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Kinematics of moving a point along an ellipse

Abstract

The vectors of velocity and acceleration of various ways of moving a point along an ellipse are investigated.

Keywords

ellipse, vector, velocity, acceleration.

Note the property of collinear vectors on the plane - Rectangles built on vectors, figure 1, should be similar:

$$\frac{BD}{AD} = \frac{B_1D_1}{A_1D_1} \qquad (1)$$

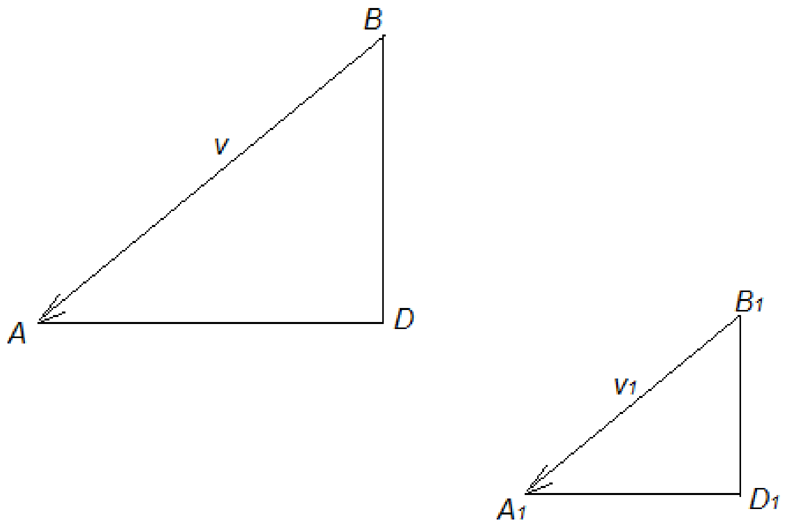

Figure 1

There is a system of equations for a parametric pendulum (2)

The parameter is the angle of rotation, independent of time.

$$\begin{cases} x = r(\varphi)\cdot cos(\varphi) \\ y = r(\varphi)\cdot sin(\varphi) \end{cases} \qquad (2)$$

Option 1. Point C moves along an ellipse relative to the center, figure 2

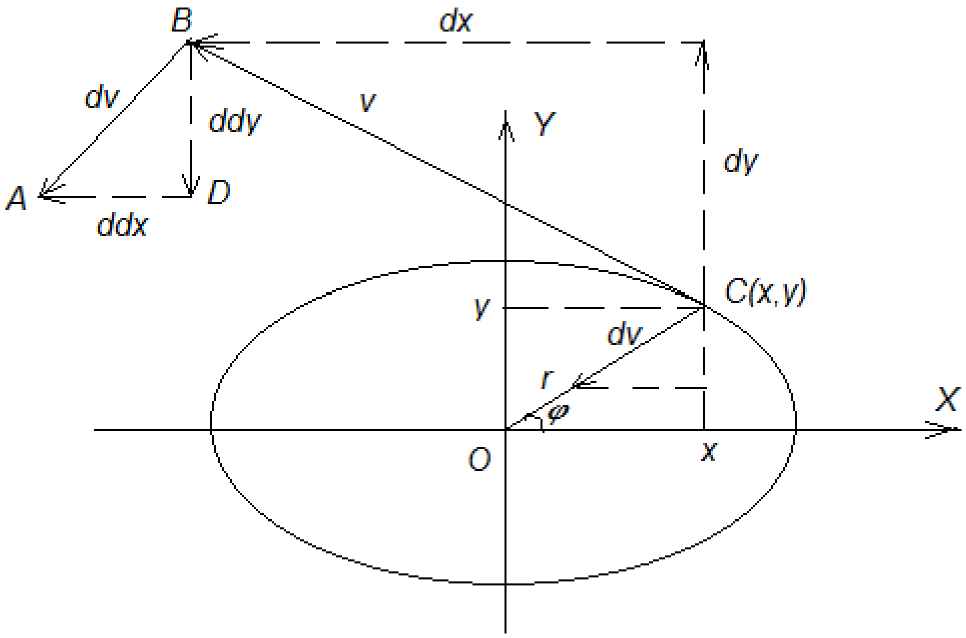

Figure 2

Let us substitute into system (2) the radius of the ellipse relative to the center

$$r(\varphi) = \frac{b}{\sqrt{1-e^2cos^2\varphi}} \tag{3}$$

$$\begin{cases} x = \frac{b}{\sqrt{1-e^2cos^2\varphi}} \cdot cos(\varphi) \\ \boldsymbol{y} = \frac{b}{\sqrt{1-e^2cos^2\varphi}} \cdot sin(\varphi) \end{cases} \tag{4}$$

Let's differentiate twice. We get the coordinates of speed and acceleration:

$$\frac{dx}{d\varphi} = \frac{d}{d\varphi}\left(\frac{b*cos(\varphi)}{\sqrt{1-e^2*cos^2\varphi}}\right) = -\frac{b*sin(\varphi)}{(1-e^2*cos^2\varphi)^{3/2}} \tag{5}$$

$$\frac{dy}{d\varphi} = \frac{d}{d\varphi}\left(\frac{bsin(\varphi)}{\sqrt{1-e^2cos^2\varphi}}\right) = \frac{b(1-e^2)cos(\varphi)}{(1-e^2*cos^2\varphi)^{3/2}} \tag{6}$$

$$\frac{d^2x}{d\varphi} = \frac{d^2}{d\varphi}\left(\frac{bcos(\varphi)}{\sqrt{1-e^2cos^2\varphi}}\right) = -\frac{b*cos(\varphi)(2e^2cos^2\varphi-3e^2+1)}{(1-e^2*cos^2\varphi)^{5/2}} \tag{7}$$

$$\frac{d^2y}{d\varphi} = \frac{d^2}{d\varphi}\left(\frac{bsin(\varphi)}{\sqrt{1-e^2cos^2\varphi}}\right) = \frac{b*sin(\varphi)(e^2-1)(2e^2cos^2\varphi+1)}{(1-e^2*cos^2\varphi)^{5/2}} \tag{8}$$

Let's compare the ratio of the coordinates of the radius and acceleration:

$$\frac{x}{y} = \frac{\cos\varphi}{\sin\varphi} \tag{9}$$

$$\frac{\frac{d^2x}{d\varphi}}{\frac{d^2y}{d\varphi}} = \frac{(-2e^2cos^2\varphi+3e^2-1)cos(\varphi)}{sin(\varphi)(e^2-1)(2e^2cos^2\varphi+1)} \tag{10}$$

If $e = 0$, then $\frac{\frac{d^2x}{d\varphi}}{\frac{d^2y}{d\varphi}} = \frac{x}{y} = \frac{\cos\varphi}{\sin\varphi}$ (11)

We got a circle, a special case of an ellipse, figure 3.

In figures 2 - 5 are marked with red lines for speed, green for acceleration.

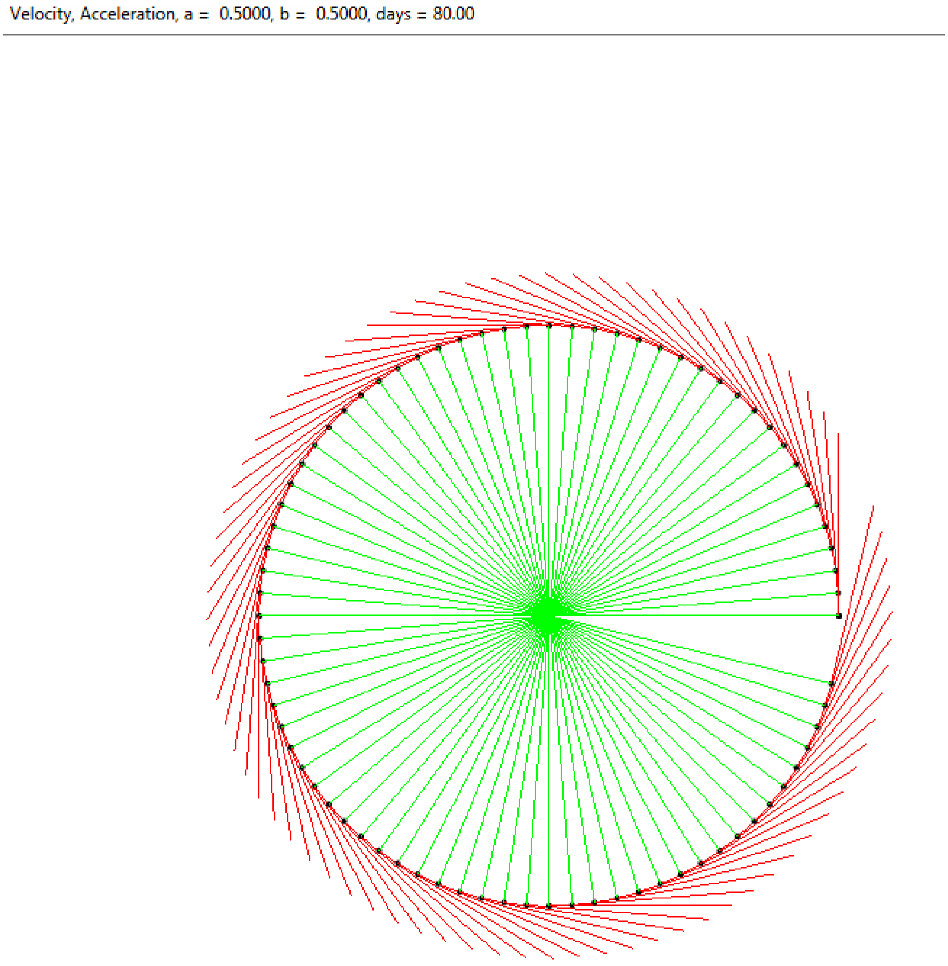

Figure 3

If $e \neq 0$, then $\frac{\frac{d^2x}{d\varphi}}{\frac{d^2y}{d\varphi}} \neq \frac{x}{y}$, figure 4. (12)

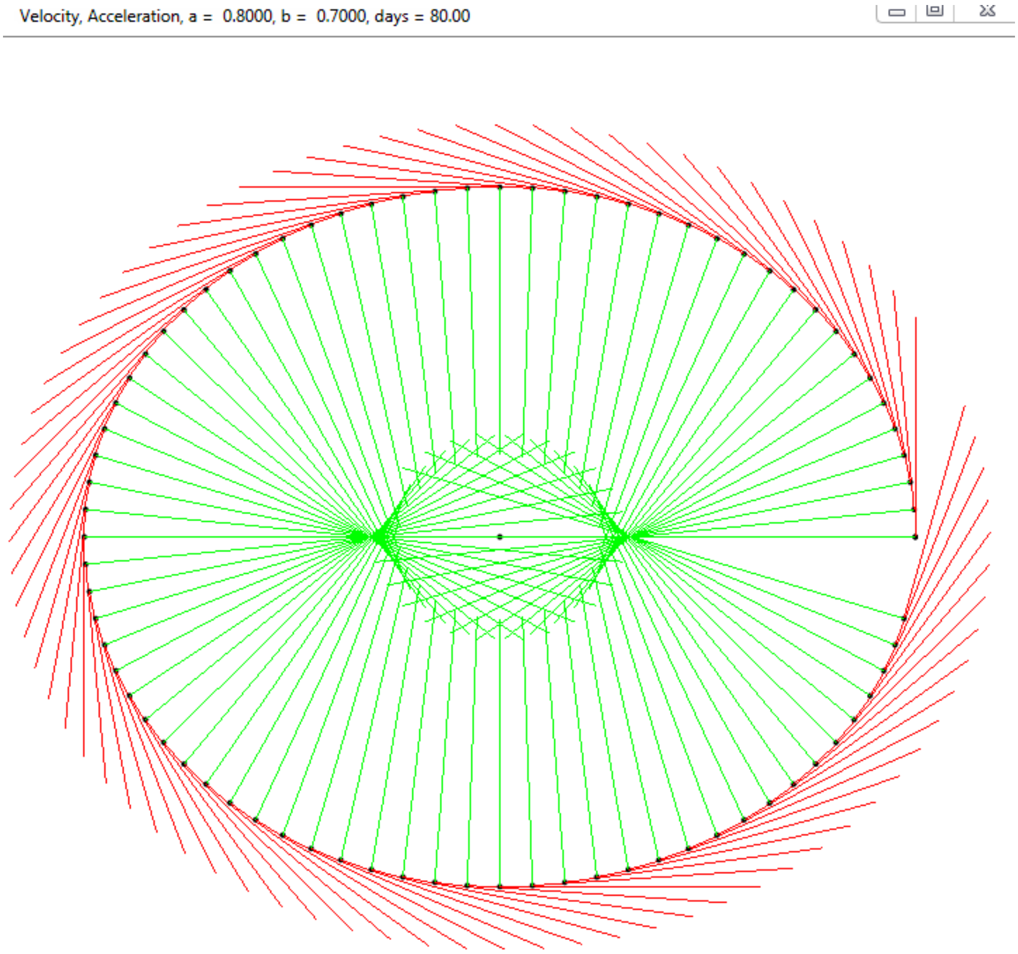

Figure 4

Option 2. Point C moves along an ellipse relative to the focus, figure 5.

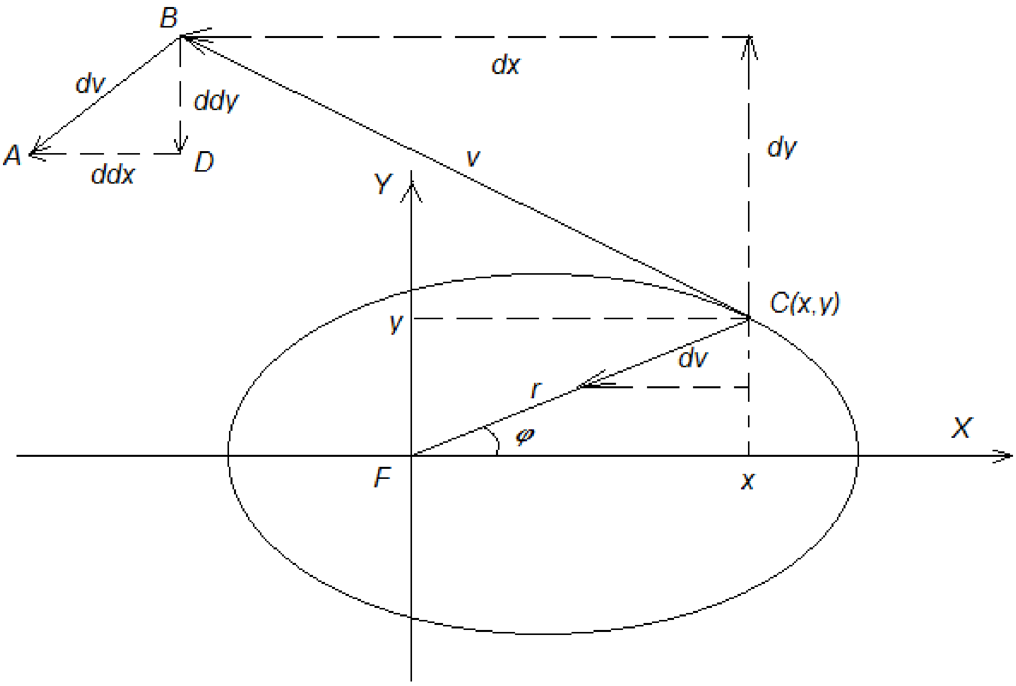

Figure 5

Let us substitute into system (2) the radius of the ellipse with respect to the focus:

$$r(\varphi) = \frac{b^2}{a\left(1-e*cos(\varphi)\right)} \tag{13}$$

$$\begin{cases} x = \frac{b^2}{a\left(1-e*cos(\varphi)\right)} \cdot cos(\varphi) \\ \boldsymbol{y} = \frac{b^2}{a\left(1-e*cos(\varphi)\right)} \cdot sin(\boldsymbol{\varphi}) \end{cases} \tag{14}$$

Let's differentiate twice. We get the coordinates of speed and acceleration:

$$\frac{dx}{d\varphi} = \frac{d}{d\varphi}\left(\frac{b^2*cos(\varphi)}{a\left(1-e*cos(\varphi)\right)}\right) = \frac{-b^2*sin(\varphi)}{a(e*cos(\varphi)-1)^2} \tag{15}$$

$$\frac{dy}{d\varphi} = \frac{d}{d\varphi}\left(\frac{b^2*sin(\varphi)}{a\left(1-e*cos(\varphi)\right)}\right) = \frac{b^2(cos(\varphi)-e)}{a(e*cos(\varphi)-1)^2} \tag{16}$$

$$\frac{d^2x}{d\varphi} = \frac{d^2}{d\varphi}\left(\frac{b^2*sin(\varphi)}{a\left(1-e*cos(\varphi)\right)}\right) = \frac{b^2\left(e*(\cos\varphi)^2+\cos\varphi-2e\right)}{a(e*cos(\varphi)-1)^3} \tag{17}$$

$$\frac{d^2y}{d\varphi} = \frac{d^2}{d\varphi}\left(\frac{b^2*sin(\varphi)}{a\left(1-e*cos(\varphi)\right)}\right) = \frac{b^2sin(\varphi)*\left(e*(\cos\varphi)^2-2e^2+1\right)}{a(e*cos(\varphi)-1)^3} \tag{18}$$

Let's compare the ratio of the coordinates of the radius and acceleration:

$$\frac{x}{y} = \frac{\cos\varphi}{\sin\varphi} \tag{19}$$

$$\frac{\frac{d^2x}{d\varphi}}{\frac{d^2y}{d\varphi}} = \frac{e*(\cos\varphi)^2+\cos\varphi-2e}{sin(\varphi)*(e*\cos\varphi-2e^2+1)} \tag{20}$$

If $e = 0$, then $\frac{\frac{d^2x}{d\varphi}}{\frac{d^2y}{d\varphi}} = \frac{x}{y} = \frac{\cos\varphi}{\sin\varphi}$ (21)

We get a circle, a special case of an ellipse, figure 6.

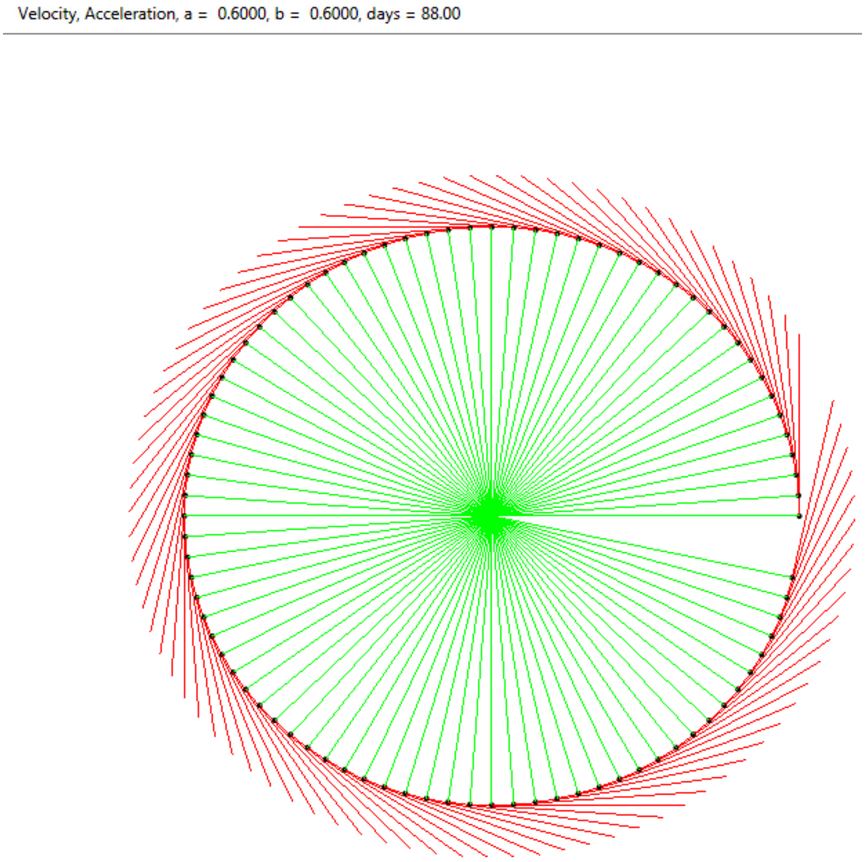

Figure 6

If $e \neq 0$, then $\frac{\frac{d^2x}{d\varphi}}{\frac{d^2y}{d\varphi}} \neq \frac{x}{y}$, figure 7. (22)

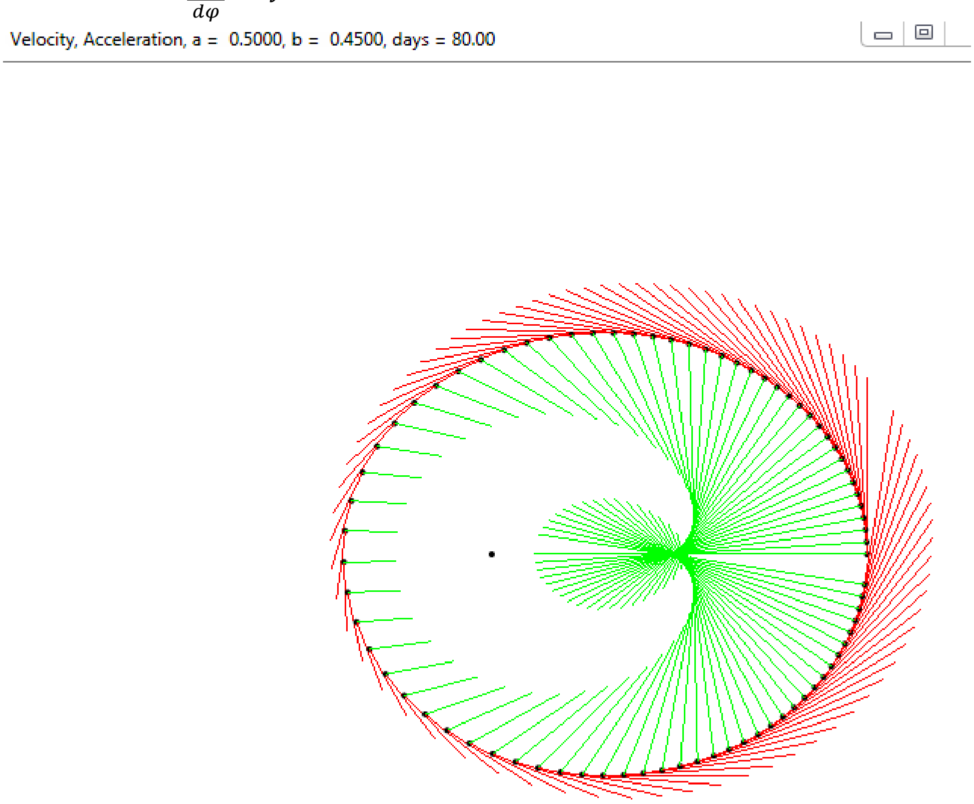

Figure 7

From the point of view of mathematics, all these examples are logical.

Contradictions arise when these calculations are applied to real experiments. This is especially clearly seen in figure 7. Velocities and accelerations at aphelion are greater than at perihelion.

Let us introduce time (t) into system (2).

$$\begin{cases} x = r(\varphi(t))\cdot cos(\varphi(t)) \\ \boldsymbol{y} = r(\varphi(t))\cdot sin(\varphi(t)) \end{cases} \tag{23}$$

We calculate the first and second time derivatives from the system of equations.

Option 1a. Point C moves along an ellipse relative to the center, figure 1.

$$r(\varphi(t)) = \frac{b}{\sqrt{1-e^2cos^2\varphi(t)}} \tag{24}$$

$$\dot{x} = \frac{d}{dt}\left(\frac{b*cos(\varphi(t))}{\sqrt{1-e^2*cos(\varphi(t))^2}}\right) = -\frac{b*\sin(\varphi)*\dot{\varphi}}{\sqrt{\left(1-e^2*cos(\varphi(t))^2\right)^3}} \tag{25}$$

$$\dot{y} = \frac{d}{dt}\left(\frac{b*sin(\varphi(t))}{\sqrt{1-e^2*cos(\varphi(t))^2}}\right) = -\frac{b*(e^2-1)*cos(\varphi(t))*\dot{\varphi}}{\sqrt{\left(1-e^2*cos(\varphi(t))^2\right)^3}} \tag{26}$$

$$\ddot{x} = \frac{d^2}{dt^2}\left(\frac{b*cos(\varphi(t))}{\sqrt{1-e^2*cos(\varphi(t))^2}}\right) = \frac{-b}{\sqrt{\left(1-e^2*cos(\varphi(t))^2\right)^5}}\left(\dot{\varphi}^2\left(2cos(\varphi(t))^3*e^2-3e^2*cos(\varphi(t))+cos(\varphi(t))\right)+\right.$$
$$\left.\ddot{\varphi}*sin(\varphi(t))\left(1-e^2*cos(\varphi(t))^2\right)\right) \tag{27}$$

$$\ddot{y} = \frac{d^2}{dt^2}\left(\frac{b*sin(\varphi(t))}{\sqrt{1-e^2*cos(\varphi(t))^2}}\right) = \frac{b}{\sqrt{\left(1-e^2cos(\varphi(t))^2\right)^5}}\left(\ddot{\varphi}\left(cos(\varphi(t))-e^2cos(\varphi(t))^3-e^2cos(\varphi(t))\right)+\right.$$
$$\left.\dot{\varphi}^2\left(-sin(\varphi(t))-2e^2sin(\varphi(t))cos(\varphi(t))^2+2sin(\varphi(t))e^4cos(\varphi(t))^2+sin(\varphi(t))e^2\right)\right) \tag{28}$$

Next, the kinematic equation is derived $\ddot{\varphi} = \frac{2*e^2*cos(\varphi)*sin(\varphi)*\dot{\varphi}^2}{1-e^2*cos(\varphi)^2}$, [1] (29)

Compare the coordinates of radius and acceleration:

$$\frac{x}{y} = \frac{\cos\varphi(t)}{\sin\varphi(t)}$$
$$\frac{\ddot{x}}{\ddot{y}} = \frac{\dot{\varphi}^2\left(2cos(\varphi(t))^3*e^2-3e^2*cos(\varphi(t))+cos(\varphi(t))\right)+\ddot{\varphi}*sin(\varphi(t))\left(1-e^2*cos(\varphi(t))^2\right)}{\ddot{\varphi}\left(cos(\varphi(t))-e^2cos(\varphi(t))^3-e^2cos(\varphi(t))\right)+\dot{\varphi}^2\left(-sin(\varphi(t))-2e^2sin(\varphi(t))cos(\varphi(t))^2+2sin(\varphi(t))e^4cos(\varphi(t))\right.}$$

(30)

If $e = 0$, then by formula (29) $\ddot{\varphi} = 0, \ \dot{\varphi} = const$
We got a circle, a special case of an ellipse, figure 8.

$$\frac{\ddot{x}}{\ddot{y}} = \frac{-\ddot{\varphi}sin(\varphi(t)) - \dot{\varphi}^2 cos(\varphi(t))}{\ddot{\varphi}cos(\varphi(t)) - \dot{\varphi}^2 sin(\varphi(t))} = \frac{-\dot{\varphi}^2 cos(\varphi(t))}{-\dot{\varphi}^2 sin(\varphi(t))} = \frac{cos(\varphi(t))}{sin(\varphi(t))} = \frac{x}{y}$$

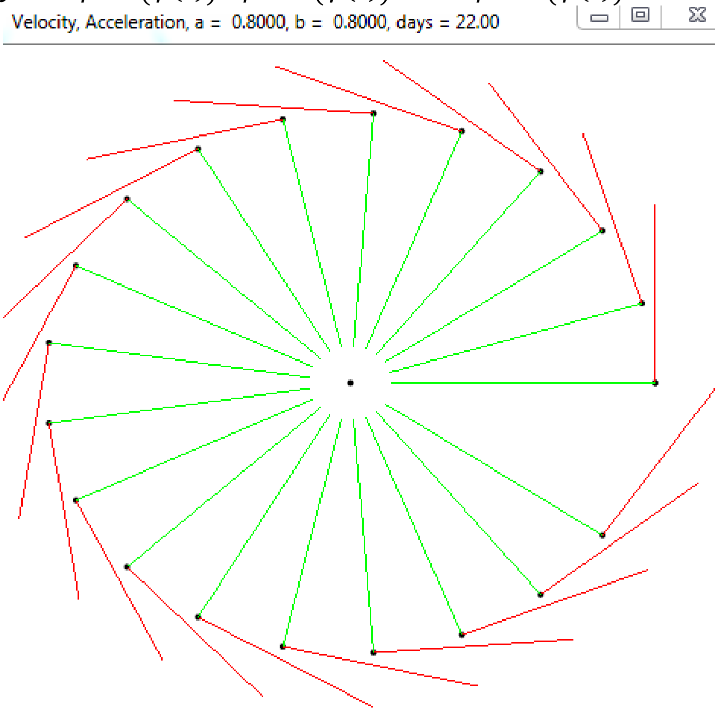

Figure 8

If $e \neq 0$, then $\frac{\ddot{x}}{\ddot{y}} \neq \frac{x}{y}$, figure 9.

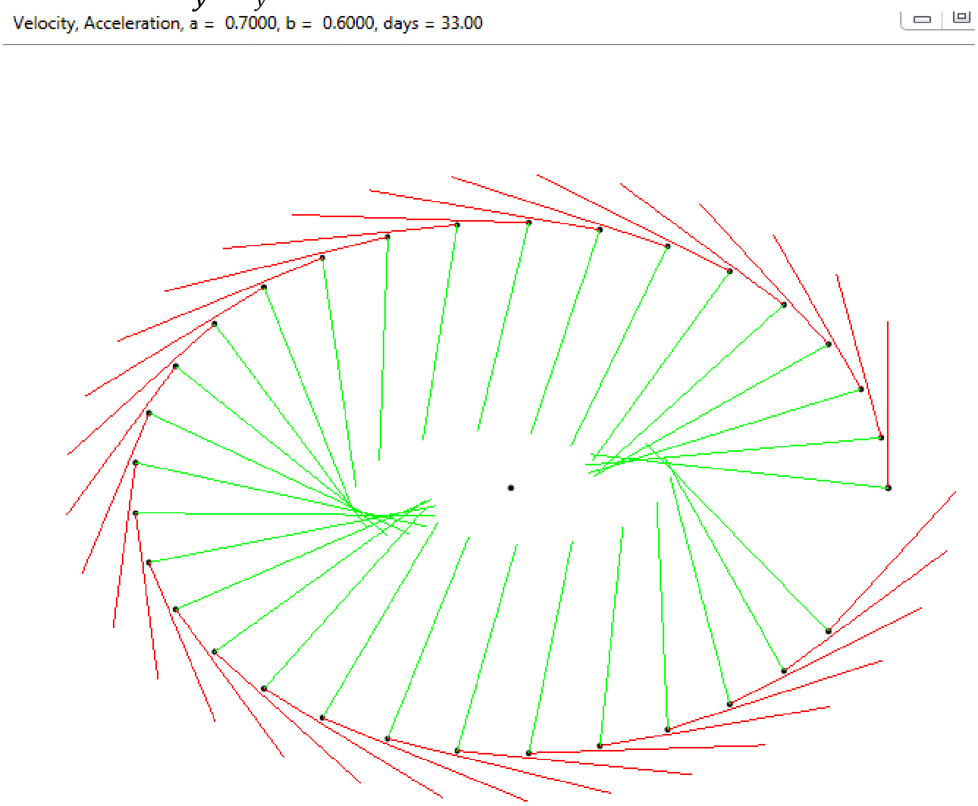

Figure 9

Option 2a. Point C moves along an ellipse relative to the focus, figure 5.

Let us substitute into system (23) the radius of the ellipse with respect to the focus:

$$r(\varphi(t)) = \frac{b^2}{a(1-e*cos(\varphi(t)))} \tag{32}$$

$$\begin{cases} x = \frac{b^2}{a(1-e*cos(\varphi(t)))} \cdot cos(\varphi(t)) \\ y = \frac{b^2}{a(1-e*cos(\varphi(t)))} \cdot sin(\varphi(t)) \end{cases} \tag{33}$$

Let's differentiate twice. We get the coordinates of speed and acceleration:

$$\dot{x} = \frac{d}{dt}\left(r(\varphi(t))cos(\varphi(t))\right) = \frac{a*r^2*\dot{\varphi}*\sin(\varphi)}{b^2} \tag{34}$$

$$\dot{y} = \frac{d}{dt}\left(\frac{p}{1-e*cos(\varphi(t))} sin(\varphi(t))\right) = \frac{a*r^2*\dot{\varphi}*(cos(\varphi(t))+e)}{b^2} \tag{35}$$

$$\ddot{x} = \frac{b^2\left(\left(-e*cos(\varphi(t))*sin(\varphi(t))+sin(\varphi(t))\right)\ddot{\varphi}+\dot{\varphi}^2\left(e*cos(\varphi(t))^2-2e+cos(\varphi(t))\right)\right)}{a(e*cos(\varphi(t))-1)^3} \tag{36}$$

$$\ddot{y} = \frac{-b^2\left((-cos(\varphi(t))(e*cos(\varphi(t))-1)+e)\ddot{\varphi}+2\dot{\varphi}^2\left(e^2-\frac{e*cos(\varphi(t))+1}{2}\right)sin(\varphi(t))\right)}{a(e*cos(\varphi(t))-1)^3} \tag{37}$$

Next, the kinematic equation is derived $\ddot{\varphi} = \frac{2*e*\sin(\varphi)*\dot{\varphi}^2}{1-e*\cos(\varphi)}$, [1] (38)

Let's compare the ratio of the coordinates of the radius and acceleration:

$$\frac{x}{y} = \frac{\cos\varphi(t)}{\sin\varphi(t)}$$

$$\frac{\ddot{x}}{\ddot{y}} = \frac{\ddot{\varphi}*sin(\varphi(t))(e*cos(\varphi(t))-1)-\dot{\varphi}^2\left(e*cos(\varphi(t))^2-2e+cos(\varphi(t))\right)}{-\ddot{\varphi}*cos(\varphi(t))(e*cos(\varphi(t))-1)+2\dot{\varphi}^2*sin(\varphi(t))\left(e^2-\frac{e*cos(\varphi(t))+1}{2}\right)} \tag{39}$$

If $e = 0$, then by formula (38) $\ddot{\varphi} = 0$, $\dot{\varphi} = const$

$$\frac{\ddot{x}}{\ddot{y}} = \frac{-\ddot{\varphi}sin(\varphi(t))-\dot{\varphi}^2cos(\varphi(t))}{\ddot{\varphi}cos(\varphi(t))-\dot{\varphi}^2sin(\varphi(t))} = \frac{-\dot{\varphi}^2cos(\varphi(t))}{-\dot{\varphi}^2sin(\varphi(t))} = \frac{cos(\varphi(t))}{sin(\varphi(t))} = \frac{x}{y} \tag{40}$$

(40)

We got a circle, a special case of an ellipse, figure 8.

If $e \neq 0$, then $\frac{\ddot{x}}{\ddot{y}} \neq \frac{x}{y}$, figure 10.

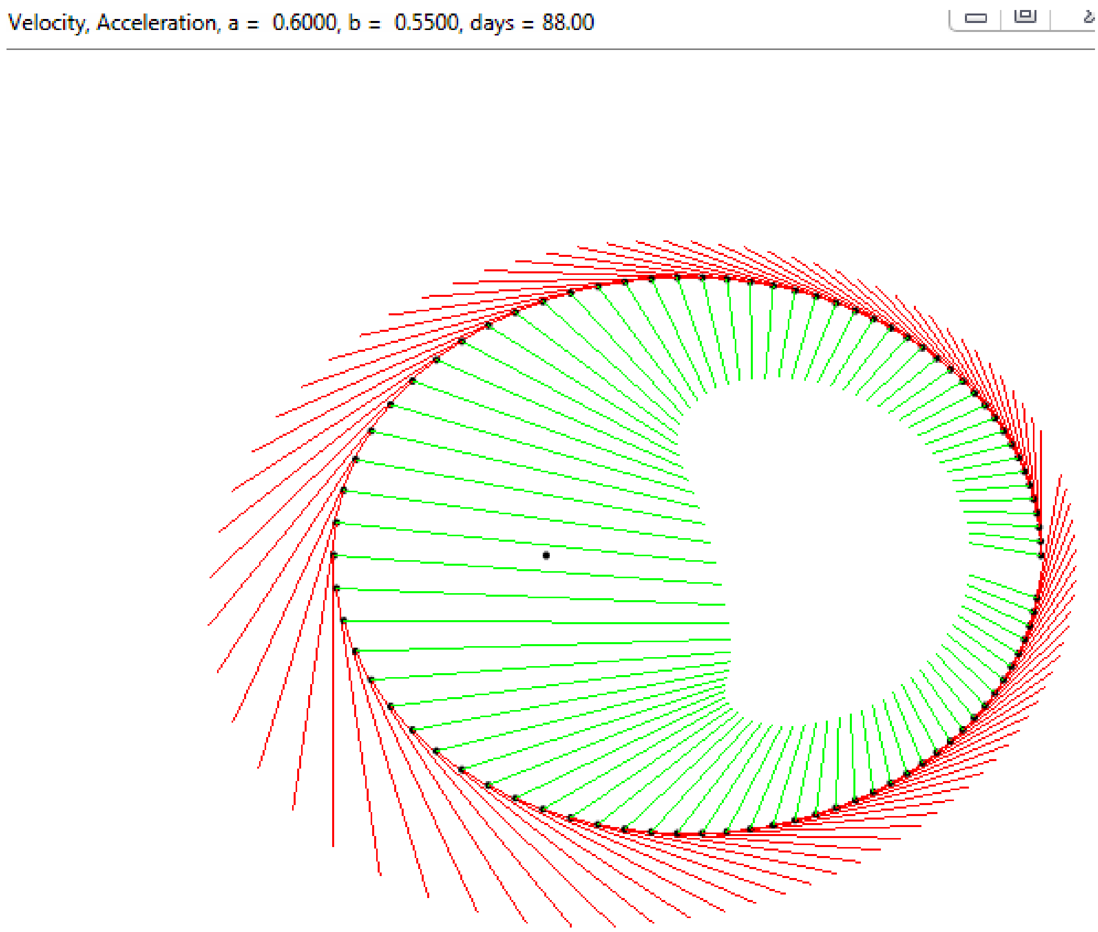

Figure 10

Option 3. Movement of the ellipsograph ruler with different speeds

Article [2] considers uniform, uniformly accelerated, elliptical motion. Velocity and acceleration vectors are calculated. We present the graphical results of the movement of a point along an ellipse at different speeds.

3.1 uniform movement, figure 11.

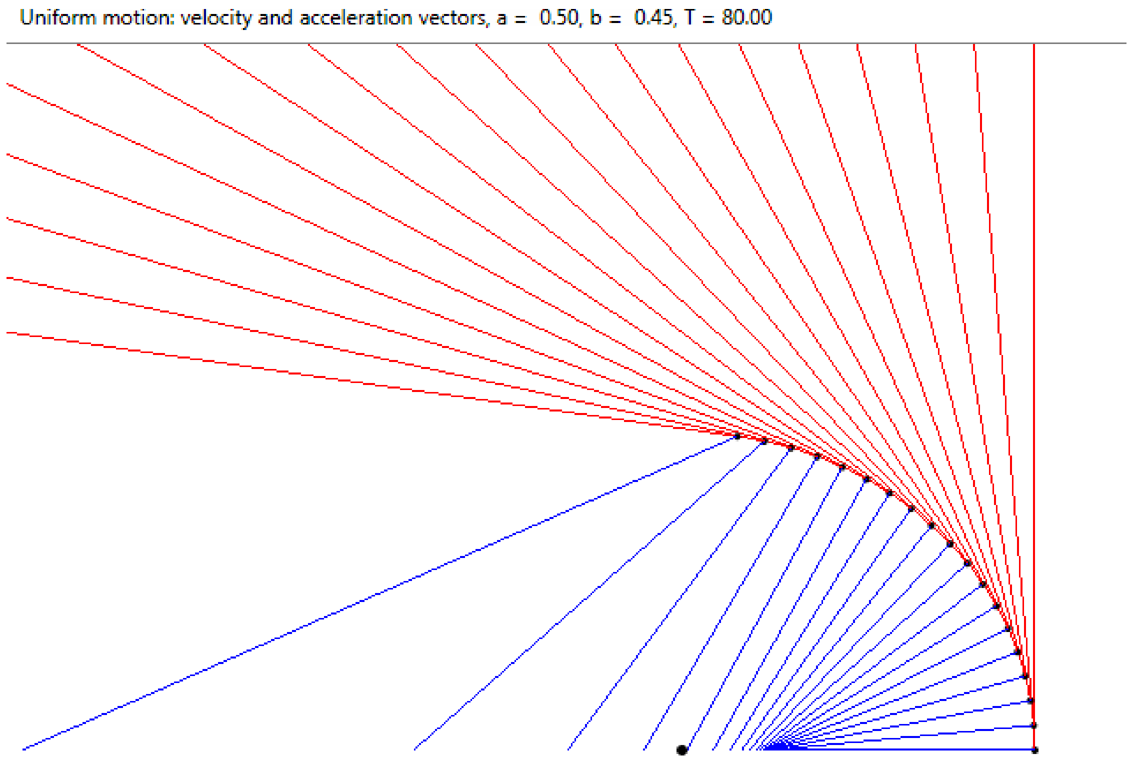

Figure 11

3.2 uniformly accelerated motion, figure 12.

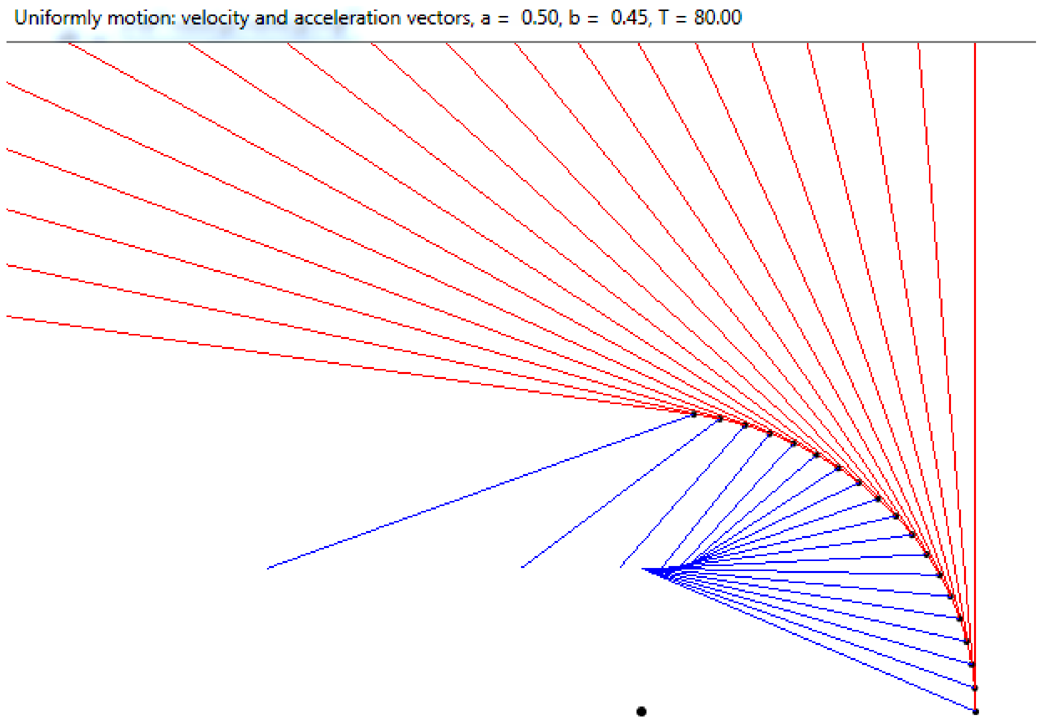

Figure 12

3.2 elliptical movement, figure 13.

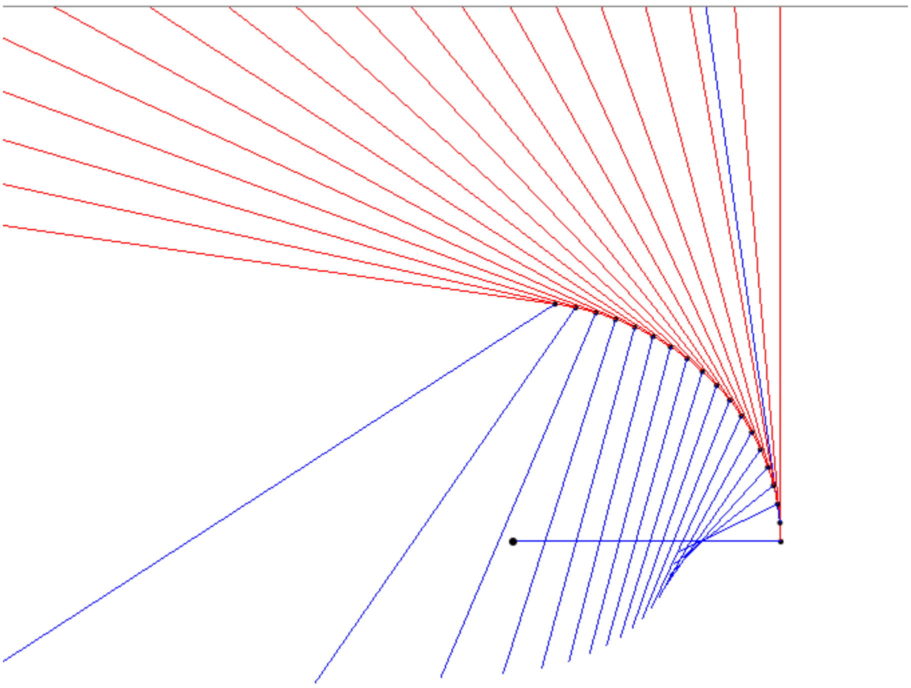

Figure 13

Conclusions

Newton's second law states that the directions of the force and acceleration vectors coincide.

Here, kinematic methods for calculating acceleration are considered. In the general case, the directions of the acceleration vectors are not directed to the point of the center of rotation.

Note

The article used materials from textbooks on mechanics. Perhaps, the derivation of the kinematic equation in the form of formulas [29, 38] is rarely given, so the article [1] is proposed. Velocity and acceleration hodographs, figures 3, 4, 6 – 10 were obtained by the program [1], link in the appendix. Velocity and acceleration hodographs of the ellipsograph, figures 11 - 13, obtained by the program [2], link in the appendix.

Literature

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2. Viktor Strohm, Movement of the ellipsograph ruler with different speeds, https://www.academia.edu/91690243/Movement_of_the_ellipsograph_ruler_with_different_speeds

Applications

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