From holographic to Wilsonian renormalisation group: the bulk/cut-off dictionary

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Abstract
We wish to systematically construct Wilsonian renormalisation group procedure of integrating infinitesimally thin momentum shells on the brane QFT directly from the bulk physics. To achieve this we will combine holographic renormalisation and establish a precise dictionary between a hard Wilsonian cut-off and quantities in the bulk.

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I. INTRODUCTION

Holographic duality or AdS/CFT [1–3] has beside its fundamental conceptual importance to physics provided us with a very useful tool for the study of strongly coupled quantum (conformal) field theories with an aid of their weakly coupled classical gravity duals.

We wish to systematically construct Wilsonian renormalisation group procedure of integrating infinitesimally thin momentum shells on the brane QFT directly from the bulk physics. To achieve this we will combine holographic renormalisation as constructed in [6] and establish a precise dictionary between a hard Wilsonian cut-off and quantities in the bulk.

II. FROM HOLOGRAPHIC TO WILSONIAN RENORMALISATION IN ASYMPTOTICALLY ANTI-DE SITTER SPACE WITH SCALARS

We begin our construction of Wilsonian renormalisation holography by considering scalar field theories in the bulk with dual holographic scalar operators, as dictated by the AdS/CFT dictionary [2, 3]. Bulk metrics g_{MN} in $d+1$ dimensions will throughout this work be taken to be asymptotically Anti-de Sitter with a boundary at $r=0$, where r is the $d+1$'th radial coordinate. Capital latin indices run over $d+1$ dimensions, whereas we reserve greek indices for d lorentzian brane dimensions with translational symmetry. Branes where dual quantum field theories live are parallel to the AdS boundary as well as a black brane horizon in the interior of the bulk. In this section we will, however, only study asymptotically AdS spaces without horizons or other kinds of singularities in the bulk. All boundary duals will therefore be at zero temperature.

To systematically construct Wilsonian renormalisation group flow of the brane theory from holographic renormalisation and integration of bulk geometry we need to first set up the language of holographic renormalisation [6]. The starting point from which we wish to flow is a field theory defined at the AdS boundary. Now since the metric at the boundary diverges, so does the boundary action from which dual correlation functions are extracted. In order to regulate this infinity, we instead of defining the boundary theory at $r=0$, define it at $r=\rho_0$ with $\rho_0 \ll 1$ very close to AdS infinity. The value ρ_0 will become dual to some initial Wilsonian UV cut-off Λ_0. We will use variable ρ to specify the radial position of the brane on which dual QFT lives. This regulated bare boundary action, $S_B[\rho_0] \equiv S_B^{\mathrm{reg}}[\rho_0]$, can be thought of as coming from the infinitesimally thin slab of geometry, the UV regime of the brane physics, where $0 \le r \le \rho_0$. In the limit when cut-off at ρ_0 is taken to 0, the bare S_B diverges. The existence of a well defined limit is, however, essential for the AdS/CFT duality, as dual quantum field theory is defined at AdS infinity. We therefore renormalise S_B using holographic renormalisation whereby a counter-term action is introduced. Counter-terms are taken to exactly equal divergent pieces of S_B as $\rho_0 \to 0$, resulting in a "minimal-subtraction" scheme, which we will be using throughout this work. Subtracting counter-terms from initial $S_B[\rho_0]$ therefore makes the overall on-shell action finite in the $\rho_0 \to 0$ limit and removes all contact terms. The subtracted boundary action is then

$$S_B^{\mathrm{sub}}[\rho_0] \equiv S_B[\rho_0] - S_B^{\mathrm{c.t.}}[\rho_0]. \tag{1}$$

A definition of the renormalised action, which is by construction finite in the $\rho_0 \to 0$ limit, naturally follows from the subtracted action via relation

$$S_B^{\mathrm{ren}} \equiv \lim_{\rho_0 \to 0} S_B^{\mathrm{sub}}[\rho_0]. \tag{2}$$

The regulated bare scalar boundary action at $r=\rho_0$ coming from the UV is

$$S_B[\rho_0] = -\frac{1}{2} \int\limits_{r=\rho_0} d^d x \sqrt{-g} g^{rr} \Phi \partial_r \Phi. \tag{3}$$

We will write the induced d-dimensional metric on a brane as $\gamma_{\mu\nu}$, which implies $\sqrt{-\gamma} = \sqrt{-g}\sqrt{g^{rr}}$. Note that we are only considering theories with two-derivative terms in the kinetic energy of the Lagrangians. In principle, however, higher derivative terms may arise in supergravity theories.

Treating the radial coordinate as time we can define a canonical conjugate to Φ by

$$\Pi \equiv \frac{\delta S_B^{\mathrm{sub}}}{\delta \Phi}. \tag{4}$$

We similarly define a bare canonical conjugate momentum as

$$\Pi_B \equiv \frac{\delta S_B}{\delta \Phi} = -\sqrt{-g} g^{rr} \partial_r \Phi. \tag{5}$$

Given that we are working with two-derivative theories, this means that counter-terms will only involve terms with quadratic powers of Φ:

$$S_B^{\mathrm{c.t.}}[\rho_0] = -\int\limits_{r=\rho_0} d^d x \sqrt{-\gamma} \left(\frac{\Delta_-}{2} \Phi^2 + \frac{1}{2} \sum_{n=1}^{\infty} c_n \Phi \,\Box_\gamma^n\, \Phi \right). \tag{6}$$

We ignore possible terms arising from conformal anomaly. Subtracted boundary action (1) is then

$$S_B^{\mathrm{sub}}[\rho_0] = \frac{1}{2} \int\limits_{r=\rho_0} d^d x \Pi \Phi, \tag{7}$$

which enables us to rewrite the original regulated bare action S_B at initial $\rho_0 = \rho_0$ as

$$S_B[\rho_0] = \frac{1}{2}\int\limits_{r=\rho_0} d^dx\Pi\Phi + S_B^{\text{c.t.}}[\rho_0] = \frac{1}{2}\int\limits_{r=\rho_0} d^dx\sqrt{-\gamma}\left[\frac{\Pi}{\sqrt{-\gamma}}\Phi - \Delta_-\Phi^2 - \sum_{n=1}^{\infty} c_n\Phi\,\Box_\gamma^n\,\Phi\right]. \tag{8}$$

We have for convenience suppressed constants in front of the action, such as Newton's constant. It is however important to keep in mind that these constant must always become large in the large N limit of the AdS/CFT duality.

We now turn our attention to showing how Wilsonian effective action on the brane arises as a result of integrating out infinitesimally thin slabs of bulk geometry [7]. Following the setup of Faulkner, Liu and Rangamani we write the total bulk action for a scalar field with a boundary at $\rho_0 = \rho_0$ as

$$S = S_B[\rho_0] + \int\limits_{r\geq\rho_0} d^{d+1}x\sqrt{-g}\mathcal{L}\left(\Phi, \partial_M\Phi\right), \tag{9}$$

where $\mathcal{L}\left(\Phi, \partial_M\Phi\right) = -\frac{1}{2}\partial_M\Phi\partial^M\Phi - V(\Phi)$. We assume a polynomial potential $V(\Phi) = \frac{1}{2}m^2\Phi^2 + \sum_{n=3}^{\infty}\frac{1}{n}b_n\Phi^n$, with the mass term kept explicit. Further following [7] we can derive a functional flow equation for the bare boundary action, which must be obeyed at any d-dimensional brane with a fixed $r = \rho$ coordinate. The equation describes a flow of the bare brane action from initial $\rho_0 = \rho_0$ into the bulk. It is obtained by varying position of the brane $\rho \to \rho + \delta\rho$ and insisting that the overall bare action (9) remains constant at any $r = \rho$:

$$\partial_\rho S_B[\rho] = -\int\limits_{r=\rho} d^dx\mathcal{H} = -\int\limits_{r=\rho} d^dx\left(\frac{\delta S_B}{\delta\Phi}\frac{\partial\Phi}{\partial r} - \sqrt{-g}\mathcal{L}\left(\Phi, \partial_M\Phi\right)\right). \tag{10}$$

The term $\partial_\rho S_B$ comes from the variation of the metric. Since we are neglecting the metric backreaction, components of g_{MN} can simply be treated as functions of r.

The procedure by which we obtained (10) is completely analogous to Wilsonian integration of momentum shells by insisting that partition function of the bare action remains constant. Since we are only working with large N theories, this is equivalent to insisting that the bare action remains constant. This is sufficient as all actions that our analysis applies to become dominated by classical saddle points of the path integral in the large N limit. (10) is the Hamiltonian evolution with time replaced by the radial coordinate.

Equation (10) can be rewritten as

$$\sqrt{g^{rr}(\rho)}\partial_\rho S_B = -\int\limits_{r=\rho} d^dx\sqrt{-\gamma}\left(\frac{1}{2\gamma}\left(\frac{\delta S_B}{\delta\Phi}\right)^2 + \frac{1}{2}\gamma^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right). \tag{11}$$

We will now impose the Dirichlet boundary conditions on the bare Φ at each step of the flow. We also keep its dependence on ρ fixed all along the bulk. This condition is also motivated by Wilsonian procedure where bare fields initially defined up to Λ_0 also remain fixed after integration. If new cut-off is now some Λ_1, then anomalous dimension only enters when rescale Λ_1 back to Λ_0 to extract the flow of parameters in the effective action. Similarly in our setup wavefunction renormalisation will enter

We use the Fourier representation

$$\Phi(x, r \geq \rho) = \int\frac{d^dk}{(2\pi)^d}e^{ik\cdot x}f_k(r)\phi_0(\rho_0, k), \text{ with } f_k(\rho) = 1, \ \forall\rho, \tag{12}$$

where $\phi_0(\rho_0, k)$ is the non-renormalised boundary value of the scalar field. It is well known that in the standard Dirichlet quantisation ϕ_0 contains explicit cut-off dependence $\phi_0(\rho_0, k) \propto \rho_0^{\Delta_-}$ in the leading term, near the AdS boundary.

Flow equation will cause the coupling constants in S_B to run as we move the boundary into the bulk and change ρ. This must occur in order for the combination of S_B and the integrated-out bulk to stay independent of the cut-off ρ, which is analogous to the procedure of Wilsonian renormalisation group. There we study the flow of an action, defined with a cut-off, by integrating out degrees of freedom just below the cut-off. The procedure results in obtaining an effective action, which must after renormalisation give physical observables independent of the new cut-off ρ. In flowing from $\rho_0 = \rho_0$ it is therefore clear that initial $S_B[\rho_0]$ as written in (3), with the addition of Dirichelt boundary conditions (12), will not be able to stay invariant under the change of geometry, unless new terms appear in the Lagrangian to compensate for that. Since we could write S_B at $\rho_0 = \rho_0$ as a combination of S_B^{sub} and $S_B^{\text{c.t.}}$ in (8) we can treat this combination as the definition of the bare S_B at an arbitrary position ρ along the radial coordinate.

$$S_B[\rho] \equiv S_B^{\text{sub}}[\rho] + S_B^{\text{c.t.}}[\rho] \tag{13}$$

Integrating out consecutive slices of geometry while keeping the total action as well as the boundary value of non-renormalised field fixed will cause couplings as well as the wavefunction renormalisation to run. So to prevent the renormalised correlation functions in the dual QFT from running, we must also take into account appropriate wavefunction renormalisation. This can be done by noticing that at $\rho_0 = \rho_0$, $\Phi(\rho_0, k) = \rho_0^{\Delta_-}\Phi^{\text{ren}}(\rho_0)$. Now if ρ is to replace the initial cut-off ρ_0 we can anticipate that $\Phi(\rho, k) \propto \rho^{\Delta_-}Z^{1/2}(\rho, k)$, where Z is the usual wavefunction renormalisation.

We define

$$\Phi(\rho, k) \equiv \rho^{\Delta_-}Z^{1/2}(\rho, k)\Phi^{\text{ren}}(\rho, k). \tag{14}$$

When we apply the AdS/CFT prescription to calculate physical renormalised correlators, we will have take functional derivatives of a finite renormalised S_B^{ren} with respect to the renormalised field.

Flow of the renormalised boundary action from $\rho_0 = 0$ along the radial coordinate will give us an effective boundary theory at $r = \rho$ where we can interpret ρ as a quantity corresponding to a Wilsonian cut-off of the dual effective field theory. Initial theory at $\rho_0 = 0$ should therefore corresponds to a UV complete QFT (CFT) with its cut-off taken to infinity, without our running into problems such as a Landau pole.

Allowing for various terms to run in (8) we now use the definition (13) to write

$$S_B[\rho] = \alpha(\rho) + \int\limits_{r=\rho} d^dx\sqrt{-\gamma}\left[\frac{1}{2}J(r,x)\Phi - \sum_{n=2}^{\infty}\frac{1}{n}\lambda_n(r)\Phi^n - \frac{1}{2}\sum_{n=1}^{\infty}\kappa_n(r)\Phi\,\Box_\gamma^n\,\Phi\right]. \tag{15}$$

with higher order polynomial terms that may arise from potential $V(\Phi)$ terms in solving the RG equation (11). We also included a possible scalar term $\alpha(\rho)$. We may interpret α an effective cosmological constant, which will only affect the vacuum energy of the theory. No derivative terms with higher powers of Φ are expected as a result of our working with two-derivative scalar actions.

We solve (11) in momentum space by imposing (12) at each boundary and matching coefficients of terms with different powers of ϕ_0 []. Various coefficients must take their initial values at $\rho_0 = \rho_0$: $\alpha(\rho_0) = 0$, $J(\rho_0, k) = \frac{\Pi(\rho_0,k)}{\sqrt{-\gamma(\rho_0)}}$, $\lambda_2(\rho_0) = \Delta_-$, $\lambda_n(\rho_0) = 0$, for $n \geq 3$, and $\kappa_n(\rho_0) = c_n$. We can then take the $\rho_0 \to 0$ limit. It should be noted that because $J(r, k)$ involves derivatives of Φ it is important to Fourier transform it as a field and not simply treat it as a coupling constant like λ_n or κ_n. This is a consequence of the Dirichlet boundary condition (12) allowing for non-trivial derivatives of Φ with respect to the radial coordinate at different ρ. From initial condition at ρ_0 and the finiteness of $\Pi(\rho_0, k)$ it is clear that $\lim_{\rho_0\to 0} J(\rho_0, k)$ must vanish. It is natural to write $J(\rho, k)$ on the flowing boundary as $\frac{\Pi(\rho,k)}{\sqrt{-\gamma(\rho)}}$ thus directly determining the value of the subtracted canonical conjugate momentum $\Pi(\rho, k)$ at each step of the RG flow.

The RG flow equations are given by the following set of differential equation

$$\sqrt{g^{rr}}\partial_\rho\Pi = -2\Pi\left(\lambda_2 + \sum_{n=1}^{\infty}\kappa_n\left(-k_\mu k^\mu\right)^n\right), \tag{16}$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-\gamma}\lambda_2\right) = &-\lambda_2^2 + k_\mu k^\mu + m^2 + \frac{2}{\sqrt{-\gamma}}\Pi\lambda_3 - 2\lambda_2\sum_{n=1}^{\infty}\kappa_n\left(-k_\mu k^\mu\right)^n \\ &- \frac{1}{\sqrt{-g}}\sum_{n=1}^{\infty}\left[\partial_\rho\left(\sqrt{-\gamma}\kappa_n\right)\left(-k_\mu k^\mu\right)^n\right] - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\kappa_n\kappa_m\left(-k_\mu k^\mu\right)^n\left(-k_\mu k^\mu\right)^m\end{aligned} \tag{17}$$

$$\frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-\gamma}\lambda_n\right) = -\frac{n}{2}\sum_{\substack{m=2\\ n-m\geq 0}}^{\infty}\lambda_m\lambda_{n+2-m} + \frac{n}{\sqrt{-\gamma}}\Pi\lambda_{n+1} - n\lambda_n\sum_{m=1}^{\infty}\kappa_m\left(-k_\mu k^\mu\right)^m + b_n \tag{18}$$

and

$$\frac{1}{\sqrt{-g}}\partial_\rho\alpha = -\frac{1}{2\gamma}\int\frac{d^dk}{(2\pi)^d}\Pi(k)\Pi(-k). \tag{19}$$

All equations are written in terms of finite renormalised quantities so we can go to r=0

On the gauge theory side the running of λ_n and κ_n corresponds to various multi-trace operators that turn on in the effective Wilsonian action. The corresponding operators will be of form $\mathcal{O}^n$ and $\mathcal{O}(\partial^2)^n\mathcal{O}$. As a result of our restriction to two-derivative bulk theory there will be no terms like $\mathcal{O}^l(\partial)^m\mathcal{O}^n$, which can in principle exist in Wilsonian effective lagrangians and would come from a complete bulk supergravity.

III. TWO-POINT CORRELATION FUNCTIONS IN PURE ANTI-DE SITTER SPACE

To make a connection between the RG equations (16) - (19) coming from the bulk and the boundary physics we use the fact that a two-point correlation function is completely determined by Π. [][Iqbal,Liu] In the standard AdS/CFT prescription we look for a flux factor $\mathcal{F}(r,k)$, which gives a subtracted action

$$S_B^{\text{sub}} = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\phi_0(\rho_0,k)\mathcal{F}(\rho,k)\phi_0(\rho_0,-k)\Big|_{\rho=\rho_0}^{\rho=r_H}, \tag{20}$$

with r_H the position of the horizon, giving us the dual correlator. Boundary condition (12) fixes $\phi_0(\rho_0,k)$ along the flow. Equation (7) implies

$$\mathcal{F}(\rho,k) = -\frac{\Pi(\rho,k)}{\phi_0(\rho_0,k)}, \tag{21}$$

where $\phi_0(\rho_0,k)$ is the non-renormalised cut-off dependent boundary field. A two-point correlation function is then given by

$$G(k) \equiv \langle \mathcal{O}(k)\mathcal{O}(0)\rangle = -\mathcal{F}(\rho,k) \tag{22}$$

where the type of the two-point function (retarded, advanced, etc.) is determined by the boundary conditions imposed on Φ and consequently on Π. When removing the cut-off by taking $\rho_0 \to 0$, the canonical conjugate momentum Π is well defined and finite by construction of holographic renormalisation. However, this does not mean that the limit is well defined for correlation functions computed via standard AdS/CFT procedure in which we take functional derivatives of S_B^{sub}. An important subtlety in a complete removal of ρ_0 dependence from the correlation functions comes precisely from the fact that $\phi_0(\rho,k)$ is not renormalised. The wavefunction renormalisation must, as discussed in section II, be taken into account to obtain cut-off independent renormalised correlators.

Let us now for simplicity look at a case of pure AdS_{d+1} spacetime with a metric in Poincare coordinates

$$ds^2 = \frac{R^2}{r^2}\left(-dt^2 + d\vec{x}_{d-1}^2 + dr^2\right), \tag{23}$$

where we henceforth set $R = 1$. A scalar field solution near the boundary can be written as $\Phi = \rho_0^{\Delta_-}\phi_1 + ...$ To completely remove cut-off dependence we need to write, as in II, $\phi_0 = \rho_0^{\Delta_-}\phi_0^{\text{ren}}$, which gives

$$G^{\text{ren}}(k) = \frac{\delta}{\delta\phi_0^{\text{ren}}}\frac{\delta}{\delta\phi_0^{\text{ren}}}S_B^{\text{ren}} = \lim_{\rho_0\to 0}\rho_0^{2\Delta_-}\frac{\delta}{\delta\phi_0}\frac{\delta}{\delta\phi_0}S_B^{\text{sub}} = \lim_{\rho_0\to 0}\rho_0^{2\Delta_-}G(\rho_0,k) \tag{24}$$

at the initial cut-off.

In a calculation of a two-point function for a scalar field in pure AdS space we know that after all the divergent contact terms are removed, the two-point function has a $G(\rho_0,k) \sim \rho_0^{-2\Delta_-}$ dependence. This is exactly cancelled by the $\rho_0^{2\Delta_-}$ above coming from renormalised ϕ_0, making the renormalisation limit $\rho_0 \to 0$ possible. A combination of counter-terms and wavefunction renormalisation therefore, as in the usual quantum field theory, renders correlation functions finite and cut-off independent. In flowing into the bulk we can then write, using (14), $\phi_0 = \rho^{\Delta_-}Z^{1/2}(\rho)\phi_0^{\text{ren}}$, which gives

$$S_B^{\text{sub}} = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\phi_0^{\text{ren}}(k)\left[\rho^{2\Delta_-}Z(\rho)\mathcal{F}(\rho,k)\right]\phi_0^{\text{ren}}(-k)\Big|_{\rho}^{r_H}. \tag{25}$$

We can define a renormalised flux factor

$$\mathcal{F}^{\text{ren}}(\rho,k) \equiv \rho^{2\Delta_-}Z(\rho)\mathcal{F}(\rho,k), \tag{26}$$

which equivalently to equation (22) gives a renormalised two-point correlation function in the dual QFT

$$G^{\text{ren}}(k) \equiv \langle \mathcal{O}(k)\mathcal{O}(0)\rangle^{\text{ren}} = -\mathcal{F}^{\text{ren}}(\rho,k). \tag{27}$$

To see how wavefunction renormalisation fits into our renormalisation group picture we solve equation (16) in AdS_{d+1} with all $\kappa_n = 0$. This is allowed when we are interested in the IR behaviour of correlation functions and because the series of counter-terms naturally stops at some c_n, depending on the operator dimension $\Delta_{\mathcal{O}}$ [][Skenderis].

We know that λ_2 can be written, using the initial conditions, as $\lambda_2(\rho) = \Delta_- + \gamma(\rho)$, where $\gamma(\rho_0) = 0$. We first rewrite (16) in a more suggestive way as

$$\left(\rho\frac{\partial}{\partial\rho} + 2\Delta_- + 2\gamma(\rho)\right)\Pi(\rho) = 0. \tag{28}$$

Its solution with some initial $\Pi(\rho_0)$ is

$$\begin{aligned}\Pi(\rho) &= \Pi(\rho_0)\exp\left\{-2\int_{r=\rho_0}^{r=\rho}(\Delta_- + \gamma(r))\,d\ln r\right\}\\ &= \Pi(\rho_0)\left(\frac{\rho_0}{\rho}\right)^{2\Delta_-}\exp\left\{-2\int_{r=\rho_0}^{r=\rho}\gamma(r)d\ln r\right\}.\end{aligned} \tag{29}$$

Using (21), (22) and the fact that the non-renormalised $\phi_0(k)$ is independent of ρ at each boundary, as specified by the Dirichlet boundary condition (12), it is clear that

$$\frac{d}{d\rho}\left(\frac{\Pi(\rho,k)}{\phi_0(\rho_0,k)}\right) = \frac{1}{\phi_0(\rho_0,k)}\frac{d}{d\rho}\Pi(\rho,k). \tag{30}$$

This implies that the same renormalisation group equation describes both the flow of Π and the non-renormalised two-point function. Its solution is similarly

$$G(\rho,k) = G(\rho_0,k)\left(\frac{\rho_0}{\rho}\right)^{2\Delta_-}\exp\left\{-2\int_{r=\rho_0}^{r=\rho}\gamma(r)d\ln r\right\}. \tag{31}$$

Now using (24) and the fact that $G(\rho_0,k) \sim \rho_0^{-2\Delta_-}$ we see that cut-off dependence will be replaced by dependence on position of the d-dimensional brane on which the dual QFT is defined. This corresponds to dependence on the physical scale Λ, which is a function of ρ. The new ρ dependence is consistent with original ρ_0 dependence at initial $\rho_0 = \rho_0$. In the limit $\rho_0 \to 0$ therefore

$$G^{\mathrm{scl}}(\rho,k) = G_0^{\mathrm{scl}}(k)\exp\left\{-2\int_{r=0}^{r=\rho}\gamma(r)d\ln r\right\}, \tag{32}$$

where we define a scale-ρ dependent correlator, $G^{\mathrm{scl}}(\rho,k) \equiv G(\rho,k)\rho^{2\Delta_-}$. However G_0^{scl}, despite being independent of ρ^{Δ_-}, cannot be the renormalised two-point function on the QFT side because of its scale/Wilsonian cut-off dependence.

To clarify this point let us look for a differential equation describing G^{scl}. Given that $G(k)$ must satisfy the same RG equation as Π, we can use $G(\rho,k) = \rho^{-2\Delta_-}G^{\mathrm{scl}}(\rho,k)$ to obtain

$$\left(\rho\frac{\partial}{\partial\rho} + 2\gamma(\rho)\right)G^{\mathrm{scl}}(\rho,k) = 0. \tag{33}$$

This is precisely the Callan-Symanzik equation we would expect from Wilsonian renormalisation group procedure for a bare two-point function with an anomalous operator dimension $\gamma(\rho)$ when G^{scl} contained no explicit dependence on coupling constants in the effective action. Position of the brane ρ plays the role of the running Wilsonian cut-off scale. Its solution is (32). All coupling dependence is either written explicitly in terms of ρ or the anomalous dimension. This is not unexpected as we are deriving running couplings on the QFT side directly from the bulk physics. No perturbative treatment of correlation functions in terms of the coupling is necessary. It is also a well known fact that a beta function for a double-trace coupling in large-N QFT is directly determined by the anomalous dimension of the operator [].

We now wish to see how physical renormalised two-point functions behave in this setup. Using the flow of Π we write

$$\frac{1}{\rho^{2\Delta_-}Z(\rho)}\rho\frac{\partial}{\partial\rho}\left[\rho^{2\Delta_-}Z(\rho)G(\rho,k)\right] = \left(\rho\frac{\partial}{\partial\rho} + 2\Delta_- + 2\gamma(\rho)\right)G(\rho,k) = 0, \tag{34}$$

where, as usual, $\gamma(\rho) \equiv \frac{1}{2}\frac{d\ln Z}{d\ln\rho}$. Note that in our case $Z(\rho)$ is expressed explicitly as a function of ρ so partial derivative with respect to ρ is the same as a total derivative. Finally using (26) and (27), equation (34) implies that for a renormalised two-point function, also expressed explicitly as a function of ρ,

$$\frac{\partial}{\partial\rho}G^{\mathrm{ren}}(k) = 0. \tag{35}$$

It is therefore, as required, independent of the renormalisation procedure and the running Wilsonian cut-off.
sectionTwo-point correlation functions in asymptotically Anti-de Sitter spaces at non-zero temperature

We now generalise the discussion from section III to asymptotically Anti-de Sitter spaces with holographic duals at finite temperature. We will focus on brane geometries with the following form of Poincare coordinate metrics and AdS infinity at $r = 0$:

$$ds^2_{d+1} = g_{tt}(r)dt^2 + g_{ij}(r)dx_i dx_j + g_{rr}(r)dr^2, \tag{36}$$

so that g_{tt} and g_{rr} include a thermal factor $f(r)$. As a specific example we first consider a near-extremal D3 brane in 10 dimensions

$$ds^2 = \frac{r_0^4}{R^2}\frac{1}{r^2}\left(-f(r)dt^2 + d\vec{x}^2\right) + \frac{R^2 r_0^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2, \text{ with } f(r) = 1 - \frac{r^4}{r_0^4}. \tag{37}$$

We set the AdS radius to $R = 1$ and neglect the spherical $d\Omega_5$ space so that we are left with a $d+1 = 5$ dimensional spacetime. Horizon of the black brane is at r_0. We are interested in behaviour of two-point Green's functions dual to a scalar field propagating in this geometry. Setting as before in low momentum regime all κ_n to zero and writing $\lambda_2 = \Delta_- + \gamma$, equation (16) becomes

$$\left(\rho\frac{\partial}{\partial\rho} + 2\frac{\Delta_- + \gamma_2(\rho)}{\sqrt{f(\rho)}}\right)\Pi(\rho, k) = 0. \tag{38}$$

A new feature, compared to pure AdS, is the appearance of the thermal factor in the anomalous dimension term. To see that this makes sense we analyse operator dimensions by solving the equation of motion of a massive scalar field in thermal AdS background. [] The equation of motion is $\frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r\Phi\right) + g^{\mu\nu}\partial_\mu\partial_\nu\Phi - m^2\Phi = 0$. We need to find solutions near the AdS boundary ($r \ll 0$). In pure AdS spacetime we assume a power-series expansion of Φ which gives us Bessel function solutions. Near boundary the leading term proportional to r^Δ gives a relation $\Delta(\Delta - d) = m^2$ where in our Dirichlet (standard) quantisation we take $d - \Delta \equiv \Delta_+$ to be the dimension of the dual operator $\mathcal{O}$ to Φ. The other, smaller, solution is our $\Delta_- \equiv \Delta$.

If we use the D3 thermal metric (37) and assume as before that the leading term is r^{Δ_-}, where Δ_- is independent of r we get the leading order condition $\Delta(\Delta - d)f(r) = m^2$. Now of course in the limit $r \to 0$, $f(r) \to 1$ so dimension of the dual operator at AdS infinity stays independent of the thermal factor. But this identity suggests that thermal factor will play a role in the scaling dimension of operators as we flow into the bulk towards the horizon. Assuming instead that $\Phi \sim r^{\Delta_-/\sqrt{f(r)}}$ for small, but non-zero r, we have $\partial_r\Phi = \frac{\Delta_-}{\sqrt{f}}r^{\Delta_-/\sqrt{f}-1} - \frac{\Delta_- \ln r \partial_r f}{2f^{3/2}}r^{\Delta_-/\sqrt{f}}$. The value of initial r is in fact $0 \neq \rho_0 \ll 1$, which is consistent with this discussion. Using $\partial_r f \sim r^3$ then the two terms scale as $r^{\Delta_- - 1}$ and $r^{\Delta_- + 3}\ln r$, respectively, showing that first term remains the leading contribution. We can therefore neglect the second term coming from the differentiation of the thermal factor near the boundary. Hence dimensions $\Delta_\pm$ rescaled to $\tilde{\Delta}_\pm = \frac{\Delta_\pm}{\sqrt{f(r)}}$ for $r \ll 1$ give the original $\tilde{\Delta}_-\tilde{\Delta}_+ = -m^2$ in the presence of non-zero temperature. It is therefore not surprising that anomalous operator dimension will receive the same thermal correction.

Using analysis of section III, we can remove the $\Delta_-/\sqrt{f}$ term from equation (38) by interpreting it as initial cut-off dependence term. We write the thermal field as

$$\Phi(\rho, k) \equiv \rho^{\Delta_-/\sqrt{f(\rho)}} Z_T^{1/2}(\rho, k)\Phi^{\text{ren}}(\rho, k). \tag{39}$$

and the new anomalous dimension with a thermal factor as $\frac{\gamma(\rho)}{\sqrt{f(\rho)}} \equiv \frac{1}{2}\frac{d\ln Z_T}{d\ln\rho}$. The scale dependent two-point correlator becomes

$$G_T^{\text{scl}}(\rho, k) = G_{0T}^{\text{scl}}(k)\exp\left\{-2\int_{r=0}^{r=\rho}\frac{\gamma(r)}{\sqrt{f(r)}}d\ln r\right\} \tag{40}$$

whereas the fully renormalised thermal correlator $G_T^{\text{ren}}(k)$ remains, as required, independent of the Wilsonian cut-off scale ρ.

This discussion can easily be generalised to Dp branes as well as M2 and M5 brane backgrounds. Specific solutions of the anomalous dimension and beta functions will of course vary depending on the specific metric.

IV. HYDRODYNAMIC TRANSPORT COEFFICIENTS

We now turn our attention to hydrodynamics and attempt to see what can be learnt about transport coefficients using our insights from the connection between Wilsonian and holographic renormalisation groups. A physical renormalised hydrodynamic transport coefficient χ is defined using Kubo formula as

$$\chi = -\lim_{\omega\to 0}\lim_{\vec{k}\to 0}\frac{1}{\omega}\mathrm{Im}G_{\mathrm{T}}^{\mathrm{ren,R}}(\omega,\vec{k}). \tag{41}$$

As the simplest example we can study the shear viscosity of a holographic CFT. The relevant correlator to be used in (41) is the retarded energy-momentum $\langle[T_{xy},T_{xy}]\rangle$. Its dual bulk excitation is the h_{xy} component of a massless graviton field, which satisfies the massless scalar field equation in the bulk. [] This enables us to directly apply the analysis from section II. The dimension of T is $\Delta = d$, which implies that $\Delta_- = 0$. As a result a new term proportional to $\ln\rho_0\Phi\,\Box\,\Phi$ enters into the counter-term action. The term vanishes in the zero-momentum limit faster then the logarithm blows up as $\rho_0\to 0$. It is also important that we take the limit of vanishing momentum before removing cut-off ρ_0. We can therefore neglect the term, leaving us with a simple RG equation for anomalous dimension

$$\frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-\gamma}\lambda_2\right) = -\lambda_2^2, \tag{42}$$

with initial condition $\lambda_2(\rho_0) = 0$, coming from the vanishing Δ_-.

Scale dependent running of a bare, non-renormalised, transport coefficient χ is

$$\chi(\rho) = -\lim_{\omega\to 0}\lim_{\vec{k}\to 0}\frac{1}{\omega}\mathrm{Im}\left[G_{0\mathrm{T}}^{\mathrm{scl,R}}(\omega,\vec{k})\exp\left\{-2\int_{r=0}^{r=\rho}\frac{\gamma(r,\omega,\vec{k})}{\sqrt{f(r)}}d\ln r\right\}\right], \tag{43}$$

where the fact that we are using a retarded two-point function will result in imposing the appropriate infalling boundary condition on $\Pi(\rho,k)$.

Entropy density is $s = -V\partial_T F$. Bare free energy on the moving brane can be viewed as scale dependent in the same way $\chi(\rho)$ is.

V. DOUBLE TRACE DEFORMATIONS

In this section we study the renormalisation group equations describing the radial evolution of anomalous operator dimension induced by effective double-trace deformations. We may safely neglect all higher-trace deformations in our study of large-N flows, as all such deformations are sub-dominant in the large-N limit [11]. Another reason is that by working in the standard (Dirichlet) quantisation, the smallest possible operator dimension is $\Delta_{\mathcal{O}} = d/2$ implying that any triple-or higher-order operator would be irrelevant. Even in the alternative (Neumann) quantisation with a scalar operator obeying the unitarity bound, the smallest dimension would be $\Delta_{\mathcal{O}} = d/2 - 1$. Such an operator could only have a marginal triple-trace deformation in $d = 6$ dimensions. The relevant renormalisation group equation is then given by

$$\frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-\gamma}\lambda_2\right) = -\lambda_2^2 - \left(\frac{d}{2} - \nu\right)\left(\frac{d}{2} + \nu\right) + k_\mu k^\mu, \tag{44}$$

which we will analyse in various asymptotically AdS backgrounds.

A. Duality in pure AdS_{d+1} spacetime

As for our first example let us find the anomalous dimension of a scalar operator dual to a massive scalar in pure AdS_{d+1} with the metric in Poincare coordinates given by

$$ds^2 = \frac{L^2}{r^2}\left(-dt^2 + d\vec{x}_{d-1}^2 + dr^2\right). \tag{45}$$

We set the AdS radius to $L = 1$ throughout this work. Writing $\lambda_2(\rho) = \Delta_- + \eta(\rho)$, equation (44) for the flow of η becomes

$$\rho\frac{\partial\eta}{\partial\rho} = -\eta\left(\eta - 2\nu\right) + k^2\rho^2. \tag{46}$$

From our construction of holographic renormalisation we know that at AdS infinity η must vanish, $\eta(0) = 0$, so that $\Delta_-\Phi^2$ term can cancel the bare divergence. All quantities appearing in renormalisation group equations (16) and (44) are finite in the limit of $\rho_0 \to 0$. The starting point of all RG flows will therefore be at $\rho = 0$.

We will impose two conditions on the behaviour of η: firstly that η be monotonically increasing as ρ runs into the bulk, and secondly that η, as well as λ_2, be real and non-singular throughout the flow. This means that $\frac{\partial\eta}{\partial\rho} \geq 0$, which immediately implies the positivity $\eta \geq 0$ for all ρ. These conditions are, as we will see, very natural for a well-behaved flow of anomalous dimension between two fixed points of the theory. In particular, flowing to a complex operator dimension would imply an unstable theory. Such dynamical symmetry breaking would inevitably break conformal symmetry [11].

For timelike physical momenta, $k^2 < 0$, we have $k^2\rho^2 \leq 0$. This in combination with the above conditions consequently imposes a strict constraint on the allowed values of η. Since $\rho\frac{\partial\eta}{\partial\rho} \geq 0$, the only way for the right-hand side of (46) to be non-negative is if $\eta - 2\nu$ remains sufficiently negative while ρ increases. To see this note that at $\rho = 0$, $\eta(\eta - 2\nu) = 0$ as well as $k^2\rho^2 = 0$. Now as ρ increases the first term $-\eta(\eta - 2\nu) \geq 0$ grows larger until $\eta > 2\nu$. At that point the overall sign of the first term flips and becomes negative. The second term however decreases monotonically into negative values and may quickly begin to dominate over the first term, running the entire righ-hand-side of (46) into negative values before $\eta = 2\nu$. Function $\eta(\rho)$ therefore reaches its maximal value $\eta_\times$ at some $\rho_\times$ when $-\eta_\times\left(\eta_\times - 2\nu\right) + k^2\rho_\times^2 = 0$. The two possible solutions, $\eta_\times = \nu \pm \sqrt{\nu^2 + k^2\rho_\times^2}$, can only be real if $\sqrt{-k^2}\rho_\times \leq \nu$. But given that we seek maximal $\rho_\times$ this inequality implies that $\rho_\times = \nu/\sqrt{-k^2}$. The largest possible value of η, given some timelike momentum and ν, is therefore $\eta_\times = \nu$ irrespective of our choice of solution. Despite this, the correct solution would be the $(-)$ one because at $\rho_\times = 0$, required when $-k^2 \to \infty$, $\eta_\times$ should vanish for it to be consistent with the initial condition. Only in this case can η and its derivative be continuous and non-singular. Furthermore, since the right-hand-side of (46) vanishes at $\rho_\times > 0$, we clearly have a maximum $\frac{\partial\eta_\times}{\partial\rho} = 0$ for any timelike operator momentum at the point where the RG flow terminates. Note that we did not impose the vanishing derivative in our conditions. The solution of equation (46) can be written in terms of Bessel functions as

$$\eta(\rho) = \sqrt{-k^2}\rho\frac{Y_{\nu-1}(\nu)J_{\nu-1}(\sqrt{-k^2}\rho) - J_{\nu-1}(\nu)Y_{\nu-1}(\sqrt{-k^2}\rho) - Y_\nu(\nu)J_{\nu-1}(\sqrt{-k^2}\rho) + J_\nu(\nu)Y_{\nu-1}(\sqrt{-k^2}\rho)}{Y_{\nu-1}(\nu)J_\nu(\sqrt{-k^2}\rho) - J_{\nu-1}(\nu)Y_\nu(\sqrt{-k^2}\rho) - Y_\nu(\nu)J_\nu(\sqrt{-k^2}\rho) + J_\nu(\nu)Y_\nu(\sqrt{-k^2}\rho)}, \tag{47}$$

with range of $0 \leq \rho \leq \frac{\nu}{\sqrt{-k^2}}$. Note that this solution is well defined for all real $\nu > 0$, above the Breitenlohner-Freedman bound [12]. For non-integer values of ν, (47) can be simplified to give

$$\eta(\rho) = \sqrt{-k^2}\rho\frac{\left[J_{-\nu-1}(\nu) + J_{-\nu}(\nu)\right] J_{\nu-1}(\sqrt{-k^2}\rho) + \left[J_{\nu-1}(\nu) - J_{\nu}(\nu)\right] J_{1-\nu}(\sqrt{-k^2}\rho)}{\left[J_{-\nu-1}(\nu) + J_{-\nu}(\nu)\right] J_{\nu}(\sqrt{-k^2}\rho) + \left[J_{\nu}(\nu) - J_{\nu-1}(\nu)\right] J_{-\nu}(\sqrt{-k^2}\rho)}. \tag{48}$$

Conditions which allowed us to find a solution to the RG flow of η also give a clear interpretation of the induced momentum cut-off. If we reverse the argument and insist on integrating out geometry between $0 \leq \rho \leq \rho_\times$, then there is a limited interval of timelike momenta that operators can take after integration. The relation is $\sqrt{-k^2} \leq \nu/\rho_\times$, which implies the presence of a hard momentum cut-off on the brane side of holographic duality, induced by the sliding brane. In fact this is exactly analogous to a hard Wilsonian UV cut-off with a Lorentzian signature $\sqrt{-k^2} \leq \Lambda$, defining the energy scale up to which an effective field theory is valid. The situation is somewhat different from the Euclidean field theory case, since we are effectively integrating out energy-momentum regions above floating hyperbolae in a light-cone diagram, down to asymptotically lightlike momenta. An exact $k^2 = 0$ can nevertheless not be reached and we need to treat that case separately. The same inequality holds for any chosen momentum scale k^2 of an operator as well as any chosen scale $\rho_\times$ where we decide to terminate the integration. We can therefore find an exact correspondence between parameters describing the bulk physics and their dual Wilsonian UV cut-off $\Lambda(\rho, d, m, ...)$. It is also important for this identification that the operator $\rho\partial/\partial\rho$ is invariant under $\rho \to a\rho$, for constant a. Hence the boundary energy scale is $\Lambda = \nu/\rho$. $\Lambda_{\text{min}} = \nu/\rho_\times$ is then the lowest possible scale down to which we can integrate from $\Lambda \to \infty$, given some momentum k at which we wish to evaluate the operator. It is important to note that the constant of proportionality ν is merely a result of bulk coordinates we used in (45) to establish the dictionary between that particular bulk space and its boundary dual. We could easily redefine $r \to \nu r$ to give us a metric

$$ds^2 = \frac{1}{\nu^2 r^2}\left(-dt^2 + d\vec{x}^2_{d-1}\right) + \frac{dr^2}{r^2}. \tag{49}$$

In this background we obtain $\sqrt{-k^2}\rho_\times \leq 1$ and hence $\Lambda = 1/\rho$. For a general Poincare-like AdS chart we can therefore conclude that the Wilsonian energy scale, and cut-off, of a boundary theory is related to the radial bulk coordinate by

$$\Lambda = \frac{C(r, d, m, ...)}{r}, \tag{50}$$

where C is a constant which depends on the bulk quantities describing the background metric and can be found exactly following the above procedure. Our analysis is thus consistent with the long anticipated relationship $\Lambda \propto 1/r$ [4, 5]. In addition, it also uniquely determines the proportionality constant for a given pair of holographically dual theories.

At the Breitenlohner-Freedman bound with $\nu = 0$, and for arbitrary momentum, the RG flow analysis breaks down unless $\rho_\times = 0$. This is also apparent from the metric (49) which is singular at $\nu = 0$. The only option to have an RG flow compatible with such an operator at the UV fixed point is when $\mathcal{O}$ has lightlike on-shell momentum $k^2 = 0$. The solution is still $\eta(\rho) = 0$ and the anomalous dimension does not run, but it is well defined for all ρ.

In a lightlike case with $k^2 = 0$, equation (46) drastically simplifies for all operator dimensions. To satisfy the monotonicity condition, the anomalous dimension must behave as $\eta_\times \to 2\nu$ when $\rho_\times \to \infty$. Solution of equation (46) that satisfies the required conditions is then

$$\eta(\rho) = \frac{2\nu\chi\rho^{2\nu}}{1 + \chi\rho^{2\nu}}, \tag{51}$$

as previously found by [7, 8]. Constant χ cannot be determined from the boundary conditions we set, but needs to be matched with the normalisation of the corresponding two-point correlator.

For spacelike momenta $k^2 > 0$ the energy scale becomes pure imaginary. Having found that radial coordinate is proportional to the energy of the dual theory, it is therefore natural to take $r \to ir$. The analysis of (46) then goes through in exactly the same way as for timelike momenta. We obtain $k^2 \leq \Lambda_{\text{min}} = \nu/\rho_\times$ and $\Lambda = C/r$.

Anomalous dimension is momentum dependent, where highest energy modes have always the same anomalous dimenision. Theory is truly conformal as it looks the same at all scales. Conformal, since we can connect the vanishing beta function to the vanishing derivative of anomalous dimenions, as has to be made clear in previous section.

B. GPPZ flow from $\mathcal{N}=4$ to $\mathcal{N}=1$

The GPPZ flow [9] describes a flow from an $\mathcal{N}=4$ theory in the UV, to an $\mathcal{N}=1$ in the IR. This is achieved by deforming the $\mathcal{N}=4$ theory with a relevant mass deformation. The theory then flows between two fixed points as shown in [9, 10]. The bulk supergravity for this construction consists of type IIB scalar modes deforming the original $AdS_5 \times S^5$ metric. This 10-dimensional type IIB theory is then truncated on S^5 to give a 5-dimensional $\mathcal{N}=8$ supergravity with 42 scalars. The scalars transform as **1**, **20** and **10** under the $\mathcal{N}=4$, $SU(4)_R$, R-symmetry. The masses of these fields are $m^2=0$, -4 and -3, respectively. The GPPZ flow then describes the metric deformation resulting from the backreaction with scalars of $m^2=-3$. This corresponds to a dual deformation of dimension 3 by scalar operator in 4 spacetime dimensions, which can be identified as a fermion bi-linear operator with coupling constant of mass dimension 1 [9]. We further truncate the theory to only account for the large-N-dominant effective double-trace deformations and consider the background as static. We also set the supergravity scalar coupling constant to a value which cancels the scalar potential contribution $V=-3$ at $\phi=0$. This results in the same scalar theory we have been considering so far with mass $m^2=-3$ and a 5-dimensional metric

$$ds^2 = \frac{1}{r^2}\left(1-\frac{r^2}{r_0^2}\right)\left(-dt^2+d\vec{x}^2\right)+\frac{dr^2}{r^2}. \tag{52}$$

By r_0 we denote the radial position where the flow terminates at the IR $\mathcal{N}=1$ fixed point. Note that this metric is simply obtained from its original form in [9] by defining $y=-\ln r$ and $\lambda=+\ln r_0$.

We can now analyse the renormalisation group flow of a scalar operator deformed by both effective double-trace and a relevant mass deformation. Mass deformation being provided by the GPPZ flow of the supergravity metric through its r dependence and double-trace deformation being ensured by the quadratic term in the bulk scalar potential. Writing as before $\lambda_2=2-\nu+\eta$, the renormalisation group equation (44) for η in background (52) becomes

$$\rho\left(1-\frac{\rho^2}{r_0^2}\right)\frac{\partial\eta}{\partial\rho}=-\left(1-\frac{\rho^2}{r_0^2}\right)\eta\left(\eta-2\nu\right)+\frac{4\rho^2}{r_0^2}\left(2-\nu+\eta\right)+\rho^2k^2, \tag{53}$$

with $\nu=1$ in the GPPZ flow. We may however keep ν, as well as m, general for now, restricting it only to relevant and marginal deformations with $\nu\le 2$ so that $(2-\nu+\eta)\ge 0$. We use the same reasoning as in the pure AdS case in section V A to determine where the flow terminates given some timelike momentum $k^2<0$. Given that the pure AdS scenario corresponds to an undeformed conformal $\mathcal{N}=4$ theory it is especially interesting to compare its RG flow with the RG flow in this section. For a monotonically increasing, positive and non-singular η, the only term which can run the right hand side of (53) into negative values for $\eta<2\nu$ is ρ^2k^2. As before, to find the maximal $\eta_\times$ at $\rho_\times$, the right-hand side of (53) has to vanish. The two solutions are

$$\eta_\times=\nu+\frac{2\rho_\times^2}{r_0^2-\rho_\times^2}\pm\frac{r_0^2}{r_0^2-\rho_\times^2}\sqrt{\nu^2\left(1-\frac{\rho_\times^2}{r_0^2}\right)^2+\rho_\times^2k^2\left(1-\frac{\rho_\times^2}{r_0^2}\right)+8\frac{\rho_\times^2}{r_0^2}\left(1-\frac{\rho_\times^2}{r_0^2}\right)+4\frac{\rho_\times^4}{r_0^4}}. \tag{54}$$

For $\eta_\times$ to be real when $\rho_\times$ is maximal, the expression under the square-root needs to be non-negative. The term $\sim\rho_\times^2k^2<0$ will however cause the expression to inevitably flow towards 0. We could at this point impose a condition whereby the entire expression under the square-root need be non-negative. This is however insufficient in case $\rho_\times=r_0$, when despite the vanishing square-root, the second term in $\eta_\times$, $2\rho_\times^2/\left(r_0^2-\rho_\times^2\right)$, blows up, thus violating the non-singularity condition we imposed on λ_2. To remedy this problem we, as before, select the $(-)$ solution and impose $\nu^2\left(1-\frac{\rho_\times^2}{r_0^2}\right)+\rho_\times^2k^2+8\frac{\rho_\times^2}{r_0^2}\ge 0$, which at maximal $\rho_\times$ turns into an equality and enables the $4\rho_\times^4/r_0^4$ term under the square-root to cancel the diverging second term in $\eta_\times$ as $\rho_\times\to r_0$. Rewriting the inequality we can determine the allowed momentum range under the scale set by some maximum ρ or equivalently $\Lambda_{\min}$. In other words we can determine the range of consecutive momentum shell integrations given some operator momentum

$$-k^2\le\frac{\nu^2}{\rho^2}+\frac{4-m^2}{r_0^2}, \tag{55}$$

where $4-m^2=8-\nu^2$ and in the case of GPPZ flow $-k^2\le 1/\rho^2+7/r_0^2$. In the limit of $\rho\to r_0$, $\sqrt{-k^2}\le 2\sqrt{2}/r_0$.

C. Black brane in AdS_{d+1}

The metric of a black brane in a $d+1$-dimensional AdS space is given by

$$ds^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + d\vec{x}_{d-1}^2 + \frac{dr^2}{f(r)}\right), \text{ with } f(r) = 1 - \frac{r^d}{r_0^d}. \tag{56}$$

It is convenient to write $\lambda_2(\rho) = \sqrt{f(\rho)}\left(\frac{d}{2} - \nu + \eta(\rho)\right)$. Note that the condition $\lambda_2(0) = \Delta_-$ is still satisfied, since $f(0) = 1$ and we, as before, require that $\eta(0) = 0$. Renormalisation group flow equation (44) then becomes

$$f\rho\frac{\partial\eta}{\partial\rho} = -\eta\left(\eta - 2\nu\right) + \frac{\rho^d}{r_0^d}\left(\frac{d}{2} - \nu + \eta\right)^2 - \frac{\rho^2\omega^2}{f} + \rho^2\vec{k}^2. \tag{57}$$

Although it may be very difficult to find an analytic solution to (57) for all energies and momenta, we can still extract a relationship between Wilsonian cut-off and the bulk. Using the same monotonicity and positivity conditions as in V A we obtain

$$\eta(\rho_{\text{UV}}) = \nu + \frac{d}{2}\frac{\rho^d}{r_0^d - \rho^d} - \frac{r_0^d}{r_0^d - \rho^d}\sqrt{\frac{d^2}{4}\frac{\rho^d}{r_0^d} + \nu^2\left(1 - \frac{\rho^d}{r_0^d}\right) - \rho^2\omega^2 + \rho^2\vec{k}^2\left(1 - \frac{\rho^d}{r_0^d}\right)} \tag{58}$$

Impose

$$\frac{d^2}{4}\frac{\rho^d}{r_0^d} + \nu^2\left(1 - \frac{\rho^d}{r_0^d}\right) - \rho^2\omega^2 + \rho^2\vec{k}^2\left(1 - \frac{\rho^d}{r_0^d}\right) \geq \frac{d^2}{4}\frac{\rho^{2d}}{r_0^{2d}} \tag{59}$$

reduces to

$$\frac{\omega^2}{f(\rho)} - \vec{k}^2 \leq \frac{\nu^2}{\rho^2} + \frac{d^2}{4}\frac{\rho^{d-2}}{r_0^2} = \frac{\nu^2}{\rho^2}\left(1 + \frac{\rho^d}{r_0^d}\right) - \frac{m^2}{\rho^2}\frac{\rho^d}{r_0^d} \tag{60}$$

VI. SUMMARY

• Find all holographic counter-terms invariant under the diffeomorphism group of bulk isometries needed to renormalise the boundary action at AdS infinity, $\rho_0 \to 0$.

• Rewrite the bare boundary action in terms of the renormalised boundary action and a complete set of holographic counter-terms. Holographic counter-terms in the bare action directly correspond to additional Wilsonian terms in the effective QFT action coming from integrating out high-momentum modes.

• Holographic counter-terms are invariant under $d+1$-dimensional bulk isometry group. Therefore all terms in Wilsonian d-dimensional effective action also transform under the same group. This is consistent with the AdS/CFT dictionary, which equates the group of bulk isometries to the group of symmetries of the dual boundary quantum field theory.

• Use the Hamiltonian evolution equation, with radial direction treated as time, to derive the renormalisation group equations for couplings and anomalous dimensions of the Wilsonian effective action.

• For a conformal field theory flowing between two fixed points find the value of the radial coordinate ρ where, in dependence of physical momentum, the evolution terminates. From this value, we can extract the exact relation between a hard Wilsonian cut-off $\sqrt{-k^2} \leq \Lambda$ and quantities describing the bulk physics $\Lambda(\rho, d, m, ...)$. Λ is as expected proportional to $1/\rho$ in Poincare coordinates.

VII. NOTES

We have proven that monotonically increasing η automatically implies a flow between two fixed points.

All theories we study are conformal in the single-trace sector. This is controlled by the large-N limit.Quantum corrections to mass in the bulk would give single-trace running. $\nu(\rho) \Rightarrow m(\rho)$

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