# THE DOMINATING NUMBER OF EXPANSIONS

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Abstract. In this paper, we study the notion of dominating number of expansions.

## 1. Introduction

Let $\mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials $f_k \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. Then by an expansion on $\mathcal{S} \in \mathcal{F} := \{\mathcal{S}_i\}_{i=1}^{\infty}$ in the direction $x_i$ for $1 \leq i \leq n$, we mean the composite map

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_i]} : \mathcal{F} \longrightarrow \mathcal{F}$$

where

$$\gamma(\mathcal{S}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \quad \text{and} \quad \beta(\gamma(\mathcal{S})) = \begin{pmatrix} 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \vdots & \cdots \vdots \\ 1 & 1 & \cdots 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

with

$$\nabla_{[x_i]}(\mathcal{S}) = \left( \frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i} \right).$$

The value of the $l$ th expansion at a given value $a$ of $x_i$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^l_{[x_i](a)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^l_{[x_i](a)}(\mathcal{S})$ is a tuple of polynomials in $\mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Similarly by an expansion in the mixed direction $\otimes_{i=1}^{l}[x_{\sigma(i)}]$ we mean

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=2}^{l}[x_{\sigma(i)}]} \circ (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{[x_{\sigma(1)}]}(\mathcal{S})$$

for any permutation $\sigma : \{1, 2, \ldots, l\} \longrightarrow \{1, 2, \ldots, l\}$. The value of this expansion on a given value $a_i$ of $x_{\sigma(i)}$ for all $i \in [\sigma(1), \sigma(l)]$ is given by

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$$

where $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}](a_i)}(\mathcal{S})$ is tuple of real numbers $\mathbb{R}$. We recall from [1] that the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_z)$ is a sub-expansion of the expansion $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^l_{[x_j]}(\mathcal{S}_t)$, denoted $(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_z) \leq (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^l_{[x_j]}(\mathcal{S}_t)$ if there exist some $0 \leq m$ such that

$$(\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^k_{[x_j]}(\mathcal{S}_z) = (\gamma^{-1} \circ \beta \circ \gamma \circ \nabla)^{k+m}_{[x_j]}(\mathcal{S}_t).$$

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We say the sub-expansion is proper if $m+k=l$. We denote this proper sub-expansion by $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_z) < (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{l}_{[x_j]}(\mathcal{S}_t)$. On the other hand, we say the sub-expansion is **ancient** if $m+k>l$. Furthermore, we say the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ is **diagonalizable** in the direction $[x_j]$ $(1\leq j\leq n)$ at the spot $\mathcal{S}_r\in\mathcal{F}$ with order $k$ with $\mathcal{S}-\mathcal{S}_r$ not a tuple of $\mathbb{R}$ if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_j]}(\mathcal{S}_r).$$

We call the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)$ the **diagonal** of the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$ of **order** $k\geq 1$. We denote with $\mathcal{O}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_r)]$ the order of the diagonal. Let $\mathcal{F}=\{\mathcal{S}_i\}_{i=1}^{\infty}$ be a collection of tuples of polynomials in the ring $\mathbb{R}[x_1,x_2,\ldots,x_n]$. Then we say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})$ is **free** with **totient** $k$, denoted $\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_i]}(\mathcal{S})]$, if

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{[x_i]}(\mathcal{S}) = \mathcal{S}_0$$

where $k>0$ is the smallest such number. We call the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k-1}_{[x_i]}(\mathcal{S})$ the **residue** of the expansion, denoted by $\Theta[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k-1}_{[x_i]}(\mathcal{S})]$. Similarly by the totient of the mixed expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})$, we mean the smallest value of $k$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{k}_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S}) = \mathcal{S}_0.$$

We denote the totient of the mixed expansion with

$$\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{\otimes_{i=1}^{l}[x_{\sigma(i)}]}(\mathcal{S})].$$

## 2. **Dominating expansions**

In this section we introduce the notion of a dominating expansions and their corresponding numbers.

**Definition 2.1.** Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_z)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t)$. Then we call the smallest number $s\in\mathbb{N}$ such that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{s}_{[x_j]}(\mathcal{S}_t)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_z)$ the dominating number of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_z)$ relative to the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t)$. We say the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_z)$ dominates the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t)$ with dominating number $s$. We denote the dominating number with

$$\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_z)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_t)] = s.$$

*Remark* 2.2. We relate the notion of domination of an expansion to the notion of sub-expansion of an expansion. In fact these two notion are somewhat equivalent.

**Proposition 2.3.** *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ *and* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$*. Then*

$$\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)] < \mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\mid (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]$$

*if and only if* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$*.*

*Proof.* Let us suppose that

$$\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]<\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]$$

and let $\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]=u$ and $\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]=v$. Then we have $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^v_{[x_j]}(\mathcal{S}_n)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^u_{[x_j]}(\mathcal{S}_n)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$. Since $v$ and $u$ are the smallest such numbers and $v>u$, we obtain the chain of sub-expansions

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^v_{[x_j]}(\mathcal{S}_n)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^u_{[x_j]}(\mathcal{S}_n)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$$

so that we obtain $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$. Conversely, suppose that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)$. Then it follows that

$$\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]<\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]$$

since $\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]$ and $\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]$ are the smallest such numbers. □

The notion of the totient introduced in [1] provides a specific time frame for an expansion to run into some sort of extinction. In other words, the totient is the time taken for an expansion to come to a complete halt. The next result controls the total dominating number of any chain of sub-expansion of an expansion by an expression involving the totient.

**Proposition 2.4.** *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq\cdots\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ *be a chain of sub-expansions of the expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$. *Then*

$$\sum_{i=1}^{n}\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i)\mid(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]\leq\frac{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]-1}{2}\times(\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]-2).$$

*Proof.* Let us insert the chain $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)\leq\cdots\leq(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ of sub-expansions of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ into the full chain of sub-expansion of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$. Then we note that the sum on the left-hand side is bounded by the sum

$$\sum_{i=2}^{\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]-1}\Bigg(\Phi[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)]-i\Bigg).$$

□

One could sense that the notion of dominating number is intimately connected to the notion of the index of an expansion. We can in most cases compare these two numbers in the following simple way.

**Proposition 2.5.** *$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)$, then*

$$\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\right] \leq \mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) \mid (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)].$$

*Proof.* Let $v = \left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\right]$ and $u = \mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) \mid (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)]$ so that we can write $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_j]}(\mathcal{S}_2)$ and $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_2) \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_j]}(\mathcal{S}_2)$. It follows that $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_2)$ is a sub-expansion of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v}_{[x_j]}(\mathcal{S}_2)$ so that there exits some $t \geq 0$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_2) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{v+t}_{[x_j]}(\mathcal{S}_2).$$

It follows that $\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) \mid (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2)] = u = v + t \geq v = \left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1)\right]$. □

*Remark* 2.6. Next we generalize the inequality in proposition 2.5 to arbitrary sub-expansions in a chain.

**Theorem 2.7.** *Let $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2) \leq \cdots \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ be a chain of sub-expansions of the expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$. Then*

$$\sum_{i=k+1}^{n}\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i-1})\right] < \mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_k) \mid (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)] + (n-k).$$

*Proof.* Let $\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_k) \mid (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)] = u$ then

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_n) \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_k)$$

so that there exists some $r_k \in \mathbb{N}$ such that

$$\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_n) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_k}_{[x_j]}(\mathcal{S}_k).$$

Again by exploiting the of sub-expansion $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_k) \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{k+1})$, it follows that there exists some $r_{k+1} \in \mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_k) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_{k+1}}_{[x_j]}(\mathcal{S}_{k+1})$$

so that

$$\begin{aligned}\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_n) &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_k}_{[x_j]}(\mathcal{S}_k)\\ &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_k+r_{k+1}-1}_{[x_j]}(\mathcal{S}_{k+1}).\end{aligned}$$

Again by exploiting further the chain $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{k+1}) \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{k+2})$, it follows that there exists some $r_{k+2}\in\mathbb{N}$ such that

$$(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{k+1}) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_{k+2}}_{[x_j]}(\mathcal{S}_{k+2})$$

so that we can write

$$\begin{aligned}\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_n) &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_k}_{[x_j]}(\mathcal{S}_k)\\ &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_k+r_{k+1}-1}_{[x_j]}(\mathcal{S}_{k+1})\\ &= (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_k+r_{k+1}+r_{k+2}-2}_{[x_j]}(\mathcal{S}_{k+2}).\end{aligned}$$

By continuing this argument, we obtain

$$\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{u}_{[x_j]}(\mathcal{S}_n) = (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)^{r_k+r_{k+1}+r_{k+2}+\cdots+r_n-(n-k)}_{[x_j]}(\mathcal{S}_n).$$

It follows that

$$\begin{aligned}u &= r_k + \sum_{i=k+1}^{n} r_i - (n-k)\\ &\geq \sum_{i=k+1}^{n} r_i - (n-k)\end{aligned}$$

with $r_i = \left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i-1})\right]$ and $\mathbb{D}[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_k) \mid (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)] = u$ and the claimed inequality follows immediately. □

**Corollary 2.8.** *Let* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_1) \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_2) \leq \cdots \leq (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$ *be a chain of sub-expansions of the expansion* $(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_n)$. *Then*

$$\sum_{i=k+1}^{n}\left[(\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_i) : (\gamma^{-1}\circ\beta\circ\gamma\circ\nabla)_{[x_j]}(\mathcal{S}_{i-1})\right] > n-k.$$

## References

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