Abraham-Lorentz force and the Dirac-sea

Matthew Stephenson
*Stanford University**
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I. INTRODUCTION

We have two kinds of equations of motion in classical electrodynamics, there are dynamical laws for the charges and the equations for the electromagnetic (EM) field. They are coupled and therefore the latter have to be taken into account when the autonomous equations governing the charges are sought. This procedure is rather usual in contemporary practice, it is the construction of an effective theory. One identifies a subsystem and the elimination of the remaining dynamical degrees of freedom generates complicated dynamical laws for the subsystem. In the context of radiation back reaction the equations of motions for point charges have been identified first in the non-relativistic case [1, 2], followed by the convariant formalism [3, 4], see refs. [5–7] for iterative solutions.

The elimination of dynamical variables generates non-local integro-differential equations for extended charge systems which are difficult to handle. Therefore one usually carries out the point like limit for the charge distribution in the hope of suppressing these non-local features. But this step which intended to simplify matters brings its own complexities which can easily be understood retroactively by realizing its similarity with quantum anomalies. The common element of these two phenomena is the Fourier integral which appears on the one hand, in the perturbative treatment of quantum systems and in the other hand, in the Fourier representation of the Green functions used in the elimination of dynamical variables in classical field theory. One finds quantum anomalies when this integral is not uniformly convergent, when the limit of removing the ultraviolet cut-off can not be carried out on the integrand, before the integration. The result is an unexpected sensitivity of the physics of finite scale on short distance phenomena.

It is advantageous to separate such anomalous forces, arising in classical electrodynamics from the singular nature of the Coulomb potential at small distances from the rest which represent the true radiation backreaction. The latter is finite for a point charge and is related to the far field produced by the accelerating charge [3]. The former is related to the near field and leads to the Schott-term with the third time derivative of the coordinates when an expansion is made in the retardation effects [6]. Another appearence of the non-radiational back reaction generated by the short distance singularity of the Coulomb interaction is the survival of the effects of smearing out a charge distribution in the point like limit. In fact, the problematic run-away solution of the equations of motion generated by the Schott-term disappear when proper smearing is applied on the charge distribution [8].

Another kind of complications arise in deriving autonomous equations of motion from the separation of the degrees of freedom to eliminate or to retain in the effective theory. The dependence of the equations of motion on this separation is handled today by the renormalization group method where the issue of respecting the symmetries of the whole system by the separation is clearly seen. In the context of classical electrodynamics special care is needed in defining the conserved energy-momentum of the charges, appearing in the ballance equation in the usual derivation the back reaction force. The point is the sensitivity on the choice of a reference frame to define the energy-momentum vector from the energy-momentum tensor in order to arrive at relativistically covariant equations of motion and to avoid the 4/3 problem [9, 10].

The goal of this paper is to asses another aspect of the choice of the degrees of freedom to eliminate in arriving at the effective theory. Our concern is a quantum effect, the dynamics of charges in the Dirac-sea. On the one hand, being a genuine quantum effect, it is made small by the Planck constant. But on the other hand, the sensitivity of the back reaction problem on the dyanmics at short distance scales raises the question whether such a quantum effect which becomes important at microscopic distances might let itself be seen at finite scales.

To find the effect of vacuum polarization on the Abraham-Lorentz force we need the Lienard-Wiechert potential, the retarded Green function in particular for charges moving in the presence of the Dirac-see. This amounts in the leading order of the loop-expansion to the Schwinger-Dyson resummation of the one-loop self energy in the photon propagator. Since there is no Wick theorem for retarded Green functions we need Schwinger's Close Time Path (CTP) scheme [11] where the reduplication of the degrees of freedom achieved by the double time axis formalism provides us the usual ingredients of the perturbation expansion for retarded Green functions [12].

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*Electronic address: `matthewjstephenson@icloud.com`

Another technical issue requiring special care is the causal structure of the retarded Green function. The one-loop self energy in the vacuum is a singular function of the four momentum due to the pair creation processes. When the characteristic scales of the radiation field is in the classical domain then one can usually ignore such processes. In order to eliminate the unnecessary complications of the threshold singularities we approximate the self energy by its gradient expansion form, by the leading order term in its expansion around vanishing four momentum. But such a truncation endangers the expected causal structure of the retarded Green function and we have to turn to the spectral representation of the CTP propagators for a proper treatment of this problem. But the spectral representation of propagators is highly singular for massless particles and a way to arrive at a regular expression is found by establishing a simple relation between the self energy and the spectral weight, realized by a Borel transformation.

invariant length scale

always there

results

structure

II. RETARDED GREEN FUNCTION

We seek in this work an improvement of the Lienard-Wiechert potential by taking into account the unavoidable vacuum polarisation effects, obtained by the Schwinger-Dyson partial resummation of the perturbation series. The obvious problem of this plan is the lack of Wick theorem for retarded Green functions. To recover the usual scheme of perturbation expansion we use the CTP formalism [11, 12]. One generalizes in this scheme the usual perturbation expansion, developed for transition amplitude for expectation values written in the Heisenberg representation. The generating functional $W[j^+, j^-]$ for the connected Green function of EM field which serves as the starting point for the perturbation expansion is defined by [13]

$$e^{\frac{i}{\hbar}W[j^+,j^-]} = \mathrm{Tr}T[e^{-\frac{i}{\hbar}\int dx[H(x)-j^{+\mu}(x)A_\mu(x)]}]|0\rangle\langle 0|\bar{T}[e^{\frac{i}{\hbar}\int dx+j^{-\mu}(x)A_\mu(x)]}]] \tag{1}$$

where T and $\bar{T}$ denote the time and and the anti-time ordered product and $H(x)$ is the Hamiltonian density of QED. For the special case of $j^- = -j^+ = j^{phys}$ this expression is the trace of the density matrix at the final time, the upper limit of the time integration in the exponent, in the presence of an external source j^{phys} coupled to the EM field. The external sources play double role, they are supposed to drive the system adiabatically to the desired initial state generate in Eq. (1) adn generate the necessary operator insertions in perturbation expansion when the free generating functional is considered, obtained by replacing $H(x)$ by the free Hamiltonian density. One redoublicates the degrees of freedom by the replacement $A \to A^+, A^-$ etc. for each field variable and writes the generating functional as

$$e^{\frac{i}{\hbar}W[j^+,j^-]} = \mathrm{Tr}\tilde{T}[e^{\frac{i}{\hbar}\int dx+j^{-\mu}(x)A^-_\mu(x)]}e^{-\frac{i}{\hbar}\int dx[H(x)-j^{+\mu}(x)A^+_\mu(x)]}]|0\rangle\langle 0|] \tag{2}$$

where $\tilde{T}$ is a generalized time ordering which places the + operators right of the - ones and puts the + or - operators in time or anti-time order, respectively. The result is an extension of the usual formalism of quantum field theory which obeys Wick theorem.

The propagators have a CTP block structure in this formalism

$$i\hbar\begin{pmatrix} D & D^{+-} \\ D^{-+} & D^{--}\end{pmatrix}_{\mu,\nu}(x,y) = \begin{pmatrix}\langle T[A_\mu(x)A_\nu(y)]\rangle & \langle A_\nu(y)A_\mu(x)\rangle \\ \langle A_\mu(x)A_\nu(y)\rangle & \langle T[A_\nu(y)A_\mu(x)]\rangle^*\end{pmatrix} \tag{3}$$

for photons in particular. The identity

$$T[\phi_a\phi_b] + \bar{T}[\phi_a\phi_b] = \phi_a\phi_b + \phi_b\phi_a \tag{4}$$

allows us to parametrise the photon propagator by three real tensors,

$$\begin{pmatrix} D & D^{+-} \\ D^{-+} & D^{--}\end{pmatrix} = \begin{pmatrix} D^n + i\Im D & -D^f + i\Im D \\ D^f + i\Im D & -D^n + i\Im D\end{pmatrix}. \tag{5}$$

The ++ block, D is clearly the causal propagator and the expressions

$$\begin{aligned} i\hbar D^n_{\mu\nu}(x,y) &= \frac{1}{2}\epsilon(x^0 - x'^0)\langle 0|[A_\mu(x), A_\nu(x')]|0\rangle, \\ i\hbar D^f_{\mu\nu}(x,y) &= \frac{1}{2}\langle 0|[A_\mu(x), A_\nu(x')]|0\rangle, \\ \hbar D^i_{\mu\nu}(x,y) &= -\frac{1}{2}\langle 0|\{A_\mu(x), A_\nu(x')\}|0\rangle, \end{aligned} \tag{6}$$

identify D^n and D^f with the near and far field Green function, respectively because the combinations

$$i\hbar D^{\overset{r}{a}}_{\mu\nu}(x,y) = i\hbar[D^n_{\mu\nu}(x,y) \pm D^f_{\mu\nu}(x,y)] = \pm\Theta(\pm(x^0 - x'^0))\langle 0|[A_\mu(x), A_\nu(x')]|0\rangle \tag{7}$$

correspond to the retarded or advanced Green functions. Notice that the identity (4) holds for any time dependent operator, therefore the block structure (5) remains valid for interactive fields, too.

The one-loop photon self energy improved proprgator can easiest be obtained in the framework path integration, where the generating functional (1) is written as

$$e^{\frac{i}{\hbar}W[\hat{j}]} = \int D[\hat{\psi}]D[\hat{\bar{\psi}}]D[\hat{A}]e^{\frac{i}{\hbar}\hat{\bar{\psi}}[\hat{G}_0 - e\sigma\hat{A\!\!\!/}]\hat{\psi} + \frac{i}{2\hbar}\hat{A}\hat{D}_0^{-1}\hat{A} + \frac{i}{\hbar}\hat{j}\hat{A} + \frac{i}{\hbar}S_{CT}} \tag{8}$$

in terms of the two component CTP doublets

$$\hat{\psi} = \begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix}, \qquad \hat{A} = \begin{pmatrix}A^+\\ A^-\end{pmatrix}, \qquad \hat{j} = \begin{pmatrix}j^+\\ j^-\end{pmatrix}, \tag{9}$$

and the matrix

$$\sigma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{10}$$

acting on the CTP indices $\pm$. The CTP propagators are

$$\begin{aligned}\hat{G}_0^{-1} &= \begin{pmatrix} i\partial\!\!\!/ - m_\tau + i\epsilon & 0\\ 0 & -\gamma^0(i\partial\!\!\!/ - m_\tau + i\epsilon)^\dagger\gamma^0\end{pmatrix} + \hat{G}^{-1}_{BC},\\ \hat{D}_0^{-1} &= \begin{pmatrix}\Box T + \xi\Box L + i\epsilon & 0\\ 0 & -\Box T - \xi\Box L + i\epsilon\end{pmatrix} + \hat{D}^{-1}_{BC},\end{aligned} \tag{11}$$

where the transverse and longitudinal projection operators

$$T^{ab} = g^{ab} - L^{ab}, \quad L^{ab} = \frac{\partial^a\partial^b}{\Box} \tag{12}$$

are introduced together with the covariant gauge fixing parameter ξ. The scalar product of space-time functions includes the the space-time integration, $fg = \int d^4x f_x g_x = \int_x f_x g_x$, vector and CTP indices being summed, too. The boundary conditions in time, the closing of the + and - time countours are handled by the parts $\hat{G}^{-1}_{BC}$ and $\hat{D}^{-1}_{BC}$ in the propagators, their explicite form is not needed in the sequal. The counterterms in S_{CT} are the usual ones and will be omitted below.

The integration over the charge fields in the path integral (8) yields

$$e^{\frac{i}{\hbar}W[\hat{j}]} = \int D[\hat{A}]e^{\mathrm{Tr}\ln[\hat{G}_0 - e\sigma\hat{A\!\!\!/}] + \frac{i}{2\hbar}\hat{A}\hat{D}_0^{-1}\hat{A} + \frac{i}{\hbar}\hat{j}\hat{A}} \tag{13}$$

and which can be written by the help of the expansion

$$\ln[G^{-1} - A] = \log G^{-1} - \sum_{n=1}^\infty \frac{1}{n}(GA)^n. \tag{14}$$

as

$$e^{\frac{i}{\hbar}W[\hat{j}]} = \int D[\hat{A}]e^{\frac{i}{2\hbar}\hat{A}D^{-1}\hat{A} + \frac{i}{\hbar}\hat{A}j + \mathcal{O}(\hat{A}^3)}. \tag{15}$$

A simple Gaussian integral gives

$$W[\hat{j}] = -\frac{1}{2}\hat{j}D\hat{j} + \mathcal{O}(j^3), \tag{16}$$

where the improved propagator

$$\hat{D} = \frac{1}{\hat{D}_0^{-1} - \hat{\Pi}} \tag{17}$$

is introduced in terms of the self energy

$$\hat{\Pi}^{\sigma\tau}_{\mu\nu}(x,y) = -ie^2\sigma\tau\hbar\mathrm{tr}[G_0^{\tau\sigma}(y,x)\gamma^\mu G_0^{\sigma\tau}(x,y)\gamma^\nu], \tag{18}$$

the CTP indices σ,τ assume the values $\pm$ and the trace is over the Dirac indices.

The Fourier transform

$$\Pi(q) = \int dx e^{iq(x-y)}\Pi(x,y) \tag{19}$$

of the block $++$ of the polarisation tensor is well known, it reads

$$\Pi^{++\mu\nu}(q) = \frac{e^2}{12\pi^2}q^2\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{q^2}\right)\left[\sqrt{\frac{4m^2}{q^2}-1}\ \mathrm{arccot}\sqrt{\frac{4m^2}{q^2}-1} - 1\right]\right\} \tag{20}$$

in mass shell subtraction scheme in units of $\hbar = c = 1$. Its expansion around $k = 0$, well below the pair creation threshold yields

$$\Pi^{++\mu\nu}(q) = -\ell^2(q^2)^2 \tag{21}$$

with $\ell^2 = \alpha\lambda_C^2/15\pi$, λ_C being the electron Compton wavelength. The calculation of the +- block is straghtforward, too [14],

$$\Pi^{+-}_{\mu\nu}(q) = 32\pi^2 ie^2m^2\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left(1+\frac{q^2}{2m^2}\right)\int_p \delta(q^2+2pq)\delta(p^2-m^2)\Theta(-p^0-q^0)\Theta(p^0). \tag{22}$$

Due to the ifrst Heaviside function in the right hand side Π^{+-} is vanishing in the vincinity of $q = 0$.

Once the self energy is found we can turn to the calculation of the inverse in Eq. (17). This is not trivial because the inverse of the free photon propagator in the second equation in (11) contains the boundary condition term which is nonvanishing for the final time only and thereby breaks the invariance of the propagator with respect to the translation in time. The careful performance of the limit when the final time is sent to infinite produces a Fourier integral for the free propagator with the Fourier transform

$$\hat{D}_{0\mu\nu}(k;\mu^2) = g_{\mu\nu}\begin{pmatrix} \frac{1}{k^2-\mu^2+i\epsilon} & -2\pi i\delta(k^2-\mu^2)\Theta(-k^0) \\ -2\pi i\delta(k^2-\mu^2)\Theta(k^0) & -\frac{1}{k^2-\mu^2-i\epsilon}\end{pmatrix} \tag{23}$$

with $\mu^2 = 0$ and continuous frequency spectrum in Feynman gauge $\xi = 1$. It contains Dirac delta in the off-diagonal CTP blocks because we find only on-shell amplitudes here according to Eq. (3). The appearance of a distribution whose inverse is ill defined indicates the same problem with the inverse propagator D_0^{-1}. The simplest way to overcome this difficulty is to use regulated distribution. When the Lorentz-shape regularization

$$\delta_\epsilon(x) = \frac{\pi}{\epsilon}\frac{1}{x^2+\epsilon^2} \tag{24}$$

is used with the limit $\epsilon \to 0$ then a simple inversion gives

$$\hat{D}^{-1}_{0\mu\nu}(k;\mu^2) = g_{\mu\nu}\left[(k^2-\mu^2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\epsilon\begin{pmatrix}1 & -2\Theta(-k^0)\\ -2\Theta(k^0) & 1\end{pmatrix}\right]. \tag{25}$$

We are now ready for the inversion of Eq. (17). But instead of carrying out the calculation explicitely we point out a general feature of the CTP self energy insertions. The one particle irreducible two point function for photons, $\hat{\hat{G}}$, displays the CTP same structure (5) as the propagators. But its insertion in the inverse propagator is due to the elementary verices which represent the contact between the free propagator and the self energy insertion which comes with an extra minus sign for the - CTP variables. Therefore the self energy in the Schwinger-Dyson resummation (17) is actually $\hat{\Pi} = \sigma\hat{\hat{G}}\sigma$. Let us consider a set of CTP matrices A_j of the structure (5). It is now a matter of simple algebra to show that the product $A_1\sigma A_2\sigma\cdots A_n$ preserves the same CTP structure and $(A_1\sigma A_2\sigma\cdots A_n)^r = A_1^r A_2^r\cdots A_n^r$. When this relation is applied to the Schwinger-Dyson resummation the relation

$$D^{\overset{r}{a}} = \left(\frac{1}{\hat{D}_0^{-1} - \hat{\Pi}}\right)^{\overset{r}{a}} = \frac{1}{D_0^{\overset{r}{a}-1} - \Pi^{\overset{r}{a}}} \tag{26}$$

follows where $\Pi^{\overset{r}{a}} = \Pi^{++} - \Pi^{\pm\mp}$. The $i\epsilon$ prescription of the free inverse propagator can usually be ignored because the self energy piece, the second term in the denominator possesses finite imaginary part in momentum space.

$$D(k) = \frac{1}{k^2 - \Pi(k^2)} = \frac{Z}{\prod_n (k^2 - m_n^2)} = \sum_n \frac{z_n}{k^2 - m_n^2} \tag{27}$$

$$z_n^{-1} = Z \frac{\partial D^{-1}(k^2)}{\partial k^2}_{|k^2 = m_n^2} \tag{28}$$

UV. finiteness:

$$\sum_n z_n = 0 \tag{29}$$

III. SPECTRAL REPRESENTATION

The gradient expanded form of the one-loop photon self energy (20)-(22) yields

$$\Pi^r_{\mu\nu}(k) = -T_{\mu\nu}\ell^2(k^2)^2. \tag{30}$$

Though the exact photon self energy generates the correct causal structure for the resummed propagator (26) the use of its one-loop approximation may lead to wrong analytic behaviour on the complex energy plan. A further approximation we made is the gradent expansion which may further complicate the situation. In fact, the application of the rule (26) for the gradient expansion result gives

$$D^r_{\mu\nu} = T^{\mu\nu} D^r + \frac{1}{\xi}\frac{L^{\mu\nu}}{\Box - i\epsilon}. \tag{31}$$

with the transverse part

$$D^r(k) = \frac{1}{k^2 + \ell^2 (k^2)^2} \tag{32}$$

where the replacement $k^0 \to k^0 + i\epsilon$ has to be carried out in the fist k^2 term of the denominator. The new poles appearing on the energy plane due to the self energy term come in complex conjugate pairs and generate singulaities on the physical sheet.

To provide corrections to the resummation, needed due to our truncation of the self energy we use the spectral representation of the resummed propagators. The spectral function

$$2\pi\rho(p^2) = \int dx e^{-ipx} i D_T^{-+}(x) = \sum_n \delta^{(4)}(p - p_n)\langle 0|A_\mu(0)|n\rangle T^{\mu\nu}\langle n|A_\nu(0)|0\rangle, \tag{33}$$

is supposed to be a tempered distribution and is introduced for the transverse part of the photon propagator. The transverse part of the full propagator can then be written as

$$\hat{D}_T(x) = \int_0^\infty d\mu^2 \rho(\mu^2) \hat{D}_0(x, \mu^2) \tag{34}$$

where the free massive scalar propagator $\hat{D}_0(x, \mu^2)$ is the matrix multiplying $g_{\mu\nu}$ on the right hand side of Eq. (25). The important consequence of this relation is that every propagator, i.e. causal, retarded and advanced canbe constructed with the same spectral function.

Eq. (34) shows the essence of the spectral representation, the exact propagator is a weighted sum of massive propagators, the interaction creates competing mass shells. This simplicity is at the same time a weakness, at least when some mechanism, like symmetry prevents mass generation in the theory. What remains from the spectral representation in that case? It is clear that the spectral integral becomes formal for massless theories but the richness of a single mass shell can be exploited by the formal Taylor series of the Dirac delta, localised on the mass shell. We therefore assume that the spectral weight can be written as

$$\rho(\mu^2) = R(\partial_{\mu^2})\delta(\mu^2), \tag{35}$$

where $R(\partial_{\mu^2})$ is defined as a power series in ∂_{μ^2} and $\partial^n_{\mu^2}$ interpreted as the n-th derivative acting on the Dirac delta. After repeated integration by parts, allowed for distributions in Eq. (34) one arrives at

$$D_T(x) = R(-\partial_{\mu^2})D_0(x, \mu^2)_{|\mu^2=0}. \tag{36}$$

To find the structure of the newly introduced function $R(z)$ we consider the Fourier transform of Eq. (34) and insert the representation (35),

$$\frac{1}{k^2 - \Pi(k^2)} = \int_0^\infty d\mu^2 R(\partial_{\mu^2})\delta(\mu^2)\frac{1}{k^2 - \mu^2}. \tag{37}$$

We use this equation for the causal propagator for simplicity and write the free retarded propagator by menas of the Schwinger representation,

$$D^{00}_T(k^2) = -i\int_0^\infty d\mu^2 \int_0^\infty ds R(\partial_{\mu^2})\delta(\mu^2)e^{is(k^2-\mu^2+i\epsilon)}, \tag{38}$$

and integrate by parts to obtain the expression

$$D^{00}_T(k^2) = -i\int_0^\infty ds R(is)e^{s(ik^2-\epsilon)}, \tag{39}$$

where the μ^2 integral had been carried out in a trivial manner. This equation shows a useful result, namely that the exact propagator is the complex Laplace transform of the function $R(iz)$. We can see that the n-th derivative $\partial^n_{\mu^2}$ acting on the Schwinger representation integral brings down $(is)^n$ before setting μ^2 to 0. This as a consequence gives sense to our treatment of $R(\partial_{\mu^2})$ as a power series

$$R(s) = \sum_{n=0}^\infty \frac{c_n}{n!}s^n. \tag{40}$$

In fact, Eq. (39) requires that $R(z)$, considered as a complex variable function, increases less fast than any exponential function for $|z| \to \infty$. The representation (40) yields immediately

$$D^{00}_T(k^2) = -\int_0^\infty ds \sum_{n=0}^\infty \frac{i^{n+1}c_n}{n!}s^n e^{s(ik^2-\epsilon)} = -\sum_{n=0}^\infty \frac{(-1)^n c_n}{(k^2+i\epsilon)^{n+1}}. \tag{41}$$

In the particular case of the self-energy

$$\Pi(k^2) = -\ell^2(k^2)^2 \tag{42}$$

one may follow a direct route to justify the power series representation (40) and to find the coefficients c_n. For this end we replace to the left hand side of Eq. (41) the expressions (37) and (42) and analytically continue to the entire complex plane in $k^2 + i\epsilon \to z \in \mathbb{C}$,

$$\frac{1}{z[1+\ell^2(z-2i\epsilon)]} = -\sum_{n=0}^\infty \frac{(-1)^n c_n}{z^{n+1}}. \tag{43}$$

Next we multiply both sides by z^n and integrate along a closed contour encircling the poles to get

$$c_n = -(-1)^n \sum_j \mathrm{Res}\left[\frac{z^n}{z[1+\ell^2(z-2i\epsilon)]}; z_j\right], \tag{44}$$

where z_j are the locations of the poles $z = 0$ and $z = -\frac{1}{\ell^2}$, ϵ being ignored. The residue at $z = 0$ is only non-zero when $n = 0$ where it is 1. The residues at the other pole are $-\ell^{-2n}$. We therefore have

$$c_n = \begin{cases} 0 & n = 0, \\ -\ell^{-2n} & n > 0, \end{cases} \tag{45}$$

and the form

$$R(z) = 1 - e^{\frac{z}{\ell^2}}. \tag{46}$$

Once the power series form (40) is confirmed then Eq. (36) is justified and leads to

$$D_T(x) = \left(1 - e^{-\ell^{-2}\partial_{\mu^2}}\right) D_0(x, \mu^2)_{|\mu^2=0} \tag{47}$$

when inserted into Eq. (34).

IV. STATIC CHARGE

Let us look at the field of a static charge as a simple excercise. First we shall calculate the one-loop corrected Coulomb potential using the standard method, and then make use of the derived operator equation for any form of vacuum polarization tensor, to show its validity. Due to relativistic covariance one may apply a Lorents boost on the resulting potential to obtain the field of a charge moving with constant velocity.

We perform first the calculation of the EM field

$$A_\mu(x) = -\int dy D_{\mu\nu}(x-y) J^\nu(y) \tag{48}$$

for the current

$$J^\mu(x) = e\delta^{\mu 0} \int dt \delta(x - (t, \mathbf{0})) \tag{49}$$

of a static charge by using the one-loop improved near field propagator, $D^n(x) = \Re D^{++}(x)$,

$$A_0(x) = -e \int dt \int \frac{d^4k}{(2\pi)^4} P \frac{e^{-ik(x-(t,\mathbf{0}))}}{k^2 + \ell^2 (k^2)^2}, \tag{50}$$

P being the principal value integral. A straightforward evalutation of the integral yields

$$A_0(x) = -\frac{e}{4\pi r}\left[\cos\left(\frac{r}{\ell}\right) - 1\right]. \tag{51}$$

Another way to obtain the static potential starts is to rely on Eq. (47) where the unperturbed field of the static charge

$$A_\mu^{(0)}(x) = \frac{e}{4\pi r} e^{-\mu r} \tag{52}$$

is induced by massive photons. According to Eq. (47) we have

$$A_\mu^0(x) = \frac{1}{2}\left(1 - e^{-\ell^{-2}\partial_{\mu^2}}\right) A_\mu^{(0)}(x)_{|\mu^2=0}. \tag{53}$$

It is easy to check that the operator ∂_{μ^2} does not mix the even and the odd part of a function of μ hence the identity

$$\begin{aligned} R(-\partial_{\mu^2}) e^{-\mu r}{}_{|\mu^2=0} &= R(-\partial_{\mu^2})[\cosh(\mu r) - \sinh(\mu r)]_{|\mu^2=0} \\ &= R(-\partial_{\mu^2}) \cosh(\mu r)_{|\mu^2=0} \end{aligned} \tag{54}$$

follows, giving

$$A_0(x) = \frac{e}{4\pi r} - \frac{e}{4\pi r} \sum_{n=0}^{\infty} \frac{(-1)^n \ell^{-2n}}{n!} \partial_{\mu^2}^n \cosh(\mu r)_{|\mu^2=0}. \tag{55}$$

The even pieces in μ can easily be resummed after taking the limit $\mu^2 \to 0$. They reproduce the Eq. (51).

FIG. 1: The electrostatic potential with vacuum polarization. The solid line, the envelope is the Coulomb potential in the trivial, empty vacuum.

The dressed static potential Eq. (51) yields displays oscillations at the Compton length scale with a Coulomb-potential envelop. In fact, the envelope can be obtained by smearing and we find in the small ℓ limit

$$\begin{aligned} A_0(r) &= \lim_{\ell,\epsilon\to 0} \frac{\int\limits_{r-\epsilon}^{r+\epsilon} \left[-\frac{e}{4\pi y}(\cos(\frac{y}{\ell})-1)\right] dy}{(r+\epsilon)-(r-\epsilon)} \\ &= \lim_{\epsilon\to 0} \frac{e}{8\pi\epsilon}[\log(r+\epsilon)-\log(r-\epsilon)] \\ &= \lim_{\epsilon\to 0} \frac{e}{4\pi}\left[\frac{1}{r}+\frac{\epsilon^2}{3r^3}+\mathcal{O}(\epsilon^4)\right] = \frac{e}{4\pi r}. \end{aligned} \tag{56}$$

Furthermore, the potential is regular at short distances and tends to zero with r. Naturally, these expressions are only reliable for $r \gg \ell$.

V. RADIATION BACKREACTION

The calculation of the static force law in the previous section and the determination of the back reaction force below are actually carried out in a classical effective theory [15, 16] for the EM field, motivated by the gradient expanded form of $\Pi^{++\mu\nu}$, Eq. (21), and defined by the Lagrangian

$$L = -\frac{1}{16\pi} F_{\mu\nu}(1-\ell^2\Box)F^{\mu\nu}. \tag{57}$$

The back reaction force after having taken into account the vacuum polarization should differ from that of the trivial vacuum because the polarization cloud of the test particle introduces a non-trivial charge distribution and modifies the self force. The traditional self force calulation is based on distributing the test charge homogeneously in a sphere of radius r_0. Such a charge distribution with discontinuous space-dependence generates unphysical high frequency modes, such as runaway solutions and pre-acceleration in the limit $r_0 \to$ [3] but such anomalies can be avoided by keeping r_0 finite, non-vanishing [8]. The higher order derivative, appearing in the effective action provides a smooth high frequency cutoff which explains the observation that the run-away solutions are absent in this theory [17].

We can now turn to our main goal, the determination of the backreaction force f^μ acting on a point charge in the effective theory (57). There are two ways to reach that goal in the traditional case when the vacuum polarizations are ignored, $\ell = 0$. One is based on the application of the Lorentz force on the charge [1, 2],

$$f_0^\mu(x) = \lim_{r_0\to 0} \int dydz K^{\mu\nu}(x,y) D^r_{\nu\rho}(y,z) j^\rho(z) \tag{58}$$

where $j^\rho(z)$ is the electric current of the charges particle, the Green function generates the Liénard-Wiechert potential, and convolution factor $K^{\mu\nu}(x,y)$ makes the Lorentz force from the vector potential and includes some form factor for the charge. This latter which contains the size of the charge $r_0 \neq 0$ as a scale factor is necessary because the force is singular for vanishing distance. The point charge limit, $r_0 \to 0$ restricts the integration over y around x. The graphical representation of the force as a loop integral is shown in Fig. 2 (a).

The limit $r_0 \to 0$ renders this construction questionable hence another, equivalent way has been sought without the applicaltion of the Lorentz force. This procedure is based on the energy-momentum conservation, imposed on a tube of radius r_0 around the world line of the charge [3]. The resulting rate of loss of the energy-momentum of the EM field can be written as

$$f^\mu(x)_0 = \lim_{r_0\to 0} \int dydzdudv L^{\mu\nu\rho}(x,y,z) D^r_{\nu\kappa}(y,u) j^\kappa(u) D^r_{\rho\sigma}(z,v) j^\sigma(v) \tag{59}$$

for a point charge, where the factor $L^{\mu\nu\rho}(x,y,z)$ converts the two Liénard-Wiechert potentials, given by the two propagator into the energy momentum tensor and produces its source at x. The two one-loop integrals are shown graphically in Fig. 2 (b).

(a) (b)

FIG. 2: Two ways of obtaining the back reaction force, (a): by means of the Lorentz force, and (b) by applying the energy-momentum conservation. The solid and dashed lines denote the world line of the charge and the retarded Green function, the symbol X stands for K or L for (a) or (b), respectively.

Notice that the loop-integrals in Eqs. (58)-(59) are ill defined for $r_0 = 0$ and the limit $r_0 \to 0$ has to be carried out after integration. In other words, the integral is not uniform convergent and the similarity with quantum anomalies, mentioned in the Introduction is shown clearly.

According to Eq. (47) the Green function with self-energy improvement can be obtained in a linear manner from free Green function. Therefore we use Eq. (58) which is linear in the Green function and write the back reaction force as

$$f = \left(1 - e^{-\ell^{-2}\partial_{\mu^2}}\right) f_0(\mu)_{|\mu=0} \tag{60}$$

wher $f_0(\mu)$ denotes the back reaction force in electrodynamics based on photons of mass μ. The validity of this equation requires that $K^{\mu\nu}(x,y)$ be independent of the photon mass, a condition satisfied trivially because the explicit structure of K is fixed by the charge distribution and the choice of the minimal charge-EM field coupling. Note that there is no place for vertex corrections which could in principle introduce photon mass-dependent form-factors in an effective theory where the conserved density-current vector of the charges is retained only.

The relation (60) establishes the reaction force in terms of that of a massive electrodynamics, the Proca model defined by the action

$$S = -m_0 \int d\tau \sqrt{\dot{z}^2(\tau)} - \frac{1}{8\pi} \int dx \left[\frac{1}{2} F^2(x) - \mu^2 A^2(x) + (\partial^\mu A_\mu(x))^2\right] - e \int d\tau \dot{z}^\mu [A_\mu(z(\tau)) + A_{ext\mu}(z(\tau))]. \tag{61}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The external field $A_{ext}(x)$ is introduced to induce acceleration which produces radiation and renders the reaction force problem well defined. When the vector potential is eliminated by its equations of motion then the generated action-at-a-distance effective theory is

$$S_{eff} = -m_0 \int d\tau \sqrt{\dot{z}^2} - \frac{4\pi e^2}{2} \int d\tau d\tau' \dot{z}^\mu(\tau) D_0^n(z(\tau) - z(\tau'), \mu) \dot{z}_\mu(\tau') - e \int d\tau \dot{z}^\mu A_{ext\mu}(z(\tau)) \tag{62}$$

where $D^n = (D^r + (D^r)^{tr})/2$ is the near-field Green function and the naive equation of motion is

$$\begin{aligned} \frac{d}{d\tau}(m_0 \dot{z}^\mu(\tau)) &= -4\pi e^2 \dot{z}^\nu(z(\tau)) \int d\tau' \partial_\nu D_0^n(z(\tau) - z(\tau'), \mu) \dot{z}_\mu(\tau') + 4\pi e^2 \int d\tau' \dot{z}^\nu(\tau) \partial_\mu D_0^n(z(\tau) - z(\tau'), \mu) \dot{z}_\nu(\tau') \\ &\quad - e\dot{z}^\nu \partial_\nu A^\mu(z(\tau)) + e\dot{z}^\nu \partial_\mu A^\nu(z(\tau)) \end{aligned} \tag{63}$$

When the solution of an initial condition problem is considered then one has to add certain homogeneous solutions to the in the equation of motion which can be taken into account by replacing the near-field Green function by the retarded one in the equation of motion [14]. The result is

$$\frac{d}{d\tau}(m_0 \dot{z}^\mu(\tau)) = -e\dot{z}^\nu (F^r_{\nu\mu} + F^{ext}_{\nu\mu}) \tag{64}$$

with

$$A^r_\nu(x) = 4\pi e \int d\tau' D_0^r(z(\tau) - z(\tau'), \mu) \dot{z}_\nu(\tau'). \tag{65}$$

The massive retarded Green function can be written as the sum of the massless one plus the rest,

$$D_0^r(x, \mu) = \frac{\Theta(x^0)}{4\pi} \left[2\delta(x^2) - \Theta(x^2) \frac{\mu}{\sqrt{x^2}} J_1(\mu\sqrt{x^2})\right], \tag{66}$$

the support of the latter being inside of the future light-cone. Hence the self force, the term continaing the retarded propagator is of the structure of a sum of the self force due to the emission of massless particles and the rest, called tail-term. The former is local in space-time because the massless Green function is restricted on the light-cone and the latter contains the past history of the charge. The equation of motion (63) is naive because the massless contribution to the self force of point charge is divergence and needs some regularization procedure. Point splitting, applied in along the world line around the charge has been proposed [18] which reproduces the self force for massless vector particle emission, arising from the energy-momentum conservation [3]. It can be incorporated as a particular choice of the form factor $K^{\mu\nu}(x,y)$ in Eq. (58). Note that the tail contribution is finite on the point charge world line and needs no smearing. Thus we can assume that the local, massless contribution to the right hand side of Eq. (64) is reproducing the Abrham-Lorentz-Dirac force and the regular tail part is not influenced by the smearing over infinitesimal distances.

What follows is the calculation of the tail contribution

$$A^r_\nu(x)_{|tail} = -e\int_{-\infty}^{\tau^r} d\tau \frac{\mu J_1(\mu\sqrt{q^2(\tau)})}{\sqrt{q^2(\tau)}}\dot{z}_\nu(\tau) \tag{67}$$

where τ^r is the latest proper time light can propagate to x from the world line to the equation of motion on the charge world line at $x = z(\tau)$.

To calculate the space-time derivatives of $A^r_\nu(x)_{|tail}$ we need the derivative of the upper limit of integration, $\partial_\mu\tau^r$. For this end we write the vector $q^r = x - z(\tau^r)$ in the form $q^r = R[\dot{z}(\tau^r) + w]$, where $\dot{z}(\tau^r)$ and w are orthogonal, $\dot{z}\cdot w = 0$. Due to $q^{r2} = 0$ w is a spatial unit vector, $w^2 = -1$ and we have the expression $R = q^r\dot{z}(\tau^r)$ scalar factor. We make infinitesimal variation $x \to x + \delta x$ of the equation $q^{r2} = 0$, yielding

$$\partial_\mu\tau^r = \frac{q^{r\mu}}{q^r\dot{z}(\tau^r)}. \tag{68}$$

This expression is used for the boundary contribution of the integral,

$$\partial_\mu A^r_\nu(z(\tau))_{|tail} = -e\frac{q^r_\mu\dot{z}_\nu(\tau^r)}{q^r\cdot\dot{z}(\tau^r)}\frac{\mu J_1(\mu\sqrt{q^2(\tau^r)})}{\sqrt{q^2(\tau^r)}} - e\int_{-\infty}^{\tau^r} d\tau'\partial_\mu\sqrt{q^2(\tau')}\frac{\partial}{\partial\sqrt{q^2}}\left[\frac{\mu J_1(\mu\sqrt{q^2})}{\sqrt{q^2}}\dot{z}_\nu(\tau)\right]_{|q^2=q^2(\tau')} \tag{69}$$

where $q(\tau') = z(\tau) - z(\tau')$. We perform the change of integral variable $\tau' \to \sqrt{q^2}$ by means of the identity resulting from the variation $\tau' \to \tau' + \delta\tau$ of the proper time

$$\delta\sqrt{q^2} = \frac{\delta q\cdot q}{\sqrt{q^2}} = -\delta\tau\frac{\dot{z}\cdot q}{\sqrt{q^2}} \tag{70}$$

with the result

$$\partial_\mu A^r_\nu(z(\tau))_{|tail} = -e\frac{q^r_\mu\dot{z}_\nu(\tau^r)}{q^r\cdot\dot{z}(\tau^r)}\frac{\mu J_1(\mu\sqrt{q^2(\tau^r)})}{\sqrt{q^2(\tau^r)}} + e\int d\sqrt{q^2}\frac{q_\mu}{\dot{z}\cdot q}\frac{\partial}{\partial\sqrt{q^2}}\left[\frac{\mu J_1(\mu\sqrt{q^2})}{\sqrt{q^2}}\dot{z}_\nu(\tau)\right]. \tag{71}$$

The next step is a partial integration where the contribution cancels the first term on the right hand side and the remaining integral is rewritten over the proper time,

$$\partial_\mu A^r_\nu(z(\tau))_{|tail} = e\int_{-\infty}^{\tau^r} d\tau'\frac{\mu J_1(\mu\sqrt{q^2(\tau')})}{\sqrt{q^2(\tau')}}\dot{z}_\nu(\tau')\left(\frac{\dot{z}_\mu(\tau')}{\dot{z}(\tau')\cdot q(\tau')} + \frac{q_\mu(\tau')[\ddot{z}(\tau')\cdot q(\tau') - 1]}{(\dot{z}(\tau')\cdot q(\tau'))^2}\right). \tag{72}$$

This expression gives

$$F^r_{\mu\nu}(z(\tau))_{|tail}\dot{z}^\nu(\tau) = e\int_{-\infty}^{\tau^r} d\tau'\frac{\mu J_1(\mu\sqrt{q^2(\tau')})}{\sqrt{q^2(\tau')}}\frac{[\ddot{z}(\tau')\cdot q(\tau') - 1]}{(\dot{z}(\tau')\cdot q(\tau'))^2}[g_{\mu\nu}\dot{z}(\tau')\cdot\dot{z}(\tau) - \dot{z}_\mu(\tau')\cdot\dot{z}_\nu(\tau)]q^\nu(\tau') \tag{73}$$

Here we write $\dot{z}(\tau') = \dot{z}(\tau) - q(\tau')$ in the square bracket and find

$$F^r_{\mu\nu}(z(\tau))_{|tail}\dot{z}^\nu(\tau) = e[g_{\mu\nu} - \dot{z}_\mu\dot{z}_\nu]\int_{-\infty}^{\tau^r} d\tau'\frac{\mu J_1(\mu\sqrt{q^2(\tau')})}{\sqrt{q^2(\tau')}}\frac{[\ddot{z}(\tau')\cdot q(\tau') - 1]}{(\dot{z}(\tau')\cdot q(\tau'))^2}q^\nu(\tau'). \tag{74}$$

The appropriately smeared equation of motion contains the Abraham-Lorentz-Dirac force as the local contribution,

$$\frac{d}{d\tau}(m_0\dot{z}^\mu)_{|loc} = -\frac{2}{3}e^2[\dddot{z}^\mu + \ddot{z}^2\dot{z}^\mu] \tag{75}$$

which gives finally by the help of the equation $\frac{d}{d\tau}\dot{z}\ddot{z} = \ddot{z}^2 + \dot{z}\,\dddot{z} = 0$

$$\frac{d}{d\tau}(m_0\dot{z}^\mu(\tau)) = e^2[g_{\mu\nu} - \dot{z}_\mu(\tau)\dot{z}_\nu(\tau)]\left[-\frac{2}{3}\dddot{z}^\nu + \int_{-\infty}^{\tau^r} d\tau' \frac{\mu J_1(\mu\sqrt{q^2(\tau')})}{\sqrt{q^2(\tau')}} \frac{[\ddot{z}(\tau')\cdot q(\tau') - 1]}{(\dot{z}(\tau')\cdot q(\tau'))^2} q^\nu(\tau')\right] + eF_{ext\mu\nu}(z(\tau))\dot{z}^\nu(\tau). \tag{76}$$

Notice that the change of $m_0\dot{z}^\mu(\tau)$ is orthogonal to $\dot{z}^\mu(\tau)$, i.e. the mass is renormalization renormalized by a constant piece only.

VI. VACUUM POLARIZATION

We evaluate

$$-\sum_{n=1}^{\infty} \frac{\left(-\ell^{-2}\right)^n}{n!} \partial_{\mu^2}^n \left(\mu J_1\left(\mu\sqrt{q^2}\right)\right) \tag{77}$$

by using

$$\mu J_1\left(\mu\sqrt{q^2}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1)!} \left(\frac{\sqrt{q^2}}{2}\right)^{2j+1} \left(\mu^2\right)^{j+1} \tag{78}$$

which gives

$$\begin{aligned} -\sum_{n=1}^{\infty} \frac{\left(-\ell^{-2}\right)^n}{n!} \sum_{j=n-1}^{\infty} \frac{(-1)^j}{j!(j-n+1)!} \left(\frac{\sqrt{q^2}}{2}\right)^{2j+1} \left(\mu^2\right)^{j-n+1} = \\ = \sum_{n=1}^{\infty} \frac{\left(\ell^{-2}\right)^n}{n!} \left(\frac{1}{2}\right)^n \left(\sqrt{q^2}\right)^n \frac{J_{n-1}\left(\mu\sqrt{q^2}\right)}{\mu^{n-1}} \end{aligned} \tag{79}$$

Taking the limit $\mu^2 \to 0$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \frac{\left(\ell^{-2}\right)^n}{n!(n-1)!} \left(\frac{\sqrt{q^2}}{2}\right)^{2n-1} \\ &= \frac{1}{\ell} I_1\left(\frac{\sqrt{q^2}}{\ell}\right) \end{aligned} \tag{80}$$

This looks very problematic as I type Bessel function diverges for large arguments. I am not sure what went wrong and what exactly is going on. This would modify the equation of motion to

$$\frac{d}{d\tau}(m_0\dot{z}^\mu(\tau)) = e^2[g_{\mu\nu} - \dot{z}_\mu(\tau)\dot{z}_\nu(\tau)] \int_{-\infty}^{\tau^r} d\tau' \frac{I_1\left(\frac{\sqrt{q^2}}{\ell}\right)}{\ell\sqrt{q^2(\tau')}} \frac{[\ddot{z}(\tau')\cdot q(\tau') - 1]}{(\dot{z}(\tau')\cdot q(\tau'))^2} q^\nu(\tau') + eF_{ext\mu\nu}(z(\tau))\dot{z}^\nu(\tau). \tag{81}$$

[1] H. A. Lorentz, in *Enzyklopädie der Mathematischen Wissenschaften*, Vol. V, No.14 (B. G. Teubner, Leipzig, 1904).
[2] M. Abraham, Ann. Phys. (Leipzig) **10**, 105 (1903).
[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[4] F. Rohrlich, *Classical Charged Particles*, Addison-Wesley (1965).
[5] J. D. Jackson, *Classical Electrodynamics*, 3rd. ed. Wiley, New-York (1999).
[6] L. Landu, E. Lifsitz, *The Classical Theory of Fields*, Pergamon, New-York (1987).
[7] A. D. Yagjian, *Relativistic Dynamics of a charged Sphere*, 2nd. ed. (Springer, berlin, 2006).
[8] E. J. Moniz, D. H. Sharp, Phys. Rev. **15**, 2850 (1977).
[9] F. Rohrlich, Am. J. Phys. **38**, 1310 (1970).

[10] J. Frenkel, Phys. Rev. **E54**, 5859 (1996).
[11] J. Schwinger, J. Math. Phys. **2**, 407 (1961); *Particles and Sources*, vol. I., II., and III., Addison-Wesley, Cambridge, Mass. 1970-73.
[12] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) (Sov. Phys. JETP **20**, 1018 (1965)).
[13] J. Polonyi, Phys. Rev. **D74**, 065014 (2006).
[14] M. Planat, J. Polonyi, *Subclassical fields and polarization in electrodynamics*, arXiv:1002.2773, to appear in Phys. Rev.
[15] B. Podolski, P. Schwed, Rev. Mod. Phys. **20**, 40 (1948).
[16] A. Pais. G. E. Uhlenbeck, Phys. rev. **79**, 145 (1950).
[17] F. Bopp, Ann. Phys. **38**, 345 (1940).
[18] A. O. Barut, Phys. Rev. **10**, 3335 (1974).
[19] H. J. Bhabha, Proc. Roy. Soc.(London) **A172**, 384 (1939).
[20] H. J. Bhabha, Harish-Chandra, Proc. Roy. Soc.(London) **A185**, 384 (1946).
[21] Harish-Chandra, Proc. Roy. Soc.(London) **A185**, 250 (1946).
[22] R. G. Cawley, E. Marx, Int. J. Theor. Phys. **1**, 153 (1968).
[23] R. G. Cawley, Ann. Phys. (NY) **54**, 122 (1969).
[24] R. G. Cawley, J. Math. Phys. **11**, 761 (1970).
[25] Y. Yaremko, Electronic J. of Theor. Phys. **6**, 211 (2009).