Resolution of the Complex Plane

Null Algebra Extension II

Introduction to Null Mathematics of Trigonometry

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It is assumed the reader has read and understood both Null Algebra and Null Algebra Extension I. These texts are available for download at (https://vixra.org/abs/2103.0131) and at (https://vixra.org/abs/2206.0135). If you have not yet read these texts and attempted the examples contained therein for yourself it is highly suggested you do so before reading further as some concepts explained in detail there, are given only cursory review here. Without reading these prerequisites you may not fully understand the reasoning behind logic used in the equations of this text.

Section 1.1—Introduction to Complex Numbers:

What are the form of complex numbers? They are commonly transcribed in the following general ways on the Cartesian style Complex Plane.

Fig. 1.

$z = a \pm bi$	This is the standard form of a complex number.
$y = \sqrt{-n} = \pm bi$	The root of a negative number is a value, positive or negative, having no real part under traditional mathematics.
$y = a \pm \sqrt{-n} = a \pm bi$	Another example of a complex number formed of a real part and a root of a negative number.

Complex numbers are marked on the Complex Plane. Figure 2 shows the traditional layout of the Complex Plane, on which Complex numbers are marked out with a real and imaginary part. The Image of Figure 2 assumes the real part is the *x*-axis.

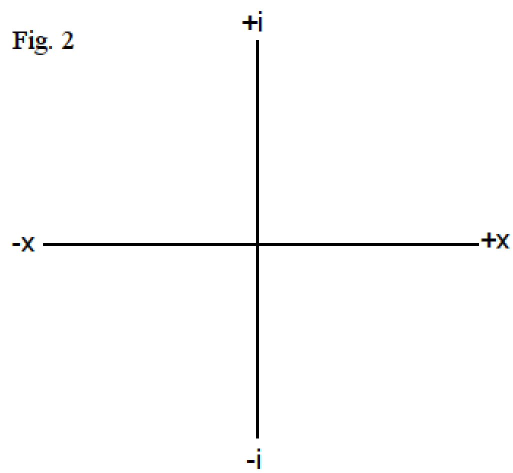


Fig. 2:

1.1.a

$z = a + bi \qquad z^* = a - bi$

We say z and z^* are complex conjugates of each other. As such squaring a complex number means to multiply it by its complex conjugate. This has the effect of always being both real and positive as a product.

1.1.b:

$$z^2 = z \cdot z^* = (a + bi)(a - bi) = a^2 + b^2$$

For the instance that $a = 0$ and $b = 1$we have:

1.1.c:

$$z^2 = z \cdot z^* = (i) \cdot (-i) = -i^2 = 1$$

Where: $z = 0 + 1 \cdot i \qquad z^* \; 0 - 1 \cdot i$

Thus 1.1.c is the very definition of the Complex Number *i* from Traditional Algebra. Namely—

<u>1.1.d:</u>

If $-i^2 = 1$ then $i^2 = -1$

1.1.e

> If the Square of a Complex Number z^2 is the multiplication of the complex number z by its complex conjugate, such that $z^2 = z \cdot z^*$, and for $z = a + bi$ defined by a = 0, b = 1 (*The complex conjugate will use b = -1*)
>
> Then $z \cdot z^*$ = (*i*)(-*i*) = $-i^2 = 1$.
>
> i must be such that $i^2 = -1$ and $i = \sqrt{-1}$

<u>1.2—Functions of Complex numbers:</u>

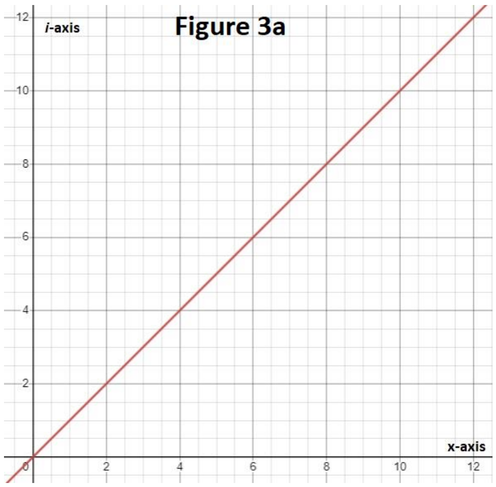
Although the Complex Plane is drawn as a standard Cartesian Plane it ignores the format of a function which when graphed has an output axis for an output variable. Thus graphs on the complex plane having labels for the *x* and *i* axis contain plots of two-dimensional output values for *y*. Consider the following equation with listed point values:

$y = 2x + xi$

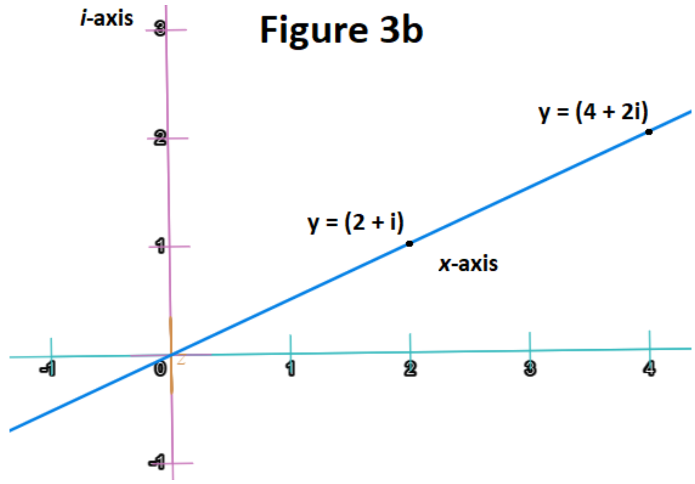
x-value	Real Part 2x	*y*-value	*Resolved y*	*i*-value	Resolved *y*-conjugate	Conjugate *y*-value	Conjugate *i*-value
1	2	2+*i*	3	*i*	1	2-*i*	-*i*
2	4	4+2*i*	6	2*i*	2	4-2*i*	-2*i*
3	6	6+3*i*	9	3*i*	3	6-3*i*	-3*i*
4	8	8+4*i*	12	4*i*	4	8-4*i*	-4*i*
5	10	10+5*i*	15	5*i*	5	10-5*i*	-5*i*
6	12	12+6*i*	18	6*i*	6	12-6*i*	-6*i*

For simplicity only the positive conjugate halves are shown below. There are two ways we could visualize the plotting of these points within traditional algebra: 1) plotting on the *xi*-plane are the actual values for *x* and *i* (the inputs), or 2) since *y* is itself composed of two-dimensional points of the form $y = a + bi$, we can plot the outputs as defined by the example equation in terms of *x* and *i*. Method 2 is the version used in traditional mathematics to plot points when looking at the complex plane. These are shown below.

Input Plot



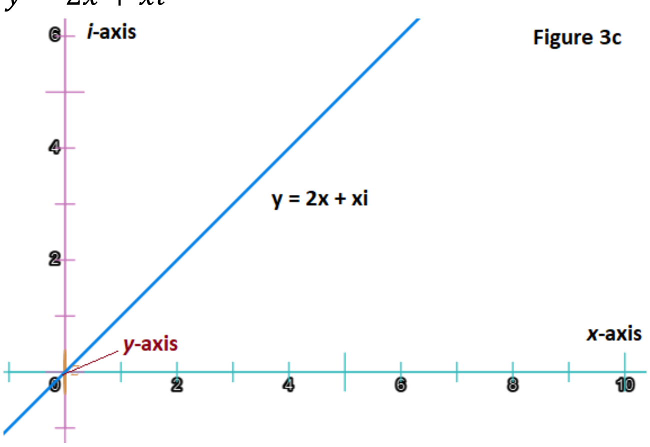
Output Plot

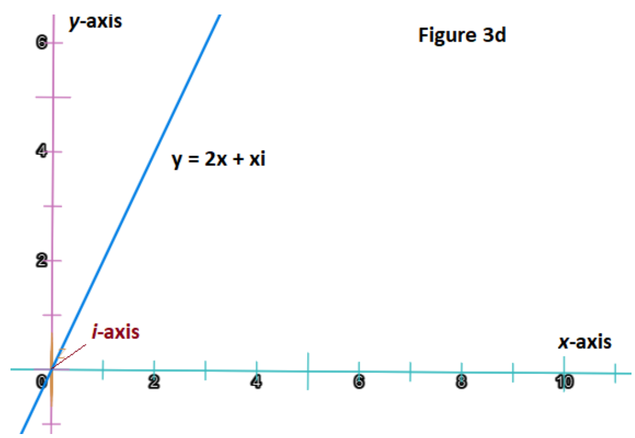


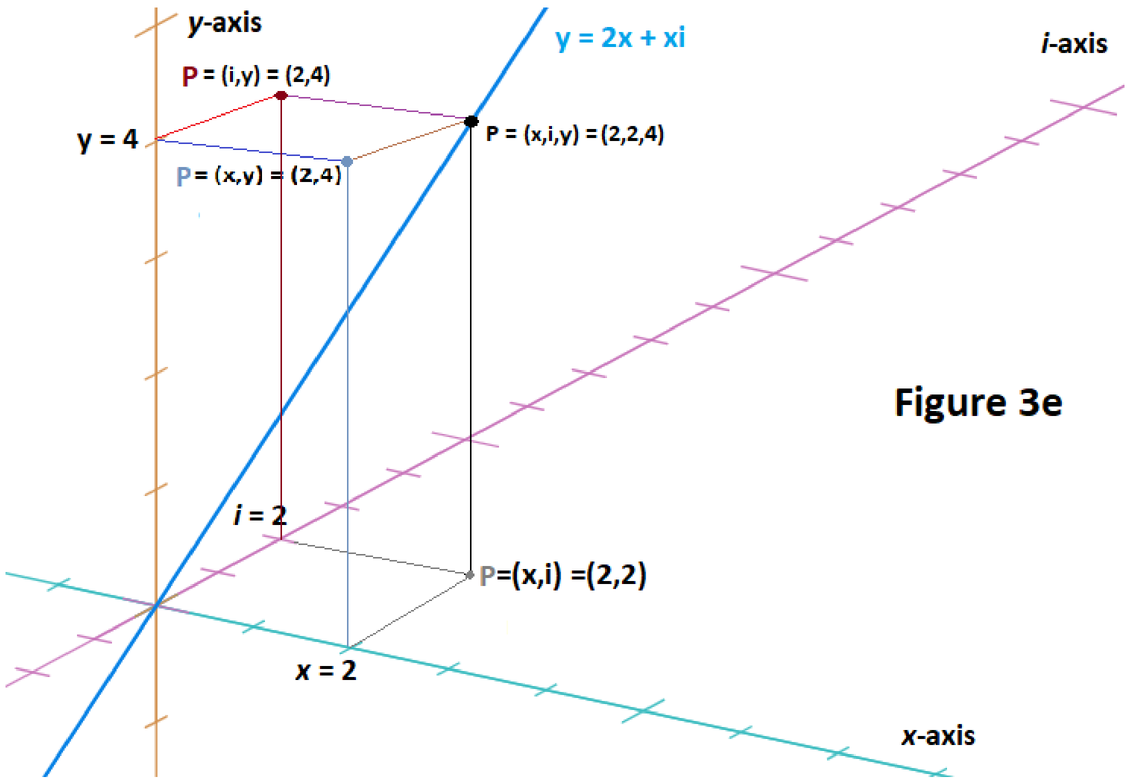
Both the input plot and output plot for the equation $y = 2x + xi$ use the x and i axis to plot points. However it is the output plot which is used in traditional mathematics to perform complex analysis. The first two y-points from the above chart and indicated in Figure 3b.

It can be beneficial to view the Complex Plane as a Three Directional-Hyperplane, which actually includes the *y*-axis. When viewing all three axis at the same time one can see additional detail about the nature of the complex points being plotted. Below are shown several graphs of the equation $y = 2x + xi$, including the *xiy*-hyperplane. Because we are exploring a three directional complex hyper-plane, the two directional graphs are all input plots; they show the input point values of the *x*-axis, while values and the *y* and *i*-axis vary by a function in terms of *x*. In three directions this equation is graphed as a series of parametric points defined by a vector function in three directions.

$y = 2x + xi$







Combined xiy-Hyperplane of $y = 2x + xi$.

<u>Figure 3 explanation:</u>

This expansion of the Complex Plane into a the three directional xiy-hyperplane is a unique application and requires a special approach. The chart in section 1.2 shows the various component values at each x input for the example equation $y = 2x + xi$. *In context of the three directional hyperplane we are treating i as an independent variable. This isn't that big of a leap as it does have its own axis. Consider the first row of the chart delineating the values for* $y = 2x + xi$ when *x* equals 1.

$y = 2x + xi$

x-value	Real Part 2x	*y*-value	*Resolved y*	*i*-value	Resolved *y*-conjugate	Conjugate *y*-value	Conjugate *i*-value
1	2	2+*i*	3	*i*	1	2-*i*	-*i*

Had there been no imaginary component to the equation, we would be plotting points on a standard Cartesian xy-Plane. The x input of 1 would provide a y axis output of 2. The presence of the imaginary component, in this example equation defined by xi, will produce a magnitude of +i units to shift the xy point along the i-axis. This is represented in the chosen points on the line through 3-Space defined by $y = 2x + xi$ *and partially plotted by the chart. Because we have not resolved the i-axis, values remain in the complex hyperplane and we must plot these points as a shift toward the positive side of the i-axis without adding them to y-axis output. This is done already when we use the xi-Complex Plane to represent single y-axis output values as two-dimensional points. The resolved values would collapse the complex xiy-hyperplane back to a real space xy-Cartesian Plane with altered values for y. The implications of this will be covered further, later in this paper.*

Also note had we been discussing the complex conjugate we would have subtracted the i values and shifted toward the negative side of the i-axis. All of this is related to how the complex value i is resolved and whether its resolved value applies to the plane of occurrence or an adjoining subspace.

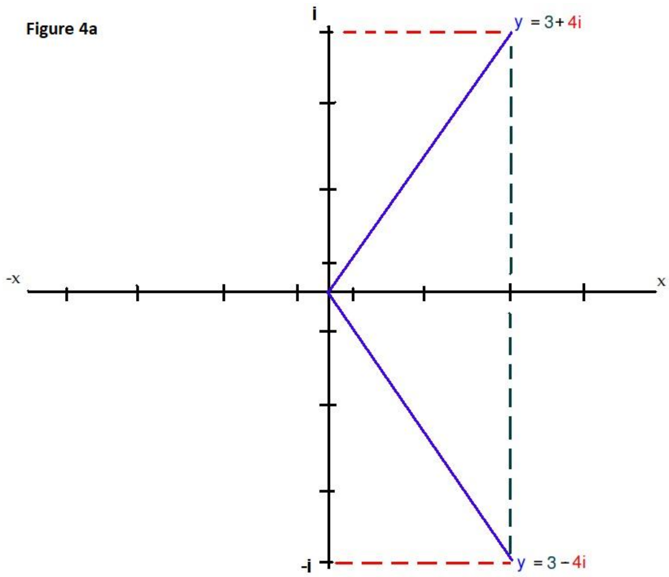
You can explore this arrangement of the xiy-Hyperplane at https://www.desmos.com/calculator/9cwevwk3jk

<u>1.2.a</u>

We need not have an equation to plot points on the complex plane; we may plot individual points. Consider the graph at right which shows the plot of an arbitrary point $y = 3 + 4i$ and its complex conjugate $y = 3 - 4i$.

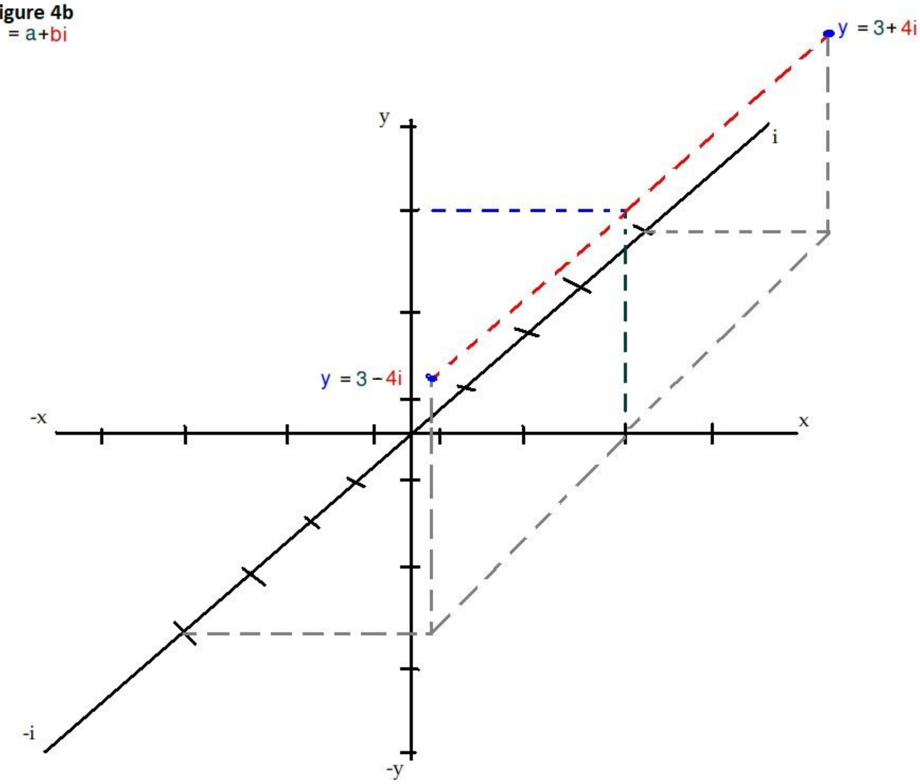
Both points could be plotted alone, or as part of an equation which includes them in it's domain and range. One such example equation containing these two given points $y = x \pm (x + 1)i$. The points defined by this equation are complex points, and each is plotted as a two-dimensional *y output plot* in terms of *x* and *i* values.

Figure 4a

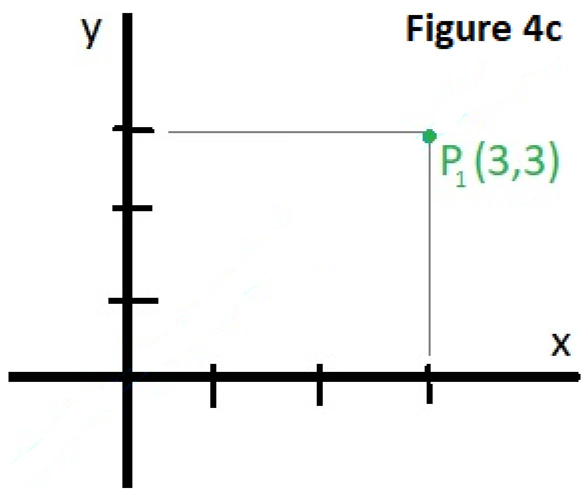


For the two points $y = 3 + 4i$ and $y = 3 - 4i$ they too can be plotted alone on a complex *xiy*-Hyperplane, as shown in Figure 4b; *showing the same two points plotted on the* xiy—*hyperplane.*

Figure 4b



In the graph of Figure 4b had we eliminated the *i*-axis we would be left with the standard Cartesian Plane; the *xy*-Plane.



The example points $y = 3 + 4i$ and $y = 3 - 4i$ would be simplified to $P_1 = (x, y) = (3,3)$ for both complex conjugate pairs. This is shown in the graph of Figure 4c.

Given the example equation, $y = x + (x + 1)i$, the equivalent equation on the *xy*-plane is given by collapsing $y = a + bi$ to just $y = a$, where a is a function of x.

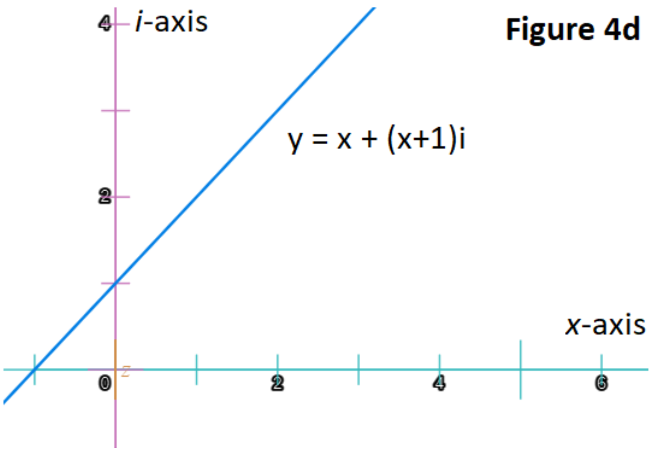
For the example equation $y = x + (x + 1)i$ it will have a real part defined by y = x, and an imaginary part defined by $\pm(x + 1)i$. The two example points exist on the line defined by this equation and are notated as $P_1 = (x, i, y) = (3,4,3)$ and $P_2 = (x, i, y) = (3, -4,3)$ for the instance of $y = f(3)$.

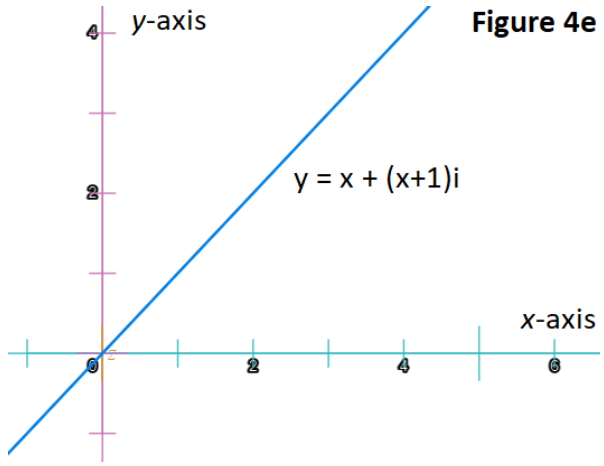
Like the previous example we can express $y = x + (x + 1)i$ and the points it defines on an *xiy*-Hyperplane. Lets begin by plotting several of the points. Remember that the values for *y* do not include addition or subtraction of the *i*-axis contributions, because being unresolved, they express a shift of the *xy* points along the *i*-axis. The resolved values are added or subtracted to the *y*-axis output values and will be covered later in the section of resolution of the value *i* and the complex plane.

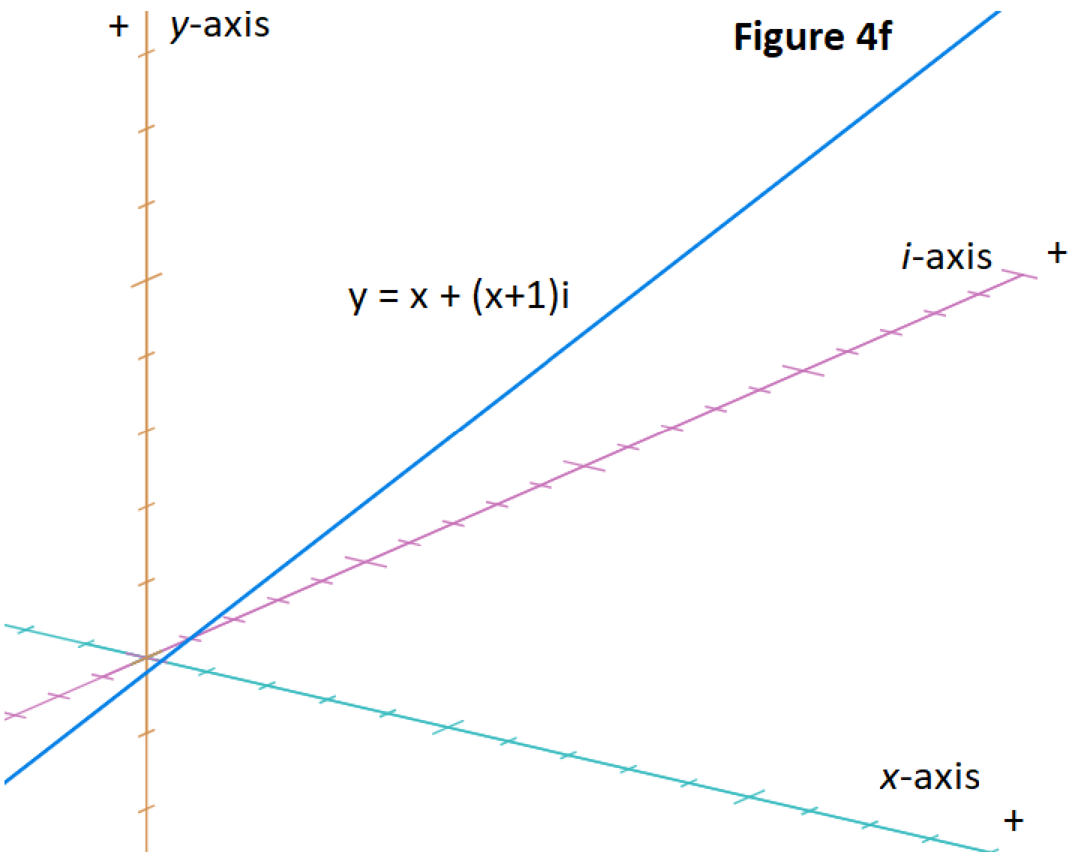
Using the equation $y = x + (x + 1)i$ from section 1.2.a, the Real portion of this equation is just the set of all linear points $y = x$. The Imaginary component for each point is a displacement on the i-axis by an amount
$\pm bi$

$y = x \pm (x + 1)i$

x-value	Real Part x	*y*-value	*Resolved y*	*i*-value	Resolved *y*-conjugate	Conjugate *y*-value	Conjugate *i*-value
1	1	1+2*i*	3	2*i*	-1	1-2*i*	-2*i*
2	2	2+3*i*	5	3i	-1	2-3*i*	-3i
3	3	3+4*i*	7	4i	-1	3-4*i*	-4i
4	4	4+5*i*	9	5i	-1	4-5*i*	-5i
5	5	5+6*i*	11	6i	-1	5-6*i*	-6i
6	6	6+7*i*	13	7i	-1	6-7*i*	-7i







1.3—The Complex Plane and Trigonometry:

The math of Trigonometry for circular functions can be summed up in three identities. Euler's Formula is of particular importance among them.

1.3.a:

$e^{ix} = cos(x) + isin(x)$ $\quad\quad$ $e^{-ix} = cos(x) - isin(x)$

$e^{ix} \cdot e^{-ix} = 1 = \big(cos(x) + isin(x)\big)\big(cos(x) - isin(x)\big) = cos^2(x) + sin^2(x)$

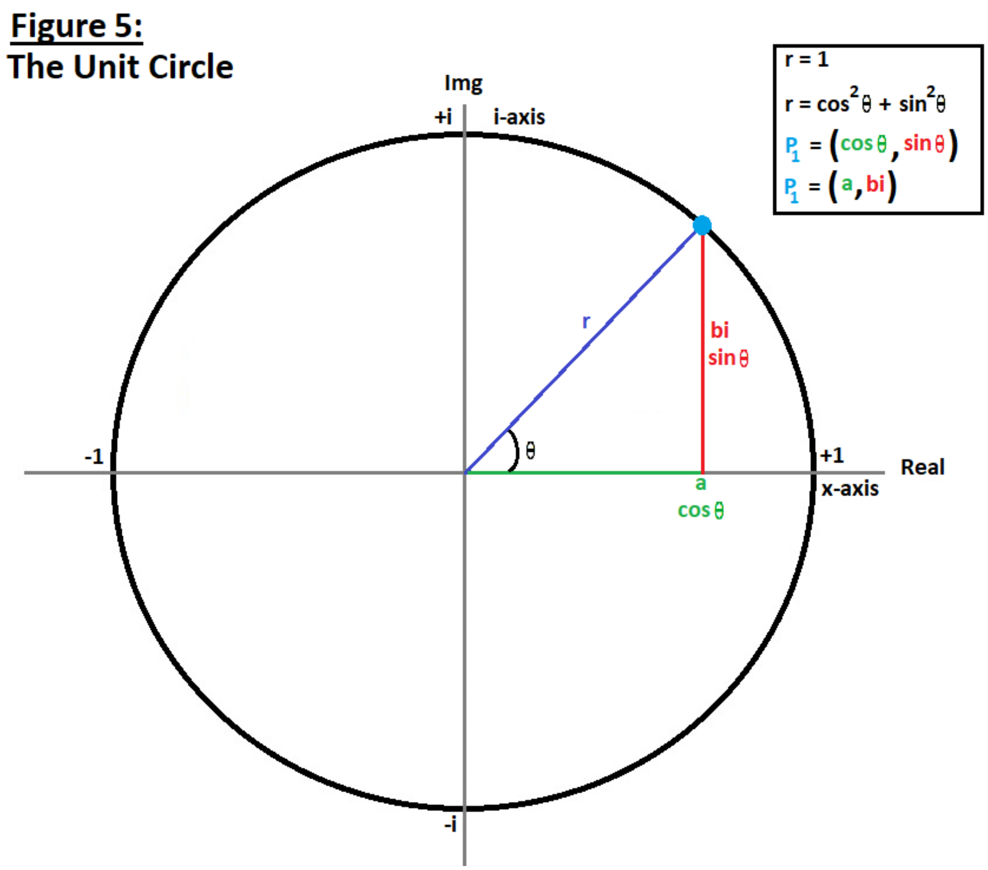
In all of these instances the real part of the equation is represented by $cos(x)$ while the imaginary part is represent by $sin(x)$. If the i is eliminated from the equation these relations shift from circular to hyperbolic. This will be returned to later in the document.

The equations follow the same pattern as Traditional Complex numbers on the Complex Plane.

Figure 5:

$e^{ix} = cos(x) + isin(x)$ $e^{-ix} = cos(x) - isin(x)$

$z = a + bi$ $z^{*}\, a - bi$



1.4—Where does **e** Originate?

Before examining the Null Algebra resolutions of *i* we must first consider where Euler's Formula derives the base value of **e**. The value of **e** is derived from a Taylor Series expansion, or rather a special case of a Taylor Series expansion which is centered around the point $x = 0$, called a Maclaurin Series.

1.4.a:
The inputs for the Maclaurin series will evaluate the value of e^xcentered around the point *x* = 0 approximated as a sum of an infinite series as shown here below.

For the function e^x this is the general series solution for any value of *x* as:

1.4.a.i:
$e^x = \sum_{n=0}^{\infty}\left(\frac{f^n(0)\cdot x^n}{n!}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots.$

1.4.a.ii:
For the specific solutions we may now evaluate $e^x$for any value of *x* simply by assigning values to it. Plugging in for *x* = 0 the expression will equal 1.

$e^o = \sum_{n=0}^{\infty}\left(\frac{f^n(0)}{n!}\right) = 1 + \frac{0}{1!} + \frac{0}{2!} + \frac{0}{3!} \ldots. = 1$

1.4.a.iii:
The evaluation of the expression at *x* = 1 we are left with the base value of **e** itself.

$e^1 = \sum_{n=0}^{\infty}\left(\frac{f^n(0)}{n!}\right) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \ldots. = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \ldots. = 2.71828\ldots$

As the number of terms added approaches infinity the value approaches ever closer to **e**.

1.4.b:
The number **e** shows up in circular trigonometry as the values e^{ix} and e^{-ix}. The value e^x has an interesting property in that the function e^x is the same value as its derivative, e^x.

1.4.b.i:
Given a function: $y = a^{f(x)}$

We find its derivative is: $\acute{y} = \left(\frac{d}{dx}f(x)\right) \cdot a^{f(x)} \cdot ln(a)$

If $y = a^{f(x)} = e^x$ we shall find $\acute{y} = e^x$

When $a = e = 2.71828\ldots.$, then y and $\acute{y}$ are equal because $ln(a)$equal 1, and the derivative of *x* is also 1.

Thus: $y = e^x$ and $\acute{y} = e^x$

1.4.c—How does this equate e^{ix} with cos(*x*) + *i* sin(*x*) ?

As was just explored we may input any function into *x* and evaluate it using the general solution provided as a sum of an infinite series in 1.4.a.i. If we input *ix* we may then evaluate for a general solution to e^{ix} or begin setting specific values for the *x* input in e^{ix}. If we set *x* = 0 the expression will still equal 1.

1.4.c.i: $e^{i(0)} = e^0 = 1$

If instead we now set *x* = 1 for the function e^{ix} we are left with e^i. Without knowing what the number *i* is we are left with seeking a method of approximating this value. Centered around the point *x* = 0, using the Maclaurin Series defined earlier in equation 1.4.a.i we can simply plug in the value of i and make our evaluation as the sum of an infinite series. We simply replace the function value with *i*.

1.4.c.ii:

$$e^i = \sum_{n=0}^{\infty}\left(\frac{f^n(0)\cdot i^n}{n!}\right) \quad = 1 + \frac{i}{1!} + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \frac{i^5}{5!} + \frac{i^6}{6!} + \frac{i^7}{7!} + \frac{i^8}{8!}\ldots.$$

$$= 1 + i - \frac{1}{2} - \frac{i}{6} + \frac{1}{24} + \frac{i}{120} - \frac{1}{720} - \frac{i}{5040} + \frac{1}{40,320}\ldots.$$

Using the complex plane, we can see what is happening to the value this infinite sum represents as more and more terms are considered. You being in the origin and add in the first term, +1. Then add in the second term; in this example adding a distance of *i*. This will move the point from the position $P_1(x, i) = (1,0)$ to the $P_2(x, i) = (1, i)$. This is repeated as far as one can till a point is reached which is accepted as the limit of the sum of this infinite series.

The Graph in Figure 6 shows this process with $e^i = 0.54027 + 0.8415i$. As additional terms are added the value will continue to grow ever closer to some value. The more terms that are added the more accurate the approximation.

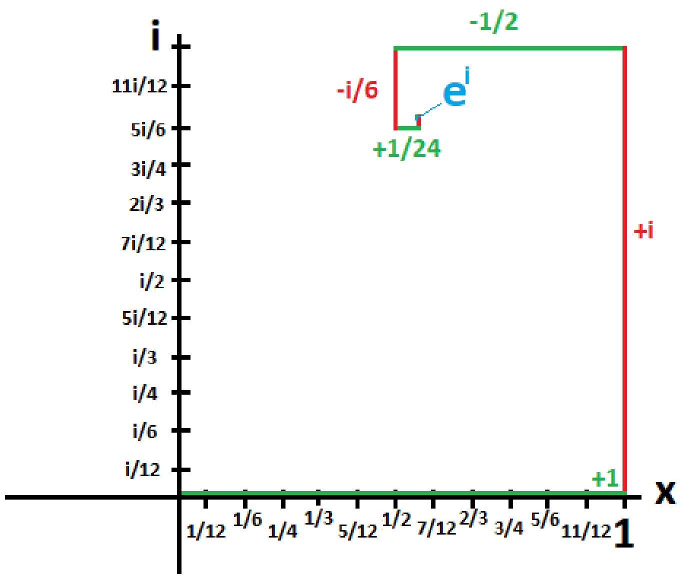


Figure 6

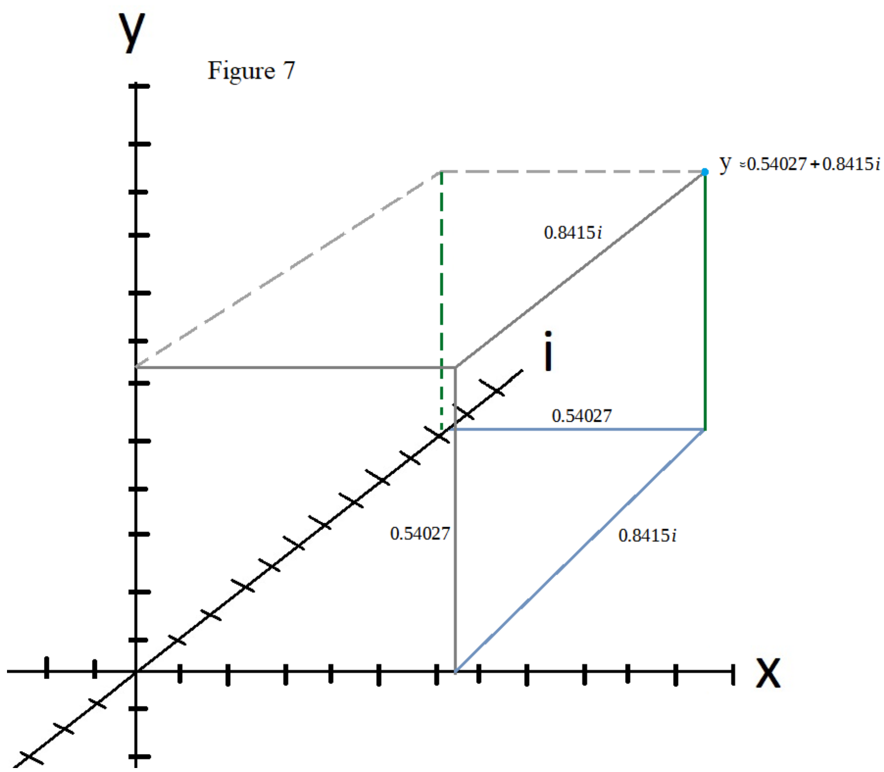
This same process can be shown on the *xiy*-hyperplane. See below here in Figure 7.
Here we have the equation:

$y = e^{ix}$ where x = 1. This will define y to be a complex number $x + bi$, and found by the same Maclaurin series approximation used above.

<u>1.4.c.iii:</u>

$$y = e^{i} = x + bi = \left[\left(1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \frac{1}{40{,}320} + \ldots\right) + \left(i - \frac{i}{6} + \frac{i}{120} - \frac{i}{5040} + \ldots\right)\right]$$
$$\approx 0.54027 + 0.8415i$$

Figure 7



1.5—The Relationship of e^x to trigonometric equations $cos(x)$ and $sin(x)$.

<u>More on Maclaurin Approximations of e^x</u>

We briefly discussed how the function e^x can be approximated using a Maclauren Series. Both the Maclauren and Taylor series are used to approximate the value of a given function around some specific x position. The Maclaurin series is just a special case of the Taylor series wherein the approximation is centered around $x = 0$. This is provided by using $a = 0$ in the below definition. Thus when $a = 0$ the below definition of a Taylor Series expansion is a Maclaurin Series expansion.

<u>1.5.a:</u>

Taylor Series Expansion:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \acute{f}(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

1.5.b:
Maclaurin Series Expansion:

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \acute{f}(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^n(0)}{n!}x^n$$

Because the function e^x is equal to its own derivative it is a unique case to test values of the Maclaurin series. Consider the below examples:

1.5.c:

For $f(x) = e^x$ centered about point x = 0

The first several derivatives for e^x

$$f(x) = e^x$$
$$f'(x) = \frac{d}{dx}e^x = e^x$$
$$f''(x) = \frac{d}{dx}e^x = e^x$$
$$f'''(x) = \frac{d}{dx}e^x = e^x$$
$$\vdots$$

Provides the Maclaurin approximation:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \acute{f}(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

with $f(x) = e^x$ and $a = 0$

$$\sum_{n=0}^{\infty} \frac{(e^x)^n(0)}{n!}(x)^n = e^{(0)} + e^{(0)}(x) + \frac{e^{(0)}}{2!}(x)^2 + \ldots + \frac{e^{(0)}}{n!}(x)^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

The value one desires to evaluate the function *x* for, is simply plugged in directly for *x*.

1.5.d—Expansion of e^1:
As this exponent implies, the number **e** is the value reached as the number of considered terms in the series approaches infinity, for $x = 1$ in the function e^x represented by the Maclaurin series infinite sum expansion $\sum_{n=0}^{\infty} \frac{(e^x)^n(0)}{n!}(x)^n$. The sum will approach ever closer to **e** = 2.71828...

For $f(x) = e^1$ Centered about point x = 0

Begin with the formula derived in 1.5.c above which provides a generalized Maclaurin series expansion for values of the function e^x.

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

We are evaluating $f(x) = e^1$. *Replace each instance of x with the value 1.*

$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320} + \frac{1}{362{,}880} \ldots \approx 2.71828\ldots$

1.5.e—Expansion of e^{ix}:

The complex number *i* can be included in these approximations.

For e^{ix} centered around the point *x* = 0 we have the approximation:

$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots$

Using the structure shown here above we can evaluate for various values of *x* of the function e^{ix}. If we now evaluate the function for *x* = 0 we get

1.5.f:

$e^{i(0)} = 1 + \frac{0}{1!} + \frac{(0)^2}{2!} + \frac{(0)^3}{3!} + \frac{(0)^4}{4!} + \ldots = 1$

Evaluating the function e^{ix} for *x* = 1

1.5.g:

$e^{i} = 1 + \frac{i}{1!} + \frac{(i)^2}{2!} + \frac{(i)^3}{3!} + \frac{(i)^4}{4!} + \ldots = 1 + i - \frac{1}{2} - \frac{i}{6} + \frac{1}{24} + \frac{i}{120} - \frac{1}{720} - \frac{i}{5040} + \frac{1}{40{,}320} \ldots$

$e^{i} \approx 0.54027 + 0.8415i$

This was shown above in sections 1.4.c.ii and 1.4.c.iii. These examples thus far, which include i, do not yet show the resolved value of i. Thus we haven't yet explored what is actually implied by the presence of an i in an equation.

Evaluating for $x = \frac{\pi}{2}$

1.5.h:

$e^{i\frac{\pi}{2}} = 1 + \frac{i\frac{\pi}{2}}{1!} + \frac{\left(i\frac{\pi}{2}\right)^2}{2!} + \frac{\left(i\frac{\pi}{2}\right)^3}{3!} + \frac{\left(i\frac{\pi}{2}\right)^4}{4!} + \ldots$

$$e^{i\frac{\pi}{2}} = 1 + 1.5708i - 1.2337 - 0.645964i + 0.2536695 + 0.0796926i - 0.02086348 - 0.00468175i + \ldots$$

If we separate out the real and imaginary components of just these few terms listed here we can list this value as a complex number *a* + *bi*, such that,

$e^{i\frac{\pi}{2}} = -0.00089398 + 0.99984685i \approx i$

As more components considered and the Maclaurin series approaches and infinite number of terms, the value will approach $e^{i\frac{\pi}{2}} = 0 + i = i$. This is consistent with the idea of each successive multiple of *i* being a rotation of 90 degrees on the Complex Plane.

Evaluating $x = \pi$

1.5.i
Pi radians trigonometrically represents 180 degrees on the unit circle. This rotation on the Complex Plane, represented by e^{ix} with $x = \pi$ should represent a value of 180 degrees, or the point on the complex plane defined by (x , *i*) = (-1 , 0).

$$e^{i\pi} = 1 + \frac{i\pi}{1!} + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \frac{(i\pi)^4}{4!} + \frac{(i\pi)^5}{5!} + \frac{(i\pi)^6}{6!} + \ldots$$

$$e^{i\pi} = 1 + 3.14159i - 4.93480 - 5.16771278i + 4.058712 + 2.550164i - 1.33526 - 0.5992645i + \ldots$$

If we extend the number of terms out to *n* = 13 in the series we get the complex number *a* + *bi* such that

$$e^{i\pi} = -0.99989 + 0.00001847847i \approx -1$$

Again as the number of terms added to the Maclaurin series approaches infinity this value will approach ever closer to $e^{i\pi} = -1 + 0i = -1$.

1.6—Connecting e^{ix} with the Unit Circle:
The points which are defined by the spiraling in of successive summation of Maclaurin Series components provide approximations of e^{ix} for various values of *x*, which are in fact points which all lay on an arc called the Unit Circle in the Complex Plane. The distance of each point from the origin is always of magnitude 1. The evaluation from 1.5.i gives us the relation

1.6.a:
$$e^{i\pi} + 1 = 0$$

Further we are given the relation known as Euler's Formula stating

1.6.b:
$$e^{ix} = cos(x) + isin(x) \qquad e^{-ix} = cos(x) - isin(x)$$

This provides an important Pythagorean identity that:

1.6.c:
$$e^{ix} \cdot e^{-ix} = 1 = cos^2(x) + sin^2(x)$$

Because the points defined by the arc e^{ix} all lay on the Unit Circle we may represent them with the Sine function (*the imaginary portions*) and the Cosine function (*the real portion*). Because we will be resolving the complex plane it is necessary to understand exactly how it is that we link e^{ix} with trigonometric functions like Sine and Cosine. Further as resolving the complex plane involves resolving *i* to real number, resolved values on the complex plane will inherently change from circular to hyperbolic functions.

Compare the Maclaurin Series approximations for e^{ix} and e^{x} shown here beside each other.

1.6.d:

$$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40{,}320}\ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24} + \frac{ix^5}{120} - \frac{x^6}{720} - \frac{ix^7}{5040} + \frac{x^8}{40{,}320}\ldots.$$

This similarities and differences can be seen even more clearly if we now evaluate for $x = 1$.

1.6.e:

$$e^{1} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}\ldots$$

$$e^{i} = 1 + i - \frac{1}{2} - \frac{i}{6} + \frac{1}{24} + \frac{i}{120} - \frac{1}{720} - \frac{i}{5040} + \frac{1}{40{,}320}\ldots.$$

Aside from the obvious difference of the missing *i*'s, the values for e^{x} are all positive whereas those for e^{ix} have alternating positive and negative values for various n^{th} components of the series.

1.7—Separation of Terms

The terms of the components for the series approximation of e^{ix} can be separated into two separate groups of terms, one real and one imaginary. The *i* can then be factored out of the imaginary sub-group of terms. We can then use the same pattern, identifying terms by their even and odd exponent value to regroup terms from the approximation of e^{x} and e^{ix}.

1.7.a.i:

$$e^{ix} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320}\right) + i\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362{,}880}\ldots.\right)$$

1.7.a.ii

$$e^{x} = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40{,}320}\right) + \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362{,}880}\right)$$

The question then is, *Is there an equation whose Maclaurin Series approximation matches the grouped portions of either or both of these above sets*? We now examine the Maclaurin series expansions of cos(*x*) , sin(*x*) and *i* sin(*x*).

1.7.b:

Because the Maclaurin Series uses derivatives of the given functions we shall first list out the successive derivatives of these several functions.

Function	$cos(x)$	$sin(x)$	$isin(x)$
1^{st} Derivative	$-sin(x)$	$cos(x)$	$icos(x)$

2nd Derivative	$-cos(x)$	$-sin(x)$	$-isin(x)$
3rd Derivative	$sin(x)$	$-cos(x)$	$-icos(x)$
4th Derivative	$cos(x)$	$sin(x)$	$isin(x)$
⋮	⋮	⋮	⋮

1.7.c:
With this information we are free to begin constructing the Maclaurin Series approximations of these functions

1.7.c.i—Maclaurin Series Approximation of Cosine:
$f(x) = cos(x)$ centered at $x = 0$ given by $a = 0$

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \acute{f}(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \acute{f}(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^n(0)}{n!}x^n$$

$$cos(x) = cos(0) - sin(0)\cdot x - \frac{cos(0)\cdot x^2}{2} + \frac{sin(0)\cdot x^3}{6} + \frac{cos(0)\cdot x^4}{24} - \frac{sin(0)\cdot x^5}{120} - \frac{cos(0)\cdot x^6}{720} + \frac{sin(0)\cdot x^7}{5040} + \cdots$$

$$cos(x) = 1 - 0x - \frac{x^2}{2} + \frac{0x^3}{6} + \frac{x^4}{24} - \frac{0x^5}{120} - \frac{x^6}{720} + \frac{0x^7}{5040} + \frac{x^8}{40{,}320} - \frac{0x^9}{362{,}880} \ldots$$

$$cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320} - \frac{x^{10}}{3{,}628{,}800} + \cdots$$

If you compare this to the values generated by the Maclaurin series approximation for e^{ix} and e^{x} above in sections 1.7.a.i and 1.7.a.ii you'll notice this pattern matches for the real portion of the expansion for e^{ix}. Its close to the even exponential powers of the expansion of e^{x}, differing only in subtraction of every other term.

1.7.c.ii—Maclaurin Series Expansion of Sine:
$f(x) = sin(x)$ centered at $x = 0$ given by $a = 0$

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \acute{f}(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \acute{f}(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^n(0)}{n!}x^n$$

$$sin(x) = sin(0) + cos(0)\cdot x - \frac{sin(0)\cdot x^2}{2} - \frac{cos(0)\cdot x^3}{6} + \frac{sin(0)\cdot x^4}{24} + \frac{cos(0)\cdot x^5}{120} - \frac{sin(0)\cdot x^6}{720} - \frac{cos(0)\cdot x^7}{5040} + \cdots$$

$$sin(x) = 0 + x - \frac{0 \cdot x^2}{2} - \frac{x^3}{6} + \frac{0 \cdot x^4}{24} + \frac{x^5}{120} - \frac{0 \cdot x^6}{720} - \frac{x^7}{5040} + \frac{0 \cdot x^8}{40{,}320} + \frac{x^9}{362{,}880} - \cdots$$

$$sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362{,}880} - \frac{x^{11}}{39{,}916{,}800} + \cdots$$

Once again if you compare this to the values generated by the Maclaurin series approximation for e^{ix} and e^x above in sections 1.7.a.i and 1.7.a.ii you'll notice this pattern matches the imaginary portion of the expansion for e^{ix} except for a missing *i* to be factored out of the set. It also closely to that of odd exponential powers of e^x, differing only in subtraction of every other term.

<u>1.7.c.iii—Maclaurin Series Expansion of $isin(x)$:</u>

Given the value of *i* is a constant it is simply a multiple of each component shown in 1.7.c.ii, and factored out of the set.

$$isin(x) =$$

$$= ix - i\frac{x^3}{6} + i\frac{x^5}{120} - i\frac{x^7}{5040} + i\frac{x^9}{362{,}880} - i\frac{x^{11}}{39{,}916{,}800} + \cdots$$

$$= i\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362{,}880} - \frac{x^{11}}{39{,}916{,}800} + \cdots\right)$$

This approximation matches the pattern found in the imaginary portion of the expansion of e^{ix} in section 1.7.a.i exactly.

This implies the following. The Expansion of e^{ix} given by the Maclaurin Series in section 1.7.a.i has a real portion which matches the Maclaurin Series approximation of Cosine detailed in section 1.7.c.i, and an imaginary portion which matches the Maclaurin Series approximation of Sine detailed section 1.7.c.iii. This means these portions of the Maclaurin Series approximation of e^{ix} can be directly replaced by the $cosx$ and $isinx$ functions respectively yielding the Euler Formula equation.

$$e^{ix} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320}\right) + i\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362{,}880} \ldots\right)$$

$$e^{ix} = cos(x) + isin(x)$$

If you take these same steps on a Maclaurin Series of expansion of e^{-ix} ultimately you will reach the conclusion that $e^{-ix} = cos(x) - isin(x)$. It is from here we obtain the Pythagorean identity:

$$e^{ix} \cdot e^{-ix} = 1 = cos^2(x) + sin^2(x)$$

<u>1.8—Making Sense of the Maclaurin Series Expansion of e^x:</u>

The Maclaurin Series expansion of e^x does not have a match in any of the terms thus far explored for the Cosine and Sine functions. These come from a Maclaurin Series expansion on a different set of functions, the Hyperbolic Sine and Hyperbolic Cosine.

If you consider the Maclaurin Series expansions of $cosh(x)$ and $sinh(x)$ and how the relate to e^x with the removal of *i* to form the trigonometric equations you will see this is indeed a departure from circular functions. We are now using hyperbolic functions.

The following chart shows the first several derivatives of e^x, $cosh(x)$ and $sinh(x)$ which are needed to create the Maclaurin Series of each:

Function	e^x	$cosh(x)$	$sinh(x)$
1st Derivative	e^x	$sinh(x)$	$cosh(x)$
2nd Derivative	e^x	$cosh(x)$	$sinh(x)$
3rd Derivative	e^x	$sinh(x)$	$cosh(x)$
4th Derivative	e^x	$cosh(x)$	$sinh(x)$
⋮	⋮	⋮	⋮

The first thing that should be obvious is the successive derivatives of cosh and sinh no longer have sign changes. They are all positive.

Here is the Maclaurin Expansion of e^x re-printed from above.

1.8.a:

$$e^x = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40{,}320}\right) + \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362{,}880}\right)$$

Here is the Maclaurin Expansion of $cosh(x)$ centered around *x* = 0 with *a* = 0:

1.8.b:

$$cosh(x) = \frac{cosh(0)x^0}{0!} + \frac{sinh(0)x^1}{1!} + \frac{cosh(0)x^2}{2!} + \frac{sinh(0)x^3}{3!} + \frac{cosh(0)x^4}{4!} + \frac{sinh(0)x^5}{5!} + \frac{cosh(0)x^6}{6!} + \cdots$$

$$cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40{,}320} \ldots$$

Here is the Maclaurin Expansion of $sinh(x)$ centered around *x* = 0 with *a* = 0:

1.8.c:

$$sinh(x) = \frac{sinh(0)x^0}{0!} + \frac{cosh(0)x^1}{1!} + \frac{sinh(0)x^2}{2!} + \frac{cosh(0)x^3}{3!} + \frac{sinh(0)x^4}{4!} + \frac{cosh(0)x^5}{5!} + \frac{sinh(0)x^6}{6!} + \cdots$$

$$sinh(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362{,}880} \ldots$$

Like was done for the circular trigonometric functions the even exponential group of e^x is replaced with $cosh(x)$ and the odd exponential group of e^x is replaced with $sinh(x)$. This yields the following:

1.8.d:

$$e^x = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40{,}320}\right) + \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362{,}880}\right)$$

$e^{x} = cosh(x) + sinh(x)$

If you conduct the same Maclaurin Series expansion on e^{-x} you will ultimately find

1.8.e:
$e^{-x} = cosh(x) - sinh(x)$

From these two equations we obtain the Hyperbolic Pythagorean identity:

1.8.f:
$e^{x} \cdot e^{-x} = 1 = cosh^2(x) - sinh^2(x)$

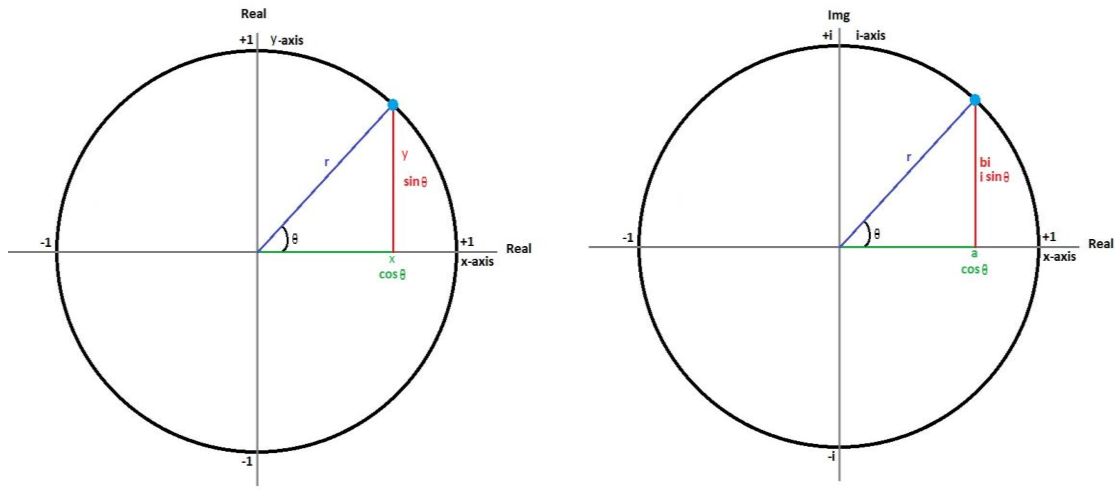
A note of caution here to reader. The usage of the ***h*** *on the ends of the hyperbolic trigonometric functions is a convention to separate them from their circular counterparts. They are pronounced cosh* and *sinch. However it is the Pythagorean relationship that* $1 = cosh^2(x) - sinh^2(x)$ *which makes them hyperbolic. If function notation is written in a form which appears to indicate circular relationships* (*i.e.* $sin(x)$ *and* $cos(x)$) *but the mathematics supports a relationship between them such that* $1 = cos^2(x) - sin^2(x)$ *then you are in fact using hyperbolic trigonometric functions which should be rewritten to include the* ***h*** *for proper nomenclature and avoid confusion.*

1.9—The relationship between the *x*, *i* and *y* axis, and the trigonometric functions.
When considering the complex plane the real components associated with the Cosine function portion of the expansion of e^{ix} is directly associated with the (*assuming we are speaking of the xy-Cartesion Plane*) *x*-axis. This axis has no *i* components and neither does the Cosine expansion. The Sine function portion of the expansion of e^{ix} matches the complex components exactly for $isin(x)$. If we ignore the presence of *i* we can assume the relationship between the two axis remains the same and apply the Sine function to the non-complex vertical *y* axis. This confirmed in that simple algebra allows us to define the Sine function as a real function in terms of Cosines:

1.9.a:
$sin(x) = \sqrt{1 - cos^2(x)}$

See Figure 8 for a graphic representation of this.

Figure 8:



2.1—Introduction to Resolving the Complex Plane:

Before actually visualizing the resolution of the complex plane one must first understand the apparent discrepancies which arise in the successive multiples of powers of *i* between traditional Algebra and Null Algebra.

Consider the first several such iterations:

2.1.a:

Traditional Algebra	i	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$	$i^5 = i$	$i^6 = -1$	$i^7 = -i$
Null Algebra	$i = \oplus 1$	$i^2 = -1$	$i^3 = 1$	$i^4 = -1$	$i^5 = 1$	$i^6 = -1$	$i^7 = 1$

Null Algebra specifies $i = \oplus 1 = \frac{0}{0}$. The squaring of *i* is the squaring of a *plus-and-minus* number which results in a negative, of the squared magnitude. Each higher power is of *i* is identical to a subsequent repeat multiplication of $\frac{0}{0}$. (*For a full explanation see Null Algebra, Section 2.b—The Negative Radical, Page 91 to 121*). In summary a negative argument for a square root is the product of the positive root of the magnitude of the argument inside the radical bar and the resolved root of $\oplus 1$, as $\overset{\uparrow}{1}$ on the axis of occurrence and $\check{1}$ on the corresponding subspace axis.

Aside from the fact that Null algebra resolves and assigns real values to the number *i* and its successive multiples, it is clear from the above list more is going on than is understandable at first glance. From the outset Null Algebra has assigned a value to *i* while traditional mathematics does not. Likewise Null Algebra provides values for the odd powers of *i* while traditional maths does not. After i^2 the two sets will agree on the value reached only for every next fourth power of *i*.

2.1.b:

Let *i* be the Traditional Algebra value of $\sqrt{-1}$.

Let $\boldsymbol{i}$ be the Null Algebra value of $\sqrt{-1}$.

Then,

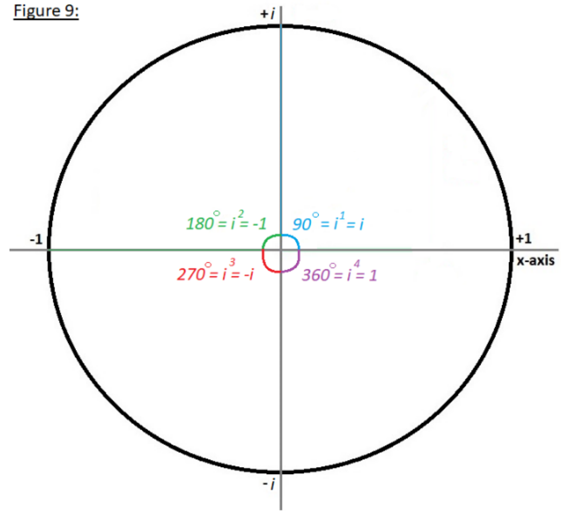
$$i^n = i^n \ \ \forall n \longrightarrow n = S_m(2+4m)$$

The two terms are equal for all n defined by the Sequence $S_m(2+4m)$ from 0 to infinity.

<u>2.2.—Powers of *i* and angular rotations:</u>
Returning to the complex plane consider the list of successive multiples of *i* for traditional mathematics. When plotting points on the complex plane there is a rotating effect, seen in the Maclaurin Series approximations, as they approach a specific value. It was already shown that when plotting each successive component of the Maclaurin Series approximation of e^{ix}, for $x = \pi$, caused a spiraling effect which provides the point $e^{i\pi} = a + bi = -1 + 0i = -1$ as the number of terms considered approaches infinity.

Each successive multiplication of *i* amounts to a rotation of 90° on the complex plane and the unit circle. The below chart and Figure 9 shows this rotation.

Figure 9:



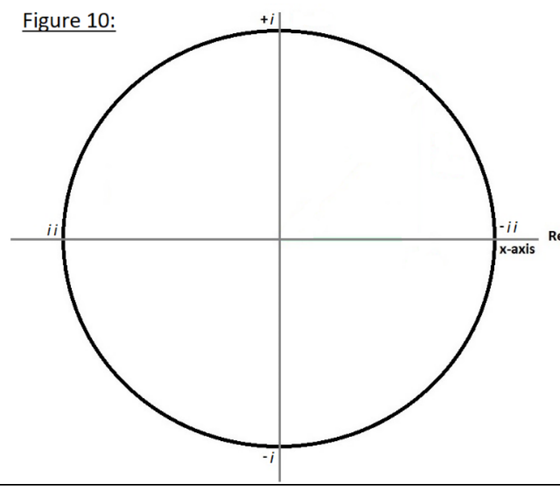
<u>2.2.a:</u>

Power of *i*	Angular Rotation	Point Notation
$i^0 = 1$	0° Rotation	$P_1(1,0i)$
$i^1 = i$	90° Rotation	$P_2(0,i)$
$i^2 = -1$	180° Rotation	$P_3(-1,0i)$
$i^3 = -i$	270° Rotation	$P_4(0,-i)$
$i^4 = 1$	360° Rotation	$P_5(1,0i)$

Pattern continues in repetition.

<u>2.2.b:</u>
To introduce the unique way of visualizing the Complex Plane within the precepts of Null Algebra and Null Calculus consider the implications of viewing the entire Complex Plane with no real part. Figure 10 shows this adaptation to the Complex Plane.

Figure 10:



<u>Key Points:</u>
In this next section consider the following key points as they are explained.

1. There is a Reversal of signs shown on what is now the *ii*-axis in Figure 10. This was previously shown as the Real Axis.

2. Squaring a complex number is multiplication by its Complex Conjugate.
3. There is only one side to the *i*-axis.
4. Any number on the i-axis is only half of a number. Their full nature is not properly conveyed in traditional descriptions of complex numbers. These numbers are paired numbers, not in the form of $a \pm bi$ but rather $a \oplus bi$

5. The Complex Plane is a composite of several planes forced into a two directional Cartesian Plane, which holds these paired values defined in Null Algebra and Null Calculus as real numbers, themselves plotted on an expanded hyperplane.

2.3—Plotting Complex numbers and Identifying *i:*

The real axis on the complex plane can be thought of as being real only because it exists as a square of imaginary components. This is significant in resolution of the complex plane to a real and subspace hyperplane.

For the moment we shall re-explore the Null Algebra resolution of *i*. The format for a Complex number is:

2.3.a:

$z = a + bi$ and $z^* = a - bi$

Where a is the real component and b is the coefficient representing the magnitude of the imaginary component. Any complex number whether adding $(+bi)$ or subtracting $(-bi)$ the imaginary component will have a complex conjugate.

Fig. 11

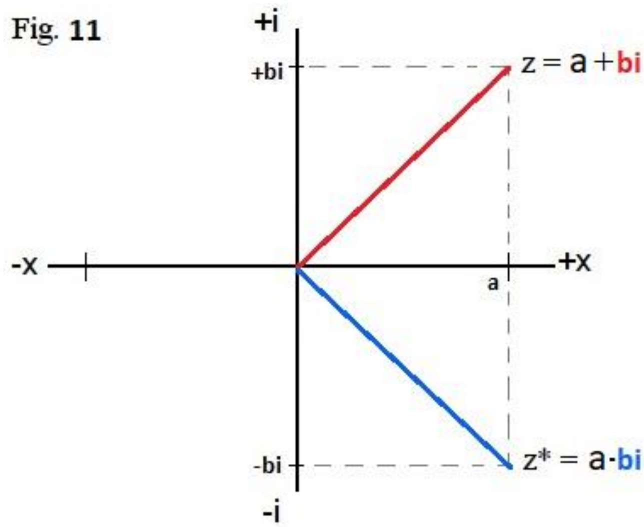


Figure 11:
Given a number $z = a + bi$ there shall exist a complex conjugate $z^* = a - bi$.

The squaring of a complex number is not conducted in the same fashion as an integer. An integer such as +4 is squared by multiplying it by itself. For example:

2.3.a.i:

$4^2 = 4 \cdot 4 = 16$

$-4^2 = -4 \cdot -4 = 16$

Complex numbers are squared by multiplying them by their complex conjugate:

2.3.a.ii:

Given $z = a + bi$

Then: $z^2 \neq z \cdot z$

$z^2 = z \cdot z^* = (a + bi)(a - bi) = a^2 + b^2$

2.3.b:

Because squaring a value is identical to multiplying a value by itself, the application of squaring complex numbers via multiplication by complex-conjugate must be equivalent to this standard process of multiplying a value by itself. This implies that the imaginary component of a complex number, regardless of whether positive $(+bi)$ or negative $(-bi)$ is only half of a number; both halves must be considered when squaring. It implies that the vertical *i*-axis does not really have separate positive and negative sides, but rather is only single sided, using + and - signs to plot both halves of a single number. This further implies that a given complex number $a \pm bi$ is not accurate. Instead this should be written as $a \oplus bi$, indicating the bicomponent is a *plus-and-minus* number whose partner halves are resolvable from one-another for accurate calculations. Because the separate halves of a given complex-number are connected its convenient to illustrate the deeper nature of the complex plane by labeling the opposite sides of the *i*-axis as $\hat{i}$ and $\check{i}$rather using + and - values.

2.3.b.i:

We can specify and isolate the complex number *i* by setting

$a = 0$ and $b = 1$

i then is a complex number which when squared, is multiplied by its complex conjugate giving it a value of a negative sign with magnitude of 1. This also implies that *i* itself must already have a magnitude of 1 as the magnitude of the squared value also equals 1; not 0 or any other value greater or less than one. See Figure 12 here.

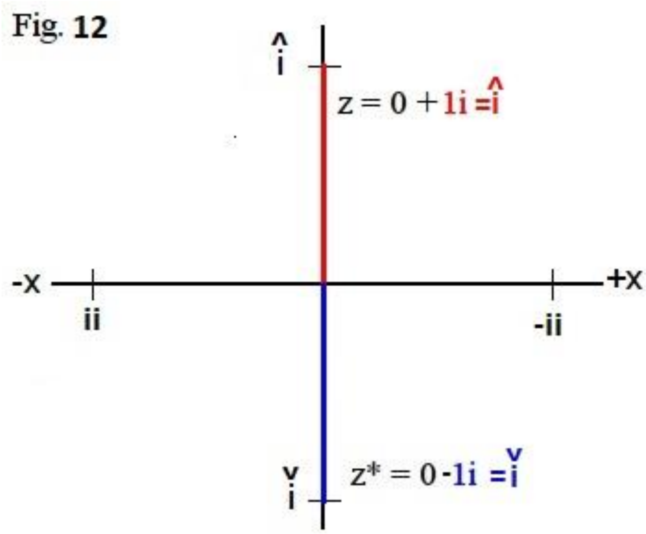


Fig. 12

Key Points:

1. Both values *i* and -*i* have a magnitude of 1.
2. They are two connected halves of they same number on a single sided *i*-axis.
3. These two values can be represented as the resolved *i* number values $\hat{i}$ and $\check{i}$.
4. i^2 is the product of $\hat{i}$ and $\check{i}$
5. The magnitude of each $\hat{i}$ and $\check{i}$are one on the complex plane
6. The magnitude of i^2 remains 1.

On Complex Plane: $i \cdot -i = -i^2 = +1$ Requiring $i^2 = -1$

↓

Resolved on real axis: $\hat{1} \cdot \check{1} = +1 \cdot -1$ Requiring Resolved $\hat{1} \cdot \check{1} = i^2 = -1$

2.4—Identifying *i:*

The next step is to identify what *i* actually is as a number. Several properties can be clearly identified from the previous section and its equations.

- *i* is a complex number
- It has two halves, $\hat{\imath}$ and $\check{\imath}$ representative of z and z^*
- The two halves of *i* are written on the complex plane as either +*i* and -*i*, or as $\hat{\imath}$ and $\check{\imath}$.
- *i* and its complex conjugate have a magnitude of 1.
- i^2 maintains a magnitude of 1.
- The single numeric value which is equal to *i* must somehow be simultaneously *positive-and-negative* to represent the two halves to the complex number *i.*
- The complex number *i* must have a magnitude of 1 and maintain a magnitude of 1 when squared despite sign change.

The only value which matches these specifications is $\frac{0}{0}$. This value is traditionally called the indeterminate form. For a full description on the resolution of this expression's various possible values see text on Null Algebra (https://vixra.org/abs/2103.0131). This section will focus on the solutions value to $\frac{0}{0}$, +1 and -1, and their relationship to the complex plane.

2.4.a—The number 0:

Zero is a unique value. Every number has an infinite set of numbers to its left and right on a number line. Zero however is the only value which has an infinite number of only negative values to its one side on a number line, and an infinite number of only positive number to its opposite side on a number line. This includes an infinite number of infinitesimally small fractions between whole number integers. Thus there are an infinite number of negative fractions -1 and 0, as well as an infinite number of positive fractions between 0 and +1.

2.4.b—Is 0 positive or negative?

Traditional Mathematics declares 0 is neither positive nor negative. However since 0 is part of both the positive and negative sides of any number-line it is actually simultaneously *positive-and-negative.* Notice that marking 0 with a positive or negative sign effectively changes nothing for traditional mathematics.

Positive 0: +0 This is a value of 0 and the sign as no effect

2.4.b.i:

$+0 = -0 \qquad +0 \not\equiv -0$

$2 + 0 = 2 \qquad 2 - 0 = 2 \qquad \frac{+0}{2} = 0 \qquad 2 \cdot (+0) = 0$

Below, even though we use -0, there is not change in value. The reader is reminded that although $+0 \not\equiv -0$, for the use of 0 in traditional mathematics $+0 = -0$.

Negative 0: -0 The magnitudinal change is 0. The sign effectively doesn't matter and is simply resolved to 0. It is still there but as 0 represents a non-change in value it also lacks the capacity to change the sign of any values its interacting with, except when in the indeterminate form.

2.4.b.ii:

$+0 = -0 \qquad +0 \not\equiv -0$

$2 + (-0) = 2 + 0 = 2 \qquad 2 - (-0) = 2 - 0 = 2 \qquad \frac{-0}{2} = 0 = \frac{0}{2} = 0$

$2 \cdot -0 = 2 \cdot 0 = 0$

Division by 0 will result in sign changes for naught. See text Null Algebra on properties of null math naught and 0.

$\frac{+2}{+0} = +\infty = +\eta_0 \qquad \frac{-2}{+0} = -\infty = -\eta_0 \qquad \frac{-2}{-0} = +\infty = +\eta_0 \qquad \frac{+2}{-0} = -\infty = -\eta_0$

Multiplication of 0 with 0 also requires no special consideration. Again there is no magnitudinal change and both +0 and -0 occupy the same point. The sign is simply resolved to positive, or unsigned.

2.4.b.iii

$0 \cdot 0 = 0 \qquad -0 \cdot 0 = -0 = 0 \qquad 0 \cdot -0 = -0 = 0 \qquad -0 \cdot -0 = 0$

2.4.c:

Where the sign of 0 matters is when it interacts with itself in division. Because the numerator and denominator have the same magnitude, though its a magnitude of 0, the expression can be interpreted as asking how many times a complete set of 0 size can fit into a complete set of size 0. The answer is 1 time as the set is already that size. The sign of the expression will depend upon the sign of both the numerator and denominator. This is different from considering the same expression as asking how many times something of size 0 can be divided into 0 parts which yields and infinity. In this situation we are considering the former situation rather than the later. For the reasoning behind this and a full explanation of the resolution of all values of the indeterminate form, as well as when each applies see Null Algebra (https://vixra.org/abs/2103.0131). We have already shown that 0 is both *positive-and-negative.* Maintaining focus on positve-and-negative status, the possible arrangement of the signs for the indeterminate and its resolved value are shown here below.

2.4.c.i:

$\frac{+0}{+0} = 1 \qquad \frac{-0}{+0} = -1 \qquad \frac{+0}{-0} = -1 \qquad \frac{-0}{-0} = 1$

So which of these is a valid interpretation of the expression? They all are simultaneously and must all be considered. Because each 0 is simultaneously *plus-and-minus* both +1 and -1 must simultaneously be held as legitimate evaluations of $\frac{0}{0}$. Thus $\frac{0}{0}$ is a *plus-and-minus* number of magnitude 1.

2.4.c.ii:

$$\frac{0}{0} = \oplus 1$$

Continuing Key Points:

1. This feature is identical to the paired halves of complex number, $\oplus\, bi$, marked on the complex plane as $+bi$ and $-bi$

2. $\left(\frac{0}{0}\right)^2 \equiv \left(\frac{+0}{0} \cdot -\frac{0}{0}\right) = -\frac{0}{0} = -\hat{1} = -1$ If the value of $\frac{0}{0}$ is not resolved to $\oplus$ 1squaring it maintains a magnitude of 1 as its form remains unchanged. It does however pick up a negative as detailed in the next key points.

3. This implies $\frac{0}{0}$ is a complex number $\oplus$ 1 resolvable to $\hat{1}$ and $\check{1}$ which are synonymous with +bi = +1i = *i* and -bi = -1i = -*i*.

4. This further implies that because $\left(\frac{0}{0}\right)^2 \equiv \left(\frac{+0}{0} \cdot -\frac{0}{0}\right) = -\frac{0}{0} = -\hat{1} = -1$, the squaring of $\frac{0}{0}$ is the squaring of a positive and negative number, requiring its complex conjugate halves must be multiplied together. Thus when RESOLVING $\frac{0}{0}$ to $\oplus$ 1 before squaring it still maintains its magnitude of 1 but now directly implies multiplication of complex conjugate halves, resulting in the same value obtained in key point note 2 above.

$$\left(\frac{0}{0}\right)^2 = (\oplus\, 1)^2 = \hat{1} \cdot \check{1} = 1 \cdot -1 = -1$$

5. All of this implies that $\frac{0}{0} = i$
6. The upper quadrants of the Complex Plane apply to equations within which the occurrence of an *i*-multiple is generated, whilst the lower quadrants apply to the subspace of the generating equation.

3.1—Resolving $\oplus$ Numbers:

If you have the presence of a $\frac{0}{0}$ in an equation you can, depending on the circumstances replace the value with a $\oplus$1. Further the key points show the properties of $\frac{0}{0}$ make it identical to the complex number *i*.

For specifics on the circumstances which dictate the appropriate resolution of the various values of $\frac{0}{0}$, *as well as a detailed description on resolution of* $\oplus$ *numbers from negative root arguments see Null*

3.1.a:

Thus the value of *i*: $\sqrt{-1} = i = \frac{0}{0} = \oplus 1$

This is the form numbers take when calculated as the root of a negative argument. For example:

3.1.b:

$$\sqrt{-4} = \oplus 2 = 2i \doteq \hat{2} \doteq 2$$

Note that, $\oplus$, means saying a number is a *plus-and-minus* number, the equivalent to saying it is the imaginary only part of a complex number, without specifying + or - but rather requiring it be both simultaneously.

3.1.c:

$$z = a + bi \quad = \quad a + (\oplus bi)$$

This is another indicator that the *i*-axis does not really have a + and - side, but rather a single side using these convenient signs to denote the positions of the paired halves of each single $\oplus$ number. To perform most operations with plus-and-minus numbers you must resolve them to their appropriate single sign value at place of occurrence. For full explanation of this process see Null Algebra section on resolving $\oplus$ numbers with subspace transformations (https://vixra.org/abs/2103.0131).

If we assume we are using equations of the form $y = f(x)$ we know we are in the *xy*-plane. Null Algebra tells us we then have the Adjoining *xu*-Subspace Plane, and the Co-Adjoining *sy*-Subspace Plane. There are additional outputs included on these planes obtained through transformations on the given $y = f(x)$ equation.

3.1.d—Example Function:

Consider the given equation:

3.1.d.i:

$$y = 2x + \sqrt{-(x)}$$

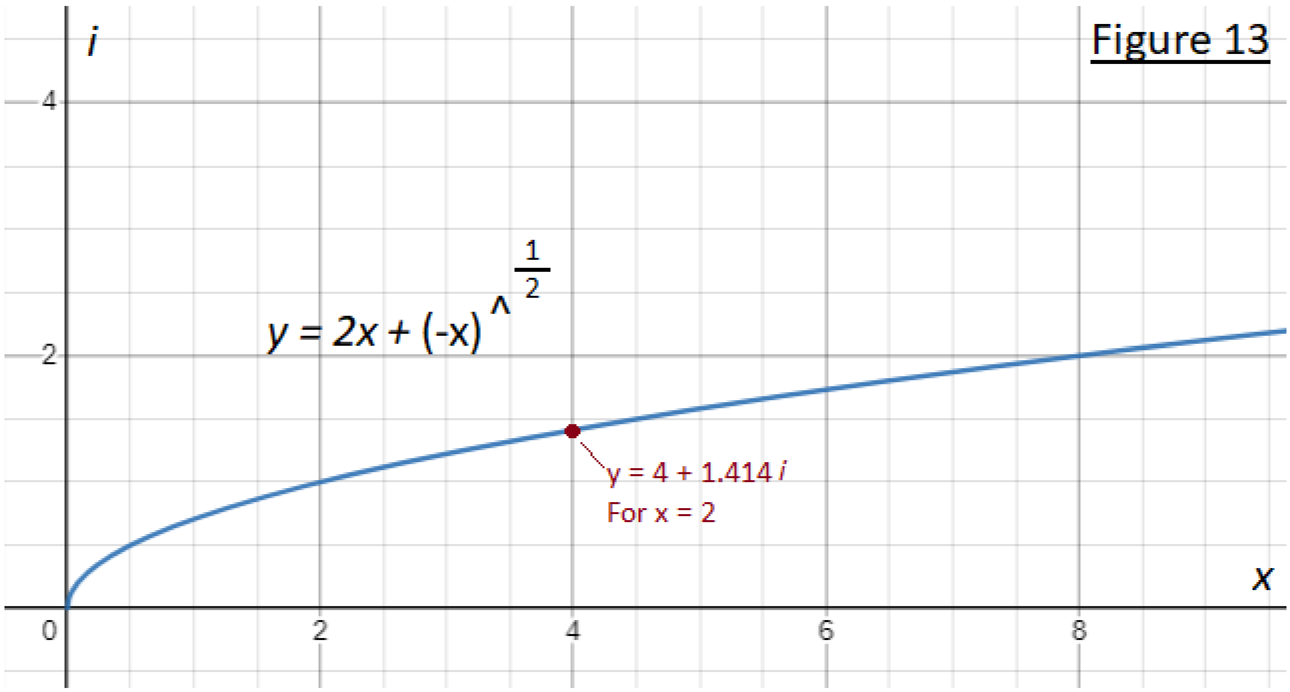
Fort his example we will consider only the positive solution to the radical (*square roots still produce ± solutions*) and both the + and - aspects of the $\oplus$ numbers which the negative argument in this example will produce. We have an equation which is structured to provide a *y*-axis output values in the form of complex numbers, having a real part defined by *2x* and an imaginary part defined by $\sqrt{-(x)}$for the traditional domain of x > 0. Lets examine several of the values of the output generated by positive integer inputs. *Note in some instances decimals in the chart below have been heavily truncated.*

$y = 2x + \sqrt{-(x)}$

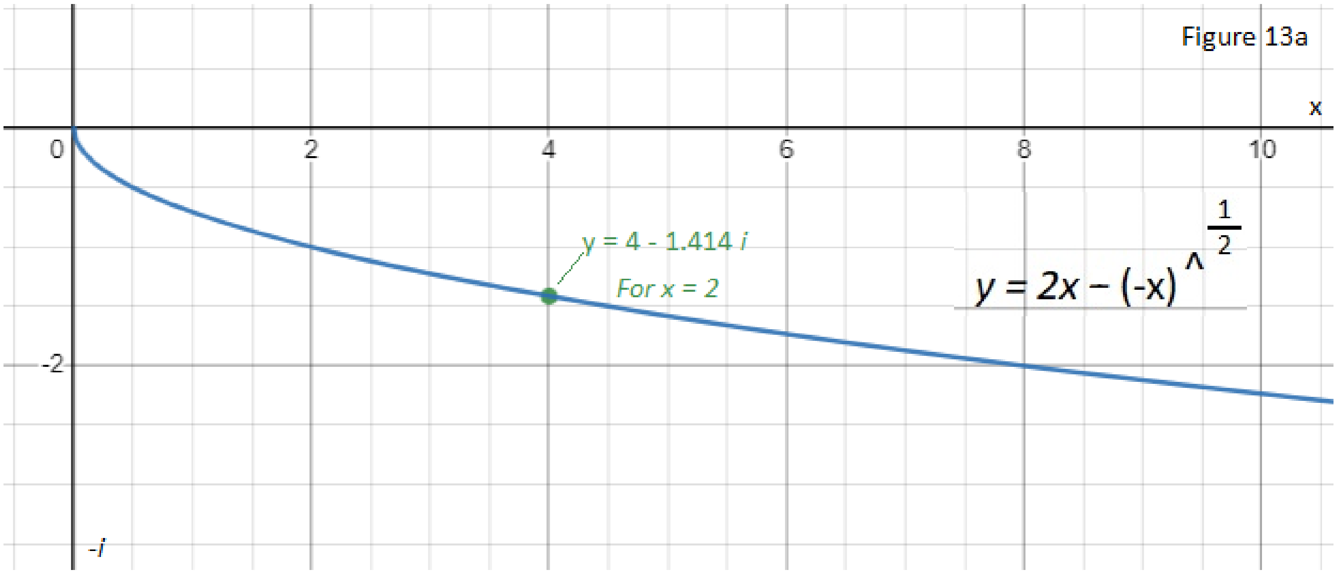
x-value	Real Part x	*y*-value	*Resolved y*	*i*-value	Resolved *y*-conjugate	Conjugate *y*-value	Conjugate *i*-value
1	2	2+i	3	i	1	2-i	-i

2	4	4+1.414i	5.414	1.414i	2.585	4-1.414i	-1.414i
3	6	6+1.732i	7.732	1.732i	4.2679	6-1.732i	-1.732i
4	8	8+2i	10	2i	6	8-2i	-2i
5	10	10+2.236i	12.236	2.236i	7.7639	10-2.236i	-2.236i
6	12	12+2.449i	14.449	2.449i	9.5505	12-2.449i	-2.449i

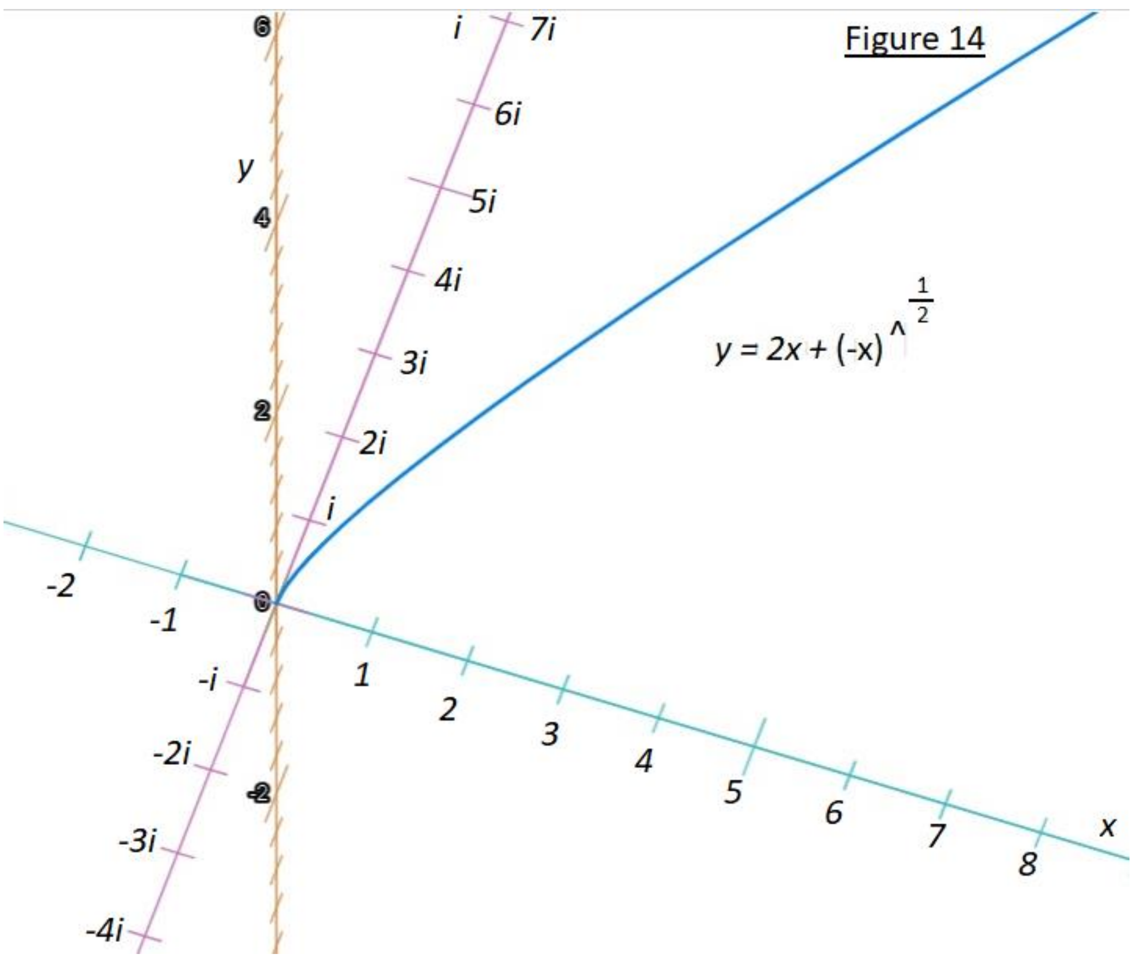
These values would traditionally be planted on the complex plane, showing the *y* output values (*an output plot*) as two dimensional points of the form $x + bi$ plotted in the graph below on he *xi*-plane.



Although we have only specified **+bi** components each has a **-bi** partner. The complex conjugate graph of $y = 2x - \sqrt{-(x)}$ shown below in Figure 13a. The subtraction of the *i* term is coming from the negative complex conjugate, not from the ± aspect of the radical bar in this example. This is only because we are limiting consideration to only the positive result of the radical solutions.



The function and its graph can be expanded to the *xyi*-hyperplane seen below in the graphs of Figures 14, 14a and 14b. As was shown in the previous examples these graphs are true input-output graphs displaying the full characteristics of the complex point's with *x*-axis inputs, and output values on the *i* and *y* axes.



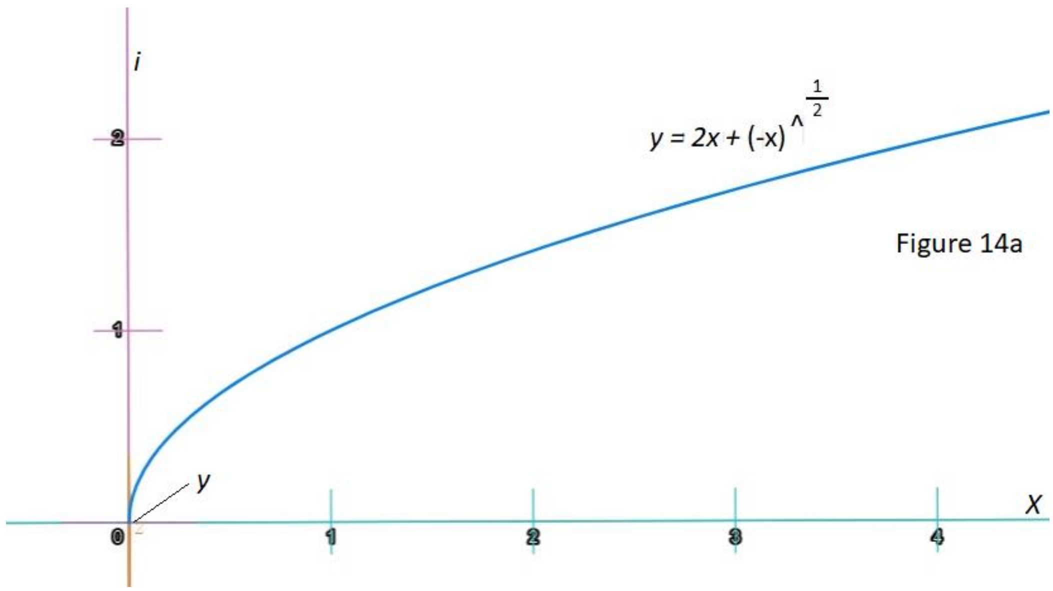


Figure 14a

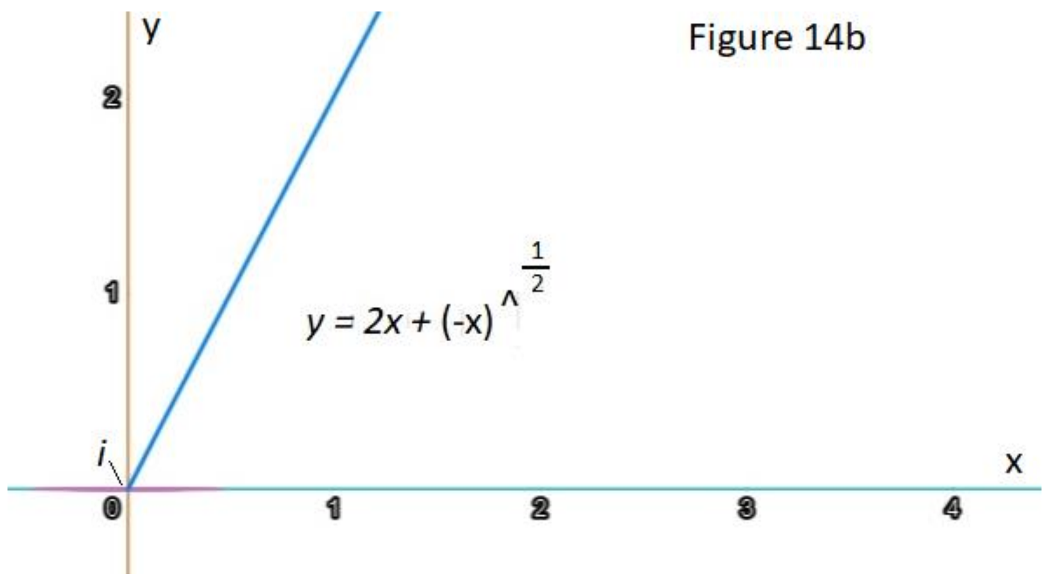


Figure 14b

3.2—Resolving the Complex numbers as plus-and-minus numbers:

We no consider resolving the ⊕ numbers which complex *i*-multiples represent. Because this equation example is of the form $y = 2x + \sqrt{-(x)} = a + bi$ the complex, imaginary values are being added to the real *a* component defined in this example by 2*x* to produce an output on the *y*-axis. The addition sign is present as we are only considering the positive solutions to the radical. The root of the negative arguments will produce ⊕ numbers. As they are occurring on the XY-Plane as a result of an $y = f(x)$ equation, they will resolve to positive, *up*, values on this plane. The process of resolving the up as well as the down values will generate line graph through an actual three-directional, xys-volume.

The imaginary *bi* components defined by $\sqrt{-(x)}$ in the given example equation, $y = 2x + \sqrt{-(x)}$ will resolve to up values on the xy-plane and simultaneously to down values on the co-adjoining subspace sy-plane.

To see this relationship we must resolve the ⊕ values. Note that the *xi*-plane is what we get when we stretch the *x*-axis into a plane whose new perpendicular axis represents additions and subtractions to *x*-inputs as complex, *i*-multiple pairs. This creates a plane representing two-dimensional numbers which can be used to represent input vales, or output values in terms of the two-dimensional inputs.

When we resolve the xi-plane inputs they will instead represent *x*-value inputs, which are paired with a subspace, *s*-value inputs.

<u>3.2.a:</u>

Given the example equation $y = 2x + \sqrt{-(x)}$ where $b = \sqrt{|x|}$

Then: $y = 2x + \sqrt{-(x)}$ = $y = 2x + bi$ ≐ $y = 2x$ + (⊕b)

≐ $y = 2x + \hat{b}$ ≐ $y = 2x + \dot{b}$ on the *yx*-plane

*The full process and reasoning behind this see Null Algebra Text (*https://vixra.org/abs/2103.0131*).*

The example equation has *i*-multiples being generated from the *x*-axis input values. This side of the equation defines the *y*-axis output variable. Becasue these inputs are two dimensional due to the nature of *i*-multiples we must account for their complex conjugate, the resolved down values which pertain to the *s*-axis in this example, and also define the *y*-axis output. The sy-equaiton is provided by the subspace transform below.

$$y = 2\frac{\Xi}{\varsigma s}x + \sqrt{-\left(\frac{\Xi}{\varsigma s}x\right)} \quad \rightarrow \quad y = -\frac{2}{s} + \sqrt{-\left(-\frac{1}{s}\right)} \rightarrow \quad y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$$

Shortly we will discuss how the equations $y = 2x + \sqrt{-(x)}$ and $y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$ are related to each other and the complex plane. We will discuss that the resolution of the set of complex points defined by $y = 2x + \sqrt{-(x)}$ on the *xiy*-hyperplane necessitates their resolved values lie on the *xys*-hyperplane.

From Null Algebra we know ⊕ numbers resolve to their positive magnitude value on the dimensional plane in which they originate. For the equation $y = 2x + \sqrt{-(x)}$ the originating dimension is the *xy*-plane, or rather the *xiy*-hyperplane. Null Algebra also provides that the resolved partner to these positive, up, vales will lie on the co-adjoining subspace plane, the *sy*-plane. Here will be the negative, down, conjugate values to the up values resolved on the *xy*-plane. They are negative in terms of the variable assigned to the dimension which is generating their presence. This is crucial to undertanding the higher multiples of i^n which appear to diverge between traditional and null algebra math disicplines. When dealing with an equation of the form $y = f(x)$ which is generating complex number outputs, the changes made in resolving the *i*-multiples will apply their positive component to the *real* portion of the *x* input first.

<u>Key Points Continued:</u>

1. One must ultimately choose how to view these types of equations. You can keep the unresolved *i* values and use transitional complex analysis which has a many important applications in fields such as Quantum Mechanics and analyzing the alternating current in circuitry. This unresolved status in not wrong in anyway. It is merely how the math looks when the subspace access containing the down components is forced into a 2D plane with an apparent two sided, + and - side, *i*-axis representing both paired halves in relation only to *x* (*assuming equations are initially evaluated on the xy-plane*).

   The resolved values of *i*-simply provide the real number outputs equivalents for the value of *i*-multiples occurring in a given equation. This is done by including the co-adjoining subspace plane attached to the central plane (*the real space plane on which the occurrence of an i-multiple arises from a given equation*), as one hyperplane surface, containing both resolved complex conjugates.

2. The *i*-axis though traditionally drawn with a + and - side is really a single sided axis plotting paired halves of bi numbers.

3. Thereby it is better to mark the *i*-axis with $\hat{\imath}$ and $\check{\imath}$.
4. The *up*-values apply to the axis where the ⊕ number originated. The *down*-value applies to the subspace of the axis from where the ⊕ originates.

5. These concepts mean if a ⊕ number originates on the *xy*-plane for equations of the form

   y = a + bi = $f_1(x) + \sqrt{-f_2(x)}$

   The upper two quadrants of *xi*-plane apply to *x*-axis, whilst the lower two quadrants apply to the *s*-axis, as its values are seen in terms of *x*. *i.e.* They require a subspace transform to be seen in terms of *s*.

6. *S is the negative reciprocal of x.* This accounts for the apparent discrepancies of higher values of *i*-multiples.

   For example the unresolved $i^3 = -i$
   Resolved this value is +1, on the *s*-axis, while its value in terms of *x* is -1.

   $-i = -\hat{\imath} = -\hat{1} = +\check{\imath} = +\check{1}$

<u>4.1—Multiples of *i*:</u>

Lets return to two concepts. We´ll re-examine the chart showing the differences between the traditional, unresolved values of various *i*-multiples, and the Null Algebra resolved values for those same instances. We will then compare those values to the traditional *xi*-plane and the resolved version of the same.

4.1.a:

i^n	i^1	i^2	i^3	i^4	$i^5 \equiv i^1$	$i^6 \equiv i^2$	$i^7 \equiv i^3$	Pattern Continues
Traditional Algebra	i	-1	-i	+1	i	-1	-i	
Null Algebra	$\oplus 1 \rightarrow \hat{1}$ $\rightarrow \check{1}$	-1	+1	-1	+1	-1	+i	

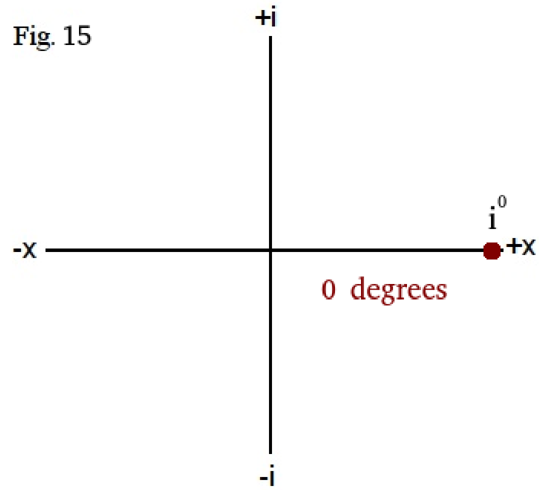
Clearly something more is going on than just saying values like $i^3 = -i$ are simply unresolved; for these higher exponential values we cannot just swap in a one for i. Values like i^4 appear to be in disagreement, equaling +1 in Traditional Algebra and -1 in Null Algebra. And yet, every fourth power of i after $n = 2$ results in identical values between the two systems.

4.1.b:

-1 = $i^2 \equiv\ i^6 \equiv\ i^{10} \equiv\ i^{14} \equiv\ \ldots$.

Recall that on the *xi*-plane each such subsequent power of *i* is identical to a 90-degree turn. We begin an exploration of this effect and how its interpretation resolves the apparent discrepancy in 4.1.a above, at the position of *x* = 1, the complex number $z = x + bi = 1 + 0i = 1$.

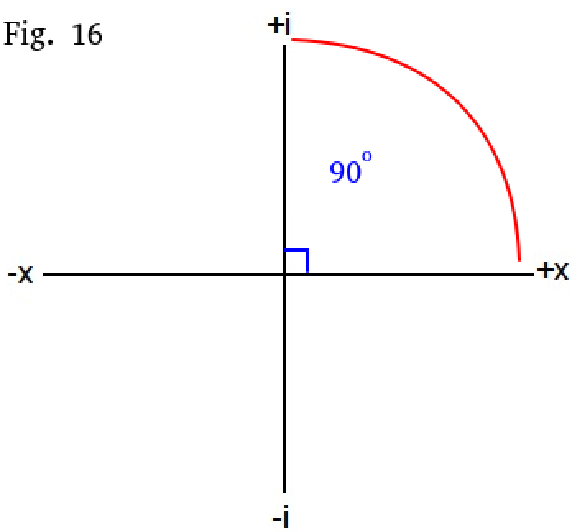
Fig. 15



This point (*x* , *bi*) = (1 , 0) is identical to *x* = 1 = i^0. This is the point *x* = 1 on an angle of 0°. It exists on the *x*-axis telling us we are on the Real axis. We shall assume these points are being generated by an equation on the *xy*-plane, of the form $y = a + bi = f_1(x) + \sqrt{-(f_2(x))}$.

A rotation of 90 is represented by $i^1$and the complex number $a + bi = 0 + 1i = i$.
This is the point (*x* , *bi*) = (0 , *i*).

Fig. 16



This point, the value $i^1 = i$ is an unresolved value. It is a positive *i* value with a magnitude of 1. This *imaginary* value is the complex number *y* = *z* = *a*+*bi* = 0 + *i* = *i*. It has a complex conjugate z^* having the same magnitude but opposite sign, given by y = z^* = *a*-*bi* = 0 - *i* = -*i*. This value is the point (*x* , *bi*) = (0 , -*i*), and is covered in a moment below.

Null Algebra has shown *i* to be associated with $\frac{0}{0}$ and its resolved value $\oplus 1$, which resolves to $\hat{1} = +1$ on the axis of occurrence of the *i*-multiple. The complex conjugate resolves to $\check{1} = -1$ but occurs on the subspace of the axis to that of occurrence. This is the value of the subspace as seen in terms of the variable representing the axis of occurrence.

The given situation for i^1 is that this represents $z = 0 + i$ on the *xi*-plane, a positive *i*-multiple added to the *x* variable. The resolved version of this value is shown here:

<u>4.1.c:</u>
Given a value $z = a+bi$ with $a = 0$ and $b = 1$ the resolved value is:
$z = x + bi$
$z = 0 + i \;\rightarrow\; z = 0 + \overset{\Uparrow}{1} \;\rightarrow z = 0 + 1 \;\rightarrow z = 1$

As we are assuming the equation generating the *i*-multipls is of the form $y = f(x)$ we can simply replace the usage of *z* with *y*. Shown here below.

$y = x + bi$
$y = 0 + i \;\rightarrow\; y = 0 + \overset{\Uparrow}{1} \rightarrow y = 0 + 1 \;\rightarrow y = 1$

Note that the complex conjugate implied to exist is the value given below:

<u>4.1.d:</u>
$z = x - bi$

Because we are discussing the instance of i^1 which is the value $y = 0+i = 0 + \overset{\Uparrow}{1} = 1$, we can obtain the complex conjugate by assigning the down value to *i* as we resolve it.

$y = z = 0 + i \;\rightarrow\; z^* = 0 + \check{1} \;\rightarrow z^* = 0 - 1 \;\rightarrow z^* = -1$

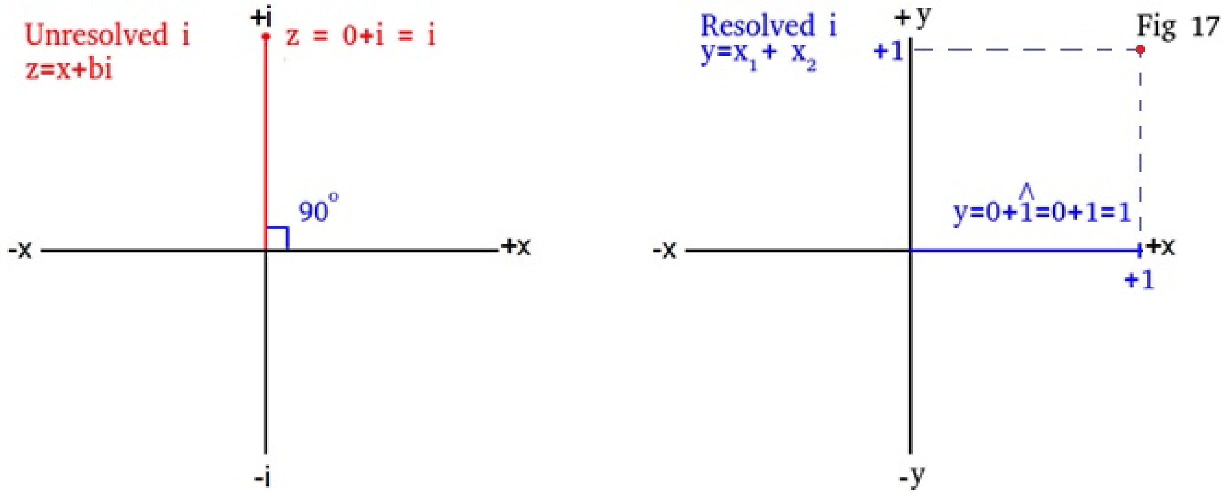
$y = 0 + i \;\rightarrow\; y^* = 0 + \check{1} \;\rightarrow y^* = 0 -1 \;\rightarrow y^* = -1$

Recall that these values, being the down component, are representative of the value added to the *s* variable but shown here in terms of *x*. Thus this value exists on the *s*-axis subspace. The Null Algebra subspace transformations show the *s*-axis is the negative reciprocate of the *x*-axis.

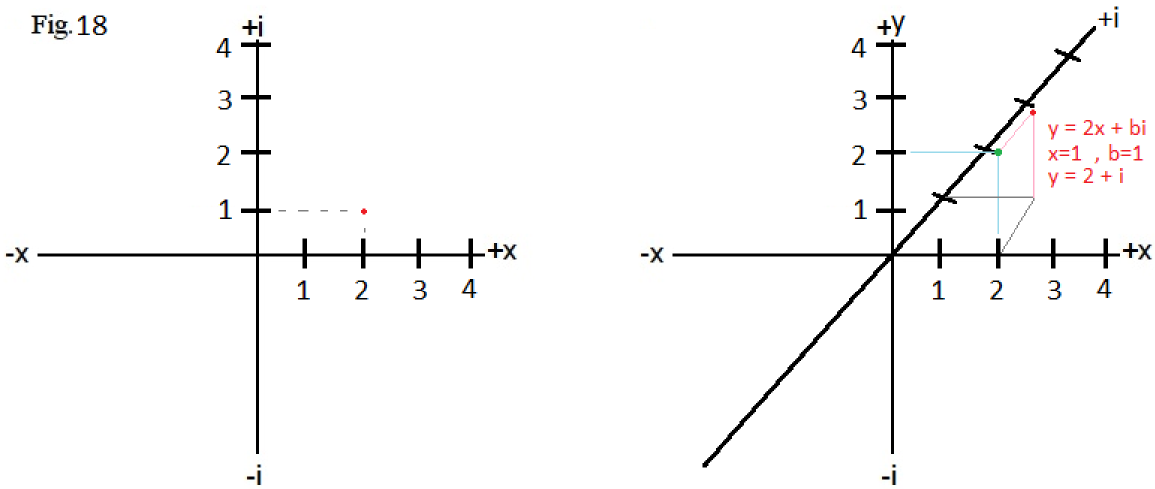
<u>4.1.e:</u>
$x = -\frac{1}{s}$ $\qquad\qquad s = -\frac{1}{x}$

If the magnitude of the down portion of the *i*-multiple is 1 we are dealing with the down portion of number *i* itself. This is the value of -1, seen from the *x*-axis in terms of *x*. On the *s*-axis, this value in terms of *s* is the negative reciprocate. Thus the same value on the *s*-axis, in terms of the *s* variable, is +1, and corresponds to the resolved i^3 value. This concept will be returned to in a moment.

See Figure 17 below.



If using the example equation from before, $y = 2x + \sqrt{-(x)}$ when $x = 2$ we have the situation shown below.



4.1.f:

$f(2) = 2(2) + \sqrt{-2} \;=\; 4 + 1.414i \;=\; 4 + (1.414) \cdot \overset{\uparrow}{1} \;=\; \dot{5}.414$

The value $4 + 1.414i$ was shown plotted in the output graph of $y = 2x + \sqrt{-(x)}$, Figure 13, in section 3.1.d above. The complex conjugate of this point is shown in the output graph of the same function of Figure 13.a. The full three direction *xiy*-hyperplane of this function was shown in Figure 14. The imaginary portion defined by $\sqrt{-(x)}$ is shown in Figure 14.a, and the real portion, 2*x*, is shown in Figure 14.b. Both Figures 14a and 14b are rotations of the three directional plane to look directly downward along the *y* and *i* axis respectively showing the corresponding two directional planes which remain seen. What we are seeing now is the resolution of the *i*-multiple. The imaginary component obtained when *x* = 2 has been resolved to its positive, up, component for the central plane and axis of occurrence defined by the format of an *y* = *f*(*x*) equation. It is then added to the real part producing an single value for *y* on the *xy* central plane. The dotted value above the 5 indicates this value is the result of a resolved *i*-multiple and is a way of tracking that *i*-multiples were present in obtaining this solution. We will graph this resolved result toward the conclusion of this paper which will show the xys-volume, as well as the xy, and sy hyperplanes.

<u>4.2—i^2, The 180-degree rotation:</u>

i^2 and each subsequent fourth multiple of it, $i^{(2+4n)}$ for $1 < n < \infty$, will equal -1 for both traditional Algebra and Null Algebra. It is identical to a rotation of 180° on the *xi*-plane.

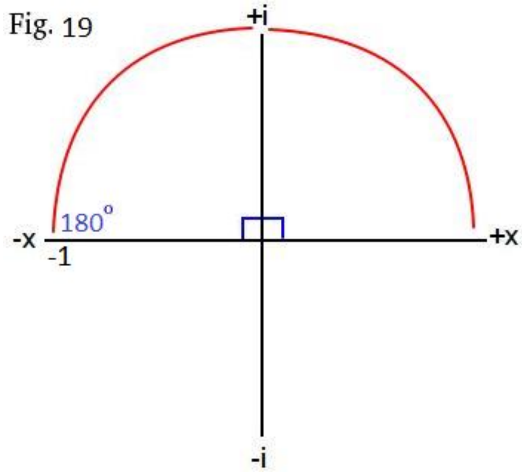


Fig. 19

The traditional Algebra interpretation of this value is from the definition which requires it exist as $i^2 = -1$. It has also been shown to be the limit of the Maclaurin approximation of $e^{i\pi}$ which details a spiraling motion that ever more closely approaches *x* = -1 on the *xi*-plane.

The Null Algebra resolution of this point equals -1 as well but for a different reason.

Because the resolved $\frac{0}{0} = \oplus 1$ represents two paired values, a $\hat{1}$ and a $\check{1}$ equivalent to the complex number and complex conjugate pairs of *z* and z^*, the square of both sets is identical to multiplying the conjugate halves together:

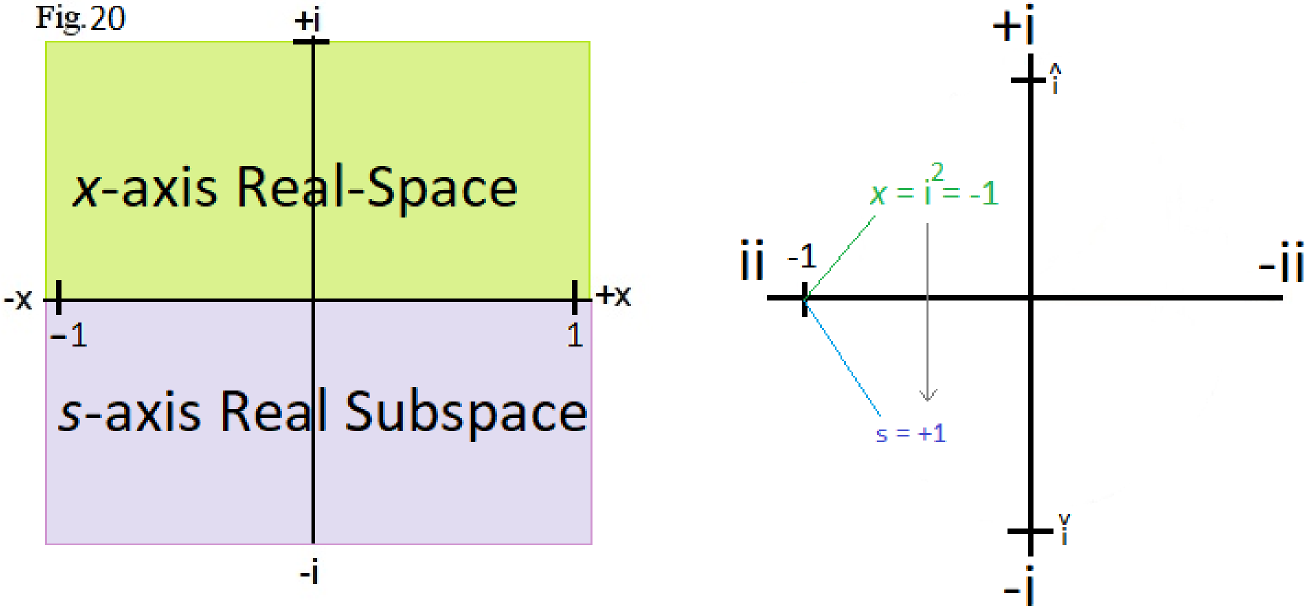
<u>4.2.a:</u>

$$i^2 = (\oplus 1)^2 = \hat{1} \cdot \check{1} = +1 \cdot -1 = -1$$

Again though the $i^2 = -1$ as a resolved value is the value seen form the *x*-axis, in terms of the *x* variable, on the *xy*-plane, the plane we have specified for these examples where equations are generating *i*-multiples. This point is special for another reason. Further rotations will move into the lower two quadrants of the *xi*-plane which we have already specified pertains the complex conjugate pairs of the *xi*-plane. These values when resolved exist on the given subspace of the generating equation. In this example that subspace is the *s*-axis. Though the value of $i^2 = -1$ as seen from the *x*-axis, the *s*-axis equivalent is its negative magnitude counterpart, +1. Consider the following resolution of the *xi*-plane viewed as only complex components. This point is a transition from the upper quadrants which represent positive resolved values added to *x* and the lower quadrants that represent the negative resolved values applied to the *s*-axis. After moving beyond the 180 degree rotation values of the resolved Null Algebra powers of *i* will be those of the values on the *s*-axis, but still viewed in terms of the *x* variable unless a transform is applied to them. Those same values seen on the unresolved *xi*-plane will show the values as they are seen in terms of *xi* variables from the *xi*-plane.

See Figure 20:

Fig. 20

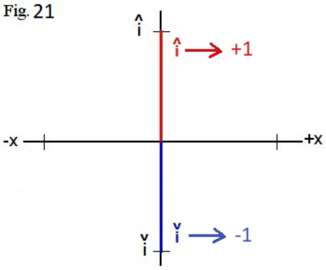


4.3: i^3—270-Degree Rotation:

We now make a rotation of 270° to the point $z^* = x - bi \rightarrow z^* = 0 - i = i^3$. The value $i^3 = -i$ on the *xi*-plane but we know this representation of z^* as a resolved number is the down component of a ⊕ number as seen from the *x*-axis, the origin point for the generation of the *i*-multiples in the given example. That value is applied to the *s*-axis when resolved; a down as seen in terms of the *x* variable, but the negative reciprocate, an up viewed in terms of the *s*-variable. The chart of values shown before in section 4.1.a, shows the traditional value of $i^3 = -i$, but also specifies the resolved Null Algebra value is $i^3 = 1$.

Key Points:

Fig. 21

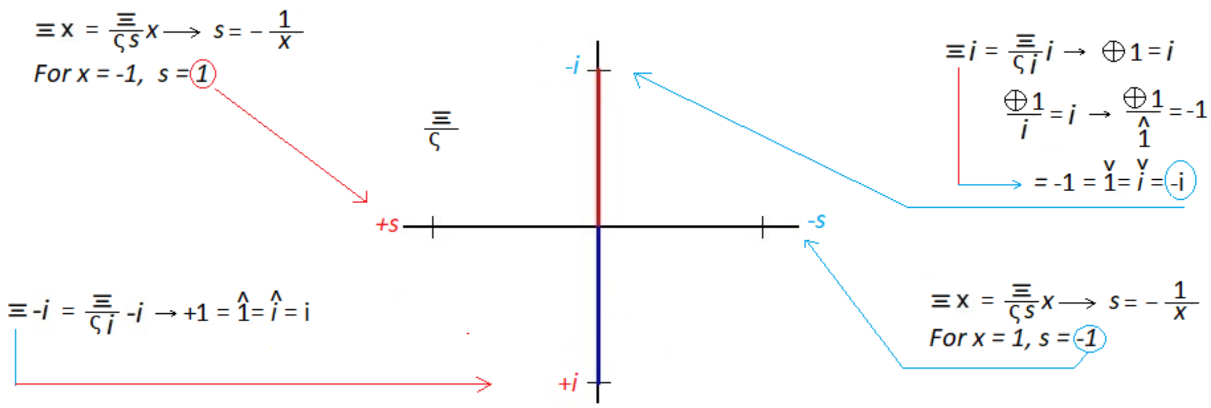


The +1 of i^3 is the +1 of the *s*-axis.

From the *xi*-plane:

As unresolved values on the *xi*-plane we may resovle both directly as the occurance of $i = \frac{0}{0} =$ ⊕1 = î on the *x*-axis. The one-up value applies to the *x* axis value. The one-down will apply to the *s*-axis, the entire lower two quadrants of the complex plane.

Fig. 22



Thus from the xi-plane, an equation of the form $y = f(x)$ generating *i*-multiple values will resolve such values to a positive-up magnitude component which pertains to the *xy*-plane, and conjugate negative-down magnitude component which pertains to the *sy*-plane. For the value $i = \frac{0}{0} = \oplus 1$, the value will resolve to $\hat{1} \doteq 1$.

For $y = \sqrt{x}$ with x = -1

$y = i = \oplus 1 = \hat{1} \doteq 1$

For the same equation, $y = \sqrt{x}$ with x = -1, the down value will pertain to the *s*-axis. From the perspective of the *xi*-plane which is used to display two-dimenisonal xi output values representing *y*, this is the complex conjugate -*i*. From the perspective of the *xi*-plane this value will resolve to -1.

Because the given function generating the *i*-multiples for is of the form $y = f(x)$ the entire *xi*-plane pertains to the *y*-axis but remains in terms of the *x-variable* whose inputs are generating the *i*-multiples. Thus down component must also pertain to the *y*-axis. But in resolving the ⊕1 to its single signed values we find from Null Algebra that the positive-up component pertains to the axis and plane where the *i*-multiple is generated, while its complex conjugate applies to a subspace. That subspace axis in this example, as the negative-down value must also pertain to the *y*-axis, is the *s*-axis subspace of the *x*-axis. Performing a transform on the up value will provide the down value seen in terms of *x* is also an *up* value in its own right seen in terms of *s* as these variables are negative reciprocals of each others. This is in keeping with any occurrence of an *i* being in fact two paired values shown unresolved as ⊕1.

The resolution of the *xi*-plane shows all all values in terms of the *xi*-plane variables, including the -*i* found at a rotation of 270 degrees defined by i^3. Because this down value must be on the *s*-axis of the *sy*-plane, to see the values of pertaining to this axis we must perform a subspace transform on the axis variables of the *xi*-plane, producing the *si*-subspace plane.

This +*i* at the bottom of the *si*-plane is the value provided when calculating the resolved value solution for i^3. This graph of the *si*-plane reveals something else as well. The value of i^2 = -1 on the *xi*-axis. It is not until after we pass 180 degrees of rotation that the *si*-plane values take over, and yet the conjugate subspace value for x = =1, remains s = +1 which is shown in the graph of Figure 22. This effect will occur again at 360 degree rotation. The value i^4=+1 applies to the *xi*-plane. Until you pass the 360 degrees of rotation you have not yet moved back into the upper quadrants which pertain to the *xi*-plane. The Null Algebra resolved solution for i^4=-1 applies to the *si*-plane and will require a subspace transform to show its corresponding conjugate value on the *xi*-plane, which is +1.

We see not only is the *s*-axis reversed of the *x*-axis, but the *i*-axis is also present and reversed. These two planes, the *xi*-plane and *si*-plane share the same *i*-axis from different perspectives. What it is that makes the complex plane, *complex,* is that the *xi* and *si*-planes are in reality the *xys*-hyperplane for any equation of the form $y = f(x)$ generating *i*-multiple components.

4.4: i^4-360 degree rotation.

The rotation to 360 moves back to the point $y = a + bi = 1 + 0i = 1$. This is the point $y = (x,i) = (1,0)$. We make this specification as a reminder that when discussing the traditional form of the complex plane we are looking at an output plot, showing two dimensional points representing the variable *y*, assuming the equation generating the *i*-multiples was of the form $y = f(x)$.

The rotation to this point proceeds through the lower quadrants of the *xi*-plane and, even when coming to a rest on this exact point, has not yet moved into the upper quadrants of the *xi*-plane. Though this point is identical in value to a 0 degree rotation, a point whose value pertained to the central plane and the *x*-axis, the 360 degree rotation will pertain to the *s*-axis and the discouraging subspace plane. Like was shown in section 4.3, the value we obtain from the Null Algebra resolved value for $i^4 = -1$ pertains to the value as it appears on the *s*-axis of the *si*-plane. As seen in terms of the *x* variable from the *xi*-plane this value is the negative reciprocate of the *s*-axis value and obtainable through a subspace transform.

Given the subspace transform for the *x*-axis is: $x = -\frac{1}{s}$

For $s = -1$, $x = 1$

This value of +1 is the value shown in traditional algebra and trigonometry for i^4. Thus we provide the following augmented chart for the multiples of *i* as they are interpreted from traditional algebra, and their resolved values.

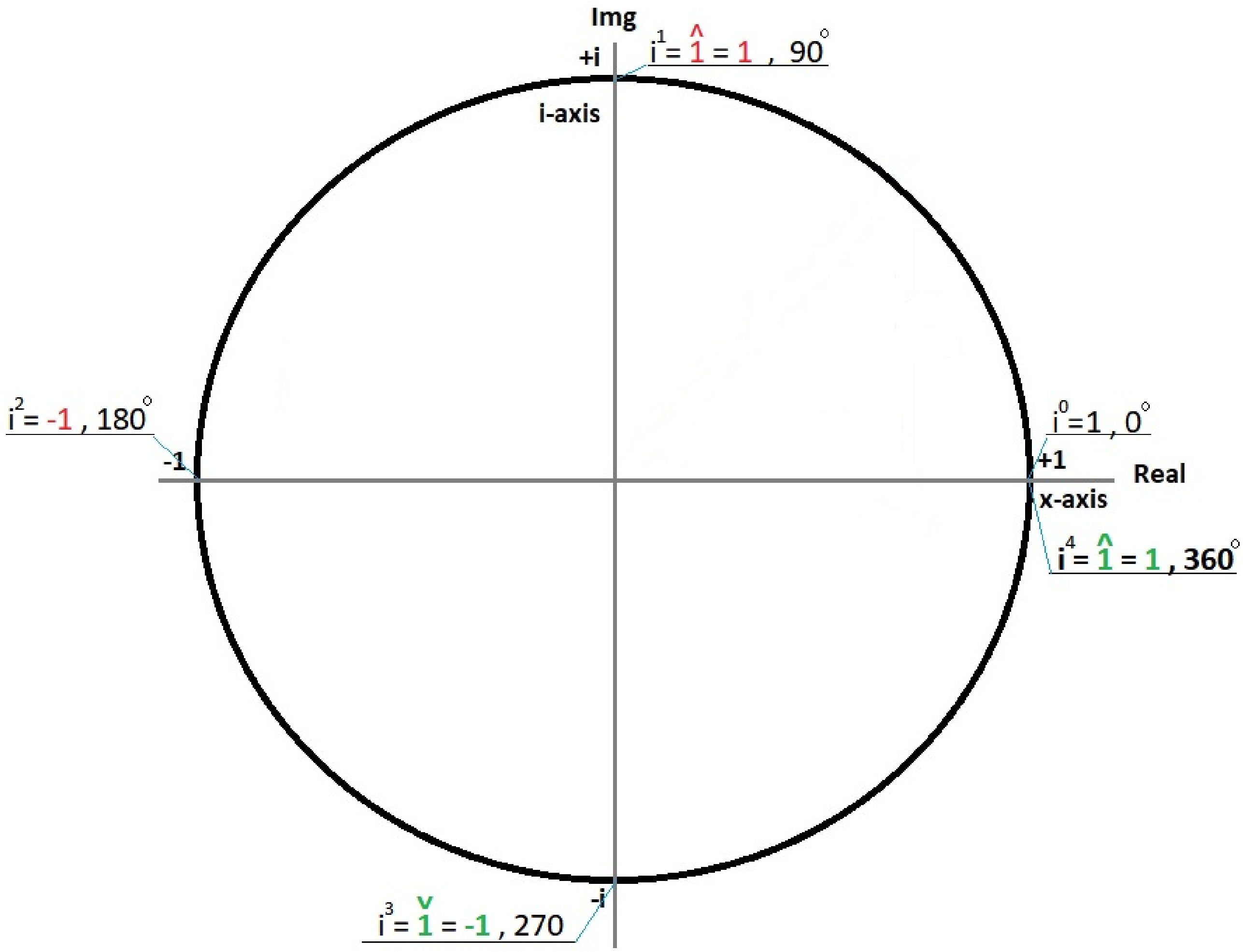
4.4.a—Values of *i* multiples and their relation to Algebra and Null Algebra:

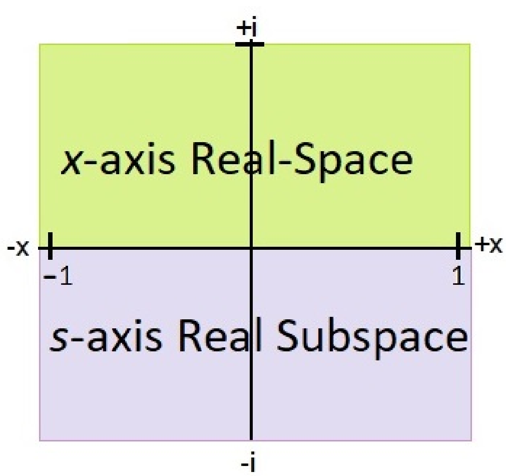
Blue Values: Values directly obtained from the Null Algebra resolutions of *i*-multiples.

Red Values: Values for *i*-multiples shown only in terms of the *x*-variable of the *xi*-plane. Those which the direct solution from Null Algebra provide a value in terms of the *s*-variable are shown in **GREEN** indicate this value has been augmented by a subspace transform to show it in term of the appropriate variable on the given *xi*-plane.

i^n	i^1	i^2	i^3	i^4	$i^5 \equiv i^1$	$i^6 \equiv i^2$	$i^7 \equiv i^3$	
Traditional Algebra	*i*	-1	-*i*	+1	*i*	-1	-*i*	Pattern Continues
Resolved Traditional Value Value in terms of *x*-axis	+1	-1	-1	+1	1	-1	-1	
Null Algebra	$\oplus 1 \rightarrow \hat{1}$ $\rightarrow \check{1}$	**-1**	**+1**	**-1**	**+1**	**-1**	**+1**	

	+1 at occurrence							
Direct result: in *x* variable	**+1**	**-1**	**-1**	**+1**	**+1**	**-1**	**-1**	
Direct result: in *s* variable	**-1**	**+1**	**+1**	**-1**	**-1**	**+1**	**+1**	





5.0—Resolution of the Complex Plane:

Using the information provided in the previous sections we are now ready to resolve an *i*-multiple generating equation of the form *y* = *f*(*x*), not to an *xiy*-hyperplane, but to the *xys*-hyperplane, a subspace volume though which the actual *xys*-points of such an equation are plotted.

5.1—The equation:

We will begin with an example equation already used to define *xiy*-points.

5.1.a:

$$y = 2x + \sqrt{-(x)}$$

We will be focusing on the *xys*-hyperplane. This equation makes clear, it is the *y* variable which is a two dimensional number, existing as the sum of a real part and imaginary part, itself composed of complex conjugate pairs. Both the real and imaginary parts vary with the input of the *x*-variable.

A simple subspace transform on the *x*-variable will generate the *y* = *f*(*s*) equation which corresponds to the example equation in 5.1.a.

5.1.b:

$$y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$$

We will begin by examining these two separate equations as they both represent the value of *y*.

5.1.c—*y* = *f*(*x*):

For simplicity will focus on only the positive values of the *x*-variable. These values constitute the extended *x*-axis domain as the structure of the equation will generate negative arguments for the radical. These arguments will produce solutions of the form:

5.1.c.i:

$\sqrt{-n} = \oplus\, b \doteq \hat{b}$ where $|b|^2 = |n|$

Radical bars still produce answers which are ±. Again for simplicity we will only concern ourselves with the positive value solution from the radical.

5.1.d:

The equation $y = 2x + \sqrt{-(x)}$ will produce for positive value of *x*, values of *y* in the form of

5.1.d.i:

y = *a* + *bi*

The resolved solutions on for the *y* = *f*(*x*) equation will plot onto the *xy*-plane as:

5.1.d.ii:

$y = a + bi = a + \hat{b} = a + \dot{b}$ Where $\dot{b}$ is the resolved positive value of magnitude *b*.

The *y* value will no longer be a two dimensional *xi* value but will instead be a single *f*(*x*) value of the form $a + \dot{b}$.

5.1.e:

The $y = 2x + \sqrt{-(x)}$ is generating *i*-multiples for the positive values of the *x*-variable, which is being resolved the to the positive-*up* value of the *plus-and-minus b*-magnitude of that *i*-multiple. The negative-*down* value of the *plus-and-minus* number, generated by $y = 2x + \sqrt{-(x)}$, for each positive *x*-variable input, are plotted on the co-adjoining subspace *sy*-plane.

This requires usage of the *y* = *f*(*s*) equation:

5.1.e.i:

$$y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$$

The conjugate value for *y*, which corresponds to each *y* = *a* + *bi* number, and is of the form *y* = *a* - *bi*, is plotted on this *sy*-plane. Because we are now dealing with *s*-axis inputs we must use the *s*-variable value which corresponds to the *x*-variable value that generated the *y* = *a* + *bi* number on the $y = 2x + \sqrt{-(x)}$ equation.

These input values of *s* are of the format

5.1.e.ii:

$$s = -\frac{1}{x}$$

This is the subspace transformation equation which relates *x* and *s* variable inputs. The chart below displays several values for all variables being considered. For *x* = 2, the corresponding *s*-axis input is $s = -\frac{1}{2}$.

When the corresponding *s*-variable is used it will generate from the equation $y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$, a value for *y* of the form:

5.1.e.iii:

$$y = a + bi = a + \check{b} = a - \dot{b}$$

$$\downarrow$$

$$= a - \hat{b} \quad = \quad a - bi$$

The value obtained when solving for *y* using the values which correspond to the given *x* inputs will generate an unresolved value for *y* which is equivalent to *y* = *a* - *bi*. The real and imaginary parts will be identical, differing only the + sign in the *y* = *f*(*x*) equation, and the - sign in the *y* = *f*(*s*) equation. Their resolved values will then differ by the addition or subtraction of the resolved *i*-multiple value. Like the *f*(*x*) equation the resolved value of the *f*(*s*) equation will produce a single value of *y* of the form

5.1.e.iv:

$y = a - \dot{b}$ Where $\dot{b}$ is the resolved positive value of magnitude *b*, *subtracted,* because it is the *down* value associated with the corresponding generating value form the *f*(*x*) equation.

5.2—Equivalence with *y*:

The values of *y* will differ based on the resolved solutions. This should not be surprising as even the unresolved values differ. Consider the single point denoted here below for the value *x* = 2.

5.2.a:

For *x* = 2 and $y = 2x + \sqrt{-(x)}$ $y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$

Then: $s = -\frac{1}{2}$

$y = 2(2) + \sqrt{-(4)} \rightarrow$ $y = 4 + 1.414i \rightarrow y = 4 + \hat{1}.414 \rightarrow y = 5.414$

$y = -\frac{2}{-\frac{1}{2}} + \sqrt{\frac{1}{-\frac{1}{2}}} \rightarrow y = 4 + 1.414i \rightarrow y = 4 + \check{1}.414 \rightarrow y = 4 - 1.414 \rightarrow y =$ 2.586

Despite this we sill have the situation that:

5.2.b:

$$2x + \sqrt{-(x)} = y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$$

This implies that for the value of *x* = 2, which produces the resolved value of *y* = 5.414, there must be some value on the *sy*-plane for the *s*-variable which will generate *y* = 5.414. An examination of the graphs generating these values will find this is true.

5.2.b.i:

For *s* = -0.2729191.... you will find *y* = 5.414.

The decimal value for s *here has been heavily truncated.*

This situation exists for all values shared between the two equations, $y = 2x + \sqrt{-(x)}$ and $y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$. Though the values for *x* and *s* inputs will likely be different, there shall exist inputs for each variable which generate the same *y* output value. This must be as both the *x* and *s* equation are related and share the output *y* axis.

Consider below the chart of several values for the various variables and the subsequent two directional graphs of the *xy*-plane and the *sy*-plane.

5.3—The *xys*-graph:

$y = 2x + \sqrt{-(x)}$ and $y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$

x Input	*x real part*	*x imaginary and resolved*	*y output for y = f(x)*	*s input from s = -1/x*	*s real part*	*s imaginary and resolved*	*y output for y = f(s)*
1	2	i 1	2 + i 3	-1	2	i -1	1
2	4	1.414i 1.414	4 + 1.414i 5.414	$-\frac{1}{2}$	4	1.414i -1.414	2.586
3	6	1.732i 1.732	6 + 1.732i 7.732	$-\frac{1}{3}$	6	1.732i -1.732	4.268
4	8	2i 2	8 + 2i 10	$-\frac{1}{4}$	8	2i -2	6
5	10	2.236i 2.236	10 + 2.236i 12.236	$-\frac{1}{5}$	10	2.236i -2.236	7.764
6	12	2.449i 2.449	12 + 2.449i 14.449	$-\frac{1}{6}$	12	2.449i -2.449	9.551
7	14	2.645i 2.645	14 + 2.645i 16.645	$-\frac{1}{7}$	14	2.645i -2.645	11.355
8	16	2.828i 2.828	16 + 2.828i 18.828	$-\frac{1}{8}$	16	2.828i -2.828	13.172
9	18	3i 3	18 + 3i 21	$-\frac{1}{9}$	18	3i -3	15
10	20	3.162i 3.162	20 + 3.162i 23.162	$-\frac{1}{10}$	20	3.162i -3.162	16.838

5.3.a

Figure 24 below shows several values for the graph of the *xy*-plane defined by the equation $y = 2x + \sqrt{-(x)}$. Consider the second row of values from the chart in section 5.3 above:

2	4	1.414i 1.414	4 + 1.414i 5.414	$-\frac{1}{2}$	4	1.414i -1.414	2.586

These are the values for the *f*(*x*) and *f*(*s*) functions for when *x* = 2. The value for *y* resolves to 5.414 on the *xy*-plane but to 2.586 on the *sy*-plane. Despite this because the *y*-axis is shard by both equations and both equations are equal to *y*, the *y* variable must have some *x*-input which also produces *y* = 2.586. Both of these points are shown below in the graphs of Figure 24, 24b and 24c.

y = *f*(*x*)
y = *f*(2) = 5.414

y = *f*(*x*)
y = *f*(0.8359) = 2.586

5.3.b:

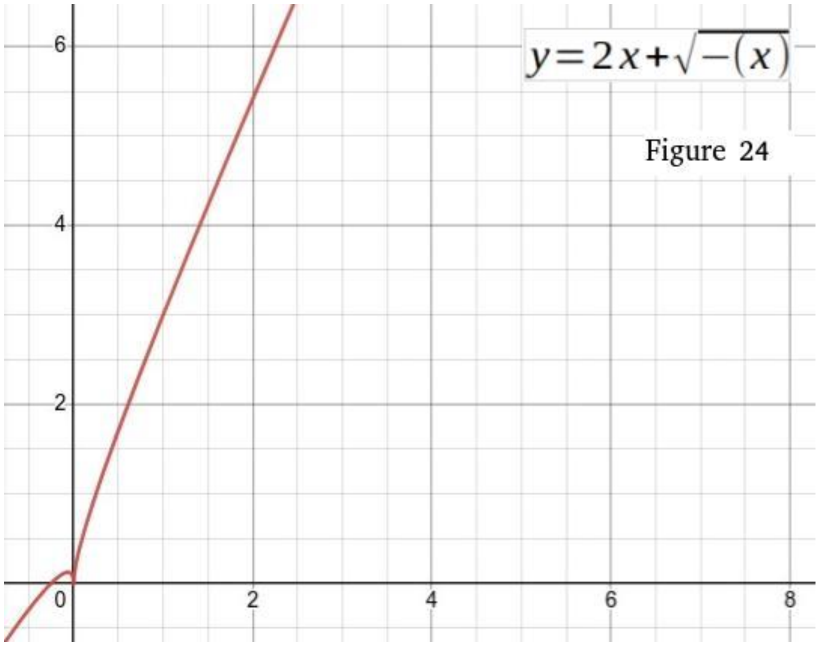


Figure 24

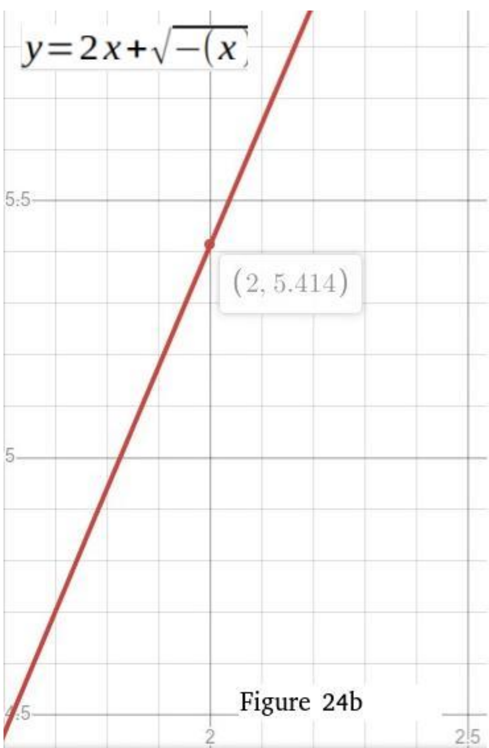


Figure 24b

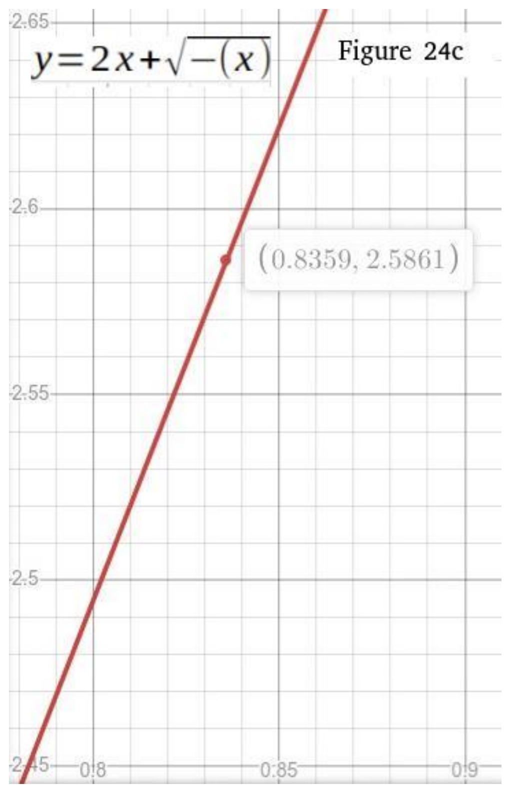


Figure 24c

Now consider the equation $y = -\frac{2}{s} + \sqrt{\frac{1}{s}}$. When x = 2 in the y = $f(x)$ equation, the corresponding value which generates the down value and complex conjugate to it for the y output is given by $s = -\frac{1}{2}$. When s = -0.5 the y-axis output is 2.586. Likewise, because the $f(x)$ and $f(s)$ equations both equal y there must be a value at which *y* equals 5.414 for some *s*-axis input. They are shown below in the Graphs of Figures 25, 25b and 25c.

<u>5.3.c:</u>

2	4	1.414i 1.414	4 + 1.414i 5.414	$-\frac{1}{2}$	4	1.414i -1.414	2.586

y = *f*(*s*)
y = *f*(-0.5) = 2.586

y = *f*(*s*)
y = *f*(0.) = 5.414

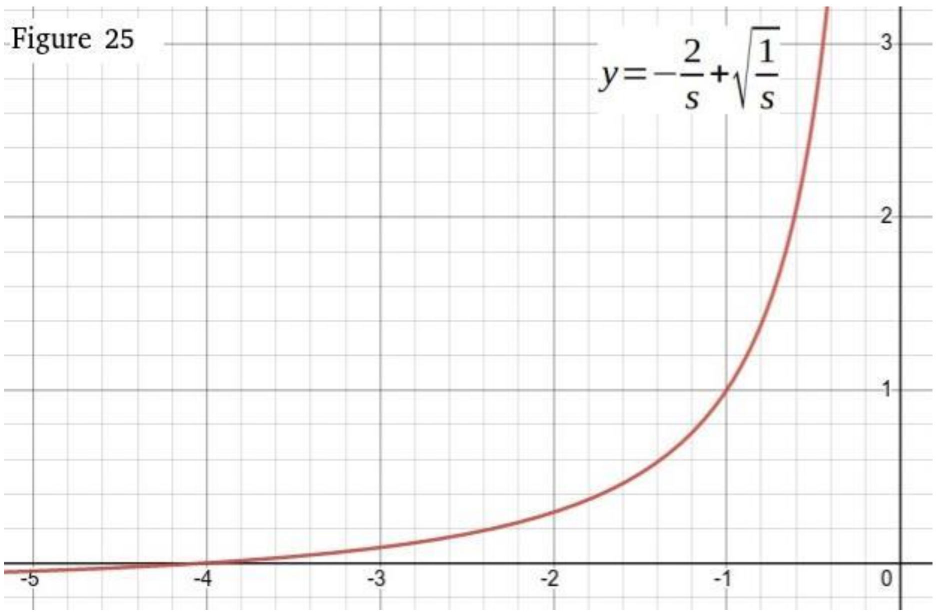


Figure 25

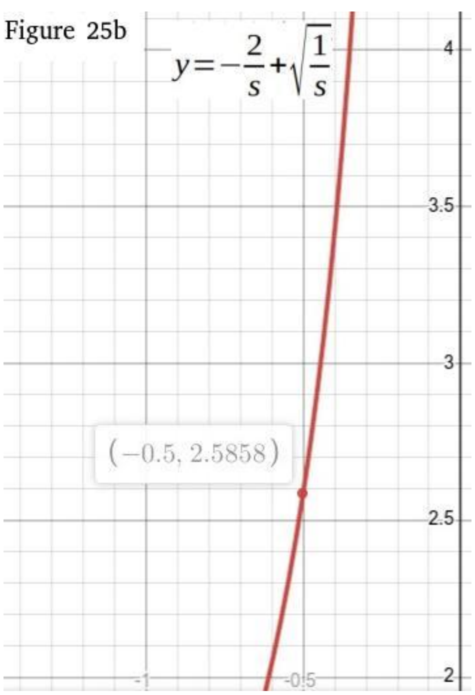


Figure 25b

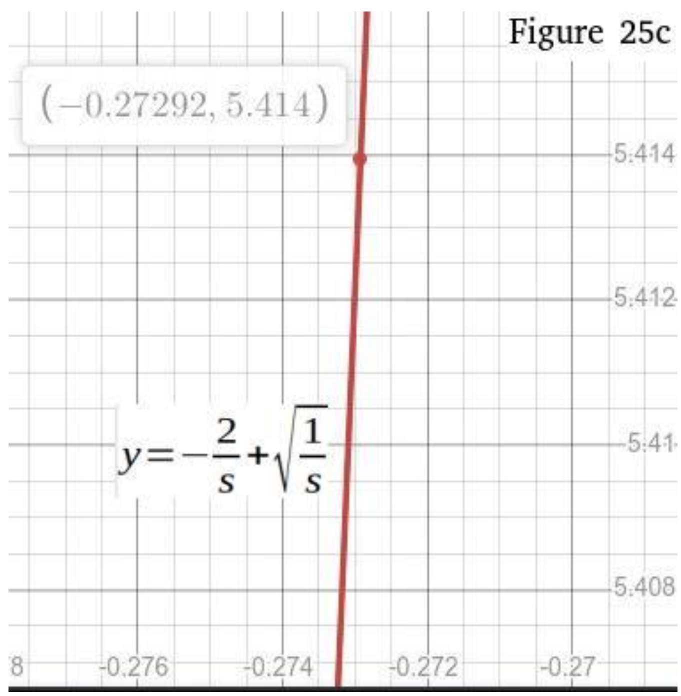


Figure 25c

More examples and further extensions of this extension to Null Algebra will appear in a later version of this paper.

—Robert S Miller 26 April, 2023.