A Boolean Algebra over a Theory

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Abstract Suppose that $\mathfrak{L}$ is a first-order language. Let $\mathfrak{L}^\dagger$ denote the union of $\mathfrak{L}$ and $\{\mathfrak{t}, \mathfrak{f}\}$ where $\mathfrak{t}$(true), $\mathfrak{f}$(false) are the nullary operations. We may define a binary relation '$\leq$' such that the sentences set Φ of the language $\mathfrak{L}^\dagger$ is a preordered set. And we may construct a boolean algebra $\Phi/\sim$, denoted $\tilde{\Phi}$, by an equivalence relation '$\sim$'. Then $\tilde{\Phi}$ is a partial ordered set. Let $\boldsymbol{A}$ be a structure of the language $\mathfrak{L}$. If $\boldsymbol{Th}(\boldsymbol{A})$ is a theory of $\boldsymbol{A}$, then $\boldsymbol{Th}^\dagger(\boldsymbol{A})$ is an ultrafilter. If $\Psi \subset \tilde{\Phi}$ is a finitely generated filter, then Ψ is principal. We may define a kernel of a homomorphism of the boolean algebra $\tilde{\Phi}$ such that the kernel is a filter. And a filter is a kernel if it is satisfied by some assumptions.

Contents

1. Introduction

Suppose that $\mathfrak{L}$ is a first-order language. Let $\mathfrak{L}^\dagger$ be the union of $\mathfrak{L}$ and $\{\mathfrak{t}, \mathfrak{f}\}$ where $\mathfrak{t}$(true), $\mathfrak{f}$(false) are two nullary operations. Then $\mathfrak{L}^\dagger$ is a first-order language. Let Φ be the set of all sentences of the language $\mathfrak{L}^\dagger$. If we define a binary relation '$\leq$', then Φ is a preordered set, see proposition 3.1 and notation 3.1. And we have that $\phi \leq \psi$ if and only if $\models \phi \to \psi$ for $\phi, \psi \in \Phi$, cf. propositions 3.4 and 3.5 and corollary 3.5.1.

If we define an equivalence relation '$\sim$', then the quotient $\tilde{\Phi} := \Phi/\sim$ is a boolean algebra, see definition 3.1, proposition 3.6, and notation 3.2 for the details. Hence $\tilde{\Phi}$ is a poset, see proposition 3.7.

Suppose that $\boldsymbol{A}$ is a structure of the language $\mathfrak{L}$. Let $\boldsymbol{Th}(\boldsymbol{A})$ be the theory of $\boldsymbol{A}$. We denote the quotient $\boldsymbol{Th}(\boldsymbol{A})/\sim$ by $\widetilde{\boldsymbol{Th}}(\boldsymbol{A})$. And let $\boldsymbol{Th}^\dagger(\boldsymbol{A})$ denote the union $\widetilde{\boldsymbol{Th}}(\boldsymbol{A}) \cup \{\mathfrak{t}\}$ where $\mathfrak{t} \in \tilde{\Phi}$, see notation 3.3. Then $\boldsymbol{Th}^\dagger(\boldsymbol{A})$ is an ultrafilter of $\tilde{\Phi}$, see proposition 3.12. Let $\mathcal{L}$ be the set of all structures of the language $\mathfrak{L}$. Then the theory $\boldsymbol{Th}^\dagger(\mathcal{M})$ is a filter for $\mathcal{M} \subset \mathcal{L}$, see proposition 3.13 and corollary 3.13.1 for the details.

If $\Psi \subset \tilde{\Phi}$ is a finitely generated filter, then Ψ has the minimum μ. Hence Ψ is principal. And the filter Ψ is consistent if and only if there exists a structure $\boldsymbol{A}$ of the language $\mathfrak{L}$ such that $A \models \mu$, cf. propositions 3.14 and 3.15 and corollary 3.14.1. For

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a finite subset $\mathcal{M} \in \mathcal{L}$, if $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is principal for every $\boldsymbol{A} \in \mathcal{M}$, then $\boldsymbol{Th}^{\dagger}(\mathcal{M})$ is principal, see proposition 3.16 and corollary 3.16.1.

We may define a kernel of a homomorphism of the boolean algebra $\tilde{\Phi}$. And the kernel is a filter, see definition 3.2 and propositions 3.17 and 3.20 for more details. And if $\phi, \psi \in \tilde{\Phi}$ and Ψ is a filter generated by $\{\phi \vee \psi\}$, then Ψ is a kernel of a homomorphism φ iff $\vDash \varphi(\phi) \vee \varphi(\psi)$ with $\nvDash \varphi(\phi)$ and $\nvDash \varphi(\psi)$, cf. propositions 3.18 and 3.19.

2. Preliminaries

2.1. **Universal Algebra.** Recall some definitions in universal algebra.

Definition 2.1 ([4, 6])**.** An n-ary **operation** on a nonempty set X is a mapping $f\colon X^n \to X$. An n-ary **relation** on X is a subset of X^n.

Definition 2.2 ([4, 6])**.** A (first-order) **language** is a nonempty set $\mathfrak{L}$ of symbols such that there exists a mapping $\sigma\colon \mathfrak{L} \to \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers. For every $f \in \mathfrak{L}$, $\sigma(f)$ is called the **arity**. If $\sigma(f) \geq 0$, then we say that f is an n-ary operation symbol. If $\sigma(f) < 0$, then f is called the n-ary relation symbol. If the arity of an operation symbol f is 0, 1 or 2, then f is said to be a **nullary**, **unary** or **binary** operation symbol, respectively. The language $\mathfrak{L}$ is said to be **algebraic** if $\mathfrak{L}$ has no relation symbols.

Definition 2.3 ([4, 6])**.** A **structure** $\boldsymbol{A}$ of a language $\mathfrak{L}$ is an ordered pair $\langle A, L\rangle$ where A is a nonempty set, and L is a mapping such that $L(f)$ is an n-ary operation(relation) f^A on A, for every n-ary operation(relation) symbol $f \in \mathfrak{L}$. If f is a nullary operation symbol in $\mathfrak{L}$, then $L(f)$ is a constant in A. If $\mathfrak{L}$ is algebraic, then $\boldsymbol{A}$ is called an **algebra**.

Definition 2.4 ([4, 6])**.** Suppose that $\mathfrak{L}$ is a language. Let $\mathfrak{L}' := \{f \in \mathfrak{L} \mid \sigma(f) \geq 0\}$. Then $\mathfrak{L}'$ is an algebraic language. Let X be a nonempty set, $\boldsymbol{T}$ an algebra of the language $\mathfrak{L}'$ generated by X. Then a member of $\boldsymbol{T}$ is called a **term**.

Definition 2.5 ([6])**.** An algebra $\langle \boldsymbol{B}, \vee, \wedge, ', 0, 1\rangle$ with two binary operations($\vee, \wedge$), one unary operation($'$), and two nullary operations(0, 1) is called a **boolean algebra** provided that

- $\langle \boldsymbol{B}, \vee, \wedge\rangle$ is a distributive lattice[6].
- $x \vee 1 = 1$ and $x \wedge 0 = 0$.
- $x \vee x' = 1$ and $x \wedge x' = 0$.

Definition 2.6 ([6])**.** Let $\boldsymbol{B}$ be a boolean algebra. A subset F of $\boldsymbol{B}$ is a **filter** if

- $1 \in F$.
- If $a, b \in F$ then $a \wedge b \in F$.
- If $a \in F$ then $x \in F$ for all $x \in \boldsymbol{B}$ with $x \geq a$.[1]

A maximal filter is called an **ultrafilter**. A filter F is said to be **principal** if F is generated by one element.

Definition 2.7 ([4, 6])**.** Suppose that $\boldsymbol{A}, \boldsymbol{B}$ are structures of a language $\mathfrak{L}$. Then a function $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ is called a **homomorphism** provided that

$$\varphi(f^A(a_1, \ldots, a_n)) = f^B(\varphi(a_1), \ldots, \varphi(a_n))$$

for all n-ary operation f, and

$$r^A(a_1, \ldots, a_m) \text{ implies } r^B(\varphi(a_1), \ldots, \varphi(a_m))$$

for all m-ary relation r. We denote the set of the homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ by $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$.

[1] Let $a \geq b$ if $a \vee b = a$. Then a lattice is a poset, cf. [4].

2.2. **Mathematical Logic.**

Definition 2.8 ([5]). The following symbols are called the **propositional connectives**.

$$\text{Equivalence} \quad \leftrightarrow \tag{2.1}$$
$$\text{Implication} \quad \rightarrow \tag{2.2}$$
$$\text{Conjunction} \quad \wedge \tag{2.3}$$
$$\text{Disjunction} \quad \vee \tag{2.4}$$
$$\text{Negation} \quad \neg \tag{2.5}$$

And the following symbols are the **quantifiers**.

$$\text{Universal} \quad \forall \tag{2.6}$$
$$\text{Existential} \quad \exists \tag{2.7}$$

Definition 2.9 ([5,6]). Suppose that $\mathfrak{L}$ is a first-order language and X is a nonempty set of variables. Let r be an n-ary relation symbol in $\mathfrak{L}$, and $t_1, \ldots, t_n$ terms[definition 2.4] over X. Then $r(t_1 \ldots, t_n)$ is said to be an **atomic formula**. An expression is called a **formula** of the language $\mathfrak{L}$ if it has one of the following forms

- an atomic formula.
- $s = t$ where s, t are terms.
- $\forall x\psi$, $\exists x\psi$ where x is a variable and ψ is a formula.
- $\psi \leftrightarrow \phi$, $\psi \rightarrow \phi$, $\psi \wedge \phi$, $\psi \vee \phi$, $\neg\psi$ where ψ, ϕ are formulas.

A formula ψ is a **subformula** of ϕ if ψ is consecutive string of symbols in the formula ϕ.

Theorem 2.1 ([5]). *Let ψ, ϕ, ω be formulas. Then we have following axiom schemata*

$$\models \phi \leftrightarrow \phi \tag{2.8}$$
$$\models \phi \vee \phi \leftrightarrow \phi \tag{2.9}$$
$$\models \phi \wedge \phi \leftrightarrow \phi \tag{2.10}$$
$$\models \psi \vee \neg\psi \tag{2.11}$$
$$\models \psi \leftrightarrow \neg\neg\psi \tag{2.12}$$
$$\models (\psi \leftrightarrow \phi) \leftrightarrow (\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi) \tag{2.13}$$
$$\models \psi \rightarrow \phi \leftrightarrow \neg\psi \vee \phi \tag{2.14}$$
$$\models (\psi \rightarrow \phi) \leftrightarrow (\neg\phi \rightarrow \neg\psi) \tag{2.15}$$
$$\models \psi \wedge \phi \leftrightarrow \neg(\neg\psi \vee \neg\phi) \tag{2.16}$$
$$\models \psi \wedge \phi \leftrightarrow \phi \wedge \psi \tag{2.17}$$
$$\models \psi \vee \phi \leftrightarrow \phi \vee \psi \tag{2.18}$$
$$\models \psi \vee (\psi \wedge \phi) \leftrightarrow \psi \tag{2.19}$$
$$\models \psi \wedge (\psi \vee \phi) \leftrightarrow \psi \tag{2.20}$$
$$\models \psi \wedge (\phi \vee \omega) \leftrightarrow (\psi \wedge \phi) \vee (\psi \wedge \omega) \tag{2.21}$$
$$\models \psi \vee (\phi \wedge \omega) \leftrightarrow (\psi \vee \phi) \wedge (\psi \vee \omega) \tag{2.22}$$
$$\models \phi \vee (\psi \vee \omega) \leftrightarrow (\phi \vee \psi) \vee \omega \tag{2.23}$$
$$\models \phi \wedge (\psi \wedge \omega) \leftrightarrow (\phi \wedge \psi) \wedge \omega \tag{2.24}$$

$$\vDash \phi \to \phi \vee \psi \tag{2.25}$$

$$\vDash \phi \to (\psi \to \phi) \tag{2.26}$$

Proof. Immediate from truth tables. □

Definition 2.10 ([6])**.** An occurrence of a variable x in a formula ψ is **bound** if a subformula of ψ has the form $\forall x\phi$ or $\exists x\phi$. Otherwise, an occurrence of x is **free** in ψ. A formula is called a **sentence** if the formula has no free variable.

Definition 2.11 ([2, 4])**.** Let $\boldsymbol{A}$ be a structure of a language $\mathfrak{L}$. Suppose that $\boldsymbol{T}$ is an algebra of terms of the language $\mathfrak{L}$. Then an **interpretation** is a member of $\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{A})$. If $\varphi \in \mathrm{Hom}(\boldsymbol{T}, \boldsymbol{A})$ is a homomorphism, then φ^x_a is the homomorphism such that $\varphi^x_a(x) = a$ and $\varphi^x_a(y) = \varphi(y)$ for all $y \neq x$. For $t \in \boldsymbol{T}$ and $\varphi \in \mathrm{Hom}(\boldsymbol{T}, \boldsymbol{A})$, the value $\varphi(t) \in \boldsymbol{A}$ is denoted $t^A[\varphi]$.

Definition 2.12 ([2])**.** Let $\boldsymbol{A}$ be a structure of a language $\mathfrak{L}$ and ϕ a sentence of the language $\mathfrak{L}$. We say that $\boldsymbol{A}$ satisfies ϕ, denoted $\boldsymbol{A} \vDash \phi$, as follows

$$\phi := (s = t)[\varphi] \qquad A \vDash \phi \text{ iff } s^A[\varphi] = t^A[\varphi]. \tag{2.27}$$

$$\phi := r(t_1 \dots t_n)[\varphi] \qquad A \vDash \phi \text{ iff } r^A(t^A_1[\varphi] \dots t^A_n[\varphi]). \tag{2.28}$$

$$\phi := \neg\psi[\varphi] \qquad A \vDash \phi \text{ iff } A \nvDash \psi[\varphi]. \tag{2.29}$$

$$\phi := (\psi \vee \omega)[\varphi] \qquad A \vDash \phi \text{ iff } A \vDash \psi[\varphi] \text{ or } A \vDash \omega[\varphi]. \tag{2.30}$$

$$\phi := (\psi \wedge \omega)[\varphi] \qquad A \vDash \phi \text{ iff } A \vDash \psi[\varphi] \text{ and } A \vDash \omega[\varphi]. \tag{2.31}$$

$$\phi := (\psi \to \omega)[\varphi] \qquad A \vDash \phi \text{ iff } A \vDash \psi[\varphi] \text{ implies } A \vDash \omega[\varphi]. \tag{2.32}$$

$$\phi := (\psi \leftrightarrow \omega)[\varphi] \qquad A \vDash \phi \text{ iff } A \vDash \psi[\varphi] \text{ if and only if } A \vDash \omega[\varphi]. \tag{2.33}$$

$$\phi := \forall x\psi[\varphi] \qquad A \vDash \phi \text{ iff } A \vDash \psi[\varphi^x_a] \text{ for all } a \in \boldsymbol{A}. \tag{2.34}$$

$$\phi := \exists x\psi[\varphi] \qquad A \vDash \phi \text{ iff } A \vDash \psi[\varphi^x_a] \text{ for some } a \in \boldsymbol{A}. \tag{2.35}$$

If $A \vDash \phi$ for all $\boldsymbol{A} \in \mathcal{L}$, then ϕ is called a **tautology**, denoted $\vDash\phi$, where $\mathcal{L}$ is the set of the structures of $\mathfrak{L}$. The set of the tautologies is denoted $\boldsymbol{Th}$.

Definition 2.13 ([2, 4])**.** Let ϕ be a sentence of a language $\mathfrak{L}$ and $\boldsymbol{A}$ a structure of the language $\mathfrak{L}$. If $A \vDash \phi$, then A is a **model** of ϕ. We denote the set of the models of ϕ by $\boldsymbol{Mod}(\phi)$. A **theory** is a set of sentences. A **theory of a model** $\boldsymbol{A}$ is the set of the sentences satisfied by $\boldsymbol{A}$. Let $\boldsymbol{Th}(\boldsymbol{A})$ denote the theory of the model $\boldsymbol{A}$.

Theorem 2.2 (cf. [2, 4, 6])**.** *Let ϕ, ψ be sentences of a language $\mathfrak{L}$. Then $\boldsymbol{Mod}(\phi \wedge \psi) = \boldsymbol{Mod}(\phi) \cap \boldsymbol{Mod}(\psi)$.*

Proof. Immediate from (2.31) of definition 2.12. □

Corollary 2.2.1 (cf. [2, 4, 6])**.** *Let ϕ, ψ be sentences of a language $\mathfrak{L}$. If $\vDash \phi \wedge \psi \leftrightarrow \phi$, then $\boldsymbol{Mod}(\phi)$ is a subset of $\boldsymbol{Mod}(\psi)$.*

Proof. Immediate from theorem 2.2. □

Definition 2.14 ([2, 4])**.** A theory Φ is said to be **consistent** if there exists at least one model $\boldsymbol{A}$ such that $A \vDash \Phi$.

Theorem 2.3 ([4])**.** *Let Φ be a consistent theory. Then we have that $\phi \in \Phi$ implies $\neg\phi \notin \Phi$.*

Proof. There exists a model $\boldsymbol{A}$ such that $A \vDash \Phi$. Hence we have that $A \vDash \phi$ for every $\phi \in \Phi$. It follows that $A \nvDash \neg\phi$ for all $\phi \in \Phi$ by axiom schemata (2.12) and (2.29). Therefore, we have that $\neg\phi \notin \Phi$ if $\phi \in \Phi$. □

3. A Boolean algebra of sentences

***Notation* 3.1.** We suppose that $\mathfrak{L}$ is a first-order language. Let $\mathfrak{L}^{\dagger} := \mathfrak{L} \cup \{\mathfrak{t}, \mathfrak{f}\}$ be a language where $\mathfrak{t}$(true) and $\mathfrak{f}$(false) are nullary[definition 2.2] operations. And We suppose that Φ is the set of all sentences of the language $\mathfrak{L}^{\dagger}$. Let $\mathcal{L}$ be the set of all structures of the language $\mathfrak{L}$.

Proposition 3.1. *Let ϕ, ψ be sentences of the language $\mathfrak{L}^{\dagger}$. Define $\phi \leq \psi$ if $\models \phi \wedge \psi \leftrightarrow \phi$. Then the set Φ is a preordered*[1] *set where Φ is defined in notation 3.1.*

Proof. By axiom schema (2.10), we have $\phi \leq \phi$. And if $\phi \leq \psi$ and $\psi \leq \omega$ then we have

1. $\models \phi \wedge \psi \leftrightarrow \phi$ — hypothesis.
2. $\models \psi \wedge \omega \leftrightarrow \psi$ — hypothesis.
3. $\models \phi \wedge \psi \wedge \omega \leftrightarrow \phi$ — modus ponens, 1 and 2.
4. $\models \phi \wedge \omega \leftrightarrow \phi$ — modus ponens, 1 and 3.

It follows that $\phi \leq \psi$ and $\psi \leq \omega$ implies $\phi \leq \omega$. Therefore, the set Φ is preordered. □

Corollary 3.1.1. *Let ϕ, ψ be sentences of the language $\mathfrak{L}^{\dagger}$. If $\psi \geq \phi$ then $\models \phi \vee \psi \leftrightarrow \psi$.*

Proof. We have that

1. $\models \phi \wedge \psi \leftrightarrow \phi$ — hypothesis.
2. $\models (\phi \wedge \psi \leftrightarrow \phi) \to (\phi \to \phi \wedge \psi)$ — axiom schema (2.13).
3. $\models \phi \to \phi \wedge \psi$ — modus ponens, 2.
4. $\models \phi \to \phi \wedge \psi \vee \psi$ — axiom schema (2.25).
5. $\models \phi \to \psi$ — axiom schema (2.19).
6. $\models \psi \to \psi$ — axiom schema (2.8).
7. $\models \phi \vee \psi \to \psi$ — modus ponens, 5 and 6.
8. $\models \psi \to \psi \vee \phi$ — axiom schema (2.25).
9. $\models \phi \vee \psi \leftrightarrow \psi$ — modus ponens, 7 and 8.

This completes the proof. □

Proposition 3.2. *Let τ be a tautology of the language $\mathfrak{L}^{\dagger}$. Then we have $\phi \leq \tau$ for all sentence ϕ of the language $\mathfrak{L}^{\dagger}$.*

Proof. For all sentence ϕ of $\mathfrak{L}^{\dagger}$,

1. $\models \phi \to (\tau \to \phi)$ — axiom schema (2.26).
2. $\models \neg\phi \vee (\neg\tau \vee \phi)$ — axiom schema (2.14).
3. $\models (\neg\phi \vee \neg\tau) \vee \phi$ — axiom schema (2.23).
4. $\models \phi \wedge \tau \to \phi$ — axiom schemata (2.14) and (2.16).
5. $\models \tau$ — hypothesis.
6. $\models \tau \to (\phi \to \tau)$ — axiom schema (2.26).
7. $\models \phi \to \tau$ — modus ponens, 5 and 6.
8. $\models \phi \to \phi$ — axiom schema (2.8).
9. $\models \phi \to \tau \wedge \phi$ — modus ponens, 7 and 8.
10. $\models \phi \leftrightarrow \phi \wedge \tau$ — modus ponens, 4 and 9.

This completes the proof. □

Proposition 3.3. *Let $\phi, \psi \in \Phi$. Then we have $\phi \leq \phi \vee \psi$ and $\phi \wedge \psi \leq \phi$.*

Proof. Immediate from axiom schemata (2.10), (2.20) and (2.24). □

Proposition 3.4. *Let $\phi, \psi \in \Phi$. Then we have that $A \models \phi$ implies $A \models \psi$ if $\phi \leq \psi$ for all structure $\boldsymbol{A}$ of the language $\mathfrak{L}$.*

Proof. This is an immediate consequence of corollary 2.2.1. □

The following proposition is the converse of the proposition 3.4.

Proposition 3.5. *Let $\phi, \psi \in \Phi$. If $\models \phi \rightarrow \psi$ then $\phi \leq \psi$.*

Proof. We have that

1. $\models \phi \rightarrow \phi$ — axiom schema (2.8).
2. $\models \phi \rightarrow \psi$ — hypothesis.
3. $\models \phi \rightarrow \phi \wedge \psi$ — modus ponens, 1 and 2.
4. $\models \phi \vee \neg\phi$ — axiom schema (2.11).
5. $\models \phi \vee \neg\phi \rightarrow \phi \vee \neg\phi \vee \neg\psi$ — axiom schema (2.25).
6. $\models \phi \vee \neg\phi \vee \neg\psi$ — modus ponens, 5.
7. $\models \phi \vee \neg\phi \vee \neg\psi \rightarrow (\phi \wedge \psi \rightarrow \phi)$ — axiom schemata (2.14) and (2.16).
8. $\models \phi \wedge \psi \rightarrow \phi$ — modus ponens, 7.
9. $\models \phi \wedge \psi \leftrightarrow \phi$ — modus ponens, 3 and 8.

This completes the proof. □

Corollary 3.5.1. *Let $\phi, \psi \in \Phi$. Then $\phi \leq \psi$ if and only if $\models \phi \rightarrow \psi$.*

Proof. Immediate from propositions 3.4 and 3.5. □

We have that Φ is a preordered set. Now we may construct a poset[4] by a equivalence relation on Φ.

Definition 3.1. Suppose that ϕ and ψ are sentences of the language $\mathfrak{L}^{\dagger}$. We define $\phi \sim \psi$ if $\models \phi \leftrightarrow \psi$. It is clear that '$\sim$' is an **equivalence relation**[3]. Let $\tilde{\Phi}$ denote the quotient[3] set of Φ by $\sim$.

***Notation* 3.2.** If $\phi \in \Phi$ then the equivalence class[3] of ϕ is also denoted ϕ. Hence if $\phi \in \Phi$ then ϕ is a sentence, and if $\phi \in \tilde{\Phi}$ then ϕ is an equivalence class.

In definition 2.8, there are five propositional connectives. But it follows from theorem 2.1 that logic calculus only need two connectives, i.e., $\neg$ and $\vee$. Hence $\tilde{\Phi}$ may form a boolean algebra.

Proposition 3.6. *We have that $\langle \tilde{\Phi}, \vee, \wedge, \neg, \mathfrak{t}, \mathfrak{f} \rangle$ is a boolean algebra[definition 2.5].*

Proof. By axiom schemata (2.17) to (2.22), we have that $\langle \tilde{\Phi}, \vee, \wedge \rangle$ is a distributive lattice. And we have

$$\models \phi \wedge \mathfrak{f} \leftrightarrow \mathfrak{f}$$
$$\models \phi \vee \mathfrak{t} \leftrightarrow \mathfrak{t}$$
$$\models \phi \vee \neg\phi \leftrightarrow \mathfrak{t}$$
$$\models \phi \wedge \neg\phi \leftrightarrow \mathfrak{f}$$

Therefore, the distributive lattice is a boolean algebra. □

A lattice is a poset, cf. [4, 6]. Hence we have the following propositions.

Proposition 3.7. *Let $\phi, \psi \in \tilde{\Phi}$. Define $\phi \leq \psi$ if $\models \phi \wedge \psi = \phi$. Then the boolean algebra $\tilde{\Phi}$ is a poset[6].*

Proof. This is an immediate consequence of definition 3.1 and proposition 3.1. □

Proposition 3.8. *Let $\phi, \psi, \omega \in \tilde{\Phi}$. Then we have that*

$$\phi \wedge \psi \leq \phi, \psi \leq \phi \vee \psi.$$

Proof. Immediate from proposition 3.3. □

Proposition 3.9. *Let $\phi, \omega, \omega' \in \tilde{\Phi}$. Then $\phi \leq \omega$ and $\phi \leq \omega'$ implies $\phi \leq \omega \wedge \omega'$.*

Proof. It is obvious. □

Proposition 3.10. *Let $\phi, \psi \in \tilde{\Phi}$. Then we have that $\phi \leq \psi$ if and only if there exists $\omega \in \tilde{\Phi}$ such that $\vDash \phi \vee \omega \leftrightarrow \psi$.*

Proof. If $\phi \leq \psi$ then $\vDash \phi \vee \psi \leftrightarrow \psi$ by corollary 3.1.1. Hence ψ is the desired sentence. On the other hand, if $\vDash \phi \vee \omega \leftrightarrow \psi$ then $\phi \leq \psi$ by proposition 3.3. □

Proposition 3.11. *Let $\phi, \psi \in \tilde{\Phi}$. Then let Ω be the set $\{\omega \in \tilde{\Phi} \mid \omega \geq \phi$ and $\omega \geq \psi\}$. Then the infimum of Ω exists and $(\inf \Omega = \phi \vee \psi) \in \Omega$.*

Proof. Let $\omega, \omega' \in \tilde{\Phi}$. If $\phi, \psi \leq \omega$ and $\phi, \psi \leq \omega'$ then $\phi, \psi \leq \omega \wedge \omega'$ by proposition 3.9. And we have $\omega \wedge \omega' \leq \omega, \omega'$ by proposition 3.8. If $\omega \geq \phi, \psi$ then $\vDash \omega \wedge (\phi \vee \psi) \leftrightarrow \phi \vee \psi$, since axiom schema (2.21). Hence $\omega \geq \phi \vee \psi$. By proposition 3.8 we have $\phi, \psi \leq \phi \vee \psi$. Therefore, $\phi \vee \psi$ is the infimum of Ω. □

Suppose that $\boldsymbol{A}$ is a structure of the language $\mathfrak{L}$. Then there exists a mapping $\varrho\colon \boldsymbol{Th}(\boldsymbol{A}) \to \tilde{\Phi}$ by sending the sentences to its equivalence classes. We shall see that the union $\varrho(\boldsymbol{Th}(\boldsymbol{A})) \cup \{\mathrm{t}\}$ is a filter[definition 2.6] in the boolean algebra $\tilde{\Phi}$ where $\mathrm{t} \in \tilde{\Phi}$.

***Notation* 3.3.** Suppose that $\boldsymbol{A}$ is a structure of the language $\mathfrak{L}$. The quotient subset $\boldsymbol{Th}(\boldsymbol{A})/{\sim}$ is denoted $\widetilde{\boldsymbol{Th}}(\boldsymbol{A})$ where '$\sim$' is defined in definition 3.1. And we denote the union $\widetilde{\boldsymbol{Th}}(\boldsymbol{A}) \cup \{\mathrm{t}\}$ by $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ where $\mathrm{t} \in \tilde{\Phi}$. It is clear that $\boldsymbol{Th}^{\dagger}(\boldsymbol{A}) \subset \tilde{\Phi}$.

Proposition 3.12. *The set $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is an ultrafilter[definition 2.6] in $\tilde{\Phi}$.*

Proof. It is obvious that $\mathrm{t} \in \boldsymbol{Th}^{\dagger}(\boldsymbol{A})$. Let $\phi, \psi \in \boldsymbol{Th}^{\dagger}(\boldsymbol{A})$. Then $\phi \wedge \psi \in \boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ by (2.31). For all $\omega \in \tilde{\Phi}$, we have that $\omega \geq \phi$ implies $\omega \in \boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ since corollary 3.5.1. By (2.29), exactly one of $A \vDash \phi$, $A \vDash \neg\phi$ is true for all $\phi \in \tilde{\Phi}$. This completes the proof. □

Remark 3.1. Let $\phi, \psi \in \Phi$ and $\boldsymbol{A}$ be a structure of the language $\mathfrak{L}$. We have $\vDash (\psi \to \phi) \vee (\psi \to \neg\phi)$. If $A \vDash \phi$ and $A \vDash \phi \to \psi$, then $A \vDash \phi \leftrightarrow \psi$. But $\vDash \phi \leftrightarrow \psi$ need not be true. Hence we have that $A \nvDash \psi \to \neg\phi$ and $\neg\phi \notin \boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ if $A \vDash \phi$ and $A \vDash \phi \to \psi$.

Proposition 3.13. *Let $\{\boldsymbol{A}_i\}_{i \in I}$ be a set of structures of the language $\mathfrak{L}$. Then we have*

$$\boldsymbol{Th}^{\dagger}(\{\boldsymbol{A}_i\}) = \bigcap_{i \in I} \boldsymbol{Th}^{\dagger}(\boldsymbol{A}_i).$$

Proof. It is clear that the intersection of the filters is a filter. And it is obvious that $A_i \vDash \phi$ if and only if $\phi \in \boldsymbol{Th}^{\dagger}(\boldsymbol{A}_i)$ for all i. □

Corollary 3.13.1. *Let $\mathcal{M} \subset \mathcal{L}$. Then the set $\boldsymbol{Th}^{\dagger}(\mathcal{M})$ is a filter.*

Proof. This is an immediate consequence of proposition 3.13. □

If a filter is finitely generated, then the filter is principal. And the intersection of finitely many principal filters is a principal filter. Hence we have the following propositions.

Proposition 3.14. *If the filter $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is finitely generated, then $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ has the minimum.*

Proof. Suppose that the filter $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is generated by the set $\{\phi_1, \ldots, \phi_n\}$. Let $\phi := \bigwedge_{1\le i\le n}\phi_i$. Then we have $\phi \in \boldsymbol{Th}^{\dagger}(\boldsymbol{A})$. Therefore, it is obvious that ϕ is the minimum of the filter by proposition 3.8. □

Corollary 3.14.1. *If the filter $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is finitely generated, then $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is a principal filter.*

Proof. Immediate from definition 2.6 and proposition 3.14. □

Proposition 3.15. *Let Ψ be a finitely generated filter of the boolean algebra $\tilde{\Phi}$, μ the minimum member of Ψ. Then Ψ is consistent[definition 2.14] if and only if there exists a structure $\boldsymbol{A}$ of the language $\mathfrak{L}$ such that $A \vDash \mu$.*

Proof. If Ψ is consistent, then there exists a structure $\boldsymbol{A}$ of the language $\mathfrak{L}$ such that $A \vDash \mu$ by definition 2.14. On the other hand, we have that $A \vDash \mu$ implies $A \vDash \psi$ for all $\psi \in \Psi$ by proposition 3.4. It follows that $A \vDash \Psi$. Hence Ψ is consistent. □

Proposition 3.16. *Let $\boldsymbol{A}, \boldsymbol{B}$ be structures of the language $\mathfrak{L}$. Suppose that $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ and $\boldsymbol{Th}^{\dagger}(\boldsymbol{B})$ are principal. Then $\boldsymbol{Th}^{\dagger}(\{\boldsymbol{A}, \boldsymbol{B}\})$ is a principal filter.*

Proof. Suppose that $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ and $\boldsymbol{Th}^{\dagger}(\boldsymbol{B})$ are generated by ϕ and ψ, respectively. By proposition 3.13 we have $\boldsymbol{Th}^{\dagger}(\{\boldsymbol{A}, \boldsymbol{B}\}) = \boldsymbol{Th}^{\dagger}(\boldsymbol{A}) \cap \boldsymbol{Th}^{\dagger}(\boldsymbol{B})$. Let Ω denote the intersection. Then Ω is the set $\{\omega \in \tilde{\Phi} \mid \omega \ge \phi \text{ and } \omega \ge \psi\}$. Hence Ω is generated by $\phi \vee \psi$ since proposition 3.11. □

Corollary 3.16.1. *Let $\mathcal{M} \subset \mathcal{L}$ be a finite subset. If $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is principal for every $\boldsymbol{A} \in \mathcal{M}$, then $\boldsymbol{Th}^{\dagger}(\mathcal{M})$ is principal.*

Proof. Immediate from corollary 3.13.1 and proposition 3.16. □

Remark 3.2. Suppose that $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is a principal filter generated by ϕ. We have known that $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is an ultrafilter by proposition 3.12. Hence one of $\psi, \neg\psi$ is in $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ for all $\psi \in \tilde{\Phi}$. It follows $\phi \le \psi$ or $\phi \le \neg\psi$, i.e., $\vDash (\phi \wedge \psi) \leftrightarrow \phi$ or $\vDash (\phi \wedge \neg\psi) \leftrightarrow \phi$. This is consistent since we have $\vDash (\phi \to \psi) \vee (\phi \to \neg\psi)$ and corollary 3.5.1.

We shall see that if φ a homomorphism[definition 2.7] of $\tilde{\Phi}$ then the subset $\varphi^{-1}(\mathrm{t})$ is a filter.

Proposition 3.17. *Let $\varphi\colon \tilde{\Phi} \to \tilde{\Phi}$ be a homomorphism. Then the subset $\varphi^{-1}(\mathrm{t})$ is a filter in $\tilde{\Phi}$.*

Proof. Let $\phi, \psi \in \varphi^{-1}(\mathrm{t})$ and k denote $\varphi^{-1}(\mathrm{t})$. Then we have

$$\varphi(\phi \wedge \psi) = \varphi(\phi) \wedge \varphi(\psi) = \mathrm{t}.$$

Hence we have $\phi \wedge \psi \in k$. So is $\phi \vee \psi$. And for all $\omega \in \tilde{\Phi}$ with $\omega \ge \phi$, we have

$$\varphi(\omega) = \varphi(\phi \vee \omega) = \varphi(\phi) \vee \varphi(\omega) = \mathrm{t}$$

since corollary 3.1.1. Hence we have $\omega \in k$. And it is clear that $\mathrm{t} \in k$. Therefore, the subset k is a filter. □

Definition 3.2 (cf. [3, 4, 6])**.** The **kernel** of a homomorphism φ, denoted $\ker\varphi$, is defined by $\varphi^{-1}(\mathfrak{t})$.

Remark 3.3. Suppose that a filter Ψ is not an ultrafilter in $\tilde{\Phi}$. If $\phi\vee\psi\in\Psi$ with $\phi,\psi\notin\Psi$, then there may *not* be a homomorphism such that Ψ is a kernel. Since if ϖ is a homomorphism with $\Psi=\ker\varpi$ then $\varpi(\phi\vee\psi)=\varpi(\phi)\vee\varpi(\psi)$. And $\varpi(\phi)\vee\varpi(\psi)$ may not be a tautology if $\varpi(\phi),\varpi(\psi)\neq\mathfrak{t}$. Hence we have a constradiction with $\phi,\psi\notin\Psi$. Therefore a kernel is a filter but a filter need *not* be a kernel.

Proposition 3.18 (cf. [2, 4, 6])**.** *Let ϕ, $\psi\in\tilde{\Phi}$ with $\phi\neq\psi$. Suppose that $\nvDash\phi$ and $\nvDash\psi$. Then we have that $\vDash\phi\vee\psi$ if and only if $\boldsymbol{Mod}(\phi)\cup\boldsymbol{Mod}(\psi)=\mathcal{L}$, that is, for all $\boldsymbol{A}\in\mathcal{L}$, either ϕ or ψ is satisfied by $\boldsymbol{A}$.*

Proof. We may assume $\phi\neq\neg\psi$ without loss of generality. If $\boldsymbol{Mod}(\phi)\cup\boldsymbol{Mod}(\psi)=\mathcal{L}$, then either $A\vDash\phi$ or $A\vDash\psi$ for all $\boldsymbol{A}\in\mathcal{L}$. Hence we have $\vDash\phi\vee\psi$ by (2.33). On the other hand, that $\vDash\phi\vee\psi$ implies $A\vDash\phi\vee\psi$ for all $\boldsymbol{A}\in\mathcal{L}$. By (2.33) we have $A\vDash\phi$ or $A\vDash\psi$ for all $\boldsymbol{A}\in\mathcal{L}$. It follows $\boldsymbol{Mod}(\phi)\cup\boldsymbol{Mod}(\psi)=\mathcal{L}$. This completes the proof. □

Proposition 3.19. *Let ϕ, $\psi\in\tilde{\Phi}$. Suppose that Ψ is a filter generated by $\{\phi\vee\psi\}$. Then there exists a homomorphism φ of $\tilde{\Phi}$ such that Ψ is a kernel of φ if and only if there exists ϕ', $\psi'\in\tilde{\Phi}$ such that $\vDash\phi'\vee\psi'$ with $\nvDash\phi'$ and $\nvDash\psi'$.*

Proof. Let $\varphi\colon\tilde{\Phi}\to\tilde{\Phi}$ be a homomorphism defined by

$$\varphi(x)=\begin{cases}\phi' & \text{if } x=\phi,\\ \psi' & \text{if } x=\psi,\\ \mathfrak{f} & \text{if } x\neq\phi,\psi \text{ and } x\notin\Psi.\end{cases}$$

Then it is clear that the statement holds. □

It is clear that an ultrafilter is a kernel, since one of $\phi,\neg\phi$ is in the ultrafilter for all $\phi\in\tilde{\Phi}$. The set $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ is an ultrafilter.

Proposition 3.20. *Suppose that $\boldsymbol{A}$ is a structure of the language $\mathfrak{L}$. Then there exists a homomorphism $\varphi\colon\tilde{\Phi}\to\tilde{\Phi}$ such that $\ker\varphi=\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$.*

Proof. Let $\phi,\psi\in\tilde{\Phi}$. If $\phi,\psi\notin\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$ then $\phi\wedge\psi\notin\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$, since $\phi\wedge\psi\leq\phi,\psi$. And if $A\vDash\phi\vee\psi$ then either ϕ or ψ is a member of $\boldsymbol{Th}^{\dagger}(\boldsymbol{A})$. Then we define a mapping φ as follows,

$$\varphi(\psi)=\begin{cases}\mathfrak{t} & \text{if } \psi\in\boldsymbol{Th}^{\dagger}(\boldsymbol{A}),\\ \mathfrak{f} & \text{if } \psi\notin\boldsymbol{Th}^{\dagger}(\boldsymbol{A}).\end{cases}$$

It is obvious that φ is the desired homomorphism. □

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