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On the Memory of Nature

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Abstract

It seems that a perfect description of the phenomena in our universe will be uniquely possible using fractional calculus and in the framework of fractional dynamics. In this note we will focus on the concept of memory effect as one of intrinsic characteristics of fractional operators and finally we discuss this important question that, is really the nature a book written in the unifying language of fractional calculus and so are we living in a fractional dimensional universe?

Keywords: Fractional dimensional universe, fractality, nonlocality, memory Effect, complexity, causality, noncausality.

1. What are the fractional calculus and the fractional dynamics?

In 1623 Galileo Galilei proposed this thought that: "The book of nature is written in the language of mathematics ". Now after about 400 years it seems that he may was right and we can say that, the nature can be considered as a book written in the unifying language of fractional calculus. Fractional calculus is in fact a generalization of the traditional calculus dealing with derivatives and integrals of arbitrary (real or even complex) orders. And fractional dynamics is a framework for investigations of phenomena which have five fundamental characteristics:

a) some degrees of fractality (i.e. being a fractal object or process (an arbitrary dimensional object) which their dimension mathematically will be described using the concept of Hausdorff dimensions)

b) some degrees of nonlocality (i.e. value of a quantity at a particular point is not exactly determined only by its value at that point but depends on other points)

c) some kind of memory effect (i.e. the dependence of an event on all of its historical states which mathematically will be described using memory kernel functions)

d) some level of complexity (i.e. microscopic or mutual interactions can lead to a considerable macroscopically changes in the whole of that system)

e) being causal or being non-causal (i.e. the dependence of a quantity or an event on the past time or on the future time respectively)

and therefore such phenomena in the best way will be naturally described using fractional operators [1-3]. When we see the universe from a cell to the cosmos we will immediately come

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to this conclusion that our real world problems really cannot be comprehensively described and understood without considering the above mentioned properties and so applying the fractional calculus. Almost all systems and phenomena in nature can be described in the framework of fractional dynamics however we should emphasize on this point that the levels of fractionality are completely different in each case, it can be a weak fractional system (i.e. a system with low level of fractionality) or in contrast a strong one (i.e. a system with high level of fractionality) or something between them.

In recent years fractional calculus and the approach of fractional dynamics have been considered in many studies from string theory [4] and quantum field theory [5], issues in biology and biomedicine [6] such as dynamics of protein folding [7] and growth of cancer tumors [8,9] to the topics in astrophysics [10] and cosmology [11]. Recently we have proposed our theory of fractional electrodynamics [12]. This theory successfully describes systems with electromagnetic memory effect.

More recently gauge invariant fractional electromagnetic fields has been proposed [13]. In this work the author have used well-known left (forward) and right (backward) Riemann–Liouville (RL) partial fractional derivatives (which respectively are nonlocal causal operators and nonlocal non-causal operators) of order α_μ , β_μ (which are positive real or even complex numbers) of a real valued function f of $d+1$ real variables $x^0, x^1, ..., x^d$ with respect to x_μ, i.e.:

$$ {}_{a_\mu}\partial_\mu^{\alpha_\mu} f(x^0,...,x^d) = \frac{1}{\Gamma(n_\mu - \alpha_\mu)} \partial_{x^\mu}^{n_\mu} \int_{a_\mu}^{x} \frac{f(x^0,...,x^{\mu-1},u,x^{\mu+1},...,x^d)}{(x^\mu - u)^{1+\alpha_\mu - n_\mu}} du \quad \text{(left RL)} \tag{1} $$

$$ {}_\mu\partial_{b_\mu}^{\beta_\mu} f(x^0,...,x^d) = \frac{(-1)^{n_\mu}}{\Gamma(n_\mu - \beta_\mu)} \partial_{x^\mu}^{n_\mu} \int_{x}^{b_\mu} \frac{f(x^0,...,x^{\mu-1},u,x^{\mu+1},...,x^d)}{(u - x^\mu)^{1+\beta_\mu - n_\mu}} du \quad \text{(right RL)} \tag{2} $$

where $\partial_{x^\mu}^{n_\mu}$ is the ordinary partial derivative of integer order n with respect to the variable x and a_μ , b_μ are real number which define the domain. It is worth mentioning that only left (forward) RL operator will be resulted in a causal theory and right (backward) RL operator will produce a non-causal theory. Based on above definitions one can introduce the left–right (two-sided backward-forward) fractional Riemann–Liouville operators i.e.:

$$ \partial_\mu^{\alpha\beta} = \frac{1}{2}\left({}_{a_\mu}\partial_\mu^{\alpha_\mu} - {}_\mu\partial_{b_\mu}^{\beta_\mu}\right) \tag{3} $$

where for the case of $\alpha_\mu = \beta_\mu$ becomes the well-known Riesz fractional derivatives which its explicit form for the special case of $1 < \alpha_i < 2$ becomes:

$$ \partial_\mu^{\alpha\alpha} f(x^0,...,x^d) = \frac{-1}{2\cos(\frac{\pi\alpha_\mu}{2})\Gamma(2-\alpha_\mu)} \partial_{x^\mu}^2 \int_{a_\mu}^{b_\mu} \frac{f(x^0,...,x^{\mu-1},u,x^{\mu+1},...,x^d)}{\left|x^\mu - \xi\right|^{\alpha_\mu - 1}} d\xi \tag{4} $$

2. Gauge invariant fractional electrodynamics

In this section at first we will briefly review the recently proposed gauge invariant fractional electrodynamics [12, 13] then we propose some new ideas. For this purpose we start from the lagrangian in the form of $L = L(A_\mu, \partial_\mu^{\alpha\beta} A_\mu, x^\mu)$ for N fields $A_\mu \equiv A_\mu(x^0, ..., x^d)$ where $\mu = 1, 2, ..., N$. So for $0 < \alpha_\mu, \beta_\mu < 1$ we will have the following fractional Euler–Lagrange equation for A_μ fields:

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu^{\beta\alpha} \frac{\partial L}{\partial(\partial_\nu^{\alpha\beta} A_\mu)} = 0 \tag{5}$$

Now suppose $A_\mu = (\varphi, -\vec{\mathrm{A}})$ as the 4-vector electromagnetic potential, then immediately we can write the standard Maxwell's field strength tensor $F_{\mu\nu}$ as: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and so its direct fractional generalization will take the form of: $F_{\mu\nu}^\alpha = {}_\mu \partial_b^\alpha A_\nu - {}_\nu \partial_b^\alpha A_\mu$ which has been written in terms of non-causal right Riemann–Liouville derivatives and with the following matrix form of:

$$F_{\mu\nu}^\alpha = \begin{pmatrix} 0 & E_x^\alpha & E_y^\alpha & E_z^\alpha \\ -E_x^\alpha & 0 & -B_z^\alpha & B_y^\alpha \\ -E_y^\alpha & B_z^\alpha & 0 & -B_x^\alpha \\ -E_z^\alpha & -B_y^\alpha & B_x^\alpha & 0 \end{pmatrix} \tag{6}$$

and its contravariant version in the form of:

$$F_\alpha^{\mu\nu} = \begin{pmatrix} 0 & E_\alpha^x & E_\alpha^y & E_\alpha^z \\ -E_\alpha^x & 0 & -B_\alpha^z & B_\alpha^y \\ -E_\alpha^y & B_\alpha^z & 0 & -B_\alpha^x \\ -E_\alpha^z & -B_\alpha^y & B_\alpha^x & 0 \end{pmatrix} \tag{7}$$

where F_{0i}^α and F_{ij}^α components are defined as the fractional electric fields fractional magnetic fields respectively (Latin and Greek indices (excluding α, β) are respectively equal to: $i, j, k = 1, 2, 3$ and $\mu, \nu = 0, 1, 2, 3$). Finally we can write the Lagrangian density of the fractional electromagnetic field in terms of the fractional Maxwell's field strength tensor $F_{\mu\nu}^\alpha$ as:

$$L_{\text{Fractional Electromagnetic Field}} = -\frac{1}{16\pi c} F_{\mu\nu}^\alpha F_\alpha^{\mu\nu} \tag{8}$$

where $F_\alpha^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}^\alpha$ and the Minkowski metric as: $\eta^{\mu\nu} = \eta_{\mu\nu} = diag(+1, -1, -1, -1)$.

Using the above lagrangian and the fractional Euler–Lagrange equations and the condition of $\partial_{[\rho}^\alpha F_{\mu\nu]}^\alpha = 0$ which $[\rho\mu\nu]$ denotes the antisymmetrized sum over permutations of the indices ρ,

μ, and ν for the fractional Maxwell's field strength tensor we can easily derive the other form of Maxwell's equations in terms of only causal left fractional derivatives as:

$$ {}_a\partial_\nu^\alpha F_\alpha^{\mu\nu} = -\frac{4\pi}{c} j^\mu \tag{9}$$

And so explicit form of the fractional Maxwell's equations in terms of the left fractional divergent and curl operators will read as [13]:

$$\begin{cases} \vec{\nabla}_{left}^\alpha . \vec{\mathrm{E}}^\alpha = 4\pi\rho \\ \vec{\nabla}_{left}^\alpha \times \vec{B}^\alpha = \frac{4\pi}{c}\vec{j} + \frac{1}{c^\alpha}\partial_t^\alpha \vec{\mathrm{E}}^\alpha \end{cases} \quad , \quad {}_a\partial_t^\alpha \equiv c^\alpha {}_a\partial_0^\alpha \quad , \vec{\nabla}_{left}^\alpha \equiv {}_a\partial_i^\alpha \hat{e}_i \tag{10}$$

Also we can write the non-causal right fractional counterparts of the above equations to complete the set of fractional Maxwell's equations in comparison to standard ones as follows:

$$\begin{cases} \vec{\nabla}_{right}^\alpha . \vec{B}^\alpha = 0 \\ \vec{\nabla}_{right}^\alpha \times \vec{\mathrm{E}}^\alpha = -\frac{1}{c^\alpha}\partial_b^\alpha \vec{B}^\alpha \end{cases} \quad , \vec{\nabla}_{right}^\alpha \equiv_i \partial_b^\alpha \hat{e} \tag{11}$$

As it is mentioned in [13] in spite of non-causality, the above set of equations have another problem which is they are not symmetric respect to space and time, so to resolve these problems one can generalize the fractional Maxwell's field strength tensor as:

$$F_{\mu\nu}^{\alpha\beta} = \partial_\mu^{\alpha\beta} A_\nu - \partial_\nu^{\alpha\beta} A_\mu \quad , \quad 0 < \alpha, \beta < 1 \tag{12}$$

so that it preserves the condition of $\partial_{[\rho}^{\alpha\beta} F_{\mu\nu]}^{\alpha\beta} = 0$ which $[\rho\mu\nu]$ denotes the antisymmetrized sum over permutations of the indices ρ, μ, and ν.

And the resulting Maxwell's equations will read as:

$$\begin{cases} \vec{\nabla}^{\beta\alpha} . \vec{\mathrm{E}}^{\alpha\beta} = 4\pi\rho \\ \vec{\nabla}^{\beta\alpha} \times \vec{B}^{\alpha\beta} = \frac{4\pi}{c}\vec{j} + \frac{1}{c^\alpha}\partial_t^{\beta\alpha} \vec{\mathrm{E}}^{\alpha\beta} \\ \vec{\nabla}^{\alpha\beta} . \vec{B}^{\alpha\beta} = 0 \\ \vec{\nabla}^{\alpha\beta} \times \vec{\mathrm{E}}^{\alpha\beta} = -\frac{1}{c^\alpha} \partial_t^{\alpha\beta} \vec{B}^{\alpha\beta} \end{cases} \quad , \quad \vec{\nabla}^{\alpha\beta} \equiv \partial_i^{\alpha\beta} \hat{e}_i \tag{13}$$

By choosing $\alpha_\mu = \beta_\mu$ and $b_\mu = -a_\mu$ the left–right operators reduce to Riesz derivatives and the fractional Maxwell's equations become spatially and time symmetric. Furthermore, when the order of time derivative are $\alpha_0 = \beta_0 = 1$, the fractional time derivatives will reduce to standard first order derivatives and in this case the above electromagnetic fields are causal [13].

In this formulation of fractional Maxwell's equations we will have Poynting vector $\vec{S}^{\alpha\beta}$ showing the directional energy flux of the fractional electromagnetic field as:

$$\vec{S}^{\alpha\beta} = \vec{E}^{\alpha\beta} \times \vec{B}^{\alpha\beta} \tag{14}$$

Also in this case we will have the fundamental concept of charge conservation in fractional nonlocal electrodynamics. The above fractional Maxwell's equations, if we consider only one dimensional space fractionality and the Riesz fractional operator, as well as the fractional Lorenz gauge in the form of $\partial_{\mu}^{\alpha\beta} A^{\mu} = 0$, then we will have the following fractional wave equation related to the spatially symmetrical and causal fields:

$$\frac{1}{c^2}\partial_t^2 u - (\partial_x^{\alpha\alpha})^2 u = \frac{1}{c^2}\partial_t^2 u - \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{u(\xi,\mathrm{t})}{(\mathrm{x}-\xi)^{\alpha-1}}\, d\xi = 0 \tag{15}$$

and using Fourier transform rule for the Riesz derivative which reads as:

$$F\left\{\frac{d^{\alpha} f(x)}{d|x|^{\alpha}}\right\} = -|\xi|^{\alpha} F\left\{ f(x)\right\} \quad , \quad 0 < \alpha < 2 \tag{16}$$

we will have the plane wave solution in the form of:

$$u(x,t) = \sum_{k=-\infty}^{\infty} f_k^1 \cos(\omega_{\alpha} t) e^{ikx} + \sum_{k=-\infty, k\neq 0}^{\infty} f_k^2 \omega_{\alpha}^{-1} \sin(\omega_{\alpha} t) e^{ikx} \tag{17}$$

where f_k^1 and f_k^2 are the coefficients of Fourier series expansions and $\omega_{\alpha} = |k|^{\alpha}\, c\, \sin(\frac{\alpha\pi}{2})$ is the fractional time frequency of plane-waves [13]. In the other hand if we consider only the time fractionality then we will have the plane wave solution as [12]:

$$u(x,t) = u_0 e^{-ikx} E_{2\beta}(-\omega_{\beta}{}^2 t^{2\beta}) \tag{18}$$

where

$$E_{\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\beta k)} \tag{19}$$

is one-parameter Mittag–Leffler function. It is worth mentioning that at asymptotically large times above solution will have the form of [12]:

$$u(x,t) \approx \left(\frac{u_0 e^{-ikx}}{\sqrt{\pi}\omega_{\beta}^2}\right)\frac{1}{\sqrt{t}} \tag{20}$$

Here we should point out that all of the above relations have been derived using fractional Riemann–Liouville and Riesz operators which use a simple form of memory kernel, however as we have mentioned in pervious section there are many different form of memory kernels and so there are several different kinds of fractional operators which certainly will result in different theoretical and phenomenological results. In [14] electromagnetic memory effect in analogy with the gravitational memory effect has been interpreted as velocity kick i.e. a residual velocity of charges because of electromagnetic fields. They have used the simple and incomplete definition for velocity kick as:

$$\Delta\vec{v} = \frac{q}{m}\int_{-\infty}^{+\infty}\vec{E}\ dt \tag{21}$$

which q and m are charge and mass, however it is obvious that a complete view of this quantity can be derived using the generalized fractional form of it using the Riesz fractional integral operator as:

$$\Delta\vec{v}^{\alpha} = \frac{q}{2m\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}\int_{-\infty}^{+\infty}\left|t-\tau\right|^{\alpha-1}\vec{E}\ dt \quad , \quad 0<\alpha\leq 1 \tag{22}$$

In fact we can generalize space-time evolution of any dynamical process using a convolution integral with a kind of memory kernel function $K(\mathrm{t},\tau)$ which can has many different forms, from its simplest singular power-law form $K(\mathrm{t},\tau)=\left|t-\tau\right|^{\alpha-1}$ to its more complex forms.

Finally, we can apply the formalism described in this work also at the fundamental level of quantum electrodynamics with proposing fractional generalized version of gauge invariant Stückelberg lagrangian as:

$$L_{Fractional\ Stückelberg\ Field} = -\frac{1}{4}F^{\alpha\beta}_{\mu\nu}F^{\mu\nu}_{\alpha\beta} - \frac{1}{2}m_{\gamma}^{2}A_{\mu}A^{\mu} - \frac{1}{2}(\partial^{\alpha\beta}_{\mu}\varphi)(\partial^{\mu}_{\alpha\beta}\varphi) - m_{\gamma}A^{\mu}\partial^{\alpha\beta}_{\mu}\varphi - j_{\mu}A^{\mu} \tag{23}$$

Based on the above proposed lagrangian one can investigate the memory effect in its most fundamental level and in the framework of fractional quantum electrodynamics.

3. Conclusion

Our observations of events and phenomena occurring in our universe tell us this fact that there is some kind of simplicity / complexity duality in our world which means that some phenomena seem to be simple at the microscopic level however they are completely complex in macroscopic level and vice versa. It seems that the reasonable and comprehensive tool and framework to deal with such dualities would be the reliable approach of fractional calculus.

For instance, one important issue of research in theoretical and phenomenological physics in recent years is effects of observed gravitational waves in our universe. Nowadays we know that gravitational memory effect is an important properties of gravitational waves which is in fact a permanent change in the spacetime geometry because of a passed gravitational wave and due to the energy of gravitons. Also there is a kind of electromagnetic memory effect in analogy with the gravitational one which its meaning is that electromagnetic field will produce velocity kick on a distant test particle. This fact that fractional calculus is the only natural tool for description of such phenomena (i.e. different kinds of memory effects in nature) and is strongly related to the underlying structure of spacetime leads us to this important notion that we may really are living in a fractional dimensional universe and this can be considered as common origin of both gravitational and electromagnetic memories.

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