Four-operator Algebra.

Juan Elías Millas Vera.
juanmillaszgz@gmail.com
Zaragoza (Spain)
November 2021

0. Abstract:

In this paper I am going to present a new algebra based in four different operators. As you can imagine the first three operators are the basic and existing: addition, product and power. The fourth is which I am going to develop based in the idea of Graham's number. Also I will to explain the inverse definition for the main three operators and introduce the concept of the inverse of the fourth. Even I will explain some characteristics of this four operator and its inverse.

1. Introduction:

First of all, I present some definitions of the algebraic constructions. For the sum or addition, we can see the concept based in two different states of quantity being in a fusion. So, if we got a minimal number n in a set K, we can add together some of this numbers and then add the results together how many times as we want, for example to add two quantities $a, b \in N^*$:

$$a + b = (n_1 + n_2 + ... + n_a) + (n_1 + n_2 + ... + n_b) \quad (1)$$

Where n is the minimal value of the set. In this example of N^* it will be n=1.

Secondly, for the product of two different quantities, we can understand multiplication as the consecutive summation of one number "a" a number "b" of times:

$$a \cdot b = a_1 + a_2 + ... + a_b \quad (2)$$

As we can see the first number of the product will be de base and the second the condition of development.

In third place, we can also look at the power of a base number a and an exponent power b, as the consecutive product of a for itself b times.

$$a^b = a \uparrow b \quad = \quad a_1 \cdot a_2 \cdot ... \cdot a_b \tag{3}$$

I used first exponential notation and secondly Knuth's up arrow notation. It is the same conclusion.

For negative operators we have the subtraction as inverse of the addition:

$$a - b \quad = \quad (n_1 + n_2 + ... + n_a) - (n_1 + n_2 + ... + n_b) \tag{4}$$

This is the generalization for any numbers a and b formed by basic particles n which belongs to a set.
To define the division we are going to explain an equivalence:

$$a = b_1 + b_2 + ... + b_c \tag{5}$$

For a division $a \div b$ the subindex c is the result of the division.

If we want to define the root as inverse operation of the power, we can write it as a classic root with the definition we get in [1]: A b-th root of a number a is a number $x = a^{1/b}$ whose b-th power x^b is equal to a:

$$a_0 x^b + ... + a_{b-1} x + a_b = 0 \tag{6}$$

We can write it as classic root or with the inverse of Knuth's up arrow, with a down arrow.

$$\sqrt[b]{a} = a \downarrow b \tag{7}$$

Before I explain the main concepts I am going to copy from [2] the definition of Graham's number: "Graham's number is a number invented by Ronald Graham. In order to explain what it is, the notation must be understood. It is called up-arrow notation, denoted by the $\uparrow$ symbol. One up-arrow just denotes that the second number is an exponent. For example,$3 \uparrow 3$ is 3^3, or 27. Using two arrow creates the fourth thing in the sequence of addition, multiplication, and exponentiation. Some call this math operation tetration.3 $\uparrow\uparrow$ 3 is 3^{3^3},3^{27}, or $7,625,597,484,987$. Using a third arrow, you can probably predict what happens. 3 $\uparrow\uparrow\uparrow$ is 3 $\uparrow\uparrow$ (3 $\uparrow\uparrow$ 3), or 3 $\uparrow\uparrow$ $7,625,597,484,987$. This means that you have $3^{3^{3^{...(3^3)...}}}$, and there are $7,625,597,484,987$ three's. For perspective, 3 $\uparrow\uparrow$ 4, or$3^{7,625,597,484,987}$, contains 3,638,334,640,024 digits. [..] And yet, despite how far blown out of proportion this thing has been, it is still not large enough. We need a fourth arrow. Don not even get me started on the size of 3 $\uparrow\uparrow\uparrow\uparrow$ 3, or3 $\uparrow\uparrow\uparrow$ (3 $\uparrow\uparrow\uparrow$ 3). And that number is called G(1). G(2) is3 $\uparrow\uparrow\uparrow$ ... $\uparrow\uparrow\uparrow$ 3. There are G(1) arrows. G(3) is 3 $\uparrow\uparrow\uparrow$ ... $\uparrow\uparrow\uparrow$ 3, with G(2) 3's. You get it now? Graham's number is defined as G(64)".

2. An expanded algebra.

2.1 New definitions: At this point we have defined the three basic operators and its inverses. Also the idea of consecutive powers. I can introduce now the fourth operator in positive, I recommend to name it as *property* and symbolize it with $\sqcap$.
The operator *property* following the logic of construction of the three previous operators will represent a chain of powers:

$$a \sqcap b \quad = \quad a_1 \uparrow a_2 \uparrow ... \uparrow a_b \tag{8}$$

The first number represents the base of the operator and the second number represent how many times we should raise the number by itself.

The inverse of the *property* operator is what I named as *scarcity*, it is represented with the symbol $\sqcup$. It represents how many times we should anti-raise a number by itself:

$$a \sqcup b \quad = \quad a_1 \downarrow a_2 \downarrow ... \downarrow a_b \tag{9}$$

2.2 Properties: You can see the properties of different operations in the next table:

	neutral element	commutative with itself	associative with itself
$a+b$	0	Yes	Yes
$a \cdot b$	1	Yes	Yes
$a \uparrow b$	1	No	No
$a \sqcap b$	1	No	No
	*	*	*
$a-b$	0	No	No
$a \div b$	1	No	No
$a \downarrow b$	1	No	No
$a \sqcup b$	1	No	No

2.3 Relations with the sets: To study a more complete set with operations we can classify the mathematical tools in two already existing concepts [3]: algebraic group $G(R,+)$ and ring $(R,+,\cdot)$, and two new concepts which I want to introduce (*council* $(R,+,\cdot,\uparrow)$ and *team* $(R,+,\cdot,\uparrow,\sqcap)$

2.4- Some numerical examples: Here we are going to show some numerical examples which I obtain with the help of WolframAlpha [4]:

1. No parenthesis:

$$3 \cdot 82 + 3 \uparrow 4 + 2 \sqcap 3 = \tag{10}$$
$$3 \cdot 82 + 3 \uparrow 4 + 16 = \tag{11}$$
$$3 \cdot 82 + 81 + 16 = 246 + 81 + 16 = \tag{12}$$
$$343 \tag{13}$$

2. No parenthesis and big number:

$$3 \cdot 2 \uparrow 4 + 5 \sqcap 3 \approx \tag{14}$$
$$3 \cdot 2 \uparrow 4 + 1.91 \cdot 10^{2184} \approx \tag{15}$$
$$1.91 \cdot 10^{2184} \tag{16}$$

3. Parenthesis:

$$3 \cdot (2+3) + 5 + 4 + 3 \uparrow 3 + 3 \sqcap (2+1) = \tag{17}$$
$$3 \cdot 5 + 3 + 4 + 5 \uparrow 3 + 3 \sqcap 3 = \tag{18}$$
$$3 \cdot 5 + 5 + 4 + 3 \uparrow 3 + 7,625,597,484,987 = \tag{19}$$
$$3 \cdot 5 + 5 + 4 + 27 + 7,625,597,484,987 = \tag{20}$$

$$15 + 5 + 4 + 27 + 7,625,597,484,987 = \tag{21}$$
$$7,625,597,485,038 \tag{22}$$

4. Inverse operations without parenthesis:

$$16 + 32 \cdot 3 - 25 \downarrow 2 + 4 \sqcup 2 = \tag{23}$$
$$16 + 32 \cdot 3 - 25 \downarrow 2 + 1.414... = \tag{24}$$
$$16 + 96 - 5 + 1.414... = \tag{25}$$
$$108.414... \tag{26}$$

5. Some of the symbols combined:

$$(3 - 5 + (3 \cdot 9 + 2)) \downarrow 2 - 5 \uparrow 5 + 3 + 5 \sqcup (2 \sqcap 2) = \tag{27}$$
$$(3 - 5 + 29) \downarrow 2 - 5 \uparrow 3 + 5 \sqcup 4 = \tag{28}$$
$$27 \downarrow 2 - 5 \uparrow 3 + 5 \sqcup 4 = \tag{29}$$
$$27 \downarrow 2 - 5 \uparrow 3 + 1.012... = \tag{30}$$
$$5.196... - 125 + 1.1.012... = \tag{31}$$
$$-118.790... \tag{32}$$

3. Conclusions

In this paper you have seen a two important new concepts in algebra (property and scarcity) and two secondary concepts (council and team), this four new concepts are in my opinion very interesting regardless of whether you use the words I have recommended or not. If you have had problems with the theory you always can read again the numerical examples. In the numerical examples I tried to be clear and concise in order to make the practical part known.

4. References

1. https://encyclopediaofmath.org/wiki/Root

2. https://www.urbandictionary.com/define.php?term=Graham

3. https://encyclopediaofmath.org/wiki/Abelian_group

4. https://www.wolframalpha.com/