A basis for set theory without the use of an axiom.
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Abstract
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Article

Things of any kind have properties and we can try to select this things on the basis of their properties. This will lead us to collections of selected things. They are called sets. In this article, the formation and existence of sets is based on consistent compliance with the requirements that exist when using the all-quantifier $\forall$. Without the use of an axiom, we can arrange the following:

1: If $\mathbf{x}$ has the property $\mathbf{P}$ then $\mathbf{P}(\mathbf{x})$ is true, otherwise false.

2: The term $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ means nothing else than to consider the collection of all things with the property $\mathbf{P}$. So, the all-quantifier $\forall$ is used implicitly when representing collections in this way.

3: Respecting the requirements of the all-quantifier $\forall$, a collection of things, also called a set, can only contain all things that belong to the collection and not the collection itself.

This means for the expression $\{x|P(x)\}$ that $\{x|P(x)\} \notin \{x|P(x)\}$ holds and therefore $\neg P(\{x|P(x)\})$ is always true if we are talking about a set $\{x|P(x)\}$. This is the decision criterion for the existence of a set. If we use square brackets to represent an attempt to form a set and curly braces to indicate the existence of a set and use the term $:\equiv$ for 'has per definition the same meaning', the criterion for the existence of a set is like follows:

$(\neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})]) :\equiv ([\mathbf{x}|\mathbf{P}(\mathbf{x})] \notin [\mathbf{x}|\mathbf{P}(\mathbf{x})])) \Leftrightarrow \exists\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}.$

This equivalence is inevitable if we want to use the all-quantifier $\forall$ for the formation of sets. Regarding the properties, a distinction can be made between the following categories:

1: The property is always true.

2: The property is always fslse.

3: There are objects with the property, but there are also objects without the property.

Let us consider the case one, where the property is always true.

This means $P([x|P(x)])$ is true and because of the decision criterion $[x|P(x)]$ cannot be a set. Below are some examples where this is the case.

$\mathbf{P} \equiv$ 'is an object' leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ is an object. This means the set of all things doesn't exist.

$\mathbf{P} \equiv$ 'is a set' leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ should be a set. This means the set of all sets doesn't exist.

$\mathbf{P} \equiv (\mathbf{x}=\mathbf{x})$ leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $[\mathbf{x}|\mathbf{P}(\mathbf{x})]=[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ is true. This means the set of all identities doesn't exist.

$\mathbf{P} \equiv (\mathbf{x}\notin\mathbf{x})$ leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $\mathbf{x}$ is in this contex a set and therefore, because of the decision criterion for a set, $\mathbf{x}\notin\mathbf{x}$ is always true. This means a set created with Russell's antinomy $\mathbf{x}\notin\mathbf{x}$ doesn't exist.

After formulating Russell's antinomy, $\boldsymbol{x}\notin\boldsymbol{x} \Rightarrow \boldsymbol{x}\in\boldsymbol{x}$ and $\boldsymbol{x}\in\boldsymbol{x} \Rightarrow \boldsymbol{x}\notin\boldsymbol{x}$ were inferred, which is formally correct, and thereupon the entire so-called "naive set theory" was discarded. In doing so, it was neglected to examine whether the object-selection $[\boldsymbol{x}|\boldsymbol{x}\notin\boldsymbol{x}]$ can be a set at all. As shown above this is not the case

Let us consider the case two, where the property is always false.

This means $\boldsymbol{P}([\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})])$ is false and therefore $\neg\boldsymbol{P}([\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})])$ is true. The decision criterion now says that the set $\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\}$ exist. Because $\boldsymbol{P}(\boldsymbol{x})$ is always false, this set cannot contain any element. It is the empty set $\emptyset$.

Let us consider the case three, where the property is sometimes true.

Here, if one can show that $\boldsymbol{P}([\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})])$ is false, then it's proven, that the set $\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\}$ exists.

If one can show that $\boldsymbol{P}([\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})])$ is true, then it's proven, that the object-selection $[\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})]$ cannot be a set.

Regarding the decision criterion, we can say:

Failure to observe the requirements of the all-quantifier $\forall$ inevitably does not lead to sets. Compliance with the requirements of the all-quantifier $\forall$ inevitably leads to sets.

As a result, we have a basis for set theory without the use of an axiom.