# Oscillons

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## I. QUANTISATION OF GENERAL OSCILLONS

### A. General Theory

We can apply semi-classical quantization to oscillon $\phi(x)$ by dividing the field into rectangles of a small width, like the Riemann integral by splitting the curve into rectangles and summing them.Each rectangle acts as a spatially homogeneous field and the results obtained with the Mathieu equation can be used for local quantization. All of these regions then have to be summed and a meaningful result is hopefully obtained.

To obtain the stability angles we need to solve a partial differential equation of form

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-\sum_{n=0}^{N}a_n\phi^n(x,t)\right]\xi(x,t)=0, \tag{1}$$

where $a_0 = m$ is the mass and $\phi$ the oscillon solution to the equations of motion.

Let us define an open-set rectangular function $\Pi(x)$ which takes the value 1 in the open-set neighbourhood $(x-\delta, x+\delta)$ for some $0<\delta\ll 1$ and is 0 elsewhere. Then we can use this function to discretize an otherwise smooth (at least twice differentiable) oscillon solution by writing

$$\phi(x,t)=\sum_{y=-\infty}^{\infty}\bar{\phi}(y,t)\Pi(y-x),\ \text{where } y\in\{-\infty,...,-2\delta,0,2\delta,4\delta,...,\infty\} \tag{2}$$

and the spatially averaged value of the field is

$$\bar{\phi}(y,t)=\frac{1}{2\delta}\int_{y-\delta}^{y+\delta}\phi(z,t)dz \tag{3}$$

This means that we can write

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}-\sum_{n,y}a_n\bar{\phi}^n(y,t)\Pi(y-x)\right]\xi(x,t)=0 \tag{4}$$

The rectangular functions are othonormal in the sense that $\Pi(x)\Pi(y)=\Pi(x)=\Pi(y)=1$ if $x=y\in(-\delta,\delta)$ and 0 otherwise. Of course $x-y=2n\delta$, where $n\in\mathbb{Z}$. The function can also be written as a combination of Heaviside Thetas:

$$\Pi(x)=\Theta(x-\delta)-\Theta(x+\delta) \tag{5}$$

Therefore we can multiply the equality by $\Pi(y_0-x)$ to separate the differential equation

$$\begin{aligned}\left[\Pi(y_0-x)\frac{\partial^2}{\partial t^2}+\sum_{n,y}a_n\bar{\phi}^n(y,t)\Pi(y-x)\Pi(y_0-x)\right]\xi(x,t) &= \Pi(y_0-x)\frac{\partial^2}{\partial x^2}\xi(x,t)\\ \left[\frac{\partial^2}{\partial t^2}+\sum_{n}a_n\bar{\phi}^n(y_0,t)\right]\xi(y_0,t) &= \Pi(y_0-x)\frac{\partial^2}{\partial x^2}\xi(x,t)\end{aligned} \tag{6}$$

Now since we defined $\Pi(y_0-x)$ on an open interval $(y_0-x-\delta, y_0-x+\delta)$ we can see that the neighbourhood of the second derivative of $\xi(x,t)$ will equal the second derivative of $\xi$ in the neighbourhood. However if we had defined $\Pi$ with a closed interval, there would be infinite boundary terms involved coming from the fact that the derivative at the boundary of an interval is ill-defined since differentiation requires the existence of more than one point. Therefore

$$\left[\frac{\partial^2}{\partial t^2}+\sum_{n}a_n\bar{\phi}^n(y_0,t)\right]\xi_{y_0}(y,t)=\frac{\partial^2}{\partial y^2}\xi_{y_0}(y,t),\ \text{for } y\in(y_0-\delta,y_0+\delta) \tag{7}$$

The differential equation is therefore separable in the neighbourhood of any point $y_0$ on the $x$ axis. Introducing a separation constant $C^2_{y_n}$ for each of the differential equations (at different points $y$) and writing $\xi_{y_0}(y,t) = \chi_{y_0}(y)\psi_{y_0}(t)$ for the neighbourhood , we get

$$\begin{aligned} \left[\frac{d^2}{dt^2} + \sum_n a_n \bar{\phi}^n(y_0,t)\right]\psi_{y_0}(t) &= -C^2_{y_0}\psi_{y_0}(t) \\ \frac{d^2}{dy^2}\chi_{y_0}(y) &= -C^2_{y_0}\chi_{y_0}(y) \end{aligned} \tag{8}$$

Boundary conditions need to be set to get the values of constants $C^2_{y_q} \equiv C^2_q$ for $q \in \{-\infty, ..., -2\delta, 0, 2\delta, ..., \infty\}$. Introducing a new, more compact notation, $y_q$ means the $x$-axis neighbourhood variable about a point $x = q$. Subscripts $q$ tells us at which discrete point the equations are based. Therefore the set of differential equation for the whole oscillon quantization reads:

$$\begin{aligned} \left[\frac{d^2}{dt^2} + \sum_n a_n \bar{\phi}^n(q,t)\right]\psi_q(t) &= -C^2_q\psi_q(t) \\ \frac{d^2}{dy_q^2}\chi_q(y_q) &= -C^2_q\chi_q(y_q) \\ &\text{for } q \in \{-\infty, ..., -2\delta, 0, 2\delta, ..., \infty\} \end{aligned} \tag{9}$$

Solutions to spatial equations are

$$\chi_q(y_q) = A_q e^{iC_q y_q} + B_q e^{-iC_q y_q}, \quad \forall q \tag{10}$$

We know that $\xi(x, t+\tau) = e^{i\nu}\xi(x,t)$ and that $\psi_q(t+\tau) = e^{2\pi i\mu_q}\psi_q(t)$. Also $\xi(x,t) = \sum_q \chi_q(y_q)\psi_q(t)\Pi(q-x)$, so

$$\begin{aligned} \xi(x,t+\tau) &= \sum_q \chi_q(y_q)\psi_q(t+\tau)\Pi(q-x) \\ e^{i\nu}\sum_q \chi_q(y_q)\psi_q(t)\Pi(q-x) &= \sum_q e^{2\pi i\mu_q}\chi_q(y_q)\psi_q(t)\Pi(q-x) \end{aligned} \tag{11}$$

This implies that each region must have the same stability angles. This makes sense since all the regions are governed by the same Mathieu equation for the time evolution, with only a different amplitude of oscillation of the background potential. The Mathieu characteristic exponent is therefore $q$ independent