Natural ways of mapping subsets to subsets

Pierre-Yves Gaillard

If X is a set, M its monoid of self-maps and P its power set, then P can be viewed as a left M-set ${}_MP$ or as a right M-set P_M. We compute the monoids $\operatorname{End}{}_MP$ and $\operatorname{End}P_M$.

Let X be a set, $M = X^X$ its monoid of self-maps (that is, $M = \{f : X \to X\}$) and P its power set (that is, $P = \{A \mid A \subset X\}$). Then P has a left M-set structure given by

$$fA = f_*A = \{fa \mid a \in A\}$$

and a right M-set structure given by

$$Af = f^*A = f^{-1}A = \{x \in X \mid fx \in A\}.$$

We denote these two M-sets by ${}_MP$ and M-set P_M respectively. Our purpose is to compute the monoids $\operatorname{End}{}_MP$ and $\operatorname{End}P_M$. In the sequel we denote fA and Af by f_*A and f^*A respectively.

Define the maps $\alpha, \beta, \gamma, \delta : P \to P$ by the formulas

$$\alpha A = A, \quad \beta A = \varnothing, \quad \gamma A = X \setminus A, \quad \delta A = A.$$

(Here $\varnothing$ is the empty set and $X \setminus A$ the complement of A in X.)

Theorem 1. *We have* $\operatorname{End}{}_MP = \{\alpha, \beta\}$ *and* $\operatorname{End}P_M = \{\alpha, \beta, \gamma, \delta\}$.

It suffices to show $\operatorname{End}{}_MP \subset \{\alpha, \beta\}$ and $\operatorname{End}P_M \subset \{\alpha, \beta, \gamma, \delta\}$. Indeed, the converse inclusions are clear. Moreover, to prove Theorem 1 we can, and do, assume that X has at least two elements.

1 The monoid $\operatorname{End}{}_MP$

The M-sets considered in this section are **left** M-sets.

Lemma 2. *If $\varepsilon : P \to P$ is a morphism of M-sets, then $\varepsilon \in \{\alpha, \beta\}$.*

Proof. This will follow immediately from the four steps below.

Step 1: We have $\varepsilon\varnothing = \varnothing$. Proof: The equalities $\varepsilon\varnothing = \varepsilon f_*\varnothing = f_*\varepsilon\varnothing$ hold for all f in M. This implies $\varepsilon\varnothing = \varnothing$.

Note: In view of Step 1 it suffices to show that we have either $\varepsilon A = \varnothing$ for all A in P, $A \neq \varnothing$, or $\varepsilon A = A$ for all A in P, $A \neq \varnothing$.

Step 2: We have $\varepsilon X \in \{\varnothing, X\}$. Proof: Since $\varepsilon f_* X = f_* \varepsilon X$ for all f in M, we get $\varepsilon X = f_* \varepsilon X$ for all surjection $f : X \twoheadrightarrow X$, and thus $\varepsilon X \in \{\varnothing, X\}$.

Step 3: If $\varepsilon X = \varnothing$, then $\varepsilon = \beta$. Proof: We have $\varepsilon f_* X = f_* \varnothing = \varnothing$ for all f in M. This entails $\varepsilon A = \varnothing$ for all A in P, $A \neq \varnothing$, hence $\varepsilon = \beta$ by the Note.

Step 4: If $\varepsilon X = X$, then $\varepsilon = \alpha$. Proof: We have $\varepsilon f_* X = f_* X$ for all f in M. This implies $\varepsilon A = A$ for all A in P, $A \neq \varnothing$, hence $\varepsilon = \alpha$ by the Note. $\square$

2 The monoid $\mathrm{End}\, P_M$

The M-sets considered in this section are **right** M-sets.

Let $\varepsilon : P \to P$ be a morphism of M-sets. We must show:

Lemma 3. *If $\varepsilon : P \to P$ is a morphism of M-sets, then $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$.*

Proof. We generalize slightly the notation used so far. Set $2 := \{0, 1\}$. For all set Y write Y^* for the set of subsets of Y and identify Y^* to the set 2^Y of all maps $Y \to 2$ by attaching to $A \subset X$ the map f defined by $fx = 1$ if and only if $x \in A$. Moreover we associate with a map $g : Z \to Y$ the map $g^* : Y^* \to Z^*$ defined by $g^* A = g^{-1}(A)$. Note that, if $f : Y \to 2$ is the map attached to A described above, then the map $Z \to 2$ attached to $g^* A$ is $f \circ g$, so that it is natural to denote this map by $g^* f$.

Claim: If A is a nonempty proper subset of X, then $\varepsilon A \in \{\varnothing, A, X \setminus A, X\}$.

Proof of the claim. Let A be as above and $f : X \to 2$ the map attached to A. Since for any $B \subset 2$ we have $f^* B \in \{\varnothing, A, X \setminus A, X\}$, it suffices to show that εA is of the form $f^* B$ with $B \subset 2$. Pick x_0 in $X \setminus A$ and x_1 in A, and define $g : 2 \to X$ by $g(i) = x_i$. Then $f \circ g$ is the identity of 2, and we get

$$\varepsilon A = \varepsilon f = \varepsilon(f \circ g \circ f) = \varepsilon((g \circ f)^*(f)) = (g \circ f)^*(\varepsilon f) = f^*(g^* \varepsilon f).$$

This proves the claim.

Recall that A is a nonempty proper subset of X. Let C be any subset of X, define $h \in M$ by $hx = x_1$ if $x \in C$ and $hx = x_0$ if $x \in X \setminus C$, and observe the equalities $h^* A = C$ and $\varepsilon C = \varepsilon h^* A = h^* \varepsilon A$. The claim implies $\varepsilon A \in \{\varnothing, A, X \setminus A, X\}$. If $\varepsilon A = \varnothing$ then $\varepsilon C = h^* \varnothing = \varnothing$. If $\varepsilon A = A$ then $\varepsilon C = h^* A = C$. The cases $\varepsilon A = X \setminus A$ and $\varepsilon A = X$ are similar. This shows $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$, as desired. $\square$

Now Theorem 1 follows from Lemmas 2 and 3.