# Hamiltonian Chaos and the Fractal Topology of Spacetime (Part 2)

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Abstract

This is Part 2 of the brief tutorial "Hamiltonian Chaos and the Fractal Topology of Spacetime" posted at https://www.researchgate.net/publication/369584882

## 4. Hamiltonian Chaos in classical gravity

Given their utility in computational analysis, Poincaré maps are frequently used in the study of *nonintegrable* gravitational systems. A textbook example is the Hénon-Heiles model, which describes the motion of stars in the galactic disk. The Hamiltonian of the model is given by [11, 18]:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{1}{3}y^3 \tag{13}$$

Because (13) is conservative, its orbits are confined to a constant energy hypersurface ($H=E$). Fig. 5 illustrates the sequence of maps in the $(\dot{y}, y)$ plane as $E$ (in dimensionless units) is progressively increased. At low energies, the orbits lie close to those computed from perturbation theory [18]

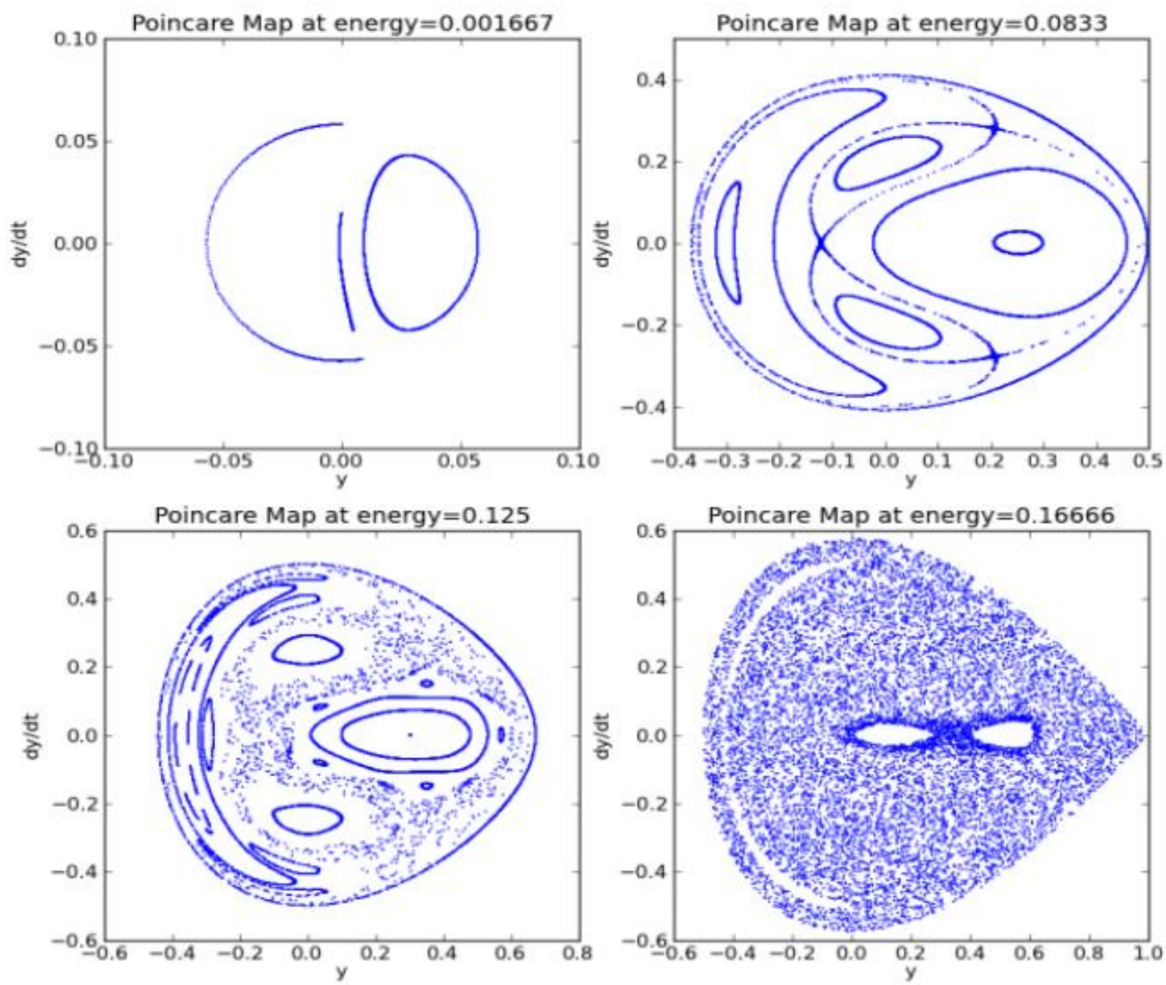


**Fig. 5:** Poincaré Maps of the Hénon-Heiles model [ref. 9]

The system appears to maintain integrability up to $E = 0.125$, at which point chaotic regions begin to develop along with sparse islands of integrability. At $E = 0.166..$, the chaotic regions are widespread and integrability is almost entirely lost. For better visualization of the transition to chaos, Fig. 6 shows a color-coded representation of the map at $E = 0.128$.

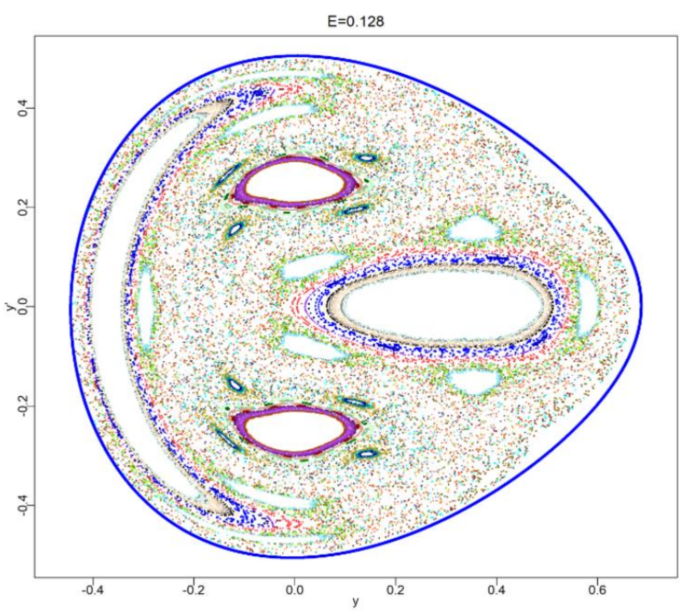


**Fig. 6:** Color coded view of the Poincaré Map at $E = 0.128$ [ref. 10]

In line with assumption A2), similar phase-space behavior occurs in many classical field systems displaying transition to chaos under continuous tuning of driving parameter(s), see e.g. [19].

## 5. Hamiltonian Chaos and fractal spacetime

It follows from these examples that a certain generality of Hamiltonian dynamics exists, based on the universal nesting of invariant tori and chaotic orbits. The distribution of regions containing invariant tori and chaotic orbits *repeats itself* on all scales and depends on the magnitude of the driving parameter(s) ($K$ in the Standard Map or conserved energy $E$ in the Hénon-Heiles model).

As chaos sets in above a critical value of the driving parameter(s), analysis shows that chaotic orbits repeatedly "stick" to the border of critical tori with a power-like distribution of sticking times [12-15]. This effect generates a long-time correlation of chaotic orbits and an *anomalous diffusion* of momentum in phase-space. If the driving parameter exceeds a critical value, the diffusion of the mean squared momentum no longer follows (12), but a power law relationship of the form [14]

$$\left\langle p^2(t) \right\rangle \approx D_\alpha t^\alpha \tag{14}$$

Here, the exponent $\alpha \neq 1$ measures the departure from standard diffusion and can be interpreted as *continuous dimension* associated with the fractal topology of Hamiltonian phase-space.

## 6. Concluding remarks

Theory and experiment alike indicate that anomalous transport/diffusion is a defining feature of many complex systems, as it links to phenomena such as fractional dynamics, Levy flights, continuous time random walks and fractional Brownian motion, to name a few. As last two decades have shown, adequate description of these phenomena requires a radically different framework in which [see e.g. 16-17, 20]

1) *Fractional differential and integral operators* replace ordinary calculus on smooth manifolds,
2) *Spacetime fractalization* makes the transition from discrete to continuous spacetime dimensions.

Given assumptions A1) and A2) and that (nearly) all classical field theories amount to Hamiltonian dynamical systems, a couple of natural questions arises, namely:

a) What is the most sensible path connecting low-energy field theory and Hamiltonian chaos?
b) Can the fractal/multifractal topology of spacetime explain some of the open issues challenging the Standard Model (SM) and strong gravitational physics?

One recalls that Quantum Field Theory (QFT) lies at the foundation of the SM, which is built in compliance with several postulates called *consistency conditions*. It can be said, in fact, that the remarkable success of SM stems from a unitary, local, renormalizable, gauge invariant and anomaly-free formulation of QFT. Thus, a sensible transition from Hamiltonian chaos to QFT can only be made via a spacetime having arbitrarily small and continuous deviations from four space dimensions, as in

$$\varepsilon(\mu) = 4 - D(\mu) = O[{m^2(\mu)}/{\Lambda_{UV}^2}] << 1 \tag{15}$$

where $\mu$ is the Renormalization flow scale, $m(\mu)$ is a mass parameter and $\Lambda_{UV}$ is the large ultraviolet cutoff of the theory. The expectation is that (15) becomes relevant in far-from-equilibrium and fully nonintegrable conditions, prone to develop far above the Fermi scale. Aside from the fragmentation of phase-space in Hamiltonian chaos, (15) arises from two other premises, namely,

1) Dimensional Regularization of QFT,

2) Emergence of nontrivial fixed points of the RG equations in statistical physics and the $\varepsilon$-expansion evaluation of critical exponents.,

Our research reveals that various aspects of $\varepsilon(\mu)$ lie behind many unsettled features of particle and gravitational physics, such as the multifractal geometry of the SM, the repetitive architecture of SM parameters, unexplained SM-related observations, the Cantor Dust structure of Dark Matter and the thermodynamic interpretation of General Relativity [20].

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