THE INDEX OF EXPANSIONS

T. AGAMA

Abstract. In this paper, we study the notion of an index of sub-expansions in an expansion. We prove the index inequality as an application.

1. Introduction

Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. Then by an expansion on $\mathcal{S} \in \mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ in the direction x_i for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \longrightarrow \mathcal{F}$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

with

$$\nabla_{[x_i]}(\mathcal{S}) = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i} \right).$$

The value of the l th expansion at a given value a of x_i is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i](a)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_i](a)}(\mathcal{S})$ is a tuple of polynomials in $\mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Similarly by an expansion in the mixed direction $\otimes_{i=1}^{l}[x_{\sigma(i)}]$ we mean

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^{l}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

for any permutation $\sigma : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, l\}$. The value of this expansion on a given value a_i of $x_{\sigma(i)}$ for all $i \in [\sigma(1), \sigma(l)]$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$ is tuple of real numbers $\mathbb{R}$. We recall from [1] that the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z)$ is a sub-expansion of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_j]}(\mathcal{S}_t)$, denoted $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{l}_{[x_j]}(\mathcal{S}_t)$ if there exist some $0 \leq m$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k}_{[x_j]}(\mathcal{S}_z) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+m}_{[x_j]}(\mathcal{S}_t).$$

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We say the sub-expansion is proper if $m+k=l$. We denote this proper sub-expansion by $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z) < (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$. On the other hand, we say the sub-expansion is **ancient** if $m+k>l$. Furthermore, we say the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is **diagonalizable** in the direction $[x_j]$ $(1\leq j\leq n)$ at the spot $\mathcal{S}_r\in\mathcal{F}$ with order k with $\mathcal{S}-\mathcal{S}_r$ not a tuple of $\mathbb{R}$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_r).$$

We call the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)$ the **diagonal** of the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ of **order** $k\geq 1$. We denote with $\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)]$ the order of the diagonal. In this paper, we study the notion of an index of a sub-expansion in an expansion. By denoting index of the sub-expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_z)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t)$ by $\Big[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_z)\Big]$, we prove the inequality

Theorem 1.1 (The index inequality). *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq\cdots\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ *- a chain of sub-expansions of the expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$*. Then*

$$\Big[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\Big] < \sum_{i=1}^{n-1}\Big[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i+1}) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i)\Big].$$

2. Sub-expansion

Definition 2.1. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. We say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)$ is a sub-expansion of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$, denoted $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$ if there exist some $0\leq m$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k+m}_{[x_j]}(\mathcal{S}_t).$$

We say the sub-expansion is proper if $m+k=l$. We denote this proper sub-expansion by $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z) < (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$. On the other hand, we say the sub-expansion is **ancient** if $m+k>l$. In general, we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)$ is a sub-expansion of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_b)$ along the directions $[x_{\sigma(1)}],\ldots,[x_{\sigma(l)}]$ each with multiplicity k_i for $1\leq i\leq l\leq n$, where $\sigma:\{1,2,\ldots,n\}\longrightarrow\{1,2,\ldots,n\}$ if and only if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{r}[x_{\sigma(i)}]^{k_i}}\circ(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_b).$$

We denote this sub-expansion by

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_k]}(\mathcal{S}_a)\leq_{[x_\sigma(1)],\ldots,[x_\sigma(l)]}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_b).$$

Definition 2.2. Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t)$ be expansions. By the index of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)$ in the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t)$, denoted $\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)\bigg]$, we mean the value of $r \in \mathbb{N}$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^r_{[x_j]}(\mathcal{S}_t)$$

and we write

$$\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)\bigg] = r.$$

We say the index is finite if and only if it exists and we write

$$\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)\bigg] < \infty.$$

On the other hand, if no such value exists then we say the index is infinite and we write

$$\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)\bigg] = \infty.$$

Proposition 2.3. *Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t)$ be expansions. Then*

$$\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z)\bigg] < \infty$$

if and only if $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_z) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_t)$.

Proof. This is a simple consequence of the notion sub-expansions of an expansion and the index of an expansion. □

Proposition 2.4. *Let $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_1)$, $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_2)$ and $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_3)$ be expansions. If $\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_3) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_2)\bigg] < \infty$ and $\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_1)\bigg] < \infty$ then*

$$\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_3) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_1)\bigg] < \infty.$$

Proof. Suppose $\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_3) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_2)\bigg] < \infty$ and $\bigg[(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_2) : (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_1)\bigg] < \infty$. Then there exist some $r, s \in \mathbb{N}$ such that we can write

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_2) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^r_{[x_j]}(\mathcal{S}_3)$$

and

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_j]}(\mathcal{S}_1) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^s_{[x_j]}(\mathcal{S}_2).$$

It follows that

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_j]}(\mathcal{S}_2\\&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r+s-1}_{[x_j]}(\mathcal{S}_3)\end{aligned}$$

so that $\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_3):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg]<\infty.$ ☐

Remark 2.5. Next we show that the index of a sub-expansion in an expansion decreases with further expansions.

Proposition 2.6. *Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)$. If there exists an $l\in\mathbb{N}$ such that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_2)$ then*

$$\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg]<\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg].$$

Proof. Suppose $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)$ then there exists some $s\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_j]}(\mathcal{S}_2).$$

Under the regularity condition $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_2)$ there exists some $u\in\mathbb{N}$ such that we can write

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l+u}_{[x_j]}(\mathcal{S}_2)$$

so that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_j]}(\mathcal{S}_2)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l+u}_{[x_j]}(\mathcal{S}_2)$$

and $u<u+l=s$. The claimed inequality follows by making the substitutions $\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg]=u$ and $\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg]=s.$ ☐

Remark 2.7. Next we relate the index of the smallest sub-expansion in a collection of chains of sub-expansion in their mother expansion to the index of other sub-expansions in other sub-expansion.

Theorem 2.8. *Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq\cdots\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ - a chain of sub-expansions of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$. Then*

$$\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg]=\sum_{i=1}^{n-1}\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i+1}):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i)\bigg]-(n-2).$$

Proof. By appealing to Proposition 2.3 then $\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i+1}):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i)\bigg]<\infty$ for all $1\leq i\leq n-1$ and there must exist some $r_1\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{n-1})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_1}(\mathcal{S}_n).$$

Again there exists some $r_2\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{n-2})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_2}(\mathcal{S}_{n-1})$$

so that

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{n-2})&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_2}(\mathcal{S}_{n-1})\\&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_1+r_2-1}(\mathcal{S}_n).\end{aligned}$$

Similarly there exists some $r_3\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{n-3})=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_3}(\mathcal{S}_{n-2})$$

so that

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{n-3})&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_3}(\mathcal{S}_{n-2})\\&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_1+r_2+r_3-2}(\mathcal{S}_n).\end{aligned}$$

By repeating this argument and taking cognisance of the fact $\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i+1}):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i)\bigg]<\infty$ for all $1\leq i\leq n-1$, we obtain

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}^{r_1+r_2+r_3+\cdots+r_{n-1}-(n-2)}(\mathcal{S}_n)$$

and it follows that

$$\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg]=\sum_{i=1}^{n-1}r_{n-i}-(n-2).$$

The claim follows since $\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i+1}):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i)\bigg]=r_{n-i}$ for $1\leq i\leq n-1$. □

We now obtain an important inequality as a consequence of Theorem 2.8 relating the index of the smallest sub-expansion in their mother expansion to local indices in each sub-expansion of the sub-expansions in the chain.

Corollary 2.9 (The index inequality)**.** *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq\cdots\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ *- a chain of sub-expansions of the expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$*. Then*

$$\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\bigg]<\sum_{i=1}^{n-1}\bigg[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i+1}):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i)\bigg].$$

Theorem 2.10. *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)$ *- a sub-expansion of the expansion. If there exists some* $s\in\mathbb{N}$ *such that* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$*, then*

$$s+1=\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\right]+\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)\right].$$

Proof. Under the condition $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)$, it follows that there exists some $l\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^l_{[x_j]}(\mathcal{S}_2)$$

so that $\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\right]=l$. Again $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$ for some $s\in\mathbb{N}$ implies that there exist some $r\in\mathbb{N}$ such that we can write

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^r_{[x_j]}(\mathcal{S}_1)$$

so that $\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)\right]=r$. It follows that we can write

$$\begin{aligned}(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^r_{[x_j]}(\mathcal{S}_1)\\&=(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r+l-1}_{[x_j]}(\mathcal{S}_2)\end{aligned}$$

and we can further write $s+1=r+l$. The claim follows by the following substitutions $\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\right]=l$ and $\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)\right]=r$. □

2.1. **Applications to additive number theory.**

Remark 2.11. Next we state a consequence of this result which one can view as an application to theory of partitions in additive number theory.

Corollary 2.12. *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)$ *such that* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$*. If* $\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\right]$ *and* $\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1):(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^s_{[x_j]}(\mathcal{S}_2)\right]$ *are both prime numbers, then* $s+1$ *can be written as a sum of two prime numbers.*

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Department of Mathematics, African Institute for mathematical sciences, Ghana.
E-mail address: Theophilus@aims.edu.gh/emperordagama@yahoo.com