EXACT EXPANSIONS

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ABSTRACT. In this paper we continue the development of multivariate expansivity theory. We introduce and study the notion of an exact expansion and exploit some applications.

1. Introduction

Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. Then by an expansion on $\mathcal{S} \in \mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ in the direction x_i for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \longrightarrow \mathcal{F}$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

with

$$\nabla_{[x_i]}(\mathcal{S}) = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i}\right).$$

The value of the l th expansion at a given value a of x_i is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i](a)}(\mathcal{S})$$

where $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_i](a)}(\mathcal{S})$ is a tuple of polynomials in $\mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Similarly by an expansion in the mixed direction $\otimes_{i=1}^{l}[x_{\sigma(i)}]$ we mean

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^{l}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

for any permutation $\sigma : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, l\}$. The value of this expansion on a given value a_i of $x_{\sigma(i)}$ for all $i \in [\sigma(1), \sigma(l)]$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$ is tuple of real numbers $\mathbb{R}$. We recall from [1] that the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z)$ is a sub-expansion of the expansion

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$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$, denoted $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$ if there exist some $0\leq m$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k+m}_{[x_j]}(\mathcal{S}_t).$$

We say the sub-expansion is proper if $m+k=l$. We denote this proper sub-expansion by $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$. On the other hand, we say the sub-expansion is **ancient** if $m+k>l$. Furthermore, we say the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is **diagonalizable** in the direction $[x_j]$ $(1\leq j\leq n)$ at the spot $\mathcal{S}_r\in\mathcal{F}$ with order k with $\mathcal{S}-\mathcal{S}_r$ not a tuple of $\mathbb{R}$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_r).$$

We call the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)$ the **diagonal** of the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ of **order** $k\geq 1$. We denote with $\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)]$ the order of the diagonal. In this paper, we explore the notion of an exactness of an expansion. This notion can be thought of an the inverse notion of diagonalization of an expansion.

2. **Exact expansion**

In this section we introduce the notion of an exact expansion.

Definition 2.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. Then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ is exact in the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ each with multiplicity 1 for $1\leq l\leq n$ and $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ at the spot $\mathcal{S}_1$ if there exists a number $s\in\mathbb{N}$, called the **degree** of the exactness, such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1).$$

In general, we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ is exact in the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ each with multiplicity $k_1,\ldots,k_l\in\mathbb{N}$ for $1\leq l\leq n$ with degree s of exactness if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]^{k_i}}(\mathcal{S}_1)$$

where $[x_{\sigma(i)}]^{k_i}=[x_{\sigma(i)}]\otimes[x_{\sigma(i)}]\cdots\otimes[x_{\sigma(i)}]$ (k_i *times*).

The following web shows the commutative diagram of a typical exact expansion

$$\begin{array}{ccc}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}_1) & \xrightarrow{\phi_2} & (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{2}[x_{\sigma(i)}]}(\mathcal{S}_1)\\ & & \downarrow{\scriptstyle\phi_3}\\ (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}) & \xrightarrow{\eta_k^2} & (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{3}[x_{\sigma(i)}]}(\mathcal{S}_1)\end{array}$$

with degree 3 of exactness, where $\phi_l=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(l)}]}$ and $\phi_l\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]\otimes[x_{\sigma(l)}]}(\mathcal{S})$ and $\eta_k^l=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_k]}$ for $1\leq l\leq n$. One can also construct more expanded commutative diagrams for exact expansion with arbitrarily large degrees. The notion of an exact expansion provides alternative paths to model an expansion in a specific direction. These type of expansion could conceivably be difficult and often delicate, so that a little distortion in the choice of directions may not guarantee the targeted expansion.

Proposition 2.2. *The expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ *is exact in the directions* $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ *for* $1\leq l\leq n$ *and* $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ *at the spot* $\mathcal{S}_1$ *with degree* $s\in\mathbb{N}$ *if and only if the expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1)$ *is diagonalizable in the direction* $[x_k]$ *at the spot* $\mathcal{S}$ *with order* s*.*

It is worth noting that Proposition 2.2 expresses the relationship between the notion of an exactness of an expansion and the diagonalization of an expansion. These two notions are quite similar except that the notion of an exactness is applied to expansions in a specific direction where as the notion of diagonalization is appropriate for expansions in a mixed directions. However one perceives these notions as different, they both can be considered as notions orthogonal to each other. Next we show that the notion of exactness in directions can be extended to other directions.

Proposition 2.3. *If the expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ *is exact in the directions* $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ *for* $1\leq l\leq n$ *and* $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ *at the spot* $\mathcal{S}_1$ *with degree* $s\in\mathbb{N}$*, then it also exact in the directions* $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}],[x_k]$ *at the spot* $\mathcal{S}_1$ *with degree* $s+1$*.*

Proof. By appealing to definition 2.1 we can write

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1).$$

The claim follows by applying an extra copy of the expansion operator $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}$ on both sides of the equation. □

Remark 2.4. Next we show that we can extend the notion of an exactness to proper sub-expansions of an expansion.

Proposition 2.5. *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ *a sub-expansion of the expansion. If* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ *is exact in the directions* $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ *for* $1\leq l\leq n$ *and* $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ *at the spot* $\mathcal{S}_1$ *with degree* $s\in\mathbb{N}$*, then there exists some* $m\in\mathbb{N}$ *such that the expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ *is exact with degree* $s+m-1$ *in the directions* $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ *for* $1\leq l\leq n$ *and* $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ *at the spot* $\mathcal{S}_1$*.*

Proof. Under the condition $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$, then there exists some fixed $m\in\mathbb{N}$ such that we can write

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{m}_{[x_k]}(\mathcal{S}_a)$$

so that by applying $(s-1)$ copies of the expansion operator $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}$ on both sides of the equation, we have

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s+m-1}_{[x_k]}(\mathcal{S}_a).$$

The claim follows since the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ is exact with degree s in the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ for $1\leq l\leq n$ at the spot $\mathcal{S}_1$. □

Although it is fairly easy to pass the notion of exactness of a sub-expansion to an expansion, the converse is actually difficult. We can only carry out this task under certain underlying condition on an expansion and their sub-expansion. The follow-up result underscores this discussion.

Proposition 2.6. *Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ a proper sub-expansion of the expansion. If $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ is exact in the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ for $1\leq l\leq n$ and $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ at the spot $\mathcal{S}_1$ with degree $s\in\mathbb{N}$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_k]}(\mathcal{S}_a)<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$, then there exists some $j\in\mathbb{N}$ such that the proper sub-expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ is exact with degree $j<s$ in the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ for $1\leq l\leq n$ and $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ at the spot $\mathcal{S}_1$.*

Proof. Suppose $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ is exact in the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ for $1\leq l\leq n$ and $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ at the spot $\mathcal{S}_1$ with degree $s\in\mathbb{N}$. Then it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_k]}(\mathcal{S}_a)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1)$$

so that under the requirement $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_k]}(\mathcal{S}_a)<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ there exists some $j\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_k]}(\mathcal{S}_a)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^j_{[x_k]}(\mathcal{S}).$$

Since $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})<(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ a proper sub-expansion of the expansion, there exists some $m\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^m_{[x_k]}(\mathcal{S}_a).$$

By combining both equations, it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_k]}(\mathcal{S}_a)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{j-1+m}_{[x_k]}(\mathcal{S}_a)$$

so that $j<s$ and the claim follows from this assertion. □

3. **Sequences of an exact expansion**

In this section we examine the structure and a commutative diagram of an exact expansion.

Definition 3.1. Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ be an exact expansion in the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ for $1\leq l\leq n$ and $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ with degree s. Then we call the chain

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}_1)\longrightarrow(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]\otimes[[x_{\sigma(2)}]}(\mathcal{S}_1)\longrightarrow\cdots$$

$$\hookrightarrow(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l-1}[x_{\sigma(i)}]}(\mathcal{S}_1)\longrightarrow(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1)$$

an exact sequence of the exact expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ - respectively, $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})$ is an exact expansion of the exact sequence - where $\phi_l=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(l)}]}$ and $\phi_l\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]\otimes[x_{\sigma(l)}]}(\mathcal{S})$ for $1\leq l\leq n$.

Definition 3.2. We say the exact sequence

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}_1)\longrightarrow(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]\otimes[[x_{\sigma(2)}]}(\mathcal{S}_1)\longrightarrow\cdots$$

$$\hookrightarrow(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l-1}[x_{\sigma(i)}]}(\mathcal{S}_1)\longrightarrow(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1)$$

is a sub-sequence of the exact sequence

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\lambda(1)}]}(\mathcal{S}_2) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\lambda(1)}]\otimes[[x_{\lambda(2)}]}(\mathcal{S}_2) \longrightarrow \cdots$$
$$\hookrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r-1}[x_{\lambda(i)}]}(\mathcal{S}_2) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r}[x_{\lambda(i)}]}(\mathcal{S}_2)$$

if the first chain is contained in the second chain.

Remark 3.3. Next we use the notion of an exactness of an expansion to study sub-expansions of an expansion. We first extend the notion of a sub-expansion of an expansion in a multivariate sense.

Definition 3.4. We say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ is a sub-expansion of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_b)$ along the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ each with multiplicity k_i for $1\leq i\leq l\leq n$, where $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ if and only if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]^{k_i}}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_b).$$

We denote this sub-expansion by

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a) \leq_{[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]} (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_b).$$

One could think of this notion as an extended notion of sub-expansions of an expansion. Indeed the underlying intuition remains, that a sub-expansion of an expansion is an outcome of several expansions on the mother expansion. This also provides some flexibility to the manner in which sub-expansions can be obtained from their mother expansion in the setting of an expansion in a mixed directions.

Theorem 3.5. *If the exact sequence*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}_1) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]\otimes[[x_{\sigma(2)}]}(\mathcal{S}_1) \longrightarrow \cdots$$
$$\hookrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l-1}[x_{\sigma(i)}]}(\mathcal{S}_1) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1)$$

of the exact expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ *with degree* u *of exactness is a sub-sequence of the exact sequence*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}_2) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]\otimes[[x_{\sigma(2)}]}(\mathcal{S}_2) \longrightarrow \cdots$$
$$\hookrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r-1}[x_{\sigma(i)}]}(\mathcal{S}_2) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(\mathcal{S}_2)$$

of the exact expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_b)$ *with degree* v *of exactness then*

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_k]}(\mathcal{S}_a) \leq_{[x_{\sigma(l+1)}],[x_{\sigma(l+2)}],\ldots,[x_{\sigma(r)}],\ldots,[x_{\lambda(s)}]} (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_j]}(\mathcal{S}_b).$$

for some $s\in\mathbb{N}$ *with* $s\leq n$ *and where* $\lambda:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$.

Proof. Under the main assumption, we can embed the chain

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}_1) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]\otimes[[x_{\sigma(2)}]}(\mathcal{S}_1) \longrightarrow \cdots$$
$$\hookrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l-1}[x_{\sigma(i)}]}(\mathcal{S}_1) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1)$$

into the chain

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]}(\mathcal{S}_2) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_{\sigma(1)}]\otimes[[x_{\sigma(2)}]}(\mathcal{S}_2) \longrightarrow \cdots$$
$$\hookrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r-1}[x_{\sigma(i)}]}(\mathcal{S}_2) \longrightarrow (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(\mathcal{S}_2)$$

so that by the commutative property of an expansion, we can write

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(\mathcal{S}_2) &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{j=1}^{s}[x_{\lambda(j)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}\\ &\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=l+1}^{r}[x_{\sigma(i)}]}(\mathcal{S}_1)\\ &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=l+1}^{r}[x_{\sigma(i)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{j=1}^{s}[x_{\lambda(j)}]}\\ &\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1)\\ &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=l+1}^{r}[x_{\sigma(i)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{j=1}^{s}[x_{\lambda(j)}]}\\ &\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_k]}(\mathcal{S}_a)\end{aligned}$$

since $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}_1) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_k]}(\mathcal{S}_a)$. Under the exactness condition $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_j]}(\mathcal{S}_b) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]}(\mathcal{S}_2)$, we obtain

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_j]}(\mathcal{S}_b) &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=l+1}^{r}[x_{\sigma(i)}]}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{j=1}^{s}[x_{\lambda(j)}]}\\ &\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_k]}(\mathcal{S}_a)\end{aligned}$$

and it follows that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_k]}(\mathcal{S}_a) \leq_{[x_{\sigma(l+1)}],[x_{\sigma(l+2)}],\ldots,[x_{\sigma(r)}],\ldots,[x_{\lambda(s)}]} (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_j]}(\mathcal{S}_b).$$

□

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