Cantor diagonal argument-for real numbers

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abstract

This analysis shows Cantor's diagonal argument published in 1891 cannot form a sequence that is not a member of a complete set.

the argument

Translation from Cantor's 1891 paper [1]:
Namely, let *m* and *n* be two different characters, and consider a set [*Inbegriff*] *M* of elements

$$E = (x_1, x_2, \dots, x_v, \dots)$$

which depend on infinitely many coordinates $x_1, x_2, \dots, x_v, \dots$, and where each of the coordinates is either *m* or *w*. Let *M* be the totality [*Gesamtheit*] of all elements *E*.
To the elements of *M* belong e.g. the following three:

$$E^{I} = (m, m, m, m, \dots),$$
$$E^{II} = (w, w, w, w, \dots),$$
$$E^{III} = (m, w, m, w, \dots).$$

I maintain now that such a manifold [*Mannigfaltigkeit*] *M* does not have the power of the series 1, 2, 3, …, *v*, ….

This follows from the following proposition:
"If $E_1, E_2, \dots, E_v, \dots$ is any simply infinite [*einfach unendliche*] series of elements of the manifold *M*, then there always exists an element E_0 of *M*, which cannot be connected with any element E_v."
For proof, let there be

$$E_1 = (a_{1.1}, a_{1.2}, \dots, a_{1,v}, \dots)$$
$$E_2 = (a_{2.1}, a_{2.2}, \dots, a_{2,v}, \dots)$$
$$E_u = (a_{u.1}, a_{u.2}, \dots, a_{u,v}, \dots)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots.$$

where the characters $a_{u,v}$ are either *m* or *w*. Then there is a series $b_1, b_2, \dots b_v, \dots$, defined so that b_v is also equal to *m* or *w* but is *different* from $a_{v,v}$.
Thus, if $a_{v,v} = m$, then $b_v = w$.
Then consider the element

$$E_0 = (b_1, b_2, b_3, \dots)$$

of *M*, then one sees straight away, that the equation

$$E_0 = E_u$$

cannot be satisfied by any positive integer u, otherwise for that u and for all values of v.

$$b_v = a_{u,v}$$

and so we would in particular have

$$b_u = a_{u,u}$$

which through the definition of b_v is impossible. From this proposition it follows immediately that the totality of all elements of *M* cannot be put into the sequence [*Reihenform*]: $E_1, E_2, \ldots, E_v, \ldots$ otherwise we would have the contradiction, that a thing [*Ding*] E0 would be both an element of M, but also not an element of M.
(end of translation)

1. analysis

Begin with M a random list of real numbers >0 and <1, without decimal points.
Each Eu is an infinite sequence of integers, formed from the set of ten (0 through 9). Each Eu must begin with one of those integers, and becomes a member of one of 10 subsets S0 to S9. Cantor's coordinate system u,v for row and column is used here.

1.1 negation

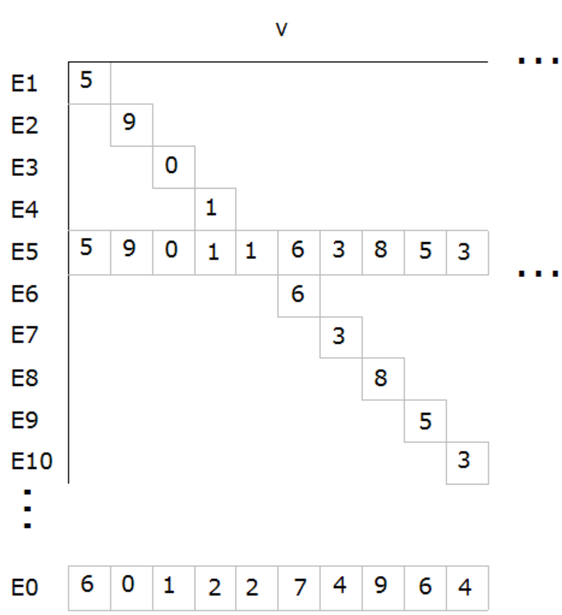

fig.1

Define a diagonal form of a sequence D composed of one integer from every row after its origin. Next form a horizontal sequence E0 that differs from D in all positions. One simple method is adding i to all integers x in D, using x'=(x+i) mod 10. With i=1, this assigns E0 to subset S6. Fig.1 shows there is nothing prohibiting the diagonal D from having a horizontal counterpart (E5). This would not be obvious since it cannot be detected with one comparison, and can occur anywhere in the list, which has no specific order of entry.

1.2 form

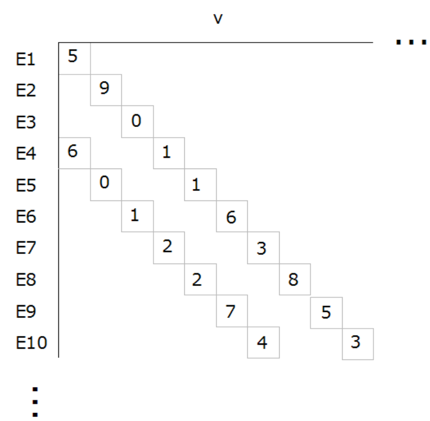

fig.2

In the original list all sequences are parallel and do not interact. If the list contained all diagonal sequences as in fig.2, they would be parallel and not interact. A problem appears when the different forms are mixed as in fig.1, where E0 could not appear in any row as a horizontal sequence, since it differs in all v.

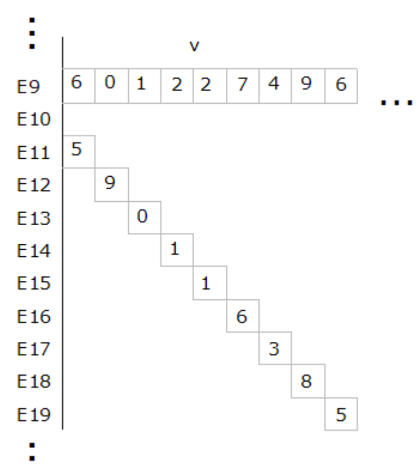

fig.3

The u,v coordinate system is relative to the current random list. There are many random lists. If D starts at row 11, E0 cannot appear in any row greater than 10 but can appear in any row less than 11 as shown.

conclusion

A random list should contain sequences that are formed and entered independently of all other sequences. The transformation applies to the integers, but not the form.
The diagonal form of a sequence by its extension across other sequences, prevents the appearance of its negation in any row that follows, effectively imposing a degree of order on an otherwise random list. Cantor has altered his random list to a semi random list.
He forms E0 as NOT D, which assigns it to any subset except S5.
E0 cannot occupy the same space in the list as D, and in his example D begins at u=1,
He misinterprets this as E0 a member and not a member of M, his contradiction.
Cantor's relative comparison of D and E0, only shows them as members of distinct subsets, which are all members of M. All sequences in M are members of one subset and not members of the other nine subsets.
Cantor's contradiction resulted from the diagonal D, and it can be removed with the removal of D.

[1] THE LOGIC MUSEUM Copyright © E.D.Buckner 2005