Received 26 April 2016; | Revised 6 June 2016; | Accepted 6 June 2016

DOI: xxx/xxxx

E. Hitzer, C. Lavor, D. Hildenbrand, Current Survey of Clifford Geometric Algebra Applications, to be published in Mathematical Methods in the Applied Sciences, 37 pages, (2022).

RESEARCH ARTICLE

Current Survey of Clifford Geometric Algebra Applications[†]

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Summary

We extensively survey applications of Clifford Geometric algebra in recent years (mainly 2019–2022). This includes engineering, electric engineering, optical fibers, geographic information systems, geometry, molecular geometry, protein structure, neural networks, artificial intelligence, encryption, physics, signal-, image- and video processing, and software.

KEYWORDS:
Clifford geometric algebra, engineering, geometry, physics, artificial intelligence, signal- and image processing, software

1 | INTRODUCTION

Clifford geometric algebra (GA) and Clifford analysis (also termed geometric calculus (GC)) is a rapidly developing field of pure and applied mathematics. In 2013 a popular survey[98] was written about its applications, followed by a more recent survey[35 1]. In this current survey we supplement these previous surveys by focusing on the period of 2019 (after the conference AGACSE 2018) up to the present (2022). We collected 200 references[2] for this survey in two ways (different from[35]). First we comprehensively used the new artificial intelligence academic publication search engine Dimensions (https://www.digital-science.com/product/dimensions/) and second we asked committee members of AGACSE 2021 to advise us on relevant publications from the same period.

The paper is structured as follows. In the next paragraph we provide standard introductory references to Clifford geometric algebra for readers still unfamiliar with this universal type of associative algebra. Section 2 deals with engineering applications at large, including electric engineering, optical fibers, robots, control, pose, material science, computer graphics and modeling. Section 3 deals with the wide field of geometry, including spinors and symmetry, computations in geometry, molecular geometry,

[†]Soli Deo Gloria. This paper is published under the terms of the Creative Peace License[99]. We wholeheartedly condemn the genocide China commits against the Uyghur minority: *Experts say that at least a million Uyghurs and other Muslims have been detained in the region and held in extra-judicial camps or sent to prisons. Former detainees and residents of Xinjiang have made allegations of torture, forced sterilization and sexual abuse.*[78].

[0]**Abbreviations:** CGA, conformal geometric algebra; GA, geometric algebra; GC, geometric calculus; NN, neural network; STA, space-time algebra.

[1]The survey[35] appeared as part of the *Topical Collection T.C. : SCIS&ISIS2020 – Modern Applications of Clifford Algebra*. Its introduction describes it as based on the conference series Applied Geometric Algebra for Computer Science and Engineering of the years 2015 and 2018, and the international conference workshops Empowering Novel Geometric Algebra for Graphics and Engineering in the years 2016 to 2020. Furthermore, the authors freely added some applications known to them.

[2]The other ca. 40 references mainly provide introductory and background information.

three-dimensional protein structures, computer algebra, curves and surfaces, and further theoretical developments in geometry. Section 4 explains applications to hypercomplex neural networks and artificial intelligence, geographic information systems, and encryption. Section 5 surveys new developments in physics of relativity, gravity, cosmology, classical physics, electromagnetism, optics, and importantly quantum physics. Section 6 guides through new developments in signal, image and video processing, in particular medical imaging, image processing and hypercomplex GA Fourier transforms, motion processing, estimation and filtering, features and detection. Section 7 on GA software surveys software libraries, computer algebra systems and software implementations. The clustering of these often very interdisciplinary works is far from trivial, and the interested reader is advised to check out subjectively less relevant sections of this survey as well. Section 8 concludes the paper followed by the extensive list of references.

GA has become popularly used in applications dealing with geometry. This framework allows to reformulate and redefine problems involving geometry in a highly intuitive and general way. GA was defined thanks to the work of W. K. Clifford[48] to unify and generalize Grassmann algebra[75] and W.R. Hamilton's quaternions[84] into a universal algebraic framework by adding the inner product to H. G. Grassmann's outer product. One of the geometric algebras that is often applied is conformal geometric algebra (CGA). It became better known through[90], is well described and illustrated in[60], and in a brief illustrated form in[96]. For standard references on GA, we refer to the following textbooks:[89,148,60]. Further in depth treatment can be found in the textbooks:[29,46,118,88,182]. A compact definition of GA is given in[69], see also[97,34]. Further noteworthy references on GA are[50,51,52,55,56,113,114,27,157,59,10]. Moreover,[11] provides a thorough study of the epistemological importance of Clifford geometric algebra.

Regarding the notation of Clifford geometric algebras, a certain variety can be found. To assist the readers we adopt the notation $Cl(p,q,r)$ for the Clifford geometric algebra of a space $\mathbb{R}^{p,q,r}$, with dimension $n = p+q+r$, with an orthonormal basis of p vectors squaring to $+1$, q vector squaring to -1, and r vectors squaring to 0. Note that for $r = 0$ it is customary to abbreviate $\mathbb{R}^{p,q} = \mathbb{R}^{p,q,0}$, and $Cl(p,q) = Cl(p,q,0)$. Furthermore, for $q = 0$, many authors abbreviate $\mathbb{R}^n = \mathbb{R}^{n,0}$, and $Cl(n) = Cl(n,0)$, e.g., the GA of three-dimensional Euclidean space $\mathbb{R}^3$ is $Cl(3,0)$, and conformal geometric algebra (CGA) for three-dimensional Euclidean space, extended by a Minkowski type plane $\mathbb{R}^{1,1}$ is $Cl(4,1)$. But note that particularly in the field of Clifford analysis, authors may instead use $\mathbb{R}^n = \mathbb{R}^{0,n}$, it is therefore advisable when reading a publication to first ascertain which notation the author uses. Moreover one often finds $\mathcal{G}_{p,q,r} = \mathbb{G}_{p,q,r} = G_{p,q,r} = Cl_{p,q,r} = \mathbb{R}_{p,q,r}$, etc.

2 | ENGINEERING APPLICATIONS

GA has proven its role in many disciplines. Indeed, whenever Euclidean transformations or $SO(n)$ symmetries appear, it is only natural to formulate and solve the problem using GA. In particular, this is typical for engineering problems. Namely, an Euclidean transformation of an element $S \in Cl(4,1)$ in CGA is realized by conjugation with an invertible element $T \in Cl(4,1)$, i.e. $S \mapsto TST^{-1}$, such that $T\tilde{T} = 1$. Note that the inverse T^{-1} can be replaced by the reverse $\tilde{T}$.

For instance, let us consider the $\mathbb{R}^{4,1}$ basis $\{e_1, e_2, e_3, e_0, e_\infty\}$ with $\{e_1, e_2, e_3\}$ representing Euclidean orthonormal basis, and e_0 and e_∞, representing the origin and point at infinity, respectively. Then translation in the direction $t = t_1e_1+t_2e_2+t_3e_3$ is realized by the multivector (translator) $T = 1 - \frac{1}{2}te_\infty$ and the rotation around the origin and the normalized axis $L = L_1e_1 + L_2e_2 + L_3e_3$ by an angle φ is realized by the multivector (rotor) $R = e^{\frac{1}{2}l\varphi} = \cos\frac{\varphi}{2} - l\sin\frac{1}{2}\varphi$, where $l = L(e_1 \wedge e_2 \wedge e_3) = L_1(e_2 \wedge e_3) + L_2(e_3 \wedge e_1) + L_3(e_1 \wedge e_2)$.

Note that this perfectly corresponds to the notation of quaternion algebra and, consequently, screw theory. The rotation around a general point and axis is given by conjugation with an element $TR\tilde{T}$. A general composition of a translator with a rotor is called a motor. Before we cite papers focused on specific problems, let us point out more complex surveys and collections. For engineering and computer science advances from 1995 to 2020, we refer to[22]. For computer vision, graphics and neurocomputing, see[17]. And finally, for the collection of applications in data engineering, see[230].

2.1 | Electric Engineering and Optical Fibers

GA has become an essential tool for alternating current (AC) analysis. The fundamental idea is to replace the phasor (a complex number) by a vector which is then treated in terms of GA. In fact, it is shown in[166] that the complex phasor representation is just a particular case (single frequency) of a more general case (multiple frequencies) in AC systems. Moreover, also fundamental transformations widely used by the power engineering community are strongly based on geometrical considerations. Indeed, in[165] it is shown that Clarke, Park, and Depenbrock's Fryze-Buchholz-Depenbrock (FBD) transformations can be derived by imposing orthogonality on the voltage and current vectors defined in a Euclidean space by using GA.

With this knowledge at hand, it is clear that major upgrades to existing power theories based on geometric algebra appear[162] and instantaneous power[158], apparent power[164], power flow[161] and non-active power[163] calculations are formulated in GA terms for non-sinusoidal circuits. In[167], the authors analyze three-phase balanced electrical circuits under sinusoidal and non-sinusoidal voltage supply.

Moreover, special treatment is devoted to compensations[43] in non-sinusoidal circuits, including passive compensation[159], and quadrature current compensation[160].

Applications of GA appear surprisingly even in optimal control of energy systems. This is due to the identification of heating and cooling loads behavior with sources of harmonics. Indeed, in[154] a new framework based on GA is presented in the definition of a multivectorial distortion power concept, which is represented by a bivector that is geometrically interpreted to distinguish the rotated distortion and distortion power bivectors in these kinds of loads. Both bivectors, and their relations to the phase angles of distorted voltage are the main subject to interpret an optimal control of building energy.

2.2 | Robotics and Control

With the model of Euclidean space and transformations as elements of the same algebra, it is only natural to model Euclidean motions in compact and coordinate-free form in GA,[18]. Indeed, forward and inverse kinematics has always been the engineering problem to be mentioned when talking about applications of GA. Let us start with holonomic mechanisms. Because the origins go back to early 2000s, we remind of Eduardo Bayro-Corrochano, Joan Lasenby and Andreas Aristidou's contribution's in this field. Furthermore, the current survey contains more specific problems of robotic systems, such as approximation of statics and kinematics of robotic manipulators[203]. Other robotic systems such as parallel mechanisms[45,207,107,131,239], Stewart platforms[238] or delta robots[82] have also been studied.

Among nonholonomic mechanism kinematics, robotic snakes[95] are of particular interest. Apart from kinematics, the solution to the problem of rigid body dynamics may be found in[81]. The problem of a robot control in general is addressed, too, from various points of view. Let us mention the approximation of a control by fitting three control poses for planar motions[171] or Newton-Euler modeling for a multi-copter[13]. A specific problem for control of a robot with human interaction is tackled in particular for medical robot vision in[20]. Let us note that apart from the use of quaternions, the algebra of dual quaternions often serves as the model space. Among many attempts to automate the robot modeling and control, we mention the library DQ Robotics,[131].

2.3 | Pose Estimation

Pose estimation is a computer vision task that infers the pose of a person or object in an image or video. We can also think of pose estimation as the problem of determining the position and orientation of a camera relative to a given person or object. This is typically done by identifying, locating, and tracking a number of keypoints on a given object or person. Humans fall into a particular category of objects that are flexible. By bending our arms or legs, keypoints will be in different positions relative to others. Most inanimate objects are rigid. Predicting the position of these objects is known as rigid pose estimation. Standard approach to this problem would include neural networks but, having the keypoints detected and knowing the technical details of the camera, one can use precise geometric operations for the pose estimate. This includes intersections of lines and planes, rotations and translations, i.e. this is exactly the place where GA comes to action.

There is also a key distinction to be made between two-dimensional (2D) and three-dimensional (3D) pose estimation. 2D pose estimation simply estimates the location of keypoints in 2D space relative to an image or video frame. The model estimates an x and y coordinate for each keypoint. 3D pose estimation works to transform an object in a 2D image into a 3D object by adding a z-dimension to the prediction. As in GA the difference between 2D and 3D model is just the algebra dimension, it is only natural that it is used for 3D human pose reconstruction[108]. To justify the use of GA more precisely, 3D pose estimation is a more challenging problem for machine learners, given the complexity required in creating data sets and algorithms that take into account a variety of factors – such as an image or video background scene, lighting conditions, and more. This is typically done by identifying, locating, and tracking a number of keypoints on a given object or person.

Another task for 3D pose estimation is the automated camera-based control of robots or other vehicles. Typically, unmanned aerial vehicle (UAV) self-location is such a problem[231].

On the other hand, the preprocessing of pictures for keypoint detection is also a subject for GA application. Indeed, for rotating space debris detection and band contour extraction see [130]. The final contour points are extracted by determining the concave and convex points in conformal space. Simulation results prove that the unified expression of the geometric features in CGA not only reduces the computational complexity but also increases the speed of the contour point extraction. In addition, based on similar approach, for pose estimation of space targets see [132].

2.4 | Material Science

The cell method (CM) is an algebraic numerical method based on the use of global variables: configuration, source and energetic global variables. Configuration variables with their topological equations, on the one hand, and source variables with their topological equations, on the other hand, define two vector spaces that are a bialgebra and its dual algebra. The operators of these topological equations are generated by the outer product of GA, for the primal vector space, and by the dual product of the dual algebra, for the dual vector space. The topological equations in the primal cell complex are coboundary processes on even exterior discrete p-forms, whereas the topological equations in the dual cell complex are coboundary processes on odd exterior discrete p-forms. Being expressed by coboundary processes in two different vector spaces, compatibility and equilibrium can be enforced at the same time, with compatibility enforced on the primal cell complex and equilibrium enforced on the dual cell complex. In [71], the CM shows its maximum potentialities right in domains made of several materials, being an algebraic approach that can easily treat any kind of domain discontinuity.

2.5 | Computer Graphics and Modeling

In computer graphics, development of GA applications is well visible in the number of computer graphics conferences with GA-oriented contributions. Let us mention SIGGRAPH 2019 [77], GAME2020 and annual CGI with its GA-focused workshop ENGAGE. A leading name in this field is currently Steven de Keninck, for his contribution to CGI 2019 on realtime rendering (see [58]).

Another standard graphics tool is ray-tracing. In [83] the authors implement a simple ray-tracer in CGA with a Blinn–Phong lighting model, before putting it to use to examine ray intersections with surfaces generated from the direct interpolation of geometric primitives. General surfaces formed from these interpolations are rendered using analytic normals. Finding the normal line to the interpolated surfaces and its use is shown in lighting calculations for the ray tracer and in generating vertex normals for exporting the evolved surfaces as polygonal meshes. A different use of GA ray-tracing for studying radio wave propagation is shown in [67].

GA contributes to graphic object modeling and manipulation, too. In [112], the authors perform real-time cuts and tears as well as drill holes on a rigged 3D model. These operations can be performed before and/or after model animation, while maintaining deformation topology. Indeed, mesh deformation is a task well suited for GA as described in [229] where a rotor estimation algorithm is proposed in CGA. On the other hand, modeling a real world object as a polygonal mesh is not a straightforward subject. To learn how to model general polyhedra as an unordered discrete and finite set of geometric numbers of a projective Clifford algebra, see [199].

3 | APPLICATIONS IN GEOMETRY

3.1 | Spinors and Symmetry

Starting from Pauli and Dirac matrices of 1928 [211] presents a friendly and unified version of the classical groups of mathematical physics as subgroups of subalgebras of real GA, created and presented by Clifford in 1879.

The paper [54] looks at the icosahedral induction of $H_3 \to H_4$ in $Cl^+(3,0)$. GA is used to perform group theoretic calculations based on the versor theorem and the Cartan–Dieudonné theorem, giving a simple construction of the Pin and Spin covers. Using this connection with H_3 via induction sheds light on geometric aspects of the H_4 root system (600-cell) as well as other related polytopes and their symmetries, such as the famous Grand Antiprism and the snub 24-cell. The uniform construction of root systems from 3D and the uniform procedure of splitting root systems with respect to subroot systems into separate invariant sets allows further systematic insight into the underlying geometry. The Coxeter plane is constructed, used for visualizing complementary pairs of invariant polytopes. This approach in GA is systematic and general for calculations concerning groups, in particular reflection groups and root systems.

How transformations in $\mathbb{R}^3$ can be described in terms of the GA $Cl(3,3)$ of quadratic space $\mathbb{R}^{3,3}$ is discussed in[209]. It is shown that this algebra describes in a unified way the operations of reflection, rotation (circular and hyperbolic), translation, shear and non-uniform scale. Moreover, using Hodge duality, cotranslation is defined, showing that perspective projection can be written in this GA as composition of translation and cotranslation. A pseudo-perspective can be implemented using the cotranslation operation. A general transformation of points can be described as well. Expressions for reflection and rotation in $Cl(3,3)$ preserve subspaces associated with the algebras $Cl(3,0)$ and $Cl(0,3)$, so that reflection and rotation can be expressed in terms of $Cl(3,0)$ or $Cl(0,3)$. All the other operations mix these subspaces in such a way that these transformations need to be expressed in terms of the full $Cl(3,3)$. Points are represented by paravectors.

The derived category of a complete intersection X of bilinear divisors in the orbifold $\mathrm{Sym}^2\mathbb{P}(V)$ is studied in[185]. Results are in the spirit of Kuznetsov's theory of homological projective duality, and describe a homological projective duality relation between $\mathrm{Sym}^2\mathbb{P}(V)$ and a category of modules over a sheaf of Clifford algebras on $\mathbb{P}(\mathrm{Sym}^2 V^\vee)$. It starts by translating $D^b(X)$ into a derived category of factorizations on a Landau–Ginzburg (LG) model, and then applies VGIT to obtain a birational LG model. Finally, the derived factorization category of the new LG model is interpreted as a Clifford module category. In some cases this Clifford module category is computed as the derived category of a variety. A new proof of a result of Hosono and Takagi is obtained, where a certain pair of non-birational Calabi–Yau 3-folds has equivalent derived categories.

3.2 | Computation in Geometry

Principal angles are used in[153] to define an angle bivector of subspaces, fully describing relative inclination. Its exponential, related to the geometric product of blades, gives rotors connecting subspaces via minimal geodesics in Grassmannians, and decomposes giving Plücker coordinates, projection factors and angles with various subspaces. This leads to new geometric interpretations for this product and its properties, and to formulas relating other blade products (scalar, inner, outer, etc., including those of Grassmann algebra) to angles between subspaces. Contractions are linked to an asymmetric angle, while commutators and anticommutators involve hyperbolic functions of the angle bivector, shedding new light on their properties.

Projection factors describe in[152] the contraction of Lebesgue measures in orthogonal projections between subspaces of a real or complex inner product space. They are connected to Grassmann algebra and GA and the Grassmann angle between subspaces, and lead to generalized Pythagorean theorems, relating measures of subsets of real or complex subspaces and their orthogonal projections on certain families of subspaces. Complex Pythagorean theorems differ from the real ones with non squared measures, and have implications for quantum theory. Projection factors of the complex line of a quantum state with the eigenspaces of an observable give the corresponding quantum probabilities. The complex Pythagorean theorem for lines corresponds to the condition of unit total probability, and may provide a way to solve the probability problem of Everettian quantum mechanics.

In[26] a simple computational approach to parametric/geometric Hermite interpolation problems by polynomial Minkowski Pythagorean hodograph (MPH) curves in $\mathbb{R}^{2,1}$ is presented and an algorithm described. First not a tangent but a normal vector space is constructed satisfying the prescribed MPH property matching given first order conditions, and then used to compute a possessing curve satisfying all the remaining interpolation conditions. Compared to other methods using special formalisms (e.g. GA), the presented approach is based only on solving systems of linear equations.

Direct linear interpolation of conformal geometric objects is shown in[80] to be of intuitive and practical use. A method that generates useful interpolations of point pairs, lines, circles, planes and spheres is presented and algorithms and proofs of interest for computer vision applications that use this direct averaging of geometric objects are described. Similarly,[120] addresses the problem of recovering covariant transformations between objects – specifically: lines, planes, circles, spheres and point pairs. Using the covariant language of CGA, such transformations are derived very simply. In CGA, rotations, translations, dilations and inversions can be written as single rotor multivectors. Rotors taking a line to a line (or plane to a plane etc.) can easily be formed and their nature is investigated. Recovering the rotor between one object and another of the same type, a useable metric which tells how close one line (plane etc) is to another, can be a function of how close this rotor is to the identity. Thus metrics can be defined for a number of common problems, including noise.

The problem of computing the inverse in GA (symbolically and numerically) of arbitrary dimension is solved in[192]. Basis-free formulas of different types (explicit and recursive) for the determinant, other characteristic polynomial coefficients, adjugate, and inverse in real GA over vector spaces of arbitrary dimension n are presented. The formulas involve only the operations of multiplication, summation, and operations of conjugation without explicit use of matrix representation. GA methods are used (including quaternion typification and operations of conjugation of special type) and generalizations to GA of numerical methods of matrix theory (the Faddeev–LeVerrier algorithm based on the Cayley–Hamilton theorem; calculating the characteristic polynomial coefficients using Bell polynomials). Construction of operations of conjugation of special type and relations between these operations and the projection operations onto fixed subspaces of GA are presented and used in the analytical proof of formulas for the determinant, other characteristic polynomial coefficients, adjugate, and inverse in GA. Basis-free inverse formulas give basis-free solutions to linear algebraic equations, widely used in computer science, image and signal processing, physics, engineering, control theory, etc.

Closed form expressions to calculate the exponential of a general multivector (MV) in GA $Cl(p, q)$ are presented in [53] for $n = p + q = 3$. The obtained exponential formulas are applied to find exact GA trigonometric and hyperbolic functions of MV arguments in agreement with series expansion of MV hyperbolic and trigonometric functions. The exponentials may be applied to solve GA differential equations, in signal and image processing, automatic control and robotics.

Villarceau circles are considered in [61] as the orbits of specific composite rotors in CGA that generate knots on nested tori. Conformal parametrizations of these circular orbits are computed by giving an equivalent, position-dependent simple rotor that generates the same parametric track for a given point. This allows compact derivation of quantitative symmetry properties of Villarceau circles. Their role in Hopf fibration and as stereographic images of isoclinic rotations on a 3-sphere of the 4D Clifford torus is derived. CGA can be used to generate 3D viewer images of the results. The compact coordinate-free CGA representation can aid in the analysis of Villarceau circles (and torus knots) occurring in Maxwell and Dirac equations.

Three different metrics are treated in [64] that may be used to compare orientations, compute the corresponding optimal averages, and relate them in a unified framework. The metrics are based on measuring differences in angular arc, axis tilt (or bivector) and rotational chord. The optimal combination of orientation estimates according to their local variances are computed, as may be employed in a Kalman filter update step. The GA characterization of rotations (by bivector angle of rotors) allows to perform computations and comparisons coordinate-free, and thus to compare and evaluate the alternative parametrizations. This extends the traditional quaternion representation of rotations.

The paper [57] introduces a novel matrix-free implementation of the Levenberg-Marquardt algorithm, in GA. The resulting algorithm is compact, geometrically intuitive, numerically stable and well suited for efficient GPU implementation. An implementation of the algorithm and the examples in this paper are publicly available.

The paper [186] proposes that derivatives of holomorphic functions can be calculated in a way similar to the Complex Step Derivative (CSD) method by taking a small step in a quaternionic direction instead. It is demonstrated that in so doing the CSD properties of high accuracy and convergence are carried over to derivatives of holomorphic functions. Numerical experiments were performed using complex quaternions, the geometric algebra of space, and a two by two matrix representation thereof.

3.3 | Molecular Geometry and 3D Protein Structures

Molecular geometry deals with problems related to the determination of 3D molecular structures. It is well known that the knowledge of such structures is fundamental to understand the function of the associated molecules, which is particularly true for proteins.

In [5], CGA is applied to calculate 3D protein structures using distance information provided by nuclear magnetic resonance (NMR) experiments. Together with the information from the chemistry of proteins, NMR data allow the definition of an atomic order such that the distances related to the pairs of atoms $\{i-3, i\}$, $\{i-2, i\}$, $\{i-1, i\}$ are available. Using additional distances associated to pairs $\{j, i\}$, with $j < i - 3$, the position of atom i ($i > 3$) can be calculated from the positions of the three predecessors and from the intersection of spheres centered at the positions for atoms $j, i-3, i-2, i-1$, with radius given by the atomic distances $d(j, i)$, $d(i-3, i)$, $d(i-2, i)$, $d(i-1, i)$.

Using orthogonal spheres, the work in [62] proposes a strategy for instances of the problem where uncertainties in NMR data must be considered. In this case, some of the distances $d(j, i)$, $j < i-2$, may not be precise, which implies that sphere intersections must be calculated considering that the centers and radius of the related spheres may not be fixed anymore.

Authors in [39] generalize the problem above considering an arbitrary dimension $n > 3$. A recent CGA approach to deal with NMR uncertainties is proposed in [124], and, in [123] and [125], the authors apply Oriented CGA (an extension of Oriented Projective Geometry) to take into account issues of arc orientation problems also as a result of sphere intersections with data uncertainties.

3.4 | GA and Computer Algebra

To make geometric reasoning simpler, more expressive, and richer in geometric meaning, [236] establishes a geometric algebraic system (point geometry built on nearly 20 basic properties/formulas about operations on points) while maintaining the advantages of the coordinate method, vector method, and particle geometry method and avoiding their disadvantages. Geometric relations in the propositions and conclusions of a geometric problem are expressed as identical equations of vector polynomials according to point geometry. Thereafter, a proof method that maintains the essence of Wu's geometric theorem proving method[3] is introduced to find the relationships between these equations. Tests were done on more than 400 geometry statements.

The paper [128] presents the practice of automated theorem proving in Euclidean geometry with null GA, a combination of CGA and Grassmann-Cayley algebra. This algebra helps generating extremely short (one- or two-termed) readable proofs. Besides, the theorems are naturally extended from qualitative description to quantitative characterization by removing one or more geometric constraints from the hypotheses.

3.5 | Curves and Surfaces in GA

Singularity theory and GA are combined in [135] to study singular ruled surfaces, taking advantage of the dual number in GA to transform singular ruled surfaces into dual singular curves on the dual unit sphere. By using research methods from singular curves, the definition of the dual evolute of the dual front in the dual unit sphere is given, and the k-th dual evolute of the dual front is provided. Moreover, the ruled surface corresponding to the dual evolute and k-th dual evolute is considered and developable conditions of these ruled surfaces are given.

An orthogonal sphere representation of arcs on spatial circles can be used to compactly perform Boolean combinations of such arcs in CGA [62], where the oriented nature allows both minor and major arcs to be treated. Easily computable quantities discriminate the cases of relative positions. An application in the first stages of a problem in Discretizable Molecular Distance Geometry is included. It is suggested how to extend this characterization by orthogonal spheres to manifolds of arcs in subsequent stages, using probabilistic eigenspheres of the distributions.

The work [104] on *geometric algebra for conics* (GAC) presents a particular GA $Cl(5,3)$ together with an embedding of two–dimensional Euclidean space such that multivectors can be interpreted as arbitrary conic sections. A full description of conic section analysis, classification and their transformations is provided, together with examples in Maple (including source code). Furthermore, [105] presents a conic fitting algorithm in this framework, introducing a novel normalization condition invariant with respect to rotations and scaling. Comparative MATLAB examples are included. The work [147] adds implementations of several conic fitting algorithms in GAC, with additional conditions for optimization problems, such as center point position at the origin of a coordinate system, axial alignment with coordinate axes, or, combinations of both. A MATLAB implementation is included with examples. Specific conic intersections in GAC using operations that may be expressed as sums of products are described in [37]. No solver for a system of quadratic equations is needed avoiding numerical errors. Specific conics connected to intersections of conics in a general mutual positions are explained. Symbolic operations may be calculated directly in GAALOP-Web software, the basis coefficients may be read off in an appropriate basis and the result immediately visualized, and compared with the Maple package Clifford.

The alternative approach called the double conformal/Darboux cyclide geometric algebra (DCGA) in $Cl(8,2)$ described in [65], can also represent conic sections, planar sections of Darboux cyclides, called cyclidic sections. Additionally, many operations on these entities are given in terms of algebraic expressions, computer code, and of computer-generated graphics. Operations include reflection, projection, rejection, and intersection with respect to spheres and planes, rotation, translation, and dilation. Applications include orthographic and perspective projections of conic sections onto view planes, of interest in computer graphics or other computational geometry subjects.

A third approach in [32] is quadric conformal geometric algebra (QCGA) $Cl(9,6)$, where three-dimensional quadratic surfaces can be defined by the outer product of CGA points in higher dimensions, or alternatively by a linear combination of $\mathbb{R}^{9,6}$ basis vectors with coefficients of the implicit quadratic equation. These multivector expressions code all types of quadratic surfaces in arbitrary scale, location, and orientation. Special attention is paid to axis aligned quadrics. The work [33] considers all previous frameworks together, covering all required operations to represent, transform, and intersect quadrics, and extract geometric properties in a unified way using either control points or implicit form coefficients.

[3] See references in [236].

Algebraic detail of how two-dimensional conics can be defined by outer products of conic CGA $Cl(5,3)$ points in higher dimensions are given in[101]. These multivector expressions code all types of conics in arbitrary scale, location and orientation. CGA of two-dimensional Euclidean geometry is fully embedded as algebraic subset. With small model preserving modifications, it is possible to consistently define in conic CGA (CCGA) versors for rotation, translation and scaling, similar to GAC, but slightly simpler, especially for translations. This is extended to three dimensional quadrics in[100] in the style of QCGA $Cl(9,6)$. Multivector expressions code all types of quadrics in arbitrary scale, location and orientation. Furthermore, a newly modified (compared to QCGA) approach now allows not only the use of the standard intersection operations, but also of versor operators (scaling, rotation, translation). The new algebraic form of the theory is explained in detail.

The paper[110] uses twisted superpotentials and twists of superpotentials in the Mori–Smith sense, checking the Artin–Schelter (AS)-regularity of GAs whose point schemes are not elliptic curves. For GAs whose point schemes are elliptic curves, a simple condition for three-dimensional quadratic AS-regular algebras is given. An application shows that every three-dimensional quadratic AS-regular algebra is graded Morita equivalent to a Calabi–Yau AS-regular algebra.

The aim of the paper[49] is to investigate a GA with advantageous features. Points, lines and planes are presented naturally by element of grades 1, 2, and 3 respectively. The self-reverse elements in the algebra form a field. This allows an equivalence relation between elements of grade 2 to be defined so that, although not every grade 2 element corresponds to a line, each equivalence class does, and vice versa. Examples are given.

3.6 | Theoretical Development of GA

Motivated by questions on orthogonality of isometries,[126] presents a new construction of the conformal model from elementary linear algebra. Pictures help the reader to understand the conformal model and useful matrix representations of isometries are obtained, e.g., for applications to computational geometry, including computer graphics, robotics, and molecular geometry.

Inner automorphisms that leave invariant fixed subspaces of real and complex GA-subspaces of fixed grades and subspaces determined by the reversion and the grade involution are considered in[193]. Groups of elements defining such inner automorphisms are studied. Some of these Lie groups can be interpreted as generalizations of Clifford, Lipschitz, and spin groups. The corresponding Lie algebras are studied. Generalizations are possible to graded central simple algebras (without or with involution).

The paper[234] presents a CGA constraint structure of seven straight lines on three adjacent planes (7L3P) and its projective invariants, obtained from a single frame image. Comparing with the use of multi-frame images to calculate invariants, the new method is more convenient. First, this paper uses the projective property of points as an index to introduce the projective transformation properties of geometric structure of three lines on two adjacent planes. Then using it as an index, the invariant of the geometric structure of 7L3P is proposed, and the invariant of the geometric structure of five lines on three adjacent planes is obtained, followed by experimental verification.

The symmetries described by Pin groups are the result of combining a finite number of discrete hyperplane reflections. The work[187] shows how an analysis using GA provides a picture complementary to that of the classic matrix Lie algebra approach, keeping information about the number of reflections in a transformation. This imposes a graded structure on Lie groups, which is not evident in their matrix representation. With this graded structure, the invariant decomposition theorem was proven: any composition of k linearly independent reflections can be decomposed into $\lceil k/2 \rceil$ commuting factors, each product of at most two reflections. This generalizes a conjecture by M. Riesz, with the Mozzi-Chasles' theorem as special 3D case. Examples such as Lorentz transformations, Wigner rotations, and screw transformations are discussed, as well as closed form formulas for the exponential and logarithmic function for all Spin groups, and one can identify planes, lines, and points, as invariants of k-reflections. Finally, a novel matrix/vector representation for GA $Cl(p,q,r)$, and its use in $E(3)$ illustrates the relationship with classic covariant, contravariant and adjoint representations for the transformation of points, planes and lines.

Projective GA (PGA), i.e. GA with degenerate signature $(n,0,1)$, is understood in[106] as a subalgebra of CGA, where flat primitives share the same representation in PGA and CGA. Particularly, duality in PGA is treated in the framework of CGA. This leads to unification of PGA and CGA primitives important for software implementation and symbolic calculations.

A guided tour to the algebra of planes PGA is undertaken in[63]. It indeed shows how computationally efficient methods are incorporated and related, and how PGA elements naturally group into blocks of four coordinates in an implementation, and how this more complete understanding of the embedding suggests handy choices to avoid extraneous computations. In the unified PGA framework, one never switches between efficient representations for subtasks, and this obviously saves time otherwise spent on data conversion.

CGA is developed further in[129] from both the algebraic aspect and the geometric viewpoint, by exploring algebraic properties and geometric interpretations of graded monomials generated by null vectors. Computation of orientations of various dimensional spheres and planes, the trigonometry of long brackets, the graded null monomial representation of angles and directions, the geometric meaning of various single-graded null monomials, etc. are all included. This helps not only in understanding the geometric implication of various advanced symbolic algebraic manipulations in CGA, but also clarifies the intrinsic machinery behind CGA induced efficiency in symbolic geometric computing.

The work[195] explores Plücker relations in GA that can be fully characterized in terms of the geometric product, without multiple different formalisms and mathematical traditions found in the literature. It suggests a general and simple (practical) algorithm (the idea of which is due to Dung B. Nguyen, who presented it 2020 (online) during the ICCA12 conference in Heifei, China) to test if a given homogeneous multivector is a blade (i.e. could be written as an outer product of vectors). In contrast other algorithms (for example[28]) appear not practical or suitable even for $n = 4$ (see[210], page 26).[4]

The GA $Cl(3, 0)$, the natural generalization of both well-known Gibbs-Heaviside vector algebra and Hamilton's quaternions, is used in[196] to study spheroidal domains, spheroidal-graphic projections, the Laplace equation, and its Lie algebra of symmetries. The Cauchy-Kovalevska extension and the Cauchy kernel function are treated in a unified way. The concept of a quasi-monogenic family of functions is introduced and studied.

A generalization of a classical Clifford algebra is discussed in[208] together with connections to a generalization of a graded Clifford algebra. A geometric approach to studying these algebras, viewed through the lens of Artin, Tate and Van den Bergh's noncommutative algebraic geometry, is also presented. In[36] various algebras of hypercomplex numbers are considered and geometric structures over them with certain applications of these structures in theoretical physics are discussed.

The paper[4] aims at evaluating GA applications and characteristics, as well as assessing the new characteristic of pinched homotopy. A new approach to pinched tensors is elaborated and the impact of such tensors on homotopy is presented. The ordinary tensor product is substituted with the pinched one. Motivated by the different properties of the two tensors, several cases for showing the differences are studied, including counter examples.

The gap problem for almost quaternion-Hermitian structures is resolved in[117], i.e. the maximal and submaximal symmetry dimensions are determined, both for Lie algebras and Lie groups, in the class of almost quaternion-Hermitian manifolds. All structures with such symmetry dimensions are classified. Geometric properties of the submaximally symmetric spaces are studied, in particular, locally conformally quaternion-Kähler structures as well as quaternion-Kähler with torsion are identified.

4 | INFORMATION PROCESSING

4.1 | Neural Networks and Artificial Intelligence

The paper[173] uses CGA in order to extract features and simultaneously reduce the dimensionality of a data set for human activity recognition using a Recurrent Neural Network (NN). Human activity data in three dimensions are pre-processed and normalized by calculating deviations from mean coordinates. Next, the data are represented as vectors in CGA with dimensions reduced to return feature vectors. Finally, a Recurrent NN model is trained with the feature vectors. An experimental Motion Capture data set shows for the new method the best test results of 92.5 %. In the case of data distributed on a hyper-sphere, the developed method can thus help to extract features and simultaneously reduce the dimensionality of a data set.

The work[205] presents a novel paradigmatic revision of traditional NN, using network theoretic methods and CGA. A unique theoretical framework called the *hyperfield cognition framework* (HCN) expands upon the mathematical foundations of NN in CGA $Cl(4, 1)$. This allows to construct a novel theoretical computational engine, similar to a standard artificial NN, but with numerous added benefits: permitting multiple training programs simultaneously, computational multiplicity in a single engine, reduced sensitivity to adversarial perturbations in training sets, broader capabilities and plasticity in training of networks, and robust and streamlined NN. Two case studies demonstrate the HCN utility and merits for automobile fuel efficiency and residual resistance modeling per unit weight of displacement of ships.

In[206], a novel methodology for modeling and forecasting of time series (and multivariate systems with known or hidden variables) using higher-dimensional networks in $\mathbb{R}^{4,1}$ is presented. Time series data are partitioned, transformed and mapped into $\mathbb{R}^{4,1}$ as a network called *Spatio-Temporal Ordinality Network* (STON). STONs are characterized using specific GA multivector

[4] We thank an anonymous reviewer for these detailed remarks on[195].

functions highly effective as featurised variables to forecast future states. STONs uniquely capture the algebra-geometric interdependencies in the historical behavior of a time series system, presenting governing expressions for forecasting. Case studies on seasonally adjusted Australian unemployment rate and the NASA Goddard Institute for Space Studies (GISS) Surface Temperature Analysis for Global Land-Ocean Temperature Index are compared against multilayer perceptrons (MLP), long short term memory (LSTM) NN, ARIMA and Holt-Winters methods.

A multidimensional local binary pattern (MDLBP) based on GA for hyperspectral images (HSI) is proposed in [138], able to extract spatial–spectral features from multiple dimensions. The geometric relationship between the local geometry of HSI in GA is calculated to realize the local multidimensional description of the local spatial–spectral information. Finally, a novel LBP coding algorithm for HSI is implemented based on the local multidimensional description to calculate the feature descriptor of HSI. Experiments show higher MDLBP accuracy than representative spatial–spectral features and existing LBP algorithms, especially for small-scale training samples.

An original unified framework to learn comprehensive shape and motion representations from skeleton sequences by using GA is proposed in [139]. A skeleton sequence space is constructed in GA to represent each skeleton sequence in spacetime. Then a rotor-based view transformation method eliminates the effect of viewpoint variation, and relative spatio-temporal relations among skeleton frames in a sequence remain. A spatio-temporal view invariant model (STVIM) collectively integrates spatial configuration and temporal dynamics of skeleton joints and bones. In STVIM, skeleton sequence shape and motion representations which mutually compensate are jointly learned to comprehensively describe skeleton-based actions. Furthermore, a selected multi-stream convolutional NN (CNN) is employed to extract and fuse deep features from mapping images of the learned representations for skeleton-based action recognition. Experiments on NTU RGB+D, Northwestern-UCLA and UTD-MHAD data sets, demonstrate efficiency and performance.

A related work [144] focuses on rotation relations, to exploit the spatial and temporal characteristics in a skeleton sequence, designing two different rotor-based feature descriptors, respectively, from one frame of skeleton and two skeletons of consecutive frames. Then an efficient feature encoding strategy is proposed to transform each kind of feature descriptor into a RGB image. Afterward, a two-stream CNN based framework learns the RGB images generated by each skeleton sequence and then fuses the scores of two networks to provide final recognition accuracy. Experiments on NTU RGB+D, Northwestern-UCLA, Gaming 3D, SYSU and UTD-MHAD data sets prove good performance.

In [42], a novel human body posture representation based on GA is proposed to extract angles and orientations of most informative body joints to describe human body postures. Motion postures are treated independently. For each posture, a new GA based skeleton posture descriptor is used to construct the feature vectors as the input for an SVM classifier to decide the motion type. To typify whole motions, most frequent classes from the sequence of predictions of the motion postures are chosen with a simple voting scheme. Tests on SYSU-3D-HIO and an in-house data set of human exercises demonstrate effectiveness.

An extended model of multilayer perceptron (MLP) based on reduced GA (RGA), namely RGA-MLP, is proposed in [133] for multi-dimensional signal processing. The RGA-MLP model treats and encodes in RGA all multi-dimensional signals as multivectors and all neuronal parameters such as inputs, connection weights, activation function, outputs, and operators. The RGA back propagation (BP) algorithm is also provided. Thanks to the commutative property of RGA, multi-dimensional signals can be processed in a holistic manner avoiding loss of multi-dimensional relationships. In experiments RGA-MLP outperforms the traditional real-valued MLP and quaternion based MLP (QMLP). In related work [221] CNN based on RGA are studied.

Clifford support vector regression (CSVR) realizes multiple outputs in GA which can be used in multistep forecasting of electric load. However, the effect of input is different from the forecasting value. Since the load forecasting value affects the energy reserve and distribution in the energy system, accuracy is important. In the study [220], a fuzzy SVM is proposed based on GA, named Clifford fuzzy support vector machine for regression (CFSVR). Through fuzzy membership, different input points have different contributions to deciding the optimal regression hyperplane. The performance of the proposed CFSVR is evaluated in fitting tasks on numerical simulation, the UCI data set and a signal data set, and in forecasting tasks on an electric load data set and the NN3 data set. The experiments indicate that CFSVR over CSVR and SVR or other algorithms, can improve the accuracy of electric load forecasting and achieve multistep forecasting.

In [140], Weyl almost automorphic solutions in distribution sense for a class of Clifford-valued stochastic NN with time-varying delays are studied. Existence and uniqueness of these solutions are investigated using the Banach fixed point theorem and the relationship between several different senses of random almost automorphy, and global exponential stability in the pth mean is proved.

In [181], a novel Visual Attentive Model (VAM) is proposed for learning from eye tracking data of adults and children, observing five paintings with similar characteristics. The paintings are selected by Cultural Heritage (CH) experts for their analogous features providing coherent visual stimuli. The new method combines a new representation of eye sequences using GA with a deep learning model for automated recognition (to identify, differentiate, or authenticate individuals) of people by the attention focus of distinctive eye movement patterns. The experiments were conducted by comparing five Deep Convolutional NN (DCNNs), and yield high accuracy (more than 80 %), demonstrating the effectiveness and suitability in identifying adults and children visitors.

The work[21] considers geometric processing in modeling of feed-forward NN using quaternion-valued NN for perception and control of robot manipulators. Quaternion spiking NN able to control robots are presented, and examples confirm the capacity to adapt on-line robots to reach desired positions. Moreover, single neuron quaternionic quantum NN for pattern recognition are introduced. Experiments validate performance. Quaternions for rotations and operation on other quaternions inspired an investigation[19] on how quantum states, quantum operator and quantum NN (QNN) can work with quaternions. A new type of QNN is developed isomorphic to the rotor subalgebra $Cl^+(3,0)$ of $Cl(3,0)$, based on the qubit neuron model. The quaternion quantum NN (QQNN) shows robust test performance.

The review[183] brings together the Lie-algebraic/group-representation perspective of quantum physics and GA, as well as their connections to complex quaternions. Altogether, this may be seen as further development of Felix Klein's Erlangen Program for symmetries and geometries. In particular, the fifteen generators of the continuous SU(4) Lie group for two qubits can be placed in one-to-one correspondence with finite projective geometries, combinatorial Steiner designs, and finite quaternionic groups. The very different perspectives that are considered provide further insight into quantum information problems and extensions for multiple qubits, as well as higher-spin or higher-dimensional qubits.

4.2 | Geographic Information Systems

For geographic information science (GIS) CGA is a new tool for unified multidimensional representation and geometric computation, from a unified perspective of multidimensional space-time. In[226], the concept of a unified representational model based on CGA for spatio-temporal objects and their spatio-temporal topological relations is formally expressed by a multi-branch decision tree, which is not only qualitative but also quantitative. The research provides theoretical and methodological support for expressing and computing spatio-temporal topological relations among any set of geographic objects. This assists expression of spatio-temporal topological relationships and enhances analytical capabilities of GIS for dealing with space and time.

The paper[190] aims to realize 3D topological analysis in the cadastral field using CGA advantages in geometric relation computation. A calculation framework is designed on the basis of the outer product to achieve the purpose of multidimensional unity for 3D cadastral topological analysis. A framework of topological relations between a boundary point (or line) and a cadastral parcel is developed. A total of 13 types of topological relations between a boundary point and a cadastral parcel and 48 types of topological relations between a boundary line and a cadastral parcel are obtained. The study indicates that the advantages of CGA in multidimensional unified representation and calculation can be used to solve problems encountered by topological models in Euclidean space.

A 3D cadastral data model in the form of boundary representation based on CGA is extended in[235] to realize the integrated representation of geometric and topological information. On the basis of this model and basic geometric operators, self-defined GA operators are designed using the advantages of GA in spatial topological calculation, and a computation framework for cadastral parcel topological error detection is put forward, and verified in a case study.

In[24] a detailed review of GA in different fields of AI and computer vision regarding its applications and the current developments in geospatial research is provided. The Clifford–Fourier transform (CFT) and quaternions (sub-algebra of GA) are also discussed because of their necessity in remote sensing image processing. The focus is on how GA helps AI and solves classification problems, as well as improving these methods using GA processing.

4.3 | Application to Encryption

With the widespread use of color images, the copyright protection of those images using watermarks is one of the latest research topics. The use of color images as watermarks has advantages over binary and irreplaceable gray scale images. Color images are intuitive, rich, and lively; they have large amounts of copyright protection information and are more easily recognized by human vision.

The authors of[23] improve the security of watermark information and embedding positions and improve the algorithm's robustness against various attacks. This is done based on a quaternion Fourier transform (QFT) algorithm, which is based on the Arnold transform and chaotic encryption.

The paper[25], also uses the QFT and additionally makes each component more secure by scrambling the pixels of the watermark and performing encryption utilizing chaotic sequencing. Different types of results utilizing Mean Square Error (MSE), Peak Signal to Noise Ratio (PSNR), Structure Similarity Index Metric (SSIM), etc. are computed to verify the performance of their algorithm and its robustness.

In[143] the traditional discrete fractional Krawtchouk transform (DFrKT) is extended to quaternion DFrKT (QDFrKT) for the purpose of color image processing. In order to evaluate the applicability of the proposed QDFrKT method, two applications are investigated, including color image encryption and color image watermarking.

The work[47] encrypts information represented by a set of GA vectors. These vectors are multiplied by multivectors that perform the rotor operation. A multivector (rotor) is used as a key. An operation corresponding to the reverse rotor is used to decrypt the information. The authors show that GA algorithms increase the security of information encryption by increasing the dimension of the algebra.

5 | APPLICATIONS IN PHYSICS

5.1 | Relativity, Gravity and Cosmology

In the 2018 book entitled *A Geometric Algebra Invitation to Space-Time Physics, Robotics and Molecular Geometry*[122], an introduction to Geometric Algebra (GA) is given, and applications: to the physics of space-time (Maxwell electromagnetism and Dirac equation), robotics (forward and inverse kinematics and singularity problem for serial robots), and molecular geometry (including 3D-protein structure calculations using NMR data).

In[3] GA is featured as a universal algebraic system applying to all branches of physics. GA brings considerable simplifications to various areas of physics (over a diversity of traditional branch specific methods), particularly in the modern formulation of special relativity in the language of GA. Moreover, in[227] the authors present a compact Baker–Campbell–Hausdorff–Dynkin formula for the composition of Lorentz transformations in spin representation (as Lorentz rotors) in terms of their generators, general to GA of dimension $n = p + q \leq 4$, generalizing Rodrigues' formula for rotations in three-dimensional Euclidean space. In particular, it applies to Lorentz rotors in Hestenes' spacetime algebra $Cl(1, 3)$[91], efficiently composing Lorentz generators. Computer implementations are possible with complex two by two Pauli spin matrices. Applied to the composition of relativistic three-velocities it yields simple expressions for the resulting boost and Wigner angle.

In[68] important properties of the special relativistic Fourier transformation (SFT), see chapter 4.4 of[102], on complex space–time algebra $Cl(3, 1)$ are investigated, such as inversion property, the Plancherel theorem, and the Hausdorff–Young inequality. Furthermore, the concept of the vector derivative in GA is used together with the notion of the space–time split to derive the Heisenberg–Pauli–Weyl inequality. Finally, SFT properties are applied for proving the Donoho–Stark uncertainty principle for $Cl(3, 1)$ multi-vector valued functions.

Alternative versions of the energy–momentum complex in general relativity are given compact new formulations with spacetime algebra (STA) $Cl(1, 3)$ in[92]. A new unitary form for Einstein's equation simplifies the derivation and analysis of gravitational superpotentials. Interpretation as a gauge field theory on flat spacetime resolves ambiguities in energy–momentum conservation laws and reveals new relations between superpotential, gauge connection and spin angular momentum with new physical possibilities.

Static energetics for gravity is treated in[15]. The authors remove gauge constraints and impose full metrical general relativity for a stress energy for linear gravity in physical space time, approximated by a flat background and find a natural generalization of the stress energy tensor to the pseudotensor of Einstein, with Møller's pseudotensor being an alternative (with tetrads adapted to Einstein-Cartan gravity). Gauge theory gravity (tetrads and spin connection as gauge fields on Minkowski space), a Poincaré gauge theory for spacetime algebra, uses GA for Einstein-Cartan gravity. The new work[15] uses a variational approach to obtain the Møller pseudotensor yielding a method for obtaining conserved currents. This is applied to static, spherical spacetimes, with a limit to Newton's $-M/r$ potential formula, and conserved static system mass M_t (which produces a local virial theorem). The gravitational energy of Einstein and Møller adds to M_t on Minkowski space to give M. Finally the theory is applied to an incompressible fluid ball. Furthermore, a systematic study of background cosmology in unitary Poincaré gauge theories (PGT) with application to emergent dark radiation and H_0 tension is undertaken in[16]. A one-parameter extension to Lambda cold dark matter (Λ CDM) is proposed, expected to strongly affect cosmological tensions. An effective dark radiation component in the early universe redshifts away as hot dark matter, therefore, tracking the dominant equation-of-state parameter and leaving a falsifiable torsion field in the current epoch. This picture results from a new PGT, one of the most promising among the latest batch of 58 PGTs found to be both power-counting renormalizable and free from ghosts and tachyons.

Again in the framework of gauge theory gravity,[119] discusses an approach to gravitational waves. Gauge Theory Gravity uses symmetries expressed within the GA of flat spacetime to derive gravitational forces as the gauge forces corresponding to localizing these symmetries. Solutions for black holes and plane gravitational waves are considered in this approach, GA simplifying both writing the solutions, and checking their properties. Then a preferred gauge emerges for gravitational plane waves, in which a 'memory effect' corresponding to non-zero velocities is left after the passage of the waves, demonstrating the physical nature of this effect. The mathematical details of the gravitational wave treatment in GA are linked with other approaches to exact waves in the literature. The author generally recommends that the general relativity metric-based version of the preferred gauge, the Brinkmann metric, be considered for use more widely by astrophysicists and others, being among others advantageous to a treatment of joint gravitational and electromagnetic plane waves. The exact solutions found for particle motion in exact impulsive gravitational waves facilitate the discussion of the possibility of backward in time motion induced by strongly non-linear waves.

Real, complex, quaternion and octonion entry Hermitian matrix Jordan algebras are proposed for universe models in[9]. The action consists of cubic terms with coefficients being the structure constants of a Jordan algebra. Coupling constants only enter the theory via symmetry breaking which also selects a physical vacuum. *Before* the symmetry breaking the universe is pre-geometric without clear notions of space or time, but time comes into existence with the symmetry breaking together with a Hamiltonian which can create space from *nothing* and in some cases propagate the space to macroscopic size in finite time. There exists symmetry breaking which results in macroscopic spacetime dimensions 3, 4, 6 and 10, using real, complex, quaternion and octonion entry Hermitian matrix Jordan algebras, respectively.

Finally,[12] explores the boundary conditions in an Einstein-Hilbert action, by considering a displacement from the Riemannian manifold to an extended one, characterized by including spinor fields into the quantum geometric description of a non-commutative spacetime. These fields are defined on the background spacetime, emerging from the expectation value of the quantum structure of spacetime generated by Clifford algebra matrices. Spinor fields thus may describe all known interactions in physics, including gravitation. The cosmological constant originates exclusively from massive fermion fields that would be the primordial components of dark energy, during the inflationary expansion of a de Sitter expansion universe.

5.2 | Classical Physics

The paper[201] formulates rigid body dynamics independent of space dimension, describing with GA states and equations of motion. Using collision detection algorithms extended to *n* dimensions resolves body collisions and contacts. A four-dimensional software implementation is given as example for dimension independent techniques: rigid bodies are displayed by taking three-dimensional slices, users can manipulate these bodies in real-time.

Next, in the work[38], the potential energy of classic particles is defined with Clifford algebra, where Clifford numbers define Euclidean space rotations, describing symmetry. Calculations are applied to Archimedean solids, their vertices are presented and energies are calculated in Cartesian coordinates.

Regarding transport and interaction of radiation with matter, a fundamental component of any Monte Carlo (MC) simulation[170] is geometrical modeling mostly based on suboptimal Euclidean representations, not being the best option for speedy geometric debugging-modeling computations of radiation transport, due to the number of operations involved in the estimation of position and direction of particles within each geometry shape. In[170] the use of CGA, for geometric modeling in MC simulation is proposed for radiation transport. Elemental CGA equations serve for microscopic modeling of positions and rotations of radiation particles and the macroscopic modeling of geometrical shapes. One can take advantage of the expression power of CGA to create and debug geometry modeling for a triboelectric X-ray application. Advantages of CGA for microscopic geometric computations are explored.

The paper[156] explores certain properties of barotropic and non-barotropic fluid flows with the help of GA over a four-dimensional Euclidean spacetime manifold. Concepts of multivectors are introduced associated with vorticity, helicity, and parity, which evolve from a four-velocity field. The fluid dynamical analogs of the Poynting theorem, Lorentz force, and Maxwell's equations are derived. The fluid's Maxwell equations can all be extracted from a single multivector equation.

Compressible Navier-Stokes equations (NSE) in cylindrical passages and general dynamics of surfaces are treated using GA in[168]: (I) flow structures and (II) biomembranes (e.g. in lungs) are analyzed under static and dynamic conditions, paying attention to NSE (reduced to a single PDE) scaling invariance, with a geometric variational calculus approach and a calculus of moving surfaces. A related method[169] solves compressible unsteady three-dimensional Navier-Stokes equations in cylindrical co-ordinates coupled to the continuity equation in cylindrical coordinates in terms of an additive solution of the three principle directions in the radial, azimuthal and z-directions of flow. A reduction to a single partial differential equation is possible and integral calculus methods are applied for the case of a body force in the direction of gravity to obtain an integral form of the Hunter-Saxton equation.

5.3 | Electro-Magnetism and Optics

The paper[178] presents a new geometric viewpoint for magnetic flux tubes (MFTs) in three-dimensional Lorentzian space through split quaternion algebra. It is shown that MFTs can be completely characterized by a split quaternion product or a homothetic motion. Moreover, magnetic field components are obtained, and flux surfaces can be generated through the magnetic vector field components at each point on a generating curve. The Frenet frame allows to derive magnetic field components, field lines and kinematic magnetic tube surface equations, in terms of the local coordinate directions. Furthermore, characterizations of the MFTs through a stretch factor are given. Analytical solutions are obtained and some examples visualized.

Using GA and GC[174] expresses electromagnetic magnitudes and relations in a gradual way for space dimensions $n = 1, 2$ and 3 and time. For $n = 1$, electric field and scalar and vector potentials get geometric interpretation in spacetime diagrams. The geometric vector derivative leads to concepts of divergence, displacement current, continuity and gauge or retarded time, with clear geometric meaning. The geometric vector derivative is invertible and it is possible to obtain retarded Liénard–Wiechert potentials propagating naturally at the speed of light. For $n = 2$ these magnitudes become more complex and a pseudsocalar magnetic field B appears. Induction reflects the relations between the electric field E and B, capacitors, electric circuits and a Poynting vector for the flow of energy arise. For $n = 2$ physical effects propagate at the speed of light with a nature absent in one or three dimensions.[5] Electromagnetic waves at the speed of light are thus a consequence of an odd number of spatial dimensions. Finally, the case $n = 3$ is treated.

The magnetotelluric impedance tensor is analyzed in[188] in Clifford algebras. The tensor is broken down into parts, each one with a particular geometric algebra meaning, the simplicity of which allows us to deduce a number of known properties and opens up many other possibilities. For example, some of the algebraic relationships involving the impedance tensor, such as rotations, Mohr diagrams and phase tensor, are shown under this theory. The galvanic distortion matrix and its transformations are also seen in Clifford algebra, allowing to recognize the galvanic distortion in a measured two/three-dimensional impedance tensor. These relationships are useful as constraints to determine the galvanic distortion parameters.

Superconducting lattices, exhibiting the Meissner–Ochsenfeld effect, at increasingly higher temperatures are addressed in[202], with a novel and generalized mathematical formulation in CGA, which maps the atomic and structural characteristics of superconducting lattice structures to their critical temperature. This formulation models many-agent systems by representing them as spatially distributed networks in CGA. We present generalized relationships between the critical temperature and basic atomic information, including crystal unit cell data, for an arbitrary lattice structure. Case studies are presented for six different kinds of substances.

In optics a formulation of the Snell–Descartes laws of reflection and refraction of optical rays by the interface of two isotropic media using GA is given in[72]. The cases of specular reflection, positive and negative refraction, and refraction in the presence of a metasurface are considered.

5.4 | Quantum Physics

Two-state quantum systems are formulated in $Cl(3, 0)$ in[8]. Both quantum states and Hermitian operators (HOs) can be written as multivectors. By writing the quantum states as elements of the minimal left ideals of this algebra, energy eigenvalues and eigenvectors for the Hamiltonian of an arbitrary two-state system can be computed. The geometric interpretation of the HOs allows to introduce an algebraic method to diagonalize them. The problem of a spin-$1/2$ particle interacting with an external

[5] By including these theoretical results in our survey, we do not make any final claims on correctness or validity, but want to point readers to what appears noteworthy.

arbitrary constant magnetic field is treated, and Clifford algebra reveals the underlying geometry of these systems, which reduces to Larmor precession in an arbitrary plane.

The vast majority of exact solutions of the Dirac equation are for highly symmetric stationary systems, but rare for time dependent dynamics. High energy electron beams interacting with a variety of quantum systems in laser fields need novel methods for finding exact solutions. The papers[40,41] present a method for building up solutions to the Dirac equation using the description of spinorial fields and their driving electromagnetic fields in terms of GA. A first family of solutions in[40] describes the shape-preserving translation of a wave packet along any desired trajectory in the $x - y$ plane. In particular, the dispersionless motion of a Gaussian wave packet along both elliptical and circular paths can be achieved with rather simple electromagnetic field configurations. A second family of solutions involves a plane electromagnetic wave combined with (generally) inhomogeneous electric and magnetic fields. The novel analytical solutions of the Dirac equation given here provide important insights into the connection between the quantum relativistic dynamics of electrons and the underlying geometry of the Lorentz group. The method is further illustrated in[41] by developing several stationary and nonstationary solutions with well defined orbital angular momentum along the electron's propagation direction. The first set of solutions describes free electron beams in terms of Bessel functions as well as stationary solutions for both homogeneous and inhomogeneous magnetic fields. The second set of new solutions involves a plane electromagnetic wave combined with a generally inhomogeneous longitudinal magnetic field. This allows to derive general physical properties of the dynamics in such field configurations, and physical predictions on dynamically induced self-consistent electromagnetic fields.

Estimates for the eigenvalues of multi-form modified Dirac operators are given in[79] constructed from a standard Dirac operator with addition of a Clifford algebra element associated to a homogenous- or multi-degree form. Necessary geometric conditions are given for such operators to admit zero modes and those for the zero modes to be parallel with respect to a suitable connection. Manifolds which admit such parallel spinors are associated with twisted covariant form hierarchies which generalize the conformal Killing–Yano forms.

Incorporating Clifford algebra into the finite part of the spectral triple, the main object that encodes the complete information of a noncommutative space, gives rise to five additional scalar fields in the basic framework. The work[14] investigates whether this can help to achieve unification. A renormalization group analysis at the one-loop level is performed, allowing possible mass values of these scalars to float from the electroweak scale to the putative unification scale, and shows that out of twenty configurations of mass hierarchy in total, no case is found that can lead to unification (of gauge constants). This confirms that the spectral action formalism for the standard model and gravity requires a model-construction scheme beyond the (modified) minimal framework.

The Composition Algebra-based Methodology (CAM)[228] provides a new model for generating the interactions of the Standard Model (SM). It is geometrically modeled for electromagnetic and weak interactions on a parallelizable sphere operator fiber bundle consisting of base space, the tangent bundle of space–time, a projection operator, parallelizable spheres conceived as operator fibers, and it has as structure group, the norm-preserving symmetry group $\mathrm{SO}(n + 1)$ for each of the division algebras which is simultaneously the isometry group of the associated unit sphere. The massless electroweak Lagrangian is shown to arise from the generation of a local coupling operation on sections of Dirac spinor and Clifford algebra bundles. Importantly, CAM is shown to be a new genre of gauge theory which subsumes Yang–Mills SM gauge theory. Local gauge symmetry is shown to be at its core a geometric phenomenon inherent to CAM gauge theory. Lastly, the higher-dimensional, topological architecture which generates CAM from within a unified eleven (1,10)-dimensional geometro-topological structure is introduced.

The paper[30] shows how to generalize the Weyl equation to include SM fermions and a dark matter fermion. Two by two complex matrices are a matrix ring. A finite group can be used to define a group algebra over this ring which is a generalization of the ring. For a group of size N, this defines N Weyl equations coupled by the group operation. The group character table is used to uncouple the equations by diagonalizing the group algebra. Using the full octahedral point symmetry group, the uncoupled Weyl equations have the symmetry of SM fermions plus a dark matter particle. The symmetry properties of dark matter are described.

A universal group theoretic description of the fermion production through any type of interaction with scalars or pseudoscalars is proposed in[155]. This approach relies on the group $\mathrm{SU}(2) \times \mathrm{U}(1)$, corresponding to the free gamma matrix choice in Clifford algebra, under which a part of the Dirac spinor function transforms like a fundamental representation. In terms of a new $\mathrm{SO}(3)(\sim \mathrm{SU}(2))$ vector constructed out of spinor functions, it is shown that the fermion production mechanism can be analogous to the classical dynamics of a vector precessing with angular velocity. In this group theoretic approach, the equation of motion takes a universal form for any system, and choosing a different type of interaction or a different basis amounts to selecting the corresponding angular velocity. The expression of the particle number density is simplified, compared to the traditional approach, and it provides a simple geometric interpretation of the fermion production dynamics. A demonstration focuses on fermion production through derivative coupling to the pseudo-scalar.

Chirality is defined in [180] in the context of chiral algebra that coincides with the more general chirality definition in the literature, which does not require the existence of a quadratic space. In this GA based approach neither matrix representation of the orthogonal group nor complex numbers are used.

The paper [150] first explores the form and action of raising and lowering operators expressed in GA. Second, it shows how increasing the number of dimensions of Euclidean space from three to four opens a new understanding for chiral asymmetry of electroweak interactions. Isomorphisms among groups represented in complex Clifford algebra, matrix algebra, and GA are applied, which allows to translate expressions for raising and lowering operators for electron and neutrino states in complex Clifford algebra into GA and elaborate them to include positrons and antineutrinos. This paper addresses such operators in the context of the electroweak sector of the SM utilizing (1) $Cl(3,0)$ for the Hestenes–Dirac equation in a Euclidean lab frame, (2) $Cl(4,0)$ to introduce chiral asymmetry, and (3) $Cl(4,1)$ to express the electroweak fermion states of the first generation of the SM and demonstrate their SU(2) relationships. Following this, [151] derives a linear, first-order, partial differential field equation (Dirac-like) in GC of $Cl(4,1)$ that has free plane-wave solutions distinct from one another that correspond to left and right chiral states of electron and neutrino. Besides the usual spacetime dependence of plane waves, the solutions have a multivector structure yielding a ladder of states with raising and lowering operators appropriate to electroweak theory and having an $\mathrm{SU}(2)_L$ relationship among the chiral electron and neutrino states. The Dirac-like equation in $Cl(4,1)$ results from a systematic review of Dirac-like equations (first-order field equations whose solutions also satisfy the Klein–Gordon equation) in GAs of lower dimension.

In [109] *Adinkra* is introduced as a supersymmetry graph in particle physics that can be adapted to study Clifford algebra representations named Cliffordinkra. It puts standard ideas in Clifford algebra representations in a visual geometric context. Recent developments in Adinkras have shown connections to error correcting codes, algebraic topology, algebraic geometry, and combinatorics. These connections also arise for Cliffordinkras.

In [87] it is shown that the Bell–Clauser–Horne–Shimony–Holt inequality that applies to local hidden-variable theories and quantum correlations, and the Tsirelson bound, depend crucially on the assumption that the values of physical magnitudes are scalars. More specifically, the assumption that these values are not scalars, but vector elements in the GA $Cl(3,0)$, makes it possible that the classical bound is violated and the quantum mechanics (QM) bound respected, even given a locality assumption. This implies, first, that the origin of the Tsirelson bound is geometrical (not physical) and that a local hidden-variable theory may not contradict QM if the values of physical magnitudes are vectors in the GA $Cl(3,0)$.

The paper [121] pursues an alternative approach to CGA with only a single extra dimension working in a constant curvature background space. It is possible to define fully covariant cost functions for geometric object matching in computer vision, invariant under rotations and translations. Applications are to matching sets of lines, replacing in GA matrix singular value decomposition. Moreover, a further application of the 1d up approach is related to claims about Bell's Theorem in quantum mechanics that are shown in [121] to reduce to combinations of rotors in $Cl(4,0)$ without connection to octonions.

Wavelike objects that are even subalgebra elements of GA in three dimensions are considered in [197]. The long existing confusion about the appearance of the imaginary unit in QM is eliminated and a clear definition of wave functions is provided. When a wave function acts through the Hopf fibration on a localized GA element, that is executing a measurement, the result can be regarded as *collapse* of the wave function. Furthermore, [198] considers the *spreons* or *sprefield* wave functions that are special g-qubit solutions of Maxwell equations in GA. The behavior of such wave functions in scattering and measurements is analyzed showing that sprefields are defined through the whole three-dimensional space at all values of the time parameter. They instantly change all their values when scattered, that is subjected to Clifford translation. In *measurements*, when a sprefield acts on a static GA element through the Hopf fibration, it collapses, creating a new non static, rotating GA element.

Using the formalism of GA and GC in which Grassmann numbers are endowed with a second associative product coming from a Clifford algebra structure, it is shown in [189] how Berezin integrals (of functions of anticommuting Grassmann variables) can be realized in the high dimensional limit as integrals in the sense of GC. Then the concepts of spinors and superspace are given in this framework.

GA and GC encode fundamental geometric relations that theories of physics must respect. The paper [76] proposes criteria given which statistics of expressions in GA are computable in quantum theory, in a way that preserves their algebraic character. One must be able to arbitrarily transform the basis of the Clifford algebra, via multiplication by elements of the algebra that act trivially on the state space, and all such elements must be neighbored by operators corresponding to factors in the original expression and not the state vectors. The consequences for a physics of dynamical multivector fields are explored.

A method of averaging in GA for computing elements of spin groups in arbitrary dimension is presented in[191]. It generalizes Hestenes' method for the case of dimension four. Explicit formulas for spin group elements that correspond to elements of orthogonal groups as two-sheeted covering are presented. These formulas allow in arbitrary dimension to compute GA rotors, which connect two different rotation related frames.

6 | SIGNAL, IMAGE AND VIDEO PROCESSING

In[218] (published in 2019), a comprehensive survey back to 1986 of 51 publications covering various GA-based algorithms is undertaken. This includes mathematics of GA and reduced GA (RGA), analysis and comparison of advanced GA-based algorithms, sparse representations, dictionary learning, Clifford SVMs, feature extraction, adaptive filtering algorithms, GA Fourier transforms, and edge detection.

6.1 | Medical Imaging

CGA has emerged as a new approach to geometric computing that offers a simple and efficient representation of geometric objects and transformations in medical imaging tasks, such as segmentation, 3D modeling, and registration of medical images. However, the practical use of CGA-based methods for big data image processing in medical imaging requires fast and efficient implementations of CGA operations. The purpose of[73] is to present a novel, practical and effective implementation of CGA-based medical imaging techniques. A new simplified formulation of CGA operators allows significantly reduced execution times while maintaining needed precision. The result is a CGA re-design of a suite of medical imaging automatic methods, including image segmentation, 3D reconstruction and registration, shown in experiments to be suitable for big data image processing. Moreover, medical imaging data coming from different acquisition modalities require automatic tools to extract useful information and support clinicians in the formulation of accurate diagnoses. GC offers a powerful mathematical and computational model for the development of fast (real time), effective, accurate and robust medical imaging algorithms. The purpose of[74] is to present the state of the art of GC techniques in this field. The use of GC-based paradigms in radiomics and deep learning, i.e. a comprehensive quantification of tumor phenotypes by applying a large number of quantitative image features and its classification, is also outlined.

Mining algorithms for Dynamic Contrast Enhanced MRI (DCE-MRI) of breast tissue are discussed in[233] for advanced current state-of-the-art computer-aided detection and analysis of breast tumors observed at various states of development. This includes image feature extraction, information fusion using radiomics, multi-parametric computer-aided classification and diagnosis using information fusion of tensorial data sets as well as GA based classification approaches and convolutional neural network deep learning methodologies. The discussion also extends to semi-supervised deep learning and self-supervised strategies as well as generative adversarial networks and algorithms using generated confrontational learning approaches. For weakly labeled tumor images, generative adversarial deep learning strategies are considered for tumor type classification. The proposed data fusion approaches provide a novel AI based framework for more robust image registration advancing early identification of heterogeneous tumor types, even when the associated imaged organs are registered as separate entities embedded in more complex geometric spaces. Finally, the general structure of a high-dimensional medical imaging analysis platform based on multi-task detection and learning is proposed, making use of novel loss functions that form the building blocks for a generated confrontation learning methodology for tensorial DCE-MRI. Conclusions on the rate of proliferation of the disease become possible. The proposed framework may reduce interpretation costs of medical images by providing automated, faster and more consistent diagnosis.

6.2 | GA Image Processing Including GA Fourier Transforms

In[115] GA enhances segmented images acquired from unmanned aerial vehicles (UAVs) of different agricultural fields. Previous image segmentation approaches depend upon color channel wise analysis. GA (including quaternions) allows genuine color space image processing, e.g. in medical image processing, and now applied to agricultural images. The image segmentation of foreground and background is enhanced using GA, hence, the results obtained are fine-tuned segmented images. This may ameliorate the condition of farmers and their livelihood. Moreover, site-specific crop mapping is essential because it helps farmers determine yield, biodiversity, energy, crop coverage, etc. GA in signaling and image processing treats multi-dimensional signals in a holistic way maintaining size relationships and preventing loss of information. The article [116] uses agricultural images acquired by UAVs to construct three-dimensional models using GA. The qualitative and quantitative performance evaluation results show that GA can generate a three-dimensional geometric statistical model directly from UAV RGB images. Through peak signal-to-noise ratio (PSNR), structural similarity index measure (SSIM), and visual comparison, the proposed algorithm's better performance is shown over other latest algorithms.

In [103], airborne laserscanning strips are combined via iterative closest point (ICP) methods to an interactive three-dimensional terrain map and CGA provides the mathematical framework for how to use the ICP method for the precise adjustment of the airborne laserscanning data strips.

GA-based geometric Fourier transforms play an increasing role in modern data analysis, in particular for color image processing. In [141] several important sharp inequalities are proved, including the Hausdorff-Young inequality and its converse, Pitt's inequality and Lieb's inequality for Clifford ambiguity functions. With these inequalities, several uncertainty principles with optimal constants are derived for the geometric Fourier transform. Most results in this paper are new even in the quaternion setting.

In [134], a novel multi-focus image fusion algorithm using a GA discrete Fourier transform (GA-DFT) is proposed. The algorithm represents the RGB color image as a GA multivector. Variance is selected as contrast measure to fuse source images in GA. Experimental results show comparable performance with conventional algorithms in terms of subjective quality evaluation, and improvements in terms of objective quality metrics in image fusion applications.

The paper [237] presents a novel computational framework of quaternion polar harmonic transforms (QPHTs): accurate QPHTs (AQPHTs). To holistically handle color images, quaternions are applied. Gaussian numerical integration is adopted for geometric and numerical error reduction. When compared with CNNs (convolutional neural networks)-based methods (i.e., VGG16) on the Oxford5K dataset, AQPHT achieves better performance for scaling invariant representation. Moreover, when evaluated on standard image retrieval benchmarks, AQPHT using smaller feature vector dimensions achieves comparable results with CNNs-based methods and outperforms hand craft-based methods by 9.6% w.r.t mAP on Holidays dataset.

Rotor-based and Prewitt-inspired Sangwine (RBS and PIS) filters are amongst the efficient GA algorithms for solving color edge detection problems. Recently, some specialized hardware architectures, called full-hardware GA implementations, are proposed for enormous GA computational loads. These architectures, such as ConformalALU co-processor, are able to execute GA algorithms in acceptable time with moderate use of computational resources. So far, all color edge detection hardwares in GA have exploited RBS filters. The paper [175] presents a full-hardware architecture for efficient execution of PIS filters, consuming less computational resources and being faster to execute, e.g., at the same speed the Gaalop pre-compiler uses twice as much resources as the proposed hardware. The latter is able to execute the edge detection algorithm almost 315 times faster than a GA co-processor, with only 2.5 times of its resources.

Hyperspectral images (HSI) embrace spectral information reflecting material radiation properties and the geometrical relationships of objects. Thus, HSI provides much more information than gray and color images. Therefore, [137] puts forward a *spatial-spectral* scale-invariant feature transform (SIFT) for HSI matching and classification by using GA. It extracts and describes the spatial-spectral SIFT HSI features. Firstly, a spatial-spectral unified model of spectral value and gradient change (UMSGC for short) is built to synthetically analyze spectral-spatial HSI information. Secondly, a scale space for HSI based on UMSGC is designed. Finally, both a new detector and descriptor of the spatial-spectral SIFT for HSI are proposed. Excellent algorithm performance is experimentally verified for HSI matching and classification.

The paper [85] proposes a new set of quaternion fractional-order generalized Laguerre orthogonal moments (QFr-GLMs) based on fractional-order generalized Laguerre polynomials. The proposed QFr-GLMs are directly constructed in Cartesian coordinate space, avoiding the need for coordinate conversion, being better image descriptors than circularly orthogonal moments constructed in polar coordinates. Unlike the latest Zernike moments based on quaternion and fractional-order transformations, which extract only global features from color images, the proposed QFr-GLMs can extract both global and local color features. A new set of invariant color-image descriptors by QFr-GLMs is derived, enabling geometric-invariant pattern recognition in color images. Finally, the performance of the proposed QFr-GLMs and moment invariants was experimentally evaluated for correlated color images, demonstrating theoretically and experimentally the value of QFr-GLMs and their geometric invariants for color image representation and recognition.

In [217], a novel multivector sparse representation model for multispectral images using GA multivectors is proposed, by fully considering spatial and spectral information. A GA dictionary learning algorithm is presented using K-GA-singular value decomposition (GASVD) (generalized K-means clustering for GASVD) method. Consequently, artifacts and blurring effects can be successfully avoided. Experiments demonstrate that the proposed sparse model surpasses existing methods for multispectral image reconstruction and denoising tasks by capturing correlations between spectral channels thoroughly being both effective and useful.

The Sylvester equation and its particular case, the Lyapunov equation, are widely used in image processing, control theory, stability analysis, signal processing, model reduction, etc. The paper[194] presents a basis-free solution to the Sylvester equation in GA of arbitrary dimension. It involves only the operations of the GA product, summation, and conjugation, and the concepts of characteristic polynomial, determinant, adjugate, and inverse in GA. For the first time, alternative formulas for the basis-free solution to the Sylvester equation for $n = 4$, the proofs for the case $n = 5$ and the case of arbitrary dimension n are given. The results can also be used in symbolic computation.

6.3 | Motion Processing

In[216] l_1-norm minimization is addressed, important in compressed sensing (CS) theory. This paper presents an algorithm using GA for solving l_1-norm minimization for multi-dimensional signals by converting it to second-order cone programming. The algorithm represents the multi-dimensional signal as GA multivector to process it holistically without losing correlations among different multi-dimensional signal components. Numerical experiments demonstrate effectiveness and robustness against noise. The proposed algorithm can also be used to guide perfect recovery of multi-dimensional signals, and may find potential applications for CS with multi-dimensional signals.

The paper[142] develops a relative state estimation method for two spacecrafts based on monocular vision measurement, where the leader spacecraft is observed by a calibrated camera fixed on the follower. A two-dimensional image of the three-dimensional leader craft with multiple geometric features is obtained by the camera. Multiple geometric features including points, lines and circles are described in dual number algebra, and observation models are proposed based on the geometric projection relationships. The geometric features of the leader craft can be employed to estimate the relative state between the two crafts. Moreover, six-DOF relative motion dynamics is developed using dual numbers. It describes the relative motion between arbitrary points on the spacecraft, considering kinematical and dynamical coupling effects. An extended Kalman filter estimates the relative two spacecraft state. An unscented Kalman filter avoids complicated derivations. Simulation results verify the effectiveness of the proposed method, and show that the relative translation and rotation estimation errors converge rapidly with high estimation accuracy. Multiple feature measurement provides superior estimation performance over methods with less features.

Spatiotemporal interest points (STIP), a local invariant in videos, have been widely used in computer vision and pattern recognition. To fully exploit motion information inherent in the temporal domain of videos,[136] aims to develop an STIP detector that uniformly captures appearance and motion information for videos, thus yielding substantial performance improvement. In GA, a spatiotemporal unified model of appearance and motion-variation information (UMAMV) is developed, and a UMAMV-based scale space of the spatiotemporal domain synthetically analyzes appearance information and motion information in videos. An STIP feature of UMAMV-SIFT embraces both appearance and motion variation information of the videos. Three experimental data set evaluation shows that the UMAMV-SIFT achieves state-of-the-art performance particularly effective for small data sets.

Human motion recognition technology in sports dance video images is treated in[70]. In GA, using instance templates, a new cfrdF method, with continuous-scale space(CSS)-based similarity of human body features, constructs an angle-adaptive and CSS template matching algorithm to calculate the similarity between horizontal plates and detected images values, matching the human body location area. Based on actual video analysis and display, using the self-similar human color structure as basic feature, GA is applied to human body video images for robustly extracting (using template convolution and adaptive template design functions) human movements from sports dance videos every 30s. The accuracy can reach 90.9%.

6.4 | Estimation and Filtering

Direction-of-arrival (DOA) estimation plays an important role in array signal processing, and the Estimating Signal Parameter via Rotational Invariance Techniques (ESPRIT) algorithm is a typical super resolution algorithm for direction finding in electromagnetic vector-sensor (EMVS) arrays. The paper[222] proposes a novel GA based (not long vectors or quaternion matrices) ESPRIT (GA-ESPRIT) algorithm to estimate 2D-DOA with double parallel uniform linear arrays. The algorithm models multi-dimensional signals in a holistic way, and then the direction angles can be calculated by different GA matrix operations to keep the correlations among multiple EMVS components. Experiments demonstrate model error robustness, lower time complexity and smaller memory requirements.

The work[172] studies equivalence of complex-valued widely linear estimation and the quaternion involution widely linear estimation with vector-valued real linear estimation counterparts, including the particular degrees of freedom and by providing matrix mappings between complex variables and isomorphic bivariate real vectors, and quaternion variables versus a quadri-variate real vectors. It is shown that the parameters in complex-valued linear estimation, complex-valued widely linear estimation, quaternion linear estimation, quaternion semi-widely linear estimation, and quaternion involution widely linear estimation, include distinct geometric structures for complex numbers and quaternions, respectively, whereas the real-valued linear estimation shows no such structure. This explains, in theory and practice, the advantage of estimation in division algebras (complex, quaternion) over their multivariate real vector counterparts. Computational complexity of the estimators of hypercomplex widely linear estimation is discussed.

The paper[146] reformulates adaptive filters (AFs) in GA (GAAFs). The minimization problem (a deterministic cost function) is formulated in GA. GC allows for applying the same derivation techniques regardless of the (subalgebra) type of the data, i.e., real, complex numbers, quaternions, etc. A deterministic quadratic cost function is posed, from which GAAFs are devised, generalizing regular AFs. From the obtained update rule, it is shown how to recover the following least mean squares (LMS) AF variants via algebraic isomorphisms: real, complex, and quaternions LMS, respectively. Mean-square analysis and simulations in a system identification scenario are provided, showing very good agreement. Real-data experiments highlight the good tracking performance of the GAAFs in joint linear prediction of different signals.

By studying the shortcoming of the traditional real-valued fixed step size adaptive filtering algorithm,[212] proposes adaptive filtering with variable step size based on Sigmoid function and GA. A multi-dimensional signal is represented as a GA multivector for vectorization. Then the contradiction between the steady-state error and the convergence rate is solved by establishing a non-linear function relationship between step size and error signal. Experimental results demonstrate better performance than existing adaptive filtering algorithms.

Correntropy is an efficient tool for analyzing higher order statistical moments in non-Gaussian noise environments, having been used with real, complex, and quaternion data. From the probabilistic view,[223] presents a novel GA correntropy (GAC) representing the similarity measure between two random GA multivector variables. Then, an adaptive filter is developed based on a maximum GAC criteria (MGACC), robust against non-Gaussian noise. Simulation verifies the advantages of the proposed adaptive algorithm in a non-Gaussian environment.

In[184], an affine projection algorithm (APA) is introduced in GA to provide fast convergence against hypercomplex colored signals. Following the principle of minimal disturbance and the orthogonal affine subspace theory, a criterion is formulated for designing a GA-APA as a constrained optimization problem, solvable with Lagrange multipliers. Then, the differentiation of the cost function is calculated using GC to get the GA-APA update formula. Stability is analyzed based on the mean-square deviation. A regularized GA-APA is also given. Simulation results show better convergence than existing algorithms for colored input signals.

Adaptive filtering algorithms based on higher-order statistics are proposed for multi-dimensional signal processing in GA with signal multivectors in[86]. The least-mean fourth (LMF) and least-mean mixed-norm (LMMN) adaptive filtering algorithms are extended to GA for multi-dimensional signal processing. Both GA-LMF and GA-LMMN algorithms need to minimize cost functions based on higher-order statistics of the error signal in GA. Simulation results show that GA-LMF performs better in terms of convergence and steady-state error with much smaller step size. The proposed GA-LMMN algorithm stabilizes the GA-LMF as the step size increases, and is more stable in mean absolute error and convergence rate.

In[219], Normalized Least Mean Fourth (NLMF) and Normalized Least Mean Square (NLMS) adaptive filtering is extended to GA for multidimensional signal processing. GA NLMF and NLMS algorithms (GA-NLMF & GA-NLMS) are proposed. GA-NLMS minimizes a normalized mean square of the error signal cost function, and remains stable as the input signal of the filter increases. GA-NLMS converges fast but has higher steady-state error. The GA-NLMF algorithm minimizes a normalized mean fourth of the error signal cost function. Simulations show that GA-NLMS adaptive filtering outperforms the conventional NLMS algorithm in convergence and steady-state error, and GA-NLMF outperforms both NLMF and GA-NLMS filtering. In experiments GA-NLMF converges faster with lower steady state error.

As complete theoretical foundation for GA-AFs, a transient behavior evaluation of the GA least mean square (GA-LMS) algorithm is undertaken in[224]. The transient mean square deviation (MSD) is obtained from GA-LMS. Furthermore, the transient excess mean square error (EMSE) is also given. The analytical results rely on independence theory commonly used in the convergence analysis of real-valued adaptive filters. Conventional analysis is not suitable for GA-LMS algorithms due to non-commutativity. A novel method is used to analyze the GA-LMS algorithm under white noise by separating the weight-error array. Theoretical results can accurately predict the transient performance of the GA-LMS, stability and steady-state performance are also analyzed. The new steady-state model is more accurate than the previous models. Numerical experiments confirm the theory.

A novel least-mean kurtosis adaptive filtering algorithm based on GA (GA-LMK) is proposed in[215] for higher dimensional signals. The GA-LMK algorithm minimizes the cost function of negated kurtosis of the error signal in GA, and provides a tradeoff between convergence rate and steady-state error. The steady-state behavior of the GA-LMK algorithm under Gaussian noise is studied to acquire conditions of misadjustment. Simulation results show that GA-LMK adaptive filtering can significantly outperform existing algorithm convergence and steady-state error.

In[149], the transient performance of the GA least mean M-estimate (GA-LMM) filtering is analyzed. Further, the variable step-size variant VSS-GA-LMM is designed to eliminate the constraint of constant step size on the performance of the GA-LMM and the optimal step size is obtained by maximizing the difference of mean square deviation (MSD) between successive iterations, effectively balancing the contradiction between convergence rate and steady-state error. Numerical simulations verify the theory and the advantages of GA-LMM and VSS-GA-LMM algorithms.

To overcome the tradeoff between the low steady state error and the fast convergence,[225] proposes a novel GA adaptive algorithm convexly combining two GA LMS (CGA-LMS) algorithms with two step sizes. The steady state CGA-LMS algorithm performance is analyzed in detail. Moreover, for relative filter lag, which slows down the overall convergence of the combined GA filter, a novel instantaneous transfer strategy is proposed, i.e. a CGA-LMS algorithm with transfer strategy (CGA-LMS-TS). To process noncircular 3D and 4D signals, a convex combination of widely linear GA-LMS (CWL-GA-LMS) algorithm is introduced and combined with a transfer strategy. Simulations verify the performance.

6.5 | Features and Detection

Existing Speeded-Up Robust Feature (SURF) algorithms cannot be directly applied to multispectral images. In[214], based on GA, a novel feature extraction algorithm named GA-SURF is proposed for multispectral images. A Hessian GA matrix is calculated for locating interest points in spatial and spectral space. GA box filters are used to simplify the calculation of the Hessian matrix and generate image pyramids. Then, with SURF, interest points are located by the image pyramids and expressed in GA. Experiments show that GA-SURF is faster, more robust and more distinctive than existing algorithms.

Existing oriented fast and rotated brief (ORB) algorithms are not capable of detecting features for multispectral images directly. In[213], a novel feature extraction method is proposed for multispectral images: GA oriented fast and rotated brief (GA-ORB). First, the scale information in both spectral and spatial spaces of multispectral images is obtained in GA, where the inherent spectral structures can be retained successfully. Then, referring to ORB, the images are computed in different scales and the interest points are detected and described in GA. Experiments outperform previous algorithms in speed, distinctiveness and robustness when extracting and matching interest points.

Currently available vertex concavity-convexity detection algorithms in computer graphics mostly use two-dimensional polygons, with limited research on vertex concavity-convexity detection algorithms for three-dimensional polyhedrons. The study[232] investigates the correlation between the outer product in GA and the topology of spatial objects with varied dimensionality. A multi-dimensional unified vertex concavity-convexity detection algorithm framework for spatial objects is proposed, and this framework is capable of detecting vertex concavity-convexity for both two and three dimensions.

7 | GEOMETRIC ALGEBRA SOFTWARE

7.1 | Software Libraries

In[200], a high-level C++ library is presented for GA. By manipulating blades and versors decomposed as vectors under a tensor structure, the library achieves high performance even in high-dimensional spaces ($\bigwedge R^n$ with $n > 256$) for $Cl(p, q)$, $n = p + q$. To simplify the use of this library, the implementation can be used both as a C++ pure library or as a back-end to a Python environment, without impact on the performance.

The paper[31] presents both a recursive scheme to perform GA operations over a prefix tree, and Garamon, a C++ library generator implementing these recursive operations. A prefix tree can support a recursive formulation of GA operations. This recursive approach presents a much better complexity than usual run time methods. A prefix tree can also efficiently represent the dual of a multivector. Based on this Garamon has been developed, a C++ library generator synthesizing efficient C++/Python libraries implementing GA in both low and higher dimensions, with arbitrary metric even in high dimensions hardly accessible with state-of-the-art software implementations. Garamon produces easy to install, easy to use, effective and numerically stable libraries.

The Robotic Template Library (RTL) of[111] deals with geometry and point cloud processing, especially in robotics. It covers vectors, line segments, quaternions, rigid transformations, etc. It proposes a segmentation module for batch or stream clustering of point clouds, a fast vectorization module for approximation of continuous point clouds by geometric objects of higher grade, and a LaTeX export module enabling automated generation of high-quality visual outputs. It is a high performance header-only library written in C++17, and uses the Eigen library as a linear algebra back-end. RTL can be used in all robotic tasks such as motion planning, map building, object recognition and many others, and it has very general point cloud processing utilities.

7.2 | Computer Algebra Systems

The paper[2] develops an algorithm for some concepts of GA using Maple programming, including the norm computation of a multivector obtained by finding the Clifford product of any two vectors of the same finite dimension.

Moreover,[94] presents GAALOPWeb for Matlab, a new easy to handle solution for GA implementations for Matlab. The usability for industrial applications is demonstrated based on a forward kinematics algorithm of a serial robot arm, and illustrated showing high run-time performance. Finally,[6] presents a new tool for Mathematica users, based on the new web-based GA algorithm optimizer (GAALOP). GAALOPWeb for Mathematica supports Mathematica users with an intuitive interface for the development, testing and visualization of GA algorithms, combining the geometric intuitiveness of GA with an efficient development of algorithms for Wolfram Mathematica. Illustration for this integration is provided for a distance geometry problem, which consists of finding three-dimensional embeddings of graphs.

7.3 | Software Implementations

The work[66] deals with optimizing software implementations of GA outermorphisms for practical prototyping applications using GA. The approach proposed here for implementing outermorphisms requires orders of magnitude less memory compared to other common approaches, while being comparable in time performance, especially for high-dimensional GA.

In[44], a parallel-pipelined VLSI architecture of GA co-processor is proposed to mitigate GA intricacies, thereby becoming proficiently competent for computing a variety of GA operations. 4 channel parallelisms and 8-stage pipelining are the main attractions of the proposed design which does not necessitate any wait-state even in the situation of concurrent accessing of memory. Supreme quality is shown in terms of number of processing cycles, latency and throughput for exemplary real time applications.

GA is a powerful language to describe quantum operations using its geometric intuitiveness[7]. Using the web-based GAALOPWeb, an online GA algorithm optimizer for computing with qubits, and new formulations for the NOT operation are described, as well as a strategy to formulate Z gates and Hadamard operations both for one and multiple qubits.

A novel hardware design for a GA coprocessor is presented in[93], called GAPPCO, which is based on GA Parallelism Programs (GAPP). GAPPCO is a design for a coprocessor combining the advantages of optimizing software with a configurable hardware able to implement arbitrary GA algorithms. The idea is to have a fixed hardware easily and fast to be configured for different algorithms. The new hardware design together with the complete tool chain for its configuration is described.

8 | CONCLUSION

In this current survey of Clifford geometric algebra applications we guide the reader through about 200 relevant publications mainly from 2019 to the present (early 2022), plus about 30 background references (the latter are mainly referred to in the introduction). We try to be rather comprehensive for this short span of years, but due to the enormous publication volume we mainly focus on articles published in reviewed journals, and nearly completely bypass preprints, books (except for background material in the introduction) and book chapters, etc. We apologize for inevitably omitting potentially important work of colleagues.

We start with engineering applications at large, including electric engineering, optical fibers, robots, control, pose, material science, computer graphics, and modeling. Furthermore, we deal with the wide field of geometry, including spinors and symmetry, computations in geometry, molecular geometry, protein structure, computer algebra, curves and surfaces, advancing theoretical developments in geometry. Moreover, we explain applications to information processing including hypercomplex neural networks, artificial intelligence, geographic information systems, encryption, and survey new developments in physics of relativity, gravity, cosmology, classical physics, electro-magnetism, optics, and importantly quantum physics. Next, we guide through new developments in signal, image and video processing, in particular medical imaging, image processing and hypercomplex GA Fourier transforms, motion processing, estimation and filtering, features and detection. As for software, we finally refer to software libraries, computer algebra systems and software implementations.

ACKNOWLEDGMENTS

Soli Deo Gloria. We thank Petr Vašík for substantially helping to arrange this survey. We thank Rafael Alves, Eduardo Bayro-Corrochano, Leo Dorst, Steven De Keninck, Jaroslav Hrdina, Anthony Lasenby, Aleš Návrat, Vincent Nozick and Salvatore Vitabile for providing additional references. We thank Thomas Hempfling and Janina Krieger of SpringerNature for introducing the search engine Dimensions (https://www.digital-science.com/product/dimensions/) to us. We finally gratefully acknowledge the constructive comments and suggestions by the anonymous reviewers.

Financial disclosure

None reported.

Conflict of interest

The authors declare no potential conflict of interests.

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How to cite this article: Hitzer E., Lavor C., Hildenbrand D., (2022), Current Survey of Clifford Geometric Algebra Applications, *Mathematical Methods in the Applied Sciences, 2022;00:1–37*.

E. Hitzer, C. Lavor, D. Hildenbrand, Current Survey of Clifford Geometric Algebra Applications, to be published in Mathematical Methods in the Applied Sciences, 37 pages, (2022).