Symmetry of the conductivity matrix

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1. *The Green's function equation*

Let us first consider the conductivity matrix on a non-compact manifold in $d = 2$ dimensions. The differential equation that describes the behaviour of α,

$$\partial_i \left[Z \sqrt{\gamma} \gamma^{ij} \left(E_j + \partial_j \alpha \right) \right] = 0, \tag{1}$$

can be formally solved by writing

$$\alpha(\vec{x}) = \int d^2 y \, G \left(\vec{x} - \vec{y} \right) \frac{\partial}{\partial y^i} \left[Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) \right] E_j, \tag{2}$$

which leads to the Green's function equation

$$\frac{\partial}{\partial x^i} \left[Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} G \left(\vec{x} - \vec{y} \right) \right] = -\delta^{(2)} \left(\vec{x} - \vec{y} \right). \tag{3}$$

On a torus, the right-hand-side needs to be modified so that the integral over the expression vanishes. Hence, we can write down a new Green's function equation

$$\frac{\partial}{\partial x^i} \left[Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} G \left(\vec{x} - \vec{y} \right) \right] = -\delta^{(2)} \left(\vec{x} - \vec{y} \right) + \frac{1}{\int d^2 x' \sqrt{\gamma \left(x' \right)}} \sqrt{\gamma(\vec{x})}. \tag{4}$$

Note that this extension of the Green's function equation is consistent with the defining differential equations as the addition is purely a function of x. Hence, we can formally define $G(x - y) = \tilde{G}(x - y) + R(x)$, so that

$$\frac{\partial}{\partial x^i} \left[Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} R(x) \right] = \frac{1}{\int d^2 x' \sqrt{\gamma \left(x' \right)}} \sqrt{\gamma(x)}, \tag{5}$$

$$\frac{\partial}{\partial x^i} \left[Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} \tilde{G} \left(\vec{x} - \vec{y} \right) \right] = -\delta^{(2)} \left(\vec{x} - \vec{y} \right). \tag{6}$$

This implies that

$$\alpha(\vec{x}) = \int d^2 y \, \tilde{G} \left(\vec{x} - \vec{y} \right) \frac{\partial}{\partial y^i} \left[Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) \right] E_j = \int d^2 y \, G \left(\vec{x} - \vec{y} \right) \frac{\partial}{\partial y^i} \left[Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) \right] E_j, \tag{7}$$

as the following divergence integral over a torus vanishes (the metric and Z are single-valued):

$$R(x) \int d^2 y \frac{\partial}{\partial y^i} \left[Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) E_j \right] = R(x) \int d^2 y \sqrt{\gamma} \, \nabla_i \left(Z(y) E^i \right) = 0. \tag{8}$$

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2. *Symmetry of the conductivity matrix*

The next important thing is to understand whether $G(x-y)$ is symmetric under the interchange of x and y. Instead of this statement, we will prove a somewhat weaker statement, which will be sufficient to show that the conductivity matrix is symmetric. Consider the integral

$$\begin{aligned} I &= \int d^2y\sqrt{\gamma(y)}\left\{G(y-x)\nabla_i\left[Z(y)\nabla^i G\left(y-x'\right)\right]-G(y-x')\nabla_i\left[Z(y)\nabla^i G\left(y-x\right)\right]\right\} \\ &= \int d^2y\nabla_i\left\{\sqrt{\gamma(y)}\,G(y-x)Z(y)\nabla^i G(y-x')-\sqrt{\gamma(y)}\,G(y-x')Z(y)\nabla^i G(y-x)\right\} \\ &= 0, \end{aligned} \tag{9}$$

where all covariant derivatives act w.r.t. y. Since the integrand is a total derivative, and we are integrating over a compact torus without a boundary (and $G(x-y)$ is single valued by construction), the integral automatically vanishes.

Consider now another integral, which also clearly vanishes,

$$I' = \frac{\partial^2}{\partial x^i \partial x'^j} I = 0. \tag{10}$$

By using Eq. (4), we find that

$$I' = \frac{\partial^2}{\partial x^i \partial x'^j}\left[G(x-x')-G(x'-x)+\frac{1}{\tilde{V}_2}\int d^2y\sqrt{\gamma(y)}\left[G(y-x)-G(y-x')\right]\right] \tag{11}$$

$$= \frac{\partial^2}{\partial x^i \partial x'^j}\left[G(x-x')-G(x'-x)\right] = 0. \tag{12}$$

Hence,

$$\frac{\partial^2}{\partial x^i \partial y^j} G(x-y) = \frac{\partial^2}{\partial x^i \partial y^j} G(y-x). \tag{13}$$

The solution for the conserved "auxiliary" current can be written as

$$e^2\mathcal{J}^i = Z\sqrt{\gamma}\gamma^{ij}\left(E_j+\partial_j\alpha\right) \tag{14}$$

$$= \left[Z(\vec{x})\sqrt{\gamma(\vec{x})}\gamma^{ij}(\vec{x}) - \int d^2y\, Z(\vec{x})Z(\vec{y})\sqrt{\gamma(\vec{x})\gamma(\vec{y})}\gamma^{ik}(\vec{x})\gamma^{jl}(\vec{y})\frac{\partial^2}{\partial x^k \partial y^l}G\left(\vec{x}-\vec{y}\right)\right]E_j, \tag{15}$$

and the conductivity matrix is given by

$$\begin{aligned} e^2\sigma^{ij} &= \frac{1}{V_2}\int d^2x\, Z(\vec{x})\sqrt{\gamma(\vec{x})}\gamma^{ij}(\vec{x}) \\ &\quad - \frac{1}{V_2}\int d^2x\, d^2y\, Z(\vec{x})Z(\vec{y})\sqrt{\gamma(\vec{x})\gamma(\vec{y})}\gamma^{ik}(\vec{x})\gamma^{jl}(\vec{y})\frac{\partial^2}{\partial x^k \partial y^l}G\left(\vec{x}-\vec{y}\right). \end{aligned} \tag{16}$$

To show that σ^{ij} is symmetric, consider

$$\begin{aligned} e^2\left(\sigma^{ij}-\sigma^{ji}\right) &= -\frac{1}{V_2}\int d^2x\, d^2y\, Z(\vec{x})Z(\vec{y})\sqrt{\gamma(\vec{x})\gamma(\vec{y})}\gamma^{ik}(\vec{x})\gamma^{jl}(\vec{y})\frac{\partial^2}{\partial x^k \partial y^l}\left[G\left(\vec{x}-\vec{y}\right)-G\left(\vec{y}-\vec{x}\right)\right] \\ &= 0, \end{aligned} \tag{17}$$

due to the symmetry of the metric tensor γ_{ij} and Eq. (13).

The analysis in other dimensions is trivial due to our ability to eliminate Z from (1) with conformal transformations and the fact that we didn't use any special properties of two dimensional spaces in the proof.