Weird fluids

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I. FLUIDS

Work in three dimensions.
Find σ^{ab}, such that

$$\partial_a \sigma^{ab} = 0 \tag{1}$$

Write

$$\sigma^{ab} = \epsilon^{bcd} \partial^a \partial_c \psi_d + \epsilon^{acd} \partial^b \partial_c \psi_d \tag{2}$$

with the additional condition that

$$\partial^2 \psi_a = \mathcal{K} \partial_a \phi. \tag{3}$$

The tensor σ^{ab} is then symmetric, traceless and divergenceless by construction.
Transverse:

$$u_a \sigma^{ab} = u_a \left(\epsilon^{bcd} \partial^a \partial_c \psi_d + \epsilon^{acd} \partial^b \partial_c \psi_d \right) = 0 \tag{4}$$

Vorticity:

$$\omega^{ab} = \Delta^{ac} \Delta^{bd} \left(\partial_c u_d - \partial_d u_c \right) \tag{5}$$

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II. VELOCITY

Velocity

$$u_a = \epsilon_{abc}\partial^b\psi^c \tag{6}$$

divergence

$$\partial_a u^a = 0 \tag{7}$$

derivative

$$\partial_a u_b = \epsilon_{bcd}\partial_a\partial^c\psi^d \tag{8}$$

$$= \frac{1}{2}\left(\epsilon_{bcd}\partial_a\partial^c\psi^d + \epsilon_{acd}\partial_b\partial^c\psi^d\right) + \frac{1}{2}\left(\epsilon_{bcd}\partial_a\partial^c\psi^d - \epsilon_{acd}\partial_b\partial^c\psi^d\right) \tag{9}$$

normalisation

$$u_a u^a = \epsilon_{abc}\epsilon^{ade}\partial^b\psi^c\partial_d\psi_e = \left(\delta^d_b\delta^e_c - \delta^e_b\delta^d_c\right)\partial^b\psi^c\partial_d\psi_e = \partial_a\psi_b\partial^a\psi^b - \partial_a\psi_b\partial^b\psi^a = -1 \tag{10}$$

$$\partial_a\sigma^{ab} = \frac{1}{2}\partial^2 u^b = \epsilon^{bcd}\partial_c\partial^2\psi_d = 0 \tag{11}$$

hence

$$\partial^2\psi^a = K^a \tag{12}$$

Green's function

$$\psi^a(x) = K^a\int d^3y G(x-y) \tag{13}$$

$$\partial^2_x G(x-y) = \delta(x-y) \tag{14}$$

III. CONFORMAL ANOMALY IN COSMOLOGY

Solution with $u^\mu = (1,0,0,0)$, $\ln s =$ const. and a conformally flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2. \tag{15}$$

Since $g_{\mu\nu}$ is conformally flat and $t^{\mu\nu}$ transform homogeneously under Weyl transformations, it is clear that

$$\int d^4x\sqrt{-g}\gamma_{\mu\nu}T^{\mu\nu} = \int d^4x\gamma_{\mu\nu}T^{\mu\nu}[\eta], \tag{16}$$

thus only conformal anomaly can contribute to fourth order hydrodynamics and the full stress-energy tensor can be written as

$$T^{\mu\nu} = P\left(3u^\mu u^\nu + \Delta^{\mu\nu}\right) + \delta\left[\frac{c-a}{48\pi^2}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \frac{2a-c}{24\pi^2}R_{\alpha\beta}R^{\alpha\beta} + \frac{c-3a}{144\pi^2}R^2\right]\Delta^{\mu\nu}, \tag{17}$$

where δ is a small parameter that keeps tract of the gradient expansion. The Einstein's equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \delta\Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{18}$$

where Λ is also being treated as being small and potentially at the same order as the conformal anomaly terms.

The solution is

$$P(t) = \frac{1}{32\pi G_N t^2} + \delta\frac{a_0\left(9G_N a - 8\pi\Lambda t^4\right) - 36\left(C_1 + C_2\right)\pi t}{576 G_N \pi^2 a_0 t^4}, \tag{19}$$

$$a(t) = a_0\sqrt{t} - \delta\frac{a_0\left(9G_N a_0 - 16\pi\Lambda t^4\right) - 72\pi t\left(C_1(1+t) + C_2(1-t)\right)}{144\pi t^{3/2}}. \tag{20}$$

The simplest solution with $C_1 = C_2 = \Lambda = 0$ is then

$$P(t) = \frac{1}{32\pi G_N t^2} + \delta\frac{a}{64\pi^2 t^4}, \tag{21}$$

$$a(t) = a_0\left(\sqrt{t} - \delta\frac{aG_N}{16\pi t^{3/2}}\right). \tag{22}$$

Each δ-dependent term is suppressed by the Newton's constant, hence we can simply write

$$P(t) = \frac{1}{32\pi G_N t^2}\left(1 + G_N\frac{a}{2\pi t^2}\right), \tag{23}$$

$$a(t) = a_0\sqrt{t}\left(1 - G_N\frac{a}{16\pi t^2}\right). \tag{24}$$