A basis for set theory without the use of an axiom.

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Abstract

Remarks on the existence of sets when they are formed with the all-quantifier.

Article

Things of any kind have properties, and we can try to select these things on the basis of their properties. This will lead us to collections of different things. This means everything in such a collection exist there exactly one times and has the selecting property, and everything that has the selecting property is present in the collection. This kind of collection is called a set in this article. See also other definitions of sets[2]. The formation and existence of sets is based on consistent compliance with the requirements that exist when using the all-quantifier $\forall$. Without the use of an axiom, we can arrange the following:

1. If $\mathbf{x}$ has the property $\mathbf{P}$, then $\mathbf{P}(\mathbf{x})$ is true, otherwise false.

2. The term $\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$ means nothing else than to consider the collection of all different things with the property $\mathbf{P}$. So, the all-quantifier $\forall$ is used implicitly when representing collections in this way.

3. Respecting the requirements of the all-quantifier $\forall$, a collection of different things, also called a set, can only contain all things that belong to the collection and not the collection itself.

This means for the expression $\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\}$ that $\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\} \notin \{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\}$ holds and therefore $\neg\boldsymbol{P}(\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\})$ is always true if we are talking about a set $\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\}$. This leads to a decision criterion for the existence of a set. If we use square brackets to represent an attempt to form a set and curly braces to indicate the existence of a set, the criterion for the existence of a set is like follows:

$\neg\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})]) \Longleftrightarrow ([\mathbf{x}|\mathbf{P}(\mathbf{x})] \notin [\mathbf{x}|\mathbf{P}(\mathbf{x})]) \Longleftrightarrow \exists\{\mathbf{x}|\mathbf{P}(\mathbf{x})\}$.

Regarding the properties, a distinction can be made between the following categories:

1. The property is always true.

2. The property is always false.

3. There are objects with the property, but there are also objects without the property.

Let us consider the case of number one, where the property is always true.

This means $\boldsymbol{P([x|P(x)])}$ *is true and because of the decision criterion* $\boldsymbol{[x|P(x)]}$ *cannot be a set. Below are some examples where this is the case.*

$\mathbf{P} \equiv$ "is an object" leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ is an object. This means the set of all things doesn't exist.

$\mathbf{P} \equiv$ "is a set" leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ should be a set. This means the set of all sets doesn't exist.

$\mathbf{P} \equiv (\mathbf{x}=\mathbf{x})$ leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $[\mathbf{x}|\mathbf{P}(\mathbf{x})]=[\mathbf{x}|\mathbf{P}(\mathbf{x})]$ is true. This means the set of all identities doesn't exist.

$\mathbf{P} \equiv (\mathbf{x}\notin\mathbf{x})$ leads to $\mathbf{P}([\mathbf{x}|\mathbf{P}(\mathbf{x})])$ since $\mathbf{x}$ is in this context a set and therefore, because of the decision criterion for a set, $\mathbf{x}\notin\mathbf{x}$ is always true. This means a set created with Russell's antinomy[1] $\mathbf{x}\notin\mathbf{x}$ doesn't exist.

After formulating Russell's antinomy, $\boldsymbol{x\notin x} \Rightarrow \boldsymbol{x\in x}$ *and* $\boldsymbol{x\in x} \Rightarrow \boldsymbol{x\notin x}$ *were inferred, which is formally correct, and thereupon the entire so-called "naive set theory" was discarded. In doing so, it was not examined whether the object selection* $\boldsymbol{[x|x\notin x]}$ *can be a set at all. As shown above, this is not the case*

Let us consider the case of number two, where the property is always false.

This means $\boldsymbol{P([x|P(x)])}$ *is false and therefore* $\neg\boldsymbol{P([x|P(x)])}$ *is true. The decision criterion now says that the set* $\boldsymbol{\{x|P(x)\}}$ *exist. Because* $\boldsymbol{P(x)}$ *is always false, this set cannot contain any element. It is the empty set* $\emptyset$.

Let us consider the case of number three, where the property is sometimes true.

Here, if one can show that $\boldsymbol{P([x|P(x)])}$ *is false, then it's proven, that the set* $\boldsymbol{\{x|P(x)\}}$ *exists.*

If one can show that $\boldsymbol{P([x|P(x)])}$ *is true, then it's proven, that the object-selection* $\boldsymbol{[x|P(x)]}$ *cannot be a set.*

Regarding the decision criterion, we can say:

Failure to observe the requirements of the all-quantifier $\forall$ does not lead to sets. Compliance with the requirements of the all-quantifier $\forall$ leads to sets.

As a result, we have a basis for set theory without the use of an axiom.

References

[1] Weisstein, Eric W. "Russell's Antinomy." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/RussellsAntinomy.html

[2] Stover, Christopher and Weisstein, Eric W. "Set." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Set.html