# The expression of binomial formula $(a+b)^n$ when n is a prime

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**Abstract**

*We give the expression of binomial formula $(a+b)^n$ when $n$ is a prime number.*

The binomial formula :
$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + ... + b^n$

If n is a prime number,$n = p > 3$, we obtain:

$$(a+b)^p = a^p + b^p + pab(a+b)(a^2+ab+b^2)M_{a,b}$$

$M_{a,b}$ : Polynomial(a,b) of degree $p-5$

For p = 5:

$(a+b)^5 = a^5 + b^5 + 5ab(a+b)(a^2+ab+b^2)$

For p = 7:

$(a+b)^7 = a^7 + b^7 + 7ab(a+b)(a^2+ab+b^2)(a^2+ab+b^2) = a^7 + b^7 + 5ab(a+b)(a^2+ab+b^2)^2$

For p = 11:

$(a+b)^{11} = a^{11} + b^{11} + 11ab(a+b)(a^2+ab+b^2)(a^6+3a^5b+7a^4b^2+9a^3b^3+7a^2b^4+3ab^5+b^6)$

$(a+b)^{11} = a^{11} + b^{11} + 11ab(a+b)(a^2+ab+b^2)((a^2+ab+b^2)^3 + a^2b^2(a+b)^2)$
For p = 13:

$(a+b)^{13} = a^{13} + b^{13} + 11ab(a+b)(a^2+ab+b^2)(a^8+4a^7b+2a^6b^2+42a^5b^3+7a^4b^4+42a^3b^5+2a^2b^6+4ab^7+b^8)$
Proof:

We have:
$(a+b)^5 = a^5 + b^5 + 5ab(a+b)(a^2+ab+b^2)$
In other words $a^2+ab+b^2$ is a factor of $(a+b)^5 - (a^5+b^5)$

1. $p = 3m+1$

$p = 3m+1 \Leftrightarrow p = 6k+7$
$(a+b)^p = (a+b)^{2(3k+1)}(a+b)^5 = (a^2+2ab+b^2)^{3k+1}(a+b)^5$

$= (a^2+ab+b^2+ab)^{3k+1}(a+b)^5 = [(a^2+ab+b^2)N + a^{3k+1}b^{3k+1}](a+b)^5 = (a^2+ab+b^2)N(a+b)^5 + a^{3k+1}b^{3k+1}(a+b)^5$
and
$a^p + b^p = a^{6k+7} + b^{6k+7} = (a^{3k+6)} - b^{3k+6)})(a^{3k+1} - b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)$

$a^p + b^p = (a^{3(k+2)} - b^{3(k+3)})(a^{3k+1} - b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)$
$a^{3(k+2)} - b^{3(k+3)} = (a^3-b^3)L$
$\Rightarrow a^p + b^p = (a^3-b^3)L(a^{3k+1} - b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)$
$\Rightarrow (a+b)^p - (a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 + a^{3k+1}b^{2k+1}(a+b)^5 - [(a^3-b^3)L(a^{3k+1} - b^{3k+1}) + a^{3k+1}b^{3k+1}(a^5+b^5)]$
$\Rightarrow (a+b)^p - (a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{3k+1} - b^{3k+1}) + a^{3k+1}b^{3k+1}(a+b)^5 - a^{3k+1}b^{3k+1}(a^5+b^5)$
$\Rightarrow (a+b)^p - (a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{3k+1} - b^{3k+1}) + a^{3k+1}b^{3k+1}[(a+b)^5 - (a^5+b^5)]$

$(a^2+ab+b^2)N(a+b)^5, (a^3-b^3)L(a^{3k+1} - b^{3k+1}), a^{3k+1}b^{3k+1}[(a+b)^5 - (a^5+b^5)]$ have a common factor $a^2+ab+b^2$, hence $(a+b)^p - (a^p+b^p)$ has the factor $a^2+ab+b^2$

2. $p = 3n+2$

$p = 3n+2 \Rightarrow p = 6k+5$
$(a+b)^p = (a+b)^{6k}(a+b)^5 = (a^2+2ab+b^2)^{3k}(a+b)^5$
$= (a^2+ab+b^2+ab)^{3k}(a+b)^5 = [(a^2+ab+b^2)N + a^{3k}b^{3k}](a+b)^5 = (a^2+ab+b^2)N(a+b)^5 + a^{3k}b^{3k}(a+b)^5$
and
$a^p + b^p = a^{5+6k} + b^{5+6k} = (a^{5+3k} - b^{5+3k})(a^{3k} - b^{3k}) + a^{3k}b^{3k}(a^5+b^5)$

$a^p + b^p = (a^{5+3k)} - b^{5+3k)})(a^{3k} - b^{3k}) + a^{3k}b^{3k}(a^5+b^5)$
$a^{3k} - b^{3k} = (a^3-b^3)L$
$\Rightarrow a^p + b^p = (a^3-b^3)L(a^{5+3k} - b^{5+3k}) + a^{3k}b^{3k}(a^5+b^5)$
$\Rightarrow (a+b)^p - (a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 + a^{3k}b^{3k}(a+b)^5 - [(a^3-b^3)L(a^{5+3k} - b^{5+3k}) + a^{3k}b^{3k}(a^5+b^5]$

$\Rightarrow (a+b)^p - (a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{5+3k} - b^{5+3k}) + a^{3k}b^{3k}(a+b)^5 - a^{3k}b^{3k}(a^5+b^5)]$

$\Rightarrow (a+b)^p - (a^p+b^p) = (a^2+ab+b^2)N(a+b)^5 - (a^3-b^3)L(a^{5+3k} - b^{5+3k}) + a^{3k}b^{3k}[(a+b)^5 - (a^5+b^5)]$

$(a^2+ab+b^2)N(a+b)^5, (a^3-b^3)L(a^{5+3k} - b^{5+3k}), a^{3k}b^{3k}[(a+b)^5 - (a^5+b^5)]$ have a common factor $a^2+ab+b^2$, hence $(a+b)^p - (a^p+b^p)$ has the factor $a^2+ab+b^2$

It is easy to show that p, a, b and (a + b) are the factors of $(a+b)^p - (a^p+b^p)$, finally we obtain:

$$(a+b)^p = a^p + b^p + pab(a+b)(a^2+ab+b^2)M_{a,b}$$

Note that if $a, b = 1$ then $a, b, (a+b), (a^2+ab+b^2) = 1$

We also get:

$a^p + b^p = (a+b)^p - pab(a+b)(a^2+ab+b^2)M_{a,b}$

$$a^p + b^p = (a+b)[(a+b)^{p-1} - pab(a^2+ab+b^2)M_{a,b}]$$

$$a^{p-1} - a^{p-2}b + a^{p-3}b^2 - ... + b^{p-1} = (a+b)^{p-1} - pab(a^2+ab+b^2)M_{a,b}$$

# References


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