Discovering and Programming the Cubic Formula

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November 28, 2022

Abstract

Solving a cubic polynomial using a formula is possible; a formula exists. In this article we connect various dots from a pre-calculus course and attempt to show how the formula could be discovered. Along the way we make a TI-84 CE menu driven program that allows for experiments, confirmations of speculations, and eventually a working program that solves all cubic polynomials.

Introduction

Blitzer's Algebra and Trigonometry [1] gives the formula for one root of $x^3 + mx = n$ at the beginning of Section 3.4: Zeros of Polynomial Functions. We'll confirm that the formula works for a few cases using a TI-84 CE (TI-83 family) program, see Figure 1.

Where did the formula come from? Can we derive it using no more than the contents of this algebra book? Using the chapter on polynomials, we can infer that one root of all cubics will be real; that's the intermediate value theorem (3.2). It is easily understood as all cubics will have negative values that map to negative values and positive values that map to positive values. As these functions are continuous these properties imply that the function's graph crosses the x-axis and hence that it has at least one real root. Finding that one real root should be enough: using synthetic division of polynomials (3.3), we can divide the original cubic by $(x - r)$, r the one found real (or actually any) root and obtain as the quotient a quadratic. The quadratic formula (1.5) can solve this quadratic and give us the remaining two roots.

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-1/2+i√(3)/2→O
-1/2-i√(3)/2→S
Menu("CUBICS","ENTER NEW COEFTS",A,"EVALUATE",B,"CONFIRM BLITZER",C,"CARDAN 3 ROOTS",D,"QUIT",Q)
Lbl A
Prompt A,B,C,D
prgmCUBICS
Lbl B
Prompt X
AX³+BX²+CX+D→Y
Disp Y
Pause
prgmCUBICS
Lbl C
Prompt M,N
³√(√((N/2)²+(M/3)³)-N/2)→F
-³√(√((N/2)²+(M/3)³)+N/2)→L
Disp -(F+L)
Pause
prgmCUBICS
Lbl D
C→M:-D→N
³√(√((N/2)²+(M/3)³)-N/2)→F
-³√(√((N/2)²+(M/3)³)+N/2)→L
Disp -(F+L)
Disp -(F*O+L*S)
Disp -(F*S+L*O)
Pause
prgmCUBICS
Lbl Q
Stop
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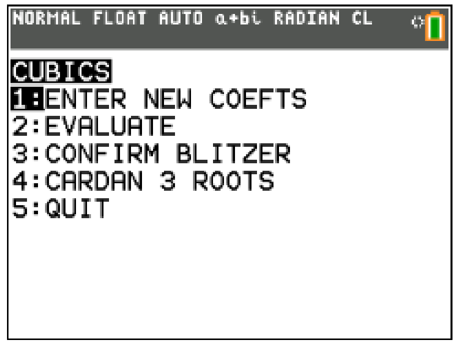

Figure 1: 1) Enter coefficients for general cubic; 2) evaluate at a specified X; 3) confirm Blitzer's formula; 4) show 3 roots per Cardan.

We do need a result not in the book: an identity. But the identity we need is close to one the book does have (P5): $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$. The book's identity can be glossed as a way to factor some cubic polynomials, our theme: for example, $x^3+2^3 = (x+2)(x^2-2x+4)$. We extend this identity to the sum of three cubics and combine it with the cube roots of unity (7.5) to give an identity that factors, hence solves a *depressed* cubic (one lacking a x^2 term). This identity is in Chrystal [2] and other books and it is referenced (obliquely) in Wikipedia's article on cubics. Our derivation might be new; this latter reference puts it in the context of a discrete Fourier transform and Chrystal makes it a problem; see Wikipedia's Lagrange's method in their article on cubics. We just use a few inferences from the factored form of a cubic to its coefficient form, nothing too fancy for a high school student: we think it is about at the level of deriving the quadratic formula, a little hard.

Other references of interest are Tignol's *Galois Theory of Algebraic Equations* and Herstein's *Topics in Algebra* [3, 4]. The first gives an historical evolution of solving the cubic and puts in a broader context of Galois theory.

Herstein presents a highly condensed, fast formulae for all three roots to a general cubic, page 251. He builds into his formulae the transformations that *depress* a general cubic. We'll touch on transformations in our last section.

Formula confirmation

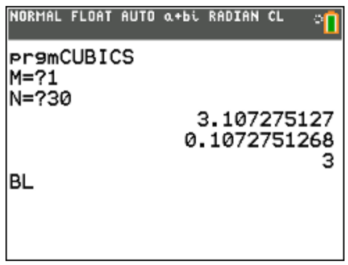

Figure 2: Option 3 of the CUBICS program with M=1 and N=30 gives a root of 3 which is correct.

Before going forward, Blitzer specifies that

$$x = \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} + \frac{n}{2}} - \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} - \frac{n}{2}} \tag{1}$$

is one root of $x^3 + mx = n$. Taking $m = 1$ and $n = 30$, we should find a root of 3.

Using our laboratory program we confirm the formula works for this case. The pertinent lines of code are 013 to 017, Figure 1. Figure 2 gives the confirmation. We've got one of three roots.

An important cubic

The easiest cubic equation that has three calculable roots is $x^3 - 1 = 0$. Its roots are $\omega, \omega^2, \omega^3 = 1$ where $\omega = (\cos 120 + i \sin 120)$. As powers rotate points by multiples of 120, we get $3{*}120 = 360$, $6{*}120 = 2{\cdot}360$ per DeMoivre's theorem (7.5), implying each is a root of unity. We can also translate these complex numbers into radical forms by solving $x^3 + (-1)^3 = (x-1)(x^2+x+1)$

using the quadratic formula and an earlier identity:

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2},\ \text{and } \omega^3 = 1.$$

We gain some experience by confirming these results with a little algebra:

$$\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - 2\left(\frac{1}{2}\right)\left(i\frac{\sqrt{3}}{2}\right) - \frac{3}{4}$$

$$-\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

and

$$\begin{aligned}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 &= -\frac{1}{8} + 3\frac{1}{4}i\frac{\sqrt{3}}{2} - 3\frac{1}{2}\left(\frac{i\sqrt{3}}{2}\right)^2 + \left(\frac{i\sqrt{3}}{2}\right)^3 \\ &= -\frac{1}{8} + \frac{3}{8}i\sqrt{3} + \frac{9}{8} - \frac{3}{8}i\sqrt{3} \\ &= 1.\end{aligned}$$

We note that

$$\omega + \omega^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2} = -1. \tag{2}$$

We'll use this later. The two values ω and ω^2 are stored as O and S; see code lines 001 and 002 of Figure 1.

A real identity

There is a relationship between the root form (left hand side of (3)) and the coefficient form (right hand side of (3)) of a cubic: $p_3(x) =$

$$(x-r_1)(x-r_2)(x-r_3) = x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_1r_3+r_2r_3)x-r_1r_2r_3. \tag{3}$$

As $p_3(r_1) = p_3(r_2) = p_3(r_3) = 0$, we can derive an identity by summing the following three equations

$$\begin{aligned}p_3(r_1) &= r_1^3 - (r_1 + r_2 + r_3)r_1^2 + (r_1r_2 + r_1r_3 + r_2r_3)r_1 - r_1r_2r_3 = 0 \\ p_3(r_2) &= r_2^3 - (r_1 + r_2 + r_3)r_2^2 + (r_1r_2 + r_1r_3 + r_2r_3)r_2 - r_1r_2r_3 = 0 \\ p_3(r_3) &= r_3^3 - (r_1 + r_2 + r_3)r_3^2 + (r_1r_2 + r_1r_3 + r_2r_3)r_3 - r_1r_2r_3 = 0.\end{aligned}$$

That sum implies

$$r_1^3 + r_2^3 + r_3^3 - 3r_1r_2r_3 = (r_1 + r_2 + r_3)(r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_1r_3 - r_2r_3). \tag{4}$$

A complex identity

We can use (4) with (2) to arrive at

$$-r_1r_2 - r_1r_3 - r_2r_3 = (\omega+\omega^2)r_1r_2 + (\omega+\omega^2)r_1r_3 + (\omega+\omega^2)r_2r_3. \quad (5)$$

Adding $r_1^2+r_2^2+r_3^2$ to the right hand side of (5) and multiplying by $r_1+r_2+r_3$ gives the complex identity:

$$r_1^3+r_2^3+r_3^3-3r_1r_2r_3 = (r_1+r_2+r_3)(r_1+\omega r_2+\omega^2 r_3)(r_1+\omega^2 r_2+\omega r_3). \quad (6)$$

It might be the easiet derivation to just note

$$(r_1+\omega r_2+\omega^2 r_3)(r_1+\omega^2 r_2+\omega r_3)$$
$$= r_1^2+r_2^2+r_3^2+(\omega+\omega^2)r_1r_2+(\omega+\omega^2)r_1r_3+(\omega+\omega^2)r_2r_3,$$

by just doing the multiplication and remembering that $\omega^3 = 1$.

Exploration

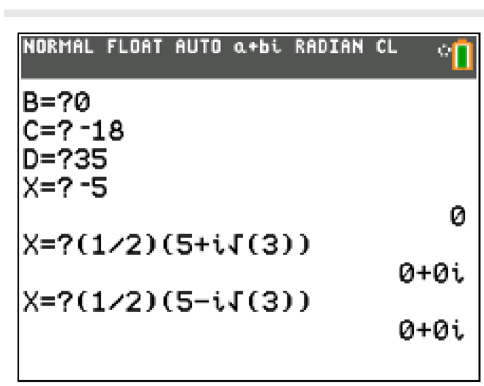

Figure 3: Using option 1) and 2) the CUBICS program confirms roots.

We can gloss the identity (6) as the factoring of a depressed cubic; just make $r_1 = x$ for

$$x^3 - 3xr_2r_3 + r_2^3 + r_3^3 = (x+r_2+r_3)(x+\omega r_2+\omega^2 r_3)(x+\omega^2 r_2+\omega r_3). \quad (7)$$

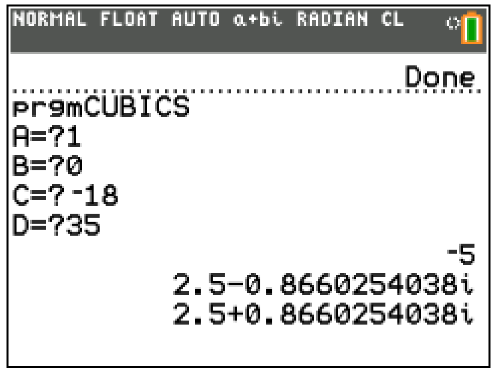

Figure 4: Caption for cubics-cardan-forMinus18and35

Let's experiment. Let $r_2 = 2$, and $r_3 = 3$ in (6) then (7) becomes

$$x^3 - 18x + 35 = (x+5)\left(x - \frac{1}{2}\left(5 + i\sqrt{3}\right)\right)\left(x - \frac{1}{2}\left(5 - i\sqrt{3}\right)\right).$$

We can confirm that $-5, 1/2(5+i\sqrt{3}), 1/2(5-i\sqrt{3})$ are roots using options from our calculator program, Figure 3.

Buoyed by this likely success, let's find the other roots by using (7) in our program: that's the code lines 021 through 026. Bingo: Figure 4.

The TI-84 did the complex arithmetic, but it dropped the symbolic, exact solutions. Maple keeps the exact form for such calculations. Figure 5 shows this.

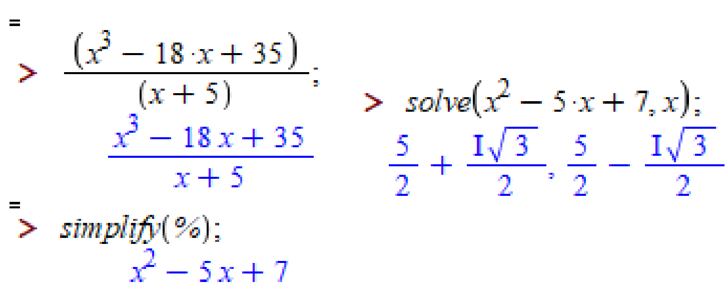

Figure 5: Given any one root r of a cubic, we can divide by $x - r$ and then solve the quotient quadratic.

Depressed cubic formula

We can infer from this example that the central puzzle to solving a cubic given in the form $x^3 + cx + d$ is to find r_2 and r_3 given that $c = -3r_2r_3$ and $d = r_2^3 + r_3^3$. If we are able to to do this we can use (7) and read off the roots: $R_1 = -r_2 - r_3$, $R_2 = -\omega r_2 - \omega^2 r_3$, and $R_3 = -\omega^2 r_2 - \omega r_3$.

Here is the "aha" moment: we can find two numbers knowing the product and sum of the numbers using a quadratic equation; and, as a bonus, quadratic solutions are complex conjugates. Solving the quadratic $x^2 + dx - (c/3)^3$ does it. If the roots are q_1 and q_2, then $\sqrt[3]{q_1} = r_2$ and $\sqrt[3]{q_2} = r_3$. For our example $x^3 - 18x + 35$, $-3r_2r_3 = -18$, so $r_2r_3 = 6$, that cubed is the quadratic's constant term and $d = r_2^3 + r_3^3$. The quadratic associated with $x^3 - 18x + 35$ is $x^2 - 35x + 216$. The quadratic formula yields

$$q_1 = \frac{35 - \sqrt{35^2 - 4(1)(216)}}{2} = 8 = r_2^3$$

and

$$q_2 = \frac{35 + \sqrt{35^2 - 4(1)(216)}}{2} = 27 = r_3^2.$$

Some algebraic manipulations of these ideas yields Blitzer's formula for the depressed cubic equation: $x^3 + mx - n = 0$. Here goes: $m = -3r_2r_3$ and $-n = r_2^3 + r_3^3$. So the quadratic associated with this cubic is $x^2 + nx - (m/3)^3$. The quadratic formula gives

$$\frac{-n + \sqrt{n^2 - 4(1)(-m^3/27)}}{2} \text{ and } \frac{-n - \sqrt{n^2 + 4(1)(-m^3/27)}}{2}$$

or

$$\frac{\sqrt{(4/4)n^2 - 4(1)(-m^3/27)} - n}{2} \text{ and } \frac{-\sqrt{(4/4)n^2 + 4(1)(-m^3/27)} - n}{2}$$

or

$$\sqrt{4}\frac{\sqrt{(1/4)n^2 - (1)(-m^3/27)} - n}{2} \text{ and } \sqrt{4}\frac{-\sqrt{(1/4)n^2 + (1)(-m^3/27)} - n}{2}$$

or

$$\sqrt{(n/2)^2 + (m/3)^3} - n/2 \text{ and } -\sqrt{(n/2)^2 + (m/3)^3} - n/2.$$

Taking the cube root of each and the negative of the sum gives (1).

General cubic formula

Given $p_3(x) = x^3 + bx^2 + cx + d$ can we eliminate the x^2 term by a transformation [1, Section 2.5], solve the resulting *depressed* cubic $d_3(x)$, and then solve the original $p_3(x)$? We will then be in a position to formulate a general solution and implement it as a calculator program. As a test case, let's take

$$p_3(x) = (x-1)(x-(2-2i))(x+(2-2i)) = x^3 - 5x^2 + 12x - 8. \tag{8}$$

Left and right horizontal shifts just move a graph left and right. So in theory, if as a result of a horizontal shift we eliminate the x^2 term, we should be able to shift roots found back. Consider

$$p_3(x+h) = (x+h)^3 + b(x+h)^2 + c(x+h) + d$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 + b(x^2 + 2hx + h^2) + cx + ch + d$$

$$3hx^2 + bx^2 + \text{ other terms} = (3h+b)x^2 + \text{ other terms.}$$

Setting $3h + b = 0$, we get $h = -b/3$ eliminates the x^2 term. We can crunch the new $d_3(x)$ polynomial using Maple, Figure 6.

$f := x \rightarrow (x-1)\cdot(x-(2-2\cdot I))\cdot(x-(2+2\cdot I));$

$$f := x \mapsto (x-1)\cdot(x-2+2\cdot \mathrm{I})\cdot(x-2-2\cdot \mathrm{I}) \tag{1}$$

$expand(f(x));$

$$x^3 - 5x^2 + 12x - 8 \tag{2}$$

$expand\left(f\left(x+\frac{5}{3}\right)\right);$

$$x^3 + \frac{74}{27} + \frac{11}{3}x \tag{3}$$

Figure 6: Maple can perform the transformation that eliminates the x^2 term.

We determine one root of $x^3 + 11/3x + 74/27$ is $x = -2/3$ using Option 3 with $M = 11/3$ and $N = -74/27$, Figure 7 at the top. Using synthetic division (or Maple) a division yields the quadratic $x^2 - 2/3x + 37/9$ and this has roots $1/3 \pm 2i$ via the quadratic formula (Maple, again). But we can get all three (in theory) using Option 1 and entering the coefficients $A = 1$, $B = 0$, $C = 11/3$, $D = 74/27$ and then Option 4. Viola: Figure 7 shows all three roots. When 5/3 is added to these roots we have a confirmation of (8); we get the correct roots.

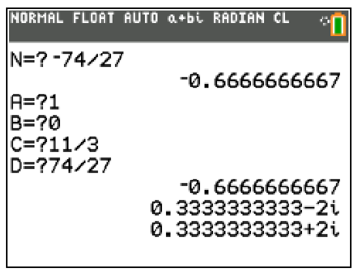

Figure 7: Option 3 gives the single root $-2/3$, and Option 4 gives all three roots.

Conclusion

What could possibly go wrong? What are the roots of $x^3 - 12x + 10$? There is some fine print to these cubic equations. Chrystal, page 549 touches on negative discriminants of the quadratic we derive from a given cubic renders the omega math we've been using *useless for purposes of numerical calculation*.

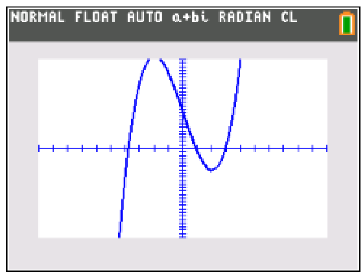

Figure 8: Graph for the cubic $x^3 - 12x + 10$ indicating it has three real roots.

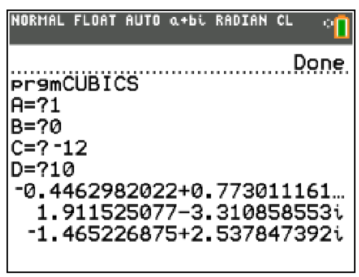

Figure 9: Option 4 gives three complex roots – not even close!

Chrystal's book was originally written in the 1800's, so maybe this statement is a little dated. But the calculator seems to falter with this example. We are supposed to get three real roots as a graph of this cubic indicates Figure 8 and we get three complex roots, Figure 9.

In the sequel to this article, we will attempt to resolve these issues. Hint: polar coordinates!

References

[1] Blitzer, R. (2010). *Algebra and Trigonometry*, 3rd ed., Pearson: Boston.

[2] Chrystal, G. (1999). *Textbook of Algebra*, Vol. 1, 7th ed., AMS: Providence, RI.

[3] Herstein, I.N. (1976). *Topics in Algebra*, 2nd ed., Wiley: New York.

[4] Tignol, J.P. (2001). *Galois' Theory of Algebraic Equations*, World Scientific: London.