DBI degeneracy from an RG flow

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1 Effective action in flat background

1.1 Construction of the RG flow

A DBI action in flat space written in the string frame is

$$S_{DBI} = -\int d^{p+1}x\, e^{-\phi}\mathcal{O}, \tag{1.1}$$

where we have defined a local operator

$$\mathcal{O} = T_p\sqrt{-\det\left[\eta_{ab} + 2\pi\alpha' F_{ab}\right]}. \tag{1.2}$$

Consider embedding the brane into a $p+2$-dimensional flat-space bulk so that the entire bulk-brane action with a dynamical dilaton takes the form

$$S = -\int d^{p+1}x\, e^{-\phi}\mathcal{O} - \frac{1}{2\kappa^2}\int d^{p+2}x\, e^{-\Phi}\partial_\mu\Phi\partial^\mu\Phi, \tag{1.3}$$

where the bulk extends along a radial coordinate $r \in [0,\infty)$. The brane is placed at $r = 0$ and $\Phi(r = 0) = \phi \equiv \phi_0$. We now discreetise the bulk by putting it on a radial lattice of an ϵ distance spacing with the dilaton defined at each lattice site,

$$\phi_n \equiv \Phi(r = n\epsilon). \tag{1.4}$$

The path integral measure then factorises into

$$Z = \int \mathcal{D}\Phi\, e^{iS} = \int \prod_{n=0}^{\infty} \mathcal{D}\phi_n\, e^{iS}, \tag{1.5}$$

and the action can be written as

$$
\begin{aligned}
S &= S_{DBI} + S^{(0)}_{bulk} \\
&= -\int d^{p+1}x\, e^{-\phi_0}\mathcal{O} - \frac{1}{2\kappa_5^2\epsilon}\int d^{p+1}x \sum_{n=0}^{\infty}\left\{e^{-\frac{\phi_n+\phi_{n+1}}{2}}\left(\phi_{n+1}-\phi_n\right)^2 + \epsilon^2 e^{-\phi_n}\partial_a\phi_n\partial^a\phi_n\right\}, \quad (1.6)
\end{aligned}
$$

where (0) in $S^{(0)}_{bulk}$ denotes the minimal radial position of the bulk action along the lattice sites. Now separate out the ϕ_0 contribution

$$
\begin{aligned}
S = &-\int d^{p+1}x e^{-\phi_0}\mathcal{O} - \frac{1}{2\kappa_5^2\epsilon}\int d^{p+1}x\left\{e^{-\frac{\phi_0+\phi_1}{2}}\left(\phi_1-\phi_0\right)^2 + \epsilon^2 e^{-\phi_0}\partial_a\phi_0\partial^a\phi_0\right\} \\
&- \frac{1}{2\kappa_5^2\epsilon}\int d^{p+1}x\sum_{n=1}^{\infty}\left\{e^{-\frac{\phi_n+\phi_{n+1}}{2}}\left(\phi_{n+1}-\phi_n\right)^2 + \epsilon^2 e^{-\phi_n}\partial_a\phi_n\partial^a\phi_n\right\}, \qquad (1.7)
\end{aligned}
$$

and shift the integral measure $\mathcal{D}\phi_0$ by $\phi_0 \to \phi_0 + \phi_1$ to find

$$
\begin{aligned}
S = &-\int d^{p+1}x e^{-\phi_1}\left\{e^{-\phi_0}\mathcal{O} + \frac{1}{2\kappa^2\epsilon}\left[e^{-\phi_0/2}\phi_0^2 + \epsilon^2 e^{-\phi_0}\left(\partial_a\phi_0\partial^a\phi_0 + 2\partial_a\phi_0\partial^a\phi_1 + \partial_a\phi_1\partial^a\phi_1\right)\right]\right\} \\
&- \frac{1}{2\kappa_5^2\epsilon}\int d^{p+1}x\sum_{n=1}^{\infty}\left\{e^{-\frac{\phi_n+\phi_{n+1}}{2}}\left(\phi_{n+1}-\phi_n\right)^2 + \epsilon^2 e^{-\phi_n}\partial_a\phi_n\partial^a\phi_n\right\}, \qquad (1.8)
\end{aligned}
$$

Now introduce $\phi_0 = \hat{\phi}_0 + \delta\tilde{\phi}_0$, where $\delta \ll 1$ and $\tilde{\phi}_0$ is a fluctuation around a dilaton value $\hat{\phi}_0$. To order δ^2,

$$
\begin{aligned}
S = &- e^{-\hat{\phi}_0}\int d^{p+1}x\, e^{-\phi_1}\left\{\left(1-\delta\tilde{\phi}_0+\frac{1}{2}\delta^2\tilde{\phi}_0^2\right)\left(\mathcal{O}+\frac{\epsilon}{2\kappa^2}\partial_a\phi_1\partial^a\phi_1\right)\right. \\
&+\frac{e^{\hat{\phi}_0/2}}{2\kappa^2\epsilon}\left(\hat{\phi}_0^2+\frac{\delta}{2}\hat{\phi}_0(4-\hat{\phi}_0)\tilde{\phi}_0+\frac{\delta^2}{8}\left(8-8\hat{\phi}_0+\hat{\phi}_0^2\right)\tilde{\phi}_0^2\right) \\
&\left.+\frac{\epsilon}{2\kappa^2}\left(\delta^2\partial_a\tilde{\phi}_0\partial^a\tilde{\phi}_0 + 2\delta\left(1-\delta\tilde{\phi}_0\right)\partial_a\tilde{\phi}_0\partial^a\phi_1\right)\right\} + S^{(1)}_{bulk} \qquad (1.9)
\end{aligned}
$$

Assuming that the fields do not have enormous derivatives, we can neglect $\mathcal{O}(\epsilon)$ terms. Defining

$$
\begin{aligned}
a_n &= \hat{\phi}_n^2 \\
b_n &= \frac{1}{2}\hat{\phi}_n(4-\hat{\phi}_n) \\
c_n &= \frac{1}{8}\left(8-8\hat{\phi}_n+\hat{\phi}_n^2\right) \\
\gamma_n &= e^{-\hat{\phi}_n/2} \qquad (1.10)
\end{aligned}
$$

and also neglect the $\delta^2\mathcal{O}$ term as $\delta^2 \ll \delta^2/\epsilon$, we find the effective action

$$
S^{(0)} = -\int d^{p+1}x\, e^{-\phi_1}\left\{\gamma_0^2\left(1-\delta\tilde{\phi}_0\right)\mathcal{O} + \frac{\gamma_0}{2\kappa^2\epsilon}\left(a_0+b_0\delta\tilde{\phi}_0+c_0\delta^2\tilde{\phi}_0^2\right)\right\} + S^{(1)}_{bulk}. \qquad (1.11)
$$

Now integrate out $\tilde{\phi}_0$ to find

$$
S^{(1)} = -\int d^{p+1}x\, e^{-\phi_1}\left\{\frac{\left(4a_0c_0-b_0^2\right)\gamma_0}{8\epsilon\kappa^2 c_0} + \left(1+\frac{b_0}{2c_0}\right)\gamma_0^2\mathcal{O} - \frac{\epsilon\kappa^2\gamma_0^3}{2c_0}\mathcal{O}^2\right\} + S^{(1)}_{bulk}. \qquad (1.12)
$$

Define

$$
\mathcal{O}_0 \equiv \mathcal{O} \qquad (1.13)
$$

$$
\mathcal{O}_1 \equiv \frac{\left(4a_0c_0-b_0^2\right)\gamma_0}{8\epsilon\kappa^2 c_0} + \left(1+\frac{b_0}{2c_0}\right)\gamma_0^2\mathcal{O} - \frac{\epsilon\kappa^2\gamma_0^3}{2c_0}\mathcal{O}^2, \qquad (1.14)
$$

so

$$S^{(1)} = -\int d^{p+1}x\, e^{-\phi_1}\mathcal{O}_1 + S^{(1)}_{bulk}. \tag{1.15}$$

We can thus recursively integrate out the entire bulk dilaton and generate a recurrence relation

$$\mathcal{O}_{n+1} = q_n + r_n\mathcal{O}_n + s_n\mathcal{O}_n^2, \tag{1.16}$$

where

$$q_n = \frac{\left(4a_nc_n - b_n^2\right)\gamma_n}{8\epsilon\kappa^2 c_n} \tag{1.17}$$

$$r_n = \left(1+\frac{b_n}{2c_n}\right)\gamma_n^2 \tag{1.18}$$

$$s_n = -\frac{\epsilon\kappa^2\gamma_n^3}{2c_n} \tag{1.19}$$

1.2 Conformal D3 brane

For a conformal D3 brane, the dilaton is constant, hence $f_n = f$, $g_n = g$ and $h_n = h$. It is convenient to introduce

$$\mathcal{O}_n = \frac{c}{\epsilon\kappa^2\gamma^3}\left\{\Psi_n - \left[1-\left(1+\frac{b}{2c}\right)\gamma^2 \pm \sqrt{\frac{c-(b+2c)\gamma^2+(a+b+c)\gamma^4}{c}}\right]\right\}. \tag{1.20}$$

The new recurrence relation is then

$$\Psi_{n+1} = f\Psi_n - \frac{1}{2}\Psi_n^2, \tag{1.21}$$

where

$$f = 1 \pm \sqrt{\frac{c-(b+2c)\gamma^2+(a+b+c)\gamma^4}{c}}. \tag{1.22}$$

Hence

$$\mathcal{O}_n = \frac{c}{\epsilon\kappa^2\gamma^3}\left(\Psi_n - f + \left(1+\frac{b}{2c}\right)\gamma^2\right). \tag{1.23}$$

Think of Ψ_n as a polynomial in the variable Ψ. Let us begin by writing it in a more convenient form

$$\Psi_{n+1} = -\frac{1}{2}\Psi_n\left(\Psi_n - 2f\right). \tag{1.24}$$

Let $\psi^p_{n,m}$ with indices $m, n, p \in \mathbb{Z}^+$ be the roots of the polynomial $\Psi_p(\Psi_n)$, where $p \geq n+1$, labeled by an integer m which counts the number of roots. Since we dealing with a quadratic recursion, clearly

$$\min[m] = 1 \qquad \text{and} \qquad \max[m] = 2^{p-n}. \tag{1.25}$$

The first step in the polynomial recurrence is

$$\Psi_1 = -\frac{1}{2}\Psi\left(\Psi - 2f\right), \tag{1.26}$$

followed by $\Psi_2(\Psi_1)$, which can be written in the form of a factorised expression

$$\Psi_2 = -\frac{1}{2}\left(\Psi_1 - \psi^2_{1,1}\right)\left(\Psi_1 - \psi^2_{1,2}\right), \tag{1.27}$$

where $\psi^2_{1,m}$ are the two roots of $\Psi_2(\Psi_1)$, i.e. $\psi^2_{1,1} = 0$ and $\psi^2_{1,2} = 2f$. Using the values of the roots gives

$$\Psi_2 = -\frac{1}{2}\left[-\frac{1}{2}\Psi\left(\Psi - 2f\right) - \psi^2_{1,1}\right]\left[-\frac{1}{2}\Psi\left(\Psi - 2f\right) - \psi^2_{1,2}\right], \tag{1.28}$$

and similarly

$$\Psi_{n+2} = -\frac{1}{2}\left[-\frac{1}{2}\Psi_n\left(\Psi_n - 2f\right) - \psi^{n+2}_{n+1,1}\right]\left[-\frac{1}{2}\Psi_n\left(\Psi_n - 2f\right) - \psi^{n+2}_{n+1,2}\right], \tag{1.29}$$

which allows us to find the roots $\psi^{n+2}_{n,m}$ from $\psi^{n+2}_{n+1,m}$:

$$\psi^{n+2}_{n,2m-1} = f + \sqrt{f^2 - 2\psi^{n+2}_{n+1,m}}, \qquad \psi^{n+2}_{n,2m} = f - \sqrt{f^2 - 2\psi^{n+2}_{n+1,m}}. \tag{1.30}$$

Finally, we want to find all $\psi^{n+2}_{0,m}$ so that we can express $\Psi_{n+2}(\Psi)$. The recurrence relation therefore runs from $n+2$ down to 0 and not the other way! Hence, all the roots of

$$\Psi_p = \left(-\frac{1}{2}\right)^{2^p-1}\prod_{n=1}^{2^p}\left(\Psi - \psi^p_{0,n}\right) \tag{1.31}$$

are given by

$$\psi^p_n = f \pm \sqrt{f^2 - 2\psi^p_{n+1}} \tag{1.32}$$

where ψ^p_n stands for all the roots $\left\{\psi^p_{n,1}, \dots \psi^p_{n,m}\right\}$.
But since $\psi^1_{0,m}$ are the same as $\psi^{n+1}_{n,m}$ we can also turn the recurrence of roots around and start the recurrence at $n = 0$. We get

$$\psi_{n+1} = f \pm \sqrt{f^2 - 2\psi_n}, \qquad \psi_0 = 0, \tag{1.33}$$

so that

$$\Psi_n = \left(-\frac{1}{2}\right)^{2^n-1}\prod_{m=1}^{2^n}\left(\Psi - \psi_m\right) \tag{1.34}$$

Furthermore

$$\Psi_n = \frac{\epsilon\kappa^2\gamma^3}{c}\mathcal{O}_n + f - \left(1 + \frac{b}{2c}\right)\gamma^2. \tag{1.35}$$

2 New duality

Two solutions for ϕ give the same f.

3 Effective action for D3 brane

The effective action for a probe D3 brane with fixed bulk value of the dilaton is

$$\begin{aligned} S^{(n)} = &- \int d^4x e^{-\phi}\frac{c}{\epsilon\kappa^2\gamma^3}\left\{\left(-\frac{1}{2}\right)^{2^n-1}\prod_{m=1}^{2^n}\left(\Psi - \psi_m\right) - f + \left(1 + \frac{b}{2c}\right)\gamma^2\right\} \\ &- \frac{1}{2\kappa_5^2}\int d^4x \int_{n\epsilon}^{\infty} e^{-\Phi}\partial_\mu\Phi\partial^\mu\Phi \end{aligned} \tag{3.1}$$

3.1 Example 1

Consider a case with $f = 2$. The recursion $\psi_{n+1} = f \pm \sqrt{f^2 - 2\psi_n}$ can then be solved exactly,

$$\psi_n = 2\left(1 - e^{-2^{-n}C}\right) \tag{3.2}$$

where C is set so that $\psi_n = 0$. Thus,

$$\psi_n = 2\left(1 - \exp\left\{2^{1-n}\pi i m\right\}\right), \quad \text{where } m \in \mathbb{Z}. \tag{3.3}$$

The exponent is a periodic function that repeats with a period of $m = 2^n$, which is exactly the number of distinct roots at order n. Hence, all roots are given by

$$\psi_n = 2\left(1 - \exp\left\{2^{1-n}\pi i m\right\}\right), \quad \text{where } m \in \{1, 2, 3, \ldots, 2^n\}. \tag{3.4}$$

The effective action is thus

$$\begin{aligned} S^{(n)} = &- \int d^4x e^{-\phi} \frac{c}{\epsilon \kappa^2 \gamma^3} \left\{ 2\left(-1\right)^{2^n+1} \prod_{m=1}^{2^n} \left[\frac{1}{2}\Psi - 1 + \exp\left\{2^{1-n}\pi i m\right\}\right] - 2 + \left(1 + \frac{b}{2c}\right)\gamma^2 \right\} \\ &- \frac{1}{2\kappa_5^2} \int d^4x \int_{n\epsilon}^{\infty} e^{-\Phi} \partial_\mu \Phi \partial^\mu \Phi. \end{aligned} \tag{3.5}$$

In the limit of $n \to \infty$ we get a continuous orbit for Ψ_n, with the roots ψ_n drawing out a circle with a radius of 2, centred around $(\Re\Psi, \Im\Psi) = (2, 0)$, on the complex Ψ plane,

$$\psi = 2\left(1 - e^{i\mu}\right), \quad \text{where } \mu \in [0, 2\pi). \tag{3.6}$$

We can now use the fact that the factorised monomic polynomial $\Psi_n(X)$, where $X = \frac{1}{2}\Psi - 1$, can be written in term of elementary symmetric polynomials

$$\prod_{m=1}^{2^n} \left[X + \exp\left\{2^{1-n}\pi i m\right\}\right] = X^{2^n} + e1 \cdot X^{2^n-1} + \ldots + e_{2^n}, \tag{3.7}$$

where

$$e_k = \sum_{1 \le m_1 < \ldots < m_k \le 2^n} \exp\left\{2^{1-n}\pi i \sum_{i=1}^{k} m_k\right\}. \tag{3.8}$$

It is easy to see that in this case $e_1 = e_2 = \ldots = e_{2^n-1} = 0$ and that $e_{2^n} = -1$ if $n \ge 0$, while $e_{2^n} = +1$ if $n = 0$. Hence,

$$\prod_{m=1}^{2^n} \left[X + \exp\left\{2^{1-n}\pi i m\right\}\right] = X^{2^n} - \left(-1\right)^{2^n}. \tag{3.9}$$

The effective action thus becomes

$$\begin{aligned} S^{(n)} = &- \int d^4x\, e^{-\phi} \frac{c}{\epsilon \kappa^2 \gamma^3} \left\{ 2\left[1 - \left(1 - \frac{1}{2}\Psi\right)^{2^n}\right] - 2 + \left(1 + \frac{b}{2c}\right)\gamma^2 \right\} \\ &- \frac{1}{2\kappa_5^2} \int d^4x \int_{n\epsilon}^{\infty} e^{-\Phi} \partial_\mu \Phi \partial^\mu \Phi, \end{aligned} \tag{3.10}$$

or in terms of $\mathcal{O}$,

$$\begin{aligned} S^{(n)} = &- \frac{2c}{\epsilon \kappa^2 \gamma^3} \int d^4x\, e^{-\phi} \left\{ \frac{\gamma^2}{2}\left(1 + \frac{b}{2c}\right) - \left[\frac{\gamma^2}{2}\left(1 + \frac{b}{2c}\right) - \frac{\epsilon \kappa^2 \gamma^3}{2c}\mathcal{O}\right]^{2^n} \right\} \\ &- \frac{1}{2\kappa_5^2} \int d^4x \int_{n\epsilon}^{\infty} e^{-\Phi} \partial_\mu \Phi \partial^\mu \Phi. \end{aligned} \tag{3.11}$$

We can use the binomial theorem to expand

$$\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)-\left[\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)-\frac{\epsilon\kappa^2\gamma^3}{2c}\mathcal{O}\right]^{2^n}=\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)-\left[\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)\right]^{2^n}$$
$$+2^n\frac{\epsilon\kappa^2\gamma^3}{2c}\left[\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)\right]^{2^n-1}\mathcal{O}+\dots \tag{3.12}$$

The effective DBI action vanishes when

$$\left(1-\frac{1}{2}\Psi\right)^{2^n}=1-\frac{1}{2}\left[f-\left(1+\frac{b}{2c}\right)\gamma^2\right] \tag{3.13}$$

hence the roots of the action polynomial $\mathcal{L}(\Psi)=\prod(\Psi-\psi_{\mathcal{L}})$, are

$$\psi_{\mathcal{L}}=2-2\left[1-\frac{f}{2}+\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)\right]^{1/2^n}\exp\left\{2^{1-n}\pi im\right\},\quad m\in\{1,2,3,\dots,2^n\}. \tag{3.14}$$

In terms of the operator $\mathcal{O}$, the Lagrangian $\mathcal{L}(\mathcal{O})=\prod(\mathcal{O}-\varphi)$,

$$\varphi_{\mathcal{L}}=\frac{c}{\epsilon\kappa^2\gamma^3}\left\{2\left[1-\frac{f}{2}+\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)\right]-2\left[1-\frac{f}{2}+\frac{\gamma^2}{2}\left(1+\frac{b}{2c}\right)\right]^{1/2^n}e^{2^{1-n}\pi im}\right\}, \tag{3.15}$$

and since $f=2$,

$$\varphi_{\mathcal{L}}=\frac{c}{\epsilon\kappa_5^2\gamma^3}\left\{\gamma^2\left(1+\frac{b}{2c}\right)-2\left[\frac{1}{2}\gamma^2\left(1+\frac{b}{2c}\right)\right]^{1/2^n}e^{2^{1-n}\pi im}\right\}. \tag{3.16}$$

Hence

$$-\det\left[\eta_{ab}+2\pi\alpha' F_{ab}\right]=\frac{c^2e^{3\phi}}{\epsilon^2\kappa_5^4T_p^2}\left[\gamma^2\left(1+\frac{b}{2c}\right)-2\left[\frac{1}{2}\gamma^2\left(1+\frac{b}{2c}\right)\right]^{1/2^n}e^{2^{1-n}\pi im}\right]^2 \tag{3.17}$$

Now turn on E_x and B_y fields so that $-\det\left[\eta_{ab}+2\pi\alpha' F_{ab}\right]=1+4\pi^2\alpha'^2\left(B_y^2-E_x^2\right)$

$$E_x^2-B_y^2=\frac{1}{4\pi^2\alpha'^2}\left\{1-\frac{c^2e^{3\phi}}{\epsilon^2\kappa_5^4T_p^2}\left[\gamma^2\left(1+\frac{b}{2c}\right)-2\left[\frac{1}{2}\gamma^2\left(1+\frac{b}{2c}\right)\right]^{1/2^n}e^{2^{1-n}\pi im}\right]^2\right\}. \tag{3.18}$$

For $f=2$,

$$c^2e^{3\phi}=0.21779\dots \tag{3.19}$$

$$\gamma^2\left(1+\frac{b}{2c}\right)=1.12172\dots \tag{3.20}$$

Now the 5-d gravity scale is

$$\kappa_5^2=\frac{g_s^2\ell_s^8}{R^5} \tag{3.21}$$

where

$$\ell_s=\sqrt{\alpha'} \tag{3.22}$$

so

$$\kappa_5^2=\frac{g_s^2\alpha'^4}{R^5}, \tag{3.23}$$

where R is the radius of each internal space dimension, assuming T^{10-d}.
The brane tension is

$$T_p = \frac{1}{(2\pi)^p \ell_s^{p+1} g_s} \tag{3.24}$$

or in our case with $p = 3$:

$$T_3 = \frac{1}{(2\pi)^3 \ell_s^4 g_s} = \frac{1}{(2\pi)^3 \alpha'^2 g_s}. \tag{3.25}$$

Finally,

$$g_s = e^{\phi_0} \tag{3.26}$$

and [EXPLAIN]

$$\epsilon = \sqrt{\alpha'} \tag{3.27}$$

so

$$\frac{c^2 e^{3\phi}}{\epsilon^2 \kappa_5^4 T_3^2} = \frac{(2\pi)^6 \alpha'^4 g_s^2 c^2 R^{10} e^{3\phi}}{\epsilon^2} = (2\pi)^6 c^2 \alpha'^3 R^{10} e^{5\phi} \tag{3.28}$$

With our dilaton

$$(2\pi)^6 c^2 e^{5\phi} \approx 1751.64\ldots \tag{3.29}$$

so

$$\frac{c^2 e^{3\phi}}{\epsilon^2 \kappa_5^4 T_3^2} \approx 1751.64 \times \alpha'^3 R^{10}. \tag{3.30}$$

Thus

$$E_x^2 - B_y^2 = \frac{1}{4\pi^2 \alpha'^2} - 44.3695 \times \alpha' R^{10} \left[\gamma^2 \left(1 + \frac{b}{2c}\right) - 2 \left[\frac{1}{2}\gamma^2 \left(1 + \frac{b}{2c}\right)\right]^{1/2^n} e^{2^{1-n}\pi i m} \right]^2. \tag{3.31}$$

Take $n \to \infty$ so that

$$E_x^2 - B_y^2 = \frac{1}{4\pi^2 \alpha'^2} - 44.3695 \times \alpha' R^{10} \left[\gamma^2 \left(1 + \frac{b}{2c}\right) - 2e^{i\mu} \right]^2, \quad \mu \in [0, 2\pi). \tag{3.32}$$

Since $E_x^2 - B_y^2 \in \mathbb{R}$, there are three possible values μ which give distinct real right-hand-sides:

$$\mu_1 = 0 : \qquad \left[\gamma^2 \left(1 + \frac{b}{2c}\right) - 2e^{i\mu_1} \right]^2 \approx 0.77137\ldots \tag{3.33}$$

$$\mu_2 = \pi : \qquad \left[\gamma^2 \left(1 + \frac{b}{2c}\right) - 2e^{i\mu_2} \right]^2 \approx 9.7451\ldots \tag{3.34}$$

$$\mu_3 = \arccos\left(\frac{\gamma^2}{2} \left(1 + \frac{b}{2c}\right)\right) \approx 0.97537\ldots : \qquad \left[\gamma^2 \left(1 + \frac{b}{2c}\right) - 2e^{i\mu_3} \right]^2 \approx -2.7417\ldots \tag{3.35}$$

Hence we find two 'magnetic' and one 'electric' values of the fields which make the action vanish:

$$E_x^2 - B_y^2 = \frac{1}{4\pi^2 \alpha'^2} - 34.2254 \times \alpha' R^{10} \tag{3.36}$$

$$E_x^2 - B_y^2 = \frac{1}{4\pi^2 \alpha'^2} - 432.387 \times \alpha' R^{10} \tag{3.37}$$

$$E_x^2 - B_y^2 = \frac{1}{4\pi^2 \alpha'^2} + 121.65 \times 10^8 \alpha' R^{10} \tag{3.38}$$

The third value is already in the region of instability, hence

$$4\pi^2 \alpha'^2 \left(E_x^2 - B_y^2\right) = 1 - 1.4 \times 10^3 \alpha'^3 R^{10} \tag{3.39}$$

$$4\pi^2 \alpha'^2 \left(E_x^2 - B_y^2\right) = 1 - 17.1 \times 10^3 \times \alpha'^3 R^{10} \tag{3.40}$$

3.2 Stress-energy tensor

The action for $f = 2$ is

$$S^{(n)} = -\frac{c}{\epsilon\kappa^2\gamma^3}\int d^4x e^{-\phi}\left\{2\left[1-\left(1-\frac{1}{2}\left[\frac{\epsilon\kappa^2\gamma^3}{c}\mathcal{O}+2-\left(1+\frac{b}{2c}\right)\gamma^2\right]\right)^{2^n}\right]-2+\left(1+\frac{b}{2c}\right)\gamma^2\right\}$$
$$-\frac{1}{2\kappa_5^2}\int d^4x\int_{n\epsilon}^{\infty} e^{-\Phi}\partial_\mu\Phi\partial^\mu\Phi \tag{3.41}$$

The stress-energy tensor is then

$$\begin{aligned} T^{ab} &\propto 2^n\left(1-\frac{1}{2}\left[\frac{\epsilon\kappa^2\gamma^3}{c}\mathcal{O}+2-\left(1+\frac{b}{2c}\right)\gamma^2\right]\right)^{2^n-1} T^{ab}_{DBI} \\ &= 2^n\left(1-\frac{1}{2}\Psi\right)^{2^n-1} T^{ab}_{DBI} \end{aligned} \tag{3.42}$$

3.3 $f = 4$

$$\Psi_{n+1} = 4\Psi_n - \frac{1}{2}\Psi_n^2, \tag{3.43}$$

Find

$$\Psi_n = 4 - 4\cos\left(2^n C\right) \tag{3.44}$$

To fix the constant

$$\Psi = \Psi_0 = 4 - 4\cos\left(C\right) \Longrightarrow C = \arccos\frac{4-\Psi}{4} \tag{3.45}$$

so

$$\Psi_n = 4 - 4\cos\left(2^n \arccos\left(\frac{4-\Psi}{4}\right)\right) \tag{3.46}$$

Use

$$\Psi_n = \frac{\epsilon\kappa^2\gamma^3}{c}\mathcal{O}_n + f - \left(1+\frac{b}{2c}\right)\gamma^2. \tag{3.47}$$

so

$$\mathcal{O}_n = \frac{\epsilon\kappa^2\gamma^3}{c}\left\{\left(1+\frac{b}{2c}\right)\gamma^2 - 4\cos\left[2^n\arccos\left(\frac{4-\Psi(\mathcal{O})}{4}\right)\right]\right\} \tag{3.48}$$

The roots are

$$\cos\left[2^n\arccos\left(\frac{4-\Psi(\mathcal{O})}{4}\right)\right] = \left(1+\frac{b}{2c}\right)\frac{\gamma^2}{4} \tag{3.49}$$

$$2^n\arccos\left(\frac{4-\Psi(\mathcal{O})}{4}\right) = \arccos\left[\left(1+\frac{b}{2c}\right)\frac{\gamma^2}{4}\right] \tag{3.50}$$

$$\tag{3.51}$$

4 Newton's Constant 0704.0777

Consider compactifying on T^{10-d} where each circle has radius R. In ten dimensions (neglect numerical factors),

$$G_{10} = g^2 \ell_s^8 \tag{4.1}$$

where ℓ_s is the string scale and g the string coupling constant. The effective Newton's constant in d dimensions is

$$G_d \equiv \ell_d^{d-2} = \frac{G_{10}}{R^{10-d}} = \frac{g^2 \ell_s^8}{R^{10-d}} \tag{4.2}$$

5 Add the metric

Use

$$\frac{\partial \det g}{\partial g^{\mu\nu}} = \det g g^{\mu\nu} \tag{5.1}$$

to find for $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$

$$\sqrt{-G} = \sqrt{-g}\left(1 + \frac{1}{2}h^\mu_\mu + \frac{1}{8}h^\mu_\mu h^\nu_\nu - \frac{1}{4}h^\mu_\nu h^\nu_\mu\right) \tag{5.2}$$

$$= \sqrt{-g}\left(1 + \frac{1}{2}g^{\mu\nu}h_{\mu\nu}\right) \tag{5.3}$$

$$\tag{5.4}$$

Hence

$$\sqrt{-\det\left[G_{ab} + 2\pi\alpha' F_{ab}\right]} = \sqrt{-\det\left[g_{ab} + 2\pi\alpha' F_{ab}\right]}\left(1 + \frac{1}{2}\left(g^{ab} + 2\pi\alpha' F^{ab}\right)h_{ab}\right) \tag{5.5}$$

The action

$$S = -T_p \int d^{p+1}x e^{-\phi}\sqrt{-\det\left[g_{ab} + 2\pi\alpha' F_{ab}\right]} - \frac{1}{2\kappa_5^2}\int d^{p+2}x\sqrt{-G}e^{-\Phi}\left(R + \partial_\mu\Phi\partial^\mu\Phi + \dots\right) \tag{5.6}$$

can be perturbed to give

$$\begin{aligned} S = &-T_p \int d^{p+1}x e^{-\phi}\sqrt{-\det\left[g_{ab} + 2\pi\alpha' F_{ab}\right]}\left(1 + \frac{1}{2}\left(g^{ab} + 2\pi\alpha' F^{ab}\right)h_{ab}\right) \\ &- \frac{1}{2\kappa_5^2}\int d^{p+2}x\sqrt{-G}e^{-\Phi}\left(R + \partial_\mu\Phi\partial^\mu\Phi + \dots\right) \end{aligned} \tag{5.7}$$

$$-\sqrt{-g}\left\{\frac{1}{4}\nabla_\mu h_{\rho\lambda}\nabla^\mu h^{\rho\lambda} - \frac{1}{2}\nabla_\mu h_{\rho\lambda}\nabla^\rho h^{\mu\lambda} + \frac{1}{2}\nabla_\mu h^{\mu\nu}\nabla_\nu h - \frac{1}{4}\nabla_\mu h\nabla^\mu h - \frac{1}{2}\Lambda\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right)\right\} \tag{5.8}$$

where

$$\nabla_\mu h_{\nu\rho} = \partial_\mu h_{\nu\rho} - \Gamma^\lambda{}_{\mu\nu}h_{\lambda\rho} - \Gamma^\lambda{}_{\mu\rho}h_{\nu\lambda} \tag{5.9}$$

$$\nabla_\mu h^{\nu\rho} = \partial_\mu h^{\nu\rho} + \Gamma^\nu{}_{\mu\lambda}h^{\lambda\rho} + \Gamma^\rho{}_{\mu\lambda}h^{\nu\lambda} \tag{5.10}$$

Assume a diagonal g, so that only diagonal $h_{\mu\nu}$ contribute to running of the DBi action

$$-\sqrt{-g}\left\{\frac{1}{4}\nabla_\mu h_{\nu\nu}\nabla^\mu h^{\nu\nu} - \frac{1}{2}\nabla_\mu h_{\mu\mu}\nabla^\mu h^{\mu\mu} + \frac{1}{2}\nabla_\mu h^{\mu\mu}\nabla_\mu h^\lambda_\lambda - \frac{1}{4}\nabla_\mu h\nabla^\mu h - \frac{1}{2}\Lambda\left(h^{\mu\mu}h_{\mu\mu} - \frac{1}{2}h^2\right)\right\} \tag{5.11}$$

5.1 Flat space

Then the action is

$$-\left\{\frac{1}{4}\partial_\mu h_{\rho\lambda}\partial^\mu h^{\rho\lambda}-\frac{1}{2}\partial_\mu h_{\rho\lambda}\partial^\rho h^{\mu\lambda}+\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h-\frac{1}{4}\partial_\mu h\partial^\mu h-\frac{1}{2}\Lambda\left(h^{\mu\nu}h_{\mu\nu}-\frac{1}{2}h^2\right)\right\} \tag{5.12}$$

$$=-\left\{\frac{1}{4}\partial_r h_{\mu\mu}\partial_r h^{\mu\mu}-\frac{1}{2}\left(\partial_r h_{rr}\right)^2+\frac{1}{2}\partial_r h_{rr}\partial_r h-\frac{1}{4}\left(\partial_r h\right)^2-\frac{1}{2}\Lambda\left(h^{\mu\mu}h_{\mu\mu}-\frac{1}{2}h^2\right)\right\} \tag{5.13}$$

$$=-\left\{\frac{1}{4}\partial_r h_{\mu\mu}\partial_r h^{\mu\mu}+\frac{1}{2}\partial_r h_{rr}\partial_r h_a^a-\frac{1}{4}\left(\partial_r h\right)^2-\frac{1}{2}\Lambda\left(h^{\mu\mu}h_{\mu\mu}-\frac{1}{2}h^2\right)\right\} \tag{5.14}$$

$$\tag{5.15}$$