Remarks on the axiom of comprehension.

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Abstract

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Introduction

The Oxford Reference contains under Quick Reference[1] the text "The unrestricted axiom of comprehension in set theory states that to every condition there corresponds a set of things meeting the condition: (∃y) (y={x : Fx}). The axiom needs restriction, since Russell's paradox shows that in this form it will lead to contradiction. For the classical repair see separation, axiom of.".

This is not true. The axiom doesn't need restriction because of
Russell's paradox.
Notation: In the term (∃y) (y={x : Fx}) we replace Fx by C(x) to make it more clear that all objects x that satisfy condition C are meant.

What kind of restriction the axiom needs.

Conditions can be divided into the following categories:

1. Every object x meet the condition.
2. There is no object x that satisfies the condition.
3. There are objects that meet the condition, but there are also objects that don't meet the condition.

Regarding the first case, where every object x meet the condition. This means C({x : C(x)}) is true and therefore $\{x : C(x)\} \in \{x : C(x)\}$. At the same time, $\{x : C(x)\} \notin \{x : C(x)\}$ must apply, since otherwise {x : C(x)} cannot be contained. This is a contradiction. Therefore, the formation of sets with the help of the condition C is only possible if ¬C({x : C(x)}) is true and therefore $\{x : C(x)\} \notin \{x : C(x)\}$. For this reason, the axiom of comprehension must be restricted so that only conditions of the second or third category are allowed.

Why Russell's paradox doesn't lead to contradiction.

Since in Russell's paradox $x \notin x$ the variable x is a set and therefore, as shown above, $x \notin x$ is always true. This means a set created with Russell's paradox $x \notin x$ doesn't exist. This is clarified by the following conclusions.

- $\exists \{x : x \notin x\} \implies \{x : x \notin x\} \notin \{x : x \notin x\} \implies \neg \exists \{x : x \notin x\}$.
- $\neg \exists \{x : x \notin x\} \implies \{x : x \notin x\} \in \{x : x \notin x\} \implies \neg \exists \{x : x \notin x\}$

Conclusion.

The axiom of comprehension must be restricted, not because of Russell's paradox, but because conditions that are satisfied by all objects create a contradiction. If such conditions are excluded, then Russell's paradox is also excluded from set formation. Conditions that are satisfied by every object must be excluded from the formation of sets. Therefore, each set has the property of not being contained within itself.

References

[1] Oxford Quick Reference "axiom of comprehension" https://www.oxfordreference.com/view/10.1093/oi/authority.20110803095628841