A Paradox of “Adjacent” Real Points and Beyond

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Abstract. We reveal adjacent real points in the real set using a concise logical reference. This raises a paradox, as the real set is believed as existing and complete. However, we prove each element in a totally ordered set has adjacent element(s). It follows that no set is dense; furthermore, since the natural numbers can also be dense under certain ordering, the set of natural numbers, which is involved with each infinite set in ZFC set theory, does not exist itself.

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Opening Words: A Paradox of “Adjacent” Real Points

It is well known that any two real points is separated by at least their midpoint. So nobody attempts to find “adjacent” real points. However, they are just there within reach of everyone.

On X-axis there is a segment, [1, 2], each point x of which is planted with a vertical segment $[0, x]$. The rightmost vertical segment is red in color and the one on X-axis is transparent whilst the others are green. When standing at zero and looking at them (in parallel perspective), a viewer sees an otherwise totally green vertical “segment” with a red top point. There is no doubt the visible red point is the point 2 of the vertical segment [0, 2].

Next remove the upper endpoint from each of the vertical segments. Then the same viewer sees an almost same “segment”. This time the new visible red point is the upper endpoint of the vertical segment [0, 2), namely, the lower “adjacent” point of the above-mentioned point 2.

This paradox defends a crucial fact, which may have been otherwise denied or obscured.

1. Introduction

We view Georg Cantor’s theory about infinity with grave suspicion [1], and have waited for an incisive proof to finally close the book on it. When encounter the paradox of “adjacent” real points, we are not surprised at all for knowing that the set $\mathbb{R}$ is full of loopholes in the first place [1, sec. 4]. However, further reflection suggests that adjacent elements are all over a totally ordered set, and as a consequence the set $\mathbb{Q}$, which we have questioned in [1, *Thought Experiment 6.1*], is illogical. Then the natural desire that follows is to have the equally questionable set $\mathbb{N}$ [1, *Thought Experiment 5.2*] involved. We achieve the goal via rearranging the natural numbers into a dense pattern, just like the rational numbers. The end result is far-reaching, eliminating a long list of paradoxes about infinity and continuum all at once.

Trivial Declaration. In this paper, all involved sets are totally ordered (linearly ordered) sets with usual order $\leq$, each with many elements. (We just take the usual $\leq$ as an example of total order, since that is enough for our purpose). In our discussion, point and number are merely different expressions of the same thing, and we switch between them for better intuition.

2. Adjacent Elements in a Totally Ordered Set

Because of the paradoxes exposed in [1, sec. 4], we are not astonished at the collapse of $\mathbb{R}$ as a set; instead, considering other paradoxes in [1], we would be disappointed if the crisis were limited to just the set $\mathbb{R}$.

2.1. Finding an Adjacent Element

Let the universe E be an arbitrarily given set with total order $\leq$. For $a_1, a_2 \in E$ with $a_1 < a_2$, we call $\{x \in E : a_1 \leq x \leq a_2\}$ an *E-segment* and denote it by $[a_1,\ a_2]_E$; and call $\{x \in E : a_1 \leq x < a_2\}$, which is denoted by $[a_1,\ a_2)_E$, an *E-segment*, too.

Side note: *An* E-segment *is a collection of ordered points in our discussion (other properties such as "denseness" or "completeness" are irrelevant). If* $E = \{1,\ 2,\ 3, \ldots, 10\}$*, then* $[2,\ 6]_E = \{2,\ 3,\ 4,\ 5,\ 6\}$ *and* $[2,\ 6)_E = \{2,\ 3,\ 4,\ 5\}$*, but* $\{2,\ 3,\ 5\}$ *is not an* E-segment.

Since, hereafter, the universal set E is always there and equally applies to horizontal and vertical directions, the subscript "E" is omitted if there is no ambiguity.

Theorem 2.1.1 *In a totally ordered set each element has at least one adjacent element. And each element that is neither the least nor greatest one has both a predecessor and a successor.*

Proof. Let x_2 be an arbitrary given element of E. We just find an adjacent element for it, and the rest is evident then.

Without loss of fairness, suppose that $0, x_1 \in E$, where $0 < x_1 < x_2$. On X-axis there is an *E-segment* $[x_1,\ x_2]$, at each point x of which roots a vertical *E-segment* $[0,\ x]$. Thus all the *E-segments* cover a vertical range of $\{y \in E : 0 \leq y \leq x_2\}$.

Side note: *We are not talking about the frequently discussed function* $f(x) = x$.

Step 1: Remove the upper endpoint from each of the vertical *E-segments*. As a consequence, the vertical range becomes $\{y \in E : 0 \leq y \leq x_2\} \setminus \{x_2\} = \{y \in E : 0 \leq y < x_2\}$, since their alignment and covering relationships determine that the vertical range is just the same as the unique tallest vertical *E-segment*. Now, the vertical *E-segment* $[0,\ x_2)$ is the unique tallest vertical one for containing the counterpart of each of the other original vertical *E-segments*, which are one point taller than their descendants respectively.

Step 2: Then remove the tallest vertical *E-segment*, namely, $[0,\ x_2)$. Hence the vertical range gets a new loss for the same reason as mentioned above.

Thus, by this time, the existence of the new loss for the vertical range is confirmed. Finally, to make it clear that the new loss is only one point, we (return to the initial conditions and) access current ending status through another path.

Step I: Limit our horizontal base to $[x_1,\ x_2)$ on X-axis, so the vertical *E-segment* $[0,\ x_2]$ is dismissed; and the vertical range becomes $\{y \in E : 0 \leq y \leq x_2\} \setminus \{x_2\} = \{y \in E : 0 \leq y < x_2\}$, since in the original vertical range the point x_2 is the only point that is exclusively contributed by the vertical *E-segment* $[0,\ x_2]$.

Step II: Remove the upper endpoint from each of the vertical *E-segments*. Here we are at the ending status again. Suppose that there are $x_3, x_4 \in E$, where $1 \leq x_3 < x_4 < x_2$, and x_3 is a newly lost point from the vertical range. As $1 < x_4 < x_2$ implies the former existence of the vertical *E-segment* $[0,\, x_4]$ and, in turn, the current existence of $[0,\, x_4)$, considering also that $1 \le x_3 < x_4$, we have that the vertical *E-segment* $[0,\, x_4)$ ensures the existence of the point x_3 for the new vertical range. A contradiction. Therefore only the lower adjacent point of x_2, if exists in E, may be a newly lost point from the range.

Combining the two paths, we conclude that the latest loss for the vertical range is a missing of one point, and the point is the lower adjacent point of x_2 in E. The final vertical range is $\{y \in E : 0 \le y < x_2\} \smallsetminus \{\text{the lower adjacent point of } x_2 \text{ in } E\}$. □

A corollary immediately follows:

Corollary 2.1.2 *No totally ordered set is a dense set.*

Aside: *The dense property implies that between any two members there is always a third one — that is our main concern in this study. The "adjacent elements" means the two elements are directly next to each other in the given set with the given ordering. It is not that we are blind to the property of a given set, but that a static set cannot afford the honor of being dense — not every set in one's imagination can really exist. We do not object the concept of denseness, but it does not apply to a set.*

(We express the above procedure more symbolically in **Appendix A**.)

2.2. The Same Story for $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$

The above reasoning is based on the total ordering of a set (that is, independent of other special characters of the set), thereby applying to the set of the real points and the set of the rational points. The set $\mathbb{R}$, which we have questioned in [1, sec. 4] for the suspicious completeness property, is the first to be trapped this time. As new evidence comes to light throughout this study, the unlikely completeness dream of the set becomes even more delusional. Certainly there are many other real points between any two adjacent rational points found in the given set E, but those in between are not in E (no matter what one might expect E to be). The set $\mathbb{Q}$ has been caught out by a paradox rising from the property of denseness [1, *Thought Experiment 6.1*] (we include the main part of that thought experiment in **Appendix B**), so it is no wonder that the set $\mathbb{Q}$ is trapped here again. Though there are many other rational points between any two adjacent rational points found in the given set E, those others are not in E. That explains the related paradox of [1, *Thought Experiment 6.1*].

It seems that the paradox of adjacent elements would not bother the set $\mathbb{N}$, as there is nothing strange for an element to have direct neighbor(s) in usual order in $\mathbb{N}$ or any of its subsets. Nevertheless the natural numbers is dense under certain ordering, and we present such an example below.

Put a decimal point and a 0 after each natural number, then reverse each expanded natural number (just like reversing a string) to make a pure decimal. Let each of the decimal fractions represent its original natural number, e.g., 0.01325 represents 52310. That defines a bijection between natural numbers and positive terminating pure decimals (in base-10 number system, for instance). When such a terminating decimal is used for representing a natural number, we call it an *N-decimal*.

Then we can make use the experience gained from dealing with normal decimals and notice that the usual ordering $\le$ of the *N-decimals* (when treated as normal decimals) is a total ordering, and between any two *N-decimals* there is always another one. It is easy to throw the *N-decimals* into trouble by simply emulating [1, *Thought Experiment 6.1*], which is partly copied as **Appendix B**. Again, the adjacent elements offer a compelling explanation.

Let the universal set E be the supposed set of $\{0.0$ and all *N-decimals*$\}$, which represents the supposed set $\{0$ and all natural numbers$\}$, so as to apply the reasoning in **Subsection 2.1** mechanically. The adjacent elements emerge in due course, thereby denying the existence of the supposed set of $\{0.0$ and all *N-decimals*$\}$ and in turn the so-called set $\mathbb{N}$ that behind it.

Aside: *Why do we eagerly disturb an otherwise "peaceful" scene? Because we have noticed something more than a suspicion that the set of the natural numbers violates the law of contradiction [1,* Thought Experiment 5.2*] [1,* p. 15, par. 1*].*

And as a consequence no "infinite set" is tenable in ZFC set theory, since there the "set of the natural numbers" is the simplest "infinite set" and every "infinite set" contains a subset equivalent with $\mathbb{N}$ (by the way, otherwise when the given set E proves cannot be interpreted as the supposed set of $\{0.0$ and "all" *N-decimals*$\}$, there would be a hope for it to be an infinite one for the possibility of having an "infinite subset" independent of $\mathbb{N}$). Therefore we conclude:

Theorem 2.2.1 *The so-called set* $\mathbb{N}$ *does not exist.*

Corollary 2.2.2 *There is no such thing as superset of a so-called countably infinite set.*

Aside: *Such being the case, where does "the set of the natural numbers" come from? So far as we know, it appears to come from nowhere but has roots in some people's personal belief. Later, in axiomatic set theory, it is introduced by* the Axiom of Infinity*, which states that* there exists an infinite set*, or in other words* there exists an inductive set*.*

3. Conclusion

At this point, things become clearer than ever before. A lot of work relating to infinity and continuum, especially the part following Cantor's ideas, needs to be rethought completely.

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Appendix

A. Expressing the Procedure in Symbols More

(For more explanation please refer to **Subsection 2.1**.)

Let the universe E be an arbitrary given set with total order $\leq$. We aim to isolate an adjacent element of a given element.

Without loss of generality, suppose that 0, x_1 and x_2 are elements of E, where $0 < x_1 < x_2$, and try to find an adjacent element of x_2.

There is a set of *E-segments* $A = \{[0,\, x] : x \in E \text{ and } x_1 \leq x \leq x_2\}$, which covers a range of $\bigcup A = \{y \in E : 0 \leq y \leq x_2\}$.

Step 1: Remove the upper endpoint of each *E-segment* of A, the stage result is $A_1 = \{[0,\, x] \setminus [x,\, x] : x \in E \text{ and } x_1 \leq x \leq x_2\} = \{[0,\, x) : x \in E \text{ and } x_1 \leq x \leq x_2\}$ and the set A_1 covers the range of $\bigcup A_1 = \{y \in E : 0 \leq y < x_2\}$; for the range, the missing element is point x_2.

Step 2: Remove the *E-segment* $[0,\, x_2)$ from A_1, the final result is $A_2 = A_1 \smallsetminus \{[0,\, x_2)\} = \{[0,\, x) : x \in E \text{ and } x_1 \le x \le x_2\} \smallsetminus \{[0,\, x) : x = x_2\} = \{[0,\, x) : x \in E \text{ and } x_1 \le x < x_2\}$ and the set A_2 covers the range of $\bigcup A_2 = \bigcup\{[0,\, x) : x \in E \text{ and } x_1 \le x < x_2\}$; for the range, there is definitely a new loss.

Now, return to the initial conditions and revisit the same final scene through another passage.

Step I: Remove the *E-segment* $[0,\, x_2]$ from A, the stage result is $A_3 = A \smallsetminus \{[0,\, x_2]\} = \{[0,\, x] : x \in E \text{ and } x_1 \le x \le x_2\} \smallsetminus \{[0,\, x] : x = x_2\} = \{[0,\, x] : x \in E \text{ and } x_1 \le x < x_2\}$ and the set A_3 covers the range of $\bigcup A_3 = \{y \in E : 0 \le y < x_2\}$; for the range, the missing element is point x_2.

Step II: Remove the upper endpoint of each *E-segment* of A_3, the final result is $A_4 = \{[0,\, x] \smallsetminus [x,\, x] : x \in E \text{ and } x_1 \le x < x_2\} = \{[0,\, x) : x \in E \text{ and } x_1 \le x < x_2\}$ and the set A_4 covers the range of $\bigcup A_4 = \bigcup\{[0,\, x) : x \in E \text{ and } x_1 \le x < x_2\}$; by logic only the lower adjacent point of x_2, if exists in E, may be a newly lost point.

The two passages together show the existence of the lower adjacent point of x_2 in E.

We denote x and its neighbors in E by $\langle x\rangle_E^{-2}$, $\langle x\rangle_E^{-1}$, x, $\langle x\rangle_E^{+1}$, $\langle x\rangle_E^{+2}$, and so on in both directions. Usually we omit the subscript "E" if the context makes it unnecessary. We have $\langle x_2\rangle^{-1} = \bigcup A_1 \smallsetminus \bigcup A_2$, or $\langle x_2\rangle^{-1} = \bigcup A_3 \smallsetminus \bigcup A_4$, that is, $\langle x_2\rangle^{-1} = \{y \in E : 0 \le y < x_2\} \smallsetminus \bigcup\{[0,\, x) : x \in E \text{ and } x_1 \le x < x_2\}$. And $\langle x_2\rangle^{-1}$ is the upper endpoint of the *E-segment* $[x_1,\, x_2)$.

Side note: *It is worth noting that* $\bigcup\{[0,\, x) : x \in E$ *and* $x_1 \le x < x_2\}$ *might be mistaken as the equivalent of* $\bigcup\{[0,\, x) : x \in E$ *and* $x_1 \le x \le x_2\}$ *or* $\bigcup\{[0,\, x] : x \in E$ *and* $x_1 \le x < x_2\}$. *(Nevertheless, the latter two equal each other.)*

Along the same line of thought, $\langle x_2\rangle^{-2}$ and others are available; and the case in the other direction is entirely analogous.

B. The Partition Paradox of Dense Set

(Below are some clips from [1, *Thought Experiment 6.1*]. As for the question mentioned, the answer is neither "Yes" nor "No".)

Containing or not?

Consider all rational numbers between (and inclusive of) 0 and 1 in the form of reduced fraction (in particular, 0 is taken as 0/1, and 1 as 1/1). We classify these rationals into two sets — set B for those have an even number as its numerator or denominator, and set D for all others. Let set $A = B \bigcup D$. We notice that between any two elements of A there are both an element of B and an element of D.

The elements of D determine a partition of A in the way that, for $d_j \in D$, the equivalence class $[d_j]$ is $\{\textit{rational number } x \in [0,\, d_j] : \forall d_i \in D\, (d_i < d_j \Rightarrow d_i < x)\}$. Of course, each equivalence class contains exactly one element of D. Our question is: Does $[d_j]$ contain any element of B?

References

[1] Zhang Ke (2021), *The Fog Covering Cantor's Paradise: Some Paradoxes on Infinity and Continuum*, figshare. Preprint. https://doi.org/10.6084/m9.figshare.16727125.v1 .