Hamiltonian Chaos and the Fractal Topology of Spacetime (Part 1)

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Abstract

Fractals and multifractals are self-similar structures endowed with *continuous dimensions*. This tutorial traces the origins of fractal spacetime to the universal features of Hamiltonian chaos, conjectured to develop far above the Fermi scale ($v = 246$ GeV). A representative signature of Hamiltonian chaos is the *fragmentation of phase-space* into islands of stability embedded within ergodic layers. Fractal topology of Hamiltonian chaos may account for the multiply connected structure of the large-scale Universe, hinted by recent cosmological data. These observations raise the possibility that gravitational physics and Quantum Field Theory emerge from the chaotic regime of the early Universe.

Key words: Hamiltonian chaos, Fractal spacetime, Continuous dimensions, Nontrivial topology, Gravitational physics, Quantum Field Theory.

1. Introduction

Recent astrophysical measurements pose a host of fresh challenges to the standard model of cosmology. It is known that, although General Relativity focuses on the local geometry of spacetime manifolds, it does not constrain the large-scale topology of the Universe [1]. There are several indications today that the dynamics of the early Universe may produce a *non-trivial (multiply connected) topology*, substantially deviating from the realm of traditional FRW cosmology [1-2]. There is a wealth of questions raised by the latest observations, among which we mention the following:

a) What is the true topology of cosmic spacetime?
b) Is the chaotic dynamics of nonlinear systems compatible with a foam-like structure of the early Universe?
c) Is the hypothetical non-trivial topology of the Universe rooted in the continuous spacetime dimensionality above the Fermi scale?

It is our belief that the methodology of Hamiltonian chaos applied to gravitational dynamics of the early Universe holds key insights into these open questions.

To facilitate communication and further analysis, the purpose of this brief tutorial is to provide an accessible introduction to Hamiltonian chaos and its phase-space topology. We base our paper on a couple of reasonable assumptions, namely:

A1) *Decoherence* of open quantum systems and the transition to classical behavior occurs far above the Fermi scale set by the electroweak vacuum.

A2) Strong Hamiltonian chaos is *nearly universal*, as reflected in the similar phase-space topology of almost all nonlinear systems, including classical gauge and gravitational fields [3].

The presentation proceeds as follows: In Part 1, section 2 surveys the definition and utility of Poincaré maps, along with the concept of nonintegrability. Chaotic dynamics of interacting fields is analyzed in section 3 via the Standard Map formulation of perturbed oscillators. In Part

2, section 4 delves into a textbook example of gravitational chaos in cosmology (the Hénon-Heiles model), which has commonalities with the properties of the Standard Map. Section 5 elaborates upon the emergence of fractional dynamics and fractal spacetime from the *anomalous diffusion* of phase-space orbits accompanying the transition to chaos. Conclusions and a short summary are presented in the last section.

2. Poincaré maps and non-integrability

In Hamiltonian mechanics, an *integrable* system with N degrees of freedom has N independent and Poisson-commuting integrals of motion. In this case, the differential equations describing the time evolution can be explicitly integrated using the *action-angle variables* (I,φ). The solutions undergo periodic phase-space motion on *invariant tori*, and display no signs of randomness or ergodicity. To illustrate this point, consider an integrable one-dimensional system, where the angle variable grows linearly in time as in

$$\varphi(t) = \omega t + \varphi(0) \tag{1}$$

where ω is the angular velocity. (1) describes the motion around a circle of fixed radius. Likewise, an integrable system in two dimensions is defined by two circular orbits orthogonal to each other, whose equations read,

$$\varphi_{1,2}(t) = \omega_{1,2} t + \varphi_{1,2}(0) \tag{2}$$

All phase-space orbits lie on the toroidal surface illustrated in Fig. A:

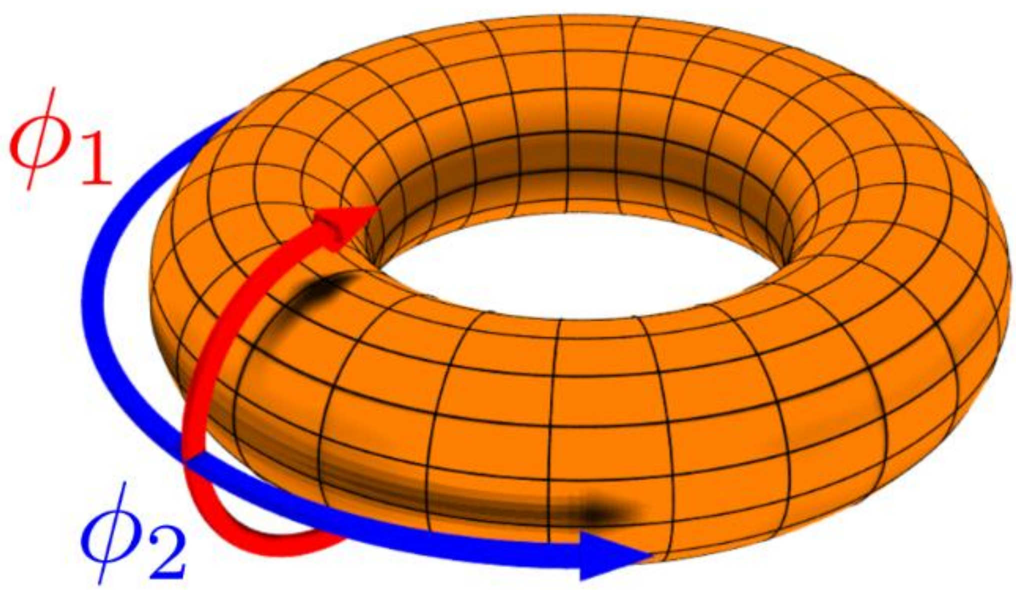

Fig. A: Phase-space motion on a 2D invariant torus [ref. 4]

Motion on invariant tori is considered *periodic* if the ratio of frequencies ω_1/ω_2 is a rational number and *quasiperiodic* otherwise. The phase-space of integrable Hamiltonian systems is *foliated* with subsets of non-intersecting tori corresponding to different initial conditions, per fig. B below:

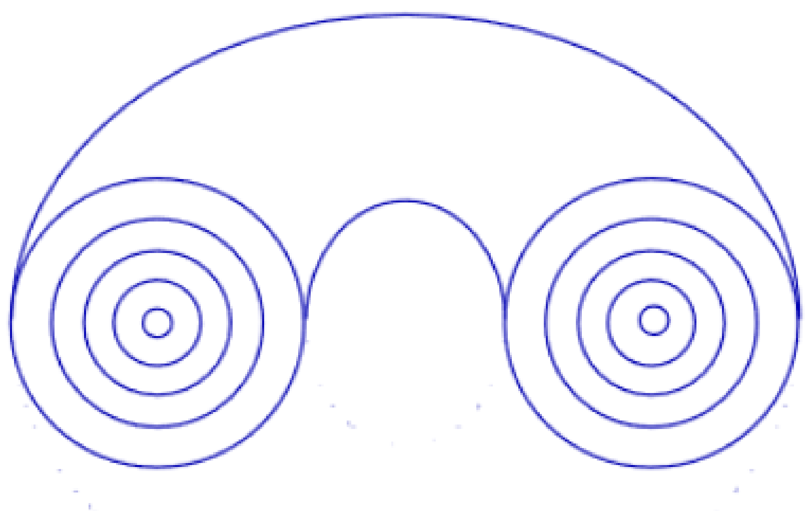

Fig B: Foliated structure of 2D invariant tori [ref. 4]

An effective tool for the analysis of Hamiltonian dynamics in phase-space is the *Poincaré map*. It is created by inserting a plane normal to the path of closed orbits and tracking their consecutive intersection points with the plane. The resulting set of iterates form a *discrete-time representation* of the original dynamics. The action of a typical Poincaré map is shown in Fig. C, where an arbitrary orbit point "z" is mapped to "$P(z)$". Closed curves define "islands of stability" and represent a dynamics regime that is *integrable*; as chaos develops, these curves become immersed in a "sea" of dots reflecting a regime that is *ergodic* and *mixing*.

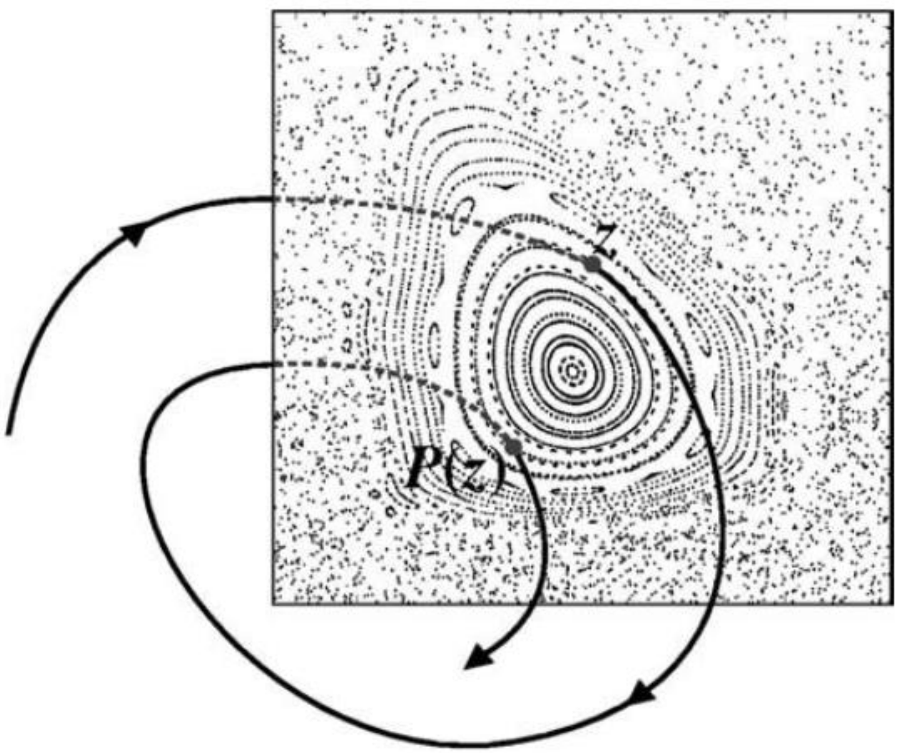

Fig. C: Poincaré Map and topology of phase-space near "z" [ref. 5]

3. Hamiltonian chaos and the dynamics of nonlinear oscillators

Nontrivial topology of Hamiltonian phase-space appears in many contexts involving nonlinear interactions. Let's consider for example, an (oversimplified) toy model of the quantum vacuum consisting of a linear array of coupled harmonic oscillators. The array is driven by impulsive fluctuations forming either a train of delta kicks or an overall periodic perturbation. According to assumption A1), if the dynamics unfolds sufficiently far from the Fermi scale, the oscillators can be considered entirely classical. Setting the model in these conditions avoids the *dynamical localization of quantum fields,* which is relevant to Quantum Chaos but not to classical (deterministic) chaos [6].

The setup is illustrated in Fig. D and is known in the literature as the *Frenkel-Kontorova model*. Here, g denotes the spring constant, a_0 represents the equilibrium separation in the absence of the driving force of period a_S and amplitude A.

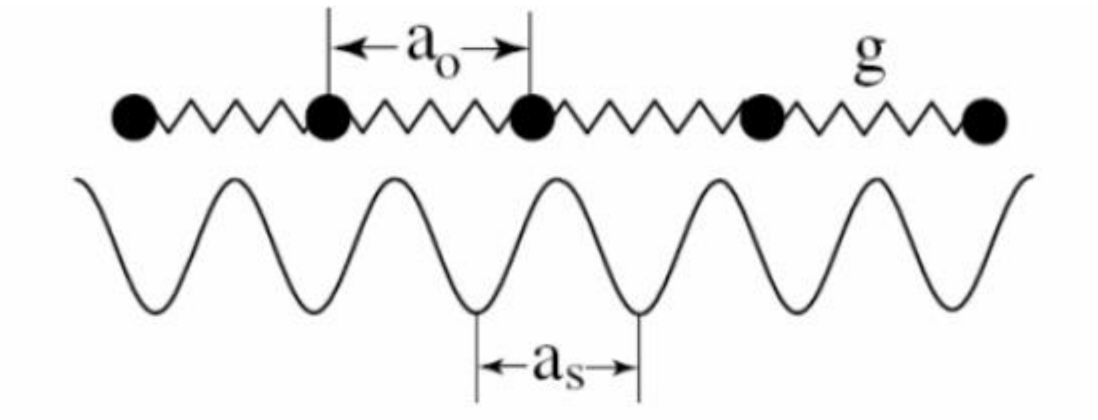

Fig. D: One-dimensional Frenkel-Kontorova model [ref. 7]

The behavior of the model is described by

$$F_j = g(x_{j+1} - x_j - a_0) - g(x_j - x_{j-1} - a_0) + A\sin(kx_j) \tag{3}$$

in which

$$k = 2\pi/a_S \tag{4}$$

Equilibrium is reached when,

$$x_{j+1} + x_{j-1} - 2x_j + \frac{A}{g}\sin(kx_j) = 0 \tag{5}$$

Using the notation

$$\theta_j = kx_j \tag{6a}$$

$$r_j = k(x_j - x_{j-1}) \tag{6b}$$

$$K = -kA/g \tag{6c}$$

turns (3)-(5) into the Standard Map equation in canonical form

$$r_{j+1} = r_j + K\sin(\theta_j) \tag{7a}$$

$$\theta_{j+1} = \theta_j + r_{j+1} \tag{7b}$$

The Standard Map (7) can be derived in a more general context from the time-dependent Hamiltonian,

$$H(p,x,t) = \frac{p^2}{2} + K\cos(x)\,\delta(t) \tag{8}$$

where p stands for the linear momentum, K a parameter quantifying the degree of chaos and $\delta(t)$ a periodic delta function expanded in series according to [6 , 8].

$$\delta(t)=\sum_m \delta(t-mT) \tag{9}$$

In this format, (9) embodies the cumulative effect of "kicks" applied at regular intervals of period T. The Hamiltonian (8) follows from integrating the canonical equations of motion,

$$\dot{p}=-\frac{\partial H}{\partial x} \tag{10a}$$

$$\dot{x}=\frac{\partial H}{\partial p} \tag{10b}$$

It can be shown that (8)-(10) lead to the iterated map equations [6]

$$p_{n+1}=p_n+K\sin(x_n) \tag{11a}$$

$$x_{n+1}=x_n+p_{n+1} \tag{11b}$$

in which n is the iteration index playing the role of discrete time. Below a critical value of parameter K (i.e. $K<K_c$), phase-space trajectories lie on invariant curves and the variation in momentum p is restricted (fig. 1). Following the Kolmogorov-Arnold-Moser (KAM) theorem, the last

invariant curve is destroyed when K is ramped up to $K_c = 0.971635$ (fig 2). Above K_c, momentum variation becomes unbounded and undergoes a *diffusive growth* linearly dependent on the number of map iterations,

$$p^2 \approx D_0(K)t \tag{12}$$

in which $D_0(K)$ represents the diffusion rate. Note that momentum diffusion (12) is a nonlocal process in phase-space, in contrast with the onset of chaos in quantum physics where the spread in momentum is highly suppressed and localized.

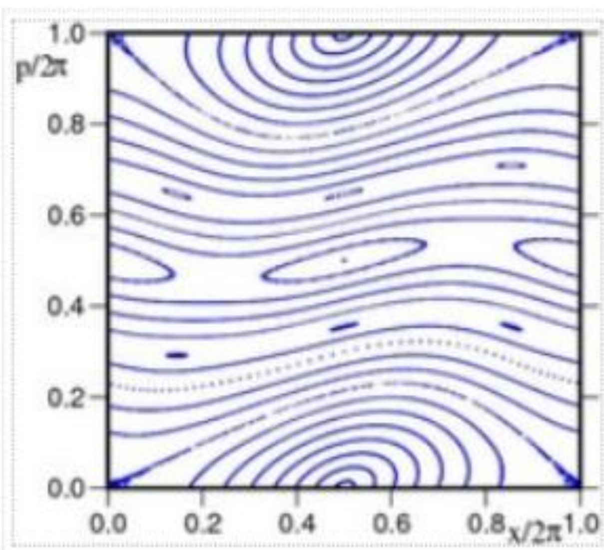

Figure 1: K=0.5

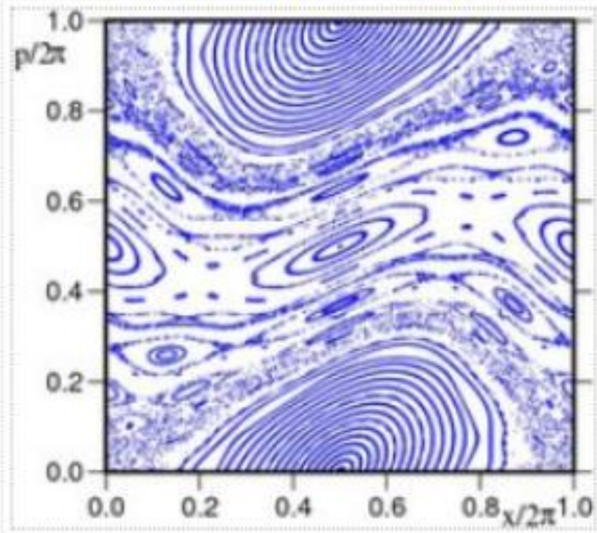

Figure 2: K=0.971635

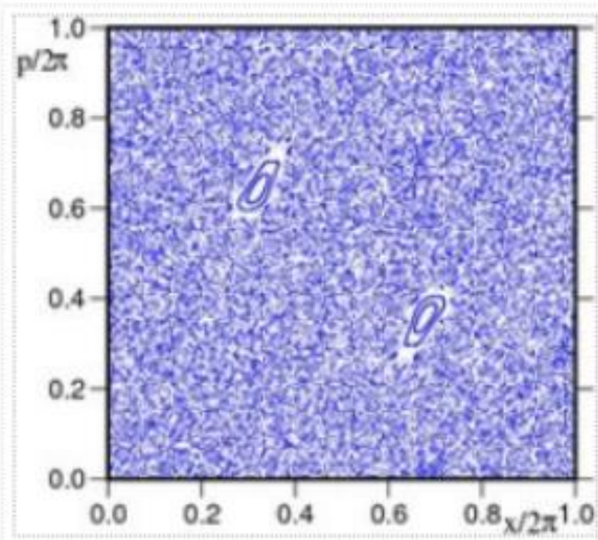

Figure 3: K=5

Fig. 1-3: Poincaré sections of the Standard Map [ref. 6]

Fig. 4 displays a magnified detail of Fig.2 showing the interlaced texture of stability islands and ergodic regions. The repetitive structure on finer scales is readily apparent.

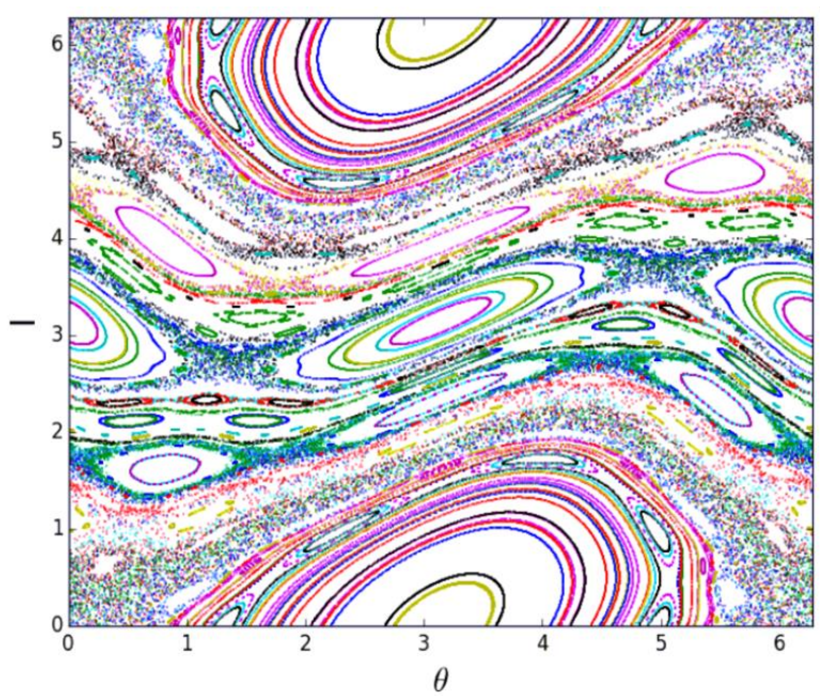

Fig. 4: Magnified detail of Fig. 2 showing repetitive structure on finer scale.

Fig. 3 depicts the Poincaré map near the onset of fully developed chaos, where the ergodic distribution of orbits nearly takes over the whole available map space.

References (Part 1)

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