A Sheaf on a Lattice

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Abstract A sheaf is constructed on a topological space. But a topological space is a bounded distributive lattice. Hence we may construct a sheaf of lattices on a bounded distributive lattice. Then we define a stalk of the sheaf at a chain in a bounded distributive lattice. And we define a morphism of the sheaves, that the morphism is induced by a homomorphism of the bounded distributive lattices. Then the kernel and image of the morphism are the subsheaves. A sheaf is obtained by gluing sheaves together.

Contents

1. Introduction

Recall the definition of sheaves(see [1, 7, 9]). Suppose that $\mathscr{F}$ is a sheaf on a topological space X. Let U be an open set of X. Then $\mathscr{F}(U)$ is a mathematical object(e.g., set, group, ring). And the sheaf $\mathscr{F}$ satisfies several properties.

Since a topological space is a bounded distributive lattice(cf. [4, 8]), we may construct a sheaf $\mathscr{L}$ of lattices on a bounded distributive lattice $\boldsymbol{L}$, see theorems 3.1 and 3.2 in subsection 3.1.

A stalk[1, 9] at $p \in X$ of the sheaf $\mathscr{F}$ is a colimit(cf. [1, 6, 7, 9]). But we define a stalk of $\mathscr{L}$ at a chain[definition 2.9]. The stalk of $\mathscr{L}$ is defined in definition 3.1.

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A morphism[1, 9] of sheaves is a natural transformation(cf. [1, 6, 7, 9]). Suppose that $\psi\colon \boldsymbol{L} \to \boldsymbol{L'}$ is a homomorphism[4, 8] of the bounded distributive lattices. Then ψ induces a morphism $\hat{\psi}\colon \mathscr{L} \to \mathscr{L}'$, see theorem 3.3. That $\hat{\psi}$ is a monomorphism(epimorphism) if ψ is a monomorphism(epimorphism), see definition 3.2.

A subsheaf[1, 9] of the $\mathscr{L}$ is a sheaf on a bounded distributive lattice which is a sublattice of the $\boldsymbol{L}$, see definition 3.3. And the kernel(cf. [1, 9]) of $\hat{\psi}$ is a subsheaf, see definition 3.4.

In subsection 3.5, we obtain a sheaf by gluing(cf. [1, 9]) sheaves together.

2. Preliminaries

2.1. Bounded Distributive Lattice.

Recall the definitions in [8].

Definition (Lattice[8])**.** A nonempty set L together with two binary operations $\vee$ and $\wedge$ is called a **lattice** if it satisfies the following identities:

(commutative laws)
$$x \vee y = y \vee x$$
$$x \wedge y = y \wedge x$$

(associative laws)
$$(x \vee y) \vee z = x \vee (y \vee z)$$
$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$

(idempotent laws)
$$x \vee x = x$$
$$x \wedge x = x$$

(absorption laws)
$$(x \vee y) \wedge x = x$$
$$(x \wedge y) \vee x = x$$

The lattice is denoted by $\boldsymbol{L}$.

Definition 2.1 (Bounded Lattice[8])**.** An algebra $\langle \boldsymbol{L}, \vee, \wedge, 0, 1\rangle$ with two binary and two nullary operations is a **bounded lattice** if it satisfies:

- $\langle \boldsymbol{L}, \vee, \wedge\rangle$ is a lattice.
- $x \wedge 0 = 0$; $x \vee 1 = 1$.

Definition 2.2 (Distributive Lattice[8])**.** A **distributive lattice** is a lattice which satisfies the distributive laws:

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$
$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

Then we have the following proposition.

Proposition 2.1 (cf. [8])**.** *Suppose that X is a topological space. Then the open subsets of X form a bounded distributive lattice. The bounded distributive lattice is denoted by $\mathfrak{L}(X)$.*

Proof. For open sets $U, V \subseteq X$, let $U \leq V$ if $U \subseteq V$. Then open subsets form a poset[8]. And every subset of the poset has the infimum and supremum. Hence the poset is a lattice by [8, definition 1.4]. Then let U, V, W be open subsets of X. Define

$$U \vee V := \sup\{U, V\}$$
$$U \wedge V := \inf\{U, V\}$$
$$0 := \varnothing$$
$$1 := X$$

And $\sup\{U,V\} = U \cup V$. Then we have that

$$\begin{aligned}(U \vee V) \wedge W &= \inf\{\sup\{U,V\},W\}\\ &= \inf\{U \cup V, W\}\\ &= \inf\{U,W\} \cup \inf\{V,W\}\\ &= \sup\{\inf\{U,W\},\inf\{V,W\}\\ &= (U \wedge W) \vee (V \wedge W)\\ (U \wedge V) \vee W &= \sup\{\inf\{U,V\},W\}\\ &= \inf\{U,V\} \cup W\\ &= \inf\{U \cup W, V \cup W\}\\ &= \inf\{\sup\{U,W\},\sup\{V,W\}\}\\ &= (U \vee W) \wedge (V \vee W)\end{aligned}$$

And,

$$\begin{aligned}U \wedge \varnothing &= \varnothing\\ U \vee X &= X\end{aligned}$$

We are done, by the definitions 2.1 and 2.2. □

2.2. **Poset.** A partial order set(briefly a poset) is a nonempty set together with a binary relation which is reflexive, transitive and antisymmetric, see [8, 10].

Definition 2.3 ([8])**.** Let $\boldsymbol{L}$ be a lattice. For $a, b \in \boldsymbol{L}$, define $a \leq b$ if $a \wedge b = a$.

Theorem 2.1 ([8])**.** *A lattice $\boldsymbol{L}$ is a poset.*

Proof. Immediate from the definition 2.3. □

Theorem 2.2 ([8])**.** *Suppose that $\langle \boldsymbol{L}, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. Then an interval $[0,a] := \{x \in \boldsymbol{L} \mid x \wedge a = x\}$ is a sublattice of $\langle \boldsymbol{L}, \vee, \wedge\rangle$ for all $a \in \boldsymbol{L}$.*

Proof. For $x, y \in [0,a]$, we have

$$\begin{aligned}x \wedge 0 &= 0\\ x \vee 0 &= x \vee (x \wedge 0)\\ &= x\\ (x \wedge y) \wedge a &= x \wedge (y \wedge a)\\ &= x \wedge y\\ (x \vee y) \wedge a &= (x \wedge a) \vee (y \wedge a)\\ &= x \vee y\end{aligned}$$

Hence $x \vee y, x \wedge y \in [0,a]$. □

Corollary 2.1.1 ([8])**.** *The interval $[0,a]$ is a bounded distributive lattice.*

Proof. It is obvious that the lattice $\langle [0,a], \vee, \wedge, 0, a\rangle$ is a bounded distributive lattice. □

Theorem 2.3 (cf. [8])**.** *Suppose that $\langle \boldsymbol{L}, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. Then $[0,1] \cong \boldsymbol{L}$.*

Lemma 2.1. *For every $a \in \boldsymbol{L}$, $a \wedge 1 = a$.*

Proof. We have $a \wedge 1 = a \wedge (a \vee 1) = a$ □

Proof of theorem 2.3. Immediate from theorem 2.2 and lemma 2.1. □

2.3. **Lattice of the Sublattices.** The intervals of a bounded distributive lattice may form a lattice.

Proposition 2.2 (cf. [8]). *Suppose that $\langle \mathbf{L}, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. Let $a, b \in \mathbf{L}$. Then the intersection $[0, a] \cap [0, b]$ is a sublattice of $\langle \mathbf{L}, \vee, \wedge\rangle$.*

Proof. The intersection is not empty, since $0 \in [0, a] \cap [0, b]$. We have $x \wedge y, x \vee y \in [0, a] \cap [0, b]$ for all $x, y \in [0, a] \cap [0, b]$. Therefore, the statement is true. □

It is obvious that $c \leq a$ implies that $[0, c]$ is a sublattice of $[0, a]$.

Corollary 2.2.1 (cf. [8]). *If $c \leq a$ and $c \leq b$, then the subset $[0, c]$ is a sublattice of $[0, a] \cap [0, b]$.*

Proof. If $x \leq c$, then $x \leq b$ and $x \leq a$. It follows $x \in [0, a] \cap [0, b]$. And we have $[0, c] \cap [0, a] \cap [0, b] = [0, c]$. By theorem 2.2, the subset $[0, c]$ is a sublattice of $[0, a] \cap [0, b]$. □

Proposition 2.3 (cf. [8]). *Suppose that $\mathbf{L}$ is a bounded distributive lattice. Let $a, b \in \mathbf{L}$. Then $[0, a] \cap [0, b]$ is the set $\{x \wedge y \mid x \in [0, a], y \in [0, b]\}$.*

Proof. We have $x \wedge y \wedge a = x \wedge y$ and $x \wedge y \wedge b = x \wedge y$. It follows

$$\{x \wedge y \mid x \in [0, a], y \in [0, b]\} \subseteq [0, a] \cap [0, b]$$

On the other hand, for every $z \in [0, a] \cap [0, b]$, we have $z = z \wedge a$ and $z = z \wedge b$. Hence $z = z \wedge z = (z \wedge a) \wedge (z \wedge b)$. So $[0, a] \cap [0, b] \subseteq \{x \wedge y \mid x \in [0, a], y \in [0, b]\}$. Therefore, $[0, a] \cap [0, b] = \{x \wedge y \mid x \in [0, a], y \in [0, b]\}$. □

Corollary 2.3.1 (cf. [8]).

$$[0, a] \cap [0, b] = [0, a \wedge b]$$

Proof. We have that $x \in [0, a] \cap [0, b]$ implies $x \leq a \wedge b \leq a, b$. Then immediate from propositions 2.2 and 2.3 and corollary 2.2.1. □

Proposition 2.4 (cf. [8]). *Suppose that $\mathbf{L}$ is a bounded distributive lattice. Let U be the set $\{x \vee y \mid x \in [0, a], y \in [0, b]\}$ for $a, b \in \mathbf{L}$. Then the set U is a sublattice of $\langle \mathbf{L}, \vee, \wedge\rangle$.*

Proof. For all $x \in [0, a]$, $y \in [0, b]$, we have $x \wedge y \in ([0, a] \cap [0, b] = [0, a \wedge b])$ by proposition 2.3 and corollary 2.3.1. And let $x, x' \in [0, a]$ and $y, y' \in [0, b]$.

$$\begin{aligned} x \vee y \vee x' \vee y' &= x \vee x' \vee y \vee y' \\ (x \vee y) \wedge (x' \vee y') &= ((x \vee y) \wedge x') \vee ((x \vee y) \wedge y') \\ &= (x \wedge x') \vee (y \wedge x') \vee (x \wedge y') \vee (y \wedge y') \\ &= ((x \wedge x') \vee (y \wedge x')) \vee ((x \wedge y') \vee (y \wedge y')) \end{aligned}$$

Since $x \wedge x', x \vee x' \in [0, a]$, $y \wedge y', y \vee y' \in [0, b]$ and $y \wedge x', x \wedge y' \in [0, a] \cap [0, b]$. Therefore, the set U is a sublattice. □

Definition 2.4 (cf. [8]). The sublattice U in proposition 2.4 is said to be **generated by** the set $[0, a] \cup [0, b]$. We denote the sublattice U by $G([0, a] \cup [0, b])$.

Corollary 2.4.1 (cf. [8])**.**

$$G([0,a]\cup[0,b])=[0,a\vee b]$$

Proof. Let $x\in[0,a], y\in[0,b]$. Then we have

$$\begin{aligned}(x\vee y)\wedge(a\vee b)&=(x\wedge(a\vee b))\vee(y\wedge(a\vee b))\\&=((x\wedge a)\vee(x\wedge b))\vee((y\wedge a)\vee(y\wedge b))\\&=(x\vee(x\wedge b))\vee((y\wedge a)\vee y)\\&=x\vee y\end{aligned}$$

Hence $U\subseteq[0,a\vee b]$. On the other hand, for every $z\in[0,a\vee b]$, we have

$$z=z\wedge(a\vee b)=(z\wedge a)\vee(z\wedge b)$$

It follows $[0,a\vee b]\subseteq U$. And $x\wedge y\in[0,a]\cap[0,b]$. Therefore, $U=[0,a\vee b]$. □

Now we may define a bounded distributive lattice by the intervals.

Theorem 2.4 (cf. [8])**.** *Suppose that $\boldsymbol{L}$ is a bounded distributive lattice. Let $\mathfrak{I}(\boldsymbol{L})$ be the set $\{[0,x]\mid x\in\boldsymbol{L}\}$. Then $\mathfrak{I}(\boldsymbol{L})$ is a bounded distributive lattice.*

Proof. For every $a,b\in\boldsymbol{L}$, define

$$\begin{aligned}[0,a]\wedge[0,b]&:=[0,a]\cap[0,b]\\ [0,a]\vee[0,b]&:=G([0,a]\cup[0,b])\\ 0&:=[0,0]\\ 1&:=\boldsymbol{L}\end{aligned}$$

Then it is a bounded lattice by propositions 2.2 to 2.4, corollarys 2.2.1 to 2.4.1, definition 2.4, and theorem 2.3.

Let $I_a=[0,a], I_b=[0,b], I_c=[0,c]$ for $a,b,c\in\boldsymbol{L}$. By corollarys 2.3.1 and 2.4.1, we have

$$\begin{aligned}(I_a\wedge I_b)\vee I_c&=I_{a\wedge b}\vee I_c\\&=I_{(a\wedge b)\vee c}\\&=I_{(a\vee c)\wedge(b\vee c)}\\&=I_{a\vee c}\wedge I_{b\vee c}\\&=(I_a\vee I_c)\wedge(I_b\vee I_c)\end{aligned}$$

and

$$\begin{aligned}(I_a\vee I_b)\wedge I_c&=I_{a\vee b}\wedge I_c\\&=I_{(a\vee b)\wedge c}\\&=I_{(a\wedge c)\vee(b\wedge c)}\\&=I_{a\wedge c}\vee I_{b\wedge c}\\&=(I_a\wedge I_c)\vee(I_b\wedge I_c)\end{aligned}$$

Therefore, the algebra $\langle\mathfrak{I}(\boldsymbol{L}),\vee,\wedge,\{0\},\boldsymbol{L}\rangle$ is a bounded distributive lattice. □

Hence we have

$$\begin{aligned}[0,a] \wedge [0,b] &= [0,a] \cap [0,b] \\ &= [0, a \wedge b] \\ G([0,a] \cup [0,b]) &= [0,a] \vee [0,b] \\ &= [0, a \vee b]\end{aligned}$$

2.4. **Homomorphism of the Lattices.** Let $\boldsymbol{L}, \boldsymbol{L'}$ be two bounded distributive lattices. A homomorphism $\varphi: \boldsymbol{L} \to \boldsymbol{L'}$ is a function compatible with the n-ary operations of the lattices for $n \geq 0$(cf. [2–4, 8]).

Theorem 2.5. *Suppose that $\boldsymbol{L}, \boldsymbol{L'}$ are two bounded distributive lattices. Let $\psi: \boldsymbol{L} \to \boldsymbol{L'}$ be a homomorphism. If $a \in \boldsymbol{L}$, then ψ induces a homomorphism $\hat{\psi}_a: [0,a] \to [0, \psi(a)]$ given by $x \mapsto \psi(x)$.*

Proof. If $x \in [0,a]$, then $x \wedge a = x$. Hence

$$\psi(x) = \psi(x \wedge a) = \psi(x) \wedge \psi(a)$$

It follows $\psi(x) \in [0, \psi(a)]$. And the subset $[0,a]$ is a sublattice of $\langle \boldsymbol{L}, \vee, \wedge \rangle$ by theorem 2.2. Hence $\hat{\psi}_a := \psi \restriction [0,a]$ is a homomorphism. □

Corollary 2.5.1 (cf. [2–4, 6–8])**.** *If ψ is a monomorphism(epimorphism, isomorphism), then $\hat{\psi}_a$ is a monomorphism(epimorphism, isomorphism) for $a \in \boldsymbol{L}$.*

Proof. Suppose that ψ is a monomorphism. Let $x, y \in [0,a]$ with $x \neq y$. Then $\psi(x) \neq \psi(y)$. It follows $\hat{\psi}(x) \neq \hat{\psi}(y)$. Hence $\hat{\psi}$ is a monomorphism.

Suppose that ψ is an epimorphism. For every $v \in [0, \psi(a)]$, there exists $u \in \boldsymbol{L}$ such that $\psi(u) = v$. And

$$v = v \wedge \psi(a) = \psi(u) \wedge \psi(a) = \psi(u \wedge a)$$

Since $u \wedge a$ is in $[0,a]$, that $\hat{\psi}$ is an epimorphism.

Suppose that ψ is an isomorphism. It follows that $\hat{\psi}$ is an isomorphism. □

Let $\psi: \boldsymbol{L} \to \boldsymbol{L'}$ be a homomorphism. Then the subset $\psi^{-1}(0)$ is special, since it has some interesting properties.

Proposition 2.5. *The subset $\psi^{-1}(0)$ is an interval. Hence it is a bounded distributive lattice.*

To prove proposition 2.5, we need the following lemma.

Lemma 2.2. *The subset $\psi^{-1}(0)$ has one maximal member.*

Proof. We have

$$\begin{aligned}\psi\Big(\bigvee_{a \in \psi^{-1}(0)} a\Big) &= \bigvee_{a \in \psi^{-1}(0)} \psi(a) \\ &= \bigvee 0 \\ &= 0\end{aligned}$$

It follows $\bigvee_{a \in \psi^{-1}(0)} a \in \psi^{-1}(0)$. And it is obvious that $\bigvee_{a \in \psi^{-1}(0)} a$ is the unique maximal member. □

Proof of proposition 2.5. Let K be the subset $\psi^{-1}(0)$, m the maximal member of K by lemma 2.2. We have $0 \in K$. For $a, b \in K$, $\psi(a \vee b) = 0$, $\psi(a \wedge b) = 0$. It follows $a \vee b, a \wedge b \in K$. Hence K is a lattice. And for every $x \leq m$, $\psi(x) = \psi(x \wedge m) = 0$. Hence $[0, m] \subseteq K$. On the other hand, for every $x \in K$, we have $x \leq m$. Hence $K \subseteq [0, m]$. Therefore, $K = [0, m]$. By corollary 2.1.1, K is a bounded distributive lattice. □

The kernel of ψ, $\ker\psi$, is a congruence relation(cf. [4, 8]), that is, $\langle a, b\rangle \in \ker\psi$ iff $\psi(a) = \psi(b)$. But we need an other definition of kernel in the case of $\hat{\psi}_\square$.

Definition 2.5 (cf. [2, 3])**.** Suppose that $\hat{\psi}_a$ is a homomorphism defined in theorem 2.5. Then the **kernel** of $\hat{\psi}_a$ is the intersection $\psi^{-1}(0) \cap [0, a]$.

Proposition 2.6. *Suppose that $\hat{\psi}_a$ is a homomorphism defined in theorem 2.5. Then the kernel $\ker\hat{\psi}_a$ is a sublattice of the lattice $[0, a]$. And the kernel is an interval.*

Proof. Immediate from propositions 2.2 and 2.5, theorem 2.2, and corollary 2.3.1 □

There exists a special homomorphism which is a mapping from an interval to its subinterval.

Theorem 2.6 ([8])**.** *Suppose that $\mathbf{L}$ is a bounded distributive lattice. Let $a, b \in \mathbf{L}$ with $a \leq b$. Then there exist a homomorphism $\varphi_{b,a} : [0, b] \to [0, a]$ given by $x \mapsto x \wedge a$. That $\varphi_{b,a}$ is a homomorphism of the bounded distributive lattices. If $a = b$, then the homomorphism $\varphi_{b,a}$ is an indentity isomorphism*[3, 4, 8].

Proof. By corollary 2.1.1, we have that the subsets $[0, a], [0, b]$ are bounded distributive lattices. For every $x \in [0, b]$, $x \wedge a = x \wedge (a \wedge a) = (x \wedge a) \wedge a$. It follows $x \wedge a \leq a$, $x \wedge a \in [0, a]$. And that $a \leq b$ implies $b \wedge a = a$. For $x, y, z \in [0, b]$,

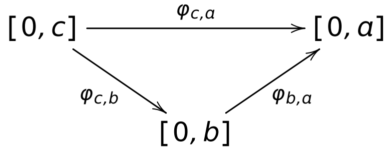
$$\begin{aligned}
(x \vee y) \wedge a &= (x \wedge a) \vee (y \wedge a) \\
(x \wedge y) \wedge a &= x \wedge y \wedge a \wedge a \\
&= (x \wedge a) \wedge (y \wedge a) \\
((x \vee y) \wedge z) \wedge a &= ((x \wedge z) \vee (y \wedge z)) \wedge a \\
&= (x \wedge z \wedge a) \vee (y \wedge z \wedge a) \\
&= (x \wedge a \wedge z \wedge a) \vee (y \wedge a \wedge z \wedge a) \\
((x \wedge y) \vee z) \wedge a &= ((x \vee z) \wedge (y \vee z)) \wedge a \\
&= ((x \vee z) \wedge a) \wedge ((y \vee z) \wedge a) \\
&= ((x \wedge a) \vee (z \wedge a)) \wedge ((y \wedge a) \vee (z \wedge a))
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi(x \vee y) &= \varphi(x) \vee \varphi(y) \\
\varphi(x \wedge y) &= \varphi(x) \wedge \varphi(y) \\
\varphi((x \vee y) \wedge z) &= (\varphi(x) \wedge \varphi(z)) \vee (\varphi(y) \wedge \varphi(z)) \\
\varphi((x \wedge y) \vee z) &= (\varphi(x) \vee \varphi(z)) \wedge (\varphi(y) \vee \varphi(z)) \\
\varphi(0) &= 0 \\
\varphi(b) &= a
\end{aligned}$$

It is obvious that $a = b$ implies that φ is an identity isomorphism. □

Proposition 2.7. *Suppose that* $\boldsymbol{L}$ *is a bounded distributive lattice. Let* $a, b, c \in \boldsymbol{L}$ *with* $a \le b \le c$. *Let* $\varphi_{b,a} : [0, b] \to [0, a]$, $\varphi_{c,b} : [0, c] \to [0, b]$, $\varphi_{c,a} : [0, c] \to [0, a]$ *be the homomorphisms which are defined in theorem 2.6. Then the following diagram is commutative.*



Proof. For every $x \in [0, c]$,

$$(x \wedge b) \wedge a = x \wedge (b \wedge a) = x \wedge a$$

It follows $\varphi_{b,a}(\varphi_{c,b}(x)) = \varphi_{c,a}(x)$. □

Theorem 2.7. *Suppose that* $\boldsymbol{L}$ *is a bounded distributive lattice. Let* $a, a_1, a_2 \in \boldsymbol{L}$ *with* $a = a_1 \vee a_2$. *If* $x, y \in [0, a]$, *then* $(x \wedge a_i = y \wedge a_i)_{i=1,2}$ *implies* $x = y$.

Proof.

$$\begin{aligned} x &= x \wedge a \\ &= x \wedge (a_1 \vee a_2) \\ &= (x \wedge a_1) \vee (x \wedge a_2) \\ &= (y \wedge a_1) \vee (y \wedge a_2) \\ &= y \wedge (a_1 \vee a_2) \\ &= y \end{aligned}$$

□

Corollary 2.7.1. *Let* $\{a_i\}_{i \in I} \subseteq \boldsymbol{L}$, $a \in \boldsymbol{L}$ *with* $\bigvee_{i \in I} a_i = a$. *If* $x \wedge a_i = y \wedge a_i$ *for all* i, *then* $x = y$.

Proof. It is obvious. □

Corollary 2.7.2. *Suppose that* φ_{a,a_i} *is a homomorphism defined in theorem 2.6 for all* i. *If* $\varphi_{a,a_i}(x) = \varphi_{a,a_i}(y)$ *for all* i, *then* $x = y$.

Proof. It is obvious. □

Theorem 2.8. *Suppose that* $\boldsymbol{L}$ *is a bounded distributive lattice. Let* $a, a_1, a_2 \in \boldsymbol{L}$ *with* $a = a_1 \vee a_2$. *If* $x_1 \in [0, a_1], x_2 \in [0, a_2]$ *with* $x_1 \wedge a_1 \wedge a_2 = x_2 \wedge a_1 \wedge a_2$, *then there exists* $x \in [0, a]$ *such that* $x \wedge a_1 = x_1$ *and* $x \wedge a_2 = x_2$.

Proof. The equation $x_1 \wedge a_1 \wedge a_2 = x_2 \wedge a_1 \wedge a_2$ implies $x_1 \wedge a_2 = x_2 \wedge a_1$. Then we have $(x_1 \wedge a_2) \vee x_1 = (x_2 \wedge a_1) \vee x_1$. It follows

$$x_1 = (x_2 \vee x_1) \wedge (a_1 \vee x_1) \tag{2.1}$$

Similarly, we have $(x_1 \wedge a_2) \vee x_2 = (x_2 \wedge a_1) \vee x_2$. It implies

$$(x_1 \vee x_2) \wedge (a_2 \vee x_2) = x_2 \tag{2.2}$$

Since $x_2 \vee x_1 \in [0, a]$, $a_1 \vee x_1 = a_1$, $a_2 \vee x_2 = a_2$, equations (2.1) and (2.2), hence the statement is true, and $x = x_1 \vee x_2$ as desired. □

Corollary 2.8.1. *Let* $\{a_i\}_{i \in I} \subseteq \boldsymbol{L}$, $a \in \boldsymbol{L}$ *with* $\bigvee_{i \in I} a_i = a$, $x_i \in [0, a_i]$. *If* $x_i \wedge a_i \wedge a_j = x_j \wedge a_i \wedge a_j$ *for all* i, j *then there exists* $x \in [0, a]$ *such that* $x_i = x \wedge a_i$.

Proof. It is obvious. □

Corollary 2.8.2. *Suppose that the following mappings $\varphi_{\square,\square}$ are the homomorphisms defined in theorem 2.6. If $\varphi_{a_i,a_i\wedge a_j}(x_i)=\varphi_{a_j,a_i\wedge a_j}(x_j)$ for all i,j then there exists $x\in[0,a]$ such that $x_i=\varphi_{a,a_i}(x)$.*

Proof. It is obvious. □

The composition of $\varphi_{\square,\square}$ and $\hat{\psi}_\square$ is commutative.

Proposition 2.8. *Suppose that $\boldsymbol{L},\boldsymbol{L}'$ are bounded distributive lattices. Let $\psi:\boldsymbol{L}\to\boldsymbol{L}'$ be a homomorphism and $a,b\in\boldsymbol{L}$ with $b\le a$. Then the following diagram is commutative where $\hat{\psi}_\square$ and $\varphi_{\square,\square}$ are defined in theorems 2.5 and 2.6, respectively.*

$$\begin{array}{ccc} [0,a] & \xrightarrow{\hat{\psi}_a} & [0,\psi(a)] \\ \downarrow{\scriptstyle \varphi_{a,b}} & & \downarrow{\scriptstyle \varphi_{\psi(a),\psi(b)}} \\ [0,b] & \xrightarrow{\hat{\psi}_b} & [0,\psi(b)] \end{array}$$

Proof. For every $x\in[0,a]$,

$$\psi(x)\wedge\psi(b)=\psi(x\wedge b)$$

Therefore, the statement is true. □

2.5. **Generated by Lattices.** By propositions 2.2 and 2.3 and corollary 2.3.1, we have $x\wedge y\in[0,a\wedge b]$ for all $x\in[0,a]$, $y\in[0,b]$. If $a\wedge b=0$, then $[0,a]\cap[0,b]=\{0\}$. Hence $x\wedge y=0$ for all $x\in[0,a]$, $y\in[0,b]$. If $a\wedge b\neq 0$, then there exists $x\in[0,a]$, $y\in[0,b]$ such that $x\wedge y\neq 0$, since $x\wedge a\wedge b$ and $y\wedge a\wedge b$ need not be 0. And we have

$$\begin{aligned}([0,a]\cap[0,b]=[0,a\wedge b])&\subseteq[0,a],[0,b]\\&\subseteq(G([0,a]\cup[0,b])=[0,a\vee b])\end{aligned}$$

Hence we have

$$\begin{aligned} x\wedge y&=x\wedge y\wedge(a\wedge b)\\ &=(x\wedge a\wedge b)\wedge(y\wedge a\wedge b)\\ &=\varphi_{a,a\wedge b}(x)\wedge\varphi_{b,a\wedge b}(y)\end{aligned}\tag{2.3}$$

where $\varphi_{\square,\square}$ is defined in theorem 2.6.

Since an interval is a bounded distributive lattice(cf. corollary 2.1.1), and a bounded distributive lattice is regarded as the 'join' of the intervals, hence we may obtain a bounded distributive lattice by other bounded distributive lattices.

Suppose that $\boldsymbol{L},\boldsymbol{L}'$ are bounded distributive lattices. Let $a\in\boldsymbol{L}$, $a'\in\boldsymbol{L}'$ with $[0,a]\cong[0,a']$. Then there exists an isomorphism $\eta:[0,a]\to[0,a']$ of bounded distributive lattices. We may define an equivalence relation[10] by η and η^{-1}.

Definition 2.6 (cf. [2,3,8,10])**.** Let '$\sim$' be an equivalence relation in $\boldsymbol{L}\cup\boldsymbol{L}'$ provided that

$$x\sim y \quad \text{if} \begin{cases} x=y & \text{for } x,y\in\boldsymbol{L}\cup\boldsymbol{L}' \\ \eta(x)=y & \text{for } x\in[0,a] \\ x=\eta^{-1}(y) & \text{for } y\in[0,a'] \end{cases}$$

A quotient(cf. [2, 3, 8]) of the set is an equivalence classes determined by an equivalence relation. Then we have a quotient $(\boldsymbol{L} \cup \boldsymbol{L}') / \sim$.

Definition 2.7. Suppose that $\boldsymbol{L}, \boldsymbol{L}'$ are bounded distributive lattices. Let $a \in \boldsymbol{L}$, $a' \in \boldsymbol{L}'$ with $[0, a] \cong [0, a']$, $\eta \colon [0, a] \to [0, a']$ an isomorphism of bounded distributive lattices. Then let $G(\boldsymbol{L} \cup \boldsymbol{L}')$ be the bounded distributive lattice generated by $(\boldsymbol{L} \cup \boldsymbol{L}') / \sim$ where $(\boldsymbol{L} \cup \boldsymbol{L}') / \sim$ is a quotient determined by an equivalence relation '$\sim$' defined by η (see definition 2.6). And let

$$1_{\boldsymbol{L}} \wedge 1_{\boldsymbol{L}'} := a (\text{or } a') \tag{2.4}$$

Then we say that $G(\boldsymbol{L} \cup \boldsymbol{L}')$ is **generated by $\boldsymbol{L} \cup \boldsymbol{L}'$ via** the isomorphism η.

Then similar to proposition 2.4, we have

Proposition 2.9. *The lattices $\boldsymbol{L}, \boldsymbol{L}'$ are the intervals of $G(\boldsymbol{L} \cup \boldsymbol{L}')$. Hence the lattice $G(\boldsymbol{L} \cup \boldsymbol{L}')$ is the set $\{x \vee x' = x' \vee x \mid x \in \boldsymbol{L}, x' \in \boldsymbol{L}'\}$*

Proof. We have $\boldsymbol{L} \cong [0, 1_{\boldsymbol{L}}]$ by theorem 2.3. For $x \in \boldsymbol{L}$, $x' \in \boldsymbol{L}'$, we have $x \wedge x' \in [0, a(a')]$ by proposition 2.3, corollary 2.3.1, and equations (2.3) and (2.4). Then similar to the proof of proposition 2.4. □

Corollary 2.9.1.

$$G(\boldsymbol{L} \cup \boldsymbol{L}') = \langle (\boldsymbol{L} \cup \boldsymbol{L}') / \sim, \vee, \wedge, 0, 1_{\boldsymbol{L}} \vee 1_{\boldsymbol{L}'} \rangle$$

Proof. It is obvious. □

2.6. **Lattice forms Category.** We seen that a partial order forms a category, see [6]. Hence a lattice forms a category, by theorem 2.1.

Definition 2.8 (cf. [6])**.** Suppose that $\boldsymbol{L}$ is a bounded distributive lattice. Let $\mathcal{L}$ be a category provided that

Objects: The members of the lattice $\boldsymbol{L}$.
Morphisms: There is at most one morphism $a \to b$ for $a, b \in \boldsymbol{L}$ with $a \leq b$.

It is obvious that $\mathcal{L}$ satisfies the definition of category.

Proposition 2.10. *A sublattice of $\boldsymbol{L}$ forms the subcategory[6] of $\mathcal{L}$.*

Proof. It is obvious. □

Proposition 2.11. *Suppose that $\boldsymbol{L}, \boldsymbol{L}'$ are bounded distributive lattices. Let $\psi \colon \boldsymbol{L} \to \boldsymbol{L}'$ be a homomorphism. If $\mathcal{L}$ and $\mathcal{L}'$ are categories define in definition 2.8, then ψ forms a functor from $\mathcal{L}$ to $\mathcal{L}'$.*

Proof. A homomorphism is compatible with the operations and the compositions of the operations. Hence the statement is true. □

2.7. **Morphism of Functors.** Recall some facts in [6]. Suppose that $\mathcal{C}, \mathcal{C}'$ are categories. Let $F, H \colon \mathcal{C} \to \mathcal{C}'$ be functors. Then a morphism from F to H is a natural transformation[6] $\tau \colon F \xrightarrow{\bullet} H$, and for every $(f \colon C \to C') \in \mathcal{C}$, the following diagram is commutative(cf. [6]).

$$\begin{array}{ccc} F(C) & \xrightarrow{\tau_C} & H(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle H(f)} \\ F(C') & \xrightarrow[\tau_{C'}]{} & H(C') \end{array} \tag{2.5}$$

Then the diagram(2.5) is regarded as the diagram(2.6) where I is an indentity functor.

(2.6)
$$\begin{array}{ccc} F(C) & \xrightarrow{\tau_C} & H(I(C)) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle H(f)} \\ F(C') & \xrightarrow[\tau_{C'}]{} & H(I(C')) \end{array}$$

Hence we may replace I by other functor.

Suppose that $\mathcal{D}$ is a category. Let $T\colon \mathcal{C} \to \mathcal{D}$, $S\colon \mathcal{D} \to \mathcal{C}'$ be functors. If η is a morphism from F to $S \circ T$, then for every $(f\colon C \to C') \in \mathcal{C}$, the morphism η makes the diagram(2.7) commutate.

(2.7)
$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & S \circ T(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle S\circ T(f)} \\ F(C') & \xrightarrow[\eta_{C'}]{} & S \circ T(C') \end{array}$$

2.8. **Chain.**

Definition 2.9 (cf. [5])**.** A **chain** $\{a_n\}$ in a bounded distributive lattice $\boldsymbol{L}$ is a nonempty subset which has the infimum, and if $a, b \in \{a_n\}$, then either $a < b$ or $b < a$.

Proposition 2.12. *A chain $\{a_n\}$ of a bounded distributive lattice $\langle \boldsymbol{L}, \vee, \wedge, 0, 1\rangle$ is a lattice $\langle \{a_n\}, \vee, \wedge \rangle$.*

To prove proposition 2.12, we need the following lemma:

Lemma 2.3 (cf. [8])**.** *Suppose that $\boldsymbol{L}$ is a lattice. Let $a, b \in \boldsymbol{L}$ with $a \leq b$. Then $a \vee b = b$.*

Proof. $a \vee b = (a \wedge b) \vee b = b$ □

Proof of proposition 2.12. Let $a, b \in \{a_n\}$. Then $a \leq b$ or $b \leq a$. By lemma 2.3, we have either $a \vee b = b$, $a \wedge b = a$ or $a \vee b = a$, $a \wedge b = b$. Therefore, the chain $\{a_n\}$ is a lattice. □

A sublattice forms a subcategory, hence we have that

Corollary 2.12.1. *A chain $\{a_n\}$ in $\boldsymbol{L}$ forms a category.*

Proof. Immediate from propositions 2.10 and 2.12. □

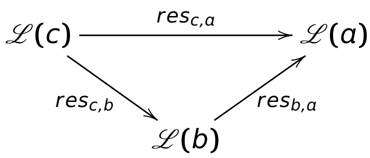
3. Sheaf

A sheaf is a contravariant functor[6] from a category $\mathfrak{Top}(X)$[1] to a category $\mathcal{C}$ where $\mathfrak{Top}(x)$ is the category of open sets in a topological space X, see [1, 9]. The open subsets in a topological space X form a distributive bounded lattice(cf. [4, 8], definition 2.1). A lattice is a poset(cf. subsection 2.2) and a poset forms a category(cf. [6]), hence a lattice forms a category(cf. definition 2.8). So we may construct a sheaf of lattices on a bounded distributive lattice.

3.1. **A Sheaf of Lattices on a Distributive Bounded Lattice.** In theorem 2.2, we have known that if $\boldsymbol{L}$ is a bounded distributive lattice and $a \in \boldsymbol{L}$, then $[0, a]$ is a sublattice of $\langle \boldsymbol{L}, \vee, \wedge \rangle$. Let $\mathcal{LAT}$ be the category of lattices, and the morphisms in $\mathcal{LAT}$ is the homomorphisms of lattices. Then we have the following theorem.

Theorem 3.1 (cf. [1, 7, 9])**.** *Suppose that $\langle \boldsymbol{L}, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice. Then let $\mathscr{L}$ be a contravariant functor from the category $\mathcal{L}$(definition 2.8) to the category $\mathcal{LAT}$ together with*

- *For every $a \in \boldsymbol{L}$, $\mathscr{L}(a) = [0, a]$(see theorem 2.2);*
- *For every $a, b \in \boldsymbol{L}$ with $a \le b$, the restriction map[9] $res_{b,a} : [0, b] \to [0, a]$ is the homomorphism $\varphi_{b,a}$(see theorem 2.6);*
- *If $a = b$, then the restriction map is the identity isomorphism(see theorem 2.6);*
- *If $a \le b \le c$, then the following diagram is commutative(see proposition 2.7);*



Then the functor $\mathscr{L}$ is a ***presheaf***[1, 7, 9] *of lattices on a bounded distributive lattice $\boldsymbol{L}$.*

Proof. Immediate from definition 2.8, theorems 2.2 and 2.6, and proposition 2.7. □

Theorem 3.2. *The presheaf $\mathscr{L}$ is a sheaf.*

Proof. The presheaf $\mathscr{L}$ satisfies identity axiom[9, subsection 2.2.6] by theorem 2.7 and corollarys 2.7.1 and 2.7.2. And the presheaf satisfies gluability axiom[9, subsection 2.2.6] by theorem 2.8 and corollarys 2.8.1 and 2.8.2. Therefore, it is a sheaf. □

We have seen that the intervals of a bounded distributive lattice form a lattice in subsection 2.3. And for all $a \in \boldsymbol{L}$, $\mathscr{L}(a)$ is an interval.

Proposition 3.1. *For $a, b \in \boldsymbol{L}$,*

$$\mathscr{L}(a \vee b) = \mathscr{L}(a) \vee \mathscr{L}(b)$$
$$\mathscr{L}(a \wedge b) = \mathscr{L}(a) \wedge \mathscr{L}(b)$$

Proof. We have

$$\mathscr{L}(a \vee b) = [0, a \vee b] = [0, a] \vee [0, b]$$
$$\mathscr{L}(a \wedge b) = [0, a \wedge b] = [0, a] \wedge [0, b]$$

by corollarys 2.1.1 to 2.4.1 and theorem 2.4. □

3.2. **Stalk of $\mathscr{L}$.** Suppose that $\mathscr{F}$ is a sheaf of sets on a topological space X. Let $p \in X$. Then the stalk at p is a colimit[6] of $\mathscr{F}(U)$ over all open sets U containing p: $\mathscr{F}_p = \varinjlim \mathscr{F}(U)$, see [1, 9].

But we may define a stalk of $\mathscr{L}$ at a chain[definition 2.9]. Suppose that $\boldsymbol{L}$ is a bounded distributive lattice. Let $\{a_n\}$ be a chain in $\boldsymbol{L}$. It is obvious that the chain $\{a_n\}$ is a subcategory of $\mathcal{L}$(definition 2.8) by propositions 2.10 and 2.12. Let $\mathcal{H}$ be the subcategory. Hence there exists a contravariant functor F from $\mathcal{H}$ to the category $\mathcal{LAT}$ of lattices such that $F = \mathscr{L}\restriction\{a_n\}$. Then we have

Definition 3.1. The **stalk** of $\mathscr{L}$ at the chain[definition 2.9] $\{a_n\}$ is the colimit of F. If $m \in \boldsymbol{L}$ is the infimum of $\{a_n\}$, then $\varinjlim F = [0, m]$. Let $\mathscr{L}_{\{a_n\}}$ denote the stalk.

Remark 3.1. We have $\mathscr{L}_{\{a_n\}} = \mathscr{L}(m)$.

And the sheaf $\mathscr{L}$ may be formed by the stalks. Let $a = \bigvee_{i \in I} a_i$ and a_i the infimum of a chain $\{s_n\}_i$ for every i. By corollary 2.4.1, theorem 2.4, and proposition 3.1, we have

$$\mathscr{L}(a) = \bigvee_{i \in I} \mathscr{L}_{\{s_n\}_i}$$

3.3. **Morphism of the Sheaves.** Suppose that $\mathscr{F}, \mathscr{F}'$ are the sheaves. Then a morphism $\pi : \mathscr{F} \to \mathscr{F}'$ is a natural transformation(cf. [1, 6, 7, 9]).

Now, we construct a morphism of sheaves $\mathscr{L}, \mathscr{L}'$.

Theorem 3.3. *Suppose that $\mathscr{L}, \mathscr{L}'$ are two sheaves defined in theorem 3.1 on bounded distributive lattices $\boldsymbol{L}, \boldsymbol{L}'$, respectively. Let $\psi : \boldsymbol{L} \to \boldsymbol{L}'$ be a homomorphism. Hence ψ forms a functor(cf. proposition 2.11). Then ψ induces a morphism $\hat{\psi} : \mathscr{L} \to \mathscr{L}'$ and the morphism $\hat{\psi}$ is the natural transformation $\mathscr{L} \xrightarrow{\bullet} \mathscr{L}' \circ \psi$. And $\hat{\psi}_a : \mathscr{L}(a) \to \mathscr{L}'(\psi(a))$ is the homomorphism defined in theorem 2.5 for all $a \in \boldsymbol{L}$.*

Proof. Immediate from definition 2.8, theorems 2.5, 3.1 and 3.2, propositions 2.8 and 2.11, and section 2.7. □

Suppose that $\boldsymbol{L}, \boldsymbol{L}'$ are two bounded distributive lattices. Let $\psi : \boldsymbol{L} \to \boldsymbol{L}'$ be a homomorphism. The image of ψ is a sublattice of $\boldsymbol{L}'$. Hence the image is a bounded distributive lattice. And the image forms a subcategory.

Proposition 3.2. *Let $\hat{\psi}$ be the morphism which is defined in theorem 3.3. Then the image of $\hat{\psi}$ is a sheaf on the bounded distributive lattice $\psi(\boldsymbol{L})$.*

Proof. By definition 2.8, proposition 2.10, and theorems 3.1 to 3.3, the image of $\hat{\psi}$ is the functor $\mathscr{L}'$ restricted to the subcategory $\psi(\mathcal{L})$ of $\mathcal{L}'$. Let $\mathscr{L}' \restriction \psi(\mathcal{L})$ denote the restricted functor. It is obvious that $\mathscr{L}' \restriction \psi(\mathcal{L})$ satisfies the definition of a sheaf. □

Definition 3.2 (cf. [1, 3, 4, 6–9])**.** Suppose that $\hat{\psi}$ is the morphism defined in theorem 3.3. Then the morphism $\hat{\psi}$ is a **monomorphism(epimorphism, isomorphism)**, if ψ is a monomorphism(epimorphism, isomorphism).

Theorem 3.4. *Suppose that $\hat{\psi}$ is the morphism defined in theorem 3.3. If $\hat{\psi}$ is a monomorphism(epimorphism, isomorphism), then $\hat{\psi}_a : \mathscr{L}(a) \to \mathscr{L}'(\psi(a))$ is a monomorphism(epimorphism, isomorphism).*

Proof. Immediate from corollary 2.5.1. □

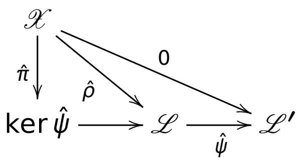
3.4. **Subsheaf.** Suppose that $\mathscr{L}', \mathscr{L}$ are sheaves defined in theorem 3.1 on bounded distributive lattices $\boldsymbol{L}', \boldsymbol{L}$, respectively. Let $\hat{\psi} : \mathscr{L}' \to \mathscr{L}$ be a monomorphism defined in definition 3.2. Then the image of $\hat{\psi}$ is isomorphic to $\mathscr{L}'$ by proposition 3.2 and definition 3.2.

Definition 3.3 (cf. [1, 9])**.** Suppose that $\mathscr{L}$ is a sheaf defined in theorem 3.1 on a bounded distributive lattice $\boldsymbol{L}$. Then the sheaf $\mathscr{L}'$ is a **subsheaf** of $\mathscr{L}$ if there exists a monomorphism $\hat{\psi} : \mathscr{L}'' \to \mathscr{L}$ such that $\mathscr{L}'$ is the image of $\hat{\psi}$. If the subsheaf $\mathscr{L}'$ is on a bounded distributive lattice $\boldsymbol{L}'$, then let $\mathscr{L} \restriction \mathcal{L}'$ denote the subsheaf where $\mathcal{L}'$ is the category which is formed by $\boldsymbol{L}'$(see definition 2.8).

A morphism $\hat{\psi}\colon \mathscr{L} \to \mathscr{L}'$ is a natural transformation(see theorem 3.3). Hence for every $a \in \mathcal{L}$, $\hat{\psi}_a$ is a homomorphism of the intervals. In definition 2.5, we defined the kernel of $\hat{\psi}_a$. And we have the fact that the subset $\psi^{-1}(0)$ is a bounded distributive lattice(cf. proposition 2.5). Hence $\psi^{-1}(0)$ is a category by definition 2.8. Now, we may define the kernel of $\hat{\psi}$.

Definition 3.4 (cf. [1,9])**.** Suppose that $\hat{\psi}$ is the morphism defined in theorem 3.3. Let a **kernel** of $\hat{\psi}$ be the subsheaf $\mathscr{L}\restriction\psi^{-1}(0)$.

Remark 3.2. The kernel $\ker\hat{\psi}$ is a subsheaf of $\mathscr{L}$ such that $\hat{\psi}_k((\ker\hat{\psi})(k)) = \{0\}$ for all $k \in \psi^{-1}(0)$. In corollary 2.1.1, we have that [0, a] is a bounded distributive lattice for $a \in \boldsymbol{L}$. Specially, let $a = 0$. Then $\{0\}$ is a bounded distributive lattice. Let $\tilde{0}$ denote the sheaf on $\{0\}$. Then the image of the morphism $\hat{\psi}\restriction\ker\hat{\psi}$ is $\tilde{0}$. Suppose that $\mathcal{S}$ is the category of the sheaves on bounded distributive lattices. Then $\tilde{0}$ is a null object[6] of $\mathcal{S}$. And for all morphism $\hat{\rho}\colon \mathscr{X} \to \mathscr{L}$ in $\mathcal{S}$, if $\hat{\psi}\circ\hat{\rho} = 0$, then there exist unique morphism $\hat{\pi}\colon \mathscr{X} \to \ker\hat{\psi}$ such that the following diagram is commutative(cf. [6]).



3.5. **Gluing Sheaves.** Suppose that $\mathscr{L}, \mathscr{L}'$ are the sheaves which are defined in theorem 3.1. In proposition 3.1, for all $a, b \in \boldsymbol{L}$, we have seen

$$\mathscr{L}(a \vee b) = \mathscr{L}(a) \vee \mathscr{L}(b) = [0, a] \vee [0, b] = [0, a \vee b]$$

And an interval $[0, a]$ is a bounded distributive lattice by corollary 2.1.1.

Definition 3.5 (cf. [1,9])**.** Suppose that $\mathscr{L}, \mathscr{L}'$ are defined on bounded distributive lattices $\boldsymbol{L}, \boldsymbol{L}'$, respectively. Let $a \in \boldsymbol{L}$, $a' \in \boldsymbol{L}'$ with $[0, a] \cong [0, a']$, $\eta\colon [0, a] \to [0, a']$ an isomorphism of the bounded distributive lattices. By definition 3.2, that η induces an isomorphism of sheaves:

$$\hat{\eta}\colon \mathscr{L}\restriction[0, a] \to \mathscr{L}'\restriction[0, a']$$

where $\mathscr{L}\restriction[0, a]$ and $\mathscr{L}'\restriction[0, a']$ are subsheaves defined in definition 3.3. Then let $\mathscr{M}$ be a sheaf on the bounded distributive lattice $G(\boldsymbol{L} \cup \boldsymbol{L}')$ which is defined in definition 2.7. We say that the sheaf $\mathscr{M}$ is obtained by **gluing** $\mathscr{L}$ and $\mathscr{L}'$ **via** an isomorphism $\hat{\eta}$. Then the sheaves $\mathscr{L}, \mathscr{L}'$ may be regarded as the subsheaves of $\mathscr{M}$.

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