Chaotic oscillations in a piecewise linear spring-mass system

Marcello Colozzo – https://www.extrabyte.info

Abstract

The dynamic evolution of a a piecewise linear spring-mass system in a viscous medium subjected to a periodic external force is characterized by the presence of bifurcations, therefore by deterministic chaos.

1 One-dimensional motion. Phase plan

For a mechanical system consisting of a particle that performs a non-relativistic motion along the x axis under the action of a force $F(x,\dot{x},t)$, the Cauchy problem relating to dynamic evolution is written:

$$\mathcal{P}:\begin{cases} m\ddot{x}=F(x,\dot{x},t) \\ x(t_0)=x_0,\ \dot{x}(t_0)=\dot{x}_0 \end{cases} \tag{1}$$

Under reasonable assumptions of regularity of the real function F, the problem (1) is compatible and determined.

As known, the dynamic evolution of the system can be studied in the phase plane. Indeed the second order differential equation $m\ddot{x}=F(x,\dot{x},t)$ is equivalent to the first order system:

$$\begin{cases} \dot{x}=y \\ \dot{y}=\frac{1}{m}F(x,\dot{x},t) \end{cases} \tag{2}$$

Denoting with $(x(t),y(t))$ the unique solution of (2) (for a given initial condition), the geometric place

$$\gamma:\ x=x(t),\quad y=y(t),\quad t\in[t_0,+\infty) \tag{3}$$

is called the *phase curve* of the system, and the corresponding cartesian plane xy is called the *phase plane*. Note that the (3) is a regular parametric representation of the place γ.

Definition 1 *A point of the x axis with abscissa ξ_0, is a* **equilibrium position** *if the particle, which at the initial time t_0 was in that position with zero velocity, remains in that position.*

In the phase plane an equilibrium position necessarily has coordinates $(\xi_0,0)$. It must then be $x(t)\equiv\xi_0$, $y(t)\equiv 0$. That is, the phase curve reduces to the point $(\xi_0,0)$. More precisely, $(x(t),y(t))\equiv(\xi_0,0)$ solves the system (2) with the initial condition $x(t_0)=\xi_0$, $y(t_0)=0$.

Definition 2 *(***Lyapunov stability** *)*

An equilibrium position ξ_0 is called **stable** *if however we take (in the phase plane) a neighborhood U of $(\xi_0,0)$, we can find a neighborhood U' of $(\xi_0,0)$ such that whatever the initial condition $(x(t_0),y(t_0))\in U'$, the solution $(x(t),y(t))$ of (2) is contained in U'. More specifically:*

$$\forall U_\varepsilon(\xi_0,0),\ \exists U'_{\delta_\varepsilon}(\xi_0,0)\mid(x(t_0),y(t_0))\in U'_{\delta_\varepsilon}(\xi_0,0)\Longrightarrow$$
$$\Longrightarrow(x(t),y(t))\in U_\varepsilon(\xi_0,0),\ \ \forall t\in[0,+\infty)$$

Numerically

$$\forall\varepsilon>0,\ \exists\delta_\varepsilon>0\mid|x(t_0)-\xi_0|<\delta_\varepsilon,\ |y(t)|<\delta_\varepsilon\Longrightarrow$$
$$\Longrightarrow|x(t)-\xi_0|<\varepsilon,\ |y(t)|<\varepsilon,\ \ \forall t\in[0,+\infty)$$

Conversely, if

$$\forall \varepsilon > 0, \ \nexists \delta_\varepsilon > 0 \mid |x(t_0) - \xi_0| < \delta_\varepsilon, \ |y(t)| < \delta_\varepsilon \Longrightarrow$$
$$\Longrightarrow |x(t) - \xi_0| < \varepsilon, \ |y(t)| < \varepsilon, \ \forall t \in [0, +\infty)$$

the equilibrium position ξ_0 is called **unstable**.

Definition 3 *(***Lyapunov asymptotic stability***)*
A stable equilibrium position ξ_0 is called **asymptotically stable**, *if in the phase plane there exists a neighborhood U of $(\xi_0, 0)$ such that however we take the initial condition $(x(t_0), y(t_0)) \in U(\xi_0, 0)$, we have*

$$\lim_{t \to +\infty} x(t) = \xi_0, \quad \lim_{t \to +\infty} y(t) = 0 \tag{4}$$

In other words, the stable equilibrium position is reached after an infinite time.

2 Conservative systems

Let us consider the special case in which the force $F(x, \dot{x}, t)$ introduced in section 1 is a positional force with potential energy $U(x)$:

$$F(x) = -\frac{dU(x)}{dx} \tag{5}$$

As is known, the corresponding mechanical system described by

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{1}{m} F(x) \end{cases} \tag{6}$$

conserves total mechanical energy:

$$W = T + U = \frac{1}{2} m \dot{x}^2 + U(x) \tag{7}$$

More precisely, for given initial conditions $x(t_0) = x_0, \ \dot{x}(t_0) = \dot{x}_0$, we have

$$W = \frac{1}{2} m \dot{x}(t)^2 + U(x(t)) = \frac{1}{2} m \dot{x}_0^2 + U(x_0) \stackrel{def}{=} W_0, \quad \forall t \in [t_0, +\infty) \tag{8}$$

From (7)

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m} [W_0 - U(x)]} \tag{9}$$

so $W_0 - U(x) \geq 0$. This suggests defining the classically accessible region:

$$\Lambda(W_0) = \{x \in \mathbb{R} \mid W_0 - U(x) \geq 0\} \subseteq \mathbb{R} \tag{10}$$

since motion is possible only in this subset of $\mathbb{R}$.

We define:

$$f(x) := \frac{2}{m} [W_0 - U(x)] \tag{11}$$

From (9):

$$\begin{cases} \dot{x}^2 = f(x) \Longrightarrow \begin{cases} \dot{x} = +\sqrt{f(x)} & \text{(moto progressivo)} \\ \dot{x} = -\sqrt{f(x)} & \text{(moto regressivo)} \end{cases} \\ x(t_0) = x_0 \end{cases} \tag{12}$$

We state the following theorem for the proof of which we refer to [1]:

Theorem 4 *For $\Lambda(W_0) = [x_1, x_2]$, the points x_1, x_2 are checkpoints with motion reversal, and the motion is periodic with period:*

$$T = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{f(x)}} \tag{13}$$

Note that the extrema of integration are singularities for the integrand, but it is easy to convince yourself that the integral converges.

3 Piecewise linear harmonic oscillator

We have the elastic force field:

$$F(x) = \begin{cases} -k_1 x, & x < 0 \\ -k_2 x, & x > 0 \end{cases}, \quad k_1 > k_2 \tag{14}$$

with potential energy:

$$U(x) = \begin{cases} \frac{1}{2}k_1 x^2, & x < 0 \\ \frac{1}{2}k_2 x^2, & x > 0 \end{cases} \tag{15}$$

whose graph consists of two arcs of parabolas connected in $(0,0)$ which is an angular point (fig. 1) since the two arcs have different slopes.

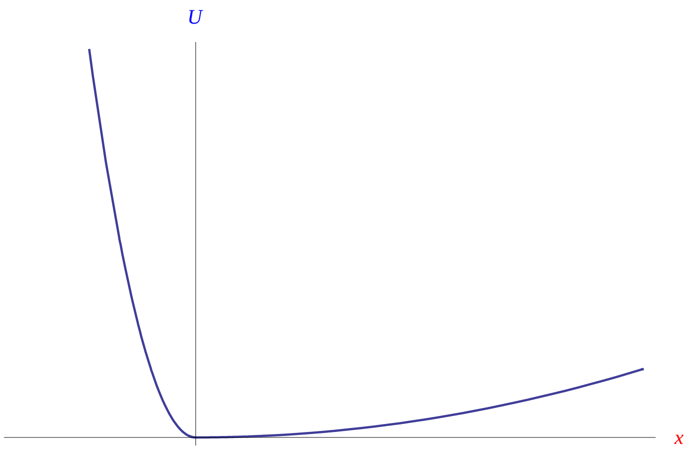

Figure 1: Potential energy trend (15).

The (14)-(15) are rewritten:

$$F(x) = -\frac{1}{2}\left[(k_1 + k_2)x - (k_1 - k_2)|x|\right]$$
$$U(x) = \frac{1}{4}\left[(k_1 + k_2)x^2 - (k_1 - k_2)x|x|\right]$$

Going from spring constant to angular frequencies $\omega_i = \sqrt{k_i/m}$

$$F(x) = -\frac{1}{2}m\left[\left(\omega_1^2 + \omega_2^2\right)x - \left(\omega_1^2 - \omega_2^2\right)|x|\right] \tag{16}$$
$$U(x) = \frac{1}{4}m\left[\left(\omega_1^2 + \omega_2^2\right)x^2 - \left(\omega_1^2 - \omega_2^2\right)x|x|\right]$$

Here it is $\omega_1 > \omega_2$ since we assumed $k_1 > k_2$. Without loss of generality if the particle is initially at rest i.e. $\dot{x}_0 = 0$, but with $x_0 \neq 0$, the mechanical energy is

$$W_0 = U(x_0)$$

and if $x_0 > 0$

$$W_0 = \frac{1}{2} m \omega_2^2 x_0^2$$

whereby the classically accessible region is

$$\Lambda(W_0) = [x_0', x_0] \tag{17}$$

as we see from fig. 2.

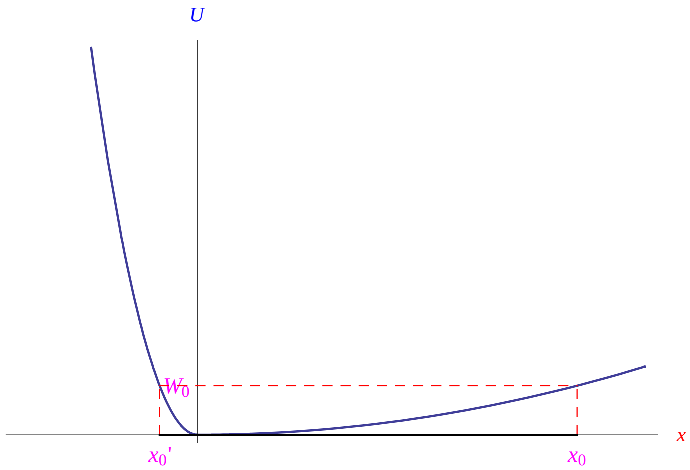

Figure 2: Tipica regione classicamente accessibile per il sistema di energia potenziale (15).

By the theorem (4) the motion is periodic in (17) with period:

$$T = 2 \int_{x_0'}^{x_0} \frac{dx}{\sqrt{\frac{2}{m}[W_0 - U(x)]}} = 2 \int_{-\frac{1}{\omega_1}\sqrt{\frac{2W_0}{m}}}^{\frac{1}{\omega_2}\sqrt{\frac{2W_0}{m}}} \frac{dx}{\sqrt{\frac{2}{m}[W_0 - U(x)]}}$$
$$= 2 \left[\int_{-\frac{1}{\omega_1}\sqrt{\frac{2W_0}{m}}}^{0} \frac{dx}{\sqrt{\frac{2}{m}\left[W_0 - \frac{1}{2} m \omega_1^2 x^2\right]}} + \int_0^{\frac{1}{\omega_2}\sqrt{\frac{2W_0}{m}}} \frac{dx}{\sqrt{\frac{2}{m}\left[W_0 - \frac{1}{2} m \omega_1^2 x^2\right]}} \right]$$

Integrals are easy to calculate. So

$$T = \pi \left(\frac{1}{\omega_1} + \frac{1}{\omega_2} \right) \tag{18}$$

By numerically integrating the differential equation of motion with the initial conditions

$$\begin{cases} \ddot{x} + \frac{1}{2}\left[(\omega_1^2 + \omega_2^2)\, x - (\omega_1^2 - \omega_2^2)\,|x|\right] = 0 \\ x(0) = 1, \ \dot{x}(0) = 0 \end{cases} \tag{19}$$

we obtain the trends for $x(t)$, $\dot{x}(t)$ shown in figs. 3, from which we see that the velocity has almost a discontinuity of the first kind in the minimum points of $x(t)$. The latter are in the region $x < 0$ i.e. where the elastic potential changes abruptly so that the particle is mechanically reflected by the quasi-potential barrier (cf. section 3.1). In fig. 5 we report the phase curve.

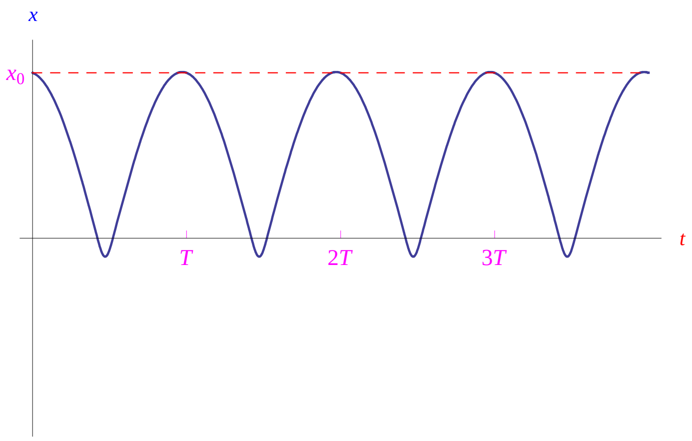

Figure 3: Evolution of the solution $x\left(t\right)$ of (19) for a particle of unit mass and $\omega_1 = 900 rad/\,\mathrm{s}$, $\omega_2 = 100 rad/\,\mathrm{s}$.

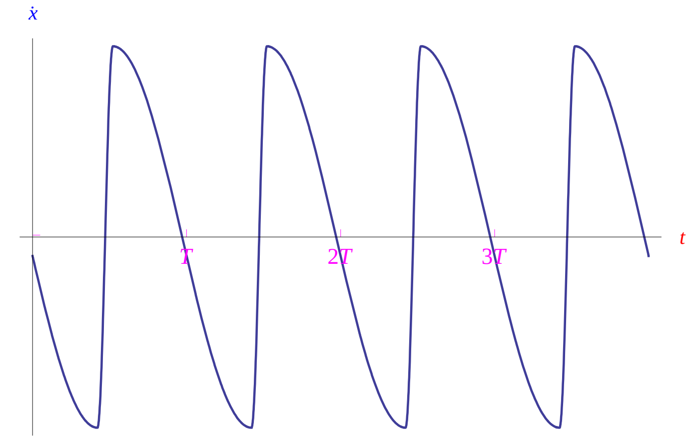

Figure 4: Evolution of the velocity $\dot{x}\left(t\right)$ of (19) for a particle of unit mass and $\omega_1 = 900 rad/\,\mathrm{s}$, $\omega_2 = 100 rad/\,\mathrm{s}$.

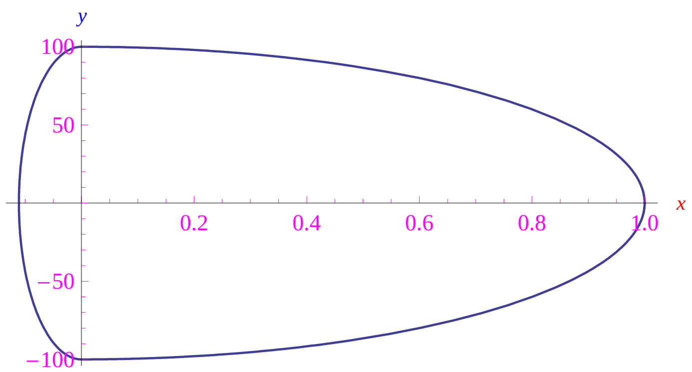

Figure 5: Phase curve trend for a particle of unit mass e $\omega_1 = 900 rad/\,\mathrm{s}$, $\omega_2 = 100 rad/\,\mathrm{s}$.

3.1 Special cases

For $\omega_2 = 0$, in the region $x > 0$ the particle is free from forces. Therefore, if the initial position is $x_0 > 0$ and the initial velocity is oriented in the direction of the positive x axis, the particle performs a progressive uniform rectilinear motion. Conversely, it approaches the origin and having mechanical energy $W_0 = \frac{1}{2} m \omega_2^2 x_0^2 > 0$ it reaches the abscissa point $\xi_0 < 0$ such that $W_0 = \frac{1}{2} m \omega_1^2 \xi_0^2$ that a stop point with motion reversal, after which performs a uniform progressive motion. In the limit $\omega_1 \to +\infty$ we have an infinitely high potential barrier which prevents the particle from penetrating into the region $x < 0$, so the predicted motion reversal is instantaneous (fig. 6). Therefore, under the same initial conditions, the particle is elastically reflected by the barrier.

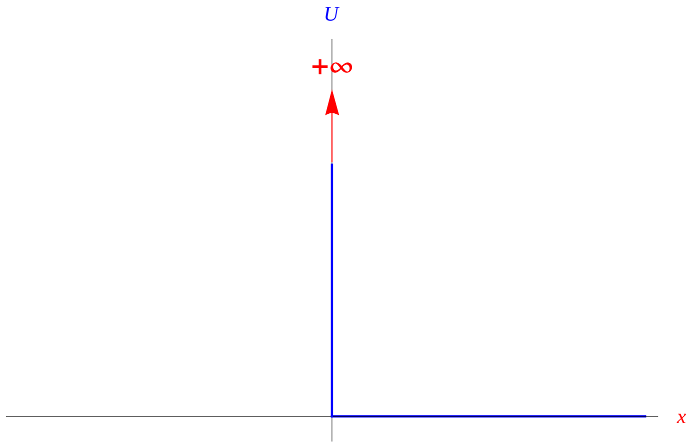

Figure 6: Potential energy trend for $\omega_1 \to +\infty$ and $\omega_2 = 0$. At $x = 0$ the potential schematizes an elastically reflecting barrier.

4 Piecewise damped linear harmonic oscillator

In the presence of a viscosity term $b > 0$, a force $-b\dot{x}$ acts on the particle which opposes the motion, so the system does not conserve mechanical energy. The differential equation of motion composes the following Cauchy problem:

$$\begin{cases} \ddot{x} + \frac{b}{m}\dot{x} + \frac{1}{2}\left[\left(\omega_1^2 + \omega_2^2\right)x - \left(\omega_1^2 - \omega_2^2\right)|x|\right] = 0 \\ x(0) = 1, \ \dot{x}(0) = 0 \end{cases} \tag{20}$$

For a particle of unit mass and $\omega_1 = 900 rad/\mathrm{s}$, $\omega_1 = 100 rad/\mathrm{s}$, $b = 4$ we obtain the phase curve of fig. 7, where we see that $x = 0$ is an asymptotically stable equilibrium point.

5 Deterministic chaos

If we apply an external force to the system of the previous section:

$$F(t) = F_{\max} \sin \Omega t - F_0, \qquad (0 < F_0 < F_{\max})$$

the Cauchy problem (20) is written:

$$\begin{cases} \ddot{x} + \frac{b}{m}\dot{x} + \frac{1}{2}\left[\left(\omega_1^2 + \omega_2^2\right)x - \left(\omega_1^2 - \omega_2^2\right)|x|\right] = \frac{F_{\max}}{m} \sin \Omega t - \frac{F_0}{m} \\ x(0) = 0, \ \dot{x}(0) = 0 \end{cases} \tag{21}$$

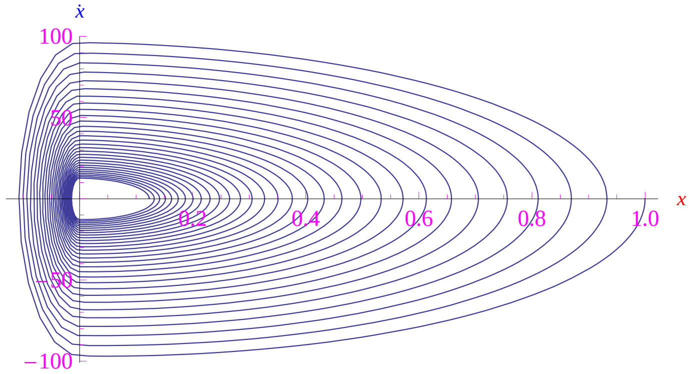

Figure 7: Phase curve trend for a particle of unit mass e $\omega_1 = 900 rad/\mathrm{s}$, $\omega_2 = 100 rad/\mathrm{s}$, $b = 4$.

where we have assumed $x(0) = \dot{x}(0) = 0$, since now there is an external force for which the system does not remain in the initial position. We integrate numerically with the following data (in SI units):

$$m = 10^{-4}\,\mathrm{kg},\ b = 60,\ \omega_1 = 5 \cdot 10^6 rad/\mathrm{s},\ \omega_2 = 3.16 \cdot 10^5 rad/\mathrm{s},\ \Omega = 4.40 \cdot 10^6 rad/\mathrm{s} \quad (22)$$
$$F_{\max} = 0.3\,\mathrm{N},\ F_0 = \frac{1}{20}\,\mathrm{N}$$

In figs. 8-9-10-11 we graph the solutions found. The bifurcation diagram (fig. 11) shows the presence of deterministic chaos.

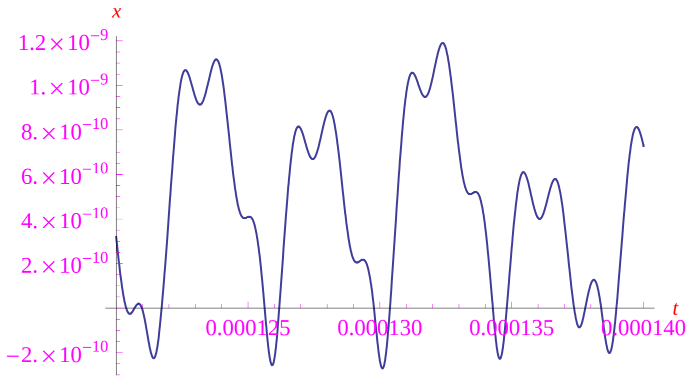

Figure 8: Trend of the abscissa $x(t)$ for the data (22).

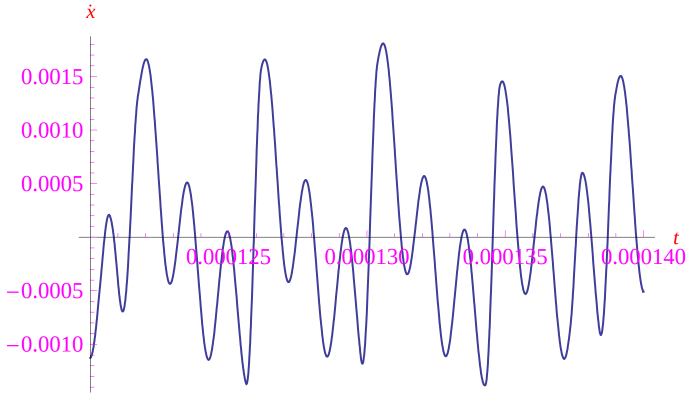

Figure 9: Speed trend for data (22).

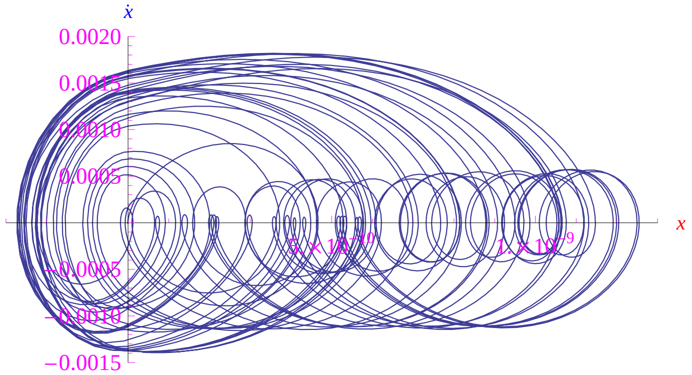

Figure 10: Phase curve trend for data (22).

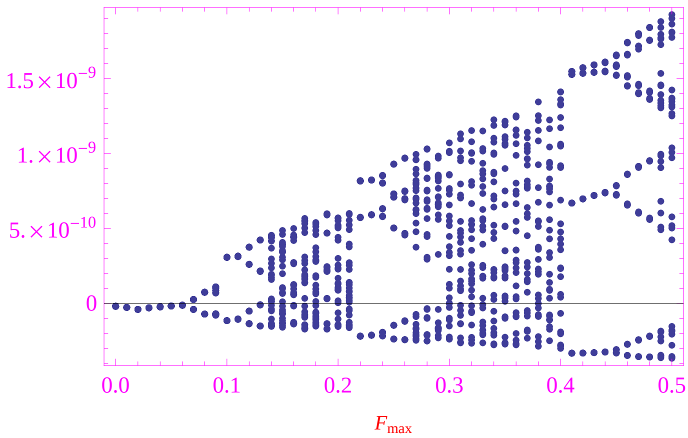

Figure 11: Bifurcation diagram.

References

[1] Colozzo M. *Appunti di Meccanica Analitica: Dinamica del punto materiale - Problemi unidimensionali*.

[2] Fasano A., Marmi S., *Meccanica analitica*. Boringhieri

[3] Gantmacher F.R., *Lezioni di meccanica analitica*. Editori Riuniti. 1980

[4] Wolfram S. *An Elementary Introduction to the Wolfram Language*.

[5] Wagon S. *Mathematica in Action*.

[6] Alligood K.T., Sauer T.D., Yorke J. A. *CHAOS: An Introduction to Dynamical Systems*. Springer