Topological structures of Möbius-strip fermions and simple-loop bosons as the fundamental massless quantum field excitations in 2D spacetime

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Abstract

In this work, we present a model to treat the relativistic quantum dynamics of massless and massive particles in a 2D Minkowski spacetime. Using a set of three independent 2x2 real-value matrices to represent a time-shift operator $E,$ a space-shift operator P, and a mass operator M, we derive operator equations for massless particles which can be classified into two types of topological structures: the symmetric type-I with commutative E and P, representing a boson, and anti-symmetric type-II with $\{\boldsymbol{E}, \boldsymbol{P}\} = 0$, representing the fermion. We illustrate their topological differences and show that the fermion wave exhibits a twist during propagation like a Möbius strip. In contrast, the type-I boson behaves like a simple loop strip without a twist. The massless boson in our model resembles a 2D photon or a Higgs boson before symmetry breaking, while the fermion resembles a massless 2D Majorana particle. Unlike conventional string theories, we use a Möbius strip and a simple loop as the most fundamental topological structures of quantum field excitations in 2D spacetime, representing fermionic and bosons. As an alternative to the string and loop quantum gravity theories, our approach could potentially serve as potential building blocks to construct elementary particles in the Standard Model, meriting an investigation into their topological properties in 4D spacetime.

Introduction

Relativity and quantum mechanics have been the two major pillars for modern physics since the dawn of the last century[1,2]. They are the most successful physics theories in human history and their predictions have been put to the test with unprecedented accuracy. However, there remain many puzzling mysteries, including some counter-intuitive quantum phenomena such as quantum entanglement[3,4], double-slit self-interference of single particles[5,6], the collapse of a wave function during measurements[7] absence of right-hand neutrinos[8], the mass oscillations of neutrinos[9,10], the causes of three generations of quarks and leptons[11], the physical origin of the Standard Model[12], dark matter[13], dark energy[14], and quantization of gravity[15], etc. In this work, we aim to improve our understanding of the spacetime fabric, its topological structures, and their effects on relativistic quantum fields and particles. We provide a topological analysis of structural deformations as represented by excited quantum fields and particles in a 2D Minkowski spacetime[16,17]. Our analysis is in 2D instead of the actual 4D case, to increase mathematical simplicity while still retaining the core concepts of physics. We present a dual-component model to describe the relativistic quantum dynamics of elementary particles. We will show how this model will naturally lead to the existence of only two kinds of elementary field excitations: fermions with a Mobius strip structure, and bosons with a simple loop structure. We will also elucidate the concepts behind Pauli's exclusion principle, fermionic statistics, and bosonic statistics. a Möbius strip.

Theory

In Newtonian mechanics or Einstein's special relativity, the dynamics of a particle are vastly different from the motion of electromagnetic waves or other types of waves governed by $\left(-\partial^2/c^2\partial t^2+\partial^2/\partial x^2\right)f(t,x)=0$. To unite the realms of particles and waves, a model with a dual-component real-value wave function is necessary to describe the quantum behavior of particles. The model is based on de Broglie's particle-wave duality and Einstein's mass-energy relation $E^2=c^2\left(m_0{}^2c^2+p^2\right)$ for a particle with a rest mass m_0. For a massless particle[18], we first consider a wave function of a dual-component wave function $\Psi(t,x)$ consisting of two real-value functions $f(t,x)$ and $g(t,x)$ which is governed by the following wave equation:

$$\left(-\frac{c\,\partial^2}{c^2\partial t^2}{}^2+\frac{\partial^2}{\partial x^2}\right)\Phi(t,x)=0,,\ \Phi(t,x)\equiv\begin{pmatrix} f(t,x) \\ g(t,x)\end{pmatrix}, \qquad \text{(1A)}$$

or equivalently after Fourier transform,

$$\left(-\omega^2+K^2\right)\Psi(\omega,K)=0,,\ \Psi(\omega,K)\equiv\begin{pmatrix} \tilde{f}(t,x) \\ \tilde{g}(t,x)\end{pmatrix}, \qquad \text{(1B)}$$

where a natural unit of $c=\hbar=1$ is used in this work. A Fourier transform of the above equation leads to $\omega^2=k^2$, a relation between the frequency ω and the wave vector k. This dispersion relation is equivalent to $\omega^2=K^2$, according to $E=\hbar\omega,\ P=\hbar K$, or $E=\omega,\ P=K$, expressed in natural units, of Einstein's energy relation to momentum for a massless particle[14], and de Broglie's particle-wave duality postulate. Eq. (1A) can be expressed in an operator form as

$$\left(-\mathbf{E}^2+\mathbf{P}^2\right)|\Psi\rangle=0. \qquad \text{(1C)}$$

Here we use Dirac's ket-vector notation $|\Psi\rangle$ to represent a 2x1 column vector of the Fourier-transformed wave function $\Psi(t,x)$. Eq. (1B) can be met if $\mathbf{E}^2|\Psi\rangle=\pm\,\omega^2|\Psi\rangle,\ \ \mathbf{P}^2|\Psi\rangle=\pm\,K^2|\Psi\rangle.$

After close examination of Eq. (1C) and the requirement of its Lorentz invariance, we have found exactly two types of solutions exist. For the type-I solution of $\left(\mathbf{E}^2-\mathbf{P}^2\right)|\Psi\rangle=0$ one has a commutative relation $[\mathbf{E},\mathbf{P}]=0$ and

$$\mathbf{E}^2|\Psi\rangle=-\omega^2|\Psi\rangle,\ \mathbf{P}^2|\Psi\rangle=-K^2|\Psi\rangle, \qquad \text{(2A)}$$

or

$$\mathbf{E}^2|\Psi\rangle=\omega^2|\Psi\rangle,\ \mathbf{P}^2|\Psi\rangle=K^2|\Psi\rangle. \tag{2B}$$

For the type-II solution of $(\mathbf{E}^2-\mathbf{P}^2)|\Psi\rangle=0$ one has a non-commutative relation with $\{\mathbf{E}\,\mathbf{P}\}=0$ and

$$(\mathbf{E}-\mathbf{P})|\Psi\rangle=0,\ \ or\ \ (\mathbf{E}+\mathbf{P})|\Psi\rangle=0. \tag{2C}$$

These two types of solutions have different physical properties and distinctive topological structures. Because $\omega^2=K^2$ we can normalize $\mathbf{E}^2,\ \mathbf{P}^2$ to become dimensionless operators and we will use this convention for the case of a massless particle.

Before we solve the operator equation in Eq. (1B) and assign these two operators to 2x2 real-value matrices, or equivalently, converting the 2^{nd}-order differential equation in Eq. (1A) involving a dual-component wave function into a set of linearly coupled 1^{st} order differential equations, we shall examine the Lorentz transform between a fixed frame and a moving frame. According to special relativity theory, the Lorentz boost[18,19] for a moving reference frame traveling along the x-axis is given in natural units by

$$\begin{pmatrix} t' \\ x' \end{pmatrix}=\begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix},\ \cosh\xi=1\Big/\sqrt{1-v^2},\ \sinh\xi=v\Big/\sqrt{1-v^2}, \tag{3A}$$

where $(t,\,x)$ are the coordinates at the fixed frame, and $(t',\,x')$ are the coordinates at the moving frame at a speed v. Therefore, one has

$$\begin{pmatrix} \partial/c\partial t' \\ \partial/\partial x' \end{pmatrix}=\begin{pmatrix} \partial t/\partial t' & \partial x/\partial t' \\ \partial t/\partial x' & \partial x/\partial x' \end{pmatrix}\begin{pmatrix} \partial/\partial t \\ \partial/\partial x \end{pmatrix}=\begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}\begin{pmatrix} \partial/\partial t \\ \partial/\partial x \end{pmatrix}, \tag{3B}$$

From the above equation, the time-shift and space-shift operators $\hat{\mathbf{E}}$ and $\hat{\mathbf{P}}$ at the moving frame become

$$\begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{P}} \end{pmatrix}=\begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}\begin{pmatrix} \mathbf{E} \\ \mathbf{P} \end{pmatrix}. \tag{3C}$$

Based on Eq. (3C) one obtains

$$\begin{aligned}
&\hat{\mathbf{E}}^2 - \hat{\mathbf{P}}^2 = \mathbf{E}^2 - \mathbf{P}^2 \\
&[\hat{\mathbf{E}}, \hat{\mathbf{P}}] = [\mathbf{E}, \mathbf{P}] \\
&\{\hat{\mathbf{E}}, \hat{\mathbf{P}}\} = (\mathbf{E}^2 + \mathbf{P}^2)\sinh 2\xi + \{\mathbf{E}, \mathbf{P}\}\cosh 2\xi.
\end{aligned} \tag{3D}$$

which confirms that $\mathbf{E}^2 - \mathbf{P}^2$ and $[\mathbf{E}, \mathbf{P}]$ are Lorentz invariants, but $\{\mathbf{E}, \mathbf{P}\}$ are not. Therefore, for type-I particles, the wave equation $\mathbf{E}^2 - \mathbf{P}^2 = 0$ and the commutative relation $[\mathbf{E}, \mathbf{P}] = 0$ are satisfied and invariant under a Lorentz boost.

However, for type-II particles with an anti-commutative relation $\{\mathbf{E}, \mathbf{P}\} = 0$, one has $[\mathbf{E}, \mathbf{P}]|\Psi\rangle \neq 0$. In addition, one also has $\{\hat{\mathbf{E}}, \hat{\mathbf{P}}\} \neq \{\mathbf{E}, \mathbf{P}\}$ which is not Lorentz invariant if one uses the Lorentz boost matrix of Eq. (3C). In order for $\{\mathbf{E}, \mathbf{P}\}$ to be Lorentz invariant one needs to generalize the transformation matrix for the type-II case by

$$\begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{P}} \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{P} \end{pmatrix}. \tag{4A}$$

From Eq. (4A), one obtains the invariance of the anti-commutator

$$\{\hat{\mathbf{E}}, \hat{\mathbf{P}}\} = (\mathbf{E}^2 - \mathbf{P}^2)\sinh 2\xi + \{\mathbf{E}, \mathbf{P}\} = \{\mathbf{E}, \mathbf{P}\} \tag{4B}$$

because $(\mathbf{E}^2 - \mathbf{P}^2)|\Psi\rangle = 0$ for type-II particle. Therefore, both $(\mathbf{E}^2 - \mathbf{P}^2)|\Psi\rangle = 0$ and $\{\hat{\mathbf{E}}, \hat{\mathbf{P}}\}|\Psi\rangle = 0$ are indeed invariant under the Lorentz boost using Eq. (4A). Based on the above analysis, we conclude an important finding: the Lorentz boost matrix for type-I particles in Eq. (3C) are different from that in Eq. (4A) for type-II particles, which therefore must be different from type-I particles. One can also define time and space operators $\mathbf{T}$ and $\mathbf{X}$ with $\mathbf{T}[\Phi(t,x)) = t\,[\Phi(t,x))$ and $\mathbf{X}[\Phi(t,x)) = x\,[\Phi(t,x))$, respectively. According to Einstein's special relativity, $t^2 - x^2$ is an invariant under a Lorentz transform. Likewise, in the operator formalism, $\mathbf{T}^2 - \mathbf{X}^2$ is also an invariant similar to $\mathbf{E}^2 - \mathbf{P}^2$. Therefore, one also has two solutions – a commutative type-I with $[\mathbf{T}, \mathbf{X}] = 0$ and an anti-commutative type-II with $\{\mathbf{T}, \mathbf{X}\} = 0$, just like $\mathbf{E}$ and $\mathbf{P}$. Together with our previous definition of $\mathbf{E}[\Psi) = \partial/\partial t\, \Phi(t, x)$ and $\mathbf{P}[\Psi) = \partial/\partial x\, \Phi(t, x)$, we can express these two types of Lorentz boosts for the transformation for the $(\mathbf{E}, \mathbf{P})$ and $(\mathbf{T}, \mathbf{X})$ pairs using 2x2 real-value matrices. For type-I scalar particles, one has

$$\begin{aligned} &\mathbf{L}_B \equiv \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} = \cosh\xi\, \mathbf{I} + \sigma_x \sinh\xi\ , \quad \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{P}} \end{pmatrix} = \mathbf{L}_B \begin{pmatrix} \mathbf{E} \\ \mathbf{P} \end{pmatrix}.. \end{aligned} \tag{5A}$$

One can show the following commutative relations are invariant under the Lorentz boost $\mathbf{L}_B$

$$\begin{aligned} &\mathbf{E}^2 - \mathbf{P}^2 = 0 \\ &[\mathbf{E}, \mathbf{P}] = 0. \end{aligned} \tag{5B}$$

As a reminder, in this work we only use real-value operators or wave functions, so the appearance of these commutators differ slightly from the conventional definition of the operators that involve a pure imaginary number.

For type-II spinor particles, one has

$$\begin{aligned} &\mathbf{L}_F \equiv \begin{pmatrix} \cosh\xi & -\sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} = \cosh\xi\, \mathbf{I} + \sigma_t \sinh\xi\ , \quad \sigma_t \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ &\begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{P}} \end{pmatrix} = \mathbf{L}_F \begin{pmatrix} \mathbf{E} \\ \mathbf{P} \end{pmatrix}. \end{aligned} \tag{6A}$$

The Lorentz boost matrix shown above for type-II particles is different from those for type-I particles. The commonly used Lorentz boost matrix in literature is only valid for type-I scalar particles where $\mathbf{E}$ and $\mathbf{P}$ commute. Such a notion has not been reported in literature. Using the above equation one can show the following commutative relations are invariant under the Lorentz boost:

$$\begin{aligned}&\mathbf{E}^2-\mathbf{P}^2=0\\&\{\mathbf{E},\,\mathbf{P}\}=0.\end{aligned} \tag{6B}$$

It is important to point out that if the traditional Lorentz boost $\mathbf{L}_F^{-1}$ were used instead of $\mathbf{L}_B^{-1}$ for $\mathbf{T}$ and $\mathbf{X}$, those commutative relations would no longer be Lorentz invariant.

Now we discuss specific assignments for the operators in both type-I and type-II cases. For the type-I case with $[\mathbf{E},\mathbf{P}]=0$ and $(\mathbf{E}^2-\mathbf{P}^2)|\Psi\rangle=0$, which are proven to follow the Lorentz boost of Eq. (3C), we first consider the choice of $\mathbf{E}^2=\mathbf{P}^2=-\mathbf{I}$ would lead to flip-flop oscillations of the f and g components in spacetime. One can assign these 2x2 real-value matrix operators to

$$\mathbf{E}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\sigma_t,\ \mathbf{P}=-\sigma_t\ ,\quad \mathbf{E}^2=\mathbf{P}^2=-\mathbf{I}. \tag{7A}$$

Using the above operator assignment in the frequency-wave vector domain, or the corresponding partial derivatives $\partial/\partial t,\partial/\partial x$ in the time-space domain, one can express the wave equation explicitly as

$$\begin{cases}\dfrac{\partial}{\partial t}f(t,x)=-g(t,x)\\ \dfrac{\partial}{\partial t}g(t,x)=f(t,x)\end{cases},\quad \begin{cases}\dfrac{\partial}{\partial x}f(t,x)=g(t,x)\\ \dfrac{\partial}{\partial x}g(t,x)=-f(t,x)\end{cases}. \tag{7B}$$

Eq. (3B) exhibits swapping behavior between f and g along both the time and space axes. The topological structure of the above coupling scheme is illustrated in Fig. 1A, displaying clockwise rotation of f and g (with red and blue arrows) around four quadrants as time evolves. Conversely, as space evolves the rotation is counterclockwise originally from the 1st quadrant to the 2nd quadrant. The solution of Eq. (3B) with $\mathbf{E}=\sigma_t,\mathbf{P}=-\sigma_t$ corresponds to a spiral wave with a right-hand chirality along the x-axis. If one assigns $\mathbf{E}=-\sigma_t,\mathbf{P}=\sigma_t$ with governing equation given by

$$\begin{cases} \frac{\partial}{\partial t} f(t,x) = g(t,x) \\ \frac{\partial}{\partial t} g(t,x) = -f(t,x) \end{cases}, \quad \begin{cases} \frac{\partial}{\partial x} f(t,x) = -g(t,x) \\ \frac{\partial}{\partial x} g(t,x) = f(t,x) \end{cases}. \tag{7C}$$

one can show both Eqs. (7B) and (7C) lead to

$$\begin{cases} \frac{\partial^2}{\partial t^2} f(t,x) = -f(t,x) & \frac{\partial^2}{\partial x^2} f(t,x) = -f(t,x) \\ \frac{\partial^2}{\partial t^2} g(t,x) = -g(t,x) & \frac{\partial^2}{\partial x^2} g(t,x) = -g(t,x) \end{cases} /. \tag{7D}$$

Both Eqs. (7B) and (7C) satisfy the conservation of the intensity $I(t,x) = f^2(t,x) + g^2(t,x)$ in time and space

$$\begin{aligned} \frac{\partial}{\partial t} I(t,x) &= 2\left(f(t,x)\frac{\partial}{\partial t} f(t,x) + g(t,x)\frac{\partial}{\partial t} f(t,x) \right) = 0 \\ \frac{\partial}{\partial x}(t,x) &= 2\left(f(t,x)\frac{\partial}{\partial x} f(t,x) + g(t,x)\frac{\partial}{\partial x} f(t,x) \right) = 0\,. \end{aligned} \tag{7E}$$

According to Eq. (7C), the quadrant rotates counterclockwise along the time axis but clockwise along the x axis as shown in Fig. 1B, and the spiral wave propagation has a left-hand chirality. For the other choice of $\mathbf{E}^2 = \mathbf{P}^2 = \mathbf{I}$, there exist two possible assignments, i.e., $\mathbf{E} = \sigma_x, \mathbf{P} = \sigma_x$ and $\mathbf{E} = -\sigma_x, \mathbf{P} = -\sigma_x$ where $\sigma_x = \sigma_1$ is the first Pauli 2x2 matrix where we define $\sigma_t = -i\,\sigma_2$ to avoid the use of an imaginary number, which might cause some confusion. While there is an isomorphism between the algebra of the 2-dimensional vector space and the complex plane, in this work we are dealing with purely real-value wave functions and 2x2 matrix operators. One can show that for $\mathbf{E}^2 = \mathbf{P}^2 = \mathbf{I}$, the wave equations governed by $\partial^2/\partial t^2\, f(t,x) = f(t,x),\ \partial^2/\partial x^2\, f(t,x) = f(t,x),$ and $\partial^2/\partial t^2\, g(t,x) = g(t,x),\ \partial^2/\partial x^2\, g(t,x) = g(t,x),$ lead to either an exponentially expanding or contracting amplitude. The topological representations of these two cases are illustrated in Fig. 1C and Fig. 1D, respectively. Therefore, the unphysical choice of $\mathbf{E}^2 = \mathbf{P}^2 = \mathbf{I}$ cannot be used here.

Only Eq. (3B) with $\mathbf{E}^2 = \mathbf{P}^2 = -\mathbf{I}$ can describe the flip-flop behavior for f and g across the lattice plane, as schematically illustrated in Fig. 1, showing two types of possible flip-flop schemes. One can obtain a plane-wave solution of $f(t,x),= A\cos(x-t),\ g(t,x),= -A\sin(x-t)$ as illustrated in Fig. (2a). If one assigns $\mathbf{E} = \sigma_t = -\mathbf{P}$ instead, its plane-wave solution becomes $f(t,x),= A\cos(x-t),\ \ g(t,x),= A\sin(x-t)$. The difference is in the propagation chirality. If one chooses $\mathbf{E} = i\sigma_2 = \mathbf{P}$ or $\mathbf{E} = \sigma_t = \mathbf{P}$, the wave propagates along the reverse direction.

For the other type-II solution of Eq. (2B) with $\mathbf{EP} \neq \mathbf{PE}$ one has

$$(\mathbf{E}-\mathbf{P})^2|\Psi\rangle = \left(\mathbf{E}^2 + \mathbf{P}^2 - (\mathbf{EP} + \mathbf{PE})\right)|\Psi\rangle = \left(-\omega^2 + K^2\right)|\Psi\rangle = 0$$

or

$$(\mathbf{E}+\mathbf{P})^2|\Psi\rangle = \left(\mathbf{E}^2 + \mathbf{P}^2 + (\mathbf{EP} + \mathbf{PE})\right)|\Psi\rangle = \left(-\omega^2 + K^2\right)|\Psi\rangle = 0. \qquad (8)$$

For the above equation to be satisfied, one has $\mathbf{E}^2|\Psi\rangle = -\omega^2|\Psi\rangle$ and $\mathbf{K}^2|\Psi\rangle = K^2|\Psi\rangle$. Because of the special case of a massless particle $\omega = K$, we normalize the operators $\mathbf{E}$ and $\mathbf{P}$ to become unitless for simplicity. Therefore, one must have the following constraints:

$$\begin{aligned}
&(\mathbf{E} - \mathbf{P})|\Psi\rangle = 0,\ or\ \ (\mathbf{E} + \mathbf{P})|\Psi\rangle = 0\\
&\{\mathbf{E}, \mathbf{P}\} = 0\\
&\mathbf{E}^2|\Psi\rangle = \mp|\Psi\rangle,\ \ \mathbf{P}^2|\Psi\rangle = \pm|\Psi\rangle.
\end{aligned} \qquad (9A)$$

The above anti-commutative property between $\mathbf{E}$ and $\mathbf{P}$ for this type-II case is characteristically from the type-I case with commutative $\mathbf{E}$ and $\mathbf{P}$. Based also on the condition of the wave function being real, one has the following assignment

$$\begin{aligned}
&\mathbf{E} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_t,\ \mathbf{E}^2 = -\mathbf{I}\\
&\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x,\ or\ \ \mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z,\ \ \mathbf{P}^2 = \mathbf{I}.
\end{aligned} \qquad (9B)$$

Using the above operator assignment, the wave equation for $\mathbf{E}=-i\sigma_2, \mathbf{P}=\sigma_1$ is given by

$$\frac{\partial}{\partial t} f(t,x)=\frac{\partial}{\partial x} g(t,x), \ \ \frac{\partial}{\partial t} g(t,x)=\frac{\partial}{\partial x} f(t,x), \tag{10A}$$

and for $\mathbf{E}=\sigma_t, \mathbf{P}=\sigma_z$

$$\frac{\partial}{\partial t} f(t,x)=-\frac{\partial}{\partial x} g(t,x), \ \frac{\partial}{\partial t} g(t,x)=-\frac{\partial}{\partial x} f(t,x). \tag{10B}$$

The coupling scheme can be schematically illustrated in Fig. 4, showing a binary exchange between the f and g components along the $x-ct$ axis. For mode-1, there is an alternate exchange between (f, g) and (g, f) like a Möbius strip[15] between the f and g components. The topological representation for these two types of wave propagation is illustrated in Fig. 4. For mode-2, the process between (f, g) and $(-g, -f)$ resembles another type of Möbius strip[20] twisting. Both above equations lead to the wave equation for each of the dual components

$$\frac{\partial^2}{\partial t^2} f(t,x)=\frac{\partial^2}{\partial x^2} f(t,x), \ \ \frac{\partial^2}{\partial t^2} g(t,x)=\frac{\partial^2}{\partial x^2} g(t,x), \ \ . \tag{10C}$$

According to the analysis in this work, both type-I and type-II massless particles all travel at the speed of light. For the type-I particle, which represents a scalar particle, $\mathbf{E}$ and $\mathbf{P}$ are commutative; however, for the type-II particle, which has an intrinsic structure, $\mathbf{E}$ and $\mathbf{P}$ are anti-commutative. In the type-I case, the recursive relation does not involve direct coupling between time and space, while in the type-II case, the recursive relation involves an intertwined coupling between space and time like a Möbius strip as shown in Fig. 3. The equation of Eq. (4B) with the choice of $\mathbf{E}|\Psi\rangle=-\mathbf{P}|\Psi\rangle$ in the continuum limit has a plane-wave solution as $f(t,x)=A\cos(k(x\mp t)), \ g(t,x)=A\sin(k(x\mp t))$ which is 90-degrees out of phase as illustrated in Fig. 2a. For the other choice of $\mathbf{E}|\Psi\rangle=\mathbf{P}|\Psi\rangle$, its plane wave solution becomes $f(t,x)=A\cos(k(x+ct)+\phi),$

and $g(t,x) = -A\sin(k(x+t))$, indicating a wave propagating along the opposite x-axis or t-axis. The wave propagation for the type-I is shown in Fig. 2a with the f and g components 90-degrees out of phase. In contrast, for the type-II case, the wave propagation is illustrated in Fig. 2b, showing the f and g components in phase or 180-degrees out of phase. The plane-wave solutions of Eq. (6B) are given by $f(t,x) = \pm g(t,x) = A\sin(k(x \mp t))$, with the same phase or 180-degrees out of phase. The type-II particle described above represents a 2D half-spin massless particle, which is a 2D analogy of a 4D Majorana particle[21]. If one uses angular momentum operators of spin-1 particles instead of Pauli's matrices in the treatment, the wave propagation of in-phase f and g components appear to be like the electric and magnetic field of a photon wave.

So far, we have considered type-I and type-II quantum lattice dynamics for massless particles. Let us now extend the treatment to particles with a rest mass. We first consider the rest frame, where there exists an internal oscillation with a frequency $m_0 c^w/\hbar$ dictated by its rest mass energy.

The wave equation in Eq. (1A) for a massless particle needs to be replaced by

$$\left(-\frac{\partial^{2}}{\partial t^{2}}^{2} + \frac{\partial^{2}}{\partial x^{2}}\right)\Psi(t,x) = \left(\frac{m_0 C}{\hbar}\right)^{2}\Psi(t,x)\,, \tag{11A}$$

which is the Klein-Gordon equation in a discrete lattice. The above equation can be expressed in an operator form like Eq. (1B) but now with a mass term as

$$E^2\mathbf{E}^2|\Psi\rangle = \left(K^2\mathbf{P}^2 + m_0{}^2\mathbf{M}^2\right)|\Psi\rangle. \tag{12B}$$

Eq. (7B) in natural units is equivalent to the de Broglie-Einstein relation of $E^2 = c^2\left(m_0{}^2c^2 + p^2\right)$ or $\omega^2 = c^2\left((m_0 c/\hbar)^2 + K^{2^2}\right)$ to satisfy the Pythagorean theorem for $\omega^2 = \left(m_0{}^2 + K^2\right)$.

Here we seek a solution for the type-II case that satisfies $\omega^2 = \left(m_0{}^2 + K^2\right)$ as an eigenvalue result of $\left(-\omega^2\mathbf{E}^2 + K^2\,\mathbf{P}^2 + m_0{}^2\mathbf{M}^2\right)|\Psi\rangle = \left(-\omega^2 + K^2 + m_0{}^2\right)|\Psi\rangle = 0.$ One can obtain a solution with $(\omega\mathbf{E} + K\mathbf{P} + m_0\mathbf{M})|\Psi\rangle = 0$ if these operators satisfy

$$\begin{aligned}&\{\mathbf{E}, \mathbf{P}\} = \{\mathbf{E}, \mathbf{M}\} = \{\mathbf{P}, \mathbf{M}\} = 0,\\ &(\omega\mathbf{E} + K\mathbf{P} + m_0\mathbf{M})|\Psi\rangle|\Psi\rangle = \left(\omega^2\mathbf{E}^2 + K^2\mathbf{P}^2 + m_0{}^2\mathbf{M}^2\right)|\Psi\rangle = \left(-\omega^2 + K^2 + m_0{}^2\right)|\Psi\rangle = 0\end{aligned} \tag{13}$$

$$\begin{aligned}&\mathbf{E} = \sigma_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \mathbf{P} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{M} = -\sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\\ &-\sigma_t^2 = \sigma_x^2 = \sigma_x^2 = \mathbf{I},\quad \sigma_x\sigma_t = -\sigma_t\sigma_x = \sigma_z,\ \sigma_t\sigma_z = -\sigma_z\,\sigma_t = \sigma_x,\ \sigma_x\sigma_t = -\sigma_z\sigma_x = \sigma_t\,.\end{aligned} \tag{14A}$$

and $-\mathbf{E}^2 = \mathbf{P}^2 = \mathbf{M}^2 = \mathbf{I}$. To satisfy $(\omega\mathbf{E} + K\mathbf{P} + m_0\mathbf{M})|\Psi\rangle = 0,$ we can assign these operators to three anti-commutative 2x2 matrices as

$$\begin{aligned}&\mathbf{E} = \sigma_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \mathbf{P} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{M} = -\sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\\ &-\sigma_t^2 = \sigma_x^2 = \sigma_x^2 = \mathbf{I},\quad \sigma_x\sigma_t = -\sigma_t\sigma_x = \sigma_z,\ \sigma_t\sigma_z = -\sigma_z\,\sigma_t = \sigma_x,\ \sigma_x\sigma_t = -\sigma_z\sigma_x = \sigma_t\,,\end{aligned} \tag{14A}$$

where $-\sigma_t^2 = \sigma_x^2 = \sigma_x^2 = \mathbf{I},\quad \sigma_x\sigma_t = -\sigma_t\sigma_x = \sigma_z,\ \sigma_t\sigma_z = -\sigma_z\,\sigma_t = \sigma_x,\ \sigma_x\sigma_t = -\sigma_z\sigma_x = \sigma_t\,.$.

The wave equation $(\omega\mathbf{E} + K\mathbf{P} + m_0\mathbf{M})|\Psi\rangle = 0,$ based on the above assigned operators, can be expressed explicitly by

$$\begin{cases}\dfrac{\partial}{\partial t} f(t,x) = -\dfrac{\partial}{\partial x} f(t,x) - m_0 g(t,x)\\ \dfrac{\partial}{\partial t} g(t,x) = -\dfrac{\partial}{\partial x} g(t,x) - m_0 f(t,x).\end{cases} \tag{14B}$$

A 2nd kind of operator assignment of $\mathbf{E} = \sigma_t,\ \mathbf{P} = -\sigma x, \mathbf{M} = -\sigma_z$ leads to

$$\begin{cases}\dfrac{\partial}{\partial t} f(t,x) = \dfrac{\partial}{\partial x} f(t,x) - m_0 g(t,x),\\ \dfrac{\partial}{\partial t} g(t,x) = -\dfrac{\partial}{\partial x} g(t,x) - m_0 f(t,x)\,.\end{cases} \tag{14C}$$

A 3rd kind of operator assignment of $\mathbf{E}=\sigma_z$, $\mathbf{P}=-\sigma_z$, $\mathbf{M}=-\sigma_x$, leads to

$$\begin{cases} \frac{\partial}{\partial t} f(t,x) = -\frac{\partial}{\partial x} g(t,x) - m_0 f(t,x), \\ \frac{\partial}{\partial t} g(t,x) = -\frac{\partial}{\partial x} f(t,x) - m_0 g(t,x) . \end{cases}$$

A 4th kind of operator assignment of $\mathbf{E}=\sigma_t$, $\mathbf{P}=\sigma_z$, $\mathbf{M}=-\sigma_x$ leads to

$$\begin{cases} \frac{\partial}{\partial t} f(t,x) = \frac{\partial}{\partial x} g(t,x) - m_0 f(t,x), \\ \frac{\partial}{\partial t} g(t,x) = \frac{\partial}{\partial x} f(t,x) - m_0 g(t,x) . \end{cases} \tag{14D}$$

Both above equations represent a massive half-spin particle, as a 2D analogy of Dirac's equation for an electron in 4D spacetime. According to our analysis, there are four possible coupling schemes as illustrated in Fig. 5, showing how the original massless spinor structures are coupled to the 2D Higgs fields with an attached spring. Such coupling leads to the spinless fermion acquiring its mass, slowing down the wave propagation from the speed of light.

The second possible solution, as an extension of the massless particle in the type-I case for a boson can be obtained if $\{\mathbf{E},\mathbf{M}\}=0, [\mathbf{P},\mathbf{E}]=[\mathbf{P},\mathbf{E}]=0, \mathbf{E}^2=\mathbf{P}^2=-\mathbf{M}^2=\mathbf{I}$

$$\begin{aligned} &(\omega\mathbf{E}-m_0\mathbf{M})|\Psi\rangle = 0, \\ &(\omega\mathbf{E}-m_0\mathbf{M})^2|\Psi\rangle = \left(\omega^2\mathbf{E}^2 + m_0{}^2\mathbf{M}^2 - \omega m_0\{\mathbf{E},\mathbf{M}\}\right)|\Psi\rangle = \left(\omega^2\mathbf{E}^2 + m_0{}^2\mathbf{M}^2\right)|\Psi\rangle = \left(E^2 - m_0{}^2\right)|\Psi\rangle, \\ &\mathbf{P}^2|\Psi\rangle = -|\Psi\rangle \ , \end{aligned}$$

where

$$\mathbf{E}, = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}, \ \mathbf{M} = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \ \mathbf{P} = \begin{pmatrix} 0 & -\sigma_t \\ \sigma_t & 0 \end{pmatrix}. \tag{15A}$$

In the above equation we need to use a tensor product of another 2x2 matrix to satisfy the constraints of $[\mathbf{P},\mathbf{E}]=[\mathbf{P},\mathbf{M}]=0$, and $\mathbf{M}^2|\Psi\rangle=-|\Psi\rangle$.

The above equations describe separate oscillations in time and space, unlike Eq. (8) which describes an intertwined link between partial differentials with respect to time and space. The third possible solution can be obtained if $\{\mathbf{P}, \mathbf{M}\} = 0, [\mathbf{E}, \mathbf{P}+\mathbf{M}] = 0, \mathbf{E}^2 = \mathbf{P}^2 = \mathbf{M}^2 = \mathbf{I}$

$$
\begin{aligned}
&(K\mathbf{P} + m_0\mathbf{M})|\Psi\rangle = 0 \\
&(K\mathbf{P} + m_0\mathbf{M})^2|\Psi\rangle = \left(K^2\mathbf{P}^2 + m_0{}^2\mathbf{M}^2 + Km_0\{\mathbf{P},\mathbf{M}\}\right)|\Psi\rangle = \; = \left(K^2 + m_0{}^2\right)|\Psi\rangle \\
&\mathbf{E}^2|\Psi\rangle = -\omega^2|\Psi\rangle
\end{aligned} \tag{15B}
$$

where

$$
\mathbf{E} = \begin{pmatrix} 0 & -\mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix}, \; \mathbf{M} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \; \mathbf{P} = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.
$$

Because in both Eq. (9A) and Eq. (9C), tensor products of two 2x2 matrices are required in order to satisfy the Klein-Gordon equation and $E^2 = c^2\left(m_0{}^2c^2 + p^2\right)$, these types of wave equations and solutions do not meet the constraints of the dual-component model with two real-value functions. Strictly speaking, Eq. (8B) with $(\mathbf{E}+\mathbf{P}+\mathbf{M})|\Psi\rangle = 0,$ is the only qualified wave equation for a particle with a rest mass.

Conclusions

In summary, we presented a dual-component model with two real-value wave functions f and g to describe the relativistic quantum dynamics of fields/particles in a 2D Minkowski spacetime. Using an operator algebra approach with a time-shift operator $\mathbf{E}$ and a space-shift operator $\mathbf{P}$, together with another mass operator represented by three independent 2x2 real-value matrices, we can construct linearly coupled 1st-order partial differential equations to describe the excitation and propagation of these quantum fields and their associated particles. We systematically analyzed all possible excitations of the 2D Minkowski spacetime fabric sheet that satisfy the de Broglie-Einstein relations between mass energy, frequency, and wave vector. From our analysis of all possible structural deformations, we have identified two types of solutions that satisfy the Lorentz invariance of $\left(\mathbf{E}^2 - \mathbf{P}^2\right)|\Psi\rangle$. For a type-I massless bosonic preon, one has commutative relations of $[\mathbf{E}, \mathbf{P}] = 0$, and for a type-II massless fermionic preon, one has anti-commutative relations of $\{\mathbf{H}, \mathbf{P}\} = 0$. This model leads naturally to only two kinds of field excitations and their associated particles as bosons and fermions. We shed light on the concepts behind Pauli's exclusion principle, fermionic statistics, and bosonic statistics. We point out that spacetime itself consists of a dual-component fabric to allow the excitation and propagation of type-I bosonic preons and type-II fermionic preons. Unlike the conventional preon modes that encounter the mass paradox[22], we have only two types of preons – anti-symmetric fermionic and symmetric bosonic preons, both of which are massless.

The Möbius strip and simple loop structures, which arise naturally from the wave equation as prescribed by Einstein's special relativity and de Broglie's wave-particle duality, could potentially be used as building blocks to construct and investigate the topological properties of elementary particles in the Standard Model. As an alternative to the string and loop quantum gravity theories, our model of fermionic and bosonic preons based on equal footing on 2D spacetime could be extended to 4D to provide insight into the topological structures of particle/field excitations in the actual 4D spacetime that we live in, so that unsolved puzzles in particle physics can someday be further explored.

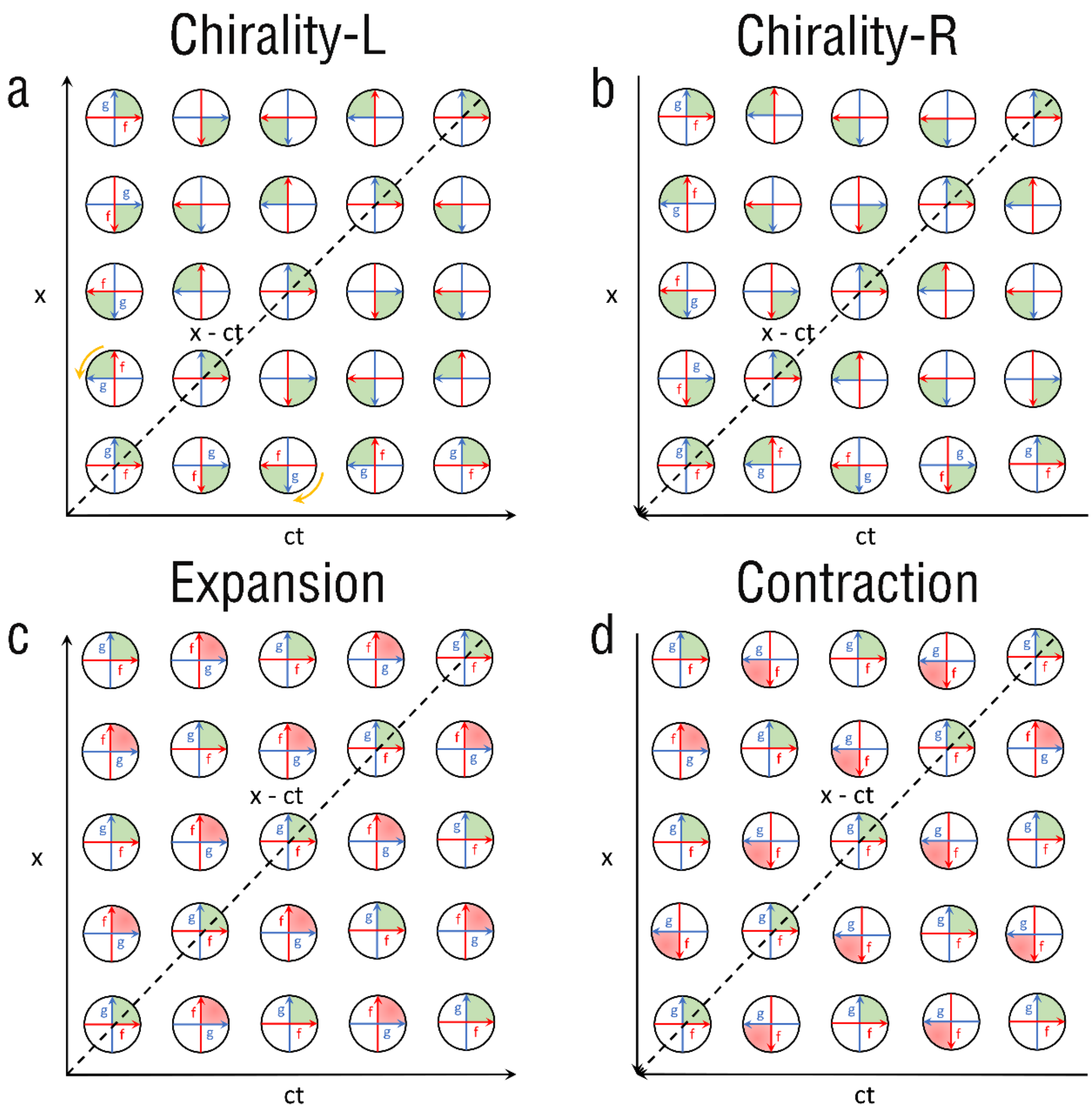

Fig. 1. Topological 2D spacetime structures of the type-I particles. In subplot (a) for mode-1, according to the recursive scheme, the quadrant formed by f (red arrow) and g (blue arrow) rotates clockwise along the t-axis and counter-clockwise along the x-axis, representing left-hand chirality for the rotation viewing along the time axis. The diagonal dot line denotes the wave propagation along $x - ct$. For mode-2 in the subplot (b) it shows a reverse rotation direction for f and g, representing right-hand chirality. In subplots (c) and (d), their specific inversion or reflection exchange schemes between f and g lead to either contraction or expansion with exponentially decreasing or increasing amplitudes.

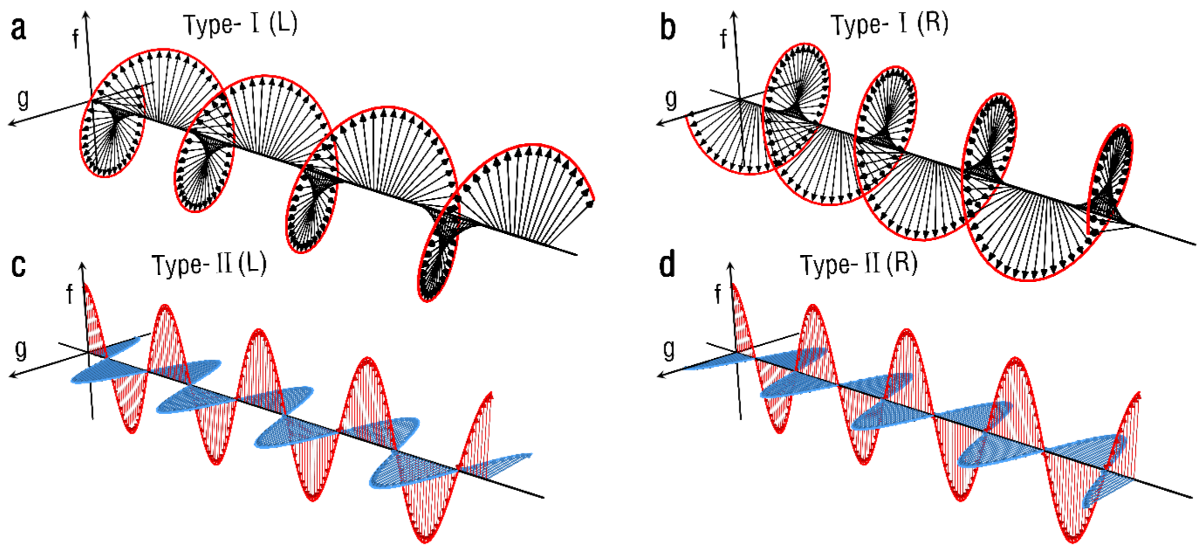

Fig 2. Two types of wave propagation for the f and g components, each with two modes. (a) The wave propagation of the type-I waves with a chirality model L. The wave oscillations for f and g, 90-degrees out of phase, are along the x-axis. They are plotted orthogonal to each other for a better view, unlike an EM wave with the electric and magnetic fields along x and y. Depending on the recursive scheme, there exists a left-hand chirality and a right-hand chirality mode as shown in subplot (b). The subplots in (c) and (d) represent type-II wave propagation with f and g in-phase or 180-degrees out of phase. There also exist two modes, called L- and R-chirality depending on the relative phase relation between f and g.

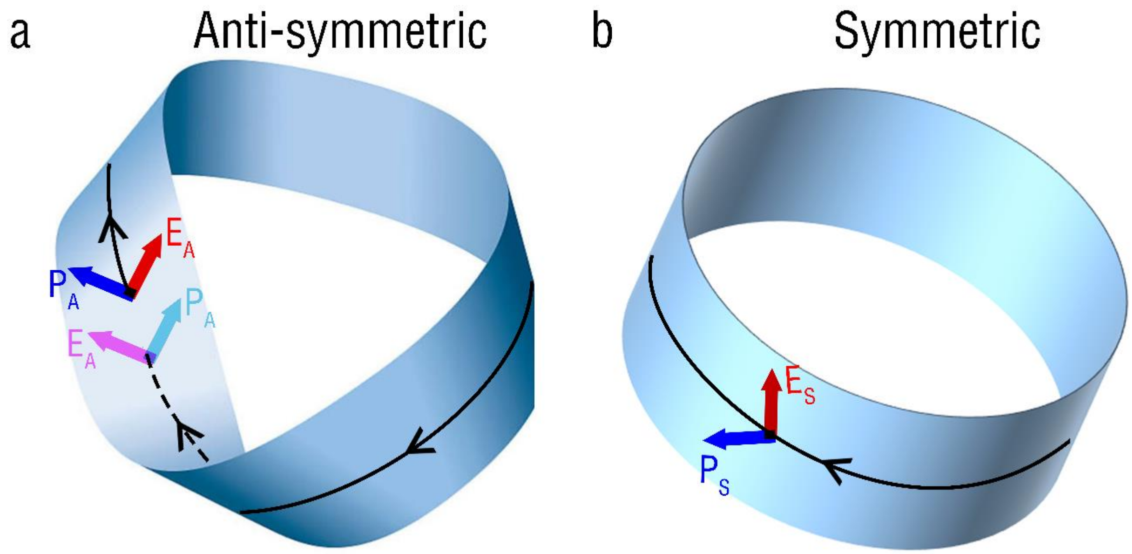

Fig. 3. | The topological structure representation of the antisymmetric type-II wave for a fermionic preon vs. the symmetric type-I bosonic preon. The fermionic preon has a topological structure like a Möbius strip, and possesses a half-spin, whereas the bosonic preon possesses a simple closed-loop strip. The operators $\mathbf{E}$ and $\mathbf{P}$ are time-shift and space-shift operators, respectively, and subscripts represent anti-symmetry (A) and symmetry (S) for the fermionic preon and bosonic preon, which could be used to construct other operators in 4D spacetime.

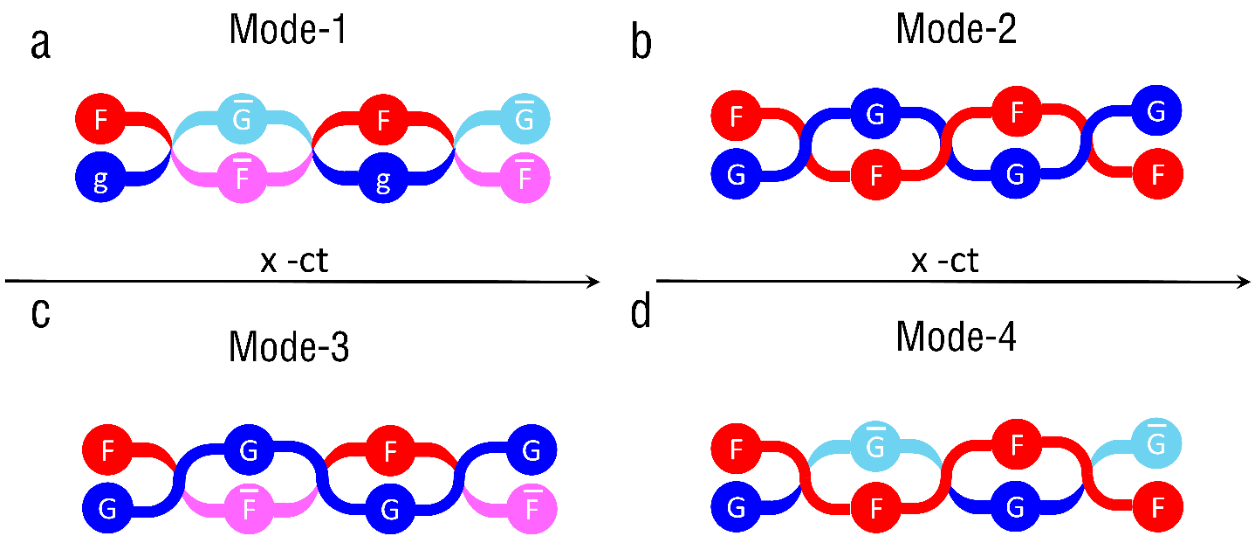

Fig 4. The topological structures for type-II spinor particles and their wave propagation along $x-ct$, exhibiting two modes of the intertwined dynamics like the twisting of a Möbius strip. (a) Mode-1 with a swapping between (F,G), representing partial derivatives in time and space for the top row and the bottom row, and (G,F). (b) Mode-2 with an exchange between (F,G) and $(\overline{G},\overline{F})$, meaning an inverse amplitude $(-G,-F)$. (c) Mode-3 with an exchange between (F,G) and (G,F). (d) Mode-4 with an exchange between (F,G) and $(\overline{G},\overline{F})$. The last two modes correspond to an exponentially decreasing or increasing amplitude.

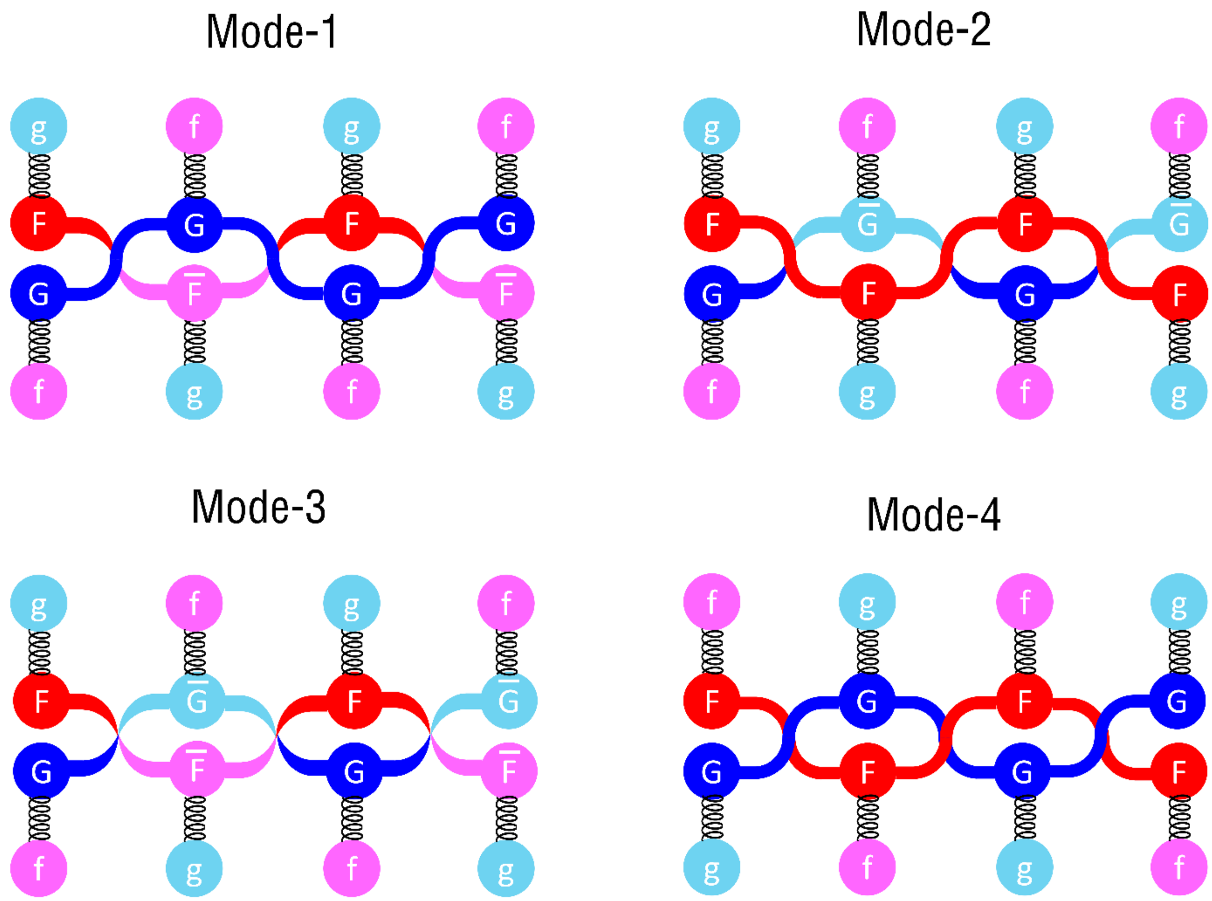

Fig. 5. Topological structures of spinor particles with a rest mass. There are four different coupling schemes between F and G, representing the dual-component wave function of type-2 particles. f and g represent the dual-component wave function of the spacetime fabric. The coupling of type-2 particles to the adjacent spacetime fabric causes the particles to acquire mass through the Higgs mechanism, but in 2D. Such couplings lead to a slower propagation velocity than the speed of light, owing to the nonlinear dispersion relation between the frequency and the wave vector.

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