Internal Set Theory IST$^{\#}$based on gyper infinitary logic with Restricted Modus Ponens Rule.

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Abstract: In this paper we deal with set theory $\mathbf{NC}_{\infty}^{\#}$ based on gyper infinitary logic with Restricted Modus Ponens Rule.Nonconservative extensions of the canonical internal set theories IST and HST are proposed.

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1.Introduction

In this paper we deal with set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$ based on gyper infinitary logic with Restricted Modus Ponens Rule [1]-[4].The main goal of this paper is to present an nonconservative extension IST$^{\#}$ of the canonical internal set theory IST.
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1.1.Set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$.

Set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$ is formulated as a system of axioms based on bivalent hyper infinitary logic ${}^{2}L_{\infty^{\#}}^{\#}$ with restricted modus ponens rule [1]-[3],see Appendix A. The language of set theory $\mathbf{NC}_{\infty^{\#}}^{\#}$ is a first-order hyper infinitary language $L_{\infty^{\#}}^{\#}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg\,(x = y)$ and $x \notin y$ for $\neg(x \in y)$. Individual variables $x,y,z,\ldots$ of $L_{\infty^{\#}}^{\#}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x), \exists! x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x) \ \& \ \forall y(\varphi(y) \Rightarrow x = y)]. L^{\#}_{\infty^{\#}}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called non-classical sets; we shall use upper case letters $A, B, \ldots$ for such sets. For each non-classical set $A = \{x|\varphi(x)\}$ the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x, A)]$ is called the defining axioms for the non-classical set A.

Remark 1.1.1. Remind that in logic $^{2}L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ does not always guarantee that

$$\alpha, \alpha \Rightarrow \beta \vdash_{RMP} \beta \tag{1.1.1}$$

since for some α and β possible

$$\alpha, \alpha \Rightarrow \beta \nvdash_{RMP} \beta \tag{1.1.2}$$

even if the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ holds [1].

Abbreviation 1.1.2. We often write for the sake of brevity instead (2.1) by

$$\alpha \Rightarrow_{s} \beta \tag{1.1.3}$$

and we often write instead (2.2) by

$$\alpha \Rightarrow_{w} \beta. \tag{1.1.4}$$

Remark 2.2. Let A be an nonclassical set. Note that in set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$ the following true formula

$$\exists A \forall x[x \in A \Leftrightarrow \varphi(x, A)] \tag{1.1.5}$$

does not always guarantee that

$$x \in A, x \in A \Rightarrow \varphi(x, A) \vdash_{RMP} \varphi(x, A) \tag{1.1.6}$$

even if $x \in A$ holds and (or)

$$\varphi(x, A), \varphi(x, A) \Rightarrow x \in A \vdash_{RMP} x \in A; \tag{1.1.7}$$

even $\varphi(x, A)$ holds, since for nonclassical set A for some y possible

$$y \in A, \ y \in A \Rightarrow \varphi(y, A) \nvdash_{RMP} \varphi(y, A) \tag{1.1.8}$$

and (or)

$$\varphi(y, A), \ \varphi(y, A) \Rightarrow y \in A \nvdash_{RMP} y \in A. \tag{1.1.9}$$

Remark 2.3. Note that in this paper the formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x) \wedge x \in u] \tag{1.1.10}$$

and more general formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x, a) \wedge x \in u] \tag{1.1.11}$$

is considered as the defining axioms for the classical set a.

Remark 2.4. Let a be a classical set. Note that in $\mathbf{NC}^{\#}_{\infty^{\#}}$: (i) the following true formula

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x, a) \wedge x \in u] \tag{1.1.12}$$

always guarantee that

$$x \in a, x \in a \Rightarrow \varphi(x, a) \vdash_{RMP} \varphi(x) \tag{1.1.13}$$

if $x \in a$ holds and

$$\varphi(x), \varphi(x) \Rightarrow x \in a \vdash_{RMP} x \in a; \tag{1.1.14}$$

if $\varphi(x)$ holds;

In order to emphasize this fact mentioned above in Remark 2.1-2.3,

we rewrite the defining axioms in general case for the nonclassical sets in the following form

$$\exists A \forall x \{[x \in A \Leftrightarrow_s \varphi(x,A)] \vee [x \in A \Leftrightarrow_w \varphi(x,A)]\} \tag{1.1.15}$$

and similarly we rewrite the defining axioms in general case for the classical sets in the following form

$$\forall x[x \in a \Leftrightarrow_s \varphi(x,a) \wedge (x \in u)]. \tag{1.1.16}$$

Abbreviation 1.1.2. We write instead (1.1.15):

$$\forall x \{[x \in A \Leftrightarrow_{s,w} \varphi(x,A)]\} \tag{1.1.17}$$

Definition 1.1.1. (1) Let A be a nonclassical set defined by formula (1.1.17).
Assum that: (i) for some y statement $\varphi(y)$ and statement $\varphi(y) \Rightarrow y \in A$ holds and
(ii) $\varphi(y), \varphi(y) \Rightarrow y \in A \nvdash_{RMP} y \in A$, $y \in A, y \in A \Rightarrow \varphi(y) \nvdash_{RMP} \varphi(y)$.
Then we say that y is a weak member of non-classical set A and abbreviate $y \in_w A$.

Abbreviation 1.1.3. Let A be a nonclassical set defined by formula (6.1) or by formula (6.2). We abbreviate $x \in_{s,w} A$ if the following statement $x \in_s A \vee x \in_w A$ holds, i.e.

$$x \in_{s,w} A \leftrightarrow_{def} (x \in_s A \vee x \in_w A). \tag{1.1.18}$$

Definition 1.1.2. (1) Two nonclassical sets A,B are defined to be equal and we write $A = B$ if $\forall x[x \in_{s,w} A \Leftrightarrow_s x \in_{s,w} B]$. (2) A is a subset of B, and we often write $A \subset_{s,v} B$, if $\forall x[x \in_{s,w} A \Rightarrow_s x \in_{s,w} B]$. (3) We also write $\mathbf{Cl.Set}(A)$ for the formula $\exists u \forall x[x \in A \Leftrightarrow x \in u]$. (4) We also write $\mathbf{NCl.Set}(A)$ for the formulas $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x)]$ and $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x,A)]$.

Remark 2.5. $\mathbf{CL.Set}(A)$) asserts that the set A is a classical set. For any classical set u,
it follows from the defining axiom for the classical set $\{x | x \in_s u \wedge \varphi(x)\}$ that $\mathbf{CL.Set}(\{x | x \in_s u \wedge \varphi(x)\})$.
We shall identify $\{x | x \in_s u\}$ with u, so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as $u \subset_s A, u \subseteq_s A$, etc.

Abbreviation 1.1.4. Let $\varphi(t)$ be a formula of $\mathbf{NC}^{\#}_{\infty^{\#}}$.
(i) $\forall x \varphi(x)$ and $\forall^{\mathbf{CL}} x \varphi(x)$ abbreviates $\forall x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(ii) $\exists x \varphi(x)$ and $\exists^{\mathbf{CL}} x \varphi(x)$ abbreviates $\forall x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(iii) $\forall X \varphi(X)$ and $\forall^{\mathbf{NCL}} X \varphi(X)$ abbreviates $\forall X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$
(iv) $\exists X \varphi(X)$ and $\exists^{\mathbf{NCL}} X \varphi(X)$ abbreviates $\exists X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$

Remark 1.1.6. If A is a nonclassical set, we write $\exists x \in A\ \varphi(x,A)$ for $\exists x[x \in A \wedge \varphi(x,A)]$ and $\forall x \in A \varphi(x,A)$ for $\forall x[x \in A \Rightarrow \varphi(x,A)]$.

We define now the following sets:
1. $\{u_1, u_2, \ldots, u_n\} = \{x | x = u_1 \vee x = u_2 \vee \ldots \vee x = u_n\}$. 2. $\{A_1, A_2, \ldots, A_n\} =$
$= \{x | x = A_1 \vee x = A_2 \vee \ldots \vee x = A_n\}$. 3. $\cup A = \{x | \exists y[y \in A \wedge x \in y]\}$.
4. $\cap A = \{x | \forall y[y \in A \Rightarrow x \in y]\}$. 5. $A \cup B = \{x | x \in A \vee x \in B\}$.
5. $A \cap B = \{x | x \in A \wedge x \in B\}$. 6. $A - B = \{x | x \in A \wedge x \notin B\}$. 7. $u^+ = u \cup \{u\}$.

8.$\mathbf{P}(A) = \{x|x \subseteq A\}$. 9.$\{x \in A|\varphi(x,A)\} = \{x|x \in A \wedge \varphi(x,A)\}$. 10.$\mathbf{V} = \{x|: x = x\}$.
11.$\varnothing = \{x|x \neq x\}$.
The system $\mathbf{NC}^{\#}_{\infty^{\#}}$ of set theory is based on the following axioms:
Extensionality1: $\forall u \forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$
Extensionality2: $\forall A \forall B[\forall x(x \in A \Leftrightarrow_{s,w} x \in B) \Rightarrow A = B]$
Universal Set: $\mathbf{NCL.Set}(\mathbf{V})$
Empty Set: $\mathbf{CL.Set}(\varnothing)$
Pairing1: $\forall u \forall v \ \mathbf{Cl.Set}(\{u,v\})$
Pairing2: $\forall A \forall B \ \mathbf{NCl.Set}(\{A,B\})$
Union1: $\forall u \ \mathbf{CL.Set}(\cup u)$
Union2: $\forall A \ \mathbf{NCL.Set}(\cup A)$
Powerset1: $\forall u \ \mathbf{CL.Set}(\mathbf{P}(u))$
Powerset2: $\forall A \ \mathbf{NCl.Set}(\mathbf{P}(A))$
Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^{+} \in a)]$
Separation1$\forall u_1 \forall u_2,\ldots \forall u_n \forall a \exists \mathbf{Cl.Set}(\{x \in_s a|\varphi(x,u_1,u_2,\ldots,u_n)\})$
Separation2$\forall u_1 \forall u_2,\ldots \forall u_n \mathbf{NCl.Set}(\{x \in_{s,w} A|\varphi(x,A;u_1,u_2,\ldots,u_n)\})$
Comprehension1$\forall u_1 \forall u_2,\ldots \forall u_n \exists A \forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x;u_1,u_2,\ldots,u_n)]$
Comprehension 2 $\forall u_1 \forall u_2,\ldots \forall u_n \exists A \forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x,A;u_1,u_2,\ldots,u_n)]$
Comprehension 3 $\forall u_1 \forall u_2,\ldots \forall u_n \exists a \forall x[x \in_s a \Leftrightarrow_s (a \subset u_1) \wedge \varphi(x,a;u_1,u_2,\ldots,u_n)]$
In particular:
Comprehension 3$'$ $\forall u \exists a \forall x[x \in_s a \Leftrightarrow_s (a \subset u) \wedge \varphi(x,a;u)]$
Hyperinfinity: see subsection 2.1.
Remark 1.1.7. Note that the axiom of hyper infinity follows from the schemata Comprehension 3.
Definition 1.1.3. The ordered pair of two sets u,v is defined as usual by

$$\langle u,v\rangle = \{\{u\},\{u,v\}\}. \tag{1.1.19}$$

Definition 1.1.4. We define the Cartesian product of two nonclassical sets A and B as usual by

$$A \times_{s,w} B = \{\langle x,y\rangle|x \in_{s,w} A \wedge y \in_{s,w} B\} \tag{1.1.20}$$

Definition 1.1.5. A binary relation between two nonclassical sets A,B is a subset $R \subseteq_{s,w} A \times_{s,w} B$. We also write $aR_{s,w}b$ for $< a,b >\in_{s,w} R$. The doman $\mathbf{dom}(R)$ and the range $\mathbf{ran}(R)$ of R are defined by

$$\mathbf{dom}(R) = \{x|\exists y(xR_{s,w}y)\}, \mathbf{ran}(R) = \{y : \exists x(xR_{s,w}y)\}. \tag{1.1.21}$$

Definition 1.1.6.A relation $F_{s,w}$ is a function, or map, written $\mathbf{Fun}(F_{s,w})$, if for each $a \in_{s,w} \mathbf{dom}(F)$ there is a unique b for which $aF_{s,w}b$. This unique b is written $F(a)$ or Fa. We write $F_{s,w} : A \to B$ for the assertion that $F_{s,w}$ s a function with $\mathbf{dom}(F_{s,w}) = A$ and $\mathbf{ran}(F_{s,w}) = B$. In this case we write $a \mapsto F_{s,w}(a)$ for $F_{s,w}a$.
Definition 1.1.7.The identity map $\mathbf{1}_A$ on A is the map $A \to A$ given by $a \mapsto a$.
If $X \subseteq_{s,w} A$, the map $x \mapsto x : X \to A$ is called the insertion map of X into A.
Definition 1.1.8.If $F_{s,w} : A \to B$ and $X \subseteq_{s,w} A$, the restriction $F_{s,w}|X$ of $F_{s,w}$ to X is the map $X \to A$ given by $x \mapsto F_{s,w}(x)$. If $Y \subseteq_{s,w} B$, the inverse image of Y under $F_{s,w}$ is the set

$$F^{-1}_{s,w}[Y] = \{x \in_{s,w} A : F_{s,w}(x) \in_{s,w} Y\}. \tag{1.1.22}$$

Given two functions $F_{s,w} : A \to B, G_{s,w} : B \to C,$ we define the composite function

$G_{s,w} \circ F_{s,w} : A \to C$ to be the function $a \mapsto G_{s,w}(F_{s,w}(a))$. If $F_{s,w} : A \to A,$ we write $F^2_{s,w}$ for $F_{s,w} \circ F_{s,w}, F^3_{s,w}$ for $F_{s,w} \circ F_{s,w} \circ F_{s,w}$ etc.

Definition 1.1.9.A function $F_{s,w} : A \to B$ is said to be monic if for all $x, y \in_{s,w} A, F_{s,w}(x) = F_{s,w}(y)$ implies $x = y,$ epi if for any $b \in_{s,w} B$ there is $a \in_{s,w} A$ for which $b = F_{s,w}(a)$, and bijective, or a bijection, if it is both monic and epi. It is easily shown that

$F_{s,w}$ is bijective if and only if $F_{s,w}$ has an inverse, that is, a map $G_{s,w} : B \to A$ such that $F_{s,w} \circ G_{s,w} = \mathbf{1}_B$ and $G_{s,w} \circ F_{s,w} = \mathbf{1}_A$.

Definition 1.1.10.Two sets X and Y are said to be equipollent, and we write $X \approx_{s,w} Y$, if there is a bijection between them.

Definition 1.1.11.Suppose we are given two sets I, A and an epi map $F_{s,w} : I \to A$. Then $A = \{F_{s,w}(i) | i \in I\}$ and so, if, for each $i \in_{s,w} I,$ we write a_i for $F_{s,w}(i)$, then A can be presented in the form of an indexed set $\{a_i : i \in_{s,w} I\}$. If A is presented as an indexed set of sets $\{X_i | i \in_{s,w} I\},$ then we write $\bigcup_{i\in I} X_i$ and $\bigcap_{i\in I} X_i$ for $\cup A$ and $\cap A$, respectively.

Definition 1.1.12.The projection maps $\pi_1 : A \times_{s,w} B \to A$ and $\pi_2 : A \times_{s,w} B \to B$ are defined to be the maps $< a, b > \mapsto$ a and $< a, b > \mapsto b$ respectively.

Definition 1.1.13.For sets $A, B,$ the exponential B^A is defined to be the set of all functions from A to B.

Axiom of nonregularity and axiom of hyperinfinity

Axiom of nonregularity

Remind that a non-empty set u is called regular iff $\forall x[x \neq \varnothing \to (\exists y \in x)(x \cap y = \varnothing)]$.

Let's investigate what it says: suppose there were a non-empty x such that $(\forall y \in x)(x \cap y \neq \varnothing)$. For any $z_1 \in x$ we would be able to get $z_2 \in z_1 \cap x$. Since $z_2 \in x$ we would be able to get $z_3 \in z_2 \cap x$. The process continues forever: $\ldots \in z_{n+1} \in z_n \ldots \in z_4 \in z_3 \in z_2 \in z_1 \in x$.Thus if we don't wish to rule out such an infinite regress we forced accept the following statement:

$$\exists x[x \neq \varnothing \to (\forall y \in x)(x \cap y \neq \varnothing)]. \quad (1.1.23)$$

Axiom of hyperinfinity.

Definition 1.1.14.(i) A non-empty transitive non regular set u is a well formed non regular set iff:

(i) there is unique countable sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\ldots \in u_{n+1} \in u_n \ldots \in u_4 \in u_3 \in u_2 \in u_1 \in u, \quad (1.1.24)$$

(ii) for any $n \in \mathbb{N}$ and any $u_{n+1} \in u_n$:

$$u_n = u^+_{n+1}, \quad (1.1.25)$$

where $a^+ = a \cup \{a\}$.

(ii) we define a function $a^{+[k]}$inductively by $a^{+[k+1]} = (a^{+[k]})^+$

Definition 1.1.15. Let u and w are well formed non regular sets. We write $w \prec u$ iff for any $n \in \mathbb{N}$

$$w \in u_n. \tag{1.1.26}$$

Definition 1.1.16. We say that an well formed non regular set u is infinite (or hyperfinite) hypernatural nuber iff:

(I) For any member $w \in u$ one and only one of the following conditions are satified:

(i) $w \in \mathbb{N}$ or

(ii) $w = u_n$ for some $n \in \mathbb{N}$ or

(iii) $w \prec u$.

(II) Let ${}_{\prec}u$ be a set ${}_{\prec}u = \{z | z \prec u\}$, then by relation $(\bullet \prec \bullet)$ a set ${}_{\prec}u$ is densely ordered with no first element.

(III) $\mathbb{N} \subset u$.

Definition 1.1.17. Assume $u \in \mathbb{N}^{\#}$, then u is infinite (hypernatural) number if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$.

Axiom of hyperinfinity

There exists unique set $\mathbb{N}^{\#}$ such that:

(i) $\mathbb{N} \subset \mathbb{N}^{\#}$

(ii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number v such that $v \prec u$

(iii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number w such that $u \prec w$

(v) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\bullet \prec \bullet)$ with no first and no last element.

1.2.Hypernaturals $\mathbb{N}^{\#}$.

In this section nonstandard arithmetic $\mathbf{A}^{\#}$ related to hypernaturals $\mathbb{N}^{\#}$ is considered axiomatically.

Axioms of the nonstandard arithmetic $\mathbf{A}^{\#}$ are:

Axiom of hyperinfinity

There exists unique set $\mathbb{N}^{\#}$ such that:

(i) $\mathbb{N} \subset \mathbb{N}^{\#}$

(ii) if u is infinite (hypernatural) number then there exists infinite (hypernatural) number v such that $v \prec u$

(iii) if u is infinite hypernatural number then there exists infinite (hypernatural) number w such that $u \prec w$

(iv) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\bullet \prec \bullet)$ with no first and no last element.

Axioms of infite ω-induction

(i)

$$\forall S(S \subset \mathbb{N})\left\{\left[\bigwedge_{n \in \omega}(n \in S \Rightarrow_s n^+ \in S)\right] \Rightarrow_s S = \mathbb{N}\right\}. \tag{1.2.1}$$

(ii) Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$, then

$$\left[\bigwedge_{n \in \omega}(F(n) \Rightarrow_s F(n^+))\right] \Rightarrow_s \forall n(n \in \omega)F(n). \tag{1.2.2}$$

Definition 1.2.1.(i) Let β be a hypernatural such that $\beta \in \mathbb{N}^{\#} \backslash \mathbb{N}$. Let $[0,\beta] \subset \mathbb{N}^{\#}$ be a set such that $\forall x[x \in [0,\beta] \Leftrightarrow 0 \leq x \leq \beta]$ and let $[0,\beta)$ be a set $[0,\beta) = [0,\beta] \backslash \{\beta\}$.

(ii) Let $\beta \in \mathbb{N}^{\#} \backslash \mathbb{N}$ and let $\beta_\infty \subset \mathbb{N}^{\#}$be a set such that

$$\forall x\{x \in \beta_\infty \Leftrightarrow \exists k(k \geq 0)[0 \leq x \leq \beta^{+[k]}]\}. \tag{1.2.3}$$

Definition 1.2.2.Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x.We will say that a wff $F(x)$ is restricted on a classical set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff the following condition is satisfied

$$\forall\alpha[\alpha \in \mathbb{N}^{\#}\backslash S \Rightarrow_s \neg F(\alpha)]. \tag{1.2.4}$$

Definition 1.2.3.Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x.We will say that a wff $F(x)$ is strictly restricted on a set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff there is no proper subset $S' \subset S$ such that a wff $F(x)$ is restricted on a set S'.

Example 1.2.1.(i)Let $\mathbf{fin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{fin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}$. Obviously wff $\mathbf{fin}(\alpha)$ is strictly restricted on a set $\mathbb{N}$ since $\forall\alpha[\alpha \in \mathbb{N}^{\#}\backslash\mathbb{N} \Rightarrow_s \neg\mathbf{fin}(\alpha)]$.
Let $\mathbf{hfin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{hfin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}^{\#}\backslash\mathbb{N}$ since $\forall\alpha[\alpha \in \mathbb{N} \Rightarrow_s \neg\mathbf{hfin}(\alpha)]$.

Definition 1.2.4. Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x.We will say that a

wff $F(x)$ is unrestricted if wff $F(x)$ is not restricted on any set S such that $S \subsetneqq \mathbb{N}^{\#}$.

Axiom of hyperfinite induction 1

$$\forall S(S \subseteq_s [0,\beta])\forall\beta(\beta \in_s \mathbb{N}^{\#}) \searrow \left\{\forall\alpha(\alpha \in_s [0,\beta))\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^{+} \in_s S)\right] \Rightarrow_s S = [0,\beta]\right\}. \tag{1.2.5}$$

Axiom of hyperfinite induction 1′

$$\forall S(S \subseteq_s [0,\beta_\infty])\forall\beta(\beta \in \mathbb{N}^{\#}) \searrow \left\{\forall\alpha(\alpha \in [0,\beta_\infty])\left[\bigwedge_{0\leq\alpha<\beta_\infty}(\alpha \in S \Rightarrow \alpha^{+} \in S)\right] \Rightarrow S = [0,\beta_\infty]\right\}. \tag{1.2.6}$$

Axiom of hyper infinite induction 1

$$\forall S(S \subset_s \mathbb{N}^{\#})\left\{\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^{+} \in_s S)\right] \Rightarrow_s S =_s \mathbb{N}^{\#}\right\}. \tag{1.2.7}$$

Definition 1.2.5.A set $S \subset_s \mathbb{N}^{\#}$is a hyper inductive if the following statement holds

$$\bigwedge_{\alpha\in\mathbb{N}^{\#}}(\alpha \in_s S \Rightarrow_s \alpha^{+} \in_s S). \tag{1.2.8}$$

Obviously a set $\mathbb{N}^{\#}$ is a hyper inductive. Thus axiom of hyper infinite induction 1 asserts that a set $\mathbb{N}^{\#}$ this is the smallest hyper inductive set.

Axioms of hyperfinite induction 2

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta]$ then

$$\left[\forall\beta(\beta \in [0,\beta])\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta])F(\alpha). \tag{1.2.9}$$

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta_\infty]$ then

$$\left[\forall\beta(\beta \in [0,\beta_\infty])\left[\bigwedge_{0\leq\alpha<\beta_\infty}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta_\infty])F(\alpha). \tag{1.2.10}$$

Axiom of hyper infinite induction 2

Let $F(x)$ be anrestricted wff of the set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$ then

$$\left[\forall\beta(\beta\in\mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha)\Rightarrow_s F(\alpha^{+}))\right]\right]\Rightarrow_s \forall\beta(\beta\in\mathbb{N}^{\#})F(\beta). \quad (1.2.11)$$

The main restricted rules of conclusion.
If $\mathbf{A}^{\#}\vdash A$ then $\neg A\nvdash B,$ where $B\in\mathcal{L}^{\#}$.
Thus if statement A holds in $\mathbf{A}^{\#}$ we cannot obtain from $\neg A$ any formula B whatsoever.

1.3. Nonconservative extension of the model theoretical NSA

Remind that Robinson nonstandard analysis (RNA) many developed using set-theoretical objects called superstructures [5-7]. A superstructure $\mathbf{V}(S)$ over a set S is defined in the following way:

$$\mathbf{V}_0(S)=S,\mathbf{V}_{n+1}(S)=\mathbf{V}_n(S)\cup(P(\mathbf{V}_n(S)),\mathbf{V}(S)=\bigcup_{n\in\mathbb{N}}\mathbf{V}_n(S). \quad (1.3.1)$$

Superstructures of the empty set consist of sets of infinite rank in the cumulative hierarchy and therefore do not satisfy the in…nity axiom. Making $S=\mathbb{R}$ will suffice for virtually any construction necessary in analysis.
Bounded formulas are formulas where all quantifiers occur in the form

$$\forall x(x\in y\Rightarrow\cdots),\exists x(x\in y\Rightarrow\cdots). \quad (1.3.2)$$

A nonstandard embedding is a mapping

$$*:\mathbf{V}(X)\to\mathbf{V}(Y) \quad (1.3.3)$$

from a superstructure $\mathbf{V}(X)$ called the standard universum, into another superstructure

$\mathbf{V}(Y),$ called nonstandard universum, satisfying the following postulates:
1. $Y={}^{*}X$
2.**Transfer Principle**.For every bounded formula $\Phi(x_1,\ldots,x_n)$ and elements $a_1,\ldots,a_n\in\mathbf{V}(X),$ the property Φ is true for $a_1,\ldots,a_n$ in the standard universum if and only if it is true for ${}^{*}a_1,\ldots,{}^{*}a_n$ in the nonstandard universum:

$$\langle\mathbf{V}(X),\in\rangle\vDash\Phi(a_1,\ldots,a_n)\Leftrightarrow\langle\mathbf{V}(Y),\in\rangle\vDash\Phi({}^{*}a_1,\ldots,{}^{*}a_n). \quad (1.3.4)$$

3.**Non-triviality**.For every infinite set A in the standard universum, the set $\{{}^{*}a|a\in A\}$ is a proper subset of ${}^{*}A$.
Definition 1.2.1.[6]-[9].A set x is <u>internal</u> if and only if x is an element of ${}^{*}A$ for some element A of $\mathbf{V}(\mathbb{R})$. Let X be a set with $A=\{A_i\}_{i\in I}$ a family of subsets of X. Then the collection A has the infinite intersection property, if any infinite subcollection $J\subset I$ has non-empty intersection. Nonstandard universum is κ-saturated if whenever $\{A_i\}_{i\in I}$is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to $\kappa,\bigcap_{i\in I}A_i\neq\varnothing$.

Remark 1.3.1.Remind that: (i) for each standard universum $U=\mathbf{V}(X)$ there exists canonical language $\mathcal{L}=\mathcal{L}_U,$(ii) for each nonstandard universum $W=\mathbf{V}(Y)$ there exists corresponding canonical nonstandard language ${}^{*}\mathcal{L}=\mathcal{L}_W$ [6]-[9].
3*.**The restricted rules of conclusion**.
If $W\vDash A$ then $\neg A\nvdash B,$where $B\in\mathcal{L}\wedge B\in{}^{*}\mathcal{L}$.

Thus if A holds in W we cannot obtain from $\neg A$ any formula B whatsoever.

Remark 1.3.2. We write $* \models A$ instead $W \models A$.

Definition 1.3.2.[2]-[4].A set $S \subset {}^*\mathbb{N}$is a <u>hyper inductive</u> if the following statement holds

$$\bigwedge_{\alpha \in {}^*\mathbb{N}} (\alpha \in S \Rightarrow \alpha^+ \in S), \tag{1.3.5}$$

where $\alpha^+ \triangleq \alpha + 1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.As we see later there is just one hyper inductive subset of ${}^*\mathbb{N}$,namely ${}^*\mathbb{N}$itself.

In this paper we apply the following hyper inductive definitions of a sets [2]-[3]

$$\exists S \forall \beta \left[\beta \in S \Leftrightarrow \bigwedge_{0 \leq \alpha < \beta} (\alpha \in S \Rightarrow \alpha^+ \in S) \right], \tag{1.3.6}$$

We extend up Robinson nonstandard analysis (**RNA**) by adding the following postulate:

4.Any hyper inductive set S is internal.

Remark 1.3.1.The statement **4** is not provable in ZFC but provable in set theory $\mathbf{NC}^{\#}_{\infty}$, see [2]-[3].Thus postulates 1-4 gives an nonconservative extension of RNA and we denote such extension by **NERNA**.

Remark 1.3.2.Note that NERNA of course based on the same gyper infinitary logic with

Restricted Modus Ponens Rule as set theory $\mathbf{NC}^{\#}_{\infty}$ [1]-[3].

Remind that in RNA the following induction principle holds.

Theorem 1.3.1.[6]. Assume that $S \subset {}^*\mathbb{N}$ is internal set, then

$$(1 \in S) \wedge \forall x [x \in S \Rightarrow x + 1] \Rightarrow S = {}^*\mathbb{N}. \tag{1.3.7}$$

In NERNA Theorem 1.1also holds.

Remark 1.3.3.It follows from postulate 4 and Theorem 1.1 that any hyper inductive set S is equivalent to ${}^*\mathbb{N}$: $S \equiv {}^*\mathbb{N}$.

Remark 1.3.4. Note that the following statement is provable in $\mathbf{NC}^{\#}_{\infty}$ [2-3]:

4′Axiom of hyper infinite induction

$$\forall S (S \subset {}^*\mathbb{N}) \left\{ \forall \beta (\beta \in {}^*\mathbb{N}) \left[\bigwedge_{0 \leq \alpha < \beta} (\alpha \in S \Rightarrow \alpha^+ \in S) \right] \Rightarrow S = {}^*\mathbb{N} \right\}. \tag{1.3.8}$$

Thus postulate **4 of the theory** NERNA is provable in $\mathbf{NC}^{\#}_{\infty^{\#}}$.

Rules of conclusion

MRR (Main Restricted rule of conclusion)

Let $\varphi(x)$ be a wff with one free variable x and such that $(n \in {}^*\mathbb{N} \backslash \mathbb{N}) \wedge \mathbf{V}(Y) \models \varphi(n)$, then$\neg\varphi(n) \nvdash B$,i.e., if statement A holds in $\mathbf{V}(Y)$ we cannot obtain from $\neg A$ any formula B whatsoever.

Remark 1.3.5.The MRR is necessarily in natural way, since by assumption $\neg\varphi(n)$ one obtains directly the apparent contradiction $\varphi(n) \wedge \neg\varphi(n)$ from which by unrestricted modus ponens rule (UMPR) one obtains $\varphi(n) \wedge \neg\varphi(n) \vdash_{UMPR} B$.

Example 1.3.1. Remind the proof of the following statement: structure $(\mathbb{N}, <)$ is a well-ordered set.

Proof.Let X be a nonempty subset of $\mathbb{N}$. Suppose X does not have a $<$ -least element. Then consider the set $\mathbb{N} \backslash X$.

Case (1) $\mathbb{N} \backslash X = \varnothing$. Then $X = \mathbb{N}$ and so 0 is a $<$ -least element. Contradiction.

Case (2) $\mathbb{N}\backslash X \neq \varnothing$.Then $1 \in \mathbb{N}\backslash X$ otherwise 1 is a $<$ -least element. Contradiction.
Case (3) $\mathbb{N}\backslash X \neq \varnothing$. Assume now that there exists an $n \in \mathbb{N}\backslash X$ such that $n \neq 1$.
Since we have supposed that X does not have a least element, thus $n+1 \notin X$.
Thus we see that for all $n : n \in \mathbb{N}\backslash X$ implies that $n+1 \in \mathbb{N}\backslash X.$ We can conclude by induction that $n \in \mathbb{N}\backslash X$ for all $n \in \mathbb{N}.$ Thus $\mathbb{N}\backslash X = \mathbb{N}$ implies $X = \varnothing$.
This is a contradiction to X being a nonempty subset of $\mathbb{N}$.
We set now $X_1 = {}^*\mathbb{N}\backslash\mathbb{N}$, thus ${}^*\mathbb{N}\backslash X_1 = \mathbb{N}$. In contrast with a set X the assumption $n \in {}^*\mathbb{N}\backslash X_1$ implies that $n+1 \in {}^*\mathbb{N}\backslash X_1$ if and only if n is finite, since for any infinite $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ the assumption $n \in {}^*\mathbb{N}\backslash X_1$ contradicts with a true statement $\mathbf{V}(Y) \models n \notin {}^*\mathbb{N}\backslash X_1 = \mathbb{N}$ and therefore in accordance with MRR we cannot obtain from $n \in {}^*\mathbb{N}\backslash X_1$ any formula B whatsoever.

2.IST$^{\#}$ and BST$^{\#}$

2.1.Internal Set Theory IST.

The axiomatics IST (Internal Set Theory) was presented in 1977 [8] and in a sense formulates within first-order language the behaviour of standard and internal sets of a nonstandard model of ZFC. This were done by adding the unary standardness predicate "st" to the language of ZFC as well as adding to the axioms of ZFC three new axiom schemes involving the predicate "st": **Idealization**, **Standardization** and **Transfer**.

Remark 2.1.1. Formulas which do not use the predicate $\mathbf{st}$ are called <u>internal</u> formulas (or $\in$-formulas) and formulas that use this new predicate are called <u>external</u> formulas (or st-$\in$-formulas).A formula φ is standard if only standard constants occur in φ.

Abbreviaion 2.1.1. We denote a set of the all naturals by $\mathbb{N}^{\#}$and a set of the all finite naturals by $\mathbb{N}$.

Abbreviaion 2.1.2. We write $\mathbf{fin}(x)$ meaning 'x is finite'. Let $\varphi(x)$ be a $\mathbf{st}$- $\in$ -formula:
1.$\forall^{\mathbf{st}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow \varphi(x))$. 2.$\exists^{\mathbf{st}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \varphi(x))$.
3.$\forall^{\mathbf{fin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \Rightarrow \varphi(x))$. 4.$\exists^{\mathbf{fin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{fin}(x) \wedge \varphi(x))$.
5.$\forall^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \Rightarrow \varphi(x))$.
6.$\exists^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \mathbf{fin}(x) \wedge \varphi(x))$.

The fundamental axioms of $\mathbf{IST}$:

(**I**) **Idealization**

$$\forall^{\mathbf{stfin}} F \exists y \forall x \in F[R(x,y) \Leftrightarrow \exists b \forall^{\mathbf{st}} x R(x,b)] \qquad (2.1.1)$$

for any internal relation R.

Remark 2.1.2. The idealization axiom obviously states that saying that for any fixed finite set F there is a y such that $R(x,y)$ holds for all $x \in F$ is the same as saying that there is a b such that for all fixed x the relation $R(x,b)$ holds.

(**II**) **Standardization**

$$\forall^{\mathbf{st}} A \exists^{\mathbf{st}} B \forall^{\mathbf{st}} x(x \in B \Leftrightarrow x \in A \wedge \varphi(x)) \qquad (2.1.2)$$

for every st-$\in$-formula φ with arbitrary (internal) parameters.

(**III**) **Transfer**

$$\forall^{\mathbf{st}} y_1,\ldots,y_n \forall^{\mathbf{st}} x[\varphi(x,y_1,\ldots,y_n)] \Rightarrow \forall x \varphi(x,y_1,\ldots,y_n) \qquad (2.1.3)$$

for all internal $\varphi(x,y_1,\ldots,y_n)$.

Remark 2.1.3. An importent consequence of (**I**) is the principle of **External Induction**, which states that for any (external or internal) formula φ,one has

$$\varphi(0) \wedge [\forall^{\mathbf{st}} n(\varphi(n) \Rightarrow \varphi(n+1))] \Rightarrow \forall^{\mathbf{st}} n\varphi(n). \quad (2.1.4)$$

Boundedness

$$\forall x \exists^{\mathbf{st}} y(x \in y) \quad (2.1.5)$$

and since (2.5) contradicts idealization the following (bounded) form is taken instead:

(**IV**) **Bounded Idealization**

For every $\in$-formula R :

$$\forall^{\mathbf{st}} Y[\forall^{\mathbf{stfin}} F \exists y \in Y(\forall x \in FR(x,y) \Leftrightarrow \exists b(b \in Y)\forall^{\mathbf{st}} xR(x,b))]. \quad (2.1.6)$$

This gives a subsystem BST, which corresponds to the bounded sets of IST.

2.2.Internal Set Theory IST$^{\#}$

The axiomatics IST$^{\#}$ formulates within infinitary first-order language the behaviour of standard and internal sets of a nonstandard model of NC$^{\#}_{\infty^{\#}}$. This done by adding the unary standardness predicate "st" to the language of NC$^{\#}_{\infty^{\#}}$ as well as adding to the axioms of NC$^{\#}_{\infty^{\#}}$ three new axiom schemes involving the predicate "st":
Idealization, **Standardization**,**Transfer** and **Axiom of internal hyper infinite induction**.

Remark 2.2.1.Formulas which do not use the predicate $\mathbf{st}$ are called internal formulas (or $\in_{sw}$ -formulas) and formulas that use this new predicate are called external formulas (or st-$\in_{sw}$ -formulas).A formula φ is standard if only standard constants occur in φ.

Abbreviaion 2.2.1.We write $\mathbf{fin}(x)$ meaning 'x is finite'. Let $\varphi(x)$ be a $\mathbf{st}$- $\in_{sw}$ -formula:

1.$\forall^{\mathbf{st}}_{s} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow_{s} \varphi(x))$.

2.$\forall^{\mathbf{st}}_{s,w} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \Rightarrow_{s,w} \varphi(x))$.

3.$\exists^{\mathbf{st}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \varphi(x))$.

4.$\forall^{\mathbf{fin}}_{s} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \Rightarrow_{s} \varphi(x))$.

5.$\forall^{\mathbf{fin}}_{s,w} x\varphi(x)$ abbreviates $\forall x(\mathbf{fin}(x)) \Rightarrow_{s,w} \varphi(x))$.

6.$\exists^{\mathbf{fin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{fin}(x) \wedge \varphi(x))$.

7.$\forall^{\mathbf{stfin}}_{s} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \Rightarrow_{s} \varphi(x))$.

8.$\forall^{\mathbf{stfin}}_{s,w} x\varphi(x)$ abbreviates $\forall x(\mathbf{st}(x) \wedge \mathbf{fin}(x)) \Rightarrow_{s,w} \varphi(x))$.

9.$\exists^{\mathbf{stfin}} x\varphi(x)$ abbreviates $\exists x(\mathbf{st}(x) \wedge \mathbf{fin}(x) \wedge \varphi(x))$.

The fundamental axioms of $\mathbf{IST}^{\#}$:

(**I**) **Idealization for classical sets**

$$\forall^{\mathbf{stfin}}_{s} F^{\mathbf{CL}} \exists y^{\mathbf{CL}} \forall x^{\mathbf{CL}} \in_{s} F[R^{\mathbf{CL}}(x,y) \Leftrightarrow_{s} \exists b^{\mathbf{CL}} \forall^{\mathbf{st}}_{s} xR^{\mathbf{CL}}(x,b)] \quad (2.2.1)$$

for any internal classical relation $R^{\mathbf{CL}}(x,y)$.

Remark 2.2.2.The idealization axiom obviously states that saying that for any fixed classical finite set F there is a classical y such that $R^{\mathbf{CL}}(x,y)$ holds for all classical $x \in_{s} F$ is the same as saying that there is a classical b such that for all fixed classical x the classical relation $R^{\mathbf{CL}}(x,b)$ holds.

(**II**) **Standardization for classical sets**

$$\forall^{\mathbf{st}} A^{\mathbf{CL}} \exists^{\mathbf{st}} B^{\mathbf{CL}} \forall^{\mathbf{st}} x^{\mathbf{CL}} (x \in B \Leftrightarrow_{s} x \in A \wedge \varphi(x)) \quad (2.2.2)$$

for every st-$\in$-formula φ with arbitrary (internal) parameters.

(**III**) **Transfer for classical sets**

$$\forall^{\mathbf{st}} y_1^{\mathbf{CL}}, \ldots, y_n^{\mathbf{CL}} \forall^{\mathbf{st}} x^{\mathbf{CL}} [\varphi(x, y_1, \ldots, y_n)] \Rightarrow_s \forall x^{\mathbf{CL}} \varphi(x, y_1, \ldots, y_n) \qquad (2.2.3)$$

for all internal $\varphi(x, y_1, \ldots, y_n)$.

Boundedness

$$\forall x^{\mathbf{CL}} \exists^{\mathbf{st}} y^{\mathbf{CL}} (x \in_s y) \qquad (2.2.4)$$

and since (2.2.4) contradicts idealization the following (bounded) form is taken instead:

(**IV**) **Bounded Idealization for classical sets**

For every $\in$-formula R :

$$\forall^{\mathbf{st}} Y^{\mathbf{CL}} [\forall^{\mathbf{stfin}} F^{\mathbf{CL}} \exists y^{\mathbf{CL}} \in Y (\forall x^{\mathbf{CL}} (x \in F) R(x, y) \Leftrightarrow_s \exists b^{\mathbf{CL}} (b \in Y) \forall^{\mathbf{st}} x R(x, b))]. \qquad (2.2.5)$$

(**V**) **Idealization for nonclassical sets**

$$\forall_{s,w}^{\mathbf{stfin}} F^{\mathbf{NCL}} \exists y^{\mathbf{NCL}} \forall x^{\mathbf{NCL}} \in_{s,w} F [R^{\mathbf{NCL}}(x, y) \Leftrightarrow_{s,w} \exists b^{\mathbf{NCL}} \forall_{s,w}^{\mathbf{st}} x R^{\mathbf{NCL}}(x, b)] \qquad (2.2.6)$$

for any internal nonclassical relation $R^{\mathbf{NCL}}(x, y)$.

Remark 2.2.3.The idealization axiom obviously states that saying that for any fixed nonclassical finite set F there is a classical y such that $R^{\mathbf{NCL}}(x, y)$ holds for all classical $x \in_s F$ is the same as saying that there is a classical b such that for all fixed classical x the nonclassical relation $R^{\mathbf{NCL}}(x, b)$ holds.

(**VI**) **Standardization for nonclassical sets**

$$\forall_{s,w}^{\mathbf{st}} A^{\mathbf{NCL}} \exists^{\mathbf{st}} B^{\mathbf{NCL}} \forall_{s,w}^{\mathbf{st}} x^{\mathbf{NCL}} (x \in_{s,w} B \Leftrightarrow_{s,w} x \in_{s,w} A \wedge \varphi(x)) \qquad (2.2.7)$$

for every st-$\in_{s,w}$ -formula φ with arbitrary (internal) parameters.

(**VII**) **Transfer for nonclassical sets**

$$\forall_{s,w}^{\mathbf{st}} y_1^{\mathbf{NCL}}, \ldots, y_n^{\mathbf{NCL}} \forall^{\mathbf{st}} x^{\mathbf{NCL}} [\varphi(x, y_1, \ldots, y_n)] \Rightarrow_{s,w} \forall_{s,w} x^{\mathbf{NCL}} \varphi(x, y_1, \ldots, y_n) \qquad (2.2.8)$$

for all internal $\varphi(x, y_1, \ldots, y_n)$.

Boundedness for nonclassical sets

$$\forall_{s,w} x^{\mathbf{NCL}} \exists^{\mathbf{st}} y^{\mathbf{NCL}} (x \in_{s,w} y) \qquad (2.2.9)$$

and since (2.2.9) contradicts idealization the following (bounded) form is taken

instead:

(**VIII**) **Bounded Idealization for nonclassical sets**

For every $\in_{s,w}$ -formula R :

$$\begin{gathered}\forall_{s,w}^{\mathbf{st}} Y^{\mathbf{NCL}} \Big[\forall_{s,w}^{\mathbf{stfin}} F^{\mathbf{NCL}} \exists y^{\mathbf{NCL}} \in_{s,w} Y (\forall_{s,w} x^{\mathbf{NCL}} (x \in F) R(x, y) \Leftrightarrow_{s,w} \\ \exists b^{\mathbf{NCL}} (b \in Y) \forall_{s,w}^{\mathbf{st}} x R(x, b))\Big].\end{gathered} \qquad (2.2.10)$$

(**IX**) **Internal Induction**

$$\forall S (S \subset_s \mathbb{N}^{\#}) \left\{ \forall \beta (\beta \in \mathbb{N}^{\#}) \left[\bigwedge_{0 \leq \alpha < \beta} (\alpha \in_s S \Rightarrow \alpha^+ \in_s S) \right] \Rightarrow_s S =_s \mathbb{N}^{\#} \right\}. \qquad (2.2.11)$$

The main restricted rules of conclusion.

If $\mathbf{IST}^{\#} \vdash A$ then $\neg A \nvdash B,$ where $B \in \mathcal{L}^{\#}$.

Thus if statement A holds in $\mathbf{IST}^{\#}$ we cannot obtain from $\neg A$ any formula B whatsoever.

3.External Set Theory HST$^{\#}$.

3.1.External Set Theory HST.

A "perfect" external set theory (a nonstandard set theory that includes external sets) should satisfy some requirements:
(**I**) It should be a conservative extension
of classical mathematics (usually ZFC) so that all classical mathematical theorems and constructions remain valid.
(**II**)The theory should also allow to perform
nonstandard constructions in its full generality and therefore include a strong version of saturation (called idealization in IST and bounded idealization in BST) and transfer principles.
(**III**) Finally it should allow to build, for any given
set, the standard set of all its standard elements. This is called standardization. This means that ideally it should be something like an extension of IST allowing external sets and quantification over external formulas. However, as pointed out by Hrbácek [10] such a theory cannot exist. In fact, the axiom of regularity cannot be extended to the external universe. To see that let $\mathbb{R}_{\infty}^{\#}$ denote the external set of infinitely large real numbers. Observe that for all ! in the (nonempty) external set $\mathbb{R}_{\infty}^{\#} \cap \mathbb{N}$, one has $\mathbb{R}_{\infty}^{\#} \cap \mathbb{N} \cap \omega \neq \varnothing$. Additionally, if one wishes to formulate a nonstandard set theory with IST-style saturation 4 , the replacement axiom in the external universe contradicts both power set and choice. Let n be a nonstandard natural number. By saturation there is a 1-1 embedding into n, for all ordinals. So by power set and transfer the class Ord is a set (see Theorem 1.3.9 and Remark 1.3.10 in [10]).

Remark 3.1.1. To be of standard size means to be an image of the set of all standard elements of a standard set (In HST,a set X is standard size if and only if X is well-ordered) .To see that choice fails, let x be well-ordered by a relation $\prec$. Consider the class of all standard ordinals ${}^{\sigma}Ord$, well-ordered by $\in$. We use the theorem that whenever two sets are well-ordered there is an order preserving embedding of one into the other (see Theorem 2.8 in [11]). Clearly ${}^{\sigma}Ord$ cannot be embedded into x, otherwise ${}^{\sigma}Ord$ would be a set. Then there is an embedding of x into Ord. In fact, to an initial segment of ${}^{\sigma}Ord$. This means that x is of standard size.

Remark 3.1.2. As a consequence, sets which are not of standard size cannot be well-ordered (see Theorem 1.3.1 in [10]). These results are known as the Hrbácek's paradoxes.

The first problem is not in fact a "real" problem because the regularity axiom is given so that every set is obtained at some level of the cumulative hierarchy over $\varnothing$ as mentioned in Section 1.2 and has no great impact on which theorems are true. This "nice picture" of the universe is contested by some mathematicians and a suitable anti-foundation axiom can be taken instead (see for example [2],[3],[12]).
In [9] Hrbácek considered already two possibilities to avoid this. The first

one was to lose both power set and choice for external sets, leading to the system NS_1 . The second one was to lose the replacement axiom for external sets,

which lead to his theory NS_2
A third possibility was developed by Kanovei
and Reeken (see Part 3 of [32] and chapter 6 of [33]). The idea is to restrict saturation by a standard infinite cardinal in order to reinstate the power set axiom. This is a system of partially saturated external sets which modifies the system HST (described below), called HST_k . This may be a solution for many practical purposes but not a solution as a foundational system for the nonstandard methods.
The theory BST possesses an extension to HST [33] [34], which formulates within first-order language essential aspects of the behaviour of standard, internal and external sets within a nonstandard model, much as in Hrbácek's system NS_1 . The system HST is conservative over ZFC [27] [32] and equiconsistent with both BST and ZFC (see Chapter 5 of [34]).
A set in HST is called internal if it is element of a standard set (see also the "Boundedness" axiom).
Remark 3.1.3. Below we use (definable) classes, they only should be interpreted as abbreviations of formulas with sets. Two important definable classes in HST are the class of standard sets

$$S = \{x|\mathbf{st}(x)\} \qquad (3.1.1)$$

and the class of internal sets

$$\mathbf{I} = x|\exists y(st(y) \wedge x \in y) \qquad (3.1.2)$$

3.2.HST Axioms

(**I**) **Axioms for all sets**.
The axioms of this group are valid for all sets. These axioms are similar to the respective ones of ZFC with the diference that in HST they are presented in the full language. This implies in particular, by the axiom of separation, that the theory HST deals with external sets; for example if X is standard and infinite, then $\{x \in X|\mathbf{st}(x)\}$ is an external set.
1.Extensionality
$\forall X\forall Y(\forall x(x \in X \Leftrightarrow x \in Y) \Rightarrow X = Y)$.
2.Pair
$\forall a\forall b\exists A\forall x(x \in A \Leftrightarrow (x = a \vee x = b))$.
3.Union
$\forall A\exists B\forall x(x \in B \Leftrightarrow \exists X \in A(x \in X))$.
4.Infinity
$\exists X(\varnothing \in X \wedge \forall x(x \in X \Rightarrow (x \cup \{x\} \in X))$.
5.Separation
$\forall X\exists Y\forall x(x \in Y \Leftrightarrow (x \in X \wedge \varphi(x)))$.
6.Collection
$\forall X\exists Y\forall x \in X(\exists y\varphi(x,y)) \Rightarrow \exists y \in Y\varphi(x,y))$.
The power set, regularity and choice axioms of ZFC are not valid in general. This is because, as mentioned above, each one of these axioms (if considered in

the full language of HST) leads to a contradiction.
(**II**) **Axioms for standard and internal sets**
In this group as well as in the next there are axioms which are not valid for all sets. The first axiom scheme states that all ZFC axioms, when restricted to standard parameters are valid in HST

3.3.Nonconservative extension of the HST. External Set Theory $HST^{\#}$.

References

[1] Foukzon, J. (2021). Set Theory $INC^{\#}$ Based on Intuitionistic Logic with Restricted Modus Ponens Rule (Part. I). Journal of Advances in Mathematics and Computer Science, 36(2), 73-88. https://doi.org/10.9734/jamcs/2021/v36i230339

[2] Foukzon, J. (2021). Set Theory $INC^{\#}_{\infty^{\#}}$ Based on Infinitary Intuitionistic Logic with Restricted Modus Ponens Rule (Part.II) Hyper Inductive Definitions. Journal of Advances in Mathematics and Computer Science, 36(4), 90-112. https://doi.org/10.9734/jamcs/2021/v36i430359

[3] Foukzon, J. (2021). Set Theory $INC^{\#}_{\infty^{\#}}$ Based on Innitary Intuitionistic Logic with Restricted Modus Ponens Rule. Hyper Inductive Denitions. Application in Transcendental Number Theory. Generalized Lindemann-Weierstrass Theorem. Journal of Advances in Mathematics and Computer Science, 36(8), 70-119. https://doi.org/10.9734/jamcs/2021/v36i830394

[4] J. Foukzon,Set Theory $NC^{\#}_{\infty^{\#}}$ Based on Bivalent Infinitary Logic with Restricted Modus Ponens Rule. Basic Analysis on External Non-Archimedean Field $\mathbb{R}^{\#}_{c}$. https://vixra.org/abs/2111.0140
https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3989960

[5] A. Robinson,Non-standard analysis. (Reviwed re-edition of the 1st edition of (1966) Princeton: Princeton University Press,1996.

[6] Stroyan, K.D., Luxemburg, W.A.J. Introduction to the theory of infinitesimals. New York: Academic Press (1st ed.), 1976

[7] S. Albeverio, J. E. Fenstad, R. Høegh-Krohn,Nonstandard Methods in Stochastic Analysis and Mathematical Physics (Dover Books on Mathematics) , February 26, 2009 Paperback : 526 pages ISBN-10 : 0486468992,ISBN-13 : 978-0486468990

[8] M. Davis, Applied Nonstandard Analysis (Dover Books on Mathematics) ISBN-13: 978-0486442297 ISBN-10: 0486442292

[9] E. Nelson, Internal Set Theory, an axiomatic approach to nonstandard analysis, Bull. Am. Math. Soc., 83:6 (1977) 1165-1198.

[10] K. Hrbáµcek, Axiomatic foundations for nonstandard analysis, Fund. Math. 98, no. 1 (1978) 1–19.

[11] V. Kanovei, M. Reeken, Nonstandard Analysis, axiomatically, Springer Monographs in Mathematics (2004).

[12] T.J. Jech, Set Theory: the third millenium, revised and expanded, Springer Monographs in Mathematics (2003).
[13] P. Aczel, Non-well-founded sets, CSLI Publications, Stanford (1988).

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Appendix A. Bivalent Hyper Infinitary first-order logic ${}^2L^{\#}_{\infty^{\#}}$ with restricted rules of conclusion.Generalized Deduction Theorem.

Hyper infinitary language $L^{\#}_{\infty^{\#}}$ are defined according to the length of hyper infinitary conjunctions/disjunctions as well as quantification it allows. In that way, assuming a supply of $\kappa < \aleph_0^{\#} = card(\mathbb{N}^{\#})$ variables to be interpreted as ranging over a nonempty domain, one includes in the inductive definition of formulas an infinitary clause for conjunctions and disjunctions, namely, whenever the hypernaturals indexed hyper infinite sequence $\{A_\delta\}_{\delta\in\mathbb{N}^{\#}}$ of formulas has length less than κ, one can form the hyperfinite conjunction/disjunction of them to produce a formula. Analogously, whenever an hypernaturals indexed sequence of variables has length less than λ, one can introduce one of the quantifiers $\forall$ or $\exists$ together with the sequence of variables in front of a formula to produce a new formula. One also stipulates that the length of any well-formed formula is less than $\aleph_0^{\#}$ itself.

The syntax of bivalent hyper infinitary first-order logics ${}^2L^{\#}_{\infty^{\#}}$ consists of a (ordered) set of sorts and a set of function and relation symbols, these latter together with the corresponding type, which is a subset with less than $\aleph_0^{\#} = card(\mathbb{N}^{\#})$ many sorts. Therefore, we assume that our signature may contain relation and function symbols on $\gamma < \aleph_0^{\#}$ many variables, and we suppose there is a supply of $\kappa < \aleph_0^{\#}$ many fresh variables of each sort. Terms and atomic formulas are defined as usual, and general formulas are defined inductively according to the following rules.

If $\phi, \psi, \{\phi_\alpha : \alpha < \gamma\}$ (for each $\gamma < \kappa$) are formulas of $L^{\#}_{\infty^{\#}}$, the following are also formulas:

(i) $\bigwedge_{\alpha<\gamma}\phi_\alpha, \bigwedge_{\alpha\leq\gamma}\phi_\alpha$,
(ii) $\bigvee_{\alpha<\gamma}\phi_\alpha, \bigvee_{\alpha\leq\gamma}\phi_\alpha$,
(iii) $\phi \to \psi, \phi \wedge \psi, \phi \vee \psi, \neg\phi$
(iv) $\forall_{\alpha<\gamma}x_\alpha\phi$ (also written $\forall\mathbf{x}_\gamma\phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),
(v) $\exists_{\alpha<\gamma}x_\alpha\phi$ (also written $\exists\mathbf{x}_\gamma\phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),
(vi) the statement $\bigwedge_{\alpha<\gamma}\phi_\alpha$ holds if and only if for any α such that $\alpha < \gamma$ the statement holds ϕ_α,
(vii) the statement $\bigvee_{\alpha<\gamma}\phi_\alpha$ holds if and only if there exist α such that $\alpha < \gamma$ the statement holds ϕ_α.

Definition 1.[7]. A valuation of a syntactic system is a function that as signs $\top$ (true) to some of its sentences, and/or $\bot$ (false) to some of its sentences.Precisely, a valuation maps a nonempty subset of the set of sentences into the set $\{\top,\bot\}$.
We call a valuation bivalent iff it maps all the sentences into $\{\top,\bot\}$.

Definition 2.[7]. L is a bivalent propositional language iff its admissible valuations

are the functions v such that for all sentences A, B of L,
(a) $v(A) \in \{\top, \bot\}$
(b) $v(\neg A) = \top$ iff $v(A) = \bot$
(c) $v(A \wedge B) = \top$ iff $v(A) = v(B) = \top$.
(d) by definition of the implication $A \Rightarrow B$ the following truth table holds

	$v(A)$	$v(B)$	$v(A \Rightarrow B)$
(1)	$\top$	$\top$	$\top$
(2)	$\top$	$\bot$	$\bot$
(3)	$\bot$	$\top$	$\top$
(4)	$\bot$	$\bot$	$\top$

Truth table 1.

Remark 1.Note that in the case (4) on a truth table 1
In this case we call implication $A \Rightarrow B$ a weak implication and abbreviate

$$A \Rightarrow_w B \tag{1}$$

We call a statement (1) as a weak statement and often abbreviate $v(A \Rightarrow B) = \top_w$ instead (1).

Definition 3.[7-8]. A is a valid (logically valid) sentence (in symbols, $\Vdash A$) in L iff every admissible valuation of L satisfies A.

The axioms of hyper infinitary first-order logic ${}^2L^{\#}_{\infty^{\#}}$ consist of the following schemata:

I. **Logical axiom**

A 1. $A \to [B \to A]$
A 2. $[A \to [B \to C] \to [[A \to B] \to [A \to C]]]$
A 3. $[\neg B \to \neg A] \to [A \to B]$
A 4. $[\bigwedge_{i<\alpha}[A \to A_i]] \to [A \to \bigwedge_{i<\alpha} A_i], \alpha \in \mathbb{N}^{\#}$
A 5. $[\bigwedge_{i<\alpha} A_i] \to A_j, \alpha \in \mathbb{N}^{\#}$
A 6. $[\forall \mathbf{x}[A \to B] \to [A \to \forall \mathbf{x} B]]$
provided no variable in $\mathbf{x}$ occurs free in A;
A 7. $\forall \mathbf{x} A(\mathbf{x}) \to S_f(A)$,
where $S_f(A)$ is a substitution based on a function f from $\mathbf{x}$ to the terms of the language; in particular:
A 7$'$. $\forall x_i[A(x_i)] \Rightarrow A(\mathbf{t})$ is a wff of ${}^2L^{\#}_{\infty^{\#}}$ and $\mathbf{t}$ is a term of ${}^2L^{\#}_{\infty^{\#}}$ that is free for x_i in $A(x_i)$. Note here that $\mathbf{t}$ may be identical with x_i; so that all wffs $\forall x_i A \Rightarrow A$ are axioms by virtue of axiom (7),see [8].
A 8.Gen (Generalization).
$\forall x_i B$ follows from B.

II.**Restricted rules of conclusion**.

Let $\mathcal{F}_{\mathbf{wff}}$ be a set of the all closed wffs of $L^{\#}_{\infty^{\#}}$.

R1.RMP (Restricted Modus Ponens).
There exist subsets $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.
From A and $A \Rightarrow B$, conclude B iff $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$.
In particular for any $A, B \in \mathcal{F}_{\mathbf{wff}} : A \Rightarrow_w B \in \Delta_2$.
If $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$ we also abbraviate by $A, A \Rightarrow B \vdash_{RMP} B$.

R2.RMT (Restricted Modus Tollens)
There exist subsets $\Delta'_1, \Delta'_2 \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.

$P \Rightarrow Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \Rightarrow Q) \notin \Delta_2'$,where $\Delta_1', \Delta_2' \subset \mathcal{F}_{\mathbf{wff}}$.
Remark 2.Note that RMP and RMT easily prevent any paradoxes of naive Cantor set theory (NC), see [1],[9].
III.**Additional derived rule of conclusion**.
Particularization rule (RPR)
Remind that canonical unrestricted particularization rule (UPR) reads
UPR: If $\mathbf{t}$ is free for x in $B(x)$, then $\forall x[B(x)] \vdash B(\mathbf{t})$,see [8].
Proof.From $\forall x[B(x)]$ and the instance $\forall x[B(x)] \Rightarrow B(\mathbf{t})$ of axiom (A7), we obtain $B(\mathbf{t})$ by unrestricted modus ponens rule.Since x is free for x in $B(x)$, a special case of unrestricted particularization rule is:$\forall xB \vdash B$.
Definition 4.Any formal theory L with a hyper infinitary lenguage $L^{\#}_{\infty^{\#}}$ is defined when the following conditions are satisfied:
1. A hyper infinite set of symbols is given as the symbols of L. A finite or hyperfinite sequence of symbols of L is called an expression of L.
2. There is a subset of the set of expressions of L called the set of well formed formulas (wffs) of L. There is usually an effective procedure to determine whether a given expression is a wff.
3. There is a set of wfs called the set of axioms of L. Most often, one can effectively decide whether a given wff is an axiom; in such a case, L is called an axiomatic theory.
4. There is a finite set $R_1,\ldots,R_n$, of relations among wffs, called rules of conclusion. For each R_i, there is a unique positive integer j such that, for every set of j wfs and each wff B, one can effectively decide whether the given j wffs are in the relation R_i to B, and, if so, B is said to follow from or to be a direct consequence of the given wffs by virtue of R_j.
Definition 5.A proof in L is a finite or hyperfinite sequence $B_1,\ldots,B_k, k \in \mathbb{N}^{\#}$ of wffs such that for each i,either B_i is an axiom of L or B_i is a direct consequence of some of the preceding wffs in the sequence by virtue of one of the rules of inference of L.
Definition 6. A theorem of L is a wff B of Y such that B is the last wff of some proof in L. Such a proof is called a proof of B in L.
Definition 7. A wff E is said to be a consequence in L of a set of Γ of wffs if and only if there is a finite or hyperfinite sequence $B_1,\ldots,B_k, k \in \mathbb{N}^{\#}$ of wffs such that E is B_k and, for each i,either B_i is an axiom or B_i is in Γ, or B_i is a direct consequence by some rule of inference of some of the preceding wffs in the sequence. Such a sequence is colled a proof (or deduction) E from Γ. The members of Γ are called the hypotheses or premisses of the proof.
We use $\Gamma \vdash E$ as an abbreviation for E as a consequence of Γ.
In order to avoid confusion when dealing with more than one theory, we write $\Gamma \vdash_L E$, adding the subscript L to indicate the theory in question.
If Γ is a finite or hyperfinite set $\{H_i\}_{1 \leq i \leq m}, m \in \mathbb{N}^{\#}$ we write $H_1,\ldots,H_m \vdash E$ instead of $\{H_i\}_{1 \leq i \leq m} \vdash E$.
Lemma 1.[8]. $\vdash B \Rightarrow B$ for all wffs B.
Theorem 1.(Generalized Deduction Theorem1). If Γ is a set of wffs and B and E are wffs, and $\Gamma, B \vdash E$, then $\Gamma \vdash B \Rightarrow_s E$. In pticular, if $B \vdash E$ then $\vdash B \Rightarrow E$.

Proof. Let $E_1,\ldots,E_n, n \in \mathbb{N}^{\#}$ be a proof of E form $\Gamma \cup \{B\}$, where E_n is E.
Let us prove, by hyperfinite induction on $j,$ that $\Gamma \vdash B \Rightarrow_s E_j$ for $1 \leq j \leq n$.
First of all, E_1 must be either in Γ or an axiom of L or B itself.
By axiom schema A1, $E_1 \Rightarrow_s (B \Rightarrow_s E_1)$ is an axiom. Hence, in the first two cases, by MP, $\Gamma \vdash B \Rightarrow_s E_1$ For the third case, when E_1 is B, we have $\vdash B \Rightarrow_s E_1$ by Lemma 1, and, therefore, $\Gamma \vdash B \Rightarrow_s E_1$. This takes care of the case $j = 1$.
Assume now that: $\vdash B \Rightarrow_s E_k$ for all $k < j, j \in \mathbb{N}^{\#}$. Either E_j is an axiom, or E_j is in Γ, or E_j is $B,$ or E_j follows by modus ponens from some E_l and E_m where $l < j$, $m < j,$ and E_m has the form $E_l \Rightarrow_s E_j$. In the first three cases, $\Gamma \vdash B \Rightarrow_s E_j$ as in the case $j = 1$ above. In the last case, we have, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s E_l$ and $\Gamma \vdash B \Rightarrow_s (E_l \Rightarrow_s E_j)$ But, by axiom schema (A2),
$\vdash B \Rightarrow_s (E_l \Rightarrow_s E_j) \Rightarrow_s ((B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j))$
Hence, by MP, $\Gamma \vdash (B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j)$ and, again by MP, $\Gamma \vdash B \Rightarrow_s E_j$.
Thus, the proof by hyperfinite induction is complete.
The case $j = n \in \mathbb{N}^{\#}$ is the desired result. Notice that, given a deduction of E from Γ and $B,$ the proof just given enables us to construct a deduction of $B \Rightarrow_s E$ from Γ. Also note that axiom schema A3 was not used in proving the generalized deduction theorem.

Remark 3.For the remainder of the chapter, unless something is said to the contrary, we shall omit the subscript L in $\vdash_L$. In addition, we shall use $\Gamma, B \vdash E$ to stand for $\Gamma \cup \{B\} \vdash E$. In general, we let $\Gamma, B_1,\ldots,B_n \vdash E$ stand for $\Gamma \cup \{B_i\}_{1 \leq i \leq n} \vdash E$.

Remark 4.We shall use the terminology proof, theorem, consequence, axiomatic, etc. and notation $\Gamma \vdash E$ introduced above.

Proposition 1. Every wff B of K that is an instance of a tautology is a theorem of $K,$ and it may be proved using only axioms A1-A3 and MP.

Proposition 2.If E does not depend upon B in a deduction showing that $\Gamma, B \vdash E$, then $\Gamma \vdash E$.

Proof.Let $D_1,\ldots,D_n$ be a deduction of E from Γ and $B,$ in which E does not depend upon B. In this deduction, D_n is E. As an inductive hypothesis, let us assume that the proposition is true for all deductions of length less than $n \in \mathbb{N}^{\#}$
If E belongs to Γ or is an axiom, then $\Gamma \vdash E$. If E is a direct consequence of one or two preceding wffs by Gen or MP, then, since E does not depend upon B, neither do these preceding wfs. By the inductive hypothesis, these preceding wfs are deducible from Γ alone. Consequently, so is E .

Theorem 2.(Generalized Deduction Theorem 2).Assume that, in some deduction showing that $\Gamma, B \vdash E,$ no application of Gen to a wff that depends upon B has as its quantified variable a free variable of B. Then $\Gamma \vdash B \Rightarrow_s E$.

Proof.Let $D_1,\ldots,D_n$ be a deduction of E from Γ and B satisfying the assumption of this theorem. In this deduction, D_n is E. Let us show by hyperfinite induction that $\Gamma \vdash B \Rightarrow_s D_i$ for each $i \leq n \in \mathbb{N}^{\#}$. If D_i is an axiom or belongs to Γ, then $\Gamma \vdash B \Rightarrow_s D_i$, since $D_i \Rightarrow_s (B \Rightarrow_s D_i)$ is an axiom. If D_i is B, then $\Gamma \vdash B \Rightarrow_s D_i,$ since, by Proposition 1, $\vdash B \Rightarrow_s B. \cdot$ If there exist j and k less than i such that D_k is $\vdash D_j \Rightarrow_s D_i,$ then, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$ and $\Gamma \vdash B \Rightarrow_s (D_j \Rightarrow_s D_i)$. Now, by axiom A2,
$\vdash B \Rightarrow_s (D_j \Rightarrow_s D_i) \Rightarrow_s ((B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s D_i))$. Hence, by MP twice, $\Gamma \vdash B \Rightarrow_s D_i$. Finally, suppose that there is some $j < i$ such that D_i is $\forall x_k D_j$.

By the inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$, and, by the hypothesis of the theorem, either D_j does not depend upon B or x_k is not a free variable of B. If D_j does not depend upon B, then, by Proposition 2, $\Gamma \vdash D_j$ and, consequently, by Gen, $\Gamma \vdash \forall x_k D_j$. Thus, $\Gamma \vdash D_i$. Now, by axiom A1, $\vdash D_i \Rightarrow_s (B \Rightarrow_s D_i)$. So, $\Gamma \vdash B \Rightarrow_s D_i$ by MP. If, on the other hand, x_k is not a free variable of B, then, by axiom A5, $\vdash \forall x_k(B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s \forall x_k D_j)$ Since $\Gamma \vdash B \Rightarrow_s D_j$, we have, by Gen,$\Gamma \vdash \forall x_k(B \Rightarrow_s D_j)$, and so, by MP,$\Gamma \vdash B \Rightarrow_s \forall x_k D_j$ that is, $\Gamma \vdash B \Rightarrow_s D_i$. This completes the induction, and our proposition is just the special case $i = n$.

Appendix B.The Generalized Recursion Theorem.

Theorem 1. Let S be a set, $c \in S$ and $G : S \to S$ is any function with $\mathbf{dom}(G) = S$ and $\mathbf{range}(G) \subseteqq S$. Let $W[G] \in \mathbb{N}^{\#} \times S$ be a binary relation such that:
(a) $(1,c) \in W[G]$ and
(b) if $(x,y) \in W[G]$ then $(\mathbf{Sc}(x), G(y)) \in W[G]$.
Then there exists a function $\mathcal{F} : \mathbb{N}^{\#} \to S$ such that:
(i) $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$and $\mathbf{range}(\mathcal{F}) \subseteqq S$;
(ii) $\mathcal{F}(1) = c$;
(iii) for all $x \in \mathbb{N}^{\#}, \mathcal{F}(\mathbf{Sc}(x)) = G(\mathcal{F}(x))$.