# *Natural sets embedded in a system of axioms.*

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## *Abstract*

To this day, the prevailing view is that in set theory, the selection of things based on their properties leads to contradictions. If the formation and existence of sets is based on consistent compliance with the requirements that exist when using the all-quanor, no contradictions can be identified. Consistent compliance with these requirements is ensured in this thesis with the help of a system of axioms. As a result, we have a basis for set theory and the recognition of ideas inherent in so-called "naive set theory".

## *Introduction*

In this treatise a set theory is developed that allows the existence of natural numbers to be proven and their properties to be deduced. We call this set theory and the sets derived from it "Natural sets".

The theory is based on a system of axioms.The system always allows the attempt to form a supposed set, but provides a criterion for each of these attempts, on the basis of which it can be clearly decided whether the set exists or does not exist. This decision criterion for the existence of a set is based on the consistent compliance with the requirements that exist when using the all-quantifier ∀. A collection of things can only contain all things that belong to the collection and not the collection itself. This is a basic requirement that the all-quantifier brings with it. Sets are formulated with the help of the all-quantifier. The term $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ means nothing else than to consider the set of all things with the property **P**. This means for the expression $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ that $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\} \notin \{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ holds and therefore $\neg P(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\})$ is always true. This is the decision criterion for the existence of a set. If we use square brackets to represent the attempt to form a set and curly brackets to indicate the existence of a set, the existence criterion for a set is as follows:

$(\ \neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})]) :\equiv ([\mathbf{x}|\mathbf{P}(\mathbf{x})] \notin [\mathbf{x}|\mathbf{P}(\mathbf{x})])\ ) \Leftrightarrow \exists\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$. It is an indispensable requirement if you want to use the all- quantifier ∀ in the formation of sets. Russell's antimony $\mathbf{x}\notin\mathbf{x}$ is therefore a property inherent in all sets. If we call this property ${}_R\mathbf{P} :\equiv \mathbf{x}\notin\mathbf{x}$ and try to build a set with it, then we get the expression $[\mathbf{x}|{}_R\mathbf{P}(\mathbf{x})]$. If this is a set, then $\neg_R\mathbf{P}([\mathbf{x}|{}_R\mathbf{P}(\mathbf{x})]) :\equiv [\mathbf{x}|{}_R\mathbf{P}(\mathbf{x})] \notin [\mathbf{x}|{}_R\mathbf{P}(\mathbf{x})]$ should hold. But this is not the case, the exact opposite is the case, namely
$\neg_R\mathbf{P}([\mathbf{x}|{}_R\mathbf{P}(\mathbf{x})]) \equiv [\mathbf{x}|{}_R\mathbf{P}(\mathbf{x})] \in [\mathbf{x}|{}_R\mathbf{P}(\mathbf{x})]$. The formation of a set with the help of the property ${}_R\mathbf{P} :\equiv \mathbf{x}\notin\mathbf{x}$ fails because of the decision criterion for the existence of a set. If one wants to build sets with the property $\mathbf{x}\notin\mathbf{x}$, then one has to do with an always true property. It applies to all sets and cannot form a set itself, since the requirements of the all-quantifier ∀ are violated in such an attempt.

All attempts to form a set that does not meet the requirements of the all-quantifier ∀ inevitably do not lead to sets.

All attempts to form a set that meet the requirements of the all-quantifier ∀ necessarily lead to sets.

The natural numbers are derived from the natural sets. Apart from the induction axiom, the properties of the natural numbers formulated in the Peano axioms (src 6.1) can be derived from the properties of the natural sets. With the natural sets the things of our perception are arranged in summaries according to their appearance characteristics. We also call these appearance features properties. They are used to select or identify things. This refers to things of any kind. We use the character **P** to represent these properties.
We call the bearers of such properties mathematical objects, or simply objects. Everything that has a property is automatically seen as a mathematical object. The terms “property” and “object” defined in this way can be used to formulate statements such as “The object **x** has the property **P**.”. We express such a statement with a functional truth relation and say: **P**(**x**) is true if the object **x**

has the property **P**, otherwise false. The summaries mentioned at the beginning can also be presented using the same functional truth relationship.
A preliminary stage of these summaries is the object-selection [**x**|**P**(**x**)]. This preliminary stage only says that **P**(**x**) is true if **x** is contained in the object-selection [**x**|**P**(**x**)]. This “to be included” is expressed as usual with the symbol ∈. So **x**∈[**x**|**P**(**x**)] only holds if **P**(**x**) is true. The axiom of object-selection guarantees the existence of the associated object-selection for each property **P**. It does not guarantee the existence of objects in the object-selection.
The set axiom defines when an object-selection can also be a set. This is the case if **x**≠[**x**|**P**(**x**)] follows from **x**∈[**x**|**P**(**x**)]. The object-selection [**x**|**P**(**x**)], a summary of objects with the property **P**, then contains all objects with the property **P**. We also refer to the objects in a summary as elements of that summary.
The possibility of forming such summaries, also called „sets“, is a natural and also an ancient concern. It goes back to Georg Cantor (src 5.1), who can be considered the author of set theory. With its definition (src 6.2):
„Unter einer ‚Menge‘ verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten in unserer Anschauung oder unseres Denkens (welche die ‚Elemente‘ von M genannt werden) zu einem Ganzen.“
he has determined what is to be understood by sets from his point of view. From this the so-called “naive set theory” developed, which came to an abrupt end

after the publication of Russell's (src 5.2) antinomy **x∉x** (src 6.3). In this document, Georg Cantor's textual definition of what sets are (src 6.2) is taken up, expanded and presented in a formal axiomatic form.

As with Georg Cantor, it is not assumed that everything around us is a set and that we only have to deal with the handling of such sets and their behavior. The existence of so-called primordial elements, i.e. objects that are not sets, is demanded in an axiom. The focus is on the definition of what sets are and the construction of sets from things of all kinds. Only then will further investigations into the behavior of sets be continued. An essential aspect in the formation of sets is that the set formed cannot itself have the set-forming characteristics of its elements. The two axioms described above, the object-selection axiom and the set axiom, are responsible for this. Another essential aspect is the identification of things based on their object-specific properties. This is a property or characteristic that exists for every thing and makes every thing unique. There are no two things with the same object-specific property and there is also such an object-specific property for every thing. This is another axiom in the underlying axiom system.

One arrives at a division of all things into so-called primordial elements on the one hand, and summaries of things on the other. As a result, one can deduce that the set of all primordial elements exists, that the set of all summaries, or the set of all sets, cannot exist. In the axiom of expansion it is demanded that there is at least a subset of the primordial set that can be expanded with primordial elements as often as desired. This then leads to the development of the term “set-size”, with the help of which the Peano axioms (src 6.1) can be derived.

When looking at the term “property” one arrives at a breakdown of the properties. With one it is possible to create sets, with the other it is fundamentally not possible. This also includes those properties that brought down the so-called “naive set theory”. In the present paper the same basic idea is pursued as in the so-called “naive set theory”, but it is clearly peeled out what sets are and what cannot be sets. The natural numbers are derived with the help of the term “set-size”. The terms “finite set” and “infinite set” are defined. The treatise ends by showing that the natural numbers are an infinite set.

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# 1 ***Table of special characters and symbols.***

| ***Character*** | ***Meaning*** | ***Example*** |
|---|---|---|
| Proposition | Formulation of supposed facts. The correctness of the facts described must be proven. This is always done immediately afterwards. | 2.1 |
| Definition | Explanation what something is or how something is used. | 2.1 |
| $\Rightarrow$ | Implication.<br>If we infer from a statement A to a statement B, then we use the symbol $\Rightarrow$ to represent this conclusion and write $A \Rightarrow B$, and by this we mean: If A holds, then B must also hold. | 2.2 |
| **P** | Property.<br>Properties are characteristic features or behavior for the selection or identification of | 2.3.1 |

| ***Character*** | ***Meaning*** | ***Example*** |
|---|---|---|
| | things. This refers to things of any kind. We use the symbol **P** to represent these properties. | |
| Object | Everything that has a property according to 2.3.1 is a mathematical object. | 2.3.2 |
| **t, w, X, y, z** | Identifier for mathematical objects. | 2.3.2 |
| **¬** | Negation. | 2.3.3 |
| **P**(**x**) | If an object **x** has the property **P**, then we say **P**(**x**) is true and ¬**P**(**x**) is false. In relation to an object **x**, every property **P** generates the statement **P**(**x**) or ¬**P**(**x**). | 2.3.3 |
| ⇔ | Equivalence. If A and B are statements and if both A⇒B and B⇒A hold, then we write A⇔B for this. We say A and B are equivalent statements, and we also say A⇔B is an equivalence. | 2.4.1 |
| = | Equal. | 2.4.1 |
| $^{R}$**P** | Root property. | 2.4.2 |

| *Character* | *Meaning* | *Example* |
|---|---|---|
| Variant | Variant of a root property. | 2.4.2 |
| ∀ | For all. | 2.4.3 |
| $^{U}\mathbf{P}$ | Universal property. | 2.4.3 |
| :≡ | Synonymous by definition. | 2.4.4.1 |
| $^{0}\mathbf{P}$ | Universal object property. $^{0}\mathbf{P}(\mathbf{x})$ applies to every object **x**. This statement cannot be negated, $\neg^{0}\mathbf{P}(\mathbf{x})$ is always false. | 2.4.4.1 |
| $^{I}\mathbf{P}$ | The quality inherent in all things, **P** :≡ "is identical with itself", is a universal property. For every object **x**, $^{I}\mathbf{P}(\mathbf{x})$ applies. This statement cannot be negated either, $\neg^{I}\mathbf{P}(\mathbf{x})$ is always false. | 2.4.4.2 |
| $_{x}\mathbf{P}$ | The object-specific property $_{x}\mathbf{P}$ exists for each object **x**. The existence of this property is axiomatically demanded in Section 2.6.1. This property is also a universal property. The | 2.4.4.3 |

| *Character* | *Meaning* | *Example* |
|---|---|---|
| | statement $\neg_x$**P**(**x**) is always false. | |
| ∧ | Logical AND. | 2.5.1 |
| ∨ | Logical OR. | 2.5.1 |
| ≡ | Identical to. | 2.5.1 |
| ∃ | It exists. | 2.6.1 |
| ≠ | Not equal to. | 2.6.1 |
| **C** | Represents all types of an object-selection. | 2.6.2 |
| Object-selection | Term for **C**. | 2.6.2 |
| $^{C}$**P** | The property of being a selection of objects. $^{C}$**P** :≡ “Is an object-selection”. | 2.6.2 |
| [**x**\|**P**(**x**)] | The object-selection belonging to property P. | 2.6.3 |
| {**x**\|**P**(**x**)} | The set belonging to property P. | 2.6.3 |
| **S** | The symbol for a set. | 2.6.3 |
| ∈ | To be contained in an object-selection [**x**\|**P**(**x**)] | 2.6.3 |

| *Character* | *Meaning* | *Example* |
|---|---|---|
| | or in a set {**x**\|**P**(**x**)}. | |
| $^{PC}$**P** | The defining property of an object-selection [**x**\|**P**(**x**)].<br>$^{PC}$**P** :≡ (**x** ∈ [**x**\|**P**(**x**)] ⇒ **P**(**x**)).<br>This definition also defines the meaning and use of the symbol ∈. | 2.6.3 |
| Element | To be contained in an object-selection [**x**\|**P**(**x**)] or in a set {**x**\|**P**(**x**)}. | 2.6.3 |
| set-forming | We are talking about set-forming properties. | 2.6.3.2 |
| $^{S}$**P** | The property of being a set,without referring to a specific object-selection.<br>$^{S}$**P** :≡ “is a set”. | 2.6.4.3 |
| $^{PS}$**P** | The property of an object-selection to be a set.<br>$^{PS}$**P** :≡ “[**x**\|**P**(**x**)] is a set”. | 2.6.4.3 |
| ∉ | Not contained in an object-selection [x\|P(x)] or in a set {x\|P(x)}. | 2.6.4.6 |

| ***Character*** | ***Meaning*** | ***Example*** |
|---|---|---|
| ***Proof*** ⇒***:*** | If A⇔B is an equivalence, then A⇒B is proved. | 2.6.4.7 |
| ***Proof*** ⇐***:*** | If A⇔B is an equivalence, then A⇐B is proved. | 2.6.4.7 |
| ∪ | Union of two object-selections. | 2.6.5.1 |
| ∩ | Intersection of two object-selections. | 2.6.5.2 |
| ⊆ | Subset, superset. | 2.6.8.1 |
| ⊂ | Proper subset, proper superset. | 2.6.8.2 |
| Primordial eement. | Mathematical object that is neither an object-selection nor a set. | 2.6.9 |
| $_{P}$**S** | Primordial set. | 2.6.10 |
| {**y**} | A set that contains only the element **y**. | 2.6.11 |
| ⇄ | The expression A ⇄ B is called an iterative process. In A there is a condition. The process step formulated in B can and will only be carried out if the condition formulated in A is met. At the end of the process step, it is | 3.2.6 |

| ***Character*** | ***Meaning*** | ***Example*** |
|---|---|---|
| | checked again whether the necessary condition formulated in A still applies. If so, the process step formulated in B is carried out again. This sequence is repeated as long as the condition formulated in A allows the process step in B to be carried out. | |
| **→** | Transfer, replace. The expression A→B transfers all properties of object A to object B. All properties that object B had before A→B was executed are lost. The name "B" remains. | 3.2.6 |
| $^{SE}$**P** | The property of a set to be expandable with primordial elements any number of times. | 3.2.6 |
| **∅** | Empty set **∅**. | 3.3.2 |
| \ | Difference between two sets. | 3.5.4 |
| := | Equal by definition. | 3.5.6 |
| **POW**(**S**) | Power set of the set **S**. | 3.9 |

| ***Character*** | ***Meaning*** | ***Example*** |
| --- | --- | ---: |
| Assignment | A triple consisting of an image set, a definition set and an assignment rule. | 3.10.1 |
| ***f***, ***g*** | Assignment rule. | 3.10.2 |
| Injective, bijective, subjective | Types of assignment. | 3.10.2 |
| **SIZ**(**S**) | Size of the set **S**. | 3.12 |
| :< | Smaller by definition. | 3.12.2 |
| $_{\mathbb{N}}\mathbf{C}$ | Representative distribution set for $\mathbb{N}$. | 4.3 |
| $\mathbb{N}$ | Set of natural numbers. | 4.3 |
| $\supseteq$ | Superset or equal. | 4.4.5 |
| **L** | Line of natural numbers. | 4.6 |
| $\exists!$ | There is exactly one ... | 4.6 |
| **n**,**b**,**e** | Natural numbers. | 4.6 |

# *2 Objects and their properties.*

## *2.1 Proposition.*

***Definition:*** Formulation of supposed facts. The correctness of the facts described must be proven. This is always done immediately afterwards.

## *2.2 Implication.*

***Definition:*** If we infer from a statement A to a statement B, then we use the symbol ⇒ to represent this conclusion and write A⇒B, and by this we mean: If A holds, then B must also hold.

## *2.3 The term „Property“ in connection with objects.*

### *2.3.1 Properties.*

***Definition:*** Properties are characteristic features or behavior for the selection or identification of things. This refers to things of any kind. We use the symbol **P** to represent these properties.

### *2.3.2 Objects*

***Definition:*** Everything that has a property according to 2.3.1 is a mathematical object. We use the letters **t**,**w**,**x**,**y** and **z** to represent objects.

### 2.3.3 Properties and objects generate statements.

***Definition:*** If an object **x** has the property **P**, then we say **P**(**x**) is true and ¬**P**(**x**) is false. In relation to an object **x**, every property **P** generates the statement **P**(**x**) or ¬**P**(**x**).

## 2.4 Definitions and axioms of properties.

### 2.4.1 Equality of properties.

***Definition:*** $(\mathbf{P}_1 \Leftrightarrow \mathbf{P}_2) \Leftrightarrow (\mathbf{P}_1 = \mathbf{P}_2)$.

In words: Two properties are the same if and only if one follows from the other.

### 2.4.2 Hierarchy of properties.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be properties and $\mathbf{P}_1 \Rightarrow \mathbf{P}_2$ .

***Definition:*** $\mathbf{P}_2$ is one of possibly several existing root properties of $\mathbf{P}_1$. If $\neg(\mathbf{P}_2 \Rightarrow \mathbf{P}_1)$ also applies, then $\mathbf{P}_1$ is called a variant of $\mathbf{P}_2$. For the property of being a root property, we reserve the symbol ${}^{R}\mathbf{P}$.

### *2.4.3 Universal properties.*

***Definition:*** A property **P** is called a universal property if all objects have this property. Then ∀**x** **P**(**x**) holds. The statement ¬**P**(**x**) is therefore always false for such properties. We reserve the sign $^{U}$**P** for this category of properties.

### *2.4.4 Examples of universal properties.*

#### *2.4.4.1 The quality of being an object.*

The property of being an object, ie **P** :≡ "is an object", adheres to all objects and is therefore a universal property. We call them $^{0}$**P** or universal object property. $^{0}$**P**(**x**) applies to every object **x**. This statement cannot be negated, ¬$^{0}$**P**(**x**) is always false.

#### *2.4.4.2 The quality of being identical to oneself.*

The quality inherent in all things, **P** :≡ "is identical with itself", is a universal property. We call it $^{I}$**P** . For every object **x**, $^{I}$**P**(**x**) applies. This statement cannot be negated either, ¬$^{I}$**P**(**x**) is always false.

#### 2.4.4.3 *Owning an object-specific property.*

Because of the axiom in 2.6.1 , every object **x** has the object-specific property $_x\mathbf{P}$. We therefore have $\forall \mathbf{x}\ {}_x\mathbf{P}(\mathbf{x})$. The statement $\neg{}_x\mathbf{P}(\mathbf{x})$ is always false.

## 2.5 *Linking of property variants.*

### 2.5.1 *The OR link.*

***Proposition:*** If $\mathbf{P}_1$ and $\mathbf{P}_2$ are variants of the root property $\mathbf{P}$, then $(\mathbf{P}_1 \lor \mathbf{P}_2)$ is also a variant of the root property $\mathbf{P}$.

***Proof:*** It applies $\neg(\mathbf{P} \Rightarrow \mathbf{P}_1) \land \neg(\mathbf{P} \Rightarrow \mathbf{P}_2)$ .

It follows $\neg( (\mathbf{P} \Rightarrow \mathbf{P}_1) \lor (\mathbf{P} \Rightarrow \mathbf{P}_2) ) \equiv \neg( \mathbf{P} \Rightarrow (\mathbf{P}_1 \lor \mathbf{P}_2) )$.

### 2.5.2 *The AND link.*

***Proposition:*** If $\mathbf{P}_1$ and $\mathbf{P}_2$ are variants of the root property $\mathbf{P}$, then $(\mathbf{P}_1 \land \mathbf{P}_2)$ is also a variant of the root property $\mathbf{P}$.

***Proof:*** It applies $\neg(\mathbf{P} \Rightarrow \mathbf{P}_1) \land \neg(\mathbf{P} \Rightarrow \mathbf{P}_2)$. If $\mathbf{P} \Rightarrow (\mathbf{P}_1 \land \mathbf{P}_2)$, then $\mathbf{P} \Rightarrow \mathbf{P}_1$ also holds, contrary to the assumption. So $\neg(\mathbf{P} \Rightarrow (\mathbf{P}_1 \land \mathbf{P}_2))$ must hold.

## 2.6 Definitions and axioms about objects.

Let **x**, **y** be objects and **P** be a property.

### 2.6.1 Object-specific property.

***Axiom:*** $\forall \mathbf{x}\ \exists \mathbf{P}\ \forall \mathbf{y} \neq \mathbf{x}\ \mathbf{P}(\mathbf{x}) \wedge \neg \mathbf{P}(\mathbf{y})$.

In a nutshell: Every object has at least one object-specific property that distinguishes it from all other objects. Each object is therefore unique. We reserve the character $_{x}\mathbf{P}$ for the object-specific property of an object **x**.

### 2.6.2 Selection of objects.

Objects can be selected in different ways. The selection could have been made by chance, for example. However, it could also have been carried out according to a predefined algorithm. It could have been made according to the characteristics of the objects, etc. For all these types of object selection, we introduce the symbol **C** and use the term object-selection for this. The property of all these object-selections represented by **C**, namely being a selection of objects, is called $^{C}\mathbf{P}$. We can also formally describe this property by saying $^{C}\mathbf{P} :\equiv$ "Is an object-selection".

### *2.6.3 The object-selection.*

***Definition:*** For a property **P** the object [**x**|**P**(**x**)] should only contain objects **x** for which, according to 2.3.3, **P**(**x**) applies. If the object [**x**|**P**(**x**)] contains all objects **x** for which **P**(**x**) holds, then we say that **P** is a property that forms a set and the object [**x**|**P**(**x**)] is a set. In order to identify the set character of the object-selection [**x**|**P**(**x**)], we replace the square brackets in the object-selection with curly brackets. The expression {**x**|**P**(**x**)} clarifies the fact that the object-selection [**x**|**P**(**x**)] contains all objects **x** for which **P**(**x**) holds. For sets we introduce the symbol **S**.

***Definition:*** The fact that an object **x** is contained in an object-selection [**x**|**P**(**x**)] or in a set {**x**|**P**(**x**)} is represented, as usual, with the character ∈. The meaning of the symbol ∈ is defined in this document as follows: ( **x** ∈ [**x**|**P**(**x**)] ∨ **x** ∈ {**x**|**P**(**x**)} ) ⇒ **P**(**x**). At the same time, this includes the defining property of an object-selection [**x**|**P**(**x**)]. We call it $^{PC}$**P**. So $^{PC}$**P** :≡ (**x** ∈ [**x**|**P**(**x**)] ⇒ **P**(**x**)).
In Words: Object-selections or sets formed with the property **P** can only contain objects **x** for which **P**(**x**) applies. For such **x** we say that **x** is an element of the object-selection [**x**|**P**(**x**)] or an element of the set {**x**|**P**(**x**)}.

#### *2.6.3.1 The existence of an object-selection.*

***Axiom:*** $\forall\mathbf{P}\ \exists[\mathbf{x}|\mathbf{P}(\mathbf{x})]\ {}^{PC}\mathbf{P} :\equiv ((\mathbf{x} \in [\mathbf{x}|\mathbf{P}(\mathbf{x})] \vee \mathbf{x} \in \{\mathbf{x}|\mathbf{P}(\mathbf{x})\}) \Rightarrow \mathbf{P}(\mathbf{x}))$.

In a nutshell: For each property $\mathbf{P}$ there is an object $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ with the property ${}^{PC}\mathbf{P}$, which means that $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ can only contain objects with the property $\mathbf{P}$.

#### *2.6.3.2 Condition for the existence of a set.*

***Definition:*** $\forall\mathbf{P}\ \neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])\quad \mathbf{P}(\mathbf{x}) \Rightarrow \mathbf{x} \in [\mathbf{x}|\mathbf{P}(\mathbf{x})]$.

In a nutshell: If the object-selection $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ belonging to a property $\mathbf{P}$ does not have the property $\mathbf{P}$, i.e. $\neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$, then this object-selection contains all objects $\mathbf{x}$ with the property by definition $\mathbf{P}$. We then speak of a set-forming property and put the object selection in curly brackets, i.e. write $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ to make it recognizable as a set. By definition, this set still has all the properties of the object-selection. In particular, $\neg\mathbf{P}(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\})$ and $\mathbf{P}(\mathbf{x}) \Rightarrow \mathbf{x}\in\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ also applies. The definition can then be presented in the following, somewhat more concise form: $\neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})]) \Rightarrow \exists\ \{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$.

### *2.6.4 The properties of an object-selection.*

#### *2.6.4.1 The object property.*

***Proposition:*** Each object-selection is an object with the property demanded in Axiom 2.6.3.1, and thus a special variant of the universal property $^{0}\mathbf{P}$ according to 2.4.2 and 2.4.4.1.

***Proof:*** The object property follows from 2.3.2 and $^{C}\mathbf{P} \Rightarrow {}^{0}\mathbf{P}$ applies. Since the property $^{0}\mathbf{P}$, which is valid for all objects, cannot be used to infer the special property demandid in Axiom 2.6.3.1, $\neg({}^{0}\mathbf{P} \Rightarrow {}^{C}\mathbf{P})$ applies. Therefore, because of 2.4.2, the property of every object-selection is a variant of the universal property $^{0}\mathbf{P}$.

#### *2.6.4.2 The selection property.*

This property is given by the axiom in Section 2.6.3.1. It is object-specific, since there is exactly one object selection $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ for each property $\mathbf{P}$ and different properties, at least formally, also generate different object-selections. As already defined in Section 2.6.3.1, we use the symbol $^{PC}\mathbf{P}$ to represent the selection property.

#### *2.6.4.3 The property of being a set.*

Because of the definition in Section 2.6.3.2, an object-selection can also be a set, namely if ¬**P**([**x**|**P**(**x**)]) holds. If this is the case, then we use the symbol $^{PS}$**P** to represent this property. If we speak of the property of being a set in general, i.e. not referring to a specific object-selection, then we use the symbol $^{S}$**P** to represent this property. The following applies: $^{S}$**P** :≡ “is a set”, $^{PS}$**P** :≡ “[**x**|**P**(**x**)] is a set”.

#### *2.6.4.4 The universal properties.*

Like every object, every object-selection [**x**|**P**(**x**)] has the universal properties $^{0}$**P**, $^{I}$**P** and $_{x}$**P** mentioned in Section 2.4.4.

#### *2.6.4.5 Axiom to the property of sets.*

***Axiom :*** ∀**P** ¬**P**([**x**|**P**(**x**)]) ⇔ ( ∃{**x**|**P**(**x**)} ¬**P**({**x**|**P**(**x**)}) )

In a nutshell: A property **P** is set-forming if and only if the object-selection belonging to **P** does not have the property **P**.

### *2.6.4.6 A set does not contain itself.*

***Proposition:*** Let **S** be a set and **P** its set-forming property, i.e. **S** = {**x**|**P**(**x**)}, then **S**∉**S** always holds.

***Proof:*** Because of 2.6.4.5 we have ¬**P**({**x**|**P**(**x**)}) ≡ ¬**P**(**S**). If **S**∈**S**, contrary to the second definition in Section 2.6.3, **P**(**S**) would follow. For every set **S** we have therfore **S**∉**S**.

### *2.6.4.7 The meaning of the symbol ∈ in relation to sets.*

***Proposition:*** Let **S** be a set and **P** its set-forming property, i.e. **S** = {**x**|**P**(**x**)}, then **P**(**x**) ⇔ **x**∈{**x**|**P**(**x**)} holds.

***Proof ⇒:*** Because of 2.6.4.5 we have ¬**P**([**x**|**P**(**x**)]) ⇔ ( ∃{**x**|**P**(**x**)} ¬**P**({**x**|**P**(**x**)}) ). Because of the definition in Section 2.6.3.2, **P**(**x**) ⇒ **x** ∈ {**x**|**P**(**x**)} applies.

***Proof ⇐:*** Because of the second definition in Section 2.6.3 we have (**x** ∈ [**x**|**P**(**x**)] ∨ **x** ∈ {**x**|**P**(**x**)}) ⇒ **P**(**x**).

### ***2.6.5 Compound properties in an object-selection.***

#### ***2.6.5.1 Union of object-selections.***

***Definition:*** Let $\mathbf{P} :\equiv (\mathbf{P}_1 \vee \mathbf{P}_2)$ be a composite property, then we define $[\mathbf{x}|\mathbf{P}(\mathbf{x})] :\equiv [\mathbf{x}|(\mathbf{P}_1 \vee \mathbf{P}_2)(\mathbf{x})] :\equiv [\mathbf{x}|\mathbf{P}_1(\mathbf{x}) \vee \mathbf{P}_2(\mathbf{x})] :\equiv [\mathbf{x}|(\mathbf{P}_1(\mathbf{x})] \vee [\mathbf{x}|\mathbf{P}_2(\mathbf{x})]$. The expressions listed stand equally for the union of the two object selections $[\mathbf{x}|\mathbf{P}_1(\mathbf{x})]$ and $[\mathbf{x}|\mathbf{P}_2(\mathbf{x})]$.

For this union we also write $[\mathbf{x}|\mathbf{P}_1(\mathbf{x})] \cup [\mathbf{x}|\mathbf{P}_2(\mathbf{x})]$.

#### **2.6.5.2** ***Intersection of object-selections.***

***Definition:*** Let $\mathbf{P} :\equiv (\mathbf{P}_1 \wedge \mathbf{P}_2)$ be a composite property, then we define $[\mathbf{x}|\mathbf{P}(\mathbf{x})] :\equiv [\mathbf{x}|(\mathbf{P}_1 \wedge \mathbf{P}_2)(\mathbf{x})] :\equiv [\mathbf{x}|(\mathbf{P}_1(\mathbf{x}) \wedge \mathbf{P}_2(\mathbf{x})]$. The expressions listed stand equally for the intersection of the two object-selections $[\mathbf{x}|\mathbf{P}_1(\mathbf{x})]$ and $[\mathbf{x}|\mathbf{P}_2(\mathbf{x})]$.

For this intersection we also write $[\mathbf{x}|\mathbf{P}_1(\mathbf{x})] \cap [\mathbf{x}|\mathbf{P}_2(\mathbf{x})]$.

### 2.6.6 *Object-specifically oriented property of a set.*

***Proposition:*** $\exists\{\mathbf{x}|\mathbf{P}(\mathbf{x})\} \Leftrightarrow (\ \forall \mathbf{y}\in[\mathbf{x}|\mathbf{P}(\mathbf{x})]\ \neg_{y}\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])\ )$.
In a nutshell: If **P** is a set-forming property, that is, if $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ exists, then $\neg_{y}\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ holds for the object-specific property $_{y}\mathbf{P}$ of each element **y** from the object-selection belonging to **P**. Conversely, if $\neg_{y}\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ also applies to the object-specific property $_{y}\mathbf{P}$ of each element **y** from the object-selection $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ belonging to **P**, then **P** is a set-forming property, it exists thus the set $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$.

***Proof*** $\Rightarrow$***:*** Since $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ exists, we have $[\mathbf{x}|\mathbf{P}(\mathbf{x})]\notin[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ because of 2.6.4.6, that means $\forall \mathbf{y}\in[\mathbf{x}|\mathbf{P}(\mathbf{x})] \Rightarrow \mathbf{y}\neq[\mathbf{x}|\mathbf{P}(\mathbf{x})]$. Because of the object-specific property $_{y}\mathbf{P}$, we have $\forall \mathbf{y}\in[\mathbf{x}|\mathbf{P}(\mathbf{x})]\ \neg_{y}\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$.

***Proof*** $\Leftarrow$***:*** By assumption we have $\forall \mathbf{y}\in[\mathbf{x}|\mathbf{P}(\mathbf{x})]\ \neg_{y}\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$. Because of the object-specific property $_{y}\mathbf{P}$, $\mathbf{y}\in[\mathbf{x}|\mathbf{P}(\mathbf{x})] \Rightarrow \mathbf{y}\neq[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ applies. This is equivalent to $[\mathbf{x}|\mathbf{P}(\mathbf{x})]\notin[\mathbf{x}|\mathbf{P}(\mathbf{x})]$. Hence $\neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ applies and by 2.6.4.5 the set $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ exists.

### *2.6.7 The set of a set.*

***Proposition:*** Let **S** be a set, then the set $\{\mathbf{S}\}$ exists.

***Proof:*** **S** is a mathematical object and, because of 2.6.1, has the object-specific property ${}_S\mathbf{P}$. The object-selection $[\mathbf{x}|{}_S\mathbf{P}(\mathbf{x})]$ therefore contains exactly one element, namely the set **S**. The object-selection $[\mathbf{x}|{}_S\mathbf{P}(\mathbf{x})]$ and the set **S** are different mathematical objects. We therefore have ${}_S\mathbf{P}(\mathbf{S})$ and $\neg{}_S\mathbf{P}([\mathbf{x}|{}_S\mathbf{P}(\mathbf{x})])$. Because of 2.6.4.5 there is therefore the set $\{\mathbf{x}|{}_S\mathbf{P}(\mathbf{x})\} :\equiv \{\mathbf{S}\}$.

## *2.6.8 Parts of a set.*

### *2.6.8.1 Subsets and supersets.*

***Axiom:*** Let $[\mathbf{x}|\mathbf{P}_1(\mathbf{x})]$ be an object-selection with its selecting property $\mathbf{P}_1$. Let $\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ be a set with its set-forming property $\mathbf{P}_2$. If $\mathbf{P}_1 \Rightarrow \mathbf{P}_2$ then $[\mathbf{x}|\mathbf{P}_1(\mathbf{x})]$ is also a set, so there is $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$.
$\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ is called a subset of $\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ and $\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ is called a superset of $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$. This relationship is noted in the form $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\} \subseteq \{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$.

***Annotation:*** Because $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ and $\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ are sets it follows, because of section 2.6.4.5, $\mathbf{x}\in\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ from the terrn $\mathbf{P}_1$ and $\mathbf{x}\in\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ from the term $\mathbf{P}_2$. So we have the equivalence
$( \mathbf{x}\in\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\} \Rightarrow \mathbf{x}\in\{\mathbf{x}|\mathbf{P}_2(\mathbf{x}) ) \Leftrightarrow ( \{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\} \subseteq \{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\} )$.

### 2.6.8.2 *Proper subsets and proper supersets.*

***Definition:*** Let $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ be a subset of $\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$. If $\neg(\mathbf{P}_2 \Rightarrow \mathbf{P}_1)$ then $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ is called a proper subset of $\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ and $\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ is called a proper superset of $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$. This relationship is noted in the form $\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\} \subset \{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$.

### 2.6.8.3 *Equivalent representation of subsets.*

***Proposition:*** If $\mathbf{S}_1$ and $\mathbf{S}_2$ are sets, then $(\mathbf{S}_1 \cup \mathbf{S}_2 = \mathbf{S}_2) \Leftrightarrow \mathbf{S}_1 \subseteq \mathbf{S}_2$ holds.

***Proof*** $\Rightarrow$***:*** Because of $\mathbf{S}_1 \cup \mathbf{S}_2 = \mathbf{S}_2$ we have $\mathbf{x} \in \mathbf{S}_1 \Rightarrow \mathbf{x} \in \mathbf{S}_2$. $\mathbf{S}_1$ therefore fulfills the condition required in Section 2.6.8.1 to be a subset of $\mathbf{S}_2$. $\mathbf{S}_1 \subseteq \mathbf{S}_2$ therefore applies.

***Proof*** $\Leftarrow$***:*** If $\mathbf{S}_1 \subseteq \mathbf{S}_2$, then $\mathbf{x} \in \mathbf{S}_1 \Rightarrow \mathbf{x} \in \mathbf{S}_2$ also applies because of Section 2.6.8.1. Therefore, $\mathbf{S}_1 \cup \mathbf{S}_2 = \mathbf{S}_2$ must apply.

## *2.6.9 Primordial elements.*

***Definition:*** We call objects $\mathbf{x}$ with the property $(\forall \mathbf{P}\ \ \mathbf{x} \neq [\mathbf{y}|\mathbf{P}(\mathbf{y})])$ primordial elements.

### 2.6.10 *There is a primordial set.*

***Proposition:*** The object-selection $[\mathbf{x}|\forall \mathbf{P}\ \mathbf{x}\neq[\mathbf{y}|\mathbf{P}(\mathbf{y})]]$ is a set.

***Proof:*** The property of the object-selection $[\mathbf{x}|\forall \mathbf{P}\ \mathbf{x}\neq[\mathbf{y}|\mathbf{P}(\mathbf{y})]]$ which forms the selection is not to be an object-selection. According to the definition in Section 2.6.3.2, the object-selection $[\mathbf{x}|\forall \mathbf{P}\ \mathbf{x}\neq[\mathbf{y}|\mathbf{P}(\mathbf{y})]]$ is a set if it is an object-selection. Since this is the case, we are dealing here with the set of all primordial elements. We call this set primordial set and reserve the symbol ${}_{P}\mathbf{S}$ for it.

### 2.6.11 *There are sets with only one element.*

***Definition:*** We represent sets also by listing all of their elements. We separate the elements from one another with commas and put the entire list in curly brackets. The list must contain all the elements in the set. As an example, the expression {**x**, **y**, **z**} represents a set that contains the elements **x** and **y** and **z**. Other objects are not included in the set.

***Proposition:*** Let **y** be a primordial element, then the set {**y**} exists.
***Proof:*** Because of 2.6.1 there is at least one object-specific property $_y$**P** of **y**, so that the object-selection [**x**|$_y$**P**(**x**)] contains exactly one element, namely the object **y**. Since **y** is a primordial element, **y**≠[**x**|$_y$**P**(**x**)] applies because of 2.6.9. The object-selection [**x**|$_y$**P**(**x**)] and the object **y** are therefore different mathematical objects. It therefore applies, also because of 2.6.1 ¬$_y$**P**([**x**|$_y$**P**(**x**)]). Because of 2.6.4.5, the object-selection [**x**|$_y$**P**(**x**)] is an existing set. Since it can only contain the object **y**, it is identical to the set {**y**}.

### 2.6.12 *Union with elements from the primordial set.*

***Proposition:*** ∀**S**⊂$_P$**S** **y**∈$_P$**S** ⇒ ∃ **S**∪{**y**}.
In a nutshell: Every subset of $_P$**S** can be combined with elements from $_P$**S** to form a set.

***Proof:*** Every set $\mathbf{S}$ and therefore every subset of $_P\mathbf{S}$ is based on a set-forming property $\mathbf{P}$, so that $\mathbf{S}=\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ applies. Because of Section 2.6.11, the single-element set $\{\mathbf{y}\}$ exists for $\mathbf{y}\in_P\mathbf{S}$ and, since $_y\mathbf{P}$ only applies object-specifically to the object $\mathbf{y}$, both $\neg_y\mathbf{P}(\{\mathbf{y}\})$ and $\neg_y\mathbf{P}(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\})$. Since $\mathbf{S}=\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ is a set, $\neg\mathbf{P}(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\})$ holds. Since by assumption $\mathbf{S}\subset_P\mathbf{S}$, $\mathbf{S}$ only contains primordial elements. Hence, $\{\mathbf{y}\}\notin\{\mathbf{x}|\mathbf{P}(\mathbf{x})\} :\equiv \neg\mathbf{P}(\{\mathbf{y}\})$ applies. In summary, therefore, $(\neg\mathbf{P}(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}) \wedge \neg_y\mathbf{P}(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\})) \equiv \neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\})$ applies as well as $(\neg\mathbf{P}(\{\mathbf{y}\}) \wedge \neg_y\mathbf{P}(\{\mathbf{y}\})) \equiv \neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{y}\})$. From this it follows $\neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}) \wedge \neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{y}\}) \equiv \neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}\vee\{\mathbf{y}\})$. Because of Section 2.6.11, we have $\{\mathbf{y}\}=\{\mathbf{x}|_y\mathbf{P}(\mathbf{x})\}$, so we can write $\neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}\vee\{\mathbf{y}\})\equiv\neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}\vee\{\mathbf{x}|_y\mathbf{P}(\mathbf{x})\})\equiv\neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\})$. According to Section 2.6.4.5, this is exactly the condition for the existence of our set $\mathbf{S}\cup\{\mathbf{y}\}$.

# 3 *Sets.*

## *3.1 Specifications for the introduction of the sets.*

Sets are about objects and their properties. The terms object, object-selection and property are described in sections 2.3, 2.4, 2.5 and 2.6. The definitions and statements stored there are used here. The axioms listed there essentially form the axiom system for the justification of sets.

## 3.2 *Axiom system for the justification of sets.*

Let **P** be a property, **S** a set and **x**, **y** objects, then the following axioms hold:

### *3.2.1 Object-specific property.*

***Axiom:*** $\forall \mathbf{x}\ \exists \mathbf{P}\ \forall \mathbf{y} \neq \mathbf{x}\ \mathbf{P}(\mathbf{x}) \land \neg \mathbf{P}(\mathbf{y})$.

In a nutshell: Every object has at least one object-specific property that distinguishes it from all other objects. Each object is therefore unique. We reserve the character ${}_{x}\mathbf{P}$ for the object-specific property of an object **x**.

### *3.2.2 The object-selection.*

***Axiom:*** ∀**P** ∃[**x**|**P**(**x**)] $^{PC}$**P** :≡ (**x** ∈ [**x**|**P**(**x**)] ⇒ **P**(**x**)).

In a nutshell: For each property **P** there is an object [**x**|**P**(**x**)] with the property $^{PC}$**P**, which means that [**x**|**P**(**x**)] can only contain objects with the property **P**.

### **3.2.3** *Sets.*

***Axiom:***

¬**P**([**x**|**P**(**x**)]) :≡ [**x**|**P**(**x**)] ∉ [**x**|**P**(**x**)] ⇔
( ∃{**x**|**P**(**x**)} ¬**P**({**x**|**P**(**x**)}) **P**(**x**)⇒(**x**∈{**x**|**P**(**x**)} ∧ **x**∈[**x**|**P**(**x**)]) ).

In a nutshell: A property **P** is a set-building property if and only if the object-selection belonging to **P** does not have the property **P**.

### *3.2.4 Subsets.*

***Axiom:*** Let **S** be a set with its set-forming property **P**, i.e. **S**={**x**|**P**(**x**)}, and let **C** be an object-selection with the property (**x**∈**C** ⇒ **x**∈**S**), then **C** is also a set . Such sets **C** are called subsets of **S** and we write **C**⊆**S** for this.

### 3.2.5 ***Primordial elements.***

***Axiom:*** $\exists \mathbf{x}\ \forall \mathbf{P}\ \mathbf{x} \neq [\mathbf{y}|\mathbf{P}(\mathbf{y})]$.

In a nutshell: There is at least one object that is not an object-selection.

### ***3.2.6 Expansion of sets.***

***Axiom:*** $\exists \mathbf{S} \subset_P \mathbf{S}$. The iterative process $\exists \mathbf{x} \in_P \mathbf{S}\ \mathbf{x} \notin \mathbf{S} \rightleftarrows \mathbf{S} \cup \{x\} \rightarrow \mathbf{S}$ never ends.

In a nutshell: There is a proper subset of the primordial set, which can be expanded as often as desired with one element from the primordial set.

***Annotation:*** The axiom assumes that the so-called subsets of a set are actually sets. This is ensured by the preceding axiom in Section 3.2.4.

The axiom presupposes the existence of the primordial set. The existence of this set is a direct consequence of the preceding set axiom in Section 3.2.3. The proof of this can be found in Section 2.6.10.

The axiom assumes the existence of sets with only one element. The proof of the existence of such sets is given in Section 2.6.11.

The axiom assumes the existence of the set $\mathbf{S} \cup \{\mathbf{x}\}$ for every subset $\mathbf{S}$ of $_P\mathbf{S}$ if $\mathbf{x} \in_P \mathbf{S}$. The proof of the existence of this set is given in Section 2.6.12.

***Definition:*** We express the property of a set to be expandable with primordial elements any number of times with the symbol ${}^{SE}\mathbf{P}$.

### *3.2.7 Comparison of sets.*

***Axiom:*** $\forall \mathbf{S}_1,\mathbf{S}_2\ (\mathbf{x}\in\mathbf{S}_1 \Leftrightarrow \mathbf{x}\in\mathbf{S}_2) \Leftrightarrow (\mathbf{S}_1=\mathbf{S}_2)$.

In a nutshell: Sets are equal if and only if they contain the same elements.

## *3.3 Basic statements about the object properties.*

In an object-selection [**x**|**P**(**x**)], exactly one of the following statements is true for the selecting property **P**.

- All objects have the selecting property **P**.
- There is no object with the selecting property **P**.
- Only some objects have the selecting property **P**.

The following consequences result for the formation of sets:

### *3.3.1 If all objects have the selecting property P.*

The following statement applies here: ∀**x** **P**(**x**). Therefore, **P**(**C**) holds for the object-selection **C**=[**x**|**P**(**x**)] and, because of the axiom in Section 3.2.3, **P** does not form a set. A set of this kind therefore doesn't exist. As we shall see later, the Antinomy $\mathbf{x}\notin\mathbf{x}$ published by Bertrand Russell also belongs to this category of properties.

### *3.3.2 If there is no object with the selecting property P.*

The following statement applies here: $\forall \mathbf{x}\ \neg \mathbf{P}(\mathbf{x})$. Therefore, $\neg \mathbf{P}(\mathbf{C})$ holds for the object-selection $\mathbf{C}=[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ and because of the axiom in Section 3.2.3, **P** is a set-forming property. Such sets therefore exist. However, since there cannot be any objects with the property **P**, all these properties lead to the empty set ∅.

### *3.3.3 If only some objects have the selecting property P.*

The following applies here: $\exists \mathbf{x} \neq \mathbf{y}\ \mathbf{P}(\mathbf{x})\ \neg \mathbf{P}(\mathbf{y})$. Here, for object-selection $\mathbf{C}=[\mathbf{x}|\mathbf{P}(\mathbf{x})]$, it must be demonstrated that $\neg \mathbf{P}(\mathbf{C})$ applies. If this succeeds, then, because of the axiom in Section 3.2.3, **P** is a set-forming property and the set belonging to the object-selection exists.

## *3.4 First conclusions from the axiom system.*

The axiom in Section 3.2.3 provides a division of the object properties into two categories. One of these categories includes all object properties that can form a set, the other includes all object properties that cannot form a set. The separation of these two categories from one another is unmistakable. It can be derived for each property from the associated object-selection. Every property that is also a property of its associated object-selection cannot form a set because of the axiom in Section 3.2.3. Every other property, i.e. one that is not a property of its associated object-selection, necessarily forms a set because of the axiom in Section 3.2.3.

### ***3.4.1 There is an empty set.***

***Proof:*** The object-selection [**x**|**P**(**x**)] with the selecting property **P** :≡ **x**≠**x** does not contain any elements, since there are no objects with such a property. However, ¬**P**([**x**|**P**(**x**)]) ≡ ([**x**|**P**(**x**)]=[**x**|**P**(**x**)]) applies. This is a true statement. By 3.2.3 we therefore have, ∃{**x**|**P**(**x**)}, that is, we are dealing with an existing set. It is called "Empty Set" and is represented by the character ∅.

### ***3.4.2 Not set-forming root properties of a selection property.***

There are, in the sense of Section 3.2.2, many selection properties ${}^{PC}\mathbf{P}$. For there, an object with the property ${}^{PC}\mathbf{P}$ is demanded for each property **P**. Because this object also has the property ${}^{O}\mathbf{P}$, namely to be an object, and can also have the property ${}^{S}\mathbf{P}$, namely to be a set, the following statements apply:

### ***3.4.2.1 The set of all object-selections does not exist.***

***Proof:*** The object-selection of all object-selections $[\mathbf{x}|{}^{C}\mathbf{P}(\mathbf{x})]$ would have to be formed. Since this is an object-selection, ${}^{C}\mathbf{P}([\mathbf{x}|{}^{C}\mathbf{P}(\mathbf{x}))$ applies. Because of the axiom in Section 3.2.3 it follows that this is not a set.

### ***3.4.2.2 The set of all objects does not exist.***

***Proof:*** The object-selection of all objects $[\mathbf{x}|{}^{0}\mathbf{P}(\mathbf{x})]$ would have to be formed. Since this is an object, ${}^{0}\mathbf{P}([\mathbf{x}|{}^{0}\mathbf{P}(\mathbf{x}))$ applies. Because of the axiom in Section 3.2.3 it follows that this is not a set.

### ***3.4.2.3 The set of all sets does not exist.***

***Proof:*** The object-selection of all sets $[\mathbf{x}|{}^{S}\mathbf{P}(\mathbf{x})]$ would have to be formed. If this were a set, then ${}^{S}\mathbf{P}([\mathbf{x}|{}^{S}\mathbf{P}(\mathbf{x})])$ would hold, in contradiction to the axiom in Section 3.2.3.

### ***3.4.3 All properties that are always true are not set-forming.***

#### ***3.4.3.1 The identity cannot form a set.***

***Proof:*** The object-selection of all identities $[\mathbf{x}|{}^{I}\mathbf{P}(\mathbf{x})]$ would have to be formed. For this object-selection, however, the identity would also apply, i.e. ${}^{I}\mathbf{P}([\mathbf{x}|{}^{I}\mathbf{P}(\mathbf{x})])$. The object selection can therefore not be a set because of the axiom in Section 3.2.3.

#### ***3.4.3.2 The antinomy published by Russell cannot form a set.***

***Proof:*** The antinomy published by Bertrand Russell reads $P :\equiv x \notin \mathbf{x}$. If one forms the object-selection $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ and demands the set property for this object-selection, then it follows, because of the axiom in Section 3.2.3, $\neg(\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}\notin\{\mathbf{x}|\mathbf{P}(\mathbf{x})\})$. This is equivalent to the statement $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}\in\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$. That is because of 2.6.4.6 a false statement. A set belonging to the antinomy published by Bertrand Russell therefore does not exist.

***Annotation:*** After formulating Russell's antinomy $\mathbf{x} \notin \mathbf{x}$, $\mathbf{x} \notin \mathbf{x} \Rightarrow \mathbf{x} \in \mathbf{x}$ and $\mathbf{x} \in \mathbf{x} \Rightarrow \mathbf{x} \notin \mathbf{x}$ were inferred, which is formally correct, and thereupon the entire so-called “naive set theory” was discarded. In doing so, it was neglected to examine whether the object-selection $[\mathbf{x}|\mathbf{x} \notin \mathbf{x}]$ can be a set at all. An object-selection $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ is defined in such a way that it should contain all objects $\mathbf{x}$ for which $\mathbf{P}(\mathbf{x})$ is true. Therefore $\neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ must be true. As a result, as shown in 2.6.4.6, a set can never contain itself. If one wants to build sets with the property $\mathbf{x} \notin \mathbf{x}$, then one has to do with an always true property. It applies to all sets and therefore cannot form a set because of the axiom in 3.2.3.

### 3.4.4 *All properties that are always false are set-forming.*

***Proof:*** For this type of property $\forall\mathbf{x}\ \neg\mathbf{P}(\mathbf{x})$ applies. It is therefore valid for the object-selection $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ belonging to such a property, also $\neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$. Because of the axiom in Section 3.2.3, such sets exist. However, since there can be no objects with the property stored in $\mathbf{P}$, all these properties lead to the empty set $\varnothing$.

## 3.5 *Manipulation of sets.*

Sets can be manipulated, for example by adding or removing elements. The operations that are commonly used for manipulating sets are shown below, and it is examined which conditions must apply so that the structures newly created by the manipulation are again existing sets.

### *3.5.1 Extension with an object-specific property.*

***Definition:*** Let $\mathbf{S}=\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ be a set with its set-forming property $\mathbf{P}$. Let $\mathbf{y}$ be an object and let $\mathbf{y}\notin\mathbf{S}$. By the set $\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\}$ we understand the extension of the set $\mathbf{S}$ with the object $\mathbf{y}$ or the addition of the object $\mathbf{y}$ to the set $\mathbf{S}$.

***Proposition:*** Let $\mathbf{S}=\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ be a set with its set-forming property $\mathbf{P}$, then the set $\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\}$ exists if and only if $\mathbf{y}\neq[\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})]$ and $[\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})]\notin\mathbf{S}$. Therefore we have the equivalence

$$\mathbf{y}\neq[\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})] \wedge [\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})]\notin\mathbf{S} \Leftrightarrow \exists\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\}.$$

***Proof*** $\Rightarrow$***:*** From $\mathbf{y}\neq[\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})]$ follows $\neg_y\mathbf{P}([\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})])$ because of the object-specific property ${}_y\mathbf{P}$. From $[\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})]\notin\mathbf{S}$ follows $\neg\mathbf{P}([\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})])$. All in all, $\neg(\mathbf{P}\vee_y\mathbf{P})([\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})])$. According to Section 3.2.3, this is the criterion for the existence of the set $\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\}$

***Proof*** $\Leftarrow$***:*** Section 3.2.3 implies $\neg(\mathbf{P}\vee_y\mathbf{P})(\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\})$ from $\exists\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\}$. We have $\neg\mathbf{P}([\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})])$ and $\neg_y\mathbf{P}([\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})])$. From $\neg\mathbf{P}([\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})])$ follows $[\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})]\notin\mathbf{S}$. From $\neg_y\mathbf{P}([\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})])$, because of the object-specific property $_y\mathbf{P}$ , it follows that $\mathbf{y}\neq[\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})]$.

### ***3.5.2 Union of sets.***

The definition below is based on the definition for object- selections in Section 2.6.5.1.
***Definition:*** The union $\mathbf{S}_1\cup\mathbf{S}_2$ of two sets $\mathbf{S}_1=\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ and $\mathbf{S}_2=\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$, with their set-forming properties $\mathbf{P}_1$ and $\mathbf{P}_2$, is understood to be the object-selection
$[\mathbf{x}|\mathbf{x}\in\mathbf{S}_1 \vee \mathbf{x}\in\mathbf{S}_2] :\equiv [\mathbf{x}|\mathbf{P}_1(\mathbf{x}) \vee \mathbf{P}_2(\mathbf{x})] :\equiv [\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]$.

#### ***3.5.2.1 The union of two sets is a set.***

***Proposition:*** Let $\mathbf{S}_1$ and $S_2$ be sets with their set-forming properties $\mathbf{P}_1$ and $\mathbf{P}_2$, i.e. $\mathbf{S}_1=\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ and $\mathbf{S}_2=\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$, then the union $\mathbf{S}_1\cup S_2$ is a set.

***Proof:*** Since $\mathbf{S}_1=\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ is a set, this set contains all objects for which $\mathbf{P}_1(\mathbf{x})$ is true. This means that the object-selection $[\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]$ must also contain all objects for which $\mathbf{P}_1(\mathbf{x})$ is true. So we have $\neg\mathbf{P}_1([\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})])$. The analogous consideration can be made for $\mathbf{S}_2=\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$. It also holds that $\neg\mathbf{P}_2([\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})])$. So overall it applies
$\neg\mathbf{P}_1([\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]) \wedge \neg\mathbf{P}_2([\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]) \equiv \neg(\mathbf{P}_1\vee\mathbf{P}_2)([\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})])$.
According to Section 3.2.3, this is the condition for the existence of the set $\{\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})\} :\equiv \mathbf{S}_1\cup\mathbf{S}_2$.

### *3.5.2.2 Typical property of elements of a union.*

***Proposition:*** Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be sets with their set-forming properties $\mathbf{P}_1$ and $\mathbf{P}_2$, i.e. $\mathbf{S}_1=\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ and $\mathbf{S}_2=\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$, then the union $\mathbf{S}_1\cup\mathbf{S}_2$ is a set if and only if $\forall\mathbf{x}\in\mathbf{S}_1\ \mathbf{x}\neq[\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]$ and $\forall\mathbf{x}\in\mathbf{S}_2\ \mathbf{x}\neq[\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]$ hold. So the equivalence applies:
$(\ (\mathbf{x}\in\mathbf{S}_1 \vee \mathbf{x}\in\mathbf{S}_2) \Rightarrow \mathbf{x}\neq[\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]\ ) \Leftrightarrow \exists\{\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})\} :\equiv \exists\mathbf{S}_1\cup\mathbf{S}_2$.

***Proof*** $\Rightarrow$***:*** Because of section 2.6.4.6, from $\mathbf{x}\in\mathbf{S}_1$ follows $\mathbf{x}\neq\mathbf{S}_1$ and from $\mathbf{x}\in\mathbf{S}_2$ follows $\mathbf{x}\neq\mathbf{S}_2$. So if $\mathbf{x}\in\mathbf{S}_1 \vee \mathbf{x}\in\mathbf{S}_2$ it follows $\mathbf{x}\neq\mathbf{S}_1 \wedge \mathbf{x}\neq\mathbf{S}_2$. Therefor we have $(\mathbf{x}\in\mathbf{S}_1 \vee \mathbf{x}\in\mathbf{S}_2) \Rightarrow \mathbf{x}\neq(\mathbf{S}_1 \vee \mathbf{S}_2)$. Because of the definition of the term „Ubion“ in section 2.6.5.1 we have $(\mathbf{S}_1 \vee \mathbf{S}_2) :\equiv [\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]$. Tue to the object-specific Property ${}_x\mathbf{P}$ we have $\forall\mathbf{x}\in(\mathbf{S}_1 \vee \mathbf{S}_2)\ \ \neg_x\mathbf{P}([\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})])$. Because of section 2.6.6 $\exists\{\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})\} :\equiv \exists\mathbf{S}_1\cup\mathbf{S}_2$.

***Proof*** $\Leftarrow$***:*** Because of 3.2.3 we have
$\exists\{\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(x)\} \Rightarrow \neg(\mathbf{P}_1\vee\mathbf{P}_2)([\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})])$. This is equivalent to
$\forall\mathbf{x}\in[\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]\ \ \mathbf{x}\neq[\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]$.
Hence $(\mathbf{x}\in\mathbf{S}_1 \vee \mathbf{x}\in\mathbf{S}_2) \Rightarrow \mathbf{x}\neq[\mathbf{x}|(\mathbf{P}_1\vee\mathbf{P}_2)(\mathbf{x})]$.

### ***3.5.3 Intersection of two sets.***

The definition below is based on the definition for object- selections in Section 2.6.5.2.

***Definition:*** The intersection $\mathbf{S}_1\cap\mathbf{S}_2$ of two sets $\mathbf{S}_1=\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ and $\mathbf{S}_2=\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$, with their set-forming properties $\mathbf{P}_1$ and $\mathbf{P}_2$, is understood to be the object-selection
$[\mathbf{x}|\mathbf{x}\in\mathbf{S}_1 \wedge \mathbf{x}\in\mathbf{S}_2] :\equiv [\mathbf{x}|\mathbf{P}_1(\mathbf{x}) \wedge \mathbf{P}_2(\mathbf{x})] :\equiv [\mathbf{x}|(\mathbf{P}_1\wedge\mathbf{P}_2)(\mathbf{x})]$.

***Proposition:*** Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be sets with their set-forming properties $\mathbf{P}_1$ and $\mathbf{P}_2$, i.e. $\mathbf{S}_1=\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ and $\mathbf{S}_2=\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$, then the intersection $\mathbf{S}_1\cap\mathbf{S}_2$ is a set.

***Proof:*** We have $\mathbf{x}\in(\mathbf{S}_1\cap\mathbf{S}_2) \Rightarrow \mathbf{x}\in\mathbf{S}_1 \wedge \mathbf{x}\in\mathbf{S}_2$. That is, according to 2.6.8.1, the intersection is a subset of both $\mathbf{S}_1$ and $\mathbf{S}_2$ and therefore an existing set because of 3.2.4.

### ***3.5.4 Difference between two sets.***

***Definition:*** The difference between two sets $\mathbf{S}_1$ and $\mathbf{S}_2$ is understood to mean the object-selection $\mathbf{S}_1\backslash\mathbf{S}_2 :\equiv [\mathbf{x}|\mathbf{x}\in\mathbf{S}_1 \wedge \neg(\mathbf{x}\in\mathbf{S}_2)]$.
***Proposition:*** If $\mathbf{S}_1$ and $\mathbf{S}_2$ are sets, then the difference $\mathbf{S}_1\backslash\mathbf{S}_2$ is an existing set.
***Proof:*** We have $\mathbf{x}\in(\mathbf{S}_1\backslash\mathbf{S}_2) \Rightarrow \mathbf{x}\in\mathbf{S}_1$. That is, the difference is a subset of $\mathbf{S}_1$ according to 2.6.8.1 and therefore an existing set because of 3.2.4.

### ***3.5.5 Set extension with primordial elements.***

***Proposition:*** If $\mathbf{S}=\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ is a set with its set-forming property $\mathbf{P}$ and if $\mathbf{S}$ does not contain all primordial elements, so if $\mathbf{S}\cap_P\mathbf{S} \subset {}_P\mathbf{S}$ holds, then $\mathbf{S}$ can be extended with a primordial element.
***Proof:*** Since $\mathbf{S}$ does not contain all primordial elements, there exists a primordial element $\mathbf{y}$ such that $\mathbf{y}\notin\mathbf{S}$ und $\mathbf{y}\in_P\mathbf{S}$. For this primordial element $\mathbf{y}$ there exists because of Section 2.6.11 the one-element set $\{\mathbf{y}\} \equiv \{\mathbf{x}|_y\mathbf{P}(\mathbf{x})\}$. The sets $\mathbf{S}$ and $\{\mathbf{y}\}$ can be united to form the set $\{\mathbf{x}|(\mathbf{P}\vee_y\mathbf{P})(\mathbf{x})\}$ according to Section 3.5.2.1.

### ***3.5.6 Union with primordial elements that can be carried out as often as desired.***

***Proposition:*** Let **x** be an primordial element, then the set $\mathbf{S}_1$ := {**x**} can be united with elements from the primordial set any number of times. The iterative process $\exists \mathbf{x} \in_P \mathbf{S}\ \mathbf{x} \notin \mathbf{S}_1 \ \rightleftharpoons \mathbf{S}_1 \cup \{\mathbf{x}\} \rightarrow \mathbf{S}_1$ never ends.

***Proof:*** The axiom in Section 3.2.6 guarantees the existence of a set $\mathbf{S} \subset_P \mathbf{S}$ such that the iterative process $\exists \mathbf{y} \in_P \mathbf{S}\ \mathbf{y} \notin \mathbf{S} \rightleftharpoons \mathbf{S} \cup \{\mathbf{y}\} \rightarrow \mathbf{S}$ never ends. If our object **x** is not contained in the set **S**, we can use this **x** to carry out the first process step in the iterative process $\exists \mathbf{y} \in_P \mathbf{S}\ \mathbf{y} \notin \mathbf{S} \rightleftharpoons \mathbf{S} \cup \{\mathbf{y}\} \rightarrow \mathbf{S}$. From then on we can assume that $\mathbf{x} \in \mathbf{S}$ holds and the iterative process $\exists \mathbf{y} \in_P \mathbf{S}\ \mathbf{y} \notin \mathbf{S} \rightleftharpoons \mathbf{S} \cup \{\mathbf{y}\} \rightarrow \mathbf{S}$ never ends. We now have $\mathbf{S}_1 \subseteq \mathbf{S}$ and $\mathbf{y} \notin \mathbf{S} \Rightarrow \mathbf{y} \notin \mathbf{S}_1$. If a process step can be carried out in the iterative process $\exists \mathbf{y} \in_P \mathbf{S}\ \mathbf{y} \notin \mathbf{S} \rightleftharpoons \mathbf{S} \cup \{\mathbf{y}\} \rightarrow \mathbf{S}$, then a process step can also be carried out in the iterative process $\exists \mathbf{x} \in_P \mathbf{S}\ \mathbf{x} \notin \mathbf{S}_1 \rightleftharpoons \mathbf{S}_1 \cup \{\mathbf{x}\} \rightarrow \mathbf{S}_1$ if we use the same element in both processes when executing the process steps. Since the first process never ends, the second process $\exists \mathbf{x} \in_P \mathbf{S}\ \mathbf{x} \notin \mathbf{S}_1 \ \rightleftharpoons \mathbf{S}_1 \cup \{\mathbf{x}\} \rightarrow \mathbf{S}_1$ can never end either. The set $\mathbf{S}_1$ := {**x**} can therefore, in the same way as the set **S** from the axiom in Section 3.2.6, be united with elements from the primordial set as often as desired.

## *3.6 Construction of sets.*

### *3.6.1 Construction using object-specific properties.*

***Proposition:*** Let **y** and **z** be objects with their object-specific properties $_{y}\mathbf{P}$ and $_{z}\mathbf{P}$, and let both **y** and **z** not be equal to the extended object-selection $[\mathbf{x}|({}_{y}\mathbf{P}\vee{}_{z}\mathbf{P})(\mathbf{x})]$, then the set $\mathbf{S} := \{\mathbf{x}|({}_{y}\mathbf{P}\vee{}_{z}\mathbf{P})(\mathbf{x})\}$ exists .

***Proof:*** $\neg({}_{y}\mathbf{P}\vee{}_{z}\mathbf{P})(\mathbf{S}) \equiv (\neg{}_{y}\mathbf{P}(\mathbf{S}) \wedge \neg{}_{z}\mathbf{P}(\mathbf{S}))$. Since **S**, **y**, **z** are different objects and both $_{y}\mathbf{P}$ and $_{z}\mathbf{P}$ are object-specific properties of the objects **y** and **z**, both $\neg{}_{y}\mathbf{P}(\mathbf{S})$ and $\neg{}_{z}\mathbf{P}(\mathbf{S})$ apply because of 3.2.1. The left side of the equality of meaning mentioned at the beginning of the proof is therefore a true statement. Because of 3.2.3 there is therefore the set $\mathbf{S} := \{\mathbf{x}|({}_{y}\mathbf{P}\vee{}_{z}\mathbf{P})(\mathbf{x})\}$.

### 3.6.2 *Construction of one-element sets*.

The axiom in section 3.2.5, together with the axiom in section 3.2.6, guarantees the existence of an inexhaustible number of primordial elements. For each of these primordial elements, we can construct the corresponding one-element set because of Section 2.6.11.

### *3.6.3 Construction by nesting.*

A set **S** can be nested arbitrarily because of 2.6.7. This creates sets as indicated below:

**S** {**S**} {{**S**}} . . . . . .

### *3.6.4 Construction by union of sets.*

As shown in 3.5.2, two sets can always be united to a set.

### *3.6.5 Construction through continued extension with primordial elements.*

***Proposition:*** Let **S** be a set and **x** a primordial element and let $\mathbf{S} \cap_P \mathbf{S} = \varnothing$ or $\mathbf{S} \cap_P \mathbf{S} = \{\mathbf{x}\}$ hold, then **S** can be extended with primordial elements any number of times.

***Proof:*** In Section 3.5.6 we saw that single-element sets can be extended with primordial elements as often as desired. In Section 3.5.5 we saw that sets can always be extended by a primordial element if they do not already contain all primordial elements. Both together guarantee that **S** can be extended with primordial elements any number of times.

### *3.6.6 Construction by pairing sets.*

***Proposition:*** If $\mathbf{S}_1$ and $\mathbf{S}_2$ are sets, then $[\mathbf{x} \mid \mathbf{x} = \mathbf{S}_1 \lor \mathbf{x} = \mathbf{S}_2]$ is also a set.

***Proof:*** The sets $\mathbf{S}_1$ and $\mathbf{S}_2$, like all objects, have object-specific properties. We will call them ${}_{s1}\mathbf{P}$ and ${}_{S2}\mathbf{P}$. We can then form the object-selection $[\mathbf{x}|({}_{S1}\mathbf{P} \lor {}_{S2}\mathbf{P})(\mathbf{x})] \equiv [\mathbf{x}|\mathbf{x}=\mathbf{S}_1 \lor \mathbf{x}=\mathbf{S}_2]$ and see, because of Section 3.6.1, that the set $\{\mathbf{x}|({}_{S1}\mathbf{P} \lor {}_{S2}\mathbf{P})(\mathbf{x})\}$ exists.

***Annotation:*** This statement can be used to construct nested sets in the manner indicated below. So let **S** be a set, then we could create any number of the following sets:

$\{\mathbf{S}\}$ $\{\{\mathbf{S}\}\ \mathbf{S}\}$ $\{\{\{\mathbf{S}\}\ \mathbf{S}\ \}\ \mathbf{S}\ \}$ . . . . . . . . .

## *3.7 Properties of the Empty Set.*

As shown in Section 3.4.1, the empty set exists. It has the following properties.

### *3.7.1 The empty set is a subset of any set.*

***Proof:*** Let **S** be a set. Since the empty set cannot contain any elements, the following applies: $\varnothing \cup \mathbf{S} = \{\mathbf{x}|\mathbf{x} \in \mathbf{S} \lor \mathbf{x} \in \varnothing\} = \{\mathbf{x}|\mathbf{x} \in \mathbf{S}\} = \mathbf{S}$. Because of Section 2.6.8.3, $\varnothing \subseteq S$ therefore applies.

### ***3.7.2 There is only one empty set.***

***Proof:*** Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be empty sets and let $\mathbf{S}_1 \neq \mathbf{S}_2$, then by 3.2.7 $\exists \mathbf{x}$ $((\mathbf{x} \in \mathbf{S}_1 \wedge \mathbf{x} \notin \mathbf{S}_2) \vee (\mathbf{x} \in \mathbf{S}_2 \wedge \mathbf{x} \notin \mathbf{S}_1))$, so that at least one of the two sets $\mathbf{S}_1$ and $\mathbf{S}_2$ is not empty.

## ***3.8 Variants of set-forming root properties are set-forming.***

***Proposition:*** Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be properties and let $\mathbf{P}_1 \Rightarrow \mathbf{P}_2$ apply. Let $\mathbf{P}_2$ be a set-forming property, i.e. if the set $\mathbf{S}_2 = \{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ exists, then $\mathbf{P}_1$ is also a set-forming property, that is, the set $\mathbf{S}_1 = \{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$ exists and $\mathbf{S}_1$ is a subset of $\mathbf{S}_2$.

***Proof:*** Since $\mathbf{P}_1 \Rightarrow \mathbf{P}_2$ we have $\forall \mathbf{x} \in [\mathbf{x}|\mathbf{P}_1(\mathbf{x})] \Rightarrow \mathbf{x} \in \{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$. The object-selection $[\mathbf{x}|\mathbf{P}_1(\mathbf{x})]$ is therefore, because of Section 2.6.8.1, a set, namely a subset of $\mathbf{S}_2 = \{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$.

## ***3.9 Power sets.***

Let $\mathbf{S}$ be a set, then we use the notation $\mathbf{POW}(\mathbf{S}) \equiv [\mathbf{x}|\mathbf{x} \subseteq \mathbf{S}]$.

***Proposition:*** Let $\mathbf{S}$ be a set with its set-forming property $\mathbf{P}$, i.e. $\mathbf{S}=\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$, then the power set $\{\mathbf{x}|\mathbf{x} \subseteq \mathbf{S}\}$ exists.

***Proof:*** Assuming the power set does not exist, then [**x**|**x**⊆**S**]⊆**S** must hold. That would mean that all elements of the object-selection [**x**|**x**⊆**S**] are both elements of **S** and subsets of **s**. However, this is not possible if **S** contains a primordial element. A primordial element is by definition not a subset. In this case we have ¬([**x**|**x**⊆**S**]⊆**S**) and the power set {**x**|**x**⊆**S**} exists because of 3.2.3. If **S** does not contain a primordial element, then by 3.5.5 we can expand **S** with a primordial element **y** to **S**∪{**y**} := $\mathbf{S}_1$. Now the power set of $\mathbf{S}_1$ exists and **Pow**(**S**) ⊂ **Pow**($\mathbf{S}_1$) applies, so that **Pow**(**S**) also exists for this case because of 3.2.4.

## *3.10 Assignment of math objects*

This section defines the already known concept of assignment in order to subsequently be able to access the definitions laid down.

### *3.10.1 Assignments.*

Mathematical objects can be assigned to one another. An assignment is made when the following specifications have been made:

#### *3.10.1.1 Image set.*

A set is defined that contains all objects to which something is assigned. This set is called the image set.

#### *3.10.1.2 Definition set.*

A set is defined that contains all objects and only objects that are to be assigned. This set is called the definition set.

#### *3.10.1.3 Assignment rule.*

A rule is defined which assigns each object from the definition set to exactly one object in the image set. This rule is called the assignment rule.

### *3.10.2 Types of assignment.*

According to 3.10.1, let $\mathbf{S}_1$ be the definition set, $\mathbf{S}_2$ be the image set and ***f*** the assignment rule of an assignment (src 5.1).

#### *3.10.2.1 Injective.*

An assignment is called injective if the following applies:

$\forall \mathbf{x},\mathbf{y} \in \mathbf{S}_1 \; \boldsymbol{f}(\mathbf{x})=\boldsymbol{f}(\mathbf{y}) \Rightarrow \mathbf{x}=\mathbf{y}$

#### *3.10.2.2 Surjective.*

An assignment is called surjective if:

$\forall \mathbf{x} \in \mathbf{S}_2 \; \exists \mathbf{y} \in \mathbf{S}_1 \; \boldsymbol{f}(\mathbf{y})=\mathbf{x}$

#### *3.10.2.3 Bijective.*

An assignment is called bijective if it is both injective and surjective.

### *3.11 Injective and non-surjective assignments.*

***Proposition:*** Let $\mathbf{S}_1$ be the definition set, (see 3.10.1.2), and $\mathbf{S}_2$ the image set, (see 3.10.1.1), of an injective and non-surjective assignment using the assignment rule $\boldsymbol{f}$, (see 3.10.1.3), then there are none for the named sets bijective assignment rule $\boldsymbol{g}$.

***Proof:*** Let $\boldsymbol{g}$ be bijective. Let $\mathbf{x}_1 \in \mathbf{S}_1$ and let $\boldsymbol{f}(\mathbf{x}_1) = \mathbf{y}$. Since $\boldsymbol{g}$ is surjective, $\mathbf{x}_2$ exists in $\mathbf{S}_1$ and $\boldsymbol{g}(\mathbf{x}_2) = \mathbf{y}$. Let $\mathbf{x}_1 \neq \mathbf{x}_2$, i.e. $\boldsymbol{g}(\mathbf{x}_1) \neq \mathbf{y}$, say $\boldsymbol{g}(\mathbf{x}_1) = \mathbf{y}_1$, then we change the assignment rule $\boldsymbol{g}$ such that: $\boldsymbol{g}(\mathbf{x}_1) = \mathbf{y}$ and $\boldsymbol{g}(\mathbf{x}_2) = \mathbf{y}_1$. Here, $\boldsymbol{g}$ remains bijective as before. We carry out this change in the assignment rule $\boldsymbol{g}$ for all $\mathbf{x} \in \mathbf{S}_1$, for which $\boldsymbol{f}(\mathbf{x}) \neq \boldsymbol{g}(\mathbf{x})$. Then: $\forall \mathbf{x} \in \mathbf{S}_1\ \boldsymbol{f}(\mathbf{x}) = \boldsymbol{g}(x)$ applies.
Since $\boldsymbol{f}$ is not surjective, we have $\exists \mathbf{y} \in \mathbf{S}_2\ \forall \mathbf{x} \in \mathbf{S}_1\ \boldsymbol{f}(\mathbf{x}) \neq \mathbf{y}$. From this it follows: $\forall \mathbf{x} \in \mathbf{S}_1\ \boldsymbol{f}(\mathbf{x}) = \boldsymbol{g}(\mathbf{x}) \neq \mathbf{y}$ in contradiction to the bijectivity of $\mathbf{g}$.

### *3.12 The size of a set.*

We define the term “size of a set” with the help of the primordial set $_P\mathbf{S}$. We assign exactly one “size” to each element from the power set of $_P\mathbf{S}$. We demand that the "size" can be manipulated without restriction by adding or removing elements, if it is not the empty set. For the empty set we only allow one manipulation by adding. Since we only refer to the power set of $_P\mathbf{S}$, or to elements of $_P\mathbf{S}$, such an unrestricted manipulation is possible. This means that we no longer need to worry about the existence of the sets that we generate through the manipulation. The manipulation is always possible, and the structures resulting from the manipulation are always sets. See the statements in Sections 3.5 and 3.6.

***Definition:*** We use the notation S**IZ**(**S**) for the size of a set **S**. The empty set does not contain any elements. We assign size 0 to it. We also say **SIZ**(∅) := 0. Because of 2.6.11 there are sets that contain exactly one element. We assign size 1 to each of these sets. So let **x** be an arbitrary object, then we also say **SIZ**({**x**}) := 1. For the size we further agree:

### *3.12.1 Each object changes the size in the same way.*

***Definition:***

(**x**∉**S** ∧ **y**∉**S**) ⇒ **SIZ**(**S**∪{**x**}) := **SIZ**(**S**∪{**y**}).

(**x**∈**S** ∧ **y**∈**S**) ⇒ **SIZ**(**S**\{**x**}) := **SIZ**(**S**\{**y**}).

Based on this definition, there are different sets with the same size, which is also true because of Section 3.12. All single-element sets have the size 1.

***Proposition:*** Let **S** be a set, let **x**∈**S** ∧ **y**∉**S**, then
**SIZ**(**S**) = **SIZ**(**S**\{**x**} ∪ {**y**}).

***Proof:*** Because of 3.12.1, **SIZ**(**S**\{**x**}∪{**y**}) = **SIZ**(**S**\{**x**}∪{**x**}) = **SIZ**(**S**).

### *3.12.2 The order of size.*

If a set does not contain all primordial elements, then we can always add an element to it because of Section 3.5.5. Because the elements of a set can be distinguished on the basis of their object-specific properties, we can always take an element from every non-empty set because of Section 3.5.4. We say the set size gets smaller if we take an element out of the set, it gets bigger if we add one to it, and it stays the same if we do nothing of the kind. To express this we write:

(**x**∈**S** ∧ **y**∉**S**) ⇒ **SIZ**(**S**\{**x**}) :< **SIZ**(**S**) :< **SIZ**(**S**∪{**y**}).

### *3.12.3 The smallest possible change in size.*

Section 3.12.2 requires at least one element to be added or removed from a set in order to change its size. We express this fact with the terms "successor" and "predecessor" and say, if ($\mathbf{x} \in \mathbf{S} \wedge \mathbf{y} \notin \mathbf{S}$):
**SIZ**($\mathbf{S} \setminus \{\mathbf{x}\}$) is the predecessor of **SIZ**(**S**), and **SIZ**($\mathbf{S} \cup \{\mathbf{y}\}$) is the successor to **SIZ**(**S**). Since we agreed in Section 3.12.1 that it is irrelevant for changing the size which element is taken from or added to the underlying set, we also write **SIZ**($\mathbf{S} \cup \{\mathbf{y}\}$):= **SIZ**(**S‘**) and **SIZ**($\mathbf{S} \setminus \{\mathbf{x}\}$):= **SIZ**(**‘S**). Based on the order of size agreed in Section 3.12.2, the following applies: **SIZ**(**‘S**) < **SIZ**(**S**) < **SIZ**(**S‘**).

### 3.12.4 *Iterative exchange process when comparing the size of two sets.*

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be sets. The iterative process
$(\exists \mathbf{x}\ \mathbf{x} \in \mathbf{S}_1 \wedge \mathbf{x} \notin \mathbf{S}_2) \wedge (\exists \mathbf{y}\ \mathbf{y} \in \mathbf{S}_2 \wedge \mathbf{y} \notin \mathbf{S}_1) \rightleftharpoons \mathbf{S}_1 := \mathbf{S}_1 \setminus \{\mathbf{x}\} \cup \{y\}$ provides the same size for $\mathbf{S}_1$ for each process step due to 3.12.1. Because of the manipulation of the set $\mathbf{S}_1$ on the right-hand side of the iteration equation and the use of the manipulated set $\mathbf{S}_1$ on the left-hand side of the iteration equation, the conditions for the iteration may be violated. The process then ends and one of the following situations occurs:

#### *3.12.4.1 Condition for "smaller".*

$\neg(\exists \mathbf{x}\ \mathbf{x} \in \mathbf{S}_1 \wedge \mathbf{x} \notin \mathbf{S}_2) \wedge (\exists \mathbf{y}\ \mathbf{y} \in \mathbf{S}_2 \wedge \mathbf{y} \notin \mathbf{S}_1)$.

For this we say: $\mathbf{SIZ}(\mathbf{S}_1) < \mathbf{SIZ}(\mathbf{S}_2)$.

Because of Section 2.6.8.2, $\mathbf{S}_2$ is a superset of $\mathbf{S}_1$.

#### *3.12.4.2 Condition for "greater".*

$(\exists \mathbf{x}\ \mathbf{x} \in \mathbf{S}_1 \wedge \mathbf{x} \notin \mathbf{S}_2) \wedge \neg(\exists \mathbf{y}\ \mathbf{y} \in \mathbf{S}_2 \wedge \mathbf{y} \notin \mathbf{S}_1)$.

For this we say: $\mathbf{SIZ}(\mathbf{S}_1) > \mathbf{SIZ}(\mathbf{S}_2)$.

Because of Section 2.6.8.2, $S_1$ is a superset of $S_2$.

#### *3.12.4.3 Condition for "equal".*

$\neg(\exists \mathbf{x}\ \mathbf{x} \in \mathbf{S}_1 \wedge \mathbf{x} \notin \mathbf{S}_2) \wedge \neg(\exists \mathbf{y}\ \mathbf{y} \in \mathbf{S}_2 \wedge \mathbf{y} \notin \mathbf{S}_1)$.

For this we say: $\mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2)$.

It is conceivable that despite the manipulation of the set $\mathbf{S}_1$ on the right-hand side of the iteration equation, the conditions for the iteration on the left-hand side of the equation are never violated. The iteration process then never ends, and we cannot make a decision on the size ratio of the two sets. But we also say for this case that the size of the sets $\mathbf{S}_1$ and $\mathbf{S}_2$ should be equal.

### *3.12.5 Preservation of the size ratio of two sets.*

#### *3.12.5.1 When expanding.*

***Proposition:*** Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be sets and $\mathbf{x} \notin \mathbf{S}_1, \mathbf{S}_2$, then:

$\mathbf{SIZ}(\mathbf{S}_1) \neq \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_1 \cup \{\mathbf{x}\}) \neq \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$

$\mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_1 \cup \{\mathbf{x}\}) = \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$

**Proof:** Because $\mathbf{x} \notin \mathbf{S}_1 \wedge \mathbf{x} \notin \mathbf{S}_2$, $\mathbf{x}$ has no influence on the iteration condition in 3.12.4. The iteration process is therefore independent of $\mathbf{x}$ and the following applies:

$\mathbf{SIZ}(\mathbf{S}_1) < \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_1 \cup \{\mathbf{x}\}) < \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$

$\mathbf{SIZ}(\mathbf{S}_1) > \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_1 \cup \{\mathbf{x}\}) > \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$

$\mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_1 \cup \{\mathbf{x}\}) = \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$

#### *3.12.5.2 With reduction.*

***Proposition:*** Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be sets and $\mathbf{x} \in \mathbf{S}_1, \mathbf{S}_2$, then:

$\mathbf{SIZ}(\mathbf{S}_1) \neq \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_{1\setminus}\{\mathbf{x}\}) \neq \mathbf{SIZ}(\mathbf{S}_{2\setminus}\{\mathbf{x}\})$

$\mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_{1\setminus}\{\mathbf{x}\}) = \mathbf{SIZ}(\mathbf{S}_{2\setminus}\{\mathbf{x}\})$

***Proof:*** Because $\mathbf{x} \in \mathbf{S}_1 \wedge \mathbf{x} \in \mathbf{S}_2$, **x** has no influence on the iteration condition in 3.12.4. The iteration process is therefore independent of **x** and the following applies:

$$\mathbf{SIZ}(\mathbf{S}_1) < \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_{1\setminus}\{\mathbf{x}\}) < \mathbf{SIZ}(\mathbf{S}_{2\setminus}\{\mathbf{x}\})$$
$$\mathbf{SIZ}(\mathbf{S}_1) > \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_{1\setminus}\{\mathbf{x}\}) > \mathbf{SIZ}(\mathbf{S}_{2\setminus}\{\mathbf{x}\})$$
$$\mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2) \Leftrightarrow \mathbf{SIZ}(\mathbf{S}_{1\setminus}\{\mathbf{x}\}) = \mathbf{SIZ}(\mathbf{S}_{2\setminus}\{\mathbf{x}\})$$

### *3.12.6 Size ratios for subsets.*

***Proposition:*** Proper Subsets are always smaller than their associated proper supersets.

***Proof:*** Let $\mathbf{S}_2=\{\mathbf{x}|\mathbf{P}_2(\mathbf{x})\}$ be a proper superset of $\mathbf{S}_1=\{\mathbf{x}|\mathbf{P}_1(\mathbf{x})\}$, i.e. because of Section 2.6.8.2 $\neg(\mathbf{P}_2 \Rightarrow \mathbf{P}_1)$, then $\exists \mathbf{y}\ \mathbf{P}_2(\mathbf{y}) \wedge \neg\mathbf{P}_1(\mathbf{y})$. Because $\mathbf{S}_1$ and $\mathbf{S}_2$ are sets it folloes that $y \in \mathbf{P}_2 \wedge \mathbf{y} \notin \mathbf{P}_1$. For the same reason it follows $\forall \mathbf{x} \in \mathbf{S}1\ \mathbf{x} \in \mathbf{S}_2$. So the case 3.12.4.1 is present and therefore $\mathbf{SIZ}(\mathbf{S}_1) < \mathbf{SIZ}(\mathbf{S}_2)$ applies.

# 4 *Numbers.*

## 4.1 *The concept of number.*

Because of Section 3.12.1 there are different sets with the same size. We say sets of the same size represent the same number. Different size represent different numbers. Each set represents exactly one number according to its size.

## 4.2 *The power set of the primordial set, divided into sets of equal size.*

In the object-selection

$${}_{\mathbb{N}}\mathbf{C}=[\mathbf{x}|\mathbf{x}\subseteq\mathbf{POW}({}_{P}\mathbf{S})\ \forall\mathbf{t}\in\mathbf{x}\ {}^{SE}\mathbf{P}(\mathbf{t})\ \forall\mathbf{y},\mathbf{z}\in\mathbf{x}\ \mathbf{SIZ}(\mathbf{y})=\mathbf{SIZ}(\mathbf{z})\ \forall\mathbf{w}\in\mathbf{POW}({}_{P}\mathbf{S})\ \wedge\ \mathbf{SIZ}(\mathbf{w})=\mathbf{SIZ}(\mathbf{y})\ \mathbf{w}\in\mathbf{x}]$$

each **x** represents all available subsets of ${}_{P}\mathbf{S}$, which match in size and are expandable with primordial elements any number of times. Each **x** of this object-selection ${}_{\mathbb{N}}\mathbf{C}$ and also each subset $\mathbf{y}\in\mathbf{x}$ thus represents exactly one and the same size.

***Proposition:*** The object-selection

$${}_{\mathbb{N}}\mathbf{C}=[\mathbf{x}|\mathbf{x}\subseteq\mathbf{POW}({}_{P}\mathbf{S})\ \forall\mathbf{t}\in\mathbf{x}\ {}^{SE}\mathbf{P}(\mathbf{t})\ \forall\mathbf{y},\mathbf{z}\in\mathbf{x}\ \mathbf{SIZ}(\mathbf{y})=\mathbf{SIZ}(\mathbf{z})\ \forall\mathbf{w}\in\mathbf{POW}({}_{P}\mathbf{S})\ \wedge\ \mathbf{SIZ}(\mathbf{w})=\mathbf{SIZ}(\mathbf{y})\ \mathbf{w}\in\mathbf{x}]$$

is a set.

***Proof:*** If $_{\mathbb{N}}\mathbf{C}$ were not a set, then, because of Section 3.2.3, the same criteria would have to apply to $_{\mathbb{N}}\mathbf{C}$ that lead to the selection of the elements of $_{\mathbb{N}}\mathbf{C}$. So $_{\mathbb{N}}\mathbf{C} \subseteq \mathbf{POW}(_{P}\mathbf{S})$ should apply. That would mean $\mathbf{x} \in {_{\mathbb{N}}\mathbf{C}} \Rightarrow \mathbf{x} \subseteq {_{P}\mathbf{S}}$, contrary to the definition of $_{\mathbb{N}}\mathbf{C}$. The object-selection $_{\mathbb{N}}\mathbf{C}$ must therefore be a set.

## 4.3 *The natural numbers.*

We equate the set of natural numbers with the partition set from Section 4.2. Each **x** of this object-selection

$_{\mathbb{N}}\mathbf{C}=[\mathbf{x}|\mathbf{x} \subseteq \mathbf{POW}(_{P}\mathbf{S})\ \forall \mathbf{t} \in \mathbf{x}\ {}^{SE}\mathbf{P}(\mathbf{t})\ \ \forall \mathbf{y},\mathbf{z} \in \mathbf{x}\ \mathbf{SIZ}(\mathbf{y})=\mathbf{SIZ}(\mathbf{z})\ \forall \mathbf{w} \in \mathbf{POW}(_{P}\mathbf{S}) \wedge \mathbf{SIZ}(\mathbf{w})=\mathbf{SIZ}(\mathbf{y})\ \mathbf{w} \in \mathbf{x}]$

and also every subset $\mathbf{y} \in \mathbf{x}$, thus represents exactly one size as well as exactly one natural number. That this representative distribution set exists has also already been shown in Section 4.2. We call it the set of natural numbers and represent it with the symbol $\mathbb{N}$.

## *4.4 Properties of the natural numbers.*

Because of Section 4.3, the properties of the natural numbers are identical to the properties of the set-size. These properties can be read from the behavior of the subsets of $_{P}\mathbf{S}$ when we manipulate these subsets.

Since the distribution set from Section 4.2 also contains the single-element subsets of $_P$**S**, and because of Section 3.5.6 these can be expanded with primordial elements as often as desired, the natural numbers can also be "expanded" as often as desired, starting from a given number. So let **S** be a subset of $_P$**S** contained in the partition set from Section 4.2 as an element of one of its partition members and let **x**, **y** be primordial elements, then the following statements hold:

### *4.4.1 The zero is included.*

If we use the iterative exchange procedure in Section 3.12.4 to compare the size of the empty set ∅ with the size of an arbitrary set not equal to ∅, then we see that the exchange process does not start and that case 3.12.4.1 occurs immediately. The empty set ∅ is therefore the smallest of all sets. We therefore define:
**SIZ**(∅) := 0 ∈ ℕ.

### 4.4.2 Every natural number has a successor.

***Proposition:*** ***S*****IZ**(**S**) ∈ ℕ ⇒ ∃**x**∈$_P$**S** **SIZ**(**S**∪{**x**}) := **SIZ**(**S‘**) ∈ ℕ.
***Proof:*** If S**IZ**(**S**) ∈ ℕ, then **S** can be expanded with primordial elements any number of times because of Section 4.2. So there must be an **x** such that **x**∈$_P$**S** and **x**∉**S**. Hence **SIZ**(**S** ∪ {**x**}):= **SIZ**(**S‘**) ∈ ℕ.

### *4.4.3 The natural numbers have a beginning.*

***Proposition:*** $\mathbf{SIZ}(\mathbf{S}) \in \mathbb{N} \Rightarrow \mathbf{SIZ}(\mathbf{S}') \neq 0$.

***Proof:*** Let $\mathbf{x} \notin \mathbf{S}$ then $\mathbf{SIZ}(\mathbf{S}') = \mathbf{SIZ}(\mathbf{S} \cup \{\mathbf{x}\}) \neq \mathbf{SIZ}(\varnothing) := 0$.

### *4.4.4 The same successors have the same predecessor.*

***Proposition:*** $\mathbf{SIZ}(\mathbf{S}_1') = \mathbf{SIZ}(\mathbf{S}_2') \Rightarrow \mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2)$.

***Proof:*** Let $\mathbf{x} \in \mathbf{S}_1$.

Because of 3.12.3, $\mathbf{x} \in \mathbf{S}_2' \wedge \mathbf{SIZ}(\mathbf{S}_1') = \mathbf{SIZ}(\mathbf{S}_2') \Rightarrow \mathbf{SIZ}(\mathbf{S}_1 \cup \{\mathbf{x}\}) = \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$.

Because of 3.12.5.1, $\mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2)$ then applies.

$\mathbf{x} \notin \mathbf{S}_2' \wedge \mathbf{SIZ}(\mathbf{S}_1') = \mathbf{SIZ}(\mathbf{S}_2') \Rightarrow \exists \mathbf{y} \in \mathbf{S}_2' \wedge \mathbf{y} \neq \mathbf{x}$

In this case, because of 3.12.1 $\mathbf{SIZ}(\mathbf{S}_2') = \mathbf{SIZ}(\mathbf{S}_2' \setminus \{\mathbf{y}\} \cup \{\mathbf{x}\}) = \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$.

From this it follows because of 3.12.3 $\mathbf{SIZ}(\mathbf{S}_1 \cup \{\mathbf{x}\} = \mathbf{SIZ}(\mathbf{S}_2 \cup \{\mathbf{x}\})$ and then because of 3.12.5.1 $\mathbf{SIZ}(\mathbf{S}_1) = \mathbf{SIZ}(\mathbf{S}_2)$.

### *4.4.5 Induction axiom.*

$( 0 \in \mathbf{C} \wedge (\mathbf{SIZ}(\mathbf{S}) \in \mathbf{C} \Rightarrow \mathbf{SIZ}(\mathbf{S}') \in \mathbf{C}) ) \Rightarrow \mathbf{C} \supseteq \mathbb{N}$

## *4.5 Note on the properties of natural numbers.*

The properties shown in 4.4 correspond to the Peano axioms and, apart from the induction axiom, are a direct consequence of the set-size developed in Section 3.12.

## *4.6 Line of natural numbers.*

***Definition:*** Let **L** be a subset of ℕ and let ∃! **b∈L ∀n∈L n‘≠b** ∃! **e∈L e‘∉L**, then we call **L** a line or an excerpt from ℕ. We call the element **b** the beginning of the line, the element **e** we call the end of the line and express this with the notation $\mathbf{L}_{b,e}$.

## *4.7 Statements about the line of natural numbers.*

### *4.7.1 Relationships at the end of a line.*

***Proposition:*** Let $\mathbf{L}_{b,e}$ be a line from ℕ, then (**e**‘)‘ ∉ $\mathbf{L}_{b,e}$.

***Proof:*** By 4.4.2 **e‘** and (**e**‘)‘ are natural numbers. Since **e‘**∉$\mathbf{L}_{b,e}$ because of 4.6, if (**e**‘)‘ ∈ $\mathbf{L}_{b,e}$, besides **b** also (**e**‘)‘ would be an element without a predecessor in $\mathbf{L}_{b,e}$, which contradicts the definition of $\mathbf{L}_{b,e}$.

### *4.7.2 Extension of a line.*

***Proposition:*** Let $\mathbf{L}_{b,e}$ be a line from ℕ, then $\mathbf{L}_{b,e}$ ∪ {**e‘**} := ($\mathbf{L}_{b,e}$)‘ := $\mathbf{L}_{b,e'}$ is also a line from ℕ and $\mathbf{L}_{b,e} \subset \mathbf{L}_{b,e'}$.

***Proof:*** First of all, **e**' exists because of 4.6 and 4.4.2.

( **e‘**∉$\mathbf{L}_{b,e}$ ∧ ($\mathbf{L}_{b,e'}$ := $\mathbf{L}_{b,e}$ ∪ {**e‘**}) ) ⇒ $\mathbf{L}_{b,e} \subset \mathbf{L}_{b,e'}$.

Since according to Section 4.7.1 (**e**‘)‘ ∉ $\mathbf{L}_{b,e}$ and **e**' is the successor of **e**, **e**' is the only element in $\mathbf{L}_{b,e'}$ without a successor.

#### *4.7.3 Equality of lines.*

***Definition:*** Two lines $\mathbf{L}_{b1,e1}$ and $\mathbf{L}_{b2,e2}$ are equal if b1 = b2 and e1 = e2.

## *4.8 Finite and infinite sets.*

#### *4.8.1 Finite sets.*

***Definition:*** A set is called finite if there is a line from ℕ such that the set can be assigned to this line bijectively.

#### *4.8.2 Infinite sets.*

***Definition:*** A set is called infinite if there is no line from ℕ such that the set can be assigned to this line bijectively.

#### *4.8.3 The set of natural numbers ℕ is infinite.*

***Proof:*** If ℕ were finite, then according to 4.8.1 there would be a line $\mathbf{L}_{b,e}$ of natural numbers and an assignment rule ***f***, which bijectively assigns $\mathbf{L}_{b,e}$ to the natural numbers ℕ. Let ***g*** be the identical assignment, then $\forall \mathbf{x} \in \mathbf{L}_{b,e}\ \mathbf{g}(\mathbf{x})=\mathbf{x}$. This assignment is injective. By 4.7.2 we can extend $\mathbf{L}_{b,e}$ to $\mathbf{L}_{b,e'}$ and we have $\mathbf{e'} \notin \mathbf{L}_{b,e} \wedge \mathbf{e'} \in \mathbb{N}$. Since ***g*** is the identical assignment, the following applies: $\forall \mathbf{x} \in \mathbf{L}_{b,e} \Rightarrow \boldsymbol{g}(\mathbf{x}) \neq \mathbf{e}'$. Thus $\boldsymbol{g}$ is injective and not surjective and because of 3.11 there is basically no bijective assignment between $\mathbf{L}_{b,e}$ and $\mathbb{N}$. Hence, by 4.8.2, $\mathbb{N}$ is an infinite set.

# *5 Persons with assigned document content.*

## *5.1 Georg Cantor*

Born on March 3rd, 1845, died on January 6th, 1918
German mathematician.

## *5.2 Bertrand Russell*

Born on May 18th, 1872, died on February 2nd, 1970.
British philosopher, mathematician, logician and writer.

## *5.3 Giuseppe Peano*

Born August 27, 1858, died April 20, 1932.
Italian mathematician.

## 6 *References.*

### *6.1 Peano‘s Axioms.*

***Weisstein, Eric W.*** ***"Peano's Axioms."***

***From MathWorld--A Wolfram Web Resource.***

***https://mathworld.wolfram.com/PeanosAxioms.html***

Quoted on 9/30/2021.

### *6.2 Cantor's definition of sets.*

***Georg Cantor.*** ***„Beiträge zur Begründung der transfiniten Mengenlehre.“***

***Mathematische Annalen, Band 46 Seite 481.***

***https://gdz.sub.uni-goettingen.de/id/PPN235181t684_0046***

Quoted on 10/5/2021.

### ***6.3 Russel‘s Antinomy.***

***Weisstein, Eric W. "Russell's Antinomy."***

***From MathWorld--A Wolfram Web Resource.***

***https://mathworld.wolfram.com/RussellsAntinomy.html***

Quoted on 10/5/2021.

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