A solution of a quartic equation

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This solution is equal to L. Ferrari's if we simply change the inner square root $\sqrt{w}$ to $\sqrt{\alpha+2y}$. This article shows the shortest way to have a resolvent cubic for a quartic equation as well as the solution of a quartic equation.

A. Derivation of a solution of a quartic equation

The solution of a quartic polynomial was discovered by Lodovico de Ferrari in 1540. Ferrari's solution is good for solving a quartic equation.This article shows a simpler way to solve the quartic equation than Ferrari.
A monic form of a quartic polynomial is written as

$$x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0 = 0. \tag{1}$$

To solve a quartic, we need to get a resolvent [1] [1] cubic. A resolvent cubic can be obtained from the above quartic equation by using the following biquadratic equation.

$$(x^2 + a_1 x + a_0)^2 - w(x + b_0)^2 = 0, \tag{2}$$

where a_1, a_0 and b_0 are arbitrary coefficients, and w is a coupling constant. Unfolding the brackets of the equation (2) and comparing to those coefficients of the equation (1), we can find

$$\begin{aligned} a_1 &= \frac{1}{2}c_3, \\ a_0 &= \frac{1}{2}c_2 + \frac{1}{2}w - \frac{1}{8}{c_3}^2, \\ b_0 &= (-c_1 + \frac{1}{2}c_2 c_3 + \frac{1}{2}c_3 w - \frac{1}{8}{c_3}^3)/(2w), \end{aligned} \tag{3}$$

and we get the remaining resolvent equation

$${a_0}^2 - {b_0}^2 w - c_0 = 0. \tag{4}$$

Solving the equation (4) with substitutions from (3), we get the resolvent cubic equation with respect to w,

$$\begin{aligned} &w^3 + (2c_2 - \frac{3}{4}{c_3}^2)w^2 + (-4c_0 + c_1 c_3 - c_2 {c_3}^2 + {c_2}^2 + \frac{3}{16}{c_3}^4)w \\ &+(c_1 c_2 c_3 - \frac{1}{4}c_1 {c_3}^3 + \frac{1}{8}c_2 {c_3}^4 - {c_1}^2 - \frac{1}{64}{c_3}^6 - \frac{1}{4}{c_2}^2 {c_3}^2) = 0. \end{aligned} \tag{5}$$

As this resolvent cubic equation is somewhat lengthy and complicated, a reduced form is applicable as follows,

$$y^3 + p_1 y + p_0 = 0, \tag{6}$$

where y represents

$$y = w + \frac{2}{3}c_2 - \frac{1}{4}{c_3}^2, \tag{7}$$

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[1] Degree $(n-1)$ form of polynomial whose roots are related to the roots of the original equation.

with p_1 and p_0 respectively

$$\begin{aligned} p_1 &= -4c_0 + c_1c_3 - \frac{1}{3}{c_2}^2, \\ p_0 &= \frac{8}{3}c_0c_2 - c_0{c_3}^2 + \frac{1}{3}c_1c_2c_3 - {c_1}^2 - \frac{2}{27}{c_2}^3. \end{aligned} \tag{8}$$

A radical solution of the cubic (6) provides

$$y = \sqrt[3]{-\frac{p_0}{2} - \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3}} + \sqrt[3]{-\frac{p_0}{2} + \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3}}. \tag{9}$$

Or we can get the solution in the form of w from the equation (5) by using the equation (7)

$$w = -\frac{2}{3}c_2 + \frac{1}{4}{c_3}^2 + \sqrt[3]{-\frac{p_0}{2} - \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3}} + \sqrt[3]{-\frac{p_0}{2} + \sqrt{\left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3}}. \tag{10}$$

with p_1 and p_0 of (8).

It is to be noted that $D_4 = (\frac{p_0}{2})^2 + (\frac{p_1}{3})^3$ represents the discriminant of the above quartic equation (1) and it can be expanded as below,

$$\begin{aligned} D_4 &= \left(\frac{p_0}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3 \\ &= -\frac{1}{108}(18c_3c_2{c_1}^3 - 80c_3{c_2}^2c_1c_0 + 144{c_3}^2c_2{c_0}^2 - 6{c_3}^2{c_1}^2c_0 - 4{c_2}^3{c_1}^2 + 16{c_2}^4c_0 \\ &\quad -192c_3c_1{c_0}^2 + 144c_2{c_1}^2c_0 - 128{c_2}^2{c_0}^2 - 27{c_1}^4 + 256{c_0}^3 \\ &\quad +{c_3}^2{c_2}^2{c_1}^2 - 4{c_3}^2{c_2}^3c_0 + 18{c_3}^3c_2c_1c_0 - 4{c_3}^3{c_1}^3 - 27{c_3}^4{c_0}^2). \end{aligned} \tag{11}$$

Now, we can derive out a radical solution of a quartic equation. It is convenient to deal with the equation (2) directly otherwise it is so complicated. It provides two quadratic solutions. One of them is

$$x^2 + (a_1 - \sqrt{w})x + a_0 - b_0\sqrt{w} = 0. \tag{12}$$

This gives two roots of the quadratic

$$x_{1,2} = -\frac{a_1}{2} + \frac{\sqrt{w}}{2} \pm \frac{1}{2}\sqrt{(a_1 - \sqrt{w})^2 - 4(a_0 - b_0\sqrt{w})}. \tag{13}$$

Substituting with the equations (3), we get

$$x_{1,2} = -\frac{c_3}{4} + \frac{\sqrt{w}}{2} \pm \frac{1}{4}\sqrt{3{c_3}^2 - 8c_2 - 4w - \frac{8c_1 - 4c_3c_2 + {c_3}^3}{\sqrt{w}}}. \tag{14}$$

with w from (10).
These are two roots of a quartic equation (1) [2]

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[2] Ferrari's solution is given from a reduced form of a quartic

$$y^4 + \alpha y^2 + \beta y + \gamma = 0,$$

which provides two kinds of solutions according to the condition of $\beta = 0$ or not.
In case $\beta \neq 0$,

$$x = -\frac{b}{4a} \pm_s \frac{\sqrt{\alpha + 2y}}{2} \pm_t \frac{1}{2}\sqrt{-(3\alpha + 2y) \pm_s \frac{2\beta}{\sqrt{\alpha + 2y}}}.$$

If $(\alpha + 2y)$ is changed to w, the above is equal to that of this article.
In case $\beta = 0$,

$$x = -\frac{b}{4a} \pm \sqrt{-\frac{\alpha}{2} \pm \sqrt{\alpha^2 - 4\gamma}}.$$

This condition is no more required in this article, but $w = 0$.

B. A full solution of a quartic equation

A general quartic equation is written as

$$c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0. \tag{15}$$

Dividing by c_4, we get a monic quartic equation

$$x^4 + \frac{c_3}{c_4}x^3 + \frac{c_2}{c_4}x^2 + \frac{c_1}{c_4}x + \frac{c_0}{c_4} = 0. \tag{16}$$

An intermediary biquadratic equation for a solution of a general monic quartic equation (16) is given as

$$(x^2 + lx + m)^2 = w(x + n)^2. \tag{17}$$

For a reduced quartic, one may use the following form

$$(x^2 + m)^2 = w(x + n)^2,$$

this is simply equal to the above (17) in case $l = 0$.
Unfolding the brackets and comparing to those coefficients of the equation (17), the coefficients are given as

$$\begin{aligned} l &= \frac{c_3}{2c_4}, \\ m &= \frac{w}{2} - \frac{{c_3}^2}{8{c_4}^2} + \frac{c_2}{2c_4}, \\ n &= -\frac{{c_3}^3}{16{c_4}^3w} + \frac{c_3c_2}{4{c_4}^2w} + \frac{c_3}{4c_4} - \frac{c_1}{2c_4w}, \end{aligned} \tag{18}$$

Substituting these coefficients to the equation (16), we get the resolvent cubic equation,

$$x^4 + \frac{c_3}{c_4}x^3 + \frac{c_2}{c_4}x^2 + \frac{c_1}{c_4}x + R(w) = 0, \tag{19}$$

where $R(w)$ provides the resolvent cubic with respect to w,

$$4wR(w) = w^3 + sw^2 + tw + u = 0, \tag{20}$$

where

$$\begin{aligned} s &= -\frac{3{c_3}^2}{4{c_4}^2} + \frac{2c_2}{c_4}, \\ t &= \frac{3{c_3}^4}{16{c_4}^4} - \frac{{c_3}^2c_2}{{c_4}^3} + \frac{c_3c_1}{{c_4}^2} + \frac{{c_2}^2}{{c_4}^2} - \frac{4c_0}{c_4}, \\ u &= -\frac{{c_3}^6}{64{c_4}^6} + \frac{{c_3}^4c_2}{8{c_4}^5} - \frac{{c_3}^2{c_2}^2}{4{c_4}^4} - \frac{{c_3}^3c_1}{4{c_4}^4} + \frac{c_3c_2c_1}{{c_4}^3} - \frac{{c_1}^2}{{c_4}^2}. \end{aligned} \tag{21}$$

In case $w = 0$, the biquadratic (17) simply becomes a perfect square of a quadratic equation $(x^2 + lx + m)^2 = 0$, which includes the case $l = 0$ when it becomes $(x^2 + m)^2 = 0$.
Therefore the equation (17) is applicable for all quartic polynomials except when $(x^2+lx+m)^2 = 0$ and $(x^2+m)^2 = 0$, which are simply solvable by factoring. To solve the resolvent cubic equation (20), we get a reduced form by substituting with $w = y - \frac{s}{3}$,

$$y^3 + py + q = 0, \tag{22}$$

where

$$p = \frac{c_3c_1}{{c_4}^2} - \frac{{c_2}^2}{3{c_4}^2} - \frac{4c_0}{c_4}, \tag{23}$$

$$q = \frac{c_3c_2c_1}{3{c_4}^3} - \frac{{c_3}^2c_0}{{c_4}^3} - \frac{2{c_2}^3}{27{c_4}^3} + \frac{8c_2c_0}{3{c_4}^2} - \frac{{c_1}^2}{{c_4}^2}. \tag{24}$$

We get a solution from (22)

$$y = \sqrt[3]{-\frac{q}{2} - \sqrt{D4}} + \sqrt[3]{-\frac{q}{2} + \sqrt{D4}}, \tag{25}$$

where D_4 is the discriminant of the quartic (15), which is given as follows,

$$\begin{aligned} D_4 &= \frac{1}{4}q^2 + \frac{1}{27}p^3 \\ &= -\frac{1}{108{c_4}^6}(18c_4c_3c_2{c_1}^3 - 80c_4c_3{c_2}^2c_1c_0 + 144c_4{c_3}^2c_2{c_0}^2 - 6c_4{c_3}^2{c_1}^2c_0 - 4c_4{c_2}^3{c_1}^2 + 16c_4{c_2}^4c_0 \\ &\quad -192{c_4}^2c_3c_1{c_0}^2 + 144{c_4}^2c_2{c_1}^2c_0 - 128{c_4}^2{c_2}^2{c_0}^2 - 27{c_4}^2{c_1}^4 + 256{c_4}^3{c_0}^3 + {c_3}^2{c_2}^2{c_1}^2 \\ &\quad -4{c_3}^2{c_2}^3c_0 + 18{c_3}^3c_2c_1c_0 - 4{c_3}^3{c_1}^3 - 27{c_3}^4{c_0}^2). \end{aligned} \tag{26}$$

With these results, we have two quadratic equations that are two factors of the quartic equation (16)

$$x^2 + (\frac{c_3}{2c_4} - \sqrt{w})x + \frac{1}{2}w + \frac{{c_3}^3}{16{c_4}^3\sqrt{w}} - \frac{c_3c_2}{4{c_4}^2\sqrt{w}} - \frac{{c_3}^2}{8{c_4}^2} - \frac{c_3\sqrt{w}}{4c_4} + \frac{c_2}{2c_4} + \frac{c_1}{2c_4\sqrt{w}}, \tag{27}$$

$$x^2 + (\frac{c_3}{2c_4} + \sqrt{w})x + \frac{1}{2}w - \frac{{c_3}^3}{16{c_4}^3\sqrt{w}} + \frac{c_3c_2}{4{c_4}^2\sqrt{w}} - \frac{{c_3}^2}{8{c_4}^2} + \frac{c_3\sqrt{w}}{4c_4} + \frac{c_2}{2c_4} - \frac{c_1}{2c_4\sqrt{w}}. \tag{28}$$

The four roots of a quartic equation are given from the above

$$x_{1,2} = -\frac{c_3}{4c_4} + \frac{\sqrt{w}}{2} \pm \frac{1}{4c_4}\sqrt{3{c_3}^2 - 8c_4c_2 - 4{c_4}^2w - \frac{{c_3}^3 - 4c_4c_3c_2 + 8{c_4}^2c_1}{c_4\sqrt{w}}}, \tag{29}$$

$$x_{3,4} = -\frac{c_3}{4c_4} - \frac{\sqrt{w}}{2} \pm \frac{1}{4c_4}\sqrt{3{c_3}^2 - 8c_4c_2 - 4{c_4}^2w + \frac{{c_3}^3 - 4c_4c_3c_2 + 8{c_4}^2c_1}{c_4\sqrt{w}}}, \tag{30}$$

and the resolvent cubic equation of w

$$\begin{aligned} w &= -\frac{2c_2}{3c_4} + \frac{{c_3}^2}{4{c_4}^2} + \frac{1}{3c_4}\sqrt[3]{{c_2}^3 - 36c_4c_2c_0 + \frac{27c_4{c_1}^2}{2} - \frac{9c_3c_2c_1}{2} + \frac{27{c_3}^2c_0}{2} + \frac{3\sqrt{3}}{2}\sqrt{-D_4}} \\ &\quad + \frac{1}{3c_4}\sqrt[3]{{c_2}^3 - 36c_4c_2c_0 + \frac{27c_4{c_1}^2}{2} - \frac{9c_3c_2c_1}{2} + \frac{27{c_3}^2c_0}{2} - \frac{3\sqrt{3}}{2}\sqrt{-D_4}}, \end{aligned} \tag{31}$$

with D_4 of (26).
A full solution of a quartic equation is consisted of three parts of the equations (29), (31) and (26).

[1] http://adsabs.harvard.edu/abs/2008arXiv0806.1927E
[2] http://en.wikipedia.org/wiki/Quartic_equation
[3] http://en.wikipedia.org/wiki/De_Moivre's_formula
[4] http://en.wikipedia.org/wiki/Discriminant
[5] http://en.wikipedia.org/wiki/Trigonometric_functions
[6] http://en.wikipedia.org/wiki/Hyperbolic_function
[7] http://www.nickalls.org/dick/papers/maths/quartic2009.pdf