Clifford algebra Cl(0,6) approach to beyond the standard model and naturalness problems

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ABSTRACT: Is there more to Dirac's gamma matrices than meets the eye? It turns out that the gamma zero operator can be split into three components. This revelation facilitates the expansion of Dirac's space-time algebra to Clifford algebra Cl(0,6). The resultant rich geometric structure can be leveraged to establish a combined framework of gravity and beyond the standard model, wherein a gravi-weak interaction between the vierbein field and the extended weak gauge field is allowed. Inspired by the composite Higgs model, we examine the vierbein field as an effective description of the fermion-antifermion condensation. The compositeness of space-time manifests itself at an energy scale which is different from the Planck scale. We propose that the regular Lagrangian terms including the cosmological constant are of quantum condensation origin, thus possibly addressing the naturalness problem. The Clifford algebra approach also permits a weaker form of charge conjugation without particle-antiparticle interchange, which leads to a Majorana-type mass that conserves lepton number. Additionally, in the context of spontaneous breaking of two global U(1) symmetries, we explore a three-Higgs-doublet model which could explain the fermion mass hierarchies.

Contents

1   Introduction

The mathematical imaginary number i is ubiquitous in physics theories. In the case of quantum mechanics, the imaginary number makes its appearance in the commutation relation of position operator $\hat{X}$ and momentum operator $\hat{P}$

$$[\hat{X}, \hat{P}] = i\hbar, \tag{1.1}$$

where $\hbar$ is the Planck constant. Consequently, the *quantum* wave function is complex-valued. On the other hand, the imaginary number also shows up in the gauge transformation of a *classical* complex field

$$\psi \to \psi e^{i\theta}, \tag{1.2}$$

which is essential in determining the electric charge property of ψ.

We customarily treat the imaginary number i for both examples as the same. It may come as a surprise that the imaginary number in the second case is different from the first one. The latter is actually a unit pseudoscalar in disguise

$$I = \gamma_0\gamma_1\gamma_2\gamma_3, \tag{1.3}$$

where γ_a are no other than the celebrated gamma operators stumbled upon by Paul Dirac in 1928. The Dirac gamma operators satisfy the Clifford algebra $Cl(1,3)$ anticommutation relations

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}, \tag{1.4}$$

where $\eta_{ab} = diag(1, -1, -1, -1)$. In view that $I^2 = -1$, the pseudoscalar I can be regarded as a surrogate for the imaginary number i. As we will learn later in this paper, replacing imaginary number i with pseudoscalar I in eq. (1.2) leads to a novel definition of charge conjugation without particle-antiparticle interchange.

Historically, the Dirac operators γ_a are represented as gamma matrices. Due to the dichotomy between fermiom states as columns and operators as matrices in the conventional formalism of quantum field theory (QFT), the aforementioned association of pseudoscalar with imaginary number would run into inconsistencies. This identification can only be achieved in an unconventional way by forgoing the traditional matrix representation and enlisting the aid of the Clifford algebra approach,[1,2,3,4,5,7,8,9,10,11,12,13,14,15,16,17,18] whereby both the algebraic spinor states and Dirac's gamma operators can be expressed in the same algebraic space.

The Clifford algebra, also known as the geometric algebra or the space-time algebra for the specific case of $Cl(1,3)$, is a potent mathematical tool that finds extensive applications in the physics arena. Remarkably, there is one more application of Clifford algebra $Cl(1,3)$ unbeknownst to Dirac. We know that gravity can be formulated as a Lorentz gauge theory[19,20] in terms of the vierbein (or tetrad) and the spin connection. The gauge approach to gravity is also known as Einstein-Cartan gravity. The spin connection, associated with the local Lorentz group $SO(1,3)$ (or $Spin(1,3)$ when fermions are involved), plays the role of the gauge fields in Yang-Mills theory.

In the Clifford algebra $Cl(1,3)$ formulation of Einstein-Cartan gravity,[21,22,23] the vierbein $\hat{e}_\mu$ and spin connection Lorentz gauge field $\hat{\omega}_\mu$ take values in the Clifford algebraic space

$$\hat{e}_\mu = e_\mu^a \gamma_a, \tag{1.5a}$$

$$\hat{\omega}_\mu = \frac{1}{4}\omega_\mu^{ab}\gamma_a\gamma_b, \tag{1.5b}$$

where $a, b, \mu = 0, 1, 2, 3, \omega_\mu^{ab} = -\omega_\mu^{ba}$. Throughout this paper the summation convention for repeated indices is adopted. The four distinct $\{\gamma_a\}$ and six distinct $\{\gamma_a\gamma_b; a < b\}$ are called vectors and bi-vectors of Clifford algebra. We denote vierbein as $\hat{e}_\mu$ and spin connection as $\hat{\omega}_\mu$ rather than e_μ and ω_μ to accentuate the fact that they are Clifford-valued. The space-time metric $g_{\mu\nu}$ is derived from the vierbein

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \tag{1.6}$$

Thus the vierbein $\hat{e}_\mu$ can be deemed as the "square root" of metric.

Given that the gravity-related fields $\hat{e}_\mu$ and $\hat{\omega}_\mu$ are vector-valued and bi-vector-valued respectively in the Clifford algebraic space of $Cl(1,3)$, it's tempting to wonder whether the other interactions in nature such as the electroweak and strong gauge fields can take values in the Clifford algebraic space as well. The answer is a resounding yes, provided that one has to go beyond the confines of the familiar Clifford algebra $Cl(1,3)$. Learning from the above experience that we arrived at $Cl(1,3)$ via splitting the imaginary number into four operators, we may go one step further by decomposing Dirac's gamma zero operator into its underlying components[13]

$$\gamma_0 = \Gamma_1 \Gamma_2 \Gamma_3, \tag{1.7}$$

where the additional trio of gamma operators $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ satisfy the anticommutation relations

$$\{\Gamma_i, \Gamma_j\} = -2\delta_{ij}, \tag{1.8}$$

and anticommute with the original trio $\{\gamma_1, \gamma_2, \gamma_3\}$

$$\{\gamma_i, \Gamma_j\} = 0. \tag{1.9}$$

Collectively, these six elements

$$\Gamma_1, \Gamma_2, \Gamma_3, \gamma_1, \gamma_2, \gamma_3, \tag{1.10}$$

constitute the orthonormal vector basis of the *real* Clifford algebra $Cl(0,6)$, which is sometimes labeled as $Cl_{0,6}$ or $Cl_{0,6}(R)$ in the literature.

Thanks to the recognition of γ_0 as a composite tri-vector, we are able to extend Dirac's Clifford algebra from $Cl(1,3)$ to $Cl(0,6)$. With it, we can define an algebraic spinor as a linear combination of all $2^6 = 64$ basis elements of $Cl(0,6)$. Considering that there are 16 Weyl fermions with $16 \times 2 = 32$ complex components (i.e. 64 real components) within each of the three fermion families including right-handed neutrinos, an algebraic spinor of the *real* $Cl(0,6)$ with 64 degrees of freedom is a perfect match for representing one generation of fermions. The geometrical wealth of Clifford algebra $Cl(0,6)$ can be exploited to establish a theory covering both the standard model and gravity.[13,15,16] The Yang-Mills gauge fields and the spin connection Lorentz gauge field which governs gravity are associated with beyond the standard model (BSM) local gauge groups

$$Spin(1,3)_L \times Spin(1,3)_R \times Spin(1,3)_{WL} \times Spin(1,1)_{WR} \times U(1)_{WR} \times SU(3)_C \times U(1)_{B-L}, \tag{1.11}$$

where $Spin(1,3)_L$ and $Spin(1,3)_R$ are left- and right-handed local Lorentz gauge groups, and $Spin(1,3)_{WL}$ subsumes the left-handed weak gauge group $SU(2)_{WL}$.

It's worth mentioning that Clifford algebra $Cl(0,6)$ is capable of accommodating some enveloping groups. For example, the real symplectic group $Sp(8,R)$, which encompasses the Pati-Salam[24] $SU(4)$, is embedded in the $Cl(0,6)$ geometric structure. The Pati-Salam

$SU(4)$, which contains $SU(3)_C \times U(1)_{B-L}$, is isomorphic to the six-dimensional rotation group $Spin(6)$ generated by all the 15 bi-vectors of $Cl(0,6)$. We settled for a parsimonious set of subgroups due to lack of experimental evidence supporting any larger unification groups such as $SU(4)$, whereas there are various clues suggestive of the BSM symmetries (1.11).

We propose that all the regular Lagrangian terms, be it the fermion kinetic term or the cosmological constant term, are of quantum condensation origin. The cosmological constant problem can thus be evaded if we take abundant precaution in the renormalization procedure that entails multiplications of divergent integrals. We regard the vierbein field $\hat{e}_\mu$ as an composite entity emerging from the fermion-antifermion quantum condensation. Hence the fermion fields are the origin of space-time metric. Since the vierbein can be viewed as the "square root" of metric, fermions can be considered as the "quarter root" of metric. Because of the chirality of the vierbein-related condensations and the chirality of spin connection fields, there are left- and right-handed gravitational interactions. Additionally, a new kind of interaction is permitted between the extended vierbeins and the extended weak gauge fields, which may have cosmological implications.

There is a cascade of spontaneous symmetry breaking (SSB) processes, resulted from nonzero vacuum expectation values (VEV) acquired by the Clifford-valued vierbeins and Higgs fields. The first stage of SSB starts with the vierbein acquiring nonzero VEVs. As a result, the local Lorentz and pseudo-weak symmetries are lost and we are left with a global Lorentz symmetry.

The next step of SSB is triggered by the Majorana Higgs field, which is a Higgs-like field in addition to the standard model Higgs field. At this stage, the Majorana Higgs field assumes a nonzero VEV and breaks the local gauge symmetries down to the standard model symmetries. Consequently, the neutrino is endowed with a lepton number-conserving Majorana mass which is much heavier than the Dirac mass. A very small effective mass can thus be derived for the neutrino via the seesaw mechanism.[25]

At the last stage of SSB, the electroweak Higgs fields acquire nonzero VEVs and break the standard model symmetries down to $SU(3)_C \times U(1)_{EM}$, where $U(1)_{EM}$ is the electromagnetic gauge symmetry. The fermion Dirac mass hierarchies can be explained by a three-Higgs-doublet model with the help from two additional *global* $U(1)_\alpha$ ($U(1)_{PQ}$) and $U(1)_\beta$ symmetries involving right-handed fermions.

One point we want to highlight is that all the ingredients of our model, such as fermions, gauge fields, vierbein, and Higgs fields, share the same Clifford algebraic space of $Cl(0,6)$. For instance, the electromagnetic gauge field $\hat{A}_\mu$ is pseudoscalar-valued

$$\hat{A}_\mu = qA_\mu I, \tag{1.12}$$

where we include charge q (such as $q = -1$ for electron) into the definition of $\hat{A}_\mu$. The pseudoscalar I is the 6-vector of $Cl(0,6)$ after expanding the tri-vector γ_0 into its constituents

$$I = \Gamma_1\Gamma_2\Gamma_3\gamma_1\gamma_2\gamma_3. \tag{1.13}$$

The electromagnetic gauge field $\hat{A}_\mu$ as shown above and the gravity-related fields $\hat{\omega}_\mu/\hat{e}_\mu$ in eq. (1.5) take values in various Clifford algebraic subspaces with the same six gamma

operators in (1.10) as the unifying building blocks. By virtue of the shared Clifford algebraic space, the gauge field of gravity and the Yang-Mills fields are connected with each other through their interactions with the common fermion fields and Higgs fields.

Therefore, the theory outlined in this paper is a cohesive fusion of the standard model and gravity. Note that the Clifford algebra approach differs from the conventional way of postulating upfront a grand unified symmetry which demands that the gauge coupling constants are unified. The Clifford algebra approach proposes a fixed Clifford algebra at the first step, which means that the fermion contents of the model are essentially predetermined by the Clifford space. At the second step, one explores the allowable symmetries that preserve the invariance properties of spinor bilinears. Hence the symmetries of the model are in a sense derived instead of postulated. The permitted symmetries usually involve a direct product of different groups, which suggests that the individual gauge coupling constants are not necessarily related to each other. As such, the fusion delineated in this paper is more of a union via spinors and less of a union via symmetry groups.

This paper is structured as follows: In Section 2, we introduce the algebraic spinors of $Cl(0,6)$ and explore beyond the standard model gauge symmetries. In Section 3, we investigate spontaneous symmetry breaking due to the non-degenerate vacuum expectation values of various bosonic fields, and study the fermion mass hierarchies and the lepton number-conserving Majorana mass. In Section 4, with the goal of addressing the Higgs mass naturalness problem and the cosmological constant problem, we propose that all the symmetry-breaking bosonic fields including the Higgs and vierbein fields are the effective representations of multi-fermionic condensations. In the last section we draw our conclusions. Throughout this paper, we adopt the units $c = \hbar = 1$.

2 Clifford algebra Cl(0,6) and symmetries

2.1 The algebraic spinor representation of fermions

As mentioned in the introduction section, one generation of the standard model fermions can be represented by the algebraic spinor of Clifford algebra $Cl(0,6)$.[13,15,16] The goal of this section is to demonstrate how individual fermions, such as electrons, neutrinos, and quarks, are linked to the algebraic spinor without resorting to the traditional column representation of fermions.

For $Cl(0,6)$, there are $\binom{6}{k}$ independent k-vectors. To wit, there are one single scalar 1 which is a 0-vector, 6 vectors (e.g. γ_1), 15 bi-vectors (e.g. $\gamma_1\gamma_2$), 20 tri-vectors (e.g. $\gamma_0 = \Gamma_1\Gamma_2\Gamma_3$), 15 4-vectors (e.g. $\gamma_1\gamma_2 I$), 6 5-vectors (e.g. $\gamma_1 I$), and finally one single pseudoscalar I which is a 6-vector. In total, there are $2^6 = 64$ independent basis elements given by the set of all k-vectors. The algebraic spinor ψ is a multivector which can be expressed as a linear combination of all the 64 basis elements

$$\psi = \psi_1 + \psi_2\Gamma_1 + \psi_3\Gamma_2 + \psi_4\Gamma_3 + \psi_5\gamma_1 + \cdots + \psi_{64}I, \qquad (2.1)$$

where the 64 linear combination coefficients $\{\psi_n; n = 1, 2, \cdots, 64\}$ are *super-real* Grassmann numbers, which satisfy the complex conjugation relation

$$\psi_n^* = \psi_n. \tag{2.2}$$

A few comments are in order at this point regarding the super-real Grassmann numbers ψ_n. First of all, due to the fermion nature of the algebraic spinor ψ, it's mandatory that ψ_n should be Grassmann-odd. This requirement is not obvious when we write down the Dirac equation for ψ, where there is no multiplication between spinors. However, the Grassmann-odd characteristic of ψ becomes essential when it comes to the Lagrangian involving multiplication between spinors. As we will learn later in this paper, the Majorana mass Lagrangian term is allowed only if the algebraic spinor is Grassmann-odd, or otherwise the Majorana mass Lagrangian term is identically zero.

Secondly, it's customary to adopt complex Grassmann numbers in the conventional QFT where super-real condition (2.2) does not hold. One way to make contact with the conventional complex Grassmann numbers is to reorganize ψ_n in pairs, such as

$$\psi_0 + \psi_{64}I, \tag{2.3}$$

where the pseudoscalar I can be a proxy for the imaginary number i. Therefore, $\psi_0 + \psi_{64}I$ is tantamount to the conventional complex Grassmann number.

It's also worth noting that there is an interesting connection between the *complex* Clifford algebra $Cl_6(C)$ and the octonions via the left-action maps.[26] Nevertheless, in our approach we will stick to the *real* $Cl(0,6)$.

The algebraic spinor ψ of the real $Cl(0,6)$ with $2^6 = 64$ components corresponds to the union of all 16 Weyl fermions in one fermion generation of the standard model (plus right-handed neutrino) endowed with $16 \times 2 = 32$ complex components (i.e. 64 real components). For most part of this paper, our discussion is restricted to one generation/family of fermions. The Clifford algebraic structure of the fermions as well as the gauge symmetries are the same for the three generations. In other words, the three families are three replicas. Section 3.4 on fermion mass hierarchy is the only exception where we propose three electroweak symmetry-breaking Higgs fields that couple to the three generations of fermions in different patterns.

How do we connect one generation of fermions, such as electrons, neutrinos, and quarks, with ψ? First of all, let's distinguish between the left-handed and right-handed fermions in the setting of Clifford algebra. We propose that fermions with left (right) chirality correspond to Clifford-odd (even) portion of ψ

$$\psi_L = \frac{1}{2}(\psi + I\psi I), \tag{2.4}$$

$$\psi_R = \frac{1}{2}(\psi - I\psi I), \tag{2.5}$$

That is to say, the left-handed ψ_L is composed of

$$\psi_L = \psi_2 \Gamma_1 + \psi_3 \Gamma_2 + \cdots + \psi_{63} \gamma_3 I, \tag{2.6}$$

whilst the right-handed ψ_R is composed of

$$\psi_R = \psi_1 + \psi_8 \Gamma_1 \Gamma_2 + \cdots + \psi_{64} I. \tag{2.7}$$

The above projection of Clifford-odd (even) portion of ψ leverages the property that the pseudoscalar I anticommutes with Clifford-odd elements, and commutes with Clifford-even elements

$$I\psi_L = -\psi_L I,$$
$$I\psi_R = \psi_R I.$$

The operation of $-I\psi I$ as in eq. (2.5) can be mapped to the traditional $\gamma^5 \psi$ operation, where $-I$ as in $-I\psi$ plays the role of the conventional pseudoscalar (more discussion will be provided on the minus sign of $-I$ in Section 3.1), while I as in ψI plays the role of the conventional imaginary number i. Note that the positioning of I relative to the algebraic spinor ψ does matter. This speaks to the fact that the electromagnetic field, as a pseudoscalar in eq. (1.12), should always be applied to the right side of a spinor so that I is equivalent to the imaginary number i in the traditional setting. And for that matter, when we attempt to link the traditional complex Grassmann numbers with the super-real Grassmann numbers as in eq. (2.3), we ought to make sure that they appear to the right side of any Clifford elements.

Now we are ready to identify ψ with electrons, neutrinos, and quarks. Specifically, the projection operators for the three colors of red, green, and blue quarks are given by

$$P_r = \frac{1}{4}(1 + I\gamma_1\Gamma_1 - I\gamma_2\Gamma_2 - I\gamma_3\Gamma_3), \tag{2.8a}$$

$$P_g = \frac{1}{4}(1 - I\gamma_1\Gamma_1 + I\gamma_2\Gamma_2 - I\gamma_3\Gamma_3), \tag{2.8b}$$

$$P_b = \frac{1}{4}(1 - I\gamma_1\Gamma_1 - I\gamma_2\Gamma_2 + I\gamma_3\Gamma_3), \tag{2.8c}$$

while the lepton projection operator is defined as

$$P_l = \frac{1}{4}(1 + I\gamma_1\Gamma_1 + I\gamma_2\Gamma_2 + I\gamma_3\Gamma_3). \tag{2.8d}$$

In the context of $SU(4)$, the lepton projection operator P_l can be regarded as the projection to the fourth color.[24] The four color projections P_l, P_r, P_g, P_b are orthogonal to each other and satisfy

$$P_l + P_r + P_g + P_b = 1. \tag{2.9}$$

Note that the bi-vectors $\gamma_i\Gamma_i$ appearing in the color projectors suggest an interesting interplay between the trialities of $\{\gamma_1,\ \gamma_2,\ \gamma_3\}/\{\Gamma_1,\ \Gamma_2,\ \Gamma_3\}$ and three colors of quarks. Figuratively speaking, the three colors of quarks are tied to the three space dimensions, whilst the lepton as the fourth color is tied to the time dimension. In Section 2.2, we will further explore the significance of these color projectors in terms of carving out the color group $SU(3)_C$ from Pati-Salam's $SU(4)$.

For the purpose of differentiating between weak isospin up-type and down-type fermions, we introduce another set of orthogonal projection operators

$$P_\pm = \frac{1}{2}(1 \pm I\Gamma_1\Gamma_2), \tag{2.10}$$

which sum up to

$$P_+ + P_- = 1. \tag{2.11}$$

We identify projections of the algebraic spinor ψ

$$\psi = \psi_L + \psi_R = (P_+ + P_-)(\psi_L + \psi_R)(P_l + P_r + P_g + P_b), \tag{2.12}$$

with left-handed neutrino, electron, and quarks

$$\begin{cases} \nu_L = P_+\psi_L P_l, \\ u_{L,r} = P_+\psi_L P_r, & u_{L,g} = P_+\psi_L P_g, & u_{L,b} = P_+\psi_L P_b, \\ e_L = P_-\psi_L P_l, \\ d_{L,r} = P_-\psi_L P_r, & d_{L,g} = P_-\psi_L P_g, & d_{L,b} = P_-\psi_L P_b, \end{cases} \tag{2.13}$$

and right-handed neutrino, electron, and quarks

$$\begin{cases} \nu_R = P_-\psi_R P_l, \\ u_{R,r} = P_-\psi_R P_r, & u_{R,g} = P_-\psi_R P_g, & u_{R,b} = P_-\psi_R P_b, \\ e_R = P_+\psi_R P_l, \\ d_{R,r} = P_+\psi_R P_r, & d_{R,g} = P_+\psi_R P_g, & d_{R,b} = P_+\psi_R P_b. \end{cases} \tag{2.14}$$

Note that the definition of isospin I_3 for a given standard model fermion ψ_f is given by

$$\psi_f(I_3 I) = \frac{1}{2}\Gamma_1\Gamma_2\psi_f. \tag{2.15}$$

With the help from the property that

$$\Gamma_1\Gamma_2 P_\pm \psi_f = \mp I P_\pm \psi_f, \tag{2.16}$$

it can thus be verified that the isospin values of the fermions in eq. (2.13) and eq. (2.14) are consistent with those of the standard model. When the I in eq. (2.16) appearing on the left side of ψ_f is moved to the right side of ψ_f as in eq. (2.15), its sign is changed for Clifford-odd left-handed fermions. This is the underlying reason why there is a flip of sign in $P_\pm$ between the left- and right-handed fermions when $P_\pm$ is assigned to the isospin up-type and down-type fermions respectively in eq. (2.13) and eq. (2.14).

It's also worth mentioning that attempts have been made to associate species of fermions with (minimal) left ideals of the Clifford algebraic spinor[10, 11] where projection operators are restricted to acting on the right-hand side of an algebraic spinor. Obviously, our fermion assignment scheme above departs from the (minimal) left ideal approach.

In summary, we have identified individual fermions with the projections of the $Cl(0,6)$ algebraic spinor without any reference to the column representation. The mappings between the $Cl(0,6)$ formulation and the conventional matrix/column representation can be worked out,[13] which will not be detailed in this paper.

2.2 Beyond the standard model symmetries

As explained in the introduction section, we strive to walk a careful line between being too ambitious and being too conservative when it comes to choosing the suitable gauge symmetries for our model. The aim of this subsection is to give an account of our thought process in selecting the BSM symmetry groups employed in this paper.

The conventional way of model building is to postulate the symmetry group upfront, and then proceed to find the fermion representation. In the case of Clifford algebra approach, it's the other way around. We choose the Clifford algebra $Cl(0,6)$ as the first step which is tantamount to staking out the fermion space. The allowable symmetry groups are thus tightly constrained by the spinor space. This is a desirable feature of the Clifford algebra approach, since the symmetry groups are in a sense derived, rather than postulated.

So how do we determine the allowable symmetries? It hinges on the spinor bilinear

$$\left\langle \bar{\psi}\psi \right\rangle_{1,I}, \tag{2.17}$$

where $\langle \ldots \rangle_{1,I}$ stands for the Clifford-scalar and -pseudoscalar parts of the enclosed expression. It is the Clifford algebraic counterpart of the conventional Dirac inner product $\bar{\psi}\psi$ between the column spinors. For the rest of the paper, we will exclusively use $\langle \ldots \rangle$ which stands for the Clifford-scalar part of the enclosed expression, since we can get the pseudoscalar part via $\langle I \ldots \rangle$ if needed. The Dirac conjugate $\bar{\psi}$ in eq. (2.17) is defined as

$$\bar{\psi} = \psi^\dagger \gamma_0, \tag{2.18}$$

and the Hermitian conjugate satisfies

$$(AB)^\dagger = B^\dagger A^\dagger, \tag{2.19}$$

for any A and B valued in Clifford algebraic space, regardless of A and B being Grassmann-even or Grassmann-odd. With the six $Cl(0,6)$ basis vectors defined as anti-Hermitian, the Hermitian conjugate of any Clifford element can thus be determined by recursively applying (2.19). For example,

$$\gamma_0^\dagger = (\Gamma_1 \Gamma_2 \Gamma_3)^\dagger = \Gamma_3^\dagger \Gamma_2^\dagger \Gamma_1^\dagger = -\Gamma_3 \Gamma_2 \Gamma_1 = \gamma_0. \tag{2.20}$$

Since the fermion Lagrangian comprises the Dirac inner product $\left\langle \bar{\psi}\psi \right\rangle$ or some variants thereof, the allowable symmetry transformations are the ones under which these sorts of Dirac inner products are invariant. Let's start with the general gauge transformation

$$\psi \quad \rightarrow \quad V\psi U, \tag{2.21}$$

where the Clifford-valued V and U are two *independent* gauge transformations. As an example, the following Dirac inner product transforms as

$$\left\langle \bar{\psi}I\psi I \right\rangle \quad \rightarrow \quad \left\langle (U^\dagger \psi^\dagger V^\dagger)\gamma_0 I(V\psi U)I \right\rangle = \left\langle \psi^\dagger (V^\dagger \gamma_0 IV)\psi(UIU^\dagger) \right\rangle, \tag{2.22}$$

which is invariant if

$$V^\dagger \gamma_0 I V = \gamma_0 I, \qquad (2.23)$$

$$U I U^\dagger = I, \qquad (2.24)$$

where we have used the property $\langle AB \rangle = \langle BA \rangle$ provided that A and B are Grassmann-even. If we restrict our discussion to gauge transformations continuously connected to identity, the general solution of the above equations are

$$V = e^{\theta^n(x) T_n}, \qquad (2.25)$$

$$U = e^{\epsilon^n(x) K_n}, \qquad (2.26)$$

where $\{T_n; n = 1, \cdots, 28\}$ are the generators of the Spin group $Spin(4,4)$ (double cover of the rotation group $SO(4,4)$) which consist of

$$\gamma_a, \gamma_a \gamma_b, \Gamma_a \Gamma_b, I \Gamma_i, \gamma_i \gamma_j \Gamma_k, \qquad (2.27)$$

and $\{K_n; n = 1, \cdots, 36\}$ are the generators of the real symplectic group $Sp(8, R)$, which comprise all 15 bi-vectors, all 20 tri-vectors, and the pseudoscalar

$$\gamma_i \gamma_j, \Gamma_i \Gamma_j, \gamma_i \Gamma_k, \gamma_0, \Gamma_0, \gamma_i \gamma_j \Gamma_k, \Gamma_i \Gamma_j \gamma_k, I, \qquad (2.28)$$

where i, j, k = 1,2,3, a, b = 0,1,2,3, i > j, a > b. The tri-vector Γ_0 is defined as $\Gamma_0 = \gamma_1 \gamma_2 \gamma_3$, thus the basis $\{\Gamma_a; a = 0, \cdots, 3\}$ parallels $\{\gamma_a; a = 0, \cdots, 3\}$. Note that the transformation parameters $\theta^n(x)$ and $\epsilon^n(x)$ are space-time dependent, since we are dealing with *local* gauge transformations. For brevity sake, we will omit the x label hereafter with the understanding that the space-time dependency is implied.

A few comments are in order. First of all, in the literature the usual candidates for gauge transformations are bi-vector-related rotations. The above T_n and K_n involve non-bi-vector Clifford elements such as γ_0 (tri-vector), γ_i (i=1,2,3, vector) and the Lorentz boosts γ_{0i} (4-vector) in T_n, which go beyond the confines of bi-vectors. While the traditional rotation transforms a vector into another vector, the generalized "rotation" via the non-bi-vector-valued Clifford elements could potentially transform vectors into multivectors. Secondly, the two gauge transformations V and U are *independent* of each other. They are applied to the left side and right side of the algebraic spinor ψ, respectively. The availability of the double-sided gauge transformations is one of the advantages of the Clifford algebra approach compared with the conventional column fermion formalism, which has historically been leveraged in Clifford algebraic models.[12]

One interesting observation is that T_n contains the 10 generators of de Sitter group $SO(1,4)$, namely

$$\gamma_a, \gamma_a \gamma_b. \qquad (2.29)$$

We know that there is another flavor of gauge gravity theory which is based on the (anti-) de Sitter group.[27] It enjoys the advantage that the vierbein $\hat{e}_\mu$ and spin connection $\hat{\omega}_\mu$ in eq. (1.5) jointly constitute the gauge fields of de Sitter group. As a comparison, in the

Lorentz gauge gravity theory,[19,20] the vierbein $\hat{e}_\mu$ is not a bona fide gauge field, albeit the spin connection $\hat{\omega}_\mu$ is indeed the gauge field of Lorentz group. The vierbein is instead regarded as an add-on to the spin connection. This incentivized us to adopt the (anti-) de Sitter gauge theory of gravity in our first paper[13] on Clifford algebra $Cl(0,6)$. However, there is a downside with this approach: given that the de Sitter group generators γ_a are Clifford-odd, the associated transformations mix the left- and right-handed spinors which are Clifford-odd and -even, respectively. Consequently, the left- and right-handed spinors have to transform in sync, which disagrees with the chirality of weak interaction. This is a major shortcoming of our first paper.

One way of circumventing the above limitation is to demand that only the Clifford-even sub-algebras of T_n and K_n are permitted, which enables us to accommodate the chirality of weak interaction by virtue of decoupling the gauge transformations of the left- and right-handed spinors. The trade off is that we have to settle for the Lorentz gauge gravity theory where vierbein $\hat{e}_\mu$ is an add-on transforming as a vector of the Lorentz symmetry. It is seemingly a disadvantage compared with the (anti-) de Sitter gauge gravity theory. However, in Sections 4.3 and 4.4 we will learn that it is a blessing in disguise, since the vierbein is never meant to be a gauge field. It's actually an emergent quantity arising from the quantum condensation of a fermion-antifermion pair.

With the restriction to the Clifford-even sub-algebras of T_n and K_n, we are left with the following symmetries[15]

$$\psi_L \quad \rightarrow \quad V_L \psi_L U_L, \tag{2.30}$$

$$\psi_R \quad \rightarrow \quad V_R \psi_R U_R, \tag{2.31}$$

where

$$V_L = e^{\theta_L^n T_n}, \quad U_L = e^{\epsilon_L^n K_n}, \tag{2.32}$$

$$V_R = e^{\theta_R^n T_n}, \quad U_R = e^{\epsilon_R^n K_n}. \tag{2.33}$$

The Clifford-even $\{T_n; n = 1, \cdots, 12\}$ comprise the generators of $Spin(1,3) \times Spin(1,3)_{Weak}$

$$\gamma_a \gamma_b; \quad \Gamma_a \Gamma_b, \tag{2.34}$$

where $a, b = 0, 1, 2, 3, a > b$, and the Clifford-even $\{K_n; n = 1, \cdots, 17\}$ comprise the generators of $Spin(6)_{Pati-Salam} \times U_I(1)$

$$\gamma_i \gamma_j, \Gamma_i \Gamma_j, \gamma_i \Gamma_k; \quad I, \tag{2.35}$$

where $i, j, k = 1, 2, 3, i > j$. Note that the gauge transformation parameters such as θ_L^n (ϵ_L^n) and θ_R^n (ϵ_R^n) are independent of each other. Therefore the left- and right-handed spinors ψ_L and ψ_R transform independently. Considering that the same copies of the symmetry groups T_n and K_n are employed for the left- and right-handed spinors, the model is left-right symmetric.

Since the vierbein $\hat{e}_\mu$ transforms as a vector under the Lorentz gauge transformation

$$\hat{e}_\mu \quad \rightarrow \quad e^{\frac{1}{4}\theta^{ab}\gamma_a\gamma_b} \, \hat{e}_\mu \, e^{-\frac{1}{4}\theta^{ab}\gamma_a\gamma_b}, \tag{2.36}$$

a spinor bilinear term such as

$$\langle \bar{\psi} \hat{e}_\mu \psi \rangle , \qquad\qquad (2.37)$$

can be verified to be gauge invariant under the above gauge transformations restricted to the Clifford-even sub-space. This invariance is of paramount importance, give that the kinetic part of the fermion Lagrangian is of similar form and should respect these symmetries.

The above gauge symmetry groups, with $(12 + 17) * 2 = 58$ group generators, are the largest ones permissible by a chiral algebraic spinor of $Cl(0,6)$. The spin group $Spin(6)_{Pati-Salam}$ is generated by all the 15 bi-vectors of $Cl(0,6)$. It is isomorphic to Pati-Salem's $SU(4)$,[24] which encompasses $SU(3)_C \times U(1)_{B-L}$. The $Spin(1,3)_{Weak}$ group comprises the regular weak group generated by $\{\Gamma_i \Gamma_j\}$ as well as the weak boosts generated by $\{\Gamma_0 \Gamma_i\}$. These are the counterparts of the spacial rotations generated by $\{\gamma_i \gamma_j\}$ and the Lorentz boosts generated by $\{\gamma_0 \gamma_i\}$. Given that there are two copies of symmetry groups, there could be two different spin connections acting on the left- and right-handed spinors, respectively. In other words, there could be two separate left- and right-handed gravitational interactions.

The BSM symmetry bonanza noted above are tantalizing. From a practical point of view, do we have any inkling of symmetries beyond the local Lorentz and standard model gauge symmetries? The observation of neutrino oscillations[28, 29, 30] provided an interesting clue. It implies that neutrinos have nonzero masses beyond the plain vanilla standard model. Curiously, the neutrino masses are much smaller than that of the other standard model fermions. The seesaw mechanism is hence proposed as an explanation,[25] which invokes the right-handed neutrinos endowed with large Majorana masses. In light of these suggestive evidences, we whittle down to a minimum subset of groups which could accommodate a Higgs-like mechanism to generate the Majorana masses.

Therefore, our choice of symmetry groups are

$$Spin(1,3) \times SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}. \qquad\qquad (2.38)$$

Note that in Section 4.4 on emergent chiral vierbeins, we will expand the above groups to accommodate the extended vierbeins and the extended weak interaction sector. But for now, we will stay with the above unextended symmetry groups. These symmetry groups are the direct product of the spin connection's $Spin(1,3)$ with six distinct generators (as in $\psi \to e^{\theta^n T_n} \psi$)

$$\frac{1}{2} \gamma_a \gamma_b, \qquad\qquad (2.39)$$

where $a, b = 0, 1, 2, 3, a > b$, and the left-handed weak interaction's $SU(2)_{WL}$ with three generators (as in $\psi_L \to e^{\theta_L^n T_n} \psi_L$)

$$\frac{1}{2} \Gamma_2 \Gamma_3, \quad \frac{1}{2} \Gamma_3 \Gamma_1, \quad \frac{1}{2} \Gamma_1 \Gamma_2, \qquad\qquad (2.40)$$

and the right-handed weak interaction's $U(1)_{WR}$ with one generator (as in $\psi_R \to e^{\theta_R^n T_n} \psi_R$)

$$\frac{1}{2} \Gamma_1 \Gamma_2, \qquad\qquad (2.41)$$

and the strong interaction's $SU(3)_C$ with eight generators (as in $\psi \rightarrow \psi e^{\epsilon^n K_n}$)

$$\begin{pmatrix} \frac{1}{4}(\gamma_1\Gamma_2 + \gamma_2\Gamma_1), & \frac{1}{4}(\Gamma_1\Gamma_2 + \gamma_1\gamma_2), & \frac{1}{4}(\Gamma_1\gamma_1 - \Gamma_2\gamma_2), \\ \frac{1}{4}(\gamma_1\Gamma_3 + \gamma_3\Gamma_1), & \frac{1}{4}(\Gamma_1\Gamma_3 + \gamma_1\gamma_3), \\ \frac{1}{4}(\gamma_2\Gamma_3 + \gamma_3\Gamma_2), & \frac{1}{4}(\Gamma_2\Gamma_3 + \gamma_2\gamma_3), \\ \frac{1}{4\sqrt{3}}(\Gamma_1\gamma_1 + \Gamma_2\gamma_2 - 2\Gamma_3\gamma_3), \end{pmatrix} \tag{2.42}$$

and the BL interaction's $U(1)_{B-L}$ with one generator (as in $\psi \rightarrow \psi e^{\epsilon^n K_n}$)

$$\frac{1}{2}J = \frac{1}{6}(\gamma_1\Gamma_1 + \gamma_2\Gamma_2 + \gamma_3\Gamma_3). \tag{2.43}$$

Note that some multipliers are applied to the generators to facilitate the gauge field definitions in Section 2.4. Due to the chirality of the weak interaction, the gauge transformation parameters for the left- and right-handed spinors (θ_L^n and θ_R^n) are kept independent for $SU(2)_{WL}$ and $U(1)_{WR}$, whereas the other gauge transformation parameters of $Spin(1,3)$, $SU(3)_C$, and $U(1)_{B-L}$ are synchronized between the left- and right-handed spinors.

Three Clifford elements are pivotal in the symmetry determination. The first element γ_0 is hard-wired into the definition of Dirac inner product $\langle \bar{\psi}\psi \rangle = \langle \psi^\dagger \gamma_0 \psi \rangle$. It facilitates pinning down the Lorentz group $Spin(1,3)$. The second element $\Gamma_1\Gamma_2$ is embedded in the definition of isospin (2.15), thus it picks out the isospin direction. The third critical Clifford element is the BL interaction's J. It is instrumental in separating out $U(3) = SU(3)_C \times U(1)_{B-L}$ from the encompassing $Spin(6)_{Pati-Salam}$ which is isomorphic to $SU(4)$. Mathematically speaking, this is a specific case of a general procedure[31, 5] of separating out $U(n)$ from $Spin(2n)$.

When J is applied to the four color projection operators (2.8), it has the nice property that

$$P_l J = -P_l I, \tag{2.44a}$$

$$P_r J = \frac{1}{3}P_r I, \quad P_g J = \frac{1}{3}P_g I, \quad P_b J = \frac{1}{3}P_b I, \tag{2.44b}$$

which means that J is tantamount to

$$J = (B - L)I, \tag{2.45}$$

where B and L are baryon and lepton numbers, respectively. Therefore, J indeed corresponds to the BL interaction. The definition of the four color projection operators (2.8) as well as the definition of the color algebra (2.42) are both predicated on how J is structured. The ansatz ensures that applying any generator in the color algebra to the lepton projector P_l is identical to zero, hence leptons are invariant (singlets) under the color gauge transformations.

If there were grand unification symmetries, the three Clifford elements γ_0, $\Gamma_1\Gamma_2$, and J could have emerged from vacuum expectation values (VEVs) of some hitherto unknown symmetry-breaking fields. That said, if we start from our choice of symmetries as a given

$$Spin(1,3) \times SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}, \tag{2.46}$$

the symmetry breaking patterns can be readily worked out and will be elaborated in Section 3.

In a nutshell, the cascade of spontaneous symmetry breaking (SSB) begins with the vierbein acquiring a nonzero VEV, which breaks the gauge symmetries down to

$$SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}. \tag{2.47}$$

As a result, the local Lorentz gauge symmetry is lost and we are left with a global Lorentz symmetry. The next step of symmetry breaking is triggered by the Majorana Higgs field, which is a Higgs-like field in addition to the standard model Higgs field. At this stage, the Majorana Higgs field assumes a nonzero VEV and breaks the local gauge symmetries down to the standard model symmetries

$$SU(3)_C \times SU(2)_{WL} \times U(1)_Y, \tag{2.48}$$

where $U(1)_Y$ is the hypercharge gauge symmetry specified by the synchronized double-sided gauge transformations

$$\psi_L \to \psi_L e^{\frac{1}{2}\epsilon_Y J}, \tag{2.49}$$

$$\psi_R \to e^{\frac{1}{2}\epsilon_Y \Gamma_1 \Gamma_2} \psi_R e^{\frac{1}{2}\epsilon_Y J}, \tag{2.50}$$

where a shared rotation angle ϵ_Y synchronizes the double-sided hypercharge gauge transformations. At the third stage of SSB, the electroweak Higgs fields acquire nonzero VEVs and break the standard model symmetries down to

$$SU(3)_C \times U(1)_{EM}, \tag{2.51}$$

where $U(1)_{EM}$ is the electromagnetic gauge symmetry characterized by the synchronized double-sided gauge transformation

$$\psi \to e^{\frac{1}{2}\epsilon_{EM} \Gamma_1 \Gamma_2} \psi e^{\frac{1}{2}\epsilon_{EM} J}, \tag{2.52}$$

where a shared rotation angle ϵ_{EM} synchronizes the double-sided gauge transformation. With the definition of the pseudoscalar-valued electromagnetic gauge field $\hat{A}_\mu$ in eq. (1.12), the electric charge q of a given standard model fermion ψ_f can thus be obtained from

$$e^{\frac{1}{2}\epsilon_{EM} \Gamma_1 \Gamma_2} \psi_f e^{\frac{1}{2}\epsilon_{EM} J} = \psi_f e^{q\epsilon_{EM} I}. \tag{2.53}$$

Thanks to the properties of $\Gamma_1 \Gamma_2$ in eq. (2.16) and J in eq. (2.44), the electric charges can be readily calculated as $q = 0, -1, \frac{2}{3}$, and $-\frac{1}{3}$ for neutrino, electron, up quarks, and down quarks according to the definitions in eq. (2.13) and eq. (2.14), which are perfectly aligned with the standard model electric charge assignments.

We shall underscore the fact that all the gauge group generators are valued in the *real* Clifford space, while the algebraic spinors are valued in the super-real Clifford space. Any reference of the imagine number i in the conventional formalism with regard to gauge transformations (and gauge fields) and spinors can be replaced by the pseudoscalar I acting on the right side of the spinor, as illustrated in the definition of electric charge above.

2.3 Charge conjugation without particle-antiparticle interchange

The charge conjugation C changes the sign of charges. In the conventional matrix formalism, the C conjugation of a fermion ψ in the Weyl basis is expressed as

$$C: \quad \psi \to \psi_c = -i\gamma_2 \psi^\star, \tag{2.54}$$

where $\psi^\star$ is the complex conjugate of ψ. Because of the complex conjugate operation, C converts a particle into its corresponding antiparticle.

Can we decouple charge conjugation from complex conjugate and thus evade particle-antiparticle interchange? Such a decoupling is indeed possible in the Clifford algebra approach, thanks to the identification of the imaginary number i with the pseudoscalar I acting on the right side of a spinor. Considering the definition of electric charge q in eq. (2.53), we can define a weaker form of charge conjugation

$$C': \quad \psi \to \psi_{c'} = (I\Gamma_2\Gamma_3)\psi\gamma_0. \tag{2.55}$$

Note that $I\Gamma_1\Gamma_2$ and γ_0 in the above definition could be replaced with the general $I\Gamma_2\Gamma_3 e^{\theta\Gamma_1\Gamma_2}$ and $\gamma_0 e^{\epsilon I}$, where θ and ϵ are two arbitrary phase factors. The weaker form of charge conjugation satisfies the property $(\psi_{c'})_{c'} = \psi$. It does not involve complex conjugate, hence there is no particle-antiparticle switching. In Section 3.2, this property of C' will be leveraged to construct a Majorana mass term that conserves lepton number.

According to eq. (2.53), it can be easily checked that $\psi_{c'}$ transforms as

$$\psi_{c'} \to \psi_{c'} e^{-q\epsilon_{EM}I}. \tag{2.56}$$

under the electromagnetic gauge transformation. Therefore, the sign of electric charge is changed. It's driven by the fact that γ_0 in the definition of C' anticommutes with the unit pseudoscalar I

$$e^{q\epsilon_{EM}I}\gamma_0 = \gamma_0 e^{-q\epsilon_{EM}I}. \tag{2.57}$$

This sort of mathematical acrobatics is otherwise impossible in the conventional formalism where the electromagnetic gauge transformation is associated with the imaginary number i as in eq. (1.2). Since i commutes with any operator, the only way to change sign of i is to invoke complex conjugate. Consequently, charge conjugation in the conventional formalism is inextricably linked to particle-antiparticle interchange.

It can be verified that C' does not change isospin (2.15) or color (2.8) of any standard model fermion. Since $I\Gamma_2\Gamma_3$ commutes with the Lorentz transformation generators $\gamma_a\gamma_b$, the Lorentz transformation properties of $\psi_{c'}$ remain the same as ψ. Hence, $\psi_{c'}$ is indeed the charge conjugation counterpart of the corresponding ψ. Note that C' changes the chirality of ψ, since C' involves the multiplication of the Clifford-odd tri-vector γ_0 and it turns a left-handed fermion into a right-handed one, and vice versa.

The charge conjugation C' of the gauge transformation parameters (and thus gauge fields) can be defined as

$$C': \quad \theta^n T_n \to \theta_{c'}^n T_n = \theta^n (I\Gamma_2\Gamma_3) T_n (I\Gamma_2\Gamma_3), \tag{2.58}$$

$$C': \quad \epsilon^n K_n \to \epsilon_{c'}^n K_n = \epsilon^n \gamma_0 K^n \gamma_0, \tag{2.59}$$

with the understanding that the left side-type T^n include the generators of $Spin(1,3) \times SU(2)_{WL} \times U(1)_{WR}$ and the right side-type K^n include the generators of $SU(3)_C \times U(1)_{B-L}$. Some notable examples are that the electromagnetic gauge field $\hat{A}_\mu$ is C'-odd, while the gravity-related spin connection Lorentz gauge field $\hat{\omega}_\mu$ is C'-even.

2.4   Lagrangian of the world

Having established the fermion representation and the symmetry structure, we are now well-positioned to investigate the Lagrangians and actions. With a view toward writing down the diffeomorphism-invariant actions, we are going to make extensive use of differential forms. We define gauge field as 1-forms and gauge forces as curvature 2-forms. Diffeomorphism-invariance can be assured if the action is expressed as an integration of 4-forms on the 4-dimensional space-time manifold.

The vierbein 1-from $\hat{e}$ and spin connection 1-form $\hat{\omega}$ of the Lorentz gauge theory of gravity are

$$\hat{e} = \hat{e}_\mu dx^\mu, \tag{2.60}$$

$$\hat{\omega} = \hat{\omega}_\mu dx^\mu, \tag{2.61}$$

where $\mu = 0, 1, 2, 3$, and the Clifford-valued $\hat{e}_\mu$ and $\hat{\omega}_\mu$ are given in eq. (1.5). The Clifford-valued forms are also called Clifforms in the literature.[21, 22]

In an similar fashion, the other gauge field 1-forms related to the gauge symmetries $SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}$ can be defined as

$$\hat{G} = \hat{G}_\mu dx^\mu, \tag{2.62}$$

$$\hat{W}_L = \hat{W}_{L\mu} dx^\mu, \tag{2.63}$$

$$\hat{W}_R = \hat{W}_{R\mu} dx^\mu, \tag{2.64}$$

$$\hat{A}_{BL} = \hat{A}_{BL\mu} dx^\mu, \tag{2.65}$$

where the strong interaction gauge field $\hat{G}_\mu$, the left-handed weak interaction gauge field $\hat{W}_{L\mu}$, the right-handed weak interaction gauge field $\hat{W}_{R\mu}$, and the BL interaction gauge field $\hat{A}_{BL\mu}$ are valued in the gauge generator space of $SU(3)_C$, $SU(2)_{WL}$, $U(1)_{WR}$, and $U(1)_{B-L}$, respectively. As an example, the left-handed weak interaction gauge field $\hat{W}_{L\mu}$ can be defined as

$$\hat{W}_{L\mu} = \frac{1}{2}(W_{L\mu}^1 \Gamma_2 \Gamma_3 + W_{L\mu}^2 \Gamma_3 \Gamma_1 + W_{L\mu}^3 \Gamma_1 \Gamma_2), \tag{2.66}$$

$$\tag{2.67}$$

The other gauge fields $\hat{G}_\mu$, $\hat{W}_R$, and $\hat{A}_{BL\mu}$ can be defined in a similar manner using the corresponding gauge group generators specified in Section 2.2. We adopt the notation convention (e.g. $\hat{G}$ rather than G) to highlight the fact that these gauge fields are *Clifford-valued* 1-forms.

The chiral gauge-covariant derivatives of the chiral spinor fields are defined as

$$D_L \psi_L = (d + \hat{\omega} + \hat{W}_L)\psi_L + \psi_L(\hat{G} + \hat{A}_{BL}), \tag{2.68}$$

$$D_R \psi_R = (d + \hat{\omega} + \hat{W}_R)\psi_R + \psi_R(\hat{G} + \hat{A}_{BL}), \tag{2.69}$$

where d is the exterior derivative. It is essential that the gauge fields should appear on the proper side of the spinor, which is dictated by how the gauge symmetries are defined in Section 2.2. Note that we follow the convention of putting the gauge coupling constant in the coefficient of the Yang-Mills Lagrangian, rather than in the gauge-covariant derivatives above.

The spin connection $\hat{\omega}$, as the gauge field of the Lorentz gauge group, is crucial in maintaining the *local* Lorentz gauge covariance of $D_L\psi_L$ and $D_R\psi_R$. On the other hand, given that the vierbein $\hat{e}$ is not a gauge field, $\hat{e}$ is conspicuously absent in the gauge-covariant derivatives of the spinor fields $\psi_{L/R}(x)$. Nonetheless, as we will learn below, $\hat{e}$ shows up in other parts of the Lagrangian and it plays a pivotal role in the model building.

The gauge interactions are formulated as curvature 2-forms, namely $\hat{R}$, $\hat{F}_G$, $\hat{F}_{WL}$, $\hat{F}_{WR}$, and $\hat{F}_{BL}$. For instance, the spin connection curvature 2-form $\hat{R}$ and the left-handed weak interaction curvature 2-form $\hat{F}_{WL}$ are expressed as

$$\hat{R} = d\hat{\omega} + \hat{\omega} \wedge \hat{\omega}, \tag{2.70}$$

$$\hat{F}_{WL} = d\hat{W}_L + \hat{W}_L \wedge \hat{W}_L = \frac{1}{2}\hat{F}_{WL\mu\nu}dx^\mu dx^\nu \tag{2.71}$$

$$= \frac{1}{4}(F_{WL\mu\nu}^1\Gamma_2\Gamma_3 + F_{WL\mu\nu}^2\Gamma_3\Gamma_1 + F_{WL\mu\nu}^3\Gamma_1\Gamma_2)dx^\mu dx^\nu, \tag{2.72}$$

where $\wedge$ stands for outer product between differential forms. For later usage we have expanded $\hat{F}_{WL}$ in more details. The other curvature 2-forms $\hat{F}_{WR}$, $\hat{F}_G$, and $\hat{F}_{BL}$ can be defined in a similar way. Note that the outer product term vanishes for abelian interactions such as $\hat{F}_{WR}$ and $\hat{F}_{BL}$.

Now we are ready to write down the local gauge- and diffeomorphism-invariant Lagrangian of the world

$$\mathcal{L}_{World} = \mathcal{L}_{Fermion} \tag{2.73}$$

$$+ \mathcal{L}_{Gravity} + \mathcal{L}_{CC} \tag{2.74}$$

$$+ \mathcal{L}_{YM-Color} + \mathcal{L}_{YM-Weak-Left} + \mathcal{L}_{YM-Weak-Right} + \mathcal{L}_{YM-BL} \tag{2.75}$$

$$+ \mathcal{L}_{Higgs-Majorana} + \mathcal{L}_{Higgs-Electroweak}, \tag{2.76}$$

where the fermion Lagrangian is of the form

$$\mathcal{L}_{Fermion} \sim i\left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \psi_L\overline{D_L\psi_L}\right\rangle + i\left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \psi_R\overline{D_R\psi_R}\right\rangle, \tag{2.77}$$

and the gravity plus cosmological constant Lagrangian terms are of the form

$$\mathcal{L}_{Gravity} + \mathcal{L}_{CC} = \frac{1}{8\pi G}\left\langle I(\hat{e} \wedge \hat{e} \wedge \hat{R} + \frac{\Lambda}{4!}\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e})\right\rangle, \tag{2.78}$$

where $\hat{R}$ is the spin connection curvature 2-form (2.70) and Λ is the cosmological constant. The left-handed weak interaction Yang-Mills Lagrangian term is of the form

$$\mathcal{L}_{YM-Weak-Left} \sim \frac{\left\langle (I\hat{e} \wedge \hat{e} \wedge \hat{F}_{WL})(I\hat{e} \wedge \hat{e} \wedge \hat{F}_{WL})\right\rangle}{\left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e}\right\rangle}. \tag{2.79}$$

The Yang-Mills-type Lagrangian terms for the other gauge interactions take a similar form. Here we adopted the generic "Yang-Mills-type" label, albeit historically the term Yang-Mills is reserved for non-abelian gauge fields only. Note that $\langle \ldots \rangle$ denotes the Clifford-scalar part of the enclosed expression, as we defined earlier. The Higgs field-related Lagrangian terms $\mathcal{L}_{Higgs-Majorana}$ and $\mathcal{L}_{Higgs-Electroweak}$ will be outlined in Sections 3.2 and 3.3 when we discuss the Higgs mechanism. Upon SSB, the Majorana and Dirac mass terms will emerge from $\mathcal{L}_{Higgs-Majorana}$ and $\mathcal{L}_{Higgs-Electroweak}$ via the Higgs mechanism. For later references, we write down the Dirac mass term in curved space-time as

$$\mathcal{L}_{Dirac-Mass} \sim im \left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e} \psi I\bar{\psi} \right\rangle, \tag{2.80}$$

where m is the Dirac mass.

It shall be reminded that in the Yang-Mills Lagrangian term, the 4-form factor $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ from one of the $I\hat{e} \wedge \hat{e} \wedge \hat{F}_{WL}$ should be canceled out by the similar 4-form factor from the denominator $I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e}$ before multiplication with the other $I\hat{e} \wedge \hat{e} \wedge \hat{F}_{WL}$. As such, the Yang-Mills Lagrangian terms, along with the other Lagrangian terms of the world, are diffeomorphism-invariant 4-forms on the 4-dimensional space-time manifold.

Also note that the Clifford algebra elements of the vierbein $\hat{e}$ and the curvature 2-forms such as $\hat{F}_G$ formally commute with each other in the Yang-Mills Lagrangian, since they transform under commuting gauge groups. Since the vierbein $\hat{e}$ transforms as a vector under Lorentz gauge transformation (2.36) and is invariant under the other gauge transformations, all the terms of the Lagrangian of the world can be proved to be local gauge-invariant under $Spin(1,3) \times SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}$.

The action of the world is

$$S_{world} = \int \mathcal{L}_{World}, \tag{2.81}$$

where the integration over $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is already embedded in the definition of the Lagrangian. We know that the space-time metric $g_{\mu\nu}$ can be derived from the vierbein. Thus the metric tensor $g_{\mu\nu}$ and the Hodge star in the conventional metric gravity can be constructed using various combinations/transformations of the vierbein. For instance, the 4-form $\frac{1}{4!}I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e}$ plays the role of the metric volume form $\sqrt{|g|}d^4x$.

We subscribe to the general notion of effective field theory,[32,33,34] which states that all the terms allowed by the symmetry requirements should be included in the Lagrangian of the world. Since one goal of this paper is to treat gravity and Yang-Mills interactions on an equal footing, we should consider Lagrangian terms that is linear in Yang-Mills fields as well, such as

$$\left\langle I\hat{e} \wedge \hat{e} \wedge \hat{F}_{WL} \right\rangle. \tag{2.82}$$

It is analogs to the gravity Lagrangian (2.78) which is linear in the spin connection curvature 2-form $\hat{R}$. However, it can be verified that such linear terms for the Yang-Mills fields are identically zero. The only allowable linear term other than the gravity Lagrangian is the

Holst term[35]

$$\left\langle \hat{e} \wedge \hat{e} \wedge \hat{R} \right\rangle, \tag{2.83}$$

which differs from the gravity Lagrangian (2.78) by removing the pseudoscalar I.

When it comes to the Lagrangian terms with two or more gauge curvature 2-forms, there is a plethora of allowable forms besides the Yang-Mills-type Lagrangian. Some examples are the topological CP-violating terms for the Yang-Mills interactions, the topological Gauss-Bonnet term and Nieh-Yan term,[36] and the higher-derivative gravity terms.[33,34] While all these higher-order terms should in principle be included in the Lagrangian, the key for model building is to recognize that the practical predictions of any model must be made within the context of separation of energy scales. The gravity and Yang-Mills Lagrangian terms happen to be amongst the first few order terms that are relevant at the energy scale accessible to experiments.

The astute reader may have noticed the presence of the imaginary number i in the fermion kinetic Lagrangian (2.77) and Dirac mass Lagrangian (2.80). As explained in the introduction section, there are two kinds of imaginary numbers. One is the genuine i, and the other can be replaced by the pseudoscalar I. We have demonstrated that the gauge group generators are valued in the real Clifford space, and the algebraic spinors are valued in the super-real Clifford space. Hence we have managed to stay away from the imaginary number i. So why do we need i in the fermion Lagrangian terms? It has to do with the requirement that the classical action of the world should be (super-)real

$$S^*_{world} = S_{world}. \tag{2.84}$$

We know that $\langle \ldots \rangle$ is employed in each Lagrangian term. By definition, it is the Clifford-scalar part of the expression, which means $\langle \ldots \rangle$ is real as long as the related fields in the expression is valued in the real Clifford space. Since the bosonic fields (gauge fields, vierbein, and Higgs fields) are valued in the real Clifford space, the reality condition is automatically satisfied for all the bosonic field-related Lagrangian terms, such as the gravity and Yang-mills Lagrangian terms.

On the other hand, since spinors are valued in the super-real Clifford space, the fermion action can essentially be reduced to a sum/integral of terms like $i\psi_x\psi_y$. Given that the Grassmann-odd coefficients of the spinors are defined to be super-real $\psi^*_x = \psi_x$ and $\psi^*_y = \psi_y$, the complex conjugate of $i\psi_x\psi_y$ can be calculated as

$$(i\psi_x\psi_y)^* = i^*\psi^*_y\psi^*_x = -i\psi_y\psi_x = i\psi_x\psi_y. \tag{2.85}$$

Hence the super-reality condition is satisfied, which is otherwise violated if i in the expression is removed. Note that the complex conjugate of the multiplication of Grassmann numbers obeys

$$(\psi_x\psi_y)^* = \psi^*_y\psi^*_x, \tag{2.86}$$

with no extra minus sign, even though both ψ_x and ψ_y are Grassmann-odd.

In summary, we are compelled to include the imaginary number i in the definition of fermion Lagrangian to enforce the super-reality condition. That said, as mentioned earlier, the imaginary number i is intimately related to the quantum theory. It will be shown later in this paper that the appearance of i in the fermion Lagrangian is the tip of the iceberg of quantum essence of almost everything.

3 Spontaneous symmetry breaking

3.1 Vierbein-induced SSB and the residual global Lorentz symmetry

The spontaneous symmetry breaking (SSB) saga of the universe starts with the vierbein field $\hat{e}$ acquiring a nonzero vacuum expectation value (VEV). As a consequence, the local Lorentz gauge symmetry $Spin(1,3)$ and the diffeomorphism symmetry are violated. The remaining local gauge symmetries are $SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}$ plus a residual *global* Lorentz symmetry.

The "ground state" of the vierbein $\hat{e}$ and spin connection $\hat{\omega}$ should satisfy the field equations, which are obtained by varying the world action S_{world} with the fields $\hat{e}$ and $\hat{\omega}$ independently. The resultant Einstein-Cartan equations read

$$\frac{1}{8\pi G}(\hat{R} \wedge \hat{e} + \hat{e} \wedge \hat{R} - \frac{\Lambda}{3!}\hat{e} \wedge \hat{e} \wedge \hat{e})I = \mathbb{T}, \tag{3.1}$$

$$\frac{1}{8\pi G}(\hat{T} \wedge \hat{e} - \hat{e} \wedge \hat{T})I = \mathbb{S}, \tag{3.2}$$

where the energy-momentum current 3-form $\mathbb{T}$ and the spin current 3-form $\mathbb{S}$ arise from the matter sector, such as the fermion, Yang-Mills, and Higgs action terms. Note that $\hat{R}$ is the spin connection curvature 2-form (2.70), and $\hat{T}$ is the torsion 2-form

$$\hat{T} = d\hat{e} + \hat{\omega} \wedge \hat{e} + \hat{e} \wedge \hat{\omega}. \tag{3.3}$$

When the spin-current $\mathbb{S}$ is zero, the second Einstein-Cartan equation (3.2) amounts to enforcing the zero-torsion condition

$$\hat{T} = 0, \tag{3.4}$$

which can be used to express the spin connection $\hat{\omega}$ in terms of the vierbein $\hat{e}$. In this case, the remaining (first) Einstein-Cartan equation can be shown to be equivalent to the regular Einstein field equations for gravity plus a cosmological constant term.

Upon SSB, the vierbein field develops a nonzero VEV

$$\hat{e} = \delta_\mu^a \gamma_a dx^\mu = \gamma_\mu dx^\mu, \tag{3.5}$$

while the spin connection remains zero

$$\hat{\omega} = 0. \tag{3.6}$$

It can be verified that both the above $\hat{e}$ and $\hat{\omega}$ satisfy the Einstein-Cartan equations, provided that $\Lambda = 0$, $\mathbb{T} = 0$, and $\mathbb{S} = 0$. Subsequently, the space-time metric $g_{\mu\nu} = \langle \hat{e}_\mu \hat{e}_\nu \rangle$ reduces to

$$g_{\mu\nu} = \langle \gamma_\mu \gamma_\nu \rangle = \eta_{\mu\nu}, \tag{3.7}$$

which is the Minkowski flat space-time metric.

With the substitution of $\hat{e}$ and $\hat{\omega}$ by the above values, the fermion action in the flat Minkowski space-time reduces to

$$S_{fermion} = \int i \langle \bar{\psi}_L \gamma^\mu D_{L\mu} \psi_L + \bar{\psi}_R \gamma^\mu D_{R\mu} \psi_R \rangle \, d^4 x, \tag{3.8}$$

where

$$D_{L\mu} \psi_L = (\partial_\mu + \hat{W}_{L\mu}) \psi_L + \psi_L (\hat{G}_\mu + \hat{A}_{BL\mu}), \tag{3.9}$$

$$D_{R\mu} \psi_R = (\partial_\mu + \hat{W}_{R\mu}) \psi_R + \psi_R (\hat{G}_\mu + \hat{A}_{BL\mu}). \tag{3.10}$$

Similarly, the Yang-Mills action(2.79) of the left-handed weak interaction $\hat{F}_{WL}$ can be rewritten as

$$S_{YM-Weak-Left} = -\frac{1}{4g_{WL}} \int F^i_{WL\mu\nu} F^{i,\mu\nu}_{WL} d^4 x, \tag{3.11}$$

where g_{WL} is the dimensionless coupling constant of the left-handed weak interaction. The Yang-Mills-type action terms of the right-handed weak, strong, and BL interactions take a similar form, with coupling constants g_{WR}, g_G, and g_{BL}, respectively. Note that

$$\gamma^\mu \eta_{\mu\nu} = \gamma_\nu, \tag{3.12}$$

$$F^{i,\mu\nu}_{WL} \eta_{\mu\alpha} \eta_{\nu\beta} = F^i_{WL\alpha\beta}, \tag{3.13}$$

where $\{\gamma^\mu\}$ is the reciprocal frame of $\{\gamma_a\}$. The reciprocal frame $\{\gamma^\mu\}$ is the avatar of the vierbein 3-form $I\hat{e} \wedge \hat{e} \wedge \hat{e}$ from the original fermion Lagrangian (2.77) when the vierbein field $\hat{e}$ acquires the nonzero VEV in eq. (3.5). Therefore, when the fermion action in the flat Minkowski space-time (3.8) is employed to derive the massless Dirac equation (the Dirac equation with nonzero mass will be discussed in Section 3.3)

$$\gamma^\mu D_{L,\mu} \psi_L = 0, \tag{3.14}$$

$$\gamma^\mu D_{R,\mu} \psi_R = 0, \tag{3.15}$$

what shows up in the Dirac equation is the reciprocal frame. As such, it's the reciprocal frame $\{\gamma^\mu\}$ that corresponds to the gamma matrices used in the conventional formalism. This fact also explains the minus sign we mentioned earlier: $-I$ plays the role of the conventional pseudoscalar when it is applied to the left side of a spinor such as $-I\psi$. The conventional pseudoscalar is defined using the reciprocal frame $\{\gamma^\mu\}$, whereas we define the pseudoscalar using the original Clifford algebra basis $\{\gamma_a\}$. Hence there is a minus sign.

It's worth mentioning that when we derive the flat space-time fermion action from the curved space-time counterpart, we have also leveraged $(\gamma^\mu)^\dagger = \gamma_0 \gamma^\mu \gamma_0$ and the following properties

$$\langle FG \rangle = -\langle GF \rangle, \tag{3.16a}$$

$$\left\langle (FG)^\dagger \right\rangle = -\langle FG \rangle, \tag{3.16b}$$

where the Grassmann-old F and G are functionals of odd multiples of $\psi(x)$ and $\bar\psi(x)$. The minus sign arises from the Grassmann-old nature of $\psi(x)$ and $\bar\psi(x)$.

Upon SSB induced by the nonzero VEV of the vierbein $\hat{e}$, both the local Lorentz gauge symmetry $Spin(1,3)$ and the diffeomorphism symmetry are violated. That said, it can be verified that the fermion (3.8) and the Yang-Mills (3.11) actions in flat Minkowski space-time is invariant under a residual *global* Lorentz transformation

$$x^\mu \quad \to \quad (\Lambda^{-1})^\mu_\nu x^\nu, \tag{3.17}$$

$$W^i_{L\mu}(x) \quad \to \quad (\Lambda^{-1})^\nu_\mu W^i_{L\nu}(\Lambda^{-1}x), \tag{3.18}$$

$$\psi(x) \quad \to \quad e^{\frac{1}{4}\theta^{ab}\gamma_a\gamma_b}\psi(\Lambda^{-1}x), \tag{3.19}$$

where the Lorentz transformation parameters $\theta^{ab} = -\theta^{ba}$ and Λ^μ_ν are related via the equation

$$e^{\frac{1}{4}\theta^{ab}\gamma_a\gamma_b}\gamma^\mu e^{-\frac{1}{4}\theta^{ab}\gamma_a\gamma_b} = \Lambda^\mu_\nu \gamma^\nu. \tag{3.20}$$

Note that the global Lorentz rotation parameters θ^{ab} above are independent of position, as opposed to the position-dependent $\theta^{ab}(x)$ for the local Lorentz gauge transformations.

The situation here parallels the Higgs mechanism where there remains a global $SU(2)$ custodial symmetry[37] after the electroweak symmetry breaking. In the case of the vierbein-induced SSB, the vestigial global Lorentz symmetry is a synchronization (enforced by eq. (3.20)) of the global portion of the local Lorentz gauge transformation ($e^{\frac{1}{4}\theta^{ab}\gamma_a\gamma_b}$) for spinors and the global volume-preserving portion of the diffeomorphism transformation (Λ^ν_μ) for space-time coordinates. The local Lorentz gauge transformation involves Clifford algebraic elements such as $\gamma_a\gamma_b$ labeled by Roman indices, while the diffeomorphism transformation involves coordinates x^μ and gauge fields (e.g. the electromagnetic field A_μ in eq. (1.12)) labeled by Greek indices. At the nexus is the VEV of the vierbein $e^a_\mu = \delta^a_\mu$ which acts as a solder form gluing together the Roman and Greek propers.

3.2 Majorana mass and absence of neutrinoless double beta decay

The second stage of SSB is triggered by the Majorana Higgs field ϕ_M that couples to the right-handed neutrinos only. It is a Higgs-like bosonic field in addition to the well known electroweak symmetry-breaking Higgs field ϕ of the standard model. The VEV of ϕ_M generates Majorana mass for the right-handed neutrinos and that's why we call it the Majorana Higgs field. It breaks the gauge symmetries from $SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}$ down to the standard model symmetries. This subsection also presents one major thesis of our paper: the neutrino Majorana mass preserves lepton number and therefore it does not lead to the neutrinoless double beta decay.

The Majorana Higgs field ϕ_M is valued in the *real* Clifford algebraic subspace spanned by two multivectors

$$\phi_M = (\phi_{M1} + \phi_{M2}I)\gamma_0 P_l, \tag{3.21}$$

where ϕ_{M1} and ϕ_{M2} are two real numbers, and P_l is the lepton projection operator (2.8). The Majorana Higgs field obeys gauge transformation rules

$$\phi_M \quad \rightarrow \quad e^{-\frac{1}{2}\theta_{WR}I - \frac{1}{2}\epsilon_{BL}J} \phi_M \ e^{\frac{1}{2}\theta_{WR}I + \frac{1}{2}\epsilon_{BL}J}, \tag{3.22}$$

where θ_{WR} and ϵ_{BL} are the right-handed weak $U(1)_{WR}$ and BL $U(1)_{B-L}$ gauge transformation parameters. As such, ϕ_M is invariant under the Lorentz $Spin(1,3)$ (Lorentz scalar), left-handed weak $SU(2)_{WL}$ (weak singlet), and color $SU(3)_C$ (color singlet) gauge transformations. Given that $JP_l = -IP_l$, one can replace J with $-I$ in the above transformation. We keep J to highlight its relevance to the BL symmetry. Note that ϕ_M is Clifford-odd, different from the Clifford-even electroweak Higgs field which will be investigated in Section 3.3.

The Majorana Higgs Lagrangian reads

$$\mathcal{L}_{Higgs-Majorana} = \left\langle (D^\mu \phi_M)^\dagger (D_\mu \phi_M) - V_M + (y_M iI)\bar{\nu}_R(I\Gamma_2\Gamma_3)\nu_R\phi_M \right\rangle, \tag{3.23}$$

where ν_R is the right-handed neutrino, y_M is the Majorana Yukawa coupling constant, and the Majorana Higgs potential V_M is

$$V_M = -\mu_M^2\phi_M^\dagger\phi_M + \lambda_M(\phi_M^\dagger\phi_M)^2. \tag{3.24}$$

The gauge-covariant derivative of ϕ_M is defined as

$$D_\mu\phi_M = (\partial_\mu - \frac{1}{2}W_{R\mu}I - \frac{1}{2}A_{BL\mu}J)\phi_M + \phi_M(\frac{1}{2}W_{R\mu}I + \frac{1}{2}A_{BL\mu}J), \tag{3.25}$$

which involves the right-handed weak gauge field $\hat{W}_{R\mu}$ and the BL gauge field $\hat{A}_{BL\mu}$. Note that the bivector $\Gamma_2\Gamma_3$ in the Yukawa term can be replaced by an arbitrary combination of $\Gamma_2\Gamma_3$ and $\Gamma_1\Gamma_3$. But it does not change the overall picture. The imaginary number i shows up in the Yukawa term. This is in compliance with the super-real condition for the Majorana Higgs Lagrangian.

Given that ϕ_M is a weak singlet, the Majorana Yukawa term couples with the right-handed fermions only, as opposed to the electroweak Higgs Yukawa term which couples with both the left- and right-handed fermions. The Majorana Higgs Lagrangian $\mathcal{L}_{Higgs-Majorana}$ can be verified to be $SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}$ gauge invariant. Due to the gauge transformation properties of ϕ_M (3.22), it can be shown that a similar Majorana Yukawa term for the right-handed electrons is prohibited since such a term violates the gauge symmetries. Therefore, the Majorana Yukawa term couples to the right-handed neutrinos exclusively.

By virtue of the Mexican hat-shaped potential V_M, the Majorana Higgs field ϕ_M acquires a nonzero VEV

$$\phi_M = \frac{1}{\sqrt{2}}\upsilon_M\gamma_0 P_l, \tag{3.26}$$

where $\upsilon_M = \frac{\mu_M}{\sqrt{\lambda_M}}$ is called the Majorana scale (or the seesaw scale).

As a result, the gauge symmetry related to the gauge field $\hat{Z}'_\mu$

$$\hat{Z}'_\mu = W_{R\mu}I + A_{BL\mu}J = (W_{R\mu} - A_{BL\mu})I, \tag{3.27}$$

is spontaneously broken. The would-be Nambu-Goldstone boson is "eaten" by the gauge field $\hat{Z}'_\mu$ which gains a mass as a consequence. The local gauge symmetries are broken down to the standard model symmetries $SU(3)_C \times SU(2)_{WL} \times U(1)_Y$, with the the hypercharge gauge symmetry $U(1)_Y$ specified by the synchronized double-sided gauge transformations (2.49). The $U(1)_Y$ gauge field $\hat{A}_Y$ remains massless and has an effective coupling constant of[15]

$$g_Y = \frac{g_{WR}g_{BL}}{\sqrt{g_{WR}^2 + g_{BL}^2}}, \tag{3.28}$$

where g_{WR} and g_{BL} are the right-handed weak and the BL coupling constants, respectively.

After replacement of ϕ_M with its VEV, the Majorana Yukawa term reduces to the Majorana mass term of the right-handed neutrino

$$Mi\langle I\bar{\nu}_R(I\Gamma_2\Gamma_3)\nu_R\gamma_0\rangle = Mi\langle I\bar{\nu}_R(\nu_R)_{C'}\rangle, \tag{3.29}$$

where $(\nu_R)_{C'}$ is the weaker form charge conjugation C' (2.55) of ν_R and the Majorana mass M is

$$M = \frac{1}{\sqrt{2}}y_M\upsilon_M. \tag{3.30}$$

It can be verified that the Majorana mass term respects all the standard model symmetries. This kind of mass is allowed for a standard model singlet such as ν_R. It can also be shown that the Majorana mass term is permitted only if ν_R is valued in the *Grassmann-odd* Clifford algebraic space. On the other hand, it would be identically zero if ν_R were valued in the real Grassmann-even Clifford algebraic space.

If we juxtapose the Majorana mass term with a typical Dirac mass term between neutrinos

$$m_\nu i\langle I\bar{\nu}\nu\rangle = m_\nu i\langle I\bar{\nu}_L\nu_R + I\bar{\nu}_R\nu_L\rangle, \tag{3.31}$$

we can see that the former couples $\bar{\nu}_R$ with $(\nu_R)_{C'}$, while the later couples $\bar{\nu}_R$ with ν_L. We know that the weaker form of charge conjugation C' (2.55) is a Clifford-odd operation. It converts the Clifford-even ν_R to Clifford-odd $(\nu_R)_{C'}$. Consequently, $(\nu_R)_{C'}$ is effectively left-handed and could be coupled to the right-handed $\bar{\nu}_R$ in a similar fashion as the Dirac mass term.

The observation of neutrino oscillations[28,29,30] indicates that neutrinos have nonzero masses which are much smaller than that of the other standard model fermions. If we assume that the neutrino Majorana mass M is much heavier than the neutrino Dirac mass m, a very small effective mass of the order of m_ν^2/M can thus be generated for the neutrino. This appealing explanation for the tiny neutrino mass is historically called the seesaw

mechanism.[25] If we could experimentally detect the Majorana nature of the neutrino mass, it would lend support to the seesaw mechanism.

Given that the traditional definition of the Majorana mass involves the charge conjugation C (2.54) that converts a particle into its corresponding antiparticle, the traditional Majorana mass term violates the conservation of lepton number and could be confirmed by the lepton number-violating process of the neutrinoless double beta decay. Therefore, it's widely believed that the observation of the neutrinoless double beta decay could be a confirmation of the Majorana mass. The lepton number-violating process can also be used to explain the origin of matter in the universe via a mechanism known as leptogenesis.[38] A slew of experiments have been commissioned to search for the neutrinoless double beta decay. As of yet, no evidence of such decay has ever been found.[39,40]

On the other hand, the weaker form of charge conjugation C' (2.55) does not invoke complex conjugate, and thus there is no particle-antiparticle interchange. Consequently, the Majorana mass term as shown in eq. (3.29) conserves lepton number, which is dissimilar to the traditional Majorana mass term that invokes the stronger form of charge conjugation C. This suggests that the absence of the neutrinoless double beta decay does not disapprove the Majorana mass as defined in eq. (3.29). We shall seek other means of the Majorana mass detection.

3.3 Scalar and antisymmetric-tensor Higgs fields

The third step of SSB concerns the well known electroweak symmetry-breaking Higgs field ϕ of the standard model which couples with both the left-handed and the right-handed fermions. The VEV of the scalar ϕ breaks the standard model symmetries down to $SU(3)_C \times U(1)_{EM}$. The SSB pattern outlined in this subsection is in many ways similar to the the conventional Higgs mechanism, only that it's transposed onto the Clifford algebraic landscape. That said, toward the end of this subsection we will touch upon a non-scalar Higgs field which could have cosmological implications.

The Higgs field ϕ is valued in the *real* Clifford algebraic subspace spanned by four Clifford-even multivectors

$$\phi = (\phi_0 + \phi_1\Gamma_2\Gamma_3 + \phi_2\Gamma_3\Gamma_1 + \phi_3\Gamma_1\Gamma_2)P_+, \tag{3.32}$$

where P_+ is the projection operator (2.10) with the property $\Gamma_1\Gamma_2 P_+ = -IP_+$. The four real coefficients $\{\phi_a; a = 0, 1, 2, 3\}$ correspond to the 2 complex components (i.e. 4 real degrees of freedom) of the traditional Higgs doublet. The Higgs field obeys gauge transformation rules

$$\phi \quad \to \quad e^{\frac{1}{2}\theta^1_{WL}\Gamma_2\Gamma_3 + \frac{1}{2}\theta^2_{WL}\Gamma_3\Gamma_1 + \frac{1}{2}\theta^3_{WL}\Gamma_1\Gamma_2} \, \phi \, e^{-\frac{1}{2}\epsilon_Y\Gamma_1\Gamma_2}, \tag{3.33}$$

where $\{\theta^1_{WL}, \theta^2_{WL}, \theta^3_{WL}\}$ and ϵ_Y are the left-handed weak $SU(2)_{WL}$ and the hypercharge $U(1)_Y$ gauge transformation parameters. As such, ϕ is a weak doublet and is invariant under the Lorentz $Spin(1,3)$ (Lorentz scalar) and the color $SU(3)_C$ (color singlet) gauge transformations. Given that $P_+\Gamma_1\Gamma_2 = -P_+I$, one can replace $-\frac{1}{2}\epsilon_Y\Gamma_1\Gamma_2$ with $\frac{1}{2}\epsilon_Y I$ in the above transformations and therefore the Higgs field has Hypercharge 1 (or $\frac{1}{2}$ depending on

the Hypercharge definition convention). We keep $-\frac{1}{2}\epsilon_Y\Gamma_1\Gamma_2$ to highlight its relevance to the Hypercharge symmetry (2.49).

Note that the Higgs field could take values in a complimentary Clifford-even subspace

$$\phi = (\phi_0 + \phi_1\Gamma_2\Gamma_3 + \phi_2\Gamma_3\Gamma_1 + \phi_3\Gamma_1\Gamma_2)P_-, \tag{3.34}$$

where the projection operator is changed from P_+ to P_-. We call the original ϕ (3.32) and the complimentary ϕ (3.34) the ϕ_+-type and ϕ_--type Higgs fields, respectively. They have opposite Hypercharges and thus their coupling patterns with the isospin up-type and down-type fermions differ from each other. This additional ϕ_--type Higgs field can be exploited in various extensions of the standard model such as the two-Higgs-doublet model (2HDM)[41,42] or the three-Higgs-doublet model (3HDM).[43] We will circle back to this point in Section 3.4. But for now, let's focus on the ϕ_+-type Higgs field only.

The standard model gauge-invariant Higgs Lagrangian reads

$$\mathcal{L}_{Higgs-Electroweak} = \left\langle (D^\mu\phi)^\dagger(D_\mu\phi) - V_M + (y_e i I \bar{l}_L \phi e_R + h.c.) \right\rangle, \tag{3.35}$$

where $l_L = \nu_L + e_L$ is the left-handed lepton doublet, e_R is the right-handed electron, y_e is the Yukawa coupling constant, and Higgs potential V is

$$V = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{3.36}$$

The gauge-covariant derivative is defined as

$$D_\mu\phi = (\partial_\mu + \frac{1}{2}W^1_{L\mu}\Gamma_2\Gamma_3 + \frac{1}{2}W^2_{L\mu}\Gamma_3\Gamma_1 + \frac{1}{2}W^3_{L\mu}\Gamma_1\Gamma_2)\phi - \phi(\frac{1}{2}A_{Y\mu}\Gamma_1\Gamma_2), \tag{3.37}$$

which involves the left-handed weak gauge field $\hat{W}_{L\mu}$ and the Hypercharge gauge field $\hat{A}_{Y\mu}$. We omit Yukawa terms for non-electron fermions which will be investigated in more detail in Section 3.4 when we tackle the issue of the fermion mass hierarchies. Note that the imaginary number i shows up in the Yukawa term. This is in compliance with the super-real condition.

By virtue of the Mexican hat-shaped potential V, the Higgs field acquires a nonzero VEV

$$\phi = \frac{1}{\sqrt{2}}\upsilon_{EW}P_+, \tag{3.38}$$

where $\upsilon_{EW} = \frac{\mu}{\sqrt{\lambda}}$ is usually called the electroweak scale. As a result, the gauge fields $\hat{W}^\pm_{L\mu}$ and $\hat{Z}_\mu$ gain masses. The standard model symmetries are broken down to $SU(3)_C \times U(1)_{EM}$. The electromagnetic $U(1)_{EM}$ gauge field $\hat{A}$ remains massless and has an effective coupling constant of[15]

$$g_{EM} = \frac{g_{WL}g_{WR}g_{BL}}{\sqrt{g^2_{WL}g^2_{WR} + g^2_{WL}g^2_{BL} + g^2_{WR}g^2_{BL}}}, \tag{3.39}$$

where g_{WL}, g_{WR} and g_{BL} are the left-handed weak, the right-handed weak and the BL coupling constants, respectively. The electromagnetic field $\hat{A}$-related gauge-covariant derivative of the algebraic spinor ψ can be cast into the form

$$D_\mu \psi = (\partial_\mu + \frac{1}{2} A_\mu \Gamma_1 \Gamma_2)\psi + \psi(\frac{1}{2} A_\mu J) = \partial_\mu \psi + q A_\mu \psi I, \qquad (3.40)$$

where q is the electric charge.

With replacement of ϕ with its VEV, the Yukawa term reduces to the Dirac mass term of electron

$$m_e i \langle I \bar{e} e \rangle = m_e i \langle I \bar{e}_L e_R + I \bar{e}_R e_L \rangle, \qquad (3.41)$$

where the electron mass m_e is

$$m_e = \frac{1}{\sqrt{2}} y_e v_{EW}. \qquad (3.42)$$

At this final stage of SSB, we are ready to write down the action of the electron

$$S_{electron} = \int \mathcal{L} \, d^4 x = i \int \left\langle \bar{\psi} \gamma^\mu (\partial_\mu \psi + q A_\mu \psi I) + m \bar{\psi} \psi I \right\rangle d^4 x, \qquad (3.43)$$

where $q = -1$ and we relabel e as ψ and m_e as m. The Clifford algebraic Dirac equation can be readily derived

$$\gamma^\mu (\partial_\mu \psi I - q A_\mu \psi) - m \psi = 0. \qquad (3.44)$$

It's similar to the conventional Dirac equation, provided that i is replaced with I positioned on the right side of ψ. It can be used to derive the equation for the C' charge conjugation (2.55) counterpart $\psi_{c'}$

$$\gamma^\mu (\partial_\mu \psi_{c'} I + q A_\mu \psi_{c'}) - m \psi_{c'} = 0. \qquad (3.45)$$

The above equation demonstrates that the weaker form of charge conjugation C' indeed changes the sign of the electric charge.

The electroweak symmetry breaking process delineated above bears close resemblance to the traditional Higgs mechanism. Curiously, the Clifford algebra framework allows for a non-scalar Higgs field ϕ_{AT} which could potentially break both the electroweak and Lorentz symmetries. The non-scalar Higgs field is valued in the *real* Clifford algebraic subspace spanned by $4 * 6 = 24$ Clifford-even multivectors[15]

$$\gamma_a \gamma_b, \quad \gamma_a \gamma_b \Gamma_i \Gamma_j, \qquad (3.46)$$

where i, j = 1, 2, 3, i>j, a, b = 0, 1, 2, 3, a>b. We have the flexibility of only considering the projected portion of it $\phi_{AT\pm} = \phi_{AT} P_\pm$ with each half having 12 independent Clifford algebraic components. The non-scalar Higgs field obeys the transformation rules

$$\phi_{AT} \quad \rightarrow \quad e^{\frac{1}{4}\theta^{ab}\gamma_a\gamma_b + \frac{1}{2}\theta^1_{WL}\Gamma_2\Gamma_3 + \frac{1}{2}\theta^2_{WL}\Gamma_3\Gamma_1 + \frac{1}{2}\theta^3_{WL}\Gamma_1\Gamma_2} \phi_{AT} \, e^{-\frac{1}{2}\epsilon_Y \Gamma_1\Gamma_2 - \frac{1}{4}\theta^{ab}\gamma_a\gamma_b}, \qquad (3.47)$$

where $\theta^{ab}\gamma_a\gamma_b$ represents Lorentz transformations. As such, ϕ_{AT} is a weak doublet as well as a Lorentz sextet (an antisymmetric tensor rather than a scalar). If ϕ_{AT} acquires a nonzero VEV

$$\phi_{AT} = \frac{1}{\sqrt{2}}v_{AT}\gamma_0\gamma_3, \tag{3.48}$$

it would break the electroweak and Lorentz symmetries at the same time, since it singles out a specific space-time direction via $\gamma_0\gamma_3$. The magnitude of this VEV could be extremely small compared with the electroweak scale $v_{AT} \ll v_{EW}$, rendering the v_{AT}-related effects unobservable in laboratories. We hypothesize that the ethereal VEV of the antisymmetric-tensor Higgs field might manifest itself as the large-scale anisotropies of the universe.[44, 45, 46, 47, 48, 49]

3.4   The 3HDM and the fermion mass hierarchy problem

Dimensionless ratios between physical constants appearing in a physical theory cannot be accidentally small. The technical naturalness principle is elegantly defined by 't Hooft:[50] a quantity should be small only if the underlying theory becomes more symmetric as that quantity tends to zero. Weakly broken symmetry ensures that the smallness of a parameter is preserved against possible large quantum corrections.

For the application of the naturalness principle, let's examine two global symmetries related to the vector $U_V(1)$ phase transformation

$$\begin{align} \psi_L &\rightarrow \psi_L e^{\theta_V I}, & \psi_R &\rightarrow \psi_R e^{\theta_V I}, \tag{3.49a}\\ \phi_M &\rightarrow \phi_M e^{2\theta_V I}, \tag{3.49b}\\ \phi &\rightarrow \phi, \tag{3.49c} \end{align}$$

and the axial $U_A(1)$ phase transformation

$$\begin{align} \psi_L &\rightarrow \psi_L e^{-\theta_A I}, & \psi_R &\rightarrow \psi_R e^{\theta_A I}, \tag{3.50a}\\ \phi_M &\rightarrow \phi_M e^{2\theta_A I}, \tag{3.50b}\\ \phi &\rightarrow \phi e^{2\theta_A I}, \tag{3.50c} \end{align}$$

where ψ_L is the left-handed spinor, ψ_R is the right-handed spinor, ϕ_M is the Majorana Higgs field, and ϕ is the regular Higgs field. The phase transformation rules for ϕ_M and ϕ may not seem intuitive. But when we consider ϕ_M and ϕ as multi-fermion condensations in Section 4.2, the reason for these transformation rules will become clear.

For later discussion, let's also introduce a $U_\alpha(1)$ phase transformation for the right-handed spinors

$$\begin{align} \psi_L &\rightarrow \psi_L, & \psi_R &\rightarrow \psi_R e^{\alpha I}, \tag{3.51a}\\ \phi_M &\rightarrow \phi_M e^{2\alpha I}, \tag{3.51b}\\ \phi &\rightarrow \phi e^{\alpha I}, \tag{3.51c} \end{align}$$

which is basically a combination of $U_V(1)$ and $U_A(1)$.

It can be checked that, when ϕ_M and ϕ are replaced by their VEVs, the Majorana (3.29) and Dirac (3.41) mass terms violate the $U_V(1)/U_A(1)/U_\alpha(1)$ and $U_A(1)/U_\alpha(1)$ symmetries, respectively. Hence the Majorana and Dirac masses are technically natural, given that these global symmetries can be restored if the Majorana and Dirac masses were set to zero. In other words, the smallness of the Majorana and Dirac masses are protected by the global symmetries against possible large quantum corrections.

Prior to the SSB induced by ϕ_M and ϕ, these two Higgs fields would transform according to the aforementioned phase transformation rules. It can be shown that all the terms of the Lagrangian of the world (2.73) respect the $U_V(1)$, $U_A(1)$, and $U_\alpha(1)$ global symmetries, with only one exception which is the $U_A(1)/U_\alpha(1)$-violating electron Yukawa term in eq. (3.35). It would be nice if we can tinker with the $U_A(1)/U_\alpha(1)$ transformation rule for ϕ, so that the whole Lagrangian of the world is invariant. Contrary to our expectation, it's not achievable. Within the confines of a single standard model Higgs field ϕ, it is impossible to make *both* the isospin up-type and down-type fermion Yukawa terms $U_A(1)/U_\alpha(1)$-invariant.

The seemingly worrisome symmetry-violating Yukawa terms can be turned into our advantage. In the spirit of the technical naturalness principle, we can exploit the global symmetry properties to explain the vast range of fermion masses which span five orders of magnitude between the heaviest top quark and the lightest electron. The key for solving the fermion mass hierarchy problem is to realize that some of the Yukawa coupling constants are actually not constants at all.[16] Embedded in the Yukawa couplings, there are six Clifford-valued bosonic scalar fields

$$\Phi_{\alpha t} = \Phi_{\alpha t1} + \Phi_{\alpha t2}I, \qquad \Phi_{\beta t} = \Phi_{\beta t1} + \Phi_{\beta t2}I, \tag{3.52}$$

$$\Phi_{\alpha\nu_\tau} = \Phi_{\alpha\nu_\tau 1} + \Phi_{\alpha\nu_\tau 2}I, \quad \Phi_{\beta\nu_\tau} = \Phi_{\beta\nu_\tau 1} + \Phi_{\beta\nu_\tau 2}I, \tag{3.53}$$

$$\Phi_{\alpha\tau} = \Phi_{\alpha\tau1} + \Phi_{\alpha\tau2}I, \qquad \Phi_{\beta\tau} = \Phi_{\beta\tau1} + \Phi_{\beta\tau2}I, \tag{3.54}$$

where all the coefficients, such as $\Phi_{\alpha t1}$ and $\Phi_{\alpha t2}$, are real numbers.

The three α-type Φ fields are tied to the $U(1)_\alpha$ symmetry, while the three β-type Φ fields are tied to a novel $U(1)_\beta$ symmetry (see table 1). The symmetry-violating effective Yukawa terms are originated from these six Φ fields acquiring nonzero VEVs via the SSB mechanism. Note that these Φ fields are gauge singlets, i.e. they are invariant under all gauge transformations. This is in contrast to the regular electroweak Higgs field which is a weak $SU(2)_{WL}$ doublet.

To account for the masses of the three families of fermions, we adopt three Higgs fields in our model (a.k.a. 3HDM), namely, the top-quark Higgs field ϕ_t, the tau-neutrino Higgs field ϕ_{ν_τ}, and the tau-lepton Higgs field ϕ_τ. The naming convention of the three Higgs fields (and the six Φ fields) will become clear when we related them to their corresponding quantum condensations in Section 4.2. Among these Higgs fields, ϕ_t and ϕ_{ν_τ} are ϕ_+-type Higgs field (3.32), while ϕ_τ is ϕ_--type Higgs field (3.34). All the three Higgs doublets obey the usual gauge transformation rules for the electroweak Higgs field (3.33).

We introduce one more global symmetry $U(1)_\beta$ which, like $U(1)_\alpha$, is related to the phases of the right-handed fermions. However, the $U(1)_\beta$ charge is not uniformly assigned to fermions. While $U_\alpha(1)$ transforms all the right-handed fermions by the same phase $e^{\alpha I}$,

$U_\beta(1)$ transforms the isospin up-type quarks (u_R, c_R, t_R) and down-type leptons (e_R, μ_R, τ_R) by the phase $e^{\beta I}$ and it transforms the down-type quarks (d_R, s_R, b_R) and up-type leptons (ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$) by the opposite phase $e^{-\beta I}$. The $U(1)_\alpha/U(1)_\beta$ charge assignments are summarized in table 1

Table 1. The $U(1)_\alpha$ and $U(1)_\beta$ charge

	u_R,c_R,t_R e_R,μ_R,τ_R	d_R,s_R,b_R $\nu_{eR},\nu_{\mu R},\nu_{\tau R}$	ϕ_t	ϕ_{ν_τ}	ϕ_τ	$\Phi_{\alpha t}$, $\Phi_{\alpha\nu_\tau}$, $\Phi_{\alpha\tau}$	$\Phi_{\beta t}$, $\Phi_{\beta\nu_\tau}$, $\Phi_{\beta\tau}$
$U(1)_\alpha$	1	1	1	1	1	-2	0
$U(1)_\beta$	1	-1	1	-1	1	0	-2

It will be shown later in this subsection that the two global symmetries $U(1)_\alpha$ and $U(1)_\beta$ are instrumental in determining the relative magnitudes of the effective Yukawa coupling constants, and consequently establishing the fermion mass hierarchies. As the $U(1)_\alpha$ charge assignment is analogous to that of the Peccei-Quine $U(1)_{PQ}$ symmetry,[51] we will use the term $U(1)_\alpha$ and $U(1)_{PQ}$ interchangeably henceforth.

Now we are ready to write down the Yukawa coupling terms for all three generations of the standard model fermions (plus right-handed neutrinos)

$$i\left\langle g_t I\bar{q}_L^3\tilde{\phi}_t t_R + g_{\nu_e}\Phi_{\beta t}^\dagger I\bar{l}_L^1\tilde{\phi}_t \nu_{eR} + g_b\Phi_{\alpha t}I\bar{q}_L^3\phi_t b_R + g_e\Phi_{\alpha t}\Phi_{\beta t}I\bar{l}_L^1\phi_t e_R\right\rangle + h.c. \tag{3.55a}$$

$$+i\left\langle g_{\nu_\tau}I\bar{l}_L^3\tilde{\phi}_{\nu_\tau}\nu_{\tau R} + g_c\Phi_{\beta\nu_\tau}I\bar{q}_L^2\phi_{\nu_\tau}^\dagger c_R + g_\mu\Phi_{\alpha\nu_\tau}I\bar{l}_L^2\phi_{\nu_\tau}\mu_R + g_d\Phi_{\alpha\nu_\tau}\Phi_{\beta\nu_\tau}^\dagger I\bar{q}_L^1\phi_{\nu_\tau}d_R\right\rangle + h.c. \tag{3.55b}$$

$$+i\left\langle g_\tau I\bar{l}_L^3\tilde{\phi}_\tau\tau_R + g_s\Phi_{\beta\tau}^\dagger I\bar{q}_L^2\tilde{\phi}_\tau s_R + g_{\nu_\mu}\Phi_{\alpha\tau}I\bar{l}_L^2\phi_\tau\nu_{\mu R} + g_u\Phi_{\alpha\tau}\Phi_{\beta\tau}I\bar{q}_L^1\phi_\tau u_R\right\rangle + h.c., \tag{3.55c}$$

where $g_t, g_{\nu_\tau}, \cdots$ are the bare Yukawa coupling constants which are dimensionless parameters of order $O(1)$. The left-handed doublets are

$$l_L^1 = \nu_{eL} + e_L, \quad q_L^1 = u_L + d_L, \tag{3.56}$$

$$l_L^2 = \nu_{\mu L} + \mu_L, \quad q_L^2 = c_L + s_L, \tag{3.57}$$

$$l_L^3 = \nu_{\tau L} + \tau_L, \quad q_L^3 = t_L + b_L, \tag{3.58}$$

where quarks stand for color triplets, such as $u_L = u_{rL} + u_{gL} + u_{bL}$. From the Yukawa coupling pattern we can tell that the Φ singlets are of mass dimension zero, different from the traditional mass dimension-one scalar fields. Alternatively, we can rewrite these singlets as the conventional mass dimension-one scalar fields, as long as they show up in the Yukawa terms as Φ/M with M being an unknown energy scale.

Note that in the literature, the ϕ_+-type and ϕ_--type Higgs field are called down-type and up-type Higgs fields, respectively. As we have mentioned before, ϕ_t is a ϕ_+-type Higgs field, which means that it can only couple to the isospin down-type fermions such as b_R and e_R, whereas direct coupling to the up-type fermions is prohibited. Only a transformed form of ϕ_t

$$\tilde{\phi}_t = \frac{1}{4}\gamma^\mu\phi_t\gamma_\mu, \tag{3.59}$$

can couple to the up-type fermions such as t_R and ν_{eR}. Similar logic applies to ϕ_{ν_τ} and ϕ_τ, which are ϕ_+-type and ϕ_--type, respectively. It's worth mentioning that the conventional formalism leverages a different transformation of the standard model Higgs field

$$\tilde{\phi} = i\sigma_2 \phi^*, \tag{3.60}$$

so that $\tilde{\phi}$ can be coupled to up-type fermions. It involves the complex conjugate, whereas the Clifford algebra version (3.59) doesn't.

The Yukawa coupling scheme (3.55) partitions fermions into three cohorts, namely

$$\phi_t \quad cohort: t, \nu_e, b, e, \tag{3.61}$$

$$\phi_{\nu_\tau} \quad cohort: \nu_\tau, c, \mu, d, \tag{3.62}$$

$$\phi_\tau \quad cohort: \tau, s, \nu_\mu, u. \tag{3.63}$$

The right-handed fermions in each cohort only couple to the designated Higgs field, thus preventing the flavor-changing neutral currents (FCNCs). Within each Higgs field cohort, only one out of four Yukawa terms is free from the Φ singlets, since it is impossible to make the relevant Yukawa terms simultaneously $U(1)_\alpha/U(1)_\beta$ invariant without including the additional Φ singlets. After inserting these Φ singlets in the Yukawa couplings, it can be verified that all the Yukawa terms respect the $U(1)_\alpha$ and $U(1)_\beta$ global symmetries.

The coupling patterns (3.55) of the three Higgs fields are predicated on an alternative generation/family assignment,

$$Generation \quad 0: \quad t, b, \nu_e, e, \tag{3.64}$$

$$Generation \quad +: \quad c, s, \nu_\tau, \tau, \tag{3.65}$$

$$Generation \quad -: \quad u, d, \nu_\mu, \mu, \tag{3.66}$$

which are tied to the flavor projection operators $\{\zeta^0, \zeta^+, \zeta^-\}$.[15,16] These flavor projection operators dictate the ζ^+/ζ^- mixing between the ϕ_{ν_τ} cohort and the ϕ_τ cohort. The projection operators can also be applied to the Majorana-type Yukawa coupling and the resultant Majorana mass can directly mix the $\nu_{\mu R}$ and $\nu_{\tau R}$ neutrinos[15,16] which is evidenced in the observation of neutrino oscillations.[28,29,30] Other flavor mixing phenomena might also be accommodated by the framework, which we leave to future research.

We assume that the Lagrangians of the three Higgs doublets and six Φ singlets are analogous to the one specified for the regular Higgs mechanism (3.35), albeit the scalar potential V could be more complicated with cross terms mixing different scalar fields.[42,43] The study of the exact form of V is beyond the scope of this paper. For our purpose here, we just postulate that these scalar fields acquire the following nonzero VEVs

$$\phi_t = \frac{1}{\sqrt{2}} v_t P_+, \quad \phi_{\nu_\tau} = \frac{1}{\sqrt{2}} v_{\nu_\tau} P_+, \quad \phi_\tau = \frac{1}{\sqrt{2}} v_\tau P_-, \tag{3.67}$$

$$\Phi_{\alpha t} = v_{\alpha t}, \quad \Phi_{\alpha \nu_\tau} = v_{\alpha \nu_\tau}, \quad \Phi_{\alpha \tau} = v_{\alpha \tau}, \tag{3.68}$$

$$\Phi_{\beta t} = v_{\beta t}, \quad \Phi_{\beta \nu_\tau} = v_{\beta \nu_\tau}, \quad \Phi_{\beta \tau} = v_{\beta \tau}. \tag{3.69}$$

The three Φ_α-type fields break the global $U(1)_\alpha/U(1)_{PQ}$ symmetry, while the three Φ_β-type fields break the global $U(1)_\beta$ symmetry. Note that the Φ fields do not break any local gauge symmetry since they are gauge singlets. Post the SSB, there will be six massive sigma modes and six Nambu-Goldstone modes. As opposed to the Higgs mechanism, the Nambu-Goldstone modes are not "eaten" by the gauge field. Due to the explicit symmetry breaking originated from the quantum anomaly and instanton effects, the otherwise massless Nambu-Goldstone bosons of the three Φ_α fields acquire masses and turn into pseudo-Nambu-Goldstone bosons in a similar fashion as the axions.[51,52,53] Since the Φ_α fields are local gauge (especially electroweak) singlets, they are more in line with the invisible axions.[54,55,56,57] The axions have historically been proposed as dark matter candidates and as a possible solution to the strong CP problem. We leave the investigation of the $U(1)_\beta$-type (pseudo-)Nambu-Goldstone boson's role as dark matter to future study.

After replacement of the three Higgs doublets and six Φ singlets with their VEVs, the Yukawa terms reduce to the Dirac mass terms

$$g_t v_t i \left\langle I\bar{t}t \right\rangle + g_{\nu_e} v_{\beta t} v_t i \left\langle I\bar{\nu}_e \nu_e \right\rangle + g_b v_{\alpha t} v_t i \left\langle I\bar{b}b \right\rangle + g_e v_{\alpha t} v_{\beta t} v_t i \left\langle I\bar{e}e \right\rangle \tag{3.70a}$$

$$+g_{\nu_\tau} v_{\nu_\tau} i \left\langle I\bar{\nu}_\tau \nu_\tau \right\rangle + g_c v_{\beta \nu_\tau} v_{\nu_\tau} i \left\langle I\bar{c}c \right\rangle + g_\mu v_{\alpha \nu_\tau} v_{\nu_\tau} i \left\langle I\bar{\mu}\mu \right\rangle + g_d v_{\alpha \nu_\tau} v_{\beta \nu_\tau} v_{\nu_\tau} i \left\langle I\bar{d}d \right\rangle \tag{3.70b}$$

$$+g_\tau v_\tau i \left\langle I\bar{\tau}\tau \right\rangle + g_s v_{\beta \tau} v_\tau i \left\langle I\bar{s}s \right\rangle + g_{\nu_\mu} v_{\alpha \tau} v_\tau i \left\langle I\bar{\nu}_\mu \nu_\mu \right\rangle + g_u v_{\alpha \tau} v_{\beta \tau} v_\tau i \left\langle I\bar{u}u \right\rangle, \tag{3.70c}$$

where for brevity sake, all terms are multiplied by $\sqrt{2}$.

Before making contact with the experimental results, we have to identify which mode of the three Higgs doublets corresponds to the 125 GeV boson observed at the Large Hadron Collider.[58,59] Generally speaking, a Higgs boson can be defined as a linear combination of the Clifford-scalar sector (with VEVs subtracted) of the three Higgs fields. For a order-of-magnitude kind of analysis, let's assume that the $125\,GeV$ Higgs boson is aligned with the top-quark Higgs field ϕ_t. Therefore the VEV of ϕ_t is approximately

$$v_t \approx 246 GeV, \tag{3.71}$$

and the bare Yukawa coupling constant g_t can be identified as the top quark Yukawa constant $y_t = g_t$. For the sake of estimation, we make the further assumption that the bare Yukawa coupling constants are almost uniform

$$y_t = g_t \approx g_{\nu_e} \approx \cdots \approx g_u. \tag{3.72}$$

The standard model Yukawa constants, except y_t, can be identified as effective coupling constants. For example, the bottom quark's effective Yukawa constant is $y_b = y_t v_{\alpha t}$, and tau neutrino's effective Yukawa constant is $y_{\nu_\tau} = y_t v_{\nu_\tau}/v_t$.

Aided by the above assumptions and the mass formula (3.70), we arrive at an estimation of the Higgs/Φ VEVs and the neutrino Dirac masses as shown in table 2, where the known fermion masses are also included for comparison.

Table 2. Higgs VEVs (MeV), Φ VEVs, and Dirac masses (MeV)

	ϕ_t Cohort		ϕ_{ν_τ} Cohort		ϕ_τ Cohort	
Higgs VEVs	v_t	246,000	v_{ν_τ}	41,900	v_τ	2,530
Φ_α VEVs	$v_{\alpha t}$	1/41	$v_{\alpha \nu_\tau}$	1/278	$v_{\alpha \tau}$	1/44
Φ_β VEVs	$v_{\beta t}$	1/8200	$v_{\beta \nu_\tau}$	1/23	$v_{\beta \tau}$	1/19
Dirac Masses	t	173,000	ν_τ	29,500	τ	1,780
	ν_e	21	c	1,280	s	96
	b	4,180	μ	106	ν_μ	40
	e	0.51	d	4.6	u	2.2

There are a couple of takeaways from the above estimations. Firstly, the magnitudes of Φ VEVs ($v_{\alpha t}, v_{\beta t}, \cdots$) are all small, albeit to varying degrees. In accordance with the technical naturalness principle of 't Hooft,[50] the weakly broken symmetries of $U(1)_\alpha/U(1)_\beta$ ensure that the smallness of the VEVs is preserved against possible quantum corrections. We have mentioned earlier that the Φ singlets can be considered as traditional mass dimension-one scalar fields characterized by an unknown energy scale M. Given that the VEVs of the Majorana Higgs field and the three electroweak Higgs fields violate the $U(1)_\alpha/U(1)_\beta$ symmetries as well, we assume that the energy scale M is higher than the Majorana scale v_M (3.26). This is to make sure that the Φ field-induced $U(1)_\alpha/U(1)_\beta$ global symmetry breaking process is decoupled from the Higgs(-like) symmetry breaking mechanism triggered by either the Majorana Higgs field ϕ_M or the electroweak Higgs fields ϕ_t, ϕ_{ν_τ}, and ϕ_τ.

The Φ VEVs play a crucial role in determining the magnitudes of the effective Yukawa constants and thus establishing fermion mass hierarchies within each of the ϕ_t, ϕ_{ν_τ}, and ϕ_τ cohorts. On the other hand, the relative fermion mass sizes between different ϕ cohorts are controlled by the Higgs field VEVs ($v_t, v_{\nu_\tau}, v_\tau$). The fermion masses within a given ϕ cohort are mostly in a descending order in each column of table 2. The only exception is the reversed order between the ν_e and b masses due to the abnormally small magnitude of the $\Phi_{\beta t}$ VEV ($v_{\beta t} \sim 1/8200$) compared with the other Φ VEVs. Note that the estimated neutrino masses are meant to be the *Dirac* masses, as opposed to the much smaller seesaw effective masses or the vastly larger Majorana masses outlined in Section 3.2. Interestingly, according to our estimation, the Dirac mass of the ν_τ neutrino ($m_{\nu_\tau} \sim 29,500 MeV$) is considerably larger than those of the ν_μ and ν_e neutrinos ($m_{\nu_\mu} \sim 40 MeV$ and $m_{\nu_e} \sim 21 MeV$).

Secondly, assuming that there is no cross term between the kinetic part of the three

Higgs Lagrangians, the masses of the $W^\pm$ and Z^0 bosons can be calculated as

$$m_{W^\pm} = \frac{1}{2} v_{total} g_{WL}, \tag{3.73}$$

$$m_{Z^0} = \frac{1}{2} v_{total} \sqrt{g_{WL}^2 + g_Y^2}, \tag{3.74}$$

where g_{WL} and g_Y are the weak and Hypercharge gauge coupling constants. The total electroweak scale v_{total} is dependent on all three Higgs VEVs

$$v_{total} = \sqrt{v_t^2 + v_{\nu_\tau}^2 + v_\tau^2}. \tag{3.75}$$

According to table 2, the estimated three Higgs VEVs $\{v_t,\, v_{\nu_\tau},\, v_\tau\}$ have a hierarchical structure

$$246\ GeV \gg 41\ GeV \gg 2.5\ GeV, \tag{3.76}$$

where the ϕ_t Higgs VEV is significantly larger than the other two. The ϕ_{ν_τ} Higgs VEV plays a non-negligible role in the electroweak scale saturation. The total electroweak scale v_{total} is dominated by the ϕ_t Higgs VEV. The ratio between them is given by

$$\frac{v_{total}}{v_t} = 1.014. \tag{3.77}$$

Given the assumption that the $125\ GeV$ Higgs boson is attributed to the top-quark Higgs field ϕ_t, this 1.4% discrepancy might be the underlying reason for the deviation of the measured W-boson mass from the standard model prediction.[60] If we tweak the uniform Yukawa coupling assumption by proposing that the bare Yukawa couplings of the ϕ_{ν_τ} cohort are 5 times larger than the other bare Yukawa couplings, then the v_{total}/v_t difference is around 0.06%, close to what is observed by the CDF Collaboration.[60]

And lastly, according to the Yukawa coupling scheme (3.55), the muon belongs to the tau-neutrino Higgs field h_{ν_τ} cohort. Given the intrinsic connection between the muon and the h_{ν_τ} Higgs field, it is worthwhile to investigate the h_{ν_τ} Higgs field's contribution to the muon anomalous magnetic moment, especially in light of the recent muon $g-2$ measurement with improved accuracy which confirms a deviation from the standard model prediction.[61]

4 Quantum emergence and the naturalness problems

4.1 Quantization via the Clifford functional integral formalism

As mentioned in the introduction section, there are two kinds of imaginary numbers. One is the genuine i which is central to the quantum theory. The other one can be replaced by the pseudoscalar I which shows up in the definition of spinors and gauge fields. Intriguingly, the imaginary number i is inextricably embedded in the *classical* fermion Lagrangian, which suggests that there might be quantum phenomenon lurking beneath the veneer of the classical Lagrangian terms. It eventually leads us to the epiphany that quantum condensations may hold the golden key to various sorts of naturalness problems.

But first, let's examine how to quantize classical spinor fields valued in the Clifford algebraic space. To this end, we have developed the Clifford functional integral formalism in our earlier paper.[17] The generating functional $Z[\eta]$ for the spinors can be represented by the Clifford functional integral

$$Z[\eta] = \int \mathcal{D}\psi e^{\frac{1}{2}\int d^4x\{i\mathcal{L}[\psi] + \langle I\bar{\eta}(x)\psi(x) + I\bar{\psi}(x)\eta(x)\rangle\}}, \tag{4.1}$$

where the imaginary number i in front of the Lagrangian is the hallmark of quantization. The super-real Grassmann-odd sources $\eta(x)$ and $\bar{\eta}(x)$ are valued in the same Clifford space as $\psi(x)$ and $\bar{\psi}(x)$. It is understood that $Z[\eta]$ satisfies the normalization condition $Z[0] = 1$.

We regard $\psi(x)$ and $\bar{\psi}(x) = \psi^\dagger(x)\gamma_0$ as dependent variables, as opposed to the traditional way of treating them as independent variables. The same logic applies to $\eta(x)$ and $\bar{\eta}(x)$. Therefore, an extra $1/2$ factor in front of the action is required to keep the calculated quantities, such as the fermion propagators, consistent with those of the conventional formalism. Note that the single-source format $\langle I\bar{\eta}(x)\psi(x)\rangle$ is employed here. Alternatively, we can adopt the bilocal-source format $\langle\bar{\eta}(x)\psi(x)\rangle\langle\eta(y)\bar{\psi}(y)\rangle$[17] which is well-suited for the non-perturbative approximations analogous to the two-particle irreducible (2PI) effective action approach.[62]

One benefit of the Clifford functional integral formalism is that we don't need to literally perform the functional integration for most cases. Rather, we resort to the property that the functional integration of a total functional derivative is zero

$$\int \mathcal{D}\psi \frac{\delta}{\delta\psi(x)} e^{\frac{1}{2}\int d^4x\{i\mathcal{L}[\psi] + \langle I\bar{\eta}(x)\psi(x) + I\bar{\psi}(x)\eta(x)\rangle\}} = 0. \tag{4.2}$$

Similar property holds for $\bar{\psi}(x)$. It means that the functional integral is invariant under a shift of $\psi(x)$.

In our earlier paper on the Clifford functional integral formalism,[17] we have provided the specific definition of the Clifford functional derivatives $\delta/\delta\psi(x)$ and $\delta/\delta\bar{\psi}(x)$. For our purpose here, we only need to know the basic Leibniz rule

$$\frac{\delta}{\delta\psi(x)}\langle\psi(y)F[\psi]\rangle = \delta(x-y)F[\psi] + \frac{\delta}{\delta\psi(x)}\left\langle\psi(y)\dot{F}[\psi]\right\rangle,$$

where the dot on $\dot{F}[\psi]$ denotes functional derivative performed on $F[\psi]$ only. Similar Leibniz rules can be applied to $\bar{\psi}(x)$, $\eta(x)$ and $\bar{\eta}(x)$. Coupled with the other two Clifford algebra properties (3.16), we are able to perform the relevant Clifford functional derivatives in this paper.

As an excise, let's apply the property (4.2) to the fermion Lagrangian (3.43) without the electromagnetic coupling. We arrive at the Schwinger-Dyson (SD) Clifford functional-differential equation

$$\gamma^\mu\partial_\mu\{\frac{\delta}{\delta\bar{\eta}(x)}Z[\eta]I\} - m\frac{\delta}{\delta\bar{\eta}(x)}Z[\eta] + \eta(x)IZ[\eta] = 0. \tag{4.3}$$

The solution to the SD equation can be readily obtained as

$$Z[\eta] = e^{-\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\langle I\bar{\eta}(p)S(p)\eta(p)\rangle}, \tag{4.4}$$

where $\eta(p) = \int d^4x \eta(x) e^{Ip \cdot x}$ and $p \cdot x = p_\mu x^\mu$. Note that $\eta(x) e^{Ip \cdot x} \neq e^{Ip \cdot x} \eta(x)$, since $\eta(x)$ has both Clifford-even and Clifford-odd parts. The Feynman propagator $S(p)$ is given by

$$S(p) = \frac{1}{\not{p} - m + i\epsilon}, \tag{4.5}$$

where $\not{p} = p_\mu \gamma^\mu$.

A few comments are in order. First of all, the imaginary number i does not explicitly show up in the DS equation (4.3). It is because the i in the Clifford functional integral (4.1) and the i in the fermion Lagrangian (3.43) cancel out. When gauge fields are included in the Lagrangian, there is no such cancellation due to the absence of i in the Yang-Mills-type Lagrangian terms.

Secondly, the Feynman propagator $S(p)$ has poles at $p^2 = m^2$. The propagator is not properly defined without a prescription on the p_0-related integral in the vicinity of the poles. A well-defined Lorentz-invariant Feynman propagator hinges on the contour integral on the p_0 complex plane prescribed by $i\epsilon$. Feynman's $i\epsilon$ trick introduces the imaginary number i through the back door. And this i is of quantum origin too, since it can be demonstrated that $i\epsilon$ is related to the proper time ordering of quantum fields in the operator formalism of field quantization.

In the subsequent subsections, the Feynman propagator will be used extensively in various calculations of quantum-loop effects. For the sake of brevity, going forward we will not explicitly write down $i\epsilon$ in the propagators.

4.2 Composite Higgs and the Higgs mass naturalness problem

The discovery of the 125 GeV Higgs boson[58,59] has renewed the interest in the possible explanation for the Higgs mass naturalness problem.[63,64,65] The 125 GeV Higgs mass is technically unnatural according to 't Hooft,[50] since even if one takes the massless Higgs boson limit, the symmetry of the standard model is not enhanced. The perturbative quantum corrections tend to draw the Higgs mass towards higher scale. This is in contrast to the case of the fermion mass, which is protected by the $U_A(1)$ global symmetry against possible large quantum corrections.

One way of addressing the Higgs mass naturalness problem is to replace the fundamental Higgs boson with a fermion-antifermion condensation, such as in the technicolor[66,67,68] and the (extended) top condensation models.[69,70,71,72,73,74,75,76,77,78,16] In these models, the Higgs sector is an effective description of the low energy physics represented by the composite Higgs field. The condensation is induced via the dynamical symmetry breaking (DSB) mechanism, which is a profound concept in physics. It is introduced into the relativistic QFT by Nambu and Jona-Lasinio (NJL),[79] inspired by the earlier Bardeen-Cooper-Schriefer (BCS) theory of superconductivity.[80]

Motivated by the proximity of top quark mass scale and the electroweak symmetry breaking scale, the top condensation model has been extensively studied. The simplest version of the top condensation model assumes the top quark-antiquark condensation only. With a view toward explaining the fermion mass hierarchies in the context of composite electroweak Higgs bosons, we have proposed the extended top condensation model in our

previous work.[16] In addition to the top quark condensation, the extended top condensation model involves the tau neutrino and tau lepton condensations as well. The 3HDM in Section 3.4 is essentially an effective representation of these three condensations.

The top condensation model in its original form is based on the NJL-like four-fermion interactions. For example, the top quark interaction term takes the form

$$V_{top-quark} \sim g \left\langle I\bar{q}_L^3 \gamma^\mu q_L^3 I\bar{t}_R \gamma_\mu t_R \right\rangle, \tag{4.6}$$

where g is the four-fermion coupling constant and $q_L^3 = t_L + b_L$. Comparing above with the ϕ_t Higgs field Yukawa coupling term (3.55), one can see that ϕ_t is an effective representation of the fermion-antifermion pair

$$i\phi_t \sim gq_L^3 I\bar{t}_R, \tag{4.7}$$

where the i multiplier will be explained later in this subsection. With the replacement of the effective Higgs field, the top quark interaction term (4.6) turns into the top quark Yukawa term

$$V_{top-quark} \sim i \left\langle I\bar{q}_L^3 \gamma^\mu \phi_t \gamma_\mu t_R \right\rangle \sim i \left\langle I\bar{q}_L^3 \tilde{\phi}_t t_R \right\rangle, \tag{4.8}$$

Now let's investigate what kind of Clifford algebraic value the effective Higgs field ϕ_t can take. First of all, ϕ_t is Clifford-even, given that q_L^3 is Clifford-odd and t_R is Clifford-even (meaning that $\bar{t}_R = t_R^\dagger \gamma_0$ is Clifford-odd). And since $t_R = P_- t_R$ (meaning that $\bar{t}_R = (P_- t)_R^\dagger \gamma_0 = \bar{t}_R P_+$), $\phi_t = \phi_t P_+$ should be the ϕ_+-type Higgs field. Therefore, there are $32/2 = 16$ components which correspond to the combination of the 4-component ϕ_+-type scalar Higgs field (3.32) and the 12-component ϕ_{AT+}-type antisymmetric-tensor Higgs field (3.46).

For a free fermion, it's straight forward to obtain the solution (4.4) to the SD equation (4.3). In the presence of interactions such as (4.6), solving the corresponding SD equation is notoriously hard. The path well trodden is to find a perturbative solution, under the assumption that a certain coupling constant is small. In our previous paper,[17] we follow a non-perturbative scheme dubbed as the bilocal-source approximation,[81,82] which effectively treats the bilocal-source term as a series expansion parameter. The zeroth-order approximation of the Clifford functional SD equation[17] is equivalent to the self-consistent Hartree mean-field approximation (a.k.a. rainbow approximation). According to the DSB mechanism, when the four-fermion interaction is strong enough, it will trigger a quantum condensation

$$iv_t \sim g \int \frac{d^4p}{(2\pi)^4} \frac{m}{p^2 - m^2}, \tag{4.9}$$

where v_t is the magnitude of the condensation and m is the emergent top quark mass.

We can see that the above integral is quadratically divergent. The integral is seemingly a real number. However, as we mentioned in Section 4.1, the fermion propagator has poles at $p^2 = m^2$. Feynman's $i\epsilon$ trick ensures that the integral on p_0 is well-defined. The proper contour integral on the complex plane of p_0 (or equivalently the Wick rotation of

time axis) would pick up an imaginary number i, thus making the quadratically divergent integral (4.9) imaginary valued. Therefore, even if there is no i in the original four-fermion interaction (4.6), the imaginary number shows up explicitly in the Higgs Yukawa coupling term (4.8).

It's well know that the calculations of QFT are plagued by divergent integrals, which need to be regularized at the intermediate stage. After renormalization, a finite and regularization scheme-independent result can be obtained for the renormalizable theories. The top condensation model's four-fermion interaction is nonrenormalizable in the conventional sense, since the four-fermion interaction is a dimension six operator. That said, as mentioned earlier, we subscribe to the general notion of effective field theory, according to which the seemingly nonrenormalizable models, including quantum gravity, are nonetheless manageable renormalization-wise and predictive quantum effect-wise, insofar as there is a separation of low energy physics from the high energy quantum perturbations.[32,33,34]

Historically the NJL model has been presented with the energy cutoff schemes,[83] which usually break the Lorentz invariance. In the presence of a cutoff scale, the four-fermion interaction coupling constant has to be fine-tuned in order to establish the hierarchy between the putative large cutoff scale and the much smaller fermion mass scale. Thus the naturalness problem seems to haunt us again in the Lorentz symmetry-violating cutoff approach.

However, there is a Lorentz symmetry-preserving implicit regularization ansatz[84,85,86] (IR) where the divergent parts of Feynman integrals are isolated in a few Lorentz-invariant primitive integrals that are independent of external momentum, whereas the remaining external momentum-dependent integrals are convergent. Because the convergent integrals are separated from the divergent ones, the finite parts can be integrated free from the effects of regularization.

In our earlier paper,[17] we applied the IR technique to the NJL-type model. Granted that the divergent primitive integrals are independent of external momentum, they can be treated as finite quantities as a result of unspecified (implicit) regularization. The central tenet of the IR approach is that no attempt whatsoever shall be made to calculate these divergent primitive integrals via any explicit regularization. The external momentum-independent divergent primitive integrals are regarded as free parameters of the model that shall be determined by comparing with measurable quantities, such as the emergent fermion mass, the composite boson mass, and the vacuum energy.

Given that no explicit regularization is required in the calculation, the traditional notion of cutoff scale and fine-tuned coupling constant are of no relevance in the IR approach. The smallness of the symmetry breaking scale of the fermion mass m is an *a priori* assumption. Once a small scale of m is settled upon at the lower order of approximation, it's ensured that the smallness of m is preserved against possible higher order disturbances due to the protection from the weakly broken axial $U_A(1)$ symmetry, which is in accordance with the technical naturalness principle.

Before preceding to examine the bosonic bound state properties of the composite Higgs model, we would like to mention some open questions. One question is how to properly calculate the vacuum energy shift due to the quantum condensation. And the other is the long-standing issue of the momentum routing ambiguity associated with the fermion bubble

diagram.[87] With the goal of tackling this issues, we propose an improved version of the IR methodology by adding two supplementary rules below.[17]

Supplementary rule No. 1: The original IR approach sets forth the rule that if an external momentum-independent primitive divergent integral A such as eq. (4.9) is isolated, at the final stage of calculation it should be replaced with a finite renormalized value so that it can be compared with the measurable quantities. We denote this renormalization procedure (R procedure) as

$$A \quad \to \quad <A>_R .\tag{4.10}$$

Rule No. 1 stipulates that if A and B are related to the same physical process, then

$$AB \quad \to \quad <AB>_R \neq <A>_R<B>_R .\tag{4.11}$$

In other words, the relationship involving the multiplication of divergent integrals such as $<AB>_R = <A>_R<B>_R$ shall be avoided if A and B are related to the same physical process. The value of $<AB>_R$ should be treated as independent of $<A>_R$ or $<B>_R$. The R procedure can be applied recursively to the multiplication of two primitive divergent integrals only if they are linked to independent physical processes, such as two independent condensations.

Supplementary rule No. 2: When a Feynman integral is convergent or logarithmically divergent, the integral is independent of the momentum routing parameter, because the parameter can be shifted away by a translation of the integration variable. When it comes to integrals that are more than logarithmically divergent, one should proceed with caution because the seemingly harmless momentum shifting changes the integral values. For example, the quadratically divergent integral corresponding to the fermion bubble diagram[79,87] in the scalar (Higgs boson) channel is

$$\Pi_s(q) = i \int \frac{d^4p}{(2\pi)^4} \left\langle S(p+(1-\alpha)q)S(p-\alpha q) \right\rangle ,\tag{4.12}$$

where S is the fermion propagator and α is an arbitrary parameter not determined by the theory.[87] Unlike the case of convergent or logarithmically divergent integrals, the seemingly innocuous momentum shifting changes the integral values. Rule No. 2 stipulates that for quadratically (or higher order) divergent integrals with momentum routing ambiguities, the momentum routing parameter α shall be set at the symmetrical value. For the above instance, the momentum routing parameter should be set at $\alpha = \frac{1}{2}$, so that $(1-\alpha)q = \alpha q$. Note that a related ambiguity problem is the triangle diagrams of the Adler-Bell-Jackiw (ABJ) anomaly,[88,89] where the integrals are linearly divergent. The ambiguity is fixed by enforcing the vector Ward identity, at the expense of the axial Ward identity.

With these two supplementary rules set, let's investigate the bosonic bound state properties of the composite Higgs model. To this end, we go beyond the zeroth-order bilocal-source approximation and turn to the first-order approximation of the Clifford functional SD equation.[17] The collective mode of the composite Higgs boson can be determined via the

pole of the composite boson propagator (a.k.a. the fermion-antifermion channel T-matrix) in the scalar channel

$$D_s(p) \sim \frac{1}{g^{-1} - \Pi_s(p)}, \tag{4.13}$$

where $\Pi_s(p)$ is the bubble function (4.12). The composite boson propagators $D_s(p)$ is the re-summation of the infinite order chain of the fermion bubble diagrams. Similar leading order calculation in the context of contact interactions goes by various names such as the random-phase approximation, ladder approximation, Bethe-Salpeter T-matrix equation, and 1/N expansion.

After setting the momentum routing parameter in eq. (4.12) to $\alpha = \frac{1}{2}$, the pole (i.e. the Higgs boson mass) of the composite boson propagator $D_s(p)$ can be calculated as[17]

$$m_h = \frac{2}{\sqrt{1 + |\Delta|}} m, \tag{4.14}$$

where m is the dynamically generated fermion mass and Δ is specified by the renormalized logarithmically divergent integral

$$\Delta^{-1} = 64\pi^2 < \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - m^2)^2} >_R . \tag{4.15}$$

Because of the $\sqrt{1 + |\Delta|}$ factor, the composite Higgs boson mass m_h is less than $2m$, which deviates from the usual first-order approximation prediction $m_h = 2m$.[79] In other words, $2m$ serves as an upper bound of the Higgs boson mass, which implies that the Higgs boson mass is also protected by the weakly broken axial symmetry, because the Higgs boson mass and the fermion mass are simultaneously generated by the same DSB mechanism. And additionally, at the electroweak scale there is no elementary Higgs mass term to be modified by any higher order quantum perturbation from external sources. Therefore, the composite Higgs mass is naturally small.

When it comes to the top quark condensation model, one phenomenological problem is related to the prediction of the Higgs-top mass ratio. Since the 2012 discovery,[58,59] the Higgs boson is known to be lighter than the top quark. According to the traditional way of Higgs mass calculation, the top condensation model appears to fail since it gives too heavy Higgs mass compared with the top quark mass. However, in our calculation the Higgs mass and the top mass relation involves an extra primitive divergent integral (4.15). According to the central rule of the IR approach, the value of such integral should not be explicitly calculated. Rather, it is determined by the experimental measurements. Therefore, the observed Higgs-top mass ratio does not falsify the top condensation model. Instead, the ratio fixes the dimensionless parameter $|\Delta|$ of the model. Based on the measured top quark mass (173Gev) and Higgs mass (125Gev), we arrive at an estimation of $|\Delta| = 6.66$ from eq. (4.14).

In the same vein as the composite electroweak Higgs field, the Majorana Higgs field ϕ_M can be regarded as a composite field representing the condensation of a right-handed neutrino-antineutrino pair[16]

$$i\phi_M \sim I\bar{\nu}_R(I\Gamma_2\Gamma_3)\nu_R. \tag{4.16}$$

Similarly, the Φ singlets can be represented by the condensation of fermion-antifermion double pairs[16] such as

$$\Phi_{\alpha t} \sim \bar{b}_R t_R \bar{q}_L^3 q_L^3, \tag{4.17}$$

$$\Phi_{\beta t} \sim \bar{l}_L^1 \gamma^\mu q_L^3 \bar{t}_R \gamma_\mu \nu_{eR}. \tag{4.18}$$

The quantum condensation details of the ϕ_M and Φ composite fields can be worked out along the lines of the aforementioned composite electroweak Higgs field. Note that there is no imaginary number i in the definition of the effective Φ field, since even numbers of i (two fermion-antifermion pairs) cancel out.

In summary, the Higgs mass could be naturally small and we have demystified the imaginary number i in the Yukawa/mass term as a vestige of the quantum condensation. Emboldened by these achievements, one might wonder whether we can also surmount the vacuum energy's naturalness problem and decipher the genesis of the imaginary number i in the fermion kinetic Lagrangian term. That is the subject of the next subsection.

4.3 Composite vierbein and the cosmological constant problem

Quantum fluctuations of the vacuum contribute to the cosmological constant Λ. The calculated vacuum energy is extremely large compared with the commonly accepted estimation of Λ.[90,91,92,93] The vacuum energy is 10^{120} times too large according to the zero-point energy calculation, or 10^{55} times too large according to the electroweak symmetry breaking calculation. The cosmological constant problem is perceived as the most severe naturalness problem in physics.[94,95,96]

Inspired by the composite Higgs model investigated in Section 4.2, we turn to the composite vierbein field[97,98,99,100,101,102,103] as a possible solution to the cosmological constant problem. Paralleling the dynamical symmetry breaking (DSB) mechanism of the composite Higgs approach, the vierbein field $\hat{e}$ can be considered as an effective representation of the fermion-antifermion condensation

$$i\hat{e} \sim E = \psi d\bar{\psi}, \tag{4.19}$$

where d is the exterior derivative, hence $E_\mu = \psi \partial_\mu \bar{\psi}$. Unlike the previous approaches to the composite vierbein, the Clifford-valued composite vierbein field above is not restricted to the vector space γ_a, albeit its VEV will congeal around γ_a. This is analogous to the Higgs mechanism where the Higgs VEV settles around the limited scalar subspace ϕ_0 out of the full Higgs doublet space of $\phi_0 + \phi_1 \Gamma_2 \Gamma_3 + \phi_2 \Gamma_3 \Gamma_1 + \phi_3 \Gamma_1 \Gamma_2$. We will delve into more details about the extended vierbein space in Section 4.4.

The composite vierbein field is to be compared with a generalized version of the composite Higgs field

$$i\phi \sim H = \psi I \bar{\psi}. \tag{4.20}$$

For simplicity reasons, we consider a generic spinor field ψ with both chirality and ignore gauge field coupling. A more accurate definition of the chiral composite vierbein fields will be provided in Section 4.4 when we examine the gauge-covariant chiral vierbeins.

There are a couple of similarities and dissimilarities between spinor the bilinears H and E. Given that there is no $\langle \ldots \rangle$ operation in the definition of H and E, both H and E take values in the entire Clifford algebraic space. The Higgs spinor bilinear H is a 0-form, whereas the vierbein spinor bilinear $E = E_\mu dx^\mu$ is a 1-form which conforms with fact that the vierbein field $\hat{e}$ is a 1-form. The Higgs spinor bilinear H acquires a Clifford-even VEV, which implies that the Higgs field ϕ describes the condensation of an opposite-handed fermion-antifermion pair ($\psi_L I \bar{\psi}_R$ or $\psi_R I \bar{\psi}_L$). On the other hand, the vierbein spinor bilinear E acquires Clifford-odd VEVs valued in $\{\gamma_a\}$, which suggests that the vierbein field $\hat{e}$ describes the condensation of a like-handed fermion-antifermion pair ($\psi_L d\bar{\psi}_L$ or $\psi_R d\bar{\psi}_R$).

In the above H and E definitions, we follow the tradition[13,104] of regarding the spinor field ψ as dimensionless, a.k.a. bare spinor field. Consequently, the Higgs spinor bilinear H is also dimensionless. The vierbein spinor bilinear $E_\mu = \psi \partial_\mu \bar{\psi}$ is endowed with mass dimension one from the partial derivative. As such, a proper differential form would remain dimensionless. For example, for $E = E_\mu dx^\mu$ the mass dimension one of E_μ is canceled out by the mass dimension minus one of dx^μ. The same logic applies to any 1-form gauge field $\hat{A} = \hat{A}_\mu dx^\mu$, provided that $\hat{A}_\mu$ is assigned mass dimension one. If we construct a Lagrangian term using the proper differential forms, the mass dimension assignment convention implies that the coefficient in front of the Lagrangian term should be of mass dimension zero. The conventional mass dimensions of parameters can be recovered when we re-scale the bare spinor field which will be discussed later in this subsection.

Leveraging the spinor bilinears E and H, we can write down the diffeomorphism-invariant Lagrangian terms of the pre-condensation primordial world

$$\mathcal{L}_{Fermion+CC} \sim \langle IE \wedge E \wedge E \wedge E \rangle, \tag{4.21a}$$

$$\mathcal{L}_{Yukawa+CC} \sim \langle IE \wedge E \wedge E \wedge EH^2 \rangle, \tag{4.21b}$$

$$\mathcal{L}_{Yang-Mills} \sim \frac{\langle (IE \wedge E \wedge \hat{F})(IE \wedge E \wedge \hat{F}) \rangle}{\langle IE \wedge E \wedge E \wedge E \rangle}, \tag{4.21c}$$

$$\mathcal{L}_{Gravity} \sim \langle IE \wedge E \wedge \hat{R} \rangle, \tag{4.21d}$$

where $\hat{F}$ stands for any Yang-Mills-type gauge field curvature 2-form and $\hat{R}$ is the spin connection curvature 2-form (2.70). Note that $\mathcal{L}_{Yukawa+CC}$ could have some variations, such as $\langle IE \wedge E \wedge E \wedge HEH \rangle$ which corresponds to the top quark-type Yukawa interaction term (4.6).

Diffeomorphism-invariance is guaranteed since all the Lagrangian terms are 4-forms on the 4-dimensional space-time manifold. As mentioned earlier, the coefficients in front of the Lagrangian terms (for brevity sake not explicitly written out) are all of mass dimension zero. And we further assume that these dimensionless coefficients should of order $O(1)$. In other words, there shouldn't be any unnaturally small or large coefficients.

It's worth mentioning that there are even numbers of fermion-antifermion pairs in each Lagrangian term. It's driven by the two imperatives of being super-real and having no imaginary number i in the pre-condensation Lagrangian. Note that there is no bare

cosmological constant term, since diffeomorphism-invariance demands that the Lagrangian terms must be 4-forms and the available gauge-invariant differential forms are either the fermion field-related E spinor bilinear 1-form or the gauge field-related curvature 2-forms.

The fermion $\mathcal{L}_{Fermion+CC}$ and the Yukawa $\mathcal{L}_{Yukawa+CC}$ Lagrangian terms are comprised of 8 and 12 Grassmann-odd fermion fields respectively. This is very different from the typical fermion Lagrangian. Upon quantum condensation, they will give rise to the conventional fermion kinetic and mass terms as well as the effective cosmological constant term. More specifically, when the three E spinor bilinears in $\mathcal{L}_{Fermion+CC}$ are replaced by their condensation values, the Lagrangian term $\mathcal{L}_{Fermion+CC}$ is left with one E spinor bilinear and is turned into the effective fermion kinetic term. Similarly, when the four E spinor bilinears and one H spinor bilinear in $\mathcal{L}_{Yukawa+CC}$ are replaced by their condensation values, the Lagrangian term $\mathcal{L}_{Yukawa+CC}$ is left with one H spinor bilinear and is turned into the effective Dirac mass term. Lastly, when all spinor bilinears in $\mathcal{L}_{Fermion+CC}$ and $\mathcal{L}_{Yukawa+CC}$ are replaced by their condensation values, these two Lagrangian terms are left with no spinor bilinear and are turned into the effective cosmological constant term.

Upon quantum condensation, the diffeomorphism-invariance is broken and the effective fermion propagator $S(p)$ assumes the form

$$S(p) = \frac{a_0}{\not{p} - m_0}, \tag{4.22}$$

where m_0 is the emergent mass arising from the $\mathcal{L}_{Yukawa+CC}$ term, and the parameter a_0 comes from the $\mathcal{L}_{Fermion+CC}$ term. The parameter a_0^{-1} is of mass dimension three, since it's related to the condensations of three mass dimension-one E_μ spinor bilinears. The parameters a_0 and m_0 can be determined self-consistently via their respective mean-field "gap" equations in a similar fashion as the NJL-type model.[79, 17]

Leveraging the Clifford generating functional $Z[\eta]$ (eq. (4.4)), the mean-field VEVs of E_μ and H can be calculated as

$$E_\mu \sim iM_0\gamma_\mu = a_0\gamma_\mu \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{p^2 - m_0^2}, \tag{4.23a}$$

$$H \sim iv_0 = a_0 \int \frac{d^4p}{(2\pi)^4} \frac{m_0}{p^2 - m_0^2}, \tag{4.23b}$$

where the VEV magnitudes M_0 and v_0 are of mass dimension one and zero, respectively. The VEV of H takes value in the Clifford-scalar space, while the VEV of E_μ takes value in the Clifford-vector space $\{\gamma_\mu\}$ as expected for an effective vierbein field. We can see that the above primitive quantum loop integrals for E_μ and H are quartically and quadratically divergent, respectively. According to the contour integral rule on the complex plane of p_0, these integrals pick up an imaginary number i factor. The VEV magnitudes M_0 and v_0 are subject to the renormalization procedure $< \cdots >_R$ as delineated in Section 4.2.

As we know, E_μ is of mass dimension one. To be consistent with the conventional formalism of the dimensionless vierbein, the correspondence between the spinor bilinear E_μ and the dimensionless vierbein $\hat{e}_\mu$ should be

$$E_\mu \sim iM_0\hat{e}_\mu, \tag{4.24}$$

which means $\hat{e}_\mu = \gamma_\mu$ according to (4.23a), as expected for the flat space-time. The above calculation of VEVs are based on the flat space-time fermion propagator $S(p)$ (4.22). In the following discussion, we will assume that the assignment of $E_\mu \sim iM_0\hat{e}_\mu$ is applicable for the general cases of curved space-time as well.

After replacement of E_μ with $iM_0\hat{e}_\mu$ and H with iv_0 and retaining the lowest order terms in the non-condensated E and H, the effective Lagrangian terms take the form

$$\mathcal{L}_{Fermion-Kinetic} \sim i < M_0^3 >_R \quad \left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge E \right\rangle, \tag{4.25a}$$

$$\mathcal{L}_{Fermion-Mass} \sim i < M_0^4 v_0 >_R \quad \left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e}H \right\rangle, \tag{4.25b}$$

$$\mathcal{L}_{Yang-Mills} \sim \frac{< M_0^4 >_R}{< M_0^4 >_R} \frac{\left\langle (I\hat{e} \wedge \hat{e} \wedge \hat{F})(I\hat{e} \wedge \hat{e} \wedge \hat{F}) \right\rangle}{\left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e} \right\rangle}, \tag{4.25c}$$

$$\mathcal{L}_{Gravity} \sim < M_0^2 >_R \quad \left\langle I\hat{e} \wedge \hat{e} \wedge \hat{R} \right\rangle, \tag{4.25d}$$

$$\mathcal{L}_{CC-Fermion} \sim < M_0^4 >_R \quad \left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e} \right\rangle, \tag{4.25e}$$

$$\mathcal{L}_{CC-Yukawa} \sim < M_0^4 v_0^2 >_R \quad \left\langle I\hat{e} \wedge \hat{e} \wedge \hat{e} \wedge \hat{e} \right\rangle, \tag{4.25f}$$

where the $\mathcal{L}_{Fermion-Kinetic}$ and $\mathcal{L}_{CC-Fermion}$ terms are derived from $\mathcal{L}_{Fermion+CC}$, while the $\mathcal{L}_{Fermion-Mass}$ and $\mathcal{L}_{CC-Yukawa}$ terms are derived from $\mathcal{L}_{Yukawa+CC}$.

Now we can trace the origin of the imaginary number i in the "classical" Lagrangian terms. The imaginary number i stems from the primitive divergent integrals (4.23) related to the quantum condensations of the E_μ and H spinor bilinears. If there are odd numbers of condensations, there is an i in the coefficient of the effective "classical" Lagrangian, such as the fermion kinetic (4.25a) and mass (4.25b) terms. On the other hand, if there are even numbers of condensations, there is no i in the coefficient of the effective "classical" Lagrangian since i squares to -1, such as the Yang-Mills (4.25c), gravity (4.25d), and cosmological constant (4.25e) (4.25f) terms.

We can verify that the fermion kinetic (4.25a) and mass (4.25b) terms conform with the corresponding terms specified in the Lagrangian of the world in Section 2.4 (see eq. (2.77) and (2.80)). The only difference is in the coefficients, since we have adopted the dimensionless spinor field in this subsection. To map to the traditional mass dimension $3/2$ spinor field (a.k.a. dressed spinor field) in Section 2.4, we can leverage the field renormalization relationship

$$\psi_{dressed} = \sqrt{< M_0^3 >_R} \ \psi_{bare}, \tag{4.26}$$

where ψ_{bare} stands for the dimensionless bare spinor field and $\psi_{dressed}$ stands for the mass dimension $3/2$ dressed spinor field. With the substitution of ψ_{bare} with $\psi_{dressed}$, we can see that the fermion kinetic term (4.25a) regains the conventional form (2.77) with the coefficient normalized to one. The same substitution in the fermion mass (4.25b) term implies that the fermion mass for the dressed spinor field is

$$m_0 \sim \frac{< M_0^4 v_0 >_R}{< M_0^3 >_R}. \tag{4.27}$$

For the effective Yang-Mills-type Lagrangian (4.25c), the $< M_0^4 >_R$ factors from the numerator and the denominator cancel out. As we mentioned earlier, the dimensionless coefficient in front of the original Yang-Mills-type Lagrangian term (4.21c) is assumed to be of order $O(1)$. Therefore we expect that the effective Yang-Mills-type Lagrangian coupling strength should not be far from being of order $O(1)$ as well. As a verification, the QED's fine-structure constant is $\alpha \approx 1/137$, which meets our expectation.

For the effective gravity Lagrangian (4.25d), the coefficient $< M_0^2 >_R$ is of mass dimension two and can be identified with the Planck mass M_{pl}

$$< M_0^2 >_R \approx M_{\mathrm{pl}}^2. \tag{4.28}$$

The effective cosmological constant Λ is of mass-dimension two. It is defined by the ratio between the $\mathcal{L}_{CC-Fermion}/\mathcal{L}_{CC-Yukawa}$ and $\mathcal{L}_{Gravity}$ Lagrangian coefficients

$$\Lambda \sim \frac{< M_0^4 >_R + < M_0^4 v_0^2 >_R}{< M_0^2 >_R} \approx \frac{< M_0^4 >_R + < M_0^4 v_0^2 >_R}{M_{\mathrm{pl}}^2}, \tag{4.29}$$

where $< M_0^4 >_R$ is the vacuum energy contribution from the fermion Lagrangian, and $< M_0^4 v_0^2 >_R$ is the Higgs ground state energy contribution from the Yukawa Lagrangian.

According to the conventional wisdom, each M_0 factor in the above equations can be identified with Planck mass M_{pl}. Resultantly, the effective fermion mass m_0 (4.27) is estimated as

$$m_0 \sim v_0 M_{\mathrm{pl}}. \tag{4.30}$$

Using the top quark mass as an example, v_0 is calculated as of order $v_0 \approx 10^{-17}$. Similarly, Λ is estimated as of order

$$\Lambda \sim (1 + v_0^2) M_{\mathrm{pl}}^2 \approx M_{\mathrm{pl}}^2, \tag{4.31}$$

which is astronomically larger than the commonly accepted estimation of $\Lambda \sim 10^{-120} M_{\mathrm{pl}}^2$.

However, there is a loophole in the above reasoning. According to the supplementary rule No. 1 of the implicit regularization set forth in Section 4.2, the renormalization procedure can not be applied recursively to the multiplication of primitive divergent integrals

$$< M_0^4 >_R \neq < M_0^2 >_R < M_0^2 >_R \approx M_{\mathrm{pl}}^4. \tag{4.32}$$

As such, $< M_0^4 >_R$ should be deemed as a parameter completely decoupled from the scale of M_{pl}^4. The magnitudes of these two could differ vastly from each other. By the same token, each of the renormalized primitive divergent integrals $< M_0^3 >_R$, $< M_0^4 v_0 >_R$, and $< M_0^4 v_0^2 >_R$ should be regarded as an individual parameter which can only be determined by comparing with measurable results.

Therefore, eq. (4.29) can not be used to predict the size of the cosmological constant. Rather, one should use the measured magnitude of Λ to impute that $< M_0^4 >_R + < M_0^4 v_0^2 >_R \sim 10^{-120} M_{\mathrm{pl}}^4$. Hence the cosmological constant problem can be evaded. It's also worth mentioning that there are other contributions to the cosmological constant, such as

the QCD-induced and the Majorana Higgs-induced phase transitions. In principal, these phase transitions can be treated in a similar way as delineated above. Note that the

In the last part of this subsection, we turn our attention to the possible experimental evidences of the composite vierbein. When we write out the effective Lagrangian (4.25), we only retain the lowest order terms of the non-condensated E/H spinor bilinears, under the assumption that the other terms are negligible at low energies. At an elevated energy level, additional terms could become relevant. As an example, let's examine the fermion Lagrangian term with two non-condensated E spinor bilinears. Therefore there are four remaining spinor fields in the effective Lagrangian

$$\frac{<M_0^2>_R}{(<M_0^3>_R)^2} \left\langle \gamma^\mu \gamma^\nu \psi \partial_\mu \bar{\psi} \, \psi \partial_\nu \bar{\psi} \right\rangle, \tag{4.33}$$

where $\mu \neq \nu$ and ψ denotes the dressed spinor field with the field renormalization (4.26). We specifically write down the Lagrangian term in flat space-time to highlight the fact that the partial derivatives ∂_μ and ∂_ν are orthogonal as opposed to being aligned, which is very different from a typical scalar field Lagrangian that involves two partial derivatives.

The Lagrangian term could be considered as two perpendicular fermion currents interacting with each other. If such an event is detected experimentally, it would be a telltale sign that one of the composite $\hat{e}$ fields in the effective fermion kenetic Lagrangian term (4.25a) is broken down into the underlying fermion fields. In other words, such an event exposes the fermion compositeness of the space-time metric. The coefficient of the above four-fermion term is of mass dimension -4. It implies an energy scale

$$M_{comp} \sim \left(\frac{(<M_0^3>_R)^2}{<M_0^2>_R} \right)^{\frac{1}{4}} \approx \left(\frac{<M_0^3>_R}{M_{\text{pl}}} \right)^{\frac{1}{2}}, \tag{4.34}$$

above which the space-time fabric disintegrates into fermionic smithereens. Note that we have no theoretical recourse to pinpoint the exact compositeness scale M_{comp}, since it involves the primitive divergent integral $<M_0^3>_R$ which could only be ascertained via empirical means as per the IR mantra. Of particular interest is the fact that the compositeness scale M_{comp} is different from the Planck scale M_{pl}. They are two unrelated scales. The compositeness scale is the scale above which there could be measurable evidences of the composite vierbeins broken down into the fermionic components, whereas the Planck scale is the scale at which higher-order gravitational Lagrangian terms become relevant. Therefore, if $M_{comp} < M_{\text{pl}}$, we could have a chance of probing the so-called Planck-scale physics at an energy level below the Planck scale.

4.4 Extended symmetries and gravi-weak interaction

In the case of the composite Higgs fields, we have benefited from various clues guiding us towards the conclusion that there are three specific fermions driving the electroweak symmetry breaking process, namely, the top quark, tau neutrino, and tau lepton condensations.[16] When it comes to the composite vierbeins, due to lack of evidences we are not able to speculate which of the standard model fermions are involved in the vierbein-related condensations. Nonetheless, we can still make progress by investigating the general symmetry properties of the composite vierbeins.

Considering the gauge transformation characteristics of the standard model fermions, we cast the effective vierbeins into three categories

$$i\hat{e}_L \sim E_L = \psi_L \overline{D_L \psi_L}, \tag{4.35}$$

$$i\hat{e}_{Ru} \sim E_u = \psi_{Ru} \overline{D_R \psi_{Ru}}, \tag{4.36}$$

$$i\hat{e}_{Rd} \sim E_d = \psi_{Rd} \overline{D_R \psi_{Rd}}, \tag{4.37}$$

where D_L and D_R are the left- and right-handed gauge-covariant derivatives, ψ_L is any left-handed doublet such as $u_L + d_L$, ψ_{Ru} is any right-handed up-type singlet such as u_R, and ψ_{Rd} is any right-handed down-type singlet such as d_R.

Given the freedom afforded by the above chiral vierbeins unconstrained by the Clifford subspace $\{\gamma_a\}$, the gauge symmetry groups (2.48) we studied earlier can be expanded to

$$Spin(1,3)_L \times Spin(1,3)_R \times Spin(1,3)_{WL} \times Spin(1,1)_{WR} \times U(1)_{WR} \times SU(3)_C \times U(1)_{B-L}, \tag{4.38}$$

where $Spin(1,3)_L$ and $Spin(1,3)_R$ are the left- and right-handed Lorentz gauge groups respectively. The $Spin(1,3)_{WL}$ group generators are

$$\Gamma_2\Gamma_3, \ \Gamma_3\Gamma_1, \ \Gamma_1\Gamma_2, \ \Gamma_0\Gamma_1, \ \Gamma_0\Gamma_2, \ \Gamma_0\Gamma_3, \tag{4.39}$$

where $\Gamma_0 = \gamma_1\gamma_2\gamma_3$ and the $Spin(1,1)_{WR}$ group generator is

$$\Gamma_0\Gamma_3. \tag{4.40}$$

Note that the $Spin(1,3)_{WL}$ group comprises the regular weak group $SU(2)_{WL}$ generated by $\{\Gamma_i\Gamma_j\}$ as well as the weak-boosts generated by $\{\Gamma_0\Gamma_i\}$. These are the counterparts of the spacial rotations generated by $\{\gamma_i\gamma_j\}$ and the Lorentz boosts generated by $\{\gamma_0\gamma_i\}$. The weak-boost is not a group on its own, it's rather the coset $Spin(1,3)_{WL}/SU(2)_{WL}$. Given the relationships such as $\Gamma_0\Gamma_3 = I\Gamma_1\Gamma_2$, we also call $\{\Gamma_0\Gamma_i\}$ the pseudo-weak generators. Henceforth, we will use the terms weak-boost and pseudo-weak interchangeably.

The vierbeins should transform as vectors under the gauge transformations of $Spin(1,3)_L \times Spin(1,3)_R \times Spin(1,3)_{WL} \times Spin(1,1)_{WR}$. Consequently, the left-handed $\hat{e}_L$ ought to be valued in the extended Clifford algebraic subspace spanned by the $4*4$ multivectors

$$\gamma_a, \quad \gamma_a\Gamma_0\Gamma_i, \tag{4.41}$$

where $a = 0, 1, 2, 3$ and $i = 1, 2, 3$. The right-handed $\hat{e}_{Ru}$ is valued in the Clifford algebraic subspace spanned by the 4 multivectors

$$\gamma_a P_+, \tag{4.42}$$

while the right-handed $\hat{e}_{Rd}$ is valued in the Clifford algebraic subspace spanned by the 4 multivectors

$$\gamma_a P_-, \tag{4.43}$$

where $P_{\pm}$ is the projection operator (2.10).

Alternatively, $\hat{e}_L$ may take values in the complimentary Clifford algebraic subspace spanned by the $4 * 4$ multivectors

$$\gamma_a I, \quad \gamma_a \Gamma_i \Gamma_j. \tag{4.44}$$

As such, $\hat{e}_L$ could develop VEVs valued in the pseudo-vector subspace $\{\gamma_a I\}$, instead of the regular vector subspace $\{\gamma_a\}$. The same logic goes for $\hat{e}_{Ru}$ and $\hat{e}_{Rd}$. Nervelessness, our discussion in this paper is concentrated on the regular vector-type $\hat{e}_L$, $\hat{e}_{Ru}$, and $\hat{e}_{Rd}$ vierbeins.

With the extended symmetries, the chiral gauge-covariant derivatives of the left- and right-handed spinor fields $\psi_{L/R}(x)$ are defined by

$$D_L \psi_L = (d + \hat{\omega}_L + \hat{W}_L + \hat{W}'_L)\psi_L + \psi_L(\hat{G} + \hat{A}_{BL}), \tag{4.45}$$

$$D_R \psi_R = (d + \hat{\omega}_R + \hat{W}_R + \hat{W}'_R)\psi_R + \psi_R(\hat{G} + \hat{A}_{BL}), \tag{4.46}$$

where the left-hand weak $SU(2)_{WL}$ gauge field $\hat{W}_L$, the right-hand weak $U(1)_{WR}$ gauge field $\hat{W}_R$, the color $SU(3)_C$ gauge field $\hat{G}$, and the BL $U(1)_{B-L}$ gauge field $\hat{A}_{BL}$ follow the same definition as specified previously (2.62). The newly introduced gauge fields are the left- and right-handed spin connections of the $Spin(1,3)_L$ and $Spin(1,3)_R$ Lorentz gauge groups

$$\hat{\omega}_L = \frac{1}{4}\omega_{L\mu}^{ab}\gamma_a\gamma_b dx^\mu, \tag{4.47}$$

$$\hat{\omega}_R = \frac{1}{4}\omega_{R\mu}^{ab}\gamma_a\gamma_b dx^\mu, \tag{4.48}$$

the pseudo-weak portion of the extended left-handed weak gauge field

$$\hat{W}'_L = \frac{1}{2}(W_{L\mu}'^1\Gamma_0\Gamma_1 + W_{L\mu}'^2\Gamma_0\Gamma_3 + W_{L\mu}'^3\Gamma_0\Gamma_3)dx^\mu, \tag{4.49}$$

and the pseudo-weak portion of the extended right-handed weak gauge field

$$\hat{W}'_R = \frac{1}{2}W_{R\mu}'^3\Gamma_0\Gamma_3 dx^\mu. \tag{4.50}$$

The combination of the regular weak $\hat{W}_L$ and the pseudo-weak $\hat{W}'_L$ constitutes the overall gauge fields of $Spin(1,3)_{WL}$

$$\hat{\omega}_{Iso-L} = \hat{W}_L + \hat{W}'_L. \tag{4.51}$$

We call $\hat{\omega}_{Iso-L}$ the isospin connection since it is in many ways analogous to the spin connection of the Lorentz group. The chiral spin connections $\hat{\omega}_L$ and $\hat{\omega}_R$ are crucial in maintaining the chiral Lorentz gauge covariance of $D_L\psi_L$ and $D_R\psi_R$, which are leveraged in conjunction with the chiral vierbeins to ensure the chiral Lorentz gauge invariance of the Lagrangian terms.

The gauge interaction curvature 2-forms for $\hat{\omega}_L$, $\hat{\omega}_R$, $\hat{W}'_L$, and $\hat{W}'_R$ are expressed as

$$\hat{R}_L = d\hat{\omega}_L + \hat{\omega}_L \wedge \hat{\omega}_L, \tag{4.52}$$

$$\hat{R}_R = d\hat{\omega}_R + \hat{\omega}_R \wedge \hat{\omega}_R, \tag{4.53}$$

$$\hat{F}_{W'L} = d\hat{W}'_L + \hat{W}'_L \wedge \hat{W}_L + \hat{W}_L \wedge \hat{W}'_L, \tag{4.54}$$

$$\hat{F}_{W'R} = d\hat{W}'_R, \tag{4.55}$$

where the outer product between gauge fields vanishes for the abelian interaction $\hat{F}_{W'R}$. The regular left-handed weak force is appended with an additional cross product term of $\hat{W}'_L$

$$\hat{F}_{WL} = d\hat{W}_L + \hat{W}_L \wedge \hat{W}_L + \hat{W}'_L \wedge \hat{W}'_L. \tag{4.56}$$

The combination of the weak $\hat{F}_{WL}$ and the pseudo-weak $\hat{F}_{W'L}$ constitutes the overall gauge curvature 2-form of the weak $Spin(1,3)_{WL}$

$$\hat{R}_{Iso-L} = \hat{F}_{WL} + \hat{F}_{W'L}. \tag{4.57}$$

We call $\hat{R}_{Iso-L}$ the isospin connection curvature 2-form (or the extended weak force) in parallel with the spin connection curvature 2-form $\hat{R}_L$ of the Lorentz group.

The local gauge- and diffeomorphism-invariant Lagrangian terms of the world are similar to the ones we inspected earlier in Section 2.4, provided that the chirality and isospin conjugations are taken care of. The following are some examples

$$\mathcal{L}_{Fermion} \sim i \left\langle I\hat{e}_L \wedge \hat{e}_L \wedge \hat{e}_L \wedge \psi_L \overline{D_L \psi_L} \right\rangle \tag{4.58a}$$

$$+ \ i \left\langle I\hat{e}_{Rd} \wedge \hat{e}_{Ru} \wedge \hat{e}_{Rd} \wedge \psi_{Ru} \overline{D_R \psi_{Ru}} \right\rangle \tag{4.58b}$$

$$+ \ i \left\langle I\hat{e}_{Ru} \wedge \hat{e}_{Rd} \wedge \hat{e}_{Ru} \wedge \psi_{Rd} \overline{D_R \psi_{Rd}} \right\rangle, \tag{4.58c}$$

$$\mathcal{L}_{Gravity-Left} \sim \left\langle I\hat{e}_L \wedge \hat{e}_L \wedge \hat{R}_L \right\rangle, \tag{4.58d}$$

$$\mathcal{L}_{Gravity-Right} \sim \left\langle I(\hat{e}_{Ru} \wedge \hat{e}_{Rd} + \hat{e}_{Rd} \wedge \hat{e}_{Ru}) \wedge \hat{R}_R \right\rangle, \tag{4.58e}$$

$$\mathcal{L}_{CC-Left} \sim \left\langle I\hat{e}_L \wedge \hat{e}_L \wedge \hat{e}_L \wedge \hat{e}_L \right\rangle, \tag{4.58f}$$

$$\mathcal{L}_{CC-Right} \sim \left\langle I\hat{e}_{Ru} \wedge \hat{e}_{Rd} \wedge \hat{e}_{Ru} \wedge \hat{e}_{Rd} \right\rangle, \tag{4.58g}$$

where the alternation between $\hat{e}_{Ru}$ and $\hat{e}_{Rd}$ is because of the properties $\hat{e}_{Ru} = P_- \hat{e}_{Ru} P_+$ and $\hat{e}_{Rd} = P_+ \hat{e}_{Rd} P_-$.

In view of the extended symmetries (4.38) of the Lagrangian of the world, let's revisit the diffeomorphism and Lorentz gauge symmetry breaking triggered by the nonzero VEV of the vierbein. It can be checked that the flat space-time VEV (3.5) of the vierbein violates the gauge symmetries $Spin(1,3)_L \times Spin(1,3)_R \times Spin(1,1)_{WR}$ and the coset $Spin(1,3)_{WL}/SU(2)_{WL}$. The remaining gauge symmetries are $SU(3)_C \times SU(2)_{WL} \times U(1)_{WR} \times U(1)_{B-L}$ plus the residual global Lorentz symmetry.

Note that the VEV magnitudes and orientations of the three vierbeins $\hat{e}_L$, $\hat{e}_{Rd}$, and $\hat{e}_{Rd}$ may not be aligned with each other. That said, we have the freedom to re-scale and re-orientate (via global Lorentz rotations) the corresponding fermions, so that the three

vierbeins take the same flat space-time VEV everywhere all at once. Hence it is assured that all fermions, regardless of their chirality and isospin stripes, share the universal Minkowski flat space time metric (3.5).

However, there are some factors that can not be re-scaled away. These are the differences between the coefficients of the chiral gravity (4.58d) (4.58e) Lagrangian terms and the differences between the coefficients of the chiral cosmological constant (4.58f) (4.58g) Lagrangian terms. As a result, the left- and right-handed fermions experience different strengths of gravitational interactions. Under normal conditions, this is not a problem since the left- and right-handed matters are usually commensurate with each other. Thus we can't really discern which chiral gravity is stronger since we routinely observe the collective gravitational interaction governed by a combined effective gravitational constant.

The only exception is when there is imbalance between the left- and right-handed matters. For example, let's assume that the left-handed gravitational constant is larger than the right-handed counterpart. We could observe an unexplainable drop of gravitational force compared with expectation if there is an excess of right-handed matter. In this regard, we would like to draw attention to the right-handed neutrinos, since they are endowed with extremely large Majorana masses. If there is a large concentration of the right-handed neutrinos in certain parts of the universe, the discrepancies between the chiral gravitational forces would possibly be revealed.

In the last part of this subsection, we turn to a novel kind of Lagrangian terms

$$\mathcal{L}_{Gravity-Weak-Left} \sim \left\langle I\hat{e}_L \wedge \hat{e}_L \wedge \hat{R}_{Iso-L} \right\rangle, \tag{4.59a}$$

$$\mathcal{L}_{Holst-Weak-Left} \sim \left\langle \hat{e}_L \wedge \hat{e}_L \wedge \hat{R}_{Iso-L} \right\rangle, \tag{4.59b}$$

where $\hat{R}_{Iso-L}$ is the left-handed isospin connection curvature 2-form (4.57). As indicated by the Lagrangian names, these terms bear close resemblance to the regular gravity (2.78) and Holst (2.83) Lagrangian terms. We mentioned earlier that under normal circumstances a Lagrangian term with a single Yang-Mills field curvature 2-form is identically zero. It's the extended symmetries (4.38) and the extended vierbein space (4.41) that make the above single-curvature terms possible.

Let's derive the field equations for the left-handed gravity Lagrangian (4.58d) and the left-handed gravi-weak Lagrangian (4.59a) by varying with $\hat{e}_L$, $\hat{\omega}_L$, and $\hat{\omega}_{Iso-L}$, respectively. The resultant extended Einstein-Cartan equations read (for brevity sake we drop the L subscripts)

$$\frac{1}{8\pi G}(\hat{R} \wedge \hat{e} + \hat{e} \wedge \hat{R})I + \frac{1}{8\pi G_{Iso}}(\hat{R}_{Iso} \wedge \hat{e} + \hat{e} \wedge \hat{R}_{Iso})I = \mathbb{T}, \tag{4.60}$$

$$\frac{1}{8\pi G}(\hat{T} \wedge \hat{e} - \hat{e} \wedge \hat{T})I = \mathbb{S}, \tag{4.61}$$

$$\frac{1}{8\pi G_{Iso}}(\hat{T}_{Iso} \wedge \hat{e} - \hat{e} \wedge \hat{T}_{Iso})I = \mathbb{S}_{Iso}, \tag{4.62}$$

where G is the regular gravitational constant, G_{Iso} is the iso-gravitation constant for the gravi-weak Lagrangian (4.59a), $\mathbb{T}$ is the regular energy-momentum current 3-form, $\mathbb{S}$ is the

regular spin current 3-form, $\mathbb{S}_{Iso}$ is the isospin current 3-form, $\hat{T}$ is the regular torsion 2-form (3.3), and $\hat{T}_{Iso}$ is the iso-torsion 2-form

$$\hat{T}_{Iso} = d\hat{e} + \hat{\omega}_{Iso} \wedge \hat{e} + \hat{e} \wedge \hat{\omega}_{Iso}. \tag{4.63}$$

Compared with the regular Einstein-Cartan equations, we have an additional equation for the iso-torsion $\hat{T}_{Iso}$. Given that the weak gauge field $\hat{W}_L$ (as part of the isospin connection $\hat{\omega}_{Iso-L}$) is susceptible to the SSB effects from the electroweak Higgs mechanism, we expect that the isospin connection $\hat{\omega}_{Iso-L}$ would have a different kind of impact on gravity than the regular spin connection $\hat{\omega}_L$.

This sort of modification to the gravitational equations could have cosmological implications. As we know, the concordance ΛCDM cosmological model is besieged with a multitude of discordances,[105,106,107] with the most acute one being the Hubble tension.[108,109,110] Various modified gravity models[111,112,113] have been proposed to remediate the shortcomings of the ΛCDM model. Some noteworthy modified gravity theories invoke either a characteristic Hubble scale h_0[14,17] or a characteristic acceleration scale a_0,[114] below which gravity changes behavior and departs from Einstein's theory of general relativity. We hope that the gravi-weak interplay delineated above could possibly shed some light on these characteristic scales.

5 Conclusions

The naturalness problems have been front and center in physics researches.[94,95,96,63,64,65] The cosmological constant problem is arguably the most severe naturalness problem in physics, with the runner-up being the Higgs mass/electroweak hierarchy problem. With the goal of addressing the naturalness problems, we propose that each and every symmetry-breaking bosonic field, such as the vierbein field or the Higgs field, is an effective representation of a unique multi-fermion condensation via the dynamical symmetry breaking (DSB) mechanism.

Our research is originated from drawing an unappreciated distinction between two imaginary numbers. The first one is the bona fide imaginary number i which governs the quantum world. The other one is the unit pseudoscalar I masquerading as imaginary number which shows up in the definition of spinors, gauge fields, and their transformations. In the Clifford algebra approach, we can manage to stay away from the genuine imaginary number i in classical field equations. This is demonstrated by the Clifford algebraic Dirac equation (3.44), where the conventional i is replaced by the pseudoscalar I as long as we stick to the regime of applying the surrogate I to the right side of the algebraic spinor.

However, when it comes to the fermion Lagrangian, the imaginary number i is irreplaceable in both the kinetic and mass terms. The conundrum of the *quantum i* enmeshed in the *classical* Lagrangian indicates that the regular classical Lagrangian terms might be of quantum origin. We propound that the imaginary number in the fermion kinetic and mass terms stems from the quantum loop integrals related to odd numbers of fermion-antifermion condensations. On the other hand, if there are even fermion-antifermion pairs

involved in the condensations, there is no i in the coefficient of the Lagrangian term, such as the Yang-Mills, gravity, and cosmological constant terms.

We would like to underscore that there is no fixed mass/energy scale in the pre-symmetry breaking world, since all the original Lagrangian coefficients are dimensionless. In other words, the pre-condensation Lagrangian terms are scale-invariant. All the mass scales of the universe, including the Planck scale, are emergent phenomena driven by the quantum condensations via the DSB mechanism. This pan-emergence paradigm of mass scales parallels Landau's symmetry breaking scheme in condensed matter physics characterized by the nonzero order parameters. One exception to this rule might be the QCD scale Λ_{QCD} possibly generated by a topological order in the quark-gluon plasma and nucleons.[18]

Let's take stock of the various symmetry-breaking quantum condensations examined in this paper. The first category of quantum condensation involves a fermion-antifermion pair of the same chirality, with an extra gauge-covariant derivative sandwiched in between. The effective representation of this sort of condensation corresponds to the vierbein field $\hat{e}$ in the Lorentz gauge theory of gravity. In the composite vierbein scenario, fermions play the dual role of interacting with the space-time metric as well as being the metric. The specific standard model fermion participants in the condensation are yet to be identified. The symmetries of the model encompass an extended weak group $Spin(1,3)_{WL} \supset SU(2)_{WL}$ allowed by the beyond-vector vierbein space. A gravi-weak interaction is thus permitted between the vierbein and the weak gauge field.

The local Lorentz and pseudo-weak symmetries are spontaneously broken when the vierbeins acquire nonzero VEVs via the DSB mechanism. One interesting implication is that there are two different energy scales. One is the compositeness scale above which there are measurable evidences of the composite vierbeins broken down into the fermionic components, whereas the other is the Planck scale at which the higher-order gravitational Lagrangian terms become relevant in quantum gravity. The second implication is that the coefficients of the effective cosmological constant and gravity Lagrangian terms are dictated by the divergent loop integrals of the quantum condensations. The cosmological constant problem can be evaded if we take abundant precaution in the renormalization procedure that entails *multiplications* of divergent integrals.

The second category of quantum condensation involves a neutrino-antineutrino pair with the same right-handed chirality. The effective description of this sort of condensation is the Majorana Higgs field ϕ_M. It is a Higgs-like field whose VEV generates mass for the Z' gauge field as well as the Majorana mass for the right-handed neutrino. The Majorana mass is capable of directly mixing neutrinos from different generations, which is evidenced in the observation of neutrino oscillations.[28, 29, 30] The Clifford algebra $Cl(0,6)$ allows for a weaker form of charge conjugation which does not invoke particle-antiparticle interchange. Consequently, the Clifford algebraic Majorana mass conserves lepton number, which is different from the traditional Majorana mass term. This might be the underlying reason that no evidence has ever been found for the neutrinoless double beta decay.[39, 40]

The third category of quantum condensation involves a fermion-antifermion pair with opposite chirality. Belonging to this category, the standard model Higgs field is an effective description of the top quark condensation ϕ_t, while the other two yet-to-be-detected com-

posite Higgs fields ϕ_{ν_τ} and ϕ_τ correspond to the tau neutrino and tau lepton condensations. The VEVs of these three Higgs fields generate masses for the $Z^0/W^\pm$ gauge fields as well as the Dirac masses for the standard model fermions. The composite Higgs mass is naturally small, since at the electroweak scale there is no elementary Higgs mass term to be modified by any higher order quantum perturbation from external sources.

The three estimated Higgs VEVs have a hierarchical structure $v_t \approx 246\ GeV$, $v_{\nu_\tau} \approx 41\ GeV$ and $v_\tau \approx 2.5\ GeV$, where the top-quark Higgs VEV v_t is much larger than the other two. Nonetheless, the tau-neutrino Higgs VEV v_{ν_τ} plays a non-negligible role in the electroweak scale saturation, which might be the root cause of the significant deviation of the measured W-boson mass from the standard model prediction.[60] Additionally, given the intrinsic connection between the muon and the tau-neutrino Higgs field ϕ_{ν_τ}, it is worthwhile to investigate the tau-neutrino Higgs field's contribution to the muon anomalous magnetic moment, especially in light of the recent muon $g-2$ measurement which confirms a deviation from the standard model prediction.[61]

The fourth category of quantum condensation involves a fermion-antifermion pair with opposite chirality, the same as the regular composite Higgs field. However, the Clifford algebra framework allows for a non-scalar antisymmetric-tensor composite Higgs field ϕ_{AT} which could potentially break both the electroweak and Lorentz symmetries. The magnitude of its VEV could be extremely small compared with the electroweak scale, rendering its effects unobservable in laboratories. The ethereal antisymmetric-tensor Higgs field VEV might manifest itself as the large-scale anisotropies of the universe.[44, 45, 46, 47, 48, 49]

The fifth category of quantum condensation involves two fermion-antifermion pairs. There are six composite Φ fields corresponding to this sort of four-fermion condensations. In contrast to the other four types of composite fields, these scalar Φ fields are invariant under the local gauge transformations. Instead, three of the Φ fields are tied to a $U_\alpha(1)$ global symmetry, which transforms all the right-handed fermions by the same phase $e^{\alpha I}$, in a manner similar to the Peccei-Quine $U(1)_{PQ}$ symmetry. The other three of the Φ fields are tied to a $U_\beta(1)$ global symmetry. It transforms the up-type quarks (u_R, c_R, t_R) and down-type leptons (e_R, μ_R, τ_R) by the phase $e^{\beta I}$, whereas it transforms the down-type quarks (d_R, s_R, b_R) and up-type leptons (ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$) by the opposite phase $e^{-\beta I}$.

Upon acquiring nonzero VEVs, these six composite fields break the $U_\alpha(1)$ and $U_\beta(1)$ global symmetries respectively. Their VEVs play a pivotal role in establishing the relative magnitudes of the effective Yukawa coupling constants, and consequently giving rise to the fermion mass hierarchies. The Dirac masses of the ν_τ, ν_μ and ν_e neutrinos are estimated as $29,500 MeV$, $40 MeV$ and $21 MeV$, respectively. Note that these estimations are meant to be the Dirac masses, as opposed to the significantly smaller seesaw effective masses.

Due to the explicit symmetry breaking originated from the quantum anomaly and instanton effects, the otherwise massless Nambu-Goldstone bosons of the α-type Φ fields acquire masses and turn into the pseudo-Nambu-Goldstone bosons in a similar fashion as the axions. Historically the axions have been proposed as a possible solution to the strong CP and dark matter problems. In this regard, we speculate that the (pseudo-)Nambu-Goldstone bosons of the β-type Φ fields could also be viable dark matter candidates.

In summary, the proposition in this paper is at the conservative end of the physics model

building spectrum. Our thesis is that the fundamental particles/fields of the universe are the garden-variety standard model fermions (plus the right-handed neutrinos) accompanied by a handful of "good old-fashioned" gauge fields. That is all there is. There are neither extra dimensions nor exotic branes. The novel bit we bring to the table is the insight that there is a kaleidoscope of quantum condensations which make the world as complex and enchanting as it is.

References

[1] D. Hestenes, *Space-Time Algebra*, (Gordon and Breach, New York, 1966).

[2] D. Hestenes and G. Sobczyk, *Clifford algebra to geometric calculus: a unified language for mathematics and physics*, (Kluwer Academic Publishers, Dordrecht, 1984).

[3] M. Pavsic, *The Landscape of Theoretical Physics: A Global View. From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle*, (Kluwer Academic Publishers, Dordrecht, 2001).

[4] P. Lounesto, *Clifford algebras and spinors*, (Cambridge University Press, Cambridge, 2001).

[5] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, (Cambridge University Press, Cambridge, 2003).

[6] R. Boudet, *Quantum Mechanics in the Geometry of Space-time: Elementary Theory*, (Springer Science & Business Media, 2011).

[7] W. E. Baylis, *Clifford (Geometric) Algebras: with applications to physics, mathematics, and engineering*, (Springer Science & Business Media, Boston, 2012).

[8] J. Vaz Jr. and R. da Rocha Jr., *An Introduction to Clifford Algebras and Spinors*, (Oxford University Press, Oxford, 2016).

[9] D. Hestenes, *Space-time structure of weak and electromagnetic interactions*, *Found. Phys.* **12** (1982) 153.

[10] J. Chisholm and R. Farwell, *J. Phys. A* **22** (1989) 1059.

[11] J. Chisholm and R. Farwell, *J. Phys. A* **32** (1999) 2805.

[12] G. Trayling and W. E. Baylis, *A geometric basis for the standard-model gauge group*, *J. Phys. A* **34** (2001) 3309.

[13] W. Lu, *Yang-Mills interactions and gravity in terms of Clifford algebra*, *Adv. Appl. Clifford Algebras* **21** (2011) 145.

[14] W. Lu, *Modified Einstein-Cartan gravity and its implications for cosmology*, [arXiv:1406.7555].

[15] W. Lu, *Electroweak and Majorana Sector Higgs Bosons and Pseudo-Nambu-Goldstone Bosons*, [arXiv:1603.0469].

[16] W. Lu, *A Clifford algebra approach to chiral symmetry breaking and fermion mass hierarchies*, *Int. J. Mod. Phys. A* **32** (2017) 1750159.

[17] W. Lu, *Dynamical symmetry breaking and negative cosmological constant*, *Int. J. Mod. Phys. A* **34** (2019) 1950136.

[18] W. Lu, *Quark–gluon plasma and nucleons à la Laughlin, Int. J. Geom. Methods Mod. Phys* **19** (2022) 2250061.

[19] R. Utiyama, *Phys. Rev.* **101** (1956) 1597.

[20] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, *Rev. Mod. Phys.* **48** (1976) 393.

[21] A. Dimakis and F. Muller-Hoissen, *Class. Quantum Grav.* **8** (1991) 2093.

[22] E. W. Mielke, *Int. J. Theor. Phys.* **40** (2001) 171.

[23] A. Randono, *Gauge gravity: a forward-looking introduction*, [arXiv:1010.5822].

[24] J. C. Pati and A. Salam, *Phys. Rev. D***10** (1974) 275.

[25] K. S. Babu, J. C. Pati, F. Wilczek, *Nucl. Phys. B***566** (2000) 33, [hep-ph/9812538].

[26] C. Furey, *Phys. Lett. B* **785**, (2018) 84.

[27] S. W. MacDowell and F. Mansouri, *Phys. Rev. Lett.***38** (1977) 739.

[28] Super-Kamiokande Collaboration, (Y. Fukuda *et al.*), *Phys. Rev. Lett.* **81** (1998) 1562.

[29] SNO Collaboration, (Q. R. Ahmad *et al.*), *Phys. Rev. Lett.* **89** (2002) 011301.

[30] KamLAND Collaboration, (K. Eguchi *et al.*). *Phys. Rev. Lett.* **90** (2003) 021802.

[31] C. Doran *et al. Lie groups as spin groups, J. Math. Phys.* **34** (1993) 3642.

[32] S. Weinberg, *Phenomenological lagrangians, Physica A* **96** (1979) 327.

[33] J. F. Donoghue, *Phys. Rev. D* **50** (1994) 3874, [gr-qc/9405057].

[34] C.P. Burgess, *Introduction to effective field theory*, (Cambridge University Press, Cambridge, 2020).

[35] S. Holst, *Phys. Rev. D***53** (1996) 5966, [gr-qc/9511026].

[36] H. T. Nieh and M. L. Yan, *J. Math. Phys.* **23** (1982) 373.

[37] P. Sikivie, L. Susskind, M. B. Voloshin and V. I. Zakharov, *Nucl. Phys. B* **173** (1980) 189.

[38] S. Davidson, E. Nardi, and Y. Nir, *Phys. Rep.* **466** (2008) 105.

[39] D.Q. Adams *et al.*, *Improved Limit on Neutrinoless Double-Beta Decay in Te 130 with CUORE, Phys. Rev. Lett.* **124** (2020) 122501.

[40] M. Agostini *et al.*, *Final results of GERDA on the search for neutrinoless double-β decay, Phys. Rev. Lett.* **125** (2020) 252502.

[41] T.D. Lee, A theory of spontaneous T violation, *Phys. Rev. D***8** (1973) 1226.

[42] G.C. Branco *et al.*, *Theory and phenomenology of two-Higgs-doublet models, Phys. Rep.* **516** (2012) 1.

[43] V. Keus , S.F. King, and S. Moretti, *Three-Higgs-doublet models: symmetries, potentials and Higgs boson masses, JHEP.* **1** (2014) 1.

[44] A. de Oliveira-Costa, M. Tegmark, M. Zaldarriaga and A. Hamilton, *Phys. Rev. D* **69** (2004) 063516.

[45] H. K. Eriksen, F. K. Hansen, A. J. Banday, K. M. Gorski and P. B. Lilje, *Astrophys. J.* **605** (2004) 14.

[46] D. J. Schwarz, G. D. Starkman, D. Huterer and C. J. Copi, *Phys. Rev. Lett.* **93** (2004) 221301.

[47] K. Land and J. Magueijo, *Phys. Rev. Lett.* **95** (2005) 071301.

[48] M.J. Longo, *Detection of a dipole in the handedness of spiral galaxies with redshifts z ∼ 0.04*, *Phys. Lett. B* **699** (2011) 224.

[49] P.K. Aluri *et al.*, *Is the observable universe consistent with the cosmological principle?*, [arXiv:2207.05765].

[50] G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking, NATO Sci. Ser. B* **59** (1980) 135.

[51] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38** (1977) 1440.

[52] S. Weinberg, *Phys. Rev. Lett.* **40** (1978) 223.

[53] F. Wilczek, *Phys. Rev. Lett.* **40** (1978) 279.

[54] J. E. Kim, *Phys. Rev. Lett.* **43** (1979) 103.

[55] M. A. Shifman, V. I. Vainstein and V. I. Zakharov, *Nucl. Phys. B* **166** (1980) 493.

[56] A. P. Zhitnitskii, *Sov. J. Nucl. Phys.* **31** (1980) 260.

[57] M. Dine, W. Fischler and M. Srednicki, *Phys. Lett. B* **104** (1981) 199.

[58] ATLAS Collaboration, (G. Aad *et al.*), *Phys. Lett. B* **716** (2012) 1.

[59] CMS Collaboration, (S. Chatrchyan *et al.*), *Phys. Lett. B* **716** (2012) 30.

[60] T. Aaltonen, *et al.* (CDF), *High-precision measurement of the W boson mass with the CDF II detector, Science* **376** (2022) 170.

[61] B. Abi, *et al.*, *Measurement of the positive muon anomalous magnetic moment to 0.46 ppm*, *Phys. Rev. Lett.* **126** (2021) 141801.

[62] J. M. Cornwall, R. Jackiw, and E. Tomboulis, *Phys. Rev. D* **10** (1974) 2428.

[63] G.F. Giudice, *Naturally speaking: the naturalness criterion and physics at the LHC. Perspectives on LHC physics* **270** (2008) 155.

[64] M. Dine, *Naturalness under stress, Annu. Rev. Nucl. Part. Sci* **65** (2015) 43.

[65] N. Craig, *Naturalness: A Snowmass White Paper*, [arXiv:2205.05708].

[66] S. Weinberg, *Phys. Rev. D* **13** (1976) 974.

[67] L. Susskind, *Phys. Rev. D* **20** (1979) 2619.

[68] E. Eichten and K.D Lane, *Phys. Let. B* **90** (1980) 125.

[69] Y. Nambu, in *New Theories in Physics, Proceedings of the XI International Symposium on Elementary Particle Physics*, eds. Z. Ajduk, S. Pokorski, and A. Trautman (World Scientific, Singapore, 1988), p. 1.

[70] V. A. Miransky, M. Tanabashi and K. Yamawaki, *Phys. Lett. B* **221** (1989) 177.

[71] W. J. Marciano, *Phys. Rev. Lett.* **62** (1989) 2793.

[72] W. A. Bardeen, C. T. Hill and M. Lindner, *Phys. Rev. D* **41** (1990) 1647.

[73] G. Cvetic, *Rev. Mod. Phys.* **71** (1999) 513.

[74] S. P. Martin, *Phys. Rev. D* **44** (1991) 2892.

[75] G. Cvetic and C. Kim, *Mod. Phys. Lett. A* **9** (1994) 289.

[76] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, *Nucl. Phys. B* **658** (2003) 203.

[77] A. Smetana, *Eur. Phys. J. C* **73** (2013) 2513.

[78] J. Krog and C. T. Hill, *Phys. Rev. D* **92** (2015) 093005.

[79] Y. Nambu and G. Jona-Lasinio, *Dynamical model of elementary particles based on an analogy with superconductivity. I*, *Phys. Rev.* **122** (1961) 345.

[80] J. Bardeen,L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108** (1957) 1175.

[81] V. E. Rochev, *J. Phys. A* **30** 3671 (1997).

[82] R. G. Jafarov and V. E. Rochev, *Central Eur. J. Phys.* **2** (2004) 367.

[83] S. P. Klevansky, *Rev. Mod. Phys.* **64** (1992) 649.

[84] O. A. Battistel, PhD Thesis, (Universidade Federal de Minas Gerais, Brazil, 1999).

[85] O. A. Battistel, A. L. Mota, and M. C. Nemes, *Mod. Phys. Lett. A* **13** (1998) 1597.

[86] O. A. Battistel, G. Dallabona, and G. Krein, *Phys. Rev. D* **77** (2008) 065025, [arXiv:0803.0537].

[87] R. Willey, *Phys. Rev. D* **48** (1993) 2877.

[88] S. L. Adler, *Phys. Rev.* **177** (1969) 2426.

[89] J. S. Bell and R. Jackiw, *Il Nuovo Cimento A* **60** (1969) 47.

[90] Supernova Search Team, (A. G. Riess *et al.*), *Astron. J.* **116** (1998) 1009.

[91] Supernova Cosmology Project, (S. Chatrchyan *et al.*), *Astrophys. J.* **517** (1999) 565.

[92] P. J. E. Peebles and B. Ratra, *The cosmological constant and dark energy*, *Rev. Mod. Phys.* **75** (2003) 559.

[93] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int. J. Mod. Phys. D* **15** (2006) 1753.

[94] S. Weinberg, *Rev. Mod. Phys.* **61** (1989) 1.

[95] J. Sola, *Cosmological constant and vacuum energy: old and new ideas*, *J. Phys. Conf. Ser.* **453** (2013) 012015.

[96] C.P. Burgess, *The cosmological constant problem: why it's hard to get dark energy from micro-physics. 100e Ecole d'Ete de Physique: Post-Planck Cosmology* (2015) 149.

[97] K. Akama, *An Attempt at Pregeometry: Gravity with Composite Metric*, *Prog. Theor. Phys.* **60** (1978) 1900.

[98] G.E. Volovik, *Superfluid 3He-B and gravity*, *Physica B: Condensed Matter* **162** (1990) 222.

[99] A. Hebecker and C. Wetterich, *Spinor gravity*, *Phys. Lett. B* **574** (2003) 269.

[100] C. Wetterich, *Gravity from spinors*, *Phys. Rev. D* **70** (2004) 105004.

[101] D. Diakonov, *Towards lattice-regularized quantum gravity*, [arXiv:1109.0091].

[102] A.A. Vladimirov and D. Diakonov, *Phase transitions in spinor quantum gravity on a lattice*, *Phys. Rev. D* **86** (2012) 104019.

[103] Y.N. Obukhov and F.W. Hehl, *Extended Einstein−Cartan theory à la Diakonov: the field equations*, *Phys. Lett. B* **713** (2012) 321.

[104] G.E. Volovik, *Dimensionless physics, J. Exp. Theor. Phys.* **132** (2021) 727.

[105] P. Bull *et al.*, *Phys. Dark Univ.* **12** (2016) 56.

[106] E. Abdalla *et al.*, *Cosmology intertwined: A review of the particle physics, astrophysics, and cosmology associated with the cosmological tensions and anomalies*, *JHEP* **34** (2022) 49.

[107] L. Perivolaropoulos and F. Skara, *Challenges for ΛCDM: An update, New Astron. Rev.* (2022) 101659.

[108] A. G. Riess *et al.*, *Astrophys. J.* **826** (2016) 56.

[109] P. A. R. Ade *et al.*, *Astron. Astrophys.* **594** (2016) A13.

[110] N. Aghanim *et al.*, *Astron. Astrophys.* **641** (2020) A6.

[111] T.P. Sotiriou and V. Faraoni, *f (R) theories of gravity, Rev. Mod. Phys.* **82** (2010) 451.

[112] S. Capozziello and M. De Laurentis, *Extended theories of gravity, Phys. Rep.* **509** (2012) 167.

[113] B.S. Haridasu, S.L. Cherkas, and V.L. Kalashnikov, *Reference level of the vacuum energy density of the Universe and astrophysical data, Fortschr. Phys.* **68** (2020) 2000047.

[114] M. Milgrom, *Astrophys. J.* **270** (1983) 365.