow \min_{a_1 \vee a_2 \vee \cdots \vee a_k \supseteq X \text{ and } X \bigcap a_i \neq \emptyset, 1 \leq i \leq k} \{a_1, a_2, \cdots, a_k \in A\}$$

4.4 The approaching descriptions of $(\wedge, \wedge, \vee)$-indescribable granules

Suppose that the granule X is $(\wedge, \wedge, \vee)$-indescribable over (U, A, I, B, J) and $f_{CN}(X) \neq \emptyset$, where A is the set of frequently used attributes, and B is the set of infrequently used attributes.

By definitions of f_{CN} and g_{CN} in Subsection 3.5, we would obtain $f_{CN}(X) = C \bigcup D$, where $C = \{a_1, a_2, \cdots, a_k\} \subseteq A$ and $D = \{b_1, b_2, \cdots, b_j\} \subseteq B$.

So,

$$g_{CN}(C \bigcup D) = a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge (b_1 \vee b_2 \vee \cdots \vee b_j)$$

is $(\wedge, \wedge, \vee)$-describable. Furthermore, if granule Y is $(\wedge, \wedge, \vee)$-describable and $X \subset Y$. Suppose that $f_{CN}(Y) = C' \bigcup D'$, by definitions of f_{CN} and g_{CN} in Subsection 3.5, $C' \subseteq C$ and $D' \supseteq D$. Thus $Y = g_{CN} f_{CN}(Y) = g(C') \bigcap D'^{\diamond} \supseteq g(C) \bigcap D^{\diamond} = g_{CN} f_{CN}(X)$, where g is defined in Definition 2.

Thus we get the following result.

Theorem 9. Given a (U, A, I, B, J), $(\wedge, \wedge, \vee)$-indescribable $X \subset U$ and $f_{CN}(X) \neq \emptyset$. Then $g_{CN} f_{CN}(X)$ must be the smallest $(\wedge, \wedge, \vee)$-describable granule and $g_{CN} f_{CN}(X) \supseteq X$.

5 Conclusions and future work

It is common that the definitions of concept operators are given in a description language. However, sometimes the logical formula can reflect the essence of the problem. Through their logical formulas, it is easy to decipher the equivalent relationship between formal concept and object oriented concept, and there is also a kind of equivalent relationship between formal concept and three-way concept. Further more, the obvious goal of granule representation is to obtain an exact logical formula for any describable granule and approaching descriptions for an indescribable granule. In this paper, the logical formulas have been given or derived for all five kinds of concepts. Thus, utilizing the logical relationship among various concepts, we have derived much concise and unified equivalent conditions of describable granules and approaching description methods of indescribable granules for all five kinds of concepts. Since it is an NP-hard problem [9] to construct the entire concept lattice, our discussions are only based on the definitions of concept-forming operations.

The concept with common attributes has been frequently studied. To investigate the granules with some special needs, we have proposed two new types of compound concepts: bipolar concept and common-and-necessary concept. Their descriptions are investigated in this paper.

In the future, we will continue to investigate other kinds of describable granules and explore their exact descriptions or approaching descriptions for indescribable granules. It is also worth to study the applications of these new types of concepts in approximate reasoning and cognitive computing.

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