New Expressions of Various Spin Particle Equations and Their Quantization

—Analysis and Application of Constant Invariant Tensors

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Preface

I have enriched, perfected, and further developed relativity, particle physics and quantum field theory in this book. Generally, a rigorous, analytical, elegant description method is adopted. I try best to impart a mathematical and physical aesthetic feeling to the entire article. The first seventeen chapters of this book are the basic parts. Several very useful mathematical tools have been proposed. Specially I have independently developed and created the constant invariant tensors analysis method for physical research. And I have restated classical physics in my own way. Most of the content belongs to the fields of classical field theory and quantum mechanics. The later chapters of Chapter 18 are the advanced parts. Most of the content belongs to the fields of quantum field theory. In particular, a new quantization program is given. According to this program, the quantization of arbitrary spin linear particles is completed in arbitrary space-time. These have greatly enriched and expanded the content of quantum field theory.

More specifically, in the first, second, and third chapters, I have independently developed and created the constant invariant tensors analysis method [1–3]. Some wonderful mathematical properties have been found and many important and useful constant invariant tensors have been proposed. It provides a very useful mathematical tool for physical research. In chapter 4, the constant invariant tensors are generalized in both high and low dimensional space-time and infinite dimensional representation. In Chapter 5, the essential relation between the constant invariant tensors and the representation transformations is pointed out. In Chapters 6, 7, 8, and 9, I used mathematical tools such as the constant tensor analysis established in the previous three chapters to reformulate the equations of electromagnetic fields, Yang-Mills fields, gravitational fields, and gravitational neutrino fields [4–14]. Various equivalent expressions have been proposed [1,2]. And I have strictly analytically proved the equivalence between various expressions. In particular, the spinor form of the bianchi identity [11–14] for the gravitational field has been obtained through analysis.

Chapter 10 is the most important part of this book. It is also my original intention to write this book at the beginning. In this chapter, I have independently and creatively proposed a new expression of the particle equations: the Spin Equation. This equation is directly constructed by using spin and the spin tensor matrix. Noting that the spin tensor is also a transformation matrix corresponding to the representation of the field. Therefore, the physical meaning of this equation is very clear. The corresponding particle equation can be simply and directly written according to the particle field quantity transformation law. It correctly describes classical equations such as neutrinos [5], electromagnetic fields [7,8], Yang Mills fields [6], and electrons [4]. And I have found that its massless representation is completely equivalent to the full symmetric Penrose spinor equation [1,2]. Of course, it is more extensive than the Penrose spinor equation and can describe more physical equations. I continue to use the idea of spin expression to further obtain a lower order derivative spin equation that correctly describes the Einstein's gravitational field and the gravitational neutrinos.In these spin representations, it is very natural to introduce a scalar field. Thus, a more interesting equation is generalized: the witch Spin Equation. When the scalar field is zero, free particles can exist. When the scalar field is not zero, free particles do not exist. This scalar field acts like a switch and controlls the generation and annihilation of particles. This provides a new physical mechanism for the generation and annihilation of particles. At the same time, it can also answer why the inflationary period of the universe [15] can be completely described only by scalar fields. And this equation itself has inherent limitations on scalar fields. Thereby the scalar field is automatically quantized. Each quantized value corresponds to a distinct physical equation, one corresponds to a classical particle equation, one corresponds to an equation similar to torsion, and one corresponds to a constant trivial solution. This provides a new idea and inspiration for the unified expression of the five superstrings.

In Chapters 11 to 12, I have conducted a comprehensive and in-depth analysis of the Penrose spinor equation [1,2], the Penrose twist equation [3], and the Bargmann-Wigner equation [16]. In a flat space-time, it is strictly proved that the Bargmann Wigner equation is equivalent to the Rarita Schwinger equation [17,18] in the case of semi integer spin [19,20] and equivalent to the Klein-Gordon equation [18,21] in the case of integer spin [20]. The profound physical connotation of the Bargmann Wigner equation has been revealed. Through comparative research, it is found that the Bargmann

Wigner equation is more suitable for describing particles with mass, while the Penrose spinor equation or the Spin Equation is more suitable for describing particles without mass.

Chapter 13 further enriches and deepens the content of the previous chapters. I study the same physical problem from the perspective of representation transformation. This provides a mathematical basis for the subsequent proof of the polynomial representation of the Lorentz transformation of various spin particles. At the same time, a fully new particle coupling theory is proposed by using representation transformation technology. In Chapter 14, I have made a detailed and in-depth analysis of the Lorentz transformation [22–24]. In particular, the polynomial representations of Lorentz transformation for various common spin particles have been obtained. It will provide another very useful mathematical tool for the future research of various spin particle physics. In Chapters 15 to 17, the mathematical analyses of helicity, spin algebra, special quasi differential operators and matrix continuous multiplication traces have been established. It provides several very useful mathematical tools for quantizing arbitrary spin massless particles successfully in the next step.

In Chapter 18, I fully utilize the four-dimensional Fourier transform technique to discuss the details of the second quantization of non relativistic particles. Because most books on quantum field theory do not discuss the quantization of Majorana particles and neutrinos in detail [25] and I have never found the corresponding content. In order to make up for this regret, I decide to deduce the calculation by myself. In Chapter 19, firstly I have given the quantization of Dirac particles [25,26] by using Lorentz push transformation. And then by using similar techniques on this basis, I have further given detailed quantizati details of Majorana particles and neutrinos. In Chapters 20 to 23, I apply the mathematical tools and constant invariant tensor analysis created in the previous chapters to quantize various massless spin particles successfully according to the new covariant quantization program. In particular, a separate chapter on scalar fields and electromagnetic fields has been discussed in detail. Thus the rationality and correctness of the new quantization program has been confirmed by comparing with classical results. On this basis, various massless spin particles have been successfully quantized by using the same and unified program.

In Chapters 24 to 29, based on the Bargmann Wigner equation and comparing the successful experience of quantizing massless particles, various massive spin particles have been quantized in the new unified formula. Several covariant commutative rules for equivalent fields or potentials have been given. In particular, a mathematical conjecture on combinatorics has been proposed. In Chapter 30, I have reorganized and analyzed the spin bases of various equations in the previous chapters. The logical deduction relation of spin base decomposition and the spin base decomposition relation under different representations have been clarified. It is further demonstrated that the spin base are the common eigenstates of general spin operators. And the spin base decomposition coefficients are just the CG coefficients. In Chapter 31, I generalized and developed the polynomial theorem for full symmetric indicators. This provides mathematical support for the previous step of unifying to quantize massive particles by using the new program. At the same time, based on the formula constructed by Behrends and Frontsdal in history, and combined with my new conclusions, I have obtained a very meaningful projection operator conjecture.

In Chapter 32, I have discussed the essential relation between CG coefficients, spin coupling, and quantum entanglement. And I have provided a wonderful mathematical description of spin entangled states based on representation transformation technology. In Chapter 33, I have ventured to speculate on a new type of interaction: internal particle component interactions. Whether it is correct that will remain to be verified by practice. In Chapter 34, I have used similar mathematical techniques to uniformly treat various symmetric and antisymmetric plane wave solutions. In Chapters 35 to 38, I have tried to unify the quantization of all particles in high and low dimensional space-time by using the new program, and conducted a lot of meaningful promotion and exploration. A series of achievements have been achieved. And the conclusions completely similar to those in four-dimensional space-time have been obtained. In particular, antisymmetric tensor fields have naturally emerged. This is a wonderful conclusion that I didn't expect.

In Chapters 39 and 40, I have discussed Bose strings, two-dimensional spinors, two-dimensional vectors, and anyons. We have discussed two-dimensional supersymmetry and superstring. This is to prepare for the application of the new quantization program to supersymmetry and superstring theory. However, no suitable entry point has yet been found. In particular, this book is actually an open topic. There are some issues that have not been resolved. For example, the new quantization program is powerless to quantize the nonlinear Yang-Mills field and Einstein gravitational field. Because it cannot be treated as a free field at this time, and it is necessary to consider self interaction and corresponding Feynman rules. This is also one of the directions to be resolved and explored in the next step.

The mathematics and physics in this book are highly original, and some mathematical and physical concepts, methods, and content are also novel. They are all strictly established by my own independent calculation and deduction step by step. More formal research can be traced back to more than

a decade ago [27]. As early as May 2004, the basic part of the current theoretical system was initially established. It lasted for several years, but it greatly affected my normal life, so I once wanted to completely forget it. So there was a break for a few years. Despite this, my interest in theoretical physics has not diminished. Later, a new round of research was launched eight years ago. Since March 2015, I have been writing continuously [28–32]. It never stopped. It consumed a lot of my time and energy. And I have used spare time to write for a long time. Specially I thank to my family for their understanding and support over the years! Because many topics in this book involve entirely new fields. It is challenging and open and have been filled with many conjectures, explorations, verifications and proofs. Some mathematical and physical problems have only been mostly solved, but they have not been completely solved. In addition due to my limited level, I can't cover all aspects. There are inevitably errors in the book and everyone is welcome to correct them!

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From March 2015 to April 2023, DanShui.

Contents

Chapter1 Constant Invariant Tensors Analysis

Self comment: In this chapter, I have developed and created a constant invariant tensors analysis method. I inspired and developed it by known constant invariant tensors. In this chapter, a large number of new fundamental constant invariant tensors have been discovered. And they are closely related to physics and have natural covariance and invariance. It is very convenient and useful to use. It is a new mathematical tool for physical research. In fact, my original intention in developing this mathematical tool is to apply it to physical research.

1 Similar Penrose abstract indices [1,2]

Symbol convention:

$\varsigma = \pm 1$.

$\sim$ means a Lorentz transformation.

$\prec$ means that the matrix is expanded into components.

$\succ$ means that the components are reduced to a matrix.

Self comment: This section has developed and promoted the Penrose abstract indices. The $\frac{1}{2}$-spin indices are extended to the general spin indices. And dual representation indices have been introduced. The dual representation indices correspond to two representations of massless particles. The advantages of this approach are in duplicate. One expression presents two representations simultaneously. One processing obtains two results simultaneously. Such abstract indices are more beautiful, complete and powerful.

1.1 Hermitian spin matrix $\sigma(s)$ under general representation

$$\sigma(s) \times \sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1), \sigma^+(s) = \sigma(s) \tag{1.1}$$

1.2 A concrete representation of hermitian spin matrix [33]

$$\sigma(s) = (\frac{1}{2}\begin{bmatrix} 0 & A_1 & 0 & 0 & 0 \\ A_1 & 0 & A_2 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix}, \frac{i}{2}\begin{bmatrix} 0 & -A_1 & 0 & 0 & 0 \\ A_1 & 0 & -A_2 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix}, \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix}) \tag{1.2a}$$

$$A_n = \sqrt{n} \cdot \sqrt{2s+1-n}, n = 1, 2, \cdots, 2s; \sigma(s) \prec \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(s) \simeq \sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s) \tag{1.2b}$$

$$\sigma(s) \times \sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1), \sigma^+(s) = \sigma(s), s = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots \tag{1.2c}$$

The metric tensor corresponding to this spin matrix is as follows:

$$\varepsilon(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & (-1)^0 \\ 0 & 0 & 0 & (-1)^1 & 0 \\ 0 & 0 & (-1)^2 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ (-1)^{2s} & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ (-1)^{2s} & 0 & 0 & 0 & 0 \end{bmatrix}, \sigma^*(s) = (-1)^{2s+1}\varepsilon(s)\sigma(s)\varepsilon(s) \tag{1.3}$$

$$\varepsilon(s) \prec \varepsilon_{k_\varsigma l_\varsigma}(s) \simeq \varepsilon^{k_\varsigma l_\varsigma}(s) \simeq \varepsilon_{k'_\varsigma l'_\varsigma}(s) \simeq \varepsilon^{k'_\varsigma l_\varsigma}(s), \varepsilon^2(s) = (-1)^{2s} \tag{1.4}$$

Self comment: Essentially, there are infinite choices for selecting a spin matrix, which can be Hermitian or not Hermitian. This book mainly uses the Hermitian spin matrix represented by the above special representation. The reason for doing so is that the observations must be Hermitian in quantum mechanics. And another reason is that under this representation of the spin matrix, there exist several perfect constant invariant tensors in the next chapter. If other representations of the spin matrix are used, such perfect constant invariant tensors can't be obtained. In fact, I initially used a full integer spin matrix. On the surface, it seems more beautiful, but has not the above two advantages. Finally, I have abandoned it and adopted the current hermitian representation.

1.3 Carding complex properties of spin matrix

$$\sigma_x^2(s)=\frac{1}{4}\begin{bmatrix} A_1^2 & 0 & A_1A_2 & 0 & 0 & 0 & 0\\ 0 & A_2^2{+}A_1^2 & 0 & A_2A_3 & 0 & 0 & 0\\ A_1A_2 & 0 & A_3^2{+}A_2^2 & 0 & \cdots & 0 & 0\\ 0 & A_2A_3 & 0 & \cdots & 0 & A_{2s-2}A_{2s-1} & 0\\ 0 & 0 & \cdots & 0 & A_{2s-1}^2{+}A_{2s-2}^2 & 0 & A_{2s-1}A_{2s}\\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s}^2{+}A_{2s-1}^2 & 0\\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \tag{1.5}$$

$$\sigma_y^2(s)=\frac{1}{4}\begin{bmatrix} A_1^2 & 0 & -A_1A_2 & 0 & 0 & 0 & 0\\ 0 & A_2^2{+}A_1^2 & 0 & -A_2A_3 & 0 & 0 & 0\\ -A_1A_2 & 0 & A_3^2{+}A_2^2 & 0 & \cdots & 0 & 0\\ 0 & -A_2A_3 & 0 & \cdots & 0 & -A_{2s-1}A_{2s} & 0\\ 0 & 0 & \cdots & 0 & A_{2s-1}^2{+}A_{2s-2}^2 & 0 & -A_{2s-1}A_{2s}\\ 0 & 0 & 0 & -A_{2s-2}A_{2s-1} & 0 & A_{2s}^2{+}A_{2s-1}^2 & 0\\ 0 & 0 & 0 & 0 & -A_{2s-1}A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \tag{1.6}$$

$$\sigma_z^2(s)=\begin{bmatrix} s^2 & 0 & 0 & 0 & 0\\ 0 & (s{-}1)^2 & 0 & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & -(s{-}1)^2 & 0\\ 0 & 0 & 0 & 0 & -s^2 \end{bmatrix} \tag{1.7}$$

$$\sigma_z(s)\sigma_x(s)=\frac{1}{2}\begin{bmatrix} 0 & sA_1 & 0 & 0 & 0\\ (s{-}1)A_1 & 0 & (s{-}1)A_2 & 0 & 0\\ 0 & (s{-}2)A_2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & -(s{-}1)A_{2s}\\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix} \quad \sigma_x(s)\sigma_z(s)=\frac{1}{2}\begin{bmatrix} 0 & (s{-}1)A_1 & 0 & 0 & 0\\ sA_1 & 0 & (s{-}2)A_2 & 0 & 0\\ 0 & (s{-}1)A_2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & -sA_{2s}\\ 0 & 0 & 0 & -(s{-}1)A_{2s} & 0 \end{bmatrix} \tag{1.8}$$

$$\sigma_z(s)\sigma_y(s)=\frac{i}{2}\begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0\\ (s{-}1)A_1 & 0 & -(s{-}1)A_2 & 0 & 0\\ 0 & (s{-}2)A_2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & (s{-}1)A_{2s}\\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix} \quad \sigma_y(s)\sigma_z(s)=\frac{i}{2}\begin{bmatrix} 0 & -(s{-}1)A_1 & 0 & 0 & 0\\ sA_1 & 0 & -(s{-}2)A_2 & 0 & 0\\ 0 & (s{-}1)A_2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & sA_{2s}\\ 0 & 0 & 0 & -(s{-}1)A_{2s} & 0 \end{bmatrix} \tag{1.9}$$

$$\sigma_x(s)\sigma_y(s)=\frac{i}{4}\begin{bmatrix} A_1^2 & 0 & -A_1A_2 & 0 & 0 & 0 & 0\\ 0 & A_2^2{-}A_1^2 & 0 & -A_2A_3 & 0 & 0 & 0\\ A_1A_2 & 0 & A_3^2{-}A_2^2 & 0 & \cdots & 0 & 0\\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-2}A_{2s-1} & 0\\ 0 & 0 & \cdots & 0 & A_{2s-1}^2{-}A_{2s-2}^2 & 0 & -A_{2s-1}A_{2s}\\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s}^2{-}A_{2s-1}^2 & 0\\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & -A_{2s}^2 \end{bmatrix} \tag{1.10}$$

$$\sigma_y(s)\sigma_x(s)=\frac{i}{4}\begin{bmatrix} -A_1^2 & 0 & -A_1A_2 & 0 & 0 & 0 & 0\\ 0 & A_1^2{-}A_2^2 & 0 & -A_2A_3 & 0 & 0 & 0\\ A_1A_2 & 0 & A_2^2{-}A_3^2 & 0 & \cdots & 0 & 0\\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-2}A_{2s-1} & 0\\ 0 & 0 & \cdots & 0 & A_{2s-2}^2{-}A_{2s-1}^2 & 0 & -A_{2s-1}A_{2s}\\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s-1}^2{-}A_{2s}^2 & 0\\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \tag{1.11}$$

The anti commutative relation of the spin matrices:

$$\{\sigma_y(s),\sigma_z(s)\}=\frac{i}{2}\begin{bmatrix} 0 & -(2s{-}1)A_1 & 0 & 0 & 0\\ (2s{-}1)A_1 & 0 & -(2s{-}3)A_2 & 0 & 0\\ 0 & (2s{-}3)A_2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & (2s{-}1)A_{2s}\\ 0 & 0 & 0 & -(2s{-}1)A_{2s} & 0 \end{bmatrix} \tag{1.12}$$

$$\{\sigma_z(s),\sigma_x(s)\}=\frac{1}{2}\begin{bmatrix} 0 & (2s{-}1)A_1 & 0 & 0 & 0\\ (2s{-}1)A_1 & 0 & (2s{-}3)A_2 & 0 & 0\\ 0 & (2s{-}3)A_2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & -(2s{-}1)A_{2s}\\ 0 & 0 & 0 & -(2s{-}1)A_{2s} & 0 \end{bmatrix} \tag{1.13}$$

$$\{\sigma_x(s),\sigma_y(s)\}=\frac{i}{2}\begin{bmatrix} 0 & 0 & -A_1A_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -A_2A_3 & 0 & 0 & 0\\ A_1A_2 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-1}A_{2s} & 0\\ 0 & 0 & \cdots & 0 & 0 & 0 & -A_{2s}A_{2s+1}\\ 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & A_{2s}A_{2s+1} & 0 & 0 \end{bmatrix} \tag{1.14}$$

$$\{\sigma_x(s),\sigma_x(s)\}=\frac{1}{2}\begin{bmatrix} 0 & 0 & A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & A_{2s-1}A_{2s} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & A_{2s}A_{2s+1} \\ 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{2s}A_{2s+1} & 0 & 0 \end{bmatrix}+s(s+1)-\sigma_z^2(s) \tag{1.15}$$

$$\{\sigma_y(s),\sigma_y(s)\}=-\frac{1}{2}\begin{bmatrix} 0 & 0 & A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & A_{2s-1}A_{2s} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & A_{2s}A_{2s+1} \\ 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{2s}A_{2s+1} & 0 & 0 \end{bmatrix}+s(s+1)-\sigma_z^2(s) \tag{1.16}$$

$$\{\sigma_z(s),\sigma_z(s)\}=2\begin{bmatrix} s^2 & 0 & 0 & 0 & 0 \\ 0 & (s-1)^2 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (s-1)^2 & 0 \\ 0 & 0 & 0 & 0 & s^2 \end{bmatrix} \tag{1.17}$$

$$A_1(s):=\frac{1}{2}\begin{bmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{2s} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_2(s):=\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{2s} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{2s+1} \end{bmatrix} \tag{1.18}$$

$$A_2^2(s)+A_1^2(s)=s(s+1)-\sigma_z^2(s), A_2^2(s)-A_1^2(s)=\sigma_z(s) \tag{1.19}$$

1.4 Lorentz transformation parameters ϑ^{ab} and spin tensor $S_{ab}(s,\varsigma)$ in orthogonal frame (The book adopts this frame.)

$\epsilon\in R$ means the velocity of O'(particle) relative to O. $\omega\in R$ means the rotation angle of O'(particle) relative to O. Self comment: In this way, the understanding of physics will not be confused. Especially over time, it is easy to confuse the correspondence between parameters and real physics. Generally, it is a different symbolIn case you forget, you can come back here and make a clean slate.

$$g_{ab}\simeq g^{ab}\succ diag(1,1,1,1), x^a\simeq x_a=(x,y,z,it), \vec{\vartheta}\equiv i\omega+\epsilon \tag{1.20a}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ it' \end{bmatrix}(=x'^a)=e^{\vartheta^a{}_b}\begin{bmatrix} x \\ y \\ z \\ it \end{bmatrix}(=x^b), \vartheta^a{}_b\succ\begin{bmatrix} 0 & \omega_z & -\omega_y & i\epsilon_x \\ -\omega_z & 0 & \omega_x & i\epsilon_y \\ \omega_y & -\omega_x & 0 & i\epsilon_z \\ -i\epsilon_x & -i\epsilon_y & -i\epsilon_z & 0 \end{bmatrix}\prec\vartheta_a{}^b\simeq\vartheta^{ab}\simeq\vartheta_{ab}\succ i\omega\cdot R+\epsilon\cdot L \tag{1.20b}$$

$$\vartheta_{ij}=\varepsilon_{ijk}\omega^k, \omega_k=\frac{1}{2}\varepsilon_{kij}\vartheta^{ij} \tag{1.20c}$$

$$x'^a=(g^a{}_b+\vartheta^a{}_b)x^b, x'^a=(g^{ab}+\vartheta^{ab})x_b, x'_a=(g_a{}^b+\vartheta_a{}^b)x_b, x'_a=(g_{ab}+\vartheta_{ab})x^b \tag{1.20d}$$

$$\begin{cases} \delta x_a=\vartheta_a{}^b x_b=\vartheta_{ab}x^b, \delta x^a=\vartheta^a{}_b x^b=\vartheta^{ab}x_b \\ \vec{S}_{abcd}=-i(g_{ac}g_{bd}-g_{ad}g_{bc})\succ\begin{bmatrix} 0 & R_z(s) & -R_y(s) & -L_x(s) \\ -R_z(s) & 0 & R_x(s) & -L_y(s) \\ R_y(s) & -R_x(s) & 0 & -L_z(s) \\ L_x(s) & L_y(s) & L_z(s) & 0 \end{bmatrix} \end{cases} \tag{1.20e}$$

$$\begin{cases} \delta\varphi(s)=\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma)\varphi(s)=\frac{i}{2}\vartheta_{ab}S^{ab}(s,\varsigma)\varphi(s) \\ S^{ab}(s,\varsigma)\succ\begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0 \end{bmatrix}\prec S_{ab}(s,\varsigma)=-i[\sigma(s),\frac{i\varsigma}{2}]_{[a}[\sigma(s),-\frac{i\varsigma}{2}]_{b]} \end{cases} \tag{1.20f}$$

$$\begin{cases} L_{ab}=x_a p_b-x_b p_a\succ\begin{bmatrix} 0 & xp_y-yp_x & -(zp_x-xp_z) & ixE-itp_x \\ -(xp_y-yp_x) & 0 & yp_z-zp_y & iyE-itp_y \\ zp_x-xp_z & -(yp_z-zp_y) & 0 & izE-itp_z \\ -(ixE-itp_x) & -(iyE-itp_y) & -(izE-itp_z) & 0 \end{bmatrix} \\ M_{ab}=L_{ab}+S_{ab}(s,\varsigma)=-i(x_a\partial_b-x_b\partial_a)+S_{ab}(s,\varsigma) \end{cases} \tag{1.20g}$$

Self comment: In essence, there are also infinite selection methods for frame selection. There are several commonly used ones. This book uses the orthogonal frame of this section. The advantage of this is that constant invariant tensors are simpler, more uniform and more regular in this frame. Of course, it can also be transformed to other frame representations through equivalent transformation.

1.5 Lorentz transformation parameters ϑ^{ab} and spin tensor $S_{ab}(s,\varsigma)$ in other frames

1.5.1 Pseudo frame

$$g_{ab}=g^{ab}=diag(1,1,1,-1), x^a=(x,y,z,t), x_a=(x,y,z,-t) \tag{1.21a}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} (= x'^a) = e^{\vartheta^a{}_b} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} (= x^b), \vartheta^a{}_b = \begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, x'_a = (g_a{}^b + \vartheta_a{}^b) x_b \tag{1.21b}$$

$$\vartheta_{ab} = g_{ac}\vartheta^c{}_b \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix}, \vartheta^{ab} = \vartheta^a{}_c g^{cb} \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, \vartheta_a{}^b = g_{ac}\vartheta^c{}_d g^{db} \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix} \tag{1.21c}$$

$$x'^a = (g^a{}_b + \vartheta^a{}_b)x^b, x'^a = (g^{ab} + \vartheta^{ab})x_b, x'_a = (g_a{}^b + \vartheta_a{}^b)x_b, x'_a = (g_{ab} + \vartheta_{ab})x^b \tag{1.21d}$$

$$\delta x^a = \vartheta_a{}^b x_b = \vartheta_{ab}x^b, \delta x_a = \vartheta^a{}_b x^b = \vartheta^{ab}x_b, \delta\psi(s) = \frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma)\psi(s) = \frac{i}{2}\vartheta_{ab}S^{ab}(s,\varsigma)\psi(s) \tag{1.21e}$$

$$\vec{S}_{abcd} \succ \begin{bmatrix} 0 & R_z(s) & -R_y(s) & L_x(s) \\ -R_z(s) & 0 & R_x(s) & L_y(s) \\ R_y(s) & -R_x(s) & 0 & L_z(s) \\ -L_x(s) & -L_y(s) & -L_z(s) & 0 \end{bmatrix} = -i(g_{ac}g_{bd} - g_{ad}g_{bc}), \delta x_a = \vartheta^{cd}\vec{S}_{abcd}x^b, \delta x^a = \vartheta_{cd}S^{abcd}x_b \tag{1.21f}$$

$$S^{ab}(s,\varsigma) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & i\varsigma\sigma_z(s) \\ -i\varsigma\sigma_x(s) & -i\varsigma\sigma_y(s) & -i\varsigma\sigma_z(s) & 0 \end{bmatrix}, S_{ab}(s,\varsigma) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -i\varsigma\sigma_z(s) \\ i\varsigma\sigma_x(s) & i\varsigma\sigma_y(s) & i\varsigma\sigma_z(s) & 0 \end{bmatrix}, \varsigma = \pm 1 \tag{1.21g}$$

$$L_{ab} = x_a p_b - x_b p_a \succ \begin{bmatrix} 0 & xp_y - yp_x & -(zp_x - xp_z) & -(xE - tp_x) \\ -(xp_y - yp_x) & 0 & yp_z - zp_y & -(yE - tp_y) \\ zp_x - xp_z & -(yp_z - zp_y) & 0 & -(zE - tp_z) \\ xE - tp_x & yE - tp_y & zE - tp_z & 0 \end{bmatrix}, M_{ab} = L_{ab} + S_{ab}(s,\varsigma) \tag{1.21h}$$

1.5.2 Negative pseudo frame

$$g_{ab} = g^{ab} = -diag(1,1,1,-1), x^a = (x,y,z,t), x_a = -(x,y,z,-t) \tag{1.22a}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} (= x'^a) = e^{\vartheta^a{}_b} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} (= x^b), \vartheta^a{}_b = \begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, x'_a = (g_a{}^b + \vartheta_a{}^b) x_b \tag{1.22b}$$

$$\vartheta_{ab} = g_{ac}\vartheta^c{}_b \succ -\begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix}, \vartheta^{ab} = \vartheta^a{}_c g^{cb} \succ -\begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, \vartheta_a{}^b = g_{ac}\vartheta^c{}_d g^{db} \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix} \tag{1.22c}$$

$$x'^a = (g^a{}_b + \vartheta^a{}_b)x^b, x'^a = (g^{ab} + \vartheta^{ab})x_b, x'_a = (g_a{}^b + \vartheta_a{}^b)x_b, x'_a = (g_{ab} + \vartheta_{ab})x^b \tag{1.22d}$$

$$\delta x^a = \vartheta_a{}^b x_b = \vartheta_{ab}x^b, \delta x_a = \vartheta^a{}_b x^b = \vartheta^{ab}x_b, \delta\psi(s) = \frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma)\psi(s) = \frac{i}{2}\vartheta_{ab}S^{ab}(s,\varsigma)\psi(s) \tag{1.22e}$$

$$\vec{S}_{abcd} \succ \begin{bmatrix} 0 & R_z(s) & -R_y(s) & L_x(s) \\ -R_z(s) & 0 & R_x(s) & L_y(s) \\ R_y(s) & -R_x(s) & 0 & L_z(s) \\ -L_x(s) & -L_y(s) & -L_z(s) & 0 \end{bmatrix} = -i(g_{ac}g_{bd} - g_{ad}g_{bc}), \delta x_a = \vartheta^{cd}\vec{S}_{abcd}x^b, \delta x^a = \vartheta_{cd}S^{abcd}x_b \tag{1.22f}$$

$$S^{ab}(s,\varsigma) \succ -\begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & i\varsigma\sigma_z(s) \\ -i\varsigma\sigma_x(s) & -i\varsigma\sigma_y(s) & -i\varsigma\sigma_z(s) & 0 \end{bmatrix}, S_{ab}(s,\varsigma) \succ -\begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -i\varsigma\sigma_z(s) \\ i\varsigma\sigma_x(s) & i\varsigma\sigma_y(s) & i\varsigma\sigma_z(s) & 0 \end{bmatrix}, \varsigma = \pm 1 \tag{1.22g}$$

$$L_{ab} = x_a p_b - x_b p_a \succ -\begin{bmatrix} 0 & xp_y - yp_x & -(zp_x - xp_z) & -(xE - tp_x) \\ -(xp_y - yp_x) & 0 & yp_z - zp_y & -(yE - tp_y) \\ zp_x - xp_z & -(yp_z - zp_y) & 0 & -(zE - tp_z) \\ xE - tp_x & yE - tp_y & zE - tp_z & 0 \end{bmatrix}, M_{ab} = L_{ab} + S_{ab}(s,\varsigma) \tag{1.22h}$$

1.6 s-spin spinor index

s-spin spinor index definition: Use lower-case Arabic letters $\{i,j,k,l,m,n,p,q,r,s\}$ to especially represent general spin cases.

$$_{k_\varsigma} \sim e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma)} = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)} \quad \Leftrightarrow_{k_+} :=_k \sim e^{(i\omega+\epsilon)\cdot\sigma(s)} \quad _{k_-} :=^{k'} \sim e^{(i\omega-\epsilon)\cdot\sigma(s)} \tag{1.23}$$

$$^{k_\varsigma} \sim e^{-\frac{i}{2}\vartheta^{ab}S^T_{ab}(s,\varsigma)} = e^{-(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)} \quad \Leftrightarrow^{k_+} :=^k \sim e^{-(i\omega+\epsilon)\cdot\sigma^T(s)} \quad ^{k_-} :=_{k'} \sim e^{-(i\omega-\epsilon)\cdot\sigma^T(s)} \tag{1.24}$$

$$ {}_{k'_\varsigma} \sim e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,-\varsigma)} = e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s)} \qquad \Leftrightarrow {}^{k'_+} := {}^{k'} \sim e^{(i\omega-\epsilon)\cdot\sigma(s)} \qquad {}^{k'_-} := {}_{k} \sim e^{(i\omega+\epsilon)\cdot\sigma(s)} \tag{1.25} $$

$$ {}_{k'_\varsigma} \sim e^{-\frac{i}{2}\vartheta^{ab}S^T_{ab}(s,-\varsigma)} = e^{-(i\omega-\varsigma\epsilon)\cdot\sigma^T(s)} \qquad \Leftrightarrow {}_{k'_+} := {}_{k'} \sim e^{-(i\omega-\epsilon)\cdot\sigma^T(s)} \qquad {}_{k'_-} := {}^{k} \sim e^{-(i\omega+\epsilon)\cdot\sigma^T(s)} \tag{1.26} $$

Indices relation: Indices are identical and spinors are also identical, which is a definition.

$$ \begin{cases} {}^{k_\varsigma} \equiv {}_{k'_{-\varsigma}} \\ {}^{k_{-\varsigma}} \equiv {}_{k'_\varsigma} \end{cases} \quad \begin{cases} {}_{k_\varsigma} \equiv {}^{k'_{-\varsigma}} \\ {}_{k_{-\varsigma}} \equiv {}^{k'_\varsigma} \end{cases} \quad \begin{cases} {}^{k_+} \equiv {}_{k'_-} \equiv {}^{k} \\ {}^{k_-} \equiv {}_{k'_+} \equiv {}_{k'} \end{cases} \quad \begin{cases} {}_{k_+} \equiv {}^{k'_-} \equiv {}_{k} \\ {}_{k_-} \equiv {}^{k'_+} \equiv {}^{k'} \end{cases} \tag{1.27} $$

Conjugate indices: If indices are equal then spinors may be equal or not. And they are equal only under the special hermitian representation.

$$ \begin{cases} ({}^{k_\varsigma})^* = {}^{k'_\varsigma} \\ ({}_{k_\varsigma})^* = {}_{k'_\varsigma} \end{cases} \quad \begin{cases} ({}^{k'_\varsigma})^* = {}^{k_\varsigma} \\ ({}_{k'_\varsigma})^* = {}_{k_\varsigma} \end{cases} \quad \begin{cases} ({}^{k})^* = {}^{k'} \\ ({}_{k})^* = {}_{k'} \end{cases} \quad \begin{cases} ({}^{k'})^* = {}^{k} \\ ({}_{k'})^* = {}_{k} \end{cases} \tag{1.28} $$

The metric tensor corresponding to the s-spin spinor index and the self consistent raising and lowering rules are as follows:

$$ \begin{cases} \varepsilon_{k_\varsigma l_\varsigma}(s) \simeq \varepsilon^{k'_\varsigma l'_\varsigma}(s) \simeq \varepsilon_{k'_\varsigma l'_\varsigma}(s) \simeq \varepsilon^{k_\varsigma l_\varsigma}(s) \succ \varepsilon(s) \\ \psi_{k_\varsigma} = (-\varsigma)^{2s}\varepsilon_{k_\varsigma l_\varsigma}(s)\psi^{l_\varsigma}, \psi^{k_\varsigma} = \varsigma^{2s}\varepsilon^{k_\varsigma l_\varsigma}(s)\psi_{l_\varsigma} \\ \psi_{k'_\varsigma} = (-\varsigma)^{2s}\varepsilon_{k'_\varsigma l'_\varsigma}(s)\psi^{l'_\varsigma}, \psi^{k'_\varsigma} = \varsigma^{2s}\varepsilon^{k'_\varsigma l'_\varsigma}(s)\psi_{l'_\varsigma} \end{cases} \tag{1.29} $$

The essence of self consistent raising and lowering rules is as follows. The rules are consistent with those of Penrose, and the Penrose indices are consistent with my indices.

$$ \begin{cases} \varepsilon_{k'l'}(s) = [\varepsilon_{kl}(s)]^* \simeq \varepsilon_{kl}(s), \varepsilon^{k'l'}(s) = [\varepsilon^{kl}(s)]^* \simeq \varepsilon^{kl}(s) \\ \psi_k = (-1)^{2s}\varepsilon_{kl}(s)\psi^l, \psi^k = \varepsilon^{kl}(s)\psi_l \\ \psi_{k'} = (-1)^{2s}\varepsilon_{k'l'}(s)\psi^{l'}, \psi^{k'} = \varepsilon^{k'l'}(s)\psi_{l'} \end{cases} \tag{1.30} $$

Self comment: Why is it necessary to specify the raising and lowering rules for new abstract indicators in this way. There are two main considerations. First tries to be consistent with the Penrose abstract indices rules and the second is the inherent self consistency requirements of the new abstract indices. It is also a summary and refinement of the later actual calculation of a large number of constant invariant tensors.

1.7 $\frac{1}{2}$-spin spinor index

$\frac{1}{2}$-spin spinor index is a special case of s-spin spinor index. In order to be consistent with the Penrose abstract index, the Arabic capital letters $\{A, B, C, \cdots\}$ are still used to specifically represent the $\frac{1}{2}$-spin case. The definitions and rules are shown in the above section($s = \frac{1}{2}$).

1.8 Complex vector index

Photon spin matrix: Greek letters $\{\alpha, \beta, \gamma, \cdots\}$ are used to specifically represent the complex vector indices.

$$ \gamma = \{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \} \tag{1.31} $$

Take $\sigma(1) = \gamma$ to obtain the complex vector index, as follows:

$$ \alpha_\varsigma \sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma} \qquad \Leftrightarrow \alpha_+ := \alpha \sim e^{(i\omega+\epsilon)\cdot\gamma} \qquad \alpha_- := \alpha' \sim e^{(i\omega-\epsilon)\cdot\gamma} \tag{1.32} $$

$$ \alpha'_\varsigma \sim e^{(i\omega-\varsigma\epsilon)\cdot\gamma} \qquad \Leftrightarrow \alpha'_+ := \alpha' \sim e^{(i\omega-\epsilon)\cdot\gamma} \qquad \alpha'_- := \alpha \sim e^{(i\omega+\epsilon)\cdot\gamma} \tag{1.33} $$

Indices relation:

$$ \alpha_\varsigma \equiv \alpha'_{-\varsigma}, \alpha_{-\varsigma} \equiv \alpha'_\varsigma; \alpha_+ \equiv \alpha'_- \equiv \alpha, \alpha_- \equiv \alpha'_+ \equiv \alpha \tag{1.34} $$

Conjugate relation:

$$ (\alpha_\varsigma)^* \equiv \alpha'_\varsigma, (\alpha'_\varsigma)^* \equiv \alpha_\varsigma; (\alpha)^* \equiv \alpha', (\alpha')^* \equiv \alpha \tag{1.35} $$

Metric tensors and raising and lowering rules corresponding to complex vector indices:

$$ \begin{cases} g_{\alpha_\varsigma\beta_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma} \succ I, g^{\alpha_\varsigma\beta_\varsigma} = \delta^{\alpha_\varsigma\beta_\varsigma} \succ I \\ g_{\alpha'_\varsigma\beta'_\varsigma} = \delta_{\alpha'_\varsigma\beta'_\varsigma} \succ I, g^{\alpha'_\varsigma\beta'_\varsigma} = \delta^{\alpha'_\varsigma\beta'_\varsigma} \succ I \end{cases}, \begin{cases} \psi_{\alpha_\varsigma} = g_{\alpha_\varsigma\beta_\varsigma}\psi^{\beta_\varsigma}, \psi^{\alpha_\varsigma} = g^{\alpha_\varsigma\beta_\varsigma}\psi_{\beta_\varsigma} \\ \psi^{\alpha'_\varsigma} = g^{\alpha'_\varsigma\beta'_\varsigma}\psi_{\beta'_\varsigma}, \psi_{\alpha'_\varsigma} = g_{\alpha'_\varsigma\beta'_\varsigma}\psi^{\beta'_\varsigma} \end{cases} \tag{1.36} $$

At this point, the metric tensor is the identity matrix. It is not necessary to distinguish between inverse and covariant tensors. Superscripts and subscripts can be synchronously exchanged at will.
Self comment: The complex vector index is derived from the description of electromagnetic fields, Yang-Mills fields, and gravitational fields, so it is also particularly suitable for describing them.

1.9 Vector index

The spatial rotation matrix R and the Lorentz boost matrix L:

$$R=\{\begin{bmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&0\\-i&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\},L=\{\begin{bmatrix}0&0&0&i\\0&0&0&0\\0&0&0&0\\-i&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}\} \quad (1.37)$$

Vector index is as follow: Lower-case Arabic letters $\{a,b,c,d,e,f,g,h,u,v,w\}$ are used to specifically represent the vector indices.

$$a_\varsigma\sim e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)} \quad \Leftrightarrow a_+:=a\sim e^{\vartheta}=e^{(i\omega\cdot R+\epsilon\cdot L)} \quad a_-:=a'\sim e^{\vartheta^*}=e^{(i\omega\cdot R-\epsilon\cdot L)} \quad (1.38)$$

$$a'_\varsigma\sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)} \quad \Leftrightarrow a'_+:=a'\sim e^{\vartheta^*}=e^{(i\omega\cdot R-\epsilon\cdot L)} \quad a'_-:=a\sim e^{\vartheta}=e^{(i\omega\cdot R+\epsilon\cdot L)} \quad (1.39)$$

Indices relation:

$$a_\varsigma\equiv a'_{-\varsigma},a_{-\varsigma}\equiv a'_\varsigma;a_+\equiv a'_-\equiv a,a_-\equiv a'_+\equiv a' \quad (1.40)$$

Conjugate relation:

$$(a_\varsigma)^*\equiv a'_\varsigma,(a'_\varsigma)^*\equiv a_\varsigma;(a)^*\equiv a',(a')^*\equiv a \quad (1.41)$$

Metric tensors and raising and lowering rules corresponding to vector indices:

$$\begin{cases}g_{a_\varsigma b_\varsigma}=\delta_{a_\varsigma b_\varsigma}\succ I,g^{a_\varsigma b_\varsigma}=\delta^{a_\varsigma b_\varsigma}\succ I\\g_{a'_\varsigma b'_\varsigma}=\delta_{a'_\varsigma b'_\varsigma}\succ I,g^{a'_\varsigma b'_\varsigma}=\delta^{a'_\varsigma b'_\varsigma}\succ I\end{cases},\begin{cases}\psi_{a_\varsigma}=g_{a_\varsigma b_\varsigma}\psi^{b_\varsigma},\psi^{a_\varsigma}=g^{a_\varsigma b_\varsigma}\psi_{b_\varsigma}\\\psi^{a'_\varsigma}=g^{a'_\varsigma b'_\varsigma}\psi_{b'_\varsigma},\psi_{a'_\varsigma}=g_{a'_\varsigma b'_\varsigma}\psi^{b'_\varsigma}\end{cases} \quad (1.42)$$

At this point, the metric tensor is the identity matrix. It is not necessary to distinguish between inverse and covariant tensors. Superscripts and subscripts can be synchronously exchanged at will.

2 Common Matrices

2.1 Pauli matrix

$$\sigma=\{\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&-i\\i&0\end{bmatrix},\begin{bmatrix}1&0\\0&-1\end{bmatrix}\},tr(\sigma_{\alpha_\varsigma})=0,tr(\sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma})=2\delta_{\alpha_\varsigma\beta_\varsigma} \quad (1.43)$$

$$[\sigma_{\alpha_\varsigma},\sigma_{\beta_\varsigma}]=2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma},\{\sigma_{\alpha_\varsigma},\sigma_{\beta_\varsigma}\}=2\delta_{\alpha_\varsigma\beta_\varsigma},\sigma^2(\tfrac{1}{2})=\tfrac{1}{2}(\tfrac{1}{2}+1) \quad (1.44)$$

2.2 Photon matrix

$$\gamma=\{\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}\},tr(\gamma_{\alpha_\varsigma})=0,tr(\gamma_{\alpha_\varsigma}\gamma_{\beta_\varsigma})=2\delta_{\alpha_\varsigma\beta_\varsigma} \quad (1.45)$$

$$[\gamma_{\alpha_\varsigma},\gamma_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\gamma_{\gamma_\varsigma},\gamma^2=1(1+1),\gamma_{\alpha_\varsigma}\prec\gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\equiv-i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \quad (1.46)$$

2.3 Rotation generator matrix

Spatial rotation generator matrix:

$$R=\{\begin{bmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&0\\-i&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\},tr(R_{\alpha_\varsigma})=0,tr(R_{\alpha_\varsigma}R_{\beta_\varsigma})=2\delta_{\alpha_\varsigma\beta_\varsigma} \quad (1.47a)$$

Lorenz boost generator matrix:

$$L=\{\begin{bmatrix}0&0&0&i\\0&0&0&0\\0&0&0&0\\-i&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}\},tr(L_{\alpha_\varsigma})=0,tr(L_{\alpha_\varsigma}L_{\beta_\varsigma})=2\delta_{\alpha_\varsigma\beta_\varsigma} \quad (1.47b)$$

$$[R_{\alpha_\varsigma},R_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}R_{\gamma_\varsigma},[L_{\alpha_\varsigma},L_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}R_{\gamma_\varsigma},[R_{\alpha_\varsigma},L_{\beta_\varsigma}]=[L_{\alpha_\varsigma},R_{\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}L_{\gamma_\varsigma} \quad (1.47c)$$

$$R^2=diag(2,2,2,1),L^2=diag(0,0,0,3) \quad (1.47d)$$

2.4 $SO(4)$ group generator matrix

The positive branch of $SO(4)$ group generator matrix:

$$\sigma_+=R+L=\{\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}\} \quad (1.48a)$$

$$\sigma_+=\{-\sigma_y\otimes\sigma_x,-I\otimes\sigma_y,\sigma_y\otimes\sigma_z\},tr(\sigma_{+_{\alpha_\varsigma}})=0,tr(\sigma_{+_{\alpha_\varsigma}}\sigma_{+_{\beta_\varsigma}})=4\delta_{\alpha_\varsigma\beta_\varsigma} \quad (1.48b)$$

The negative branch of $SO(4)$ group generator matrix:

$$\sigma_-=R-L=\{\begin{bmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix}\} \quad (1.49a)$$

$$\sigma_-=\{\sigma_x\otimes\sigma_y,-\sigma_z\otimes\sigma_y,\sigma_y\otimes I\},tr(\sigma_{-_{\alpha_\varsigma}})=0,tr(\sigma_{-_{\alpha_\varsigma}}\sigma_{-_{\beta_\varsigma}})=4\delta_{\alpha_\varsigma\beta_\varsigma} \quad (1.49b)$$

The relation between two branches of $SO(4)$ group generator matrix:

$$[\sigma_{+\alpha_\varsigma},\sigma_{+\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{+\gamma_\varsigma},\{\sigma_{+\alpha_\varsigma},\sigma_{+\beta_\varsigma}\}=2\delta_{\alpha_\varsigma\beta_\varsigma},\sigma_+^2=\frac{1}{2}(\frac{1}{2}+1) \tag{1.50a}$$

$$[\sigma_{-\alpha_\varsigma},\sigma_{-\beta_\varsigma}]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{-\gamma_\varsigma},\{\sigma_{-\alpha_\varsigma},\sigma_{-\beta_\varsigma}\}=2\delta_{\alpha_\varsigma\beta_\varsigma},\sigma_-^2=\frac{1}{2}(\frac{1}{2}+1) \tag{1.50b}$$

$$[\sigma_{+\alpha_\varsigma},\sigma_{-\beta_\varsigma}]=[\sigma_{-\alpha_\varsigma},\sigma_{+\beta_\varsigma}]=0 \tag{1.50c}$$

Unified representation of $SO(4)$ group generator matrices

$$\sigma_\varsigma=\{\begin{bmatrix}0&0&0&i\varsigma\\0&0&-i&0\\0&i&0&0\\-i\varsigma&0&0&0\end{bmatrix},\begin{bmatrix}0&0&i&0\\0&0&0&i\varsigma\\-i&0&0&0\\0&-i\varsigma&0&0\end{bmatrix},\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\varsigma\\0&0&-i\varsigma&0\end{bmatrix}\} \tag{1.51a}$$

$$[\sigma_{\kappa\alpha_\varsigma},\sigma_{\tau\beta_\varsigma}]=i\delta_{\kappa\tau}\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\kappa\gamma_\varsigma},\{\sigma_{\kappa\alpha_\varsigma},\sigma_{\tau\beta_\varsigma}\}=2\delta_{\kappa\tau}\delta_{\alpha_\varsigma\beta_\varsigma},\sigma_\varsigma^2=\frac{1}{2}(\frac{1}{2}+1) \tag{1.51b}$$

Cor. 2.4.1. $\sigma^{ab}_{\varsigma\beta_\varsigma}=\sigma^a_\varsigma|_{\beta_\varsigma}{}^b=\{\begin{bmatrix}0&0&0&i\varsigma\\0&0&i&0\\0&-i&0&0\end{bmatrix},\begin{bmatrix}0&0&-i&0\\0&0&0&i\varsigma\\i&0&0&0\end{bmatrix},\begin{bmatrix}0&i&0&0\\-i&0&0&0\\0&0&0&i\varsigma\end{bmatrix},\begin{bmatrix}-i\varsigma&0&0&0\\0&-i\varsigma&0&0\\0&0&-i\varsigma&0\end{bmatrix}\}$

Cor. 2.4.2. $\sigma^{ab}_{-\varsigma\beta'_\varsigma}=\sigma^a_{-\varsigma}|_{\beta'_\varsigma}{}^b=\{\begin{bmatrix}0&0&0&-i\varsigma\\0&0&i&0\\0&-i&0&0\end{bmatrix},\begin{bmatrix}0&0&-i&0\\0&0&0&-i\varsigma\\i&0&0&0\end{bmatrix},\begin{bmatrix}0&i&0&0\\-i&0&0&0\\0&0&0&-i\varsigma\end{bmatrix},\begin{bmatrix}i\varsigma&0&0&0\\0&i\varsigma&0&0\\0&0&i\varsigma&0\end{bmatrix}\}$

Self comment: This pair of constant invariant tensors is essentially two generator matrices of $SO(4)$. Gerard't Hooft had also used it, called Gerard't Hooft η matrix. This pair of constant invariant tensors also appeared in the construction of Ashtekar action in loop quantum gravity. But they don't be called constant invariant tensors. Here it appears in more places, is everywhere and is widely used. It is a very useful fundamental constant invariant tensor, closely related to the spin matrix. By using them, I can also obtain a new expression of the electromagnetic field and Yang-Mills field equation: the integral spinor expression.

3 Discovery and proof of constant invariant tensors

3.1 Various common metric constant invariant tensors

Self comment: The constant invariant tensors in this section had already existed in mathematics and physics before I developed this mathematical theory. It was inspired by them and Penrose spinor analysis. So that I wanted to develop a general constant invariant tensors theory. And I apply it to physics. It provides a useful mathematical tool for physical research.

3.1.1 Four vector metric constant invariant tensors $\delta_{a_\varsigma b_\varsigma},\delta^{a_\varsigma b_\varsigma},\delta_{a'_\varsigma b'_\varsigma},\delta^{a'_\varsigma b'_\varsigma}$

Thm. 3.1.1. $I_4=e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}I_4e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)^T}$*;* $\delta_{a_\varsigma b_\varsigma},\delta^{a_\varsigma b_\varsigma}$ *are constant invariant tensors.*

Cor. 3.1.1. $I_4=e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}I_4e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)^T}$*;* $\delta_{a'_\varsigma b'_\varsigma},\delta^{a'_\varsigma b'_\varsigma}$ *are constant invariant tensors.*

3.1.2 Complex vector metric constant invariant tensors $\delta_{\alpha_\varsigma\beta_\varsigma},\delta^{\alpha_\varsigma\beta_\varsigma},\delta_{\alpha'_\varsigma\beta'_\varsigma},\delta^{\alpha'_\varsigma\beta'_\varsigma}$

Thm. 3.1.2. $I_3=e^{(i\omega+\varsigma\varepsilon)\cdot\gamma}I_3e^{(i\omega+\varsigma\epsilon)\cdot\gamma^T}$*;* $\delta_{\alpha_\varsigma\beta_\varsigma},\delta^{\alpha_\varsigma\beta_\varsigma}$ *are constant invariant tensors.*

Cor. 3.1.2. $I_3=e^{(i\omega-\varsigma\varepsilon)\cdot\gamma}I_3e^{(i\omega-\varsigma\epsilon)\cdot\gamma^T}$*;* $\delta_{\alpha'_\varsigma\beta'_\varsigma},\delta^{\alpha'_\varsigma\beta'_\varsigma}$ *are constant invariant tensors.*

3.1.3 s-spinor metric constant invariant tensors $\varepsilon^{k_\varsigma l_\varsigma}(s),\varepsilon_{k_\varsigma l_\varsigma}(s),\varepsilon_{k'_\varsigma l'_\varsigma}(s),\varepsilon^{k'_\varsigma l'_\varsigma}(s)$

Lem. 3.1.1. $\sigma^T(s)=(-1)^{2s+1}\varepsilon(s)\sigma(s)\varepsilon(s)$

Thm. 3.1.3. $\varepsilon(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\varepsilon(s)e^{(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)}$*;* $\varepsilon^{k_\varsigma l_\varsigma}(s)$ *are constant invariant tensors.*

Cor. 3.1.3. $\varepsilon(s)=e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s)}\varepsilon(s)e^{(i\omega-\varsigma\epsilon)\cdot\sigma^T(s)}$*;* $\varepsilon_{k'_\varsigma l'_\varsigma}(s)$ *are constant invariant tensors.*

Cor. 3.1.4. $\varepsilon(s)=e^{-(i\omega+\varsigma\epsilon)\cdot\sigma^T(s)}\varepsilon(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$*;* $\varepsilon_{k_\varsigma l_\varsigma}(s)$ *are constant invariant tensors.*

Cor. 3.1.5. $\varepsilon(s)=e^{-(i\omega-\varsigma\epsilon)\cdot\sigma^T(s)}\varepsilon(s)e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s)}$*;* $\varepsilon^{k'_\varsigma l'_\varsigma}(s)$ *are constant invariant tensors.*

3.1.4 Antisymmetric $\frac{1}{2}$-spinor metric tensors $\varepsilon^{A_\varsigma B_\varsigma},\varepsilon_{A_\varsigma B_\varsigma},\varepsilon_{A'_\varsigma B'_\varsigma},\varepsilon^{A'_\varsigma B'_\varsigma}$

In the previous section, take $s=\frac{1}{2}$ to obtain: $\varepsilon^{A_\varsigma B_\varsigma},\varepsilon_{A_\varsigma B_\varsigma},\varepsilon_{A'_\varsigma B'_\varsigma},\varepsilon^{A'_\varsigma B'_\varsigma}$ are constant invariant tensors.

3.2 Fundamental theorem 1 and its relevant constant invariant tensors

3.2.1 Lemma

Lem. 3.2.1. $\vartheta_a{}^b(\Gamma,i\varsigma)_b\equiv(-\omega\times\Gamma-\varsigma\epsilon,-i\epsilon\cdot\Gamma)_a,\vartheta_a{}^b\succ\vartheta\equiv(i\omega\cdot R+\epsilon\cdot L)$

Lem. 3.2.2. $\frac{1}{2}i\omega\cdot[\Gamma,\Gamma_{\alpha_\varsigma}]=(\omega\times\Gamma)_{\alpha_\varsigma},\forall\omega\to0\Leftrightarrow[\Gamma_{\alpha_\varsigma},\Gamma_{\beta_\varsigma}]=2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$

Lem. 3.2.3. $\frac{1}{2}\epsilon\cdot\{\Gamma,\Gamma_{\alpha_\varsigma}\}=\epsilon_{\alpha_\varsigma},\forall\epsilon\to0\Leftrightarrow\{\Gamma_{\alpha_\varsigma},\Gamma_{\beta_\varsigma}\}=2\delta_{\alpha_\varsigma\beta_\varsigma}$

Lem. 3.2.4. $\vartheta_{ij}=\varepsilon_{ijk}\omega^k,\vartheta_{i\pi}=i\epsilon_i,\vartheta_{\pi j}=-i\epsilon_j,\omega_k=\frac{1}{2}\varepsilon_{kij}\vartheta^{ij}$

3.2.2 Fundamental theorem 1

The following theorem exists in 4-dimensional space-time.

Thm. 3.2.1. $(\Gamma, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\Gamma}(\Gamma, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\Gamma} \Leftrightarrow \begin{cases}[\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma} \\ \{\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}\} = 2\delta_{\alpha_\varsigma\beta_\varsigma}\end{cases}$

Proof: $(\Gamma, i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\Gamma}(\Gamma, i\varsigma)_b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\Gamma}, \forall\omega, \forall\epsilon$
$\Leftrightarrow (\Gamma, i\varsigma)_a = (\delta_a{}^b + \vartheta_a{}^b)(1 + (i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\Gamma)(\Gamma, i\varsigma)_b(1 - (i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\Gamma), \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow 0 = \vartheta_a{}^b(\Gamma, i\varsigma)_b + \frac{1}{2}i\omega\cdot[\Gamma, (\Gamma, i\varsigma)_a] + \frac{1}{2}\epsilon\cdot\{\Gamma, (\Gamma, i\varsigma)_a\}, \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow 0 = (-\omega\times\Gamma - \varsigma\epsilon, -i\epsilon\cdot\Gamma)_a + \frac{1}{2}i\omega\cdot[\Gamma, (\Gamma, i\varsigma)_a] + \frac{1}{2}\epsilon\cdot\{\Gamma, (\Gamma, i\varsigma)_a\}, \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow 0 = (-\omega\times\Gamma - \varsigma\epsilon)_{\alpha_\varsigma} + \frac{1}{2}i\omega\cdot[\Gamma, \Gamma_{\alpha_\varsigma}] + \frac{1}{2}\epsilon\cdot\{\Gamma, \Gamma_{\alpha_\varsigma}\}, \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow \frac{1}{2}i\omega\cdot[\Gamma, \Gamma_{\alpha_\varsigma}] = (\omega\times\Gamma)_{\alpha_\varsigma}, \frac{1}{2}\epsilon\cdot\{\Gamma, \Gamma_{\alpha_\varsigma}\} = \epsilon_{\alpha_\varsigma}, \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}, \{\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}\} = 2\delta_{\alpha_\varsigma\beta_\varsigma}$ □

The above theorem indicates: The commutative relation $[\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$ and anti commutative relation $\{\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}\} = 2\delta_{\alpha_\varsigma\beta_\varsigma}$ mean that $(\Gamma, i\varsigma)_a$ is a constant invariant tensor, vice versa.

3.2.3 Constant invariant tensors $(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}, (\sigma, -i\varsigma)_a{}^{A'_\varsigma A_\varsigma}$ [1, 2]

Cor. 3.2.1. $(\sigma, i\varsigma)^a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]^a{}_b e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, i\varsigma)^b e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$*; $(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}$ are constant invariant tensors.*

Cor. 3.2.2. $(\sigma, -i\varsigma)_a = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}(\sigma, -i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$*; $(\sigma, -i\varsigma)_a{}^{A'_\varsigma A_\varsigma}$ are constant invariant tensors.*

Self comment: This pair of constant invariant tensors is the protagonist of Penrose's spin analysis [1, 2]. This is also one of the reasons that inspired me to develop the theory of general constant invariant tensors. I'm just rediscovering it in my way here.

3.2.4 Generalization of fundamental theorem 1

The following theorem exists in any N+1 dimensional space-time.(Finally to be generalized successfully.)

Thm. 3.2.2. $[\Gamma, i\varsigma]^a = [e^{\vartheta}]^a{}_b e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i,\Gamma_j]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma}[\Gamma, i\varsigma]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i,\Gamma_j]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma} \Leftrightarrow \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}; S_{ij} := -\frac{i}{4}[\Gamma_i, \Gamma_j]$

Proof: $[\Gamma, i\varsigma]^a = [e^{\vartheta}]^a{}_b e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i,\Gamma_j]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma}[\Gamma, i\varsigma]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i,\Gamma_j]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma}$
$\Leftrightarrow [\Gamma, i\varsigma]^a = [e^{\vartheta}]^a{}_b e^{\frac{i}{2}\vartheta^{ij}S_{ij}+\varsigma\epsilon\cdot\frac{1}{2}\Gamma}[\Gamma, i\varsigma]^b e^{-\frac{i}{2}\vartheta^{ij}S_{ij}+\varsigma\epsilon\cdot\frac{1}{2}\Gamma}$
$\Leftrightarrow [\Gamma, i\varsigma]^a = (\delta^a{}_b + \vartheta^a{}_b)$
$[1 + \frac{i}{2}\vartheta^{ij}S_{ij} + \varsigma\epsilon\cdot\frac{1}{2}\Gamma][\Gamma, i\varsigma]^b[1 - \frac{i}{2}\vartheta^{ij}S_{ij} + \varsigma\epsilon\cdot\frac{1}{2}\Gamma]$
$\Leftrightarrow 0 = \vartheta^a{}_b[\Gamma, i\varsigma]^b$
$+ [\frac{i}{2}\vartheta^{ij}S_{ij} + \varsigma\epsilon\cdot\frac{1}{2}\Gamma][\Gamma, i\varsigma]^a + [\Gamma, i\varsigma]^a[-\frac{i}{2}\vartheta^{ij}S_{ij} + \varsigma\epsilon\cdot\frac{1}{2}\Gamma]$
$\Leftrightarrow 0 = \vartheta^a{}_b[\Gamma, i\varsigma]^b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\varsigma]^a] + \frac{1}{2}\varsigma\{[\Gamma, i\varsigma]^a, \epsilon\cdot\Gamma\}$
$\Leftrightarrow 0 = \vartheta^{ab}[\Gamma, i\varsigma]_b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\varsigma]^a] + \frac{1}{2}\varsigma\{[\Gamma, i\varsigma]^a, \epsilon\cdot\Gamma\}$
$\Leftrightarrow \begin{cases}0 = \vartheta^{kb}[\Gamma, i\varsigma]_b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\varsigma]^k] + \frac{1}{2}\varsigma\{[\Gamma, i\varsigma]^k, \epsilon\cdot\Gamma\} \\ 0 = \vartheta^{\pi b}[\Gamma, i\varsigma]_b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\varsigma]^\pi] + \frac{1}{2}\varsigma\{i\varsigma, \epsilon\cdot\Gamma\}\end{cases}$
$\Leftrightarrow \begin{cases}0 = \vartheta^{kj}\Gamma_j + i\varsigma\vartheta^{k\pi} + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k] + \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} \\ 0 = \vartheta^{\pi j}\Gamma_j + i\epsilon\cdot\Gamma\end{cases}$
$\Leftrightarrow \begin{cases}0 = \vartheta^{kj}\Gamma_j + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k] + \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k \\ 0 = 0\end{cases}$
$\Leftrightarrow 0 = \vartheta^{ij}\delta^k{}_i\Gamma_j + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k] + \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k$
$\Leftrightarrow \begin{cases}0 = \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k \\ 0 = \vartheta^{ij}\delta^k{}_i\Gamma_j + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k]\end{cases}$
$\Leftrightarrow \begin{cases}0 = \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k \\ 0 = \vartheta^{ij}\{\frac{1}{2}\delta_{k[i}\Gamma_{j]} + \frac{i}{2}[S_{ij}, \Gamma_k]\}\end{cases}$
$\Leftrightarrow \begin{cases}\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \\ i[\Gamma_k, S_{ij}] = \delta_{k[i}\Gamma_{j]}\end{cases}$
$\Leftrightarrow \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$ □

The above theorem indicates: The anti commutative relation $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$ means that $(\Gamma, i\varsigma)_a$ is a constant invariant tensor, vice versa. The fundamental theorem 1 is just a special case.

3.2.5 Constant invariant tensors $[\Gamma(N), i\varsigma]^a_{A_\varsigma A'_\varsigma}, [\Gamma(N), -i\varsigma]_a^{A'_\varsigma A_\varsigma}$

Cor. 3.2.3.
$\{\Gamma_i(N), \Gamma_j(N)\} = 2\delta_{ij} \Rightarrow [\Gamma(N), i\varsigma]^a = [e^{\vartheta}]^a{}_b e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i(N),\Gamma_j(N)]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma(N)}[\Gamma(N), i\varsigma]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i(N),\Gamma_j(N)]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma(N)}$

Self comment: Therefore $[\Gamma(N), i\varsigma]^a_{A_\varsigma A'_\varsigma}$ and $[\Gamma(N), -i\varsigma]_a^{A'_\varsigma A_\varsigma}$ are constant invariant tensors. It is a generalization of Penrose spinors in high and low dimensional space-time.

3.3 Fundamental theorem 2 and its relevant constant invariant tensors

3.3.1 Fundamental theorem 2

Thm. 3.3.1. $\Gamma_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\Gamma}\Gamma_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\Gamma} \Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$

Proof: $\Gamma_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\Gamma}\Gamma_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\Gamma}, \forall\omega, \forall\epsilon$
$\Leftrightarrow \Gamma_{\alpha_\varsigma} = [\delta_{\alpha_\varsigma}{}^{\beta_\varsigma} + (i\omega+\varsigma\epsilon)\cdot\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}][1+(i\omega+\varsigma\epsilon)\cdot\Gamma]\Gamma_{\beta_\varsigma}[1-(i\omega+\varsigma\epsilon)\cdot\Gamma], \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow 0 = (i\omega+\varsigma\epsilon)\cdot\{\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma} + [\Gamma, \Gamma_{\alpha_\varsigma}]\}, \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow \gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma} + [\Gamma, \Gamma_{\alpha_\varsigma}] = 0$
$\Leftrightarrow \gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma} + [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 0 (\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \equiv i\gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma})$
$\Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$ □

The above theorem indicates: The commutative relation $[\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$ means that $\Gamma_{\alpha_\varsigma}$ a constant invariant tensor, vice versa.

3.3.2 Generalized fundamental theorem 2

Thm. 3.3.2. $T_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\Gamma}T_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\Gamma} \Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\Gamma_{\gamma_\varsigma}$

Proof: $T_{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\Gamma}T_{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\Gamma}, \forall\omega, \forall\epsilon$
$\Leftrightarrow T_{\alpha_\varsigma} = [\delta_{\alpha_\varsigma}{}^{\beta_\varsigma} + (i\omega+\varsigma\epsilon)\cdot\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}][1+(i\omega+\varsigma\epsilon)\cdot\Gamma]T_{\beta_\varsigma}[1-(i\omega+\varsigma\epsilon)\cdot\Gamma], \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow 0 = (i\omega+\varsigma\epsilon)\cdot\{\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}T_{\beta_\varsigma} + [\Gamma, T_{\alpha_\varsigma}]\}, \forall\omega\to 0, \forall\epsilon\to 0$
$\Leftrightarrow \gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}T_{\beta_\varsigma} + [\Gamma, T_{\alpha_\varsigma}] = 0$
$\Leftrightarrow \gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}T_{\gamma_\varsigma} + [\Gamma_{\alpha_\varsigma}, T_{\beta_\varsigma}] = 0 (\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \equiv i\gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma})$
$\Leftrightarrow [\Gamma_{\alpha_\varsigma}, T_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}T_{\gamma_\varsigma}$ □

3.3.3 Generalization: constant invariant tensor operators [40]

Thm. 3.3.3.
$$\begin{cases} \hat{T}(j,m) = \sum\limits_{m'=j}^{-j} \langle j,m|U^+(R)|j,m'\rangle U(R)\hat{T}(j,m')U^+(R) \\ U(R)\hat{T}(j,m)U^+(R) = \sum\limits_{m'=j}^{-j} \hat{T}(j,m')\langle j,m'|U(R)|j,m\rangle \\ U(R)|j,m\rangle = \sum\limits_{m'=j}^{-j} |j,m'\rangle\langle j,m'|U(R)|j,m\rangle \end{cases}$$

3.3.4 Constant invariant tensors $\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s), \sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)$, $\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}, \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}$

Cor. 3.3.1. $\sigma^{\alpha_\varsigma}(s) = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]^{\alpha_\varsigma}{}_{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\sigma^{\beta_\varsigma}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$; $\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)$ *are constant invariant tensors.*

Cor. 3.3.2. $\sigma^{\alpha_\varsigma} = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]^{\alpha_\varsigma}{}_{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}\sigma^{\beta_\varsigma} e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$; $\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}$ *are constant invariant tensors.*

Cor. 3.3.3. $\sigma_{\alpha'_\varsigma}(s) = [e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s)}\sigma_{\beta'_\varsigma}(s)e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s)}$; $\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)$ *are constant invariant tensors.*

Cor. 3.3.4. $\sigma_{\alpha'_\varsigma} = [e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}\sigma_{\beta'_\varsigma} e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma}$; $\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}$ *are constant invariant tensors.*

Self comment: This theorem indicates that the spin matrix is a constant invariant tensor. Combining the theorem in the previous section, it is found that Pauli matrix can be combined into two types of constant invariant tensors. This is interesting, but other spin matrices do not have this property, so Pauli matrix is very special.

Cor. 3.3.5. $\sigma_{\alpha'_\varsigma}(s) = [e^{i\omega\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma} e^{i\omega\cdot\sigma(s)}\sigma_{\beta'_\varsigma}(s)e^{-i\omega\cdot\sigma(s)}[\Rightarrow]\sigma(s)\cdot\hat{p} = e^{i\omega\cdot\sigma(s)}\sigma_z(s)e^{-i\omega\cdot\sigma(s)}$

3.3.5 Antisymmetric constant invariant tensors $\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \varepsilon_{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma}$

Cor. 3.3.6. $\gamma_{\alpha_\varsigma}(s) = [e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega+\varsigma\epsilon)\cdot\gamma}\gamma_{\beta_\varsigma}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\gamma}$; $\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(\equiv i\gamma_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma})$ *are constant invariant tensors.*

Cor. 3.3.7. $\gamma_{\alpha'_\varsigma}(s) = [e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma} e^{(i\omega-\varsigma\epsilon)\cdot\gamma}\gamma_{\beta'_\varsigma}(s)e^{-(i\omega-\varsigma\epsilon)\cdot\gamma}$; $\varepsilon_{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma}(\equiv i\gamma_{\alpha'_\varsigma\beta'_\varsigma\gamma'_\varsigma})$ *are constant invariant tensors.*

Self comment: The above shows that the three dimensional antisymmetric tensor is a four dimensional Lorentz constant invariant tensor.

3.3.6 Transition

Cor. 3.3.8. $\sigma_{+\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}\sigma_{+\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}$

Cor. 3.3.9. $\sigma_{-\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}\sigma_{-\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}$

3.3.7 Constant invariant tensors $\sigma^{ab}_{+\alpha_\varsigma},\sigma^{ab}_{-\alpha'},\sigma^{ab}_{\varsigma\alpha_\varsigma},\sigma^{ab}_{-\varsigma\alpha'_\varsigma}$

Cor. 3.3.10. $\sigma_{+\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\sigma_{\varsigma\beta_\varsigma}e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)}$*; $\sigma^{a_\varsigma b_\varsigma}_{\varsigma\alpha_\varsigma}$ are constant invariant tensors.*

Proof: $\sigma_{+\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}\sigma_{+\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}$
$\Leftrightarrow\sigma_{+\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}[e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}\sigma_{+\beta_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}]e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}$
$\Leftrightarrow\sigma_{\varsigma\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\sigma_{\varsigma\beta_\varsigma}e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)}$; $\sigma^{a_\varsigma b_\varsigma}_{+\alpha_\varsigma}$ are constant invariant tensors. □

Cor. 3.3.11. $\sigma_{-\varsigma\alpha'_\varsigma}=[e^{(i\omega-\varsigma\epsilon)\cdot\gamma}]_{\alpha'_\varsigma}{}^{\beta'_\varsigma}e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\sigma_{-\varsigma\beta'_\varsigma}e^{-(i\omega\cdot R+\varsigma\epsilon\cdot L)},\sigma^{a_\varsigma b_\varsigma}_{-\varsigma\alpha'_\varsigma}$ *are constant invariant tensors.*

Proof: $\sigma_{-\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}\sigma_{-\beta_\varsigma}e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}$
$\Leftrightarrow\sigma_{-\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}[e^{(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}\sigma_{-\beta_\varsigma}e^{-(i\omega-\varsigma\epsilon)\cdot\frac{1}{2}\sigma_+}]e^{-(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma_-}$
$\Leftrightarrow\sigma_{-\alpha_\varsigma}=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)}\sigma_{\varsigma\beta_\varsigma}e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}$; $\sigma^{a'_\varsigma b'_\varsigma}_{-\alpha_\varsigma}$ are constant invariant tensors. □

Combining the two $\sigma^{a_\varsigma b_\varsigma}_{+\alpha_\varsigma},\sigma^{a'_\varsigma b'_\varsigma}_{-\alpha_\varsigma}$, it can be seen that $\sigma^{ab}_{+\alpha},\sigma^{ab}_{-\alpha'}$ are constant invariant tensors.
It is further known that $\sigma^{ab}_{\varsigma\alpha_\varsigma},\sigma^{ab}_{-\varsigma\alpha'_\varsigma}$ are constant invariant tensors.
Self comment: The above strictly proves that the two generator matrices of SO(4) are constant invariant tensors.

3.3.8 Generalization of fundamental theorem 2

Lem. 3.3.1. $[\Gamma_i,\Gamma_j]=-2\gamma^k{}_{ij}\Gamma_k\Rightarrow[\gamma_i,\gamma_j]=-\gamma^k{}_{ij}\gamma_k$

Thm. 3.3.4. $\Gamma_{\alpha_\varsigma}=[e^{\frac{i}{2}\omega^{ij}\vec{S}_{ij}+\varsigma\epsilon^k\gamma_k}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]+\frac{1}{2}\varsigma\epsilon^k\Gamma_k}\Gamma_{\beta_\varsigma}e^{-(\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]+\frac{1}{2}\varsigma\epsilon^k\Gamma_k)}$
$\Leftrightarrow[\Gamma_i,\Gamma_j]=-2\gamma^k{}_{ij}\Gamma_k,\frac{1}{4}[[\Gamma_i,\Gamma_j],\Gamma_k]=-i[\vec{S}_{ij}]_k{}^l\Gamma_l$

Proof: $\Gamma_{\alpha_\varsigma}=[e^{\frac{i}{2}\omega^{ij}\vec{S}_{ij}+\varsigma\epsilon^k\gamma_k}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]+\frac{1}{2}\varsigma\epsilon^k\Gamma_k}\Gamma_{\beta_\varsigma}e^{-(\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]+\frac{1}{2}\varsigma\epsilon^k\Gamma_k)},\forall\omega,\forall\epsilon$
$\Leftrightarrow\Gamma_{\alpha_\varsigma}=[\delta_{\alpha_\varsigma}{}^{\beta_\varsigma}+(\frac{i}{2}\omega^{ij}\vec{S}_{ij}+\varsigma\epsilon^k\gamma_k)_{\alpha_\varsigma}{}^{\beta_\varsigma}][1+\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]+\frac{1}{2}\varsigma\epsilon^k\Gamma_k]\Gamma_{\beta_\varsigma}[1-\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]-\frac{1}{2}\varsigma\epsilon^k\Gamma_k],\forall\omega\to0,\forall\epsilon\to0$
$\Leftrightarrow0=(\frac{i}{2}\omega^{ij}\vec{S}_{ij}+\varsigma\epsilon^k\gamma_k)_{\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma}+[\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]+\frac{1}{2}\varsigma\epsilon^k\Gamma_k,\Gamma_{\alpha_\varsigma}],\forall\omega\to0,\forall\epsilon\to0$
$\Leftrightarrow0=\epsilon^k\gamma_{k\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma}+[\epsilon^k\Gamma_k,\Gamma_{\alpha_\varsigma}],0=\frac{i}{2}\omega^{ij}[\vec{S}_{ij}]_{\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma}+[\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j],\Gamma_{\alpha_\varsigma}]$
$\Leftrightarrow[\Gamma_k,\Gamma_{\alpha_\varsigma}]=-2\gamma_{k\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma},\frac{1}{4}[[\Gamma_i,\Gamma_j],\Gamma_{\alpha_\varsigma}]=-i[\vec{S}_{ij}]_{\alpha_\varsigma}{}^{\beta_\varsigma}\Gamma_{\beta_\varsigma}$
$\Leftrightarrow[\Gamma_i,\Gamma_j]=-2\gamma^k{}_{ij}\Gamma_k,\frac{1}{4}[[\Gamma_i,\Gamma_j],\Gamma_k]=-i[\vec{S}_{ij}]_k{}^l\Gamma_l$
$\Leftrightarrow[\Gamma_i,\Gamma_j]=-2\gamma^k{}_{ij}\Gamma_k,\frac{1}{4}[[\Gamma_i,\Gamma_j],\Gamma_k]=\gamma^m{}_{ij}\gamma^l{}_{mk}\Gamma_l=\gamma_{ij}{}^m\gamma_{mk}{}^l\Gamma_l=-i[\vec{S}_{ij}]_k{}^l\Gamma_l$ □

The above $\gamma^k{}_{ij}$ is a four dimensional analogue. Generally $\vec{S}_{ij}$ is not directly related to $[\Gamma_i,\Gamma_j]$. It seems that it can't be promoted, and only the four dimensional situation can be satisfied.

Cor. 3.3.12. $\Gamma_{\alpha_\varsigma}=[e^{\frac{i}{2}\omega^{ij}\vec{S}_{ij}}]_{\alpha_\varsigma}{}^{\beta_\varsigma}e^{\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]}\Gamma_{\beta_\varsigma}e^{-\frac{1}{8}\omega^{ij}[\Gamma_i,\Gamma_j]}\Leftrightarrow\frac{1}{4}[[\Gamma_i,\Gamma_j],\Gamma_k]=-i[\vec{S}_{ij}]_k{}^l\Gamma_l$

3.4 Fundamental theorem 3 and its relevant constant invariant tensors
3.4.1 Fundamental theorem 3

The following theorem exists in any N+1 dimensional space-time.

Thm. 3.4.1. $\Gamma_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^de^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}\Gamma_{cd}e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}},\vartheta_{ab}=-\vartheta_{ba}$
$\Leftrightarrow i[\Gamma_{ab},\Gamma_{cd}]=\delta_{ad}\Gamma_{bc}-\delta_{ac}\Gamma_{bd}+\delta_{bc}\Gamma_{ad}-\delta_{bd}\Gamma_{ac}$

Proof: $\Gamma_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^de^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}\Gamma_{cd}e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}},\forall\vartheta^{ef}$
$\Leftrightarrow\Gamma_{ab}=[\delta_a{}^c+\vartheta_a{}^c][\delta_b{}^d+\vartheta_b{}^d](1+\frac{i}{2}\vartheta^{ef}\Gamma_{ef})\Gamma_{cd}(1-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}),\forall\vartheta^{ef}\to0$
$\Leftrightarrow0=\vartheta_a{}^c\Gamma_{cb}-\vartheta_b{}^d\Gamma_{da}-\frac{i}{2}\vartheta^{cd}[\Gamma_{ab},\Gamma_{cd}],\forall\vartheta^{cd}\to0$
$\Leftrightarrow i\vartheta^{cd}[\Gamma_{ab},\Gamma_{cd}]=2(\vartheta_a{}^c\Gamma_{cb}-\vartheta_b{}^d\Gamma_{da}),\forall\vartheta^{cd}\to0$
$\Leftrightarrow i\vartheta^{cd}[\Gamma_{ab},\Gamma_{cd}]=\vartheta^{cd}(\delta_{ad}\Gamma_{bc}-\delta_{ac}\Gamma_{bd}+\delta_{bc}\Gamma_{ad}-\delta_{bd}\Gamma_{ac}),\forall\vartheta^{cd}\to0$
$\Leftrightarrow i[\Gamma_{ab},\Gamma_{cd}]=\delta_{ad}\Gamma_{bc}-\delta_{ac}\Gamma_{bd}+\delta_{bc}\Gamma_{ad}-\delta_{bd}\Gamma_{ac}$ □

The above theorem indicates: The commutative relation $i[\Gamma_{ab},\Gamma_{cd}]=\delta_{ad}\Gamma_{bc}-\delta_{ac}\Gamma_{bd}+\delta_{bc}\Gamma_{ad}-\delta_{bd}\Gamma_{ac}$ means that Γ_{ab} is a constant invariant tensor, and vice versa.

3.4.2 Generalized fundamental theorem 3

The following theorem exists in any N+1 dimensional space-time.

Thm. 3.4.2. $T_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}T_{cd}e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}},\vartheta_{ab}=-\vartheta_{ba}$
$\Leftrightarrow i[T_{ab},\Gamma_{cd}]=\delta_{ac}T_{db}-\delta_{ad}T_{cb}+\delta_{bc}T_{ad}-\delta_{bd}T_{ac}$

Proof: $T_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}T_{cd}e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}},\forall\vartheta^{ef}$
$\Leftrightarrow T_{ab}=[\delta_a{}^c+\vartheta_a{}^c][\delta_b{}^d+\vartheta_b{}^d](1+\frac{i}{2}\vartheta^{ef}\Gamma_{ef})T_{cd}(1-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}),\forall\vartheta^{ef}\to 0$
$\Leftrightarrow 0=\vartheta_a{}^cT_{cb}+\vartheta_b{}^dT_{ad}-\frac{i}{2}\vartheta^{cd}[T_{ab},\Gamma_{cd}],\forall\vartheta^{cd}\to 0$
$\Leftrightarrow i\vartheta^{cd}[T_{ab},\Gamma_{cd}]=2(\vartheta_a{}^cT_{cb}+\vartheta_b{}^dT_{ad}),\forall\vartheta^{cd}\to 0$
$\Leftrightarrow i\vartheta^{cd}[T_{ab},\Gamma_{cd}]=2\vartheta^{cd}(-\delta_{ad}T_{cb}+\delta_{bc}T_{ad}),\forall\vartheta^{cd}\to 0$
$\Leftrightarrow i[T_{ab},\Gamma_{cd}]=\delta_{ac}T_{db}-\delta_{ad}T_{cb}+\delta_{bc}T_{ad}-\delta_{bd}T_{ac}$ □

3.4.3 Spin constant invariant tensor [8] $S_{ab\mathcal{A}}{}^{\mathcal{B}}$

Cor. 3.4.1. $S_{ab}=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}S_{ef}}S_{cd}e^{-\frac{i}{2}\vartheta^{ef}S_{ef}}$*;* $S_{ab\mathcal{A}}{}^{\mathcal{B}}$ *are constant invariant tensors.*

Self comment: The above shows that the spin tensor of physics is a constant invariant tensor in any space-time.

3.4.4 Spin constant invariant tensors $S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma),S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)$

Cor. 3.4.2. $S_{ab}(s,\varsigma)=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}S_{ef}(s,\varsigma)}S_{cd}(s,\varsigma)e^{-\frac{i}{2}\vartheta^{ef}S_{ef}(s,\varsigma)}$*;* $S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma)$ *are constant invariant tensors.*

Cor. 3.4.3. $S_{ab}(s,-\varsigma)=[e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}S_{ef}(s,-\varsigma)}S_{cd}(s,-\varsigma)e^{-\frac{i}{2}\vartheta^{ef}S_{ef}(s,-\varsigma)}$*;* $S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)$ *are constant invariant tensors.*

3.4.5 Spin constant invariant tensors $S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2},\varsigma),S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}(\frac{1}{2},-\varsigma)$

Previous section take $s=\frac{1}{2}$ **to obtain:** $S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2},\varsigma),S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}(\frac{1}{2},-\varsigma)$ **are constant invariant tensors.**

3.5 Fundamental theorem 4 and its relevant constant invariant tensors

3.5.1 Fundamental theorem 4

Lem. 3.5.1. $[\Gamma_a,[\Gamma_c,\Gamma_d]]=\frac{1}{2}(\{\{\Gamma_a,\Gamma_c\},\Gamma_d\}-\{\Gamma_c,\{\Gamma_d,\Gamma_a\}\})$

The following theorem exists in any N+1 dimensional space-time.

Thm. 3.5.1. $\Gamma_a=[e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}\Gamma_{cd}}\Gamma_b e^{-\frac{i}{2}\vartheta^{cd}\Gamma_{cd}}\Leftrightarrow i[\Gamma_a,\Gamma_{cd}]=\delta_{a[c}\Gamma_{d]}$

Proof: $\Gamma_a=[e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}\Gamma_{cd}}\Gamma_b e^{-\frac{i}{2}\vartheta^{cd}\Gamma_{cd}},\forall\vartheta^{cd}$
$\Leftrightarrow\Gamma_a=[\delta_a{}^b+\vartheta_a{}^b](1+\frac{i}{2}\vartheta^{cd}\Gamma_{cd})\Gamma_b(1-\frac{i}{2}\vartheta^{cd}\Gamma_{cd}),\forall\vartheta^{cd}\to 0$
$\Leftrightarrow 0=\vartheta_a{}^b\Gamma_b-\frac{i}{2}\vartheta^{cd}[\Gamma_a,\Gamma_{cd}],\forall\vartheta^{cd}\to 0$
$\Leftrightarrow i\vartheta^{cd}[\Gamma_a,\Gamma_{cd}]=2\vartheta_a{}^b\Gamma_b,\forall\vartheta^{cd}\to 0$
$\Leftrightarrow i\vartheta^{cd}[\Gamma_a,\Gamma_{cd}]=\vartheta^{cd}(\delta_{ac}\Gamma_d-\delta_{ad}\Gamma_c),\forall\vartheta^{cd}\to 0$
$\Leftrightarrow i[\Gamma_a,\Gamma_{cd}]=\delta_{a[c}\Gamma_{d]}$ □

The following theorem exists in any N+1 dimensional space-time.

Thm. 3.5.2. $\Gamma_a=[e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}}\Gamma_b e^{-\frac{i}{2}\vartheta^{cd}S_{cd}},S_{cd}=-\frac{i}{4}[\Gamma_c,\Gamma_d]\Leftrightarrow\frac{1}{4}[[\Gamma_c,\Gamma_d],\Gamma_a]=\Gamma_{[c}\delta_{d]a}$

Proof: $\Gamma_a=[e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}}\Gamma_b e^{-\frac{i}{2}\vartheta^{cd}S_{cd}}$
$\Leftrightarrow\Gamma_a=(1+\vartheta)_a{}^b(1+\frac{i}{2}\vartheta^{cd}S_{cd})\Gamma_b(1-\frac{i}{2}\vartheta^{cd}S_{cd})$
$\Leftrightarrow 0=\vartheta_a{}^b\Gamma_b+\frac{i}{2}\vartheta^{cd}[S_{cd},\Gamma_a]$
$\Leftrightarrow 0=-\frac{1}{2}\vartheta^{cd}\Gamma_{[c}\delta_{d]a}+\frac{i}{2}\vartheta^{cd}[S_{cd},\Gamma_a]$
$\Leftrightarrow i[\Gamma_a,S_{cd}]=\delta_{a[c}\Gamma_{d]}$
$\Leftrightarrow\frac{1}{4}[\Gamma_a,[\Gamma_c,\Gamma_d]]=\delta_{a[c}\Gamma_{d]}$ □

Cor. 3.5.1. $\Gamma_k=[e^{\vartheta}]_k{}^l e^{\frac{i}{2}\vartheta^{ij}S_{ij}}\Gamma_l e^{-\frac{i}{2}\vartheta^{ij}S_{ij}},S_{ij}=-\frac{i}{4}[\Gamma_i,\Gamma_j]\Leftrightarrow\frac{1}{4}[[\Gamma_i,\Gamma_j],\Gamma_k]=\Gamma_{[i}\delta_{j]k}$

3.5.2 Constant invariant tensor $\Gamma^a{}_{\lambda_\varsigma}{}^{\mu_\varsigma}(n)$ **in** $n=N+1$ **dimensional space-time**

Lem. 3.5.2. $\{\Gamma_a,\Gamma_b\}=2\delta_{ab}\Rightarrow\frac{1}{4}[\Gamma_a,[\Gamma_c,\Gamma_d]]=\delta_{a[c}\Gamma_{d]}$

The following theorem exists in any N+1 dimensional space-time.

Thm. 3.5.3. $\{\Gamma_a,\Gamma_b\}=2\delta_{ab}\Rightarrow\Gamma_a=[e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}}\Gamma_b e^{-\frac{i}{2}\vartheta^{cd}S_{cd}},S_{cd}=-\frac{i}{4}[\Gamma_c,\Gamma_d]$

Thm. 3.5.4. $\{\Gamma_a,\Gamma_b\}=2\delta_{ab}\Rightarrow\begin{cases}\Gamma_{N+1}=e^{\frac{i}{2}\vartheta^{*ab}S_{ab}}\Gamma_{N+1}e^{-\frac{i}{2}\vartheta^{ab}S_{ab}},S_{ab}=-\frac{i}{4}[\Gamma_a,\Gamma_b]\\ \Gamma_{N+1}=e^{\frac{i}{2}\vartheta^{ab}S_{ab}}\Gamma_{N+1}e^{-\frac{i}{2}\vartheta^{*ab}S_{ab}},S_{ab}=-\frac{i}{4}[\Gamma_a,\Gamma_b]\end{cases}$

Thm. 3.5.5. $\{\Gamma_a,\Gamma_b\}=2\delta_{ab}\Rightarrow\begin{cases}\Gamma_{N+2}=e^{\frac{i}{2}\vartheta^{ab}S_{ab}}\Gamma_{N+2}e^{-\frac{i}{2}\vartheta^{ab}S_{ab}},S_{ab}=-\frac{i}{4}[\Gamma_a,\Gamma_b],\Gamma_{N+2}=\Gamma_1\cdots\Gamma_{N+1}\\ \Gamma_{N+2}=e^{\frac{i}{2}\vartheta^{*ab}S_{ab}}\Gamma_{N+2}e^{-\frac{i}{2}\vartheta^{*ab}S_{ab}},S_{ab}=-\frac{i}{4}[\Gamma_a,\Gamma_b],\Gamma_{N+2}=\Gamma_1\cdots\Gamma_{N+1}\end{cases}$

Self comment: In any N+1 dimensional space-time, Dirac matrices Γ_a and $\Gamma_{N+1},\Gamma_{N+2}$ are constant invariant tensors.

3.5.3 Constant invariant tensors [4] $\gamma_{a\lambda_\varsigma}{}^{\mu_\varsigma}(\varsigma),\gamma_{5\lambda_\varsigma}{}^{\mu_\varsigma}(\varsigma),\delta_{\lambda_\varsigma}{}^{\mu_\varsigma},\gamma_4^{\lambda'_\varsigma\lambda_\varsigma}$

Def. 3.5.1. $\gamma_5(\varsigma)\equiv\gamma_x(\varsigma)\gamma_y(\varsigma)\gamma_z(\varsigma)\gamma_\pi(\varsigma),S_{ab}(e,\varsigma)\equiv-\frac{i}{4}[\gamma_a(\varsigma),\gamma_b(\varsigma)]$

Def. 3.5.2. $_{\lambda_\varsigma}\sim e^{\frac{i}{2}\vartheta^{ab}S_{ab}(e,\varsigma)},^{\mu_\varsigma}\sim e^{-\frac{i}{2}\vartheta^{ab}S^T_{ab}(e,\varsigma)}$

Def. 3.5.3. *A special representation:* $<\gamma_a(\varsigma),\gamma_5(\varsigma)>=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z]$

Cor. 3.5.2. $\gamma_a(\varsigma)=[e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}(e,\varsigma)}\gamma_b(\varsigma)e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(e,\varsigma)}$; $\gamma_{a\lambda_\varsigma}{}^{\mu_\varsigma}(\varsigma)$ *are constant invariant tensors.*

Cor. 3.5.3. $\gamma_5(\varsigma)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}\gamma_5(\varsigma)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}$; $\gamma_{5\lambda_\varsigma}{}^{\mu_\varsigma}(\varsigma)$ *are constant invariant tensors.*

Cor. 3.5.4. $I_4=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}I_4e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}$; $\delta_{\lambda_\varsigma}{}^{\mu_\varsigma}$ *are constant invariant tensors.*

Cor. 3.5.5. $\gamma_4=e^{\frac{i}{2}\vartheta^{*ab}S_{ab}(e,\varsigma)}\gamma_4e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}$; $\gamma_4^{\lambda'_\varsigma\lambda_\varsigma}$ *are constant invariant tensors.*

Self comment: The above shows Dirac matrix and γ_4,γ_5 matrices are all constant invariant tensors in four dimensional space-time.

3.6 Fundamental theorem 5 and its relevant constant invariant tensors

3.6.1 Fundamental theorem 5

Thm. 3.6.1. $T_\alpha=[e^{\theta^\gamma f_\gamma}]_\alpha{}^\beta e^{i\theta^\gamma T_\gamma}T_\beta e^{-i\theta^\gamma T_\gamma}\Leftrightarrow[T_\alpha,T_\beta]=if_{\alpha\beta}{}^\gamma T_\gamma,f_\alpha\prec f_{\alpha\beta}{}^\gamma$

Proof: $T_\alpha=[e^{\theta^\gamma f_\gamma}]_\alpha{}^\beta e^{i\theta^\gamma T_\gamma}T_\beta e^{-\theta^{i\gamma}T_\gamma},\forall\theta^\gamma$
$\Leftrightarrow T_\alpha=(\delta_\alpha{}^\beta+\theta^\gamma f_{\gamma\alpha}{}^\beta)(1+i\theta^\gamma T_\gamma)T_\beta(1-i\theta^\gamma T_\gamma),\forall\theta^\gamma\to0$
$\Leftrightarrow 0=\theta^\gamma(f_{\gamma\alpha}{}^\beta T_\beta+i[T_\gamma,T_\alpha]),\forall\theta^\gamma\to0$
$\Leftrightarrow 0=f_{\gamma\alpha}{}^\beta T_\beta+i[T_\gamma,T_\alpha]$
$\Leftrightarrow[T_\alpha,T_\beta]=if_{\alpha\beta}{}^\gamma T_\gamma$ □

The above theorem indicates: The commutative relation $[T_\alpha,T_\beta]=if_{\alpha\beta}{}^\gamma T_\gamma$ means that Yang-Mills basis T_α is a constant invariant tensor, vice versa. And fundamental theorem 2 is a special case of this theorem. ($f_{\alpha\beta}{}^\gamma=\varepsilon_{\alpha\beta}{}^\gamma,i\theta^\gamma=i\omega+\varsigma\epsilon,T_\alpha=\Gamma_\alpha$). This theorem is more general and can describe the covariance of internal and external spaces. And there is no requirement for the linear independence of T_α. If T_α satisfies linear independence, it can be seen from the group structure equation that the group structure constant $f_{\alpha\beta}{}^\gamma$ also satisfies a commutative relation similar to the Yang-Mills basis $f_{\alpha\beta}{}^\gamma$. That is, there is the following inference.

Cor. 3.6.1. $[-if_\alpha,-if_\beta]=if_{\alpha\beta}{}^\gamma(-if_\gamma)\Leftrightarrow-if_\alpha=[e^{\theta^\gamma f_\gamma}]_\alpha{}^\beta e^{i\theta^\gamma(-if_\gamma)}(-if_\beta)e^{-i\theta^\gamma(-if_\gamma)}$

The left side of the above equation is the structural equation of the group. So if the basis T_α satisfies linear independence, then the group structure constant $f_{\alpha\beta}{}^\gamma$ is a constant invariant tensor too.
Self comment: The above shows that the Yang-Mills basis and group structure constant are constant invariant tensors of the internal space.

3.7 Fundamental theorem 6 and its relevant constant invariant tensors

3.7.1 Fundamental theorem 6

Thm. 3.7.1. $\Gamma=e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\Gamma e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)}\Leftrightarrow[R,\Gamma]=0,\{L,\Gamma\}=0$

Proof: $\Gamma=e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\Gamma e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)},\forall\omega,\forall\epsilon$
$\Leftrightarrow\Gamma=[1+(i\omega\cdot R+\varsigma\epsilon\cdot L)]\Gamma[1-(i\omega\cdot R-\varsigma\epsilon\cdot L)],\forall\omega\to0,\forall\epsilon\to0$
$\Leftrightarrow 0=(i\omega\cdot R+\varsigma\epsilon\cdot L)]\Gamma-\Gamma(i\omega\cdot R-\varsigma\epsilon\cdot L),\forall\omega\to0,\forall\epsilon\to0$
$\Leftrightarrow 0=i\omega\cdot[R,\Gamma]+\varsigma\epsilon\cdot\{L,\Gamma\},\forall\omega\to0,\forall\epsilon\to0$
$\Leftrightarrow[R,\Gamma]=0,\{L,\Gamma\}=0$ □

Cor. 3.7.1. $\eta=e^{(i\omega\cdot R+\varsigma\epsilon\cdot L)}\eta e^{-(i\omega\cdot R-\varsigma\epsilon\cdot L)},\eta=diag(1,1,1,-1)$; $\eta^{a_\varsigma}{}_{b'_\varsigma},\eta^a{}_{b'},\eta^{a'}{}_b,\eta^{ab'},\eta^{a'b}$ *are constant invariant tensors.*

Self comment: This constant invariant tensor can complete the mutual conversion between vectors and prime vectors. The form is the same as the Minskoff metric.

3.8 Various methods for obtaining new constant invariant tensors

Method 1: $\epsilon \leftrightarrow -\epsilon$
Method 2: $\varsigma \leftrightarrow -\varsigma$
Method 3: Matrix operations such as complex conjugation, transposition, similarity transformation and representation transformation.
Method 4: Operations such as direct product, direct sum, contraction, addition, subtraction, multiplication, and division.
In addition, the above methods have been used in various corollaries of the six basic theorems to obtain various constant invariant tensors. The proof of various corollaries is basically obvious, and the proof process is mostly omitted for the sake of compactness of the content.

3.9 Reviews on six fundamental theorems

Mathematical proof of six fundamental theorems for transformation parameters $\omega, \epsilon, \vartheta^{ab}, \theta^{\alpha}$ are no special restrictions and can take any complex number. Therefore, the constant invariant tensors obtained have great mathematical universality. However, for specific physics, various parameters of the internal gauge transformation can still take complex numbers. However, for external spatiotemporal transformations, due to physical self consistency requirements, the transformation must satisfy the Lorentz group representation [12]. Therefore, there are restrictions on the transform matrix and transformation parameters, and ω, ϵ can only take real numbers. In particular, it should be pointed out that the generalized form of Fundamental Theorem 1, Fundamental Theorem 3, and Fundamental Theorem 4 not only hold true in four-dimensional space-time, but also in any N+1-dimensional space-time. This provides a mathematical analysis tool for the physical study of high and low dimensional space-time. The following enlightenment can also be obtained from the proof of the six basic theorems: The commutative and anti commutative relations of matrices imply the existence of corresponding constant invariant tensor. Conversely, a constant invariant tensor implies the existence of corresponding commutative or anti commutative relations. Starting from this idea, we can find more meaningful constant invariant tensors. It can also be seen from the above that the commutative and anti commutative relations of matrices imply their own covariance. That is, the commutative and anti commutative relations of the matrix imply that it holds true in any reference system. This is a very interesting and wonderful bootstrap mathematical property, that is memorable.

4 Properties of several intuitive basic constant invariant tensors

4.1 Properties of basic constant invariant tensors $\varepsilon^{A_\varsigma B_\varsigma}, \varepsilon_{A_\varsigma B_\varsigma}, \varepsilon^{A'_\varsigma B'_\varsigma}, \varepsilon_{A'_\varsigma B'_\varsigma}$ [1,2]

4.1.1 Important properties

$$\varepsilon^{A_\varsigma B_\varsigma}\varepsilon_{C_\varsigma D_\varsigma} = \delta^{[A_\varsigma}_{C_\varsigma}\delta^{B_\varsigma]}_{D_\varsigma} = \delta^{A_\varsigma}_{[C_\varsigma}\delta^{B_\varsigma}_{D_\varsigma]} \qquad \varepsilon_{A'_\varsigma B'_\varsigma}\varepsilon^{C'_\varsigma D'_\varsigma} = \delta^{C'_\varsigma}_{[A'_\varsigma}\delta^{D'_\varsigma}_{B'_\varsigma]} = \delta^{[C'_\varsigma}_{A'_\varsigma}\delta^{D'_\varsigma]}_{B'_\varsigma} \tag{1.52}$$

Comparison: $S_{abcd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}, S_{ab}{}^{cd} = \delta^{[c}_{a}\delta^{d]}_{b}$ (1.53)

$$\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{C_\varsigma D_\varsigma} = \varepsilon_{A_\varsigma C_\varsigma}\varepsilon_{B_\varsigma D_\varsigma} - \varepsilon_{A_\varsigma D_\varsigma}\varepsilon_{B_\varsigma C_\varsigma} \qquad \varepsilon_{A'_\varsigma B'_\varsigma}\varepsilon_{C'_\varsigma D'_\varsigma} = \varepsilon_{A'_\varsigma C'_\varsigma}\varepsilon_{B'_\varsigma D'_\varsigma} - \varepsilon_{A'_\varsigma D'_\varsigma}\varepsilon_{B'_\varsigma C'_\varsigma} \tag{1.54}$$

$$\varepsilon^{A_\varsigma B_\varsigma}\varepsilon^{C_\varsigma D_\varsigma} = \varepsilon^{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} - \varepsilon^{A_\varsigma D_\varsigma}\varepsilon^{B_\varsigma C_\varsigma} \qquad \varepsilon^{A'_\varsigma B'_\varsigma}\varepsilon^{C'_\varsigma D'_\varsigma} = \varepsilon^{A'_\varsigma C'_\varsigma}\varepsilon^{B'_\varsigma D'_\varsigma} - \varepsilon^{A'_\varsigma D'_\varsigma}\varepsilon^{B'_\varsigma C'_\varsigma} \tag{1.55}$$

$$\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{C_\varsigma D_\varsigma} + \varepsilon_{A_\varsigma C_\varsigma}\varepsilon_{D_\varsigma B_\varsigma} + \varepsilon_{A_\varsigma D_\varsigma}\varepsilon_{B_\varsigma C_\varsigma} = 0 \qquad \varepsilon_{A_\varsigma[B_\varsigma}\varepsilon_{C_\varsigma D_\varsigma]} = 0 \tag{1.56}$$

$$\varepsilon_{A_\varsigma}{}^{B_\varsigma} = \delta_{A_\varsigma}{}^{B_\varsigma} = -\varepsilon^{B_\varsigma}{}_{A_\varsigma} \qquad \varepsilon^{A'_\varsigma}{}_{B'_\varsigma} = \delta^{A'_\varsigma}{}_{B'_\varsigma} = -\varepsilon_{B'_\varsigma}{}^{A'_\varsigma} \tag{1.57}$$

4.1.2 Complex conjugation

$$[\varepsilon^{A_\varsigma B_\varsigma}]^* = \varepsilon^{A'_\varsigma B'_\varsigma} \qquad [\varepsilon_{A_\varsigma B_\varsigma}]^* = \varepsilon_{A'_\varsigma B'_\varsigma} \tag{1.58}$$

4.2 Properties of basic constant invariant tensors $(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}, (\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}$ [1,2]

4.2.1 Transformability

Transformability

$$\frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} = [\varsigma\varepsilon^{A_\varsigma B_\varsigma}][\varsigma\varepsilon^{A'_\varsigma B'_\varsigma}]\frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)_{aB_\varsigma B'_\varsigma} \tag{1.59}$$

$$\frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma} = [-\varsigma\varepsilon_{A'_\varsigma B'_\varsigma}][-\varsigma\varepsilon_{A_\varsigma B_\varsigma}]\frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{aB'_\varsigma B_\varsigma} \tag{1.60}$$

$$\frac{(-i\varsigma)}{\sqrt{2}}[\sigma, -i(-\varsigma)]_a^{A'_{-\varsigma}A_{-\varsigma}} \simeq \frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma} \tag{1.61}$$

Non redundant version: $\frac{i}{\sqrt{2}}(\sigma, -i)_a^{A'A}, \frac{i}{\sqrt{2}}(-\sigma, -i)^a_{AA'}$
The above shows that the two constant invariant tensors are not independent, and there is only one truly independent tensor.

4.2.2 Orthogonality

Reduce a pair of vector indices: (On the right is Penrose abridged notation. Expressed as $\overset{P}{=}$.)

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\delta^a_b\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}=\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma} \qquad \delta^a_b\overset{P}{=}\delta^A_B\delta^{A'}_{B'} \tag{1.62}$$

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\delta^{ab}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}=\varepsilon^{AB}\varepsilon^{A'B'} \qquad \delta^{ab}\overset{P}{=}\varepsilon^{AB}\varepsilon^{A'B'} \tag{1.63}$$

$$\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\delta_{ab}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}=\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma} \qquad \delta_{ab}\overset{P}{=}\varepsilon_{AB}\varepsilon_{A'B'} \tag{1.64}$$

iPenrose corresponding rules under various frames:

$$\frac{i}{\sqrt{2}}(\sigma^*,-i)_a^{AA'}x^a|_{++++}=\frac{i}{\sqrt{2}}(\sigma^*,-1)_a^{AA'}x^a|_{+++-}=\frac{i}{\sqrt{2}}(-\sigma^*,1)_a^{AA'}x^a|_{---+}=x^{AA'} \tag{1.65}$$

$$\frac{i}{\sqrt{2}}(-\sigma,-i)^a_{AA'}x_a|_{++++}=\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}x_a|_{+++-}=\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}x_a|_{---+}=x_{AA'} \tag{1.66}$$

$$\frac{i}{\sqrt{2}}(\sigma^*,-i)_a^{AA'}\partial^a|_{++++}=\frac{i}{\sqrt{2}}(\sigma^*,-1)_a^{AA'}\partial^a|_{+++-}=\frac{i}{\sqrt{2}}(-\sigma^*,1)_a^{AA'}\partial^a|_{---+}=\partial_{x_{AA'}}=\nabla^{AA'} \tag{1.67}$$

$$\frac{i}{\sqrt{2}}(-\sigma,-i)^a_{AA'}x_a|_{++++}=\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}x_a|_{+++-}=\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}x_a|_{---+}=\partial_{x^{AA'}}=\nabla_{AA'} \tag{1.68}$$

Rules under $(++++)$ **frame: Vector superscript** ^{a} **converts to** $^{AA'}$ **by using** $\frac{i}{\sqrt{2}}(\sigma^*,-i)_a^{AA'}$**, Vector subscript** $_a$ **converts to** $_{AA'}$ $\frac{i}{\sqrt{2}}(-\sigma,-i)^a_{AA'}$**.**
Rules under $(+++-)$ **frame: Vector superscript** ^{a} **converts to** $^{AA'}$ **by using** $\frac{i}{\sqrt{2}}(\sigma^*,-1)_a^{A'A}$**, Vector subscript** $_a$ **converts to** $_{AA'}$ **by using** $\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}$**.**
Rules under $(---+)$ **frame: Vector superscript** ^{a} **converts to** $^{AA'}$ **by using** $\frac{i}{\sqrt{2}}(-\sigma^*,1)_a^{A'A}$**, Vector subscript** $_a$ **converts to** $_{AA'}$ **by using** $\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}$**.**

$$\frac{i}{\sqrt{2}}(\sigma^*,-i)_a^{AA'}\frac{i}{\sqrt{2}}(-\sigma,-i)^a_{AA'}=\frac{i}{\sqrt{2}}(\sigma^*,-1)_a^{A'A}\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}=-\frac{i}{\sqrt{2}}(-\sigma^*,1)_a^{A'A}\frac{i}{\sqrt{2}}(-\sigma,1)^a_{AA'}=\delta^A_B\delta^{A'}_{B'} \tag{1.69}$$

Reduce a pair of spinor indices:

$$(\sigma,i\varsigma)_{aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_b^{A'_\varsigma B_\varsigma}=\delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma}+2iS_{abA_\varsigma}{}^{B_\varsigma} \tag{1.70}$$

$$(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}(\sigma,i\varsigma)_{bA_\varsigma B'_\varsigma}=\delta_{ab}\delta^{A'_\varsigma}{}_{B'_\varsigma}+2iS_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} \tag{1.71}$$

Reduce two pairs of indices:

$$(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_b^{A'_\varsigma A_\varsigma}=2\delta^a_b \qquad tr[(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_b]=2\delta_{ab} \tag{1.72}$$

$$(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_a^{A'_\varsigma B_\varsigma}=4\delta_{A_\varsigma}{}^{B_\varsigma} \qquad (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}(\sigma,i\varsigma)^a_{A_\varsigma B'_\varsigma}=4\delta^{A'_\varsigma}{}_{B'_\varsigma} \tag{1.73}$$

Reduce all indices:

$$(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}=8 \tag{1.74}$$

4.2.3 Complex conjugation

$$[(\sigma,i\varsigma)^a_{A_\varsigma B'_\varsigma}\partial_a]^*=(\sigma,i\varsigma)^a_{B_\varsigma A'_\varsigma}\partial_a \qquad [(\sigma,-i\varsigma)_a^{A'_\varsigma B_\varsigma}\partial^a]^*=(\sigma,-i\varsigma)_a^{B'_\varsigma A_\varsigma}\partial^a \tag{1.75}$$

$$[(\sigma,i\varsigma)^a\partial_a]^+=(\sigma,i\varsigma)^a\partial_a \qquad [(\sigma,-i\varsigma)_a\partial^a]^+=(\sigma,-i\varsigma)_a\partial^a \tag{1.76}$$

4.3 Properties of basic constant invariant tensors $\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma},\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}$

4.3.1 Orthogonality

Three-dimensional spin tensor:

$$S_{\alpha'_\varsigma\beta'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}=\frac{i}{2}\gamma_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma_{\gamma'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}=\frac{1}{2}\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma_{\gamma'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma} \qquad S^{\alpha_\varsigma\beta_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}=\frac{i}{2}\gamma^{\alpha_\varsigma\beta_\varsigma}{}_{\gamma_\varsigma}\sigma^{\gamma_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}=\frac{1}{2}\varepsilon^{\alpha_\varsigma\beta_\varsigma}{}_{\gamma_\varsigma}\sigma^{\gamma_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma} \tag{1.77}$$

Reduce a pair of complex vector indices:

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\sigma^{\alpha'_\varsigma C'_\varsigma}{}_{D'_\varsigma}=\delta^{A'_\varsigma}_{D'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma}-\varepsilon^{A'_\varsigma C'_\varsigma}\varepsilon_{B'_\varsigma D'_\varsigma} \qquad \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\sigma_{\alpha_\varsigma C_\varsigma}{}^{D_\varsigma}=\delta^{D_\varsigma}_{A_\varsigma}\delta^{B_\varsigma}_{C_\varsigma}-\varepsilon_{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} \tag{1.78}$$

Reduce a pair of spinor indices:

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{C'_\varsigma}\sigma_{\beta'_\varsigma}{}^{C'_\varsigma}{}_{B'_\varsigma}=\delta_{\alpha'_\varsigma\beta'_\varsigma}\delta^{A'_\varsigma}{}_{B'_\varsigma}+2iS_{\alpha'_\varsigma\beta'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma} \qquad \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{C_\varsigma}\sigma^{\beta_\varsigma}{}_{C_\varsigma}{}^{B_\varsigma}=\delta^{\alpha_\varsigma\beta_\varsigma}\delta_{A_\varsigma}{}^{B_\varsigma}+2iS^{\alpha_\varsigma\beta_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma} \tag{1.79}$$

Reduce two pairs of indices:

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\sigma_{\beta'_\varsigma}{}^{B'_\varsigma}{}_{A'_\varsigma} = 2\delta_{\alpha'_\varsigma\beta'_\varsigma} \qquad \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{A_\varsigma} = 2\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.80}$$

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{C'_\varsigma}\sigma^{\alpha'_\varsigma C'_\varsigma}{}_{B'_\varsigma} = 3\delta^{A'_\varsigma}{}_{B'_\varsigma} \qquad \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{C_\varsigma}\sigma_{\alpha_\varsigma C_\varsigma}{}^{B_\varsigma} = 3\delta_{A_\varsigma}{}^{B_\varsigma} \tag{1.81}$$

$$tr[\sigma_{\alpha'_\varsigma}\sigma_{\beta'_\varsigma}] = 2\delta_{\alpha'_\varsigma\beta'_\varsigma} \qquad tr[\sigma^{\alpha_\varsigma}\sigma^{\beta_\varsigma}] = 2\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.82}$$

Reduce all indices:

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\sigma^{\alpha'_\varsigma B'_\varsigma}{}_{A'_\varsigma} = 6 \qquad \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\sigma_{\alpha_\varsigma B_\varsigma}{}^{A_\varsigma} = 6 \tag{1.83}$$

4.3.2 Tracelessness

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{A'_\varsigma} = 0 \qquad tr[\sigma_{\alpha_\varsigma}] = 0 \qquad \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{A_\varsigma} = 0 \qquad tr[\sigma^{\alpha'_\varsigma}] = 0 \tag{1.84}$$

4.3.3 Complex conjugation

$$[\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}]^* = \sigma_{\alpha_\varsigma B_\varsigma}{}^{A_\varsigma} \qquad [\sigma_{\alpha_\varsigma A_\varsigma}{}^{B_\varsigma}]^* = \sigma_{\alpha'_\varsigma}{}^{B'_\varsigma}{}_{A'_\varsigma} \tag{1.85}$$

4.4 Properties of basic constant invariant tensors $\sigma^{\alpha}_{+ab}, \sigma^{\alpha'}_{-ab}, \sigma^{\alpha_\varsigma}_{\varsigma ab}, \sigma^{\alpha'_\varsigma}_{-\varsigma ab}$

4.4.1 Hidden complexity

Complexity: (In fact, the following formula can be used as a definition.)

$$\sigma^{\alpha_\varsigma}_{\varsigma a}{}^{b} = \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma} \tag{1.86}$$

$$\sigma_{-\varsigma\alpha'_\varsigma}{}^{a}{}_{b} = \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\delta^{A_\varsigma}{}_{B_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b \tag{1.87}$$

Proof: $(\sigma,i\varsigma)_{aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)^{A'_\varsigma B_\varsigma}_b = \delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma} + 2iS_{abA_\varsigma}{}^{B_\varsigma}$

$\Rightarrow (\sigma,i\varsigma)_{aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)^{A'_\varsigma B_\varsigma}_b\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{A_\varsigma} = (\delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma} - \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma A_\varsigma}{}^{B_\varsigma})\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{A_\varsigma}$

$\Rightarrow (\sigma,i\varsigma)_{aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)^{A'_\varsigma B_\varsigma}_b\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{A_\varsigma} = -2\sigma^{\alpha_\varsigma}_{\varsigma ab}\delta^{\alpha_\varsigma\beta_\varsigma}$

$\Rightarrow 2\sigma^{\alpha_\varsigma}_{\varsigma ab} = (\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\sigma,i\varsigma)_{bB_\varsigma A'_\varsigma}$

$\Rightarrow \sigma^{\alpha_\varsigma}_{\varsigma ab} = \frac{1}{2}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma}(\sigma,i\varsigma)_{bB_\varsigma B'_\varsigma}$

$\Rightarrow \sigma^{\alpha_\varsigma}_{\varsigma a}{}^{b} = \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}$ □

The relation between basic constant invariant tensors:

$$\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\sigma^{\alpha_\varsigma}_{\varsigma a}{}^{b}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b = \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma} \qquad \sigma^{\alpha}_{+a}{}^{b} \overset{P}{=} \sigma^{\alpha}{}_{A}{}^{B}\delta_{A'}{}^{B'} \tag{1.88}$$

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma}{}^{a}{}_{b}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma} = \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\delta^{A_\varsigma}{}_{B_\varsigma} \qquad \sigma_{-\alpha'}{}^{a}{}_{b} \overset{P}{=} \sigma_{\alpha'}{}^{A'}{}_{B'}\delta^{A}{}_{B} \tag{1.89}$$

Self comment: The above indicates that the two basic constant invariant tensors $\sigma^{\alpha_\varsigma}_{\varsigma ab}$ and $\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}$ are not independent but interrelated.

4.4.2 Orthogonality

The relation between three dimensional spin tensors and four dimensional spin tensors:

$$S^{\alpha_\varsigma\beta_\varsigma}{}_{ab}(\tfrac{1}{2}\sigma^{\alpha_\varsigma}_\varsigma) = \tfrac{i}{2}\gamma^{\alpha_\varsigma\beta_\varsigma}{}_{\gamma_\varsigma}\sigma^{\gamma_\varsigma}_{\varsigma ab} = \tfrac{1}{2}i\sigma^{\gamma_\varsigma}_{\varsigma ab}\gamma_{\gamma_\varsigma}{}^{\alpha_\varsigma\beta_\varsigma} = \tfrac{1}{2}S_{ab}{}^{\alpha_\varsigma\beta_\varsigma}(\gamma,\varsigma) \tag{1.90}$$

Reduce a pair of complex vector indices:

$$\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} = -\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varsigma\varepsilon_{abcd} \tag{1.91}$$

$$S_{abcd} = -\tfrac{1}{2}(\sigma^{\alpha'}_{-ab}\sigma_{-\alpha' cd} + \sigma^{\alpha}_{+ab}\sigma_{+\alpha cd}) = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \qquad \varepsilon_{abcd} = -\tfrac{1}{2}(\sigma^{\alpha'}_{-ab}\sigma_{-\alpha' cd} - \sigma^{\alpha}_{+ab}\sigma_{+\alpha cd}) \tag{1.92}$$

Reduce a pair of vector indices:

$$\sigma^{\beta_\varsigma}_{\varsigma ac}\delta^{cd}\sigma^{\gamma_\varsigma}_{\varsigma db} = \delta_{ab}\delta^{\beta_\varsigma\gamma_\varsigma} - \sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma\gamma_\varsigma} = \delta^{\beta_\varsigma\gamma_\varsigma}\delta_{ab} + \tfrac{i}{2}S^{\beta_\varsigma\gamma_\varsigma}{}_{ab}(\tfrac{1}{2}\sigma^{\alpha_\varsigma}_\varsigma) = \delta_{ab}\delta^{\beta_\varsigma\gamma_\varsigma} + iS_{ab}{}^{\beta_\varsigma\gamma_\varsigma}(\gamma,\varsigma) \tag{1.93}$$

Reduce a pair of vector and a pair of complex vector indices:

$$\sigma^{\alpha_\varsigma}_{\varsigma ac}\sigma^{\varsigma cb}_{\alpha_\varsigma} = 3\delta_a{}^b \tag{1.94}$$

Reduce two pairs of vector indices:

$$\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{ab}_{\varsigma\beta_\varsigma} = -4\delta^{\alpha_\varsigma}_{\beta_\varsigma} \qquad \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{ab}_{-\varsigma\beta'_\varsigma} = 0 \qquad \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa ab}_{\kappa} = -4\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa} \tag{1.95}$$

$$tr(\sigma^{\alpha_\varsigma}_\varsigma\sigma^{\beta_\varsigma}_\varsigma) = 4\delta^{\alpha_\varsigma\beta_\varsigma} \qquad tr(\sigma^{\alpha_\varsigma}_\varsigma\sigma^{\beta'_\varsigma}_{-\varsigma}) = 0 \qquad tr(\sigma^{\alpha_\varsigma}_\varsigma\sigma^{\beta_\kappa}_\kappa) = 4\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa} \tag{1.96}$$

Reduce all indices:

$$\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{ab}_{\varsigma\alpha_\varsigma}=-12 \qquad \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}=0 \tag{1.97}$$

4.4.3 Duality

$$\sigma^{\alpha}_{+ab}=-*\sigma^{\alpha}_{+ab} \qquad \sigma^{\alpha'}_{-ab}=*\sigma^{\alpha'}_{-ab} \qquad \sigma^{\alpha_\varsigma}_{\varsigma ab}=-\varsigma*\sigma^{\alpha_\varsigma}_{\varsigma ab} \tag{1.98}$$

4.4.4 Complex conjugation

$$(\sigma^{\alpha_\varsigma}_{\varsigma ab}\partial^a\hat{\partial}^b)^*=-\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\partial^a\hat{\partial}^b \qquad (\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\partial^a\hat{\partial}^b)^*=-\sigma^{\alpha_\varsigma}_{\varsigma ab}\partial^a\hat{\partial}^b \tag{1.99}$$

4.5 Properties of basic constant invariant tensors $\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s),\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)$

4.5.1 Mathematical preparation

Using formulas: $\sum_{k=1}^{2s}k^2=\frac{8}{3}s(s+\frac{1}{2})(s+\frac{1}{4})$**, get**

$$tr[\sigma_x^2(s)]=tr[\sigma_y^2(s)]=\frac{1}{4}\sum_{k=1}^{2s}2k(2s+1-k)=\frac{2}{3}s(s+\frac{1}{2})(s+1) \tag{1.100}$$

$$tr[\sigma_z^2(s)]=\frac{1}{4}\sum_{k=1}^{2s}(2s-2k)^2=\frac{2}{3}s(s+\frac{1}{2})(s+1) \tag{1.101}$$

$$tr[\sigma_x^2(s)]=tr[\sigma_y^2(s)]=tr[\sigma_z^2(s)]=\frac{2}{3}s(s+\frac{1}{2})(s+1) \tag{1.102}$$

4.5.2 Orthogonality

From a spinor perspective:

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)\sigma_{\beta'_\varsigma}{}^{l'_\varsigma}{}_{k'_\varsigma}(s)=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha'_\varsigma\beta'_\varsigma} \qquad \sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{m'_\varsigma}(s)\sigma^{\alpha'_\varsigma m'_\varsigma}{}_{l'_\varsigma}(s)=s(s+1)\delta^{k'_\varsigma}{}_{l'_\varsigma} \tag{1.103}$$

$$\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)\sigma^{\beta_\varsigma}{}_{l_\varsigma}{}^{k_\varsigma}(s)=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha_\varsigma\beta_\varsigma} \qquad \sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{m_\varsigma}(s)\sigma_{\alpha_\varsigma m_\varsigma}{}^{l_\varsigma}(s)=s(s+1)\delta_{k_\varsigma}{}^{l_\varsigma} \tag{1.104}$$

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)\sigma^{\alpha'_\varsigma l'_\varsigma}{}_{m'_\varsigma}(s)=2s(s+\frac{1}{2})(s+1) \qquad \sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)\sigma_{\alpha_\varsigma l_\varsigma}{}^{k_\varsigma}(s)=2s(s+\frac{1}{2})(s+1) \tag{1.105}$$

From a matrix perspective:

$$tr[\sigma_{\alpha'_\varsigma}(s)\sigma_{\beta'_\varsigma}(s)]=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha'_\varsigma\beta'_\varsigma} \qquad tr[\sigma^{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma}(s)]=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.106}$$

4.5.3 Orthogonality

Reduce a pair of complex vector indices:

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)\sigma^{\alpha'_\varsigma m'_\varsigma}{}_{n'_\varsigma}(s)=?\delta^{k'_\varsigma}{}_{l'_\varsigma}\delta^{m'_\varsigma}{}_{n'_\varsigma}?-2\varepsilon^{k'_\varsigma m'_\varsigma}(s)\varepsilon_{l'_\varsigma n'_\varsigma}(s) \tag{1.107}$$

$$\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)\sigma_{\alpha_\varsigma m_\varsigma}{}^{n_\varsigma}(s)=?\delta_{k_\varsigma}{}^{l_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma}?-2\varepsilon_{k_\varsigma m_\varsigma}(s)\varepsilon^{l_\varsigma n_\varsigma}(s) \tag{1.108}$$

Reduce a pair of spinor indices:

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{m'_\varsigma}(s)\sigma_{\beta'_\varsigma}{}^{m'_\varsigma}{}_{l'_\varsigma}(s)=?\delta_{\alpha'_\varsigma\beta'_\varsigma}\delta^{k'_\varsigma}{}_{l'_\varsigma}?+\frac{i}{2}S_{\alpha'_\varsigma\beta'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s) \tag{1.109}$$

$$\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{m_\varsigma}(s)\sigma^{\beta_\varsigma}{}_{m_\varsigma}{}^{l_\varsigma}(s)=?\delta^{\alpha_\varsigma\beta_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}?+\frac{i}{2}S^{\alpha_\varsigma\beta_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s) \tag{1.110}$$

Reduce two pairs of indices:

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)\sigma_{\beta'_\varsigma}{}^{l'_\varsigma}{}_{k'_\varsigma}(s)=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha'_\varsigma\beta'_\varsigma} \qquad \sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{m'_\varsigma}(s)\sigma^{\alpha'_\varsigma m'_\varsigma}{}_{l'_\varsigma}(s)=s(s+1)\delta^{k'_\varsigma}{}_{l'_\varsigma} \tag{1.111}$$

$$\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)\sigma^{\beta_\varsigma}{}_{l_\varsigma}{}^{k_\varsigma}(s)=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha_\varsigma\beta_\varsigma} \qquad \sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{m_\varsigma}(s)\sigma_{\alpha'_\varsigma m_\varsigma}{}^{l_\varsigma}(s)=s(s+1)\delta_{k_\varsigma}{}^{l_\varsigma} \tag{1.112}$$

$$tr[\sigma_{\alpha'_\varsigma}(s)\sigma_{\beta'_\varsigma}(s)]=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha'_\varsigma\beta'_\varsigma} \qquad tr[\sigma^{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma}(s)]=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.113}$$

Reduce all indices:

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)\sigma^{\alpha'_\varsigma l'_\varsigma}{}_{k'_\varsigma}(s)=2s(s+\frac{1}{2})(s+1) \qquad \sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)\sigma_{\alpha_\varsigma l_\varsigma}{}^{k_\varsigma}(s)=2s(s+\frac{1}{2})(s+1) \tag{1.114}$$

4.5.4 Tracelessness

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{k'_\varsigma}(s)=0 \qquad tr[\sigma_{\alpha'_\varsigma}(s)]=0 \qquad \sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{k_\varsigma}(s)=0 \qquad tr[\sigma^{\alpha_\varsigma}(s)]=0 \tag{1.115}$$

4.5.5 Complex conjugation

$$[\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)]^*=\sigma_{\alpha_\varsigma l_\varsigma}{}^{k_\varsigma}(s) \qquad [\sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(s)]^*=\sigma_{\alpha'_\varsigma}{}^{l'_\varsigma}{}_{k'_\varsigma}(s) \tag{1.116}$$

4.5.6 Complex properties I

$$\sigma_{\alpha_\varsigma}(s)\sigma(s)\sigma^{\alpha_\varsigma}(s) = [s(s+1)-1]\sigma(s) \qquad \sigma_{\alpha'_\varsigma}(s)\sigma(s)\sigma^{\alpha'_\varsigma}(s) = [s(s+1)-1]\sigma(s) \tag{1.117}$$

$$[\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)]\sigma^{\beta_\varsigma}(s) = \sigma_{\alpha_\varsigma}(s) \qquad -i\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma}(s) = \sigma_{\alpha_\varsigma}(s) \tag{1.118}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\varepsilon^{\gamma_\varsigma}{}_{\rho_\varsigma\sigma_\varsigma} = \delta_{\alpha_\varsigma\rho_\varsigma}\delta_{\beta_\varsigma\sigma_\varsigma} - \delta_{\alpha_\varsigma\sigma_\varsigma}\delta_{\beta_\varsigma\rho_\varsigma} \qquad \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\varepsilon^{\beta_\varsigma\gamma_\varsigma}{}_{\rho_\varsigma} = 2\delta_{\alpha_\varsigma\rho_\varsigma} \tag{1.119}$$

Cor. 4.5.1. $\sigma_\alpha(s)\sigma_i(s)\sigma^\alpha(s) = [s(s+1)-1]\sigma_i(s)$

Cor. 4.5.2. $\sigma_\alpha(s)\sigma_i(s)\sigma_j(s)\sigma^\alpha(s) = s(s+1)\delta_{ij} - \sigma_j(s)\sigma_i(s) + [s(s+1)-2]\sigma_i(s)\sigma_j(s)$

Proof: $\sigma_\alpha(s)\sigma_i(s)\sigma_j(s)\sigma^\alpha(s)$
$= [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma^\alpha(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma^\alpha(s)$
$= [\sigma_\alpha(s), \sigma_i(s)][\sigma_j(s), \sigma^\alpha(s)] + [\sigma_\alpha(s), \sigma_i(s)]\sigma^\alpha(s)\sigma_j(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma^\alpha(s)$
$= \varepsilon_{jl}{}^\alpha\varepsilon_{\alpha ik}\sigma^k(s)\sigma^l(s) + [s(s+1)-2]\sigma_i(s)\sigma_j(s)$
$= s(s+1)\delta_{ij} - \sigma_j(s)\sigma_i(s) + [s(s+1)-2]\sigma_i(s)\sigma_j(s)$ □

Cor. 4.5.3. $\sigma_\alpha(s)\sigma_{\{i}(s)\sigma_{j\}}(s)\sigma^\alpha(s) = s(s+1)\delta_{\{ij\}} + [s(s+1)-3]\sigma_{\{i}(s)\sigma_{j\}}(s)$

Cor. 4.5.4. $[\sigma_\alpha(s), \sigma_{\{i}(s)]\sigma_{j\}}(s)\sigma^\alpha(s) = s(s+1)\delta_{\{ij\}} - 2\sigma_{\{i}(s)\sigma_{j\}}(s)$

Cor. 4.5.5. $\sigma_\alpha(s)\sigma_{\{i}(s)\sigma_j(s)\sigma_{k\}}(s)\sigma^\alpha(s) = [3s(s+1)-1]\delta_{\{ij}\sigma_{k\}}(s) + [s(s+1)-6]\sigma_{\{i}(s)\sigma_j(s)\sigma_{k\}}(s)$

Cor. 4.5.6. $[\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma^\alpha(s) = s(s+1)\delta_{ij} - \sigma_j(s)\sigma_i(s) - \sigma_i(s)\sigma_j(s)$

Proof: $\sigma_\alpha(s)\sigma_i(s)\sigma_j(s)\sigma_k(s)\sigma^\alpha(s)$
$= [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma_k(s)\sigma^\alpha(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma^\alpha(s)$
$= [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)[\sigma_k(s), \sigma^\alpha(s)] + [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma^\alpha(s)\sigma_k(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma^\alpha(s)$
$= \delta_{ik}\sigma_\alpha(s)\sigma_j(s)\sigma^\alpha(s) - \sigma_k(s)\sigma_j(s)\sigma_i(s) + [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma^\alpha(s)\sigma_k(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma^\alpha(s)$
$= \delta_{ik}[s(s+1)-1]\sigma_j(s) - \sigma_k(s)\sigma_j(s)\sigma_i(s)$
$+ s(s+1)\delta_{ij}\sigma_k(s) - \sigma_j(s)\sigma_i(s)\sigma_k(s) - \sigma_i(s)\sigma_j(s)\sigma_k(s)$
$+ s(s+1)\delta_{jk}\sigma_i(s) - \sigma_i(s)\sigma_k(s)\sigma_j(s) + [s(s+1)-2]\sigma_i(s)\sigma_j(s)\sigma_k(s)$
$= \delta_{ik}\sigma_\alpha(s)\sigma_j(s)\sigma^\alpha(s) - \sigma_k(s)\sigma_j(s)\sigma_i(s) + [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma^\alpha(s)\sigma_k(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma^\alpha(s)$
$= s(s+1)[\delta_{jk}\sigma_i(s) + \delta_{ij}\sigma_k(s) + \delta_{ik}\sigma_j(s)] - \delta_{ik}\sigma_j(s) - \sigma_k(s)\sigma_j(s)\sigma_i(s) - \sigma_j(s)\sigma_i(s)\sigma_k(s) - \sigma_i(s)\sigma_k(s)\sigma_j(s) + [s(s+1) - 3]\sigma_i(s)\sigma_j(s)\sigma_k(s)$ □

Proof: $\sigma_\alpha(s)\sigma_i(s)\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s)$
$= [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s) + |||\sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s)$
$= [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma_k(s)[\sigma_l(s), \sigma^\alpha(s)] + |||[\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma_k(s)\sigma^\alpha(s)\sigma_l(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s)$ □

4.5.7 Complex properties II

$$\sigma_{\alpha_\varsigma}(s)\sigma(s)\sigma^{\alpha_\varsigma}(s) = [s(s+1)-1]\sigma(s) \qquad \sigma_{\alpha'_\varsigma}(s)\sigma(s)\sigma^{\alpha'_\varsigma}(s) = [s(s+1)-1]\sigma(s) \tag{1.120}$$

Cor. 4.5.7. $\sigma_i(s)\sigma_\alpha(s) = i\varepsilon_{i\alpha\beta}\sigma^\beta(s) + \sigma_\alpha(s)\sigma_i(s)$

5 Properties of several intuitive composite constant invariant tensors

5.1 Properties of extended constant invariant tensors$(\sigma, i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}, (\sigma, i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}$

5.1.1 Transformability

Transformability:

$$(\sigma, i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma} = -\varepsilon^{A'_\varsigma D'_\varsigma}\varepsilon_{B'_\varsigma C'_\varsigma}(\sigma, -i\kappa)_{\alpha'_\varsigma}{}^{C'_\varsigma}{}_{D'_\varsigma} \qquad (\sigma, i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma} = -\varepsilon_{A_\varsigma D_\varsigma}\varepsilon^{B_\varsigma C_\varsigma}(\sigma, -i\kappa)^{\alpha_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma} \tag{1.121}$$

5.1.2 Orthogonality

Reduce a pair of complex vector indices:

$$(\sigma, i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}(\sigma, -i\kappa)^{\alpha'_\varsigma C'_\varsigma}{}_{D'_\varsigma} = 2\delta^{A'_\varsigma}_{D'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma} \qquad (\sigma, i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\sigma, -i\kappa)_{\alpha_\varsigma C_\varsigma}{}^{D_\varsigma} = 2\delta^{D_\varsigma}_{A_\varsigma}\delta^{B_\varsigma}_{C_\varsigma} \tag{1.122}$$

$$(\sigma, i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}(\sigma, i\kappa)^{\alpha'_\varsigma C'_\varsigma}{}_{D'_\varsigma} = -2\varepsilon^{A'_\varsigma C'_\varsigma}\varepsilon_{B'_\varsigma D'_\varsigma} \qquad (\sigma, i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\sigma, i\kappa)_{\alpha_\varsigma C_\varsigma}{}^{D_\varsigma} = -2\varepsilon_{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} \tag{1.123}$$

Reduce a pair of spinor indices:

$$(\sigma, i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{C'_\varsigma}(\sigma, -i\kappa)_{\beta'_\varsigma}{}^{C'_\varsigma}{}_{B'_\varsigma} = \delta_{\alpha'_\varsigma\beta'_\varsigma}\delta^{A'_\varsigma}{}_{B'_\varsigma} + 2iS_{\alpha'_\varsigma\beta'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}(\kappa) \tag{1.124}$$

$$(\sigma, i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{C_\varsigma}(\sigma, -i\kappa)^{\beta_\varsigma}{}_{C_\varsigma}{}^{B_\varsigma} = \delta^{\alpha_\varsigma\beta_\varsigma}\delta_{A_\varsigma}{}^{B_\varsigma} + 2iS^{\alpha_\varsigma\beta_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\kappa) \tag{1.125}$$

Reduce two pairs of indices:

$$(\sigma,i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}(\sigma,-i\kappa)_{\beta'_\varsigma}{}^{B'_\varsigma}{}_{A'_\varsigma}=2\delta_{\alpha'_\varsigma\beta'_\varsigma} \qquad (\sigma,i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\sigma,-i\kappa)^{\beta_\varsigma}{}_{B_\varsigma}{}^{A_\varsigma}=2\delta^{\alpha_\varsigma\beta_\varsigma} \tag{1.126}$$

$$(\sigma,i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{C'_\varsigma}(\sigma,-i\kappa)^{\alpha'_\varsigma C'_\varsigma}{}_{B'_\varsigma}=4\delta^{A'_\varsigma}{}_{B'_\varsigma} \qquad (\sigma,i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{C_\varsigma}(\sigma,-i\kappa)_{\alpha_\varsigma C_\varsigma}{}^{B_\varsigma}=4\delta_{A_\varsigma}{}^{B_\varsigma} \tag{1.127}$$

Reduce all indices:

$$(\sigma,i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}(\sigma,-i\kappa)^{\alpha'_\varsigma B'_\varsigma}{}_{A'_\varsigma}=8 \qquad (\sigma,i\kappa)^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\sigma,-i\kappa)_{\alpha_\varsigma B_\varsigma}{}^{A_\varsigma}=8 \tag{1.128}$$

5.1.3 Complex conjugation

$$[(\sigma,i\kappa)_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}]^*=(\sigma,-i\kappa)_{\alpha_\varsigma B_\varsigma}{}^{A_\varsigma} \qquad [(\sigma,i\kappa)_{\alpha_\varsigma A_\varsigma}{}^{B_\varsigma}]^*=(\sigma,-i\kappa)_{\alpha'_\varsigma}{}^{B'_\varsigma}{}_{A'_\varsigma} \tag{1.129}$$

5.2 Properties of extended constant invariant tensors $(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab},(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}$

5.2.1 Orthogonality

Reduce a pair of complex vector indices:

$$(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}(\sigma_\varsigma,i\kappa)_{\alpha_\varsigma cd}=-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}+\varsigma\varepsilon_{abcd}-\delta_{ab}\delta_{cd} \tag{1.130}$$

$$(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}(\sigma_\varsigma,-i\kappa)_{\alpha_\varsigma cd}=-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}+\varsigma\varepsilon_{abcd}+\delta_{ab}\delta_{cd} \tag{1.131}$$

$$(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}(\sigma_{-\varsigma},i\kappa)_{\alpha'_\varsigma cd}=-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}-\varsigma\varepsilon_{abcd}-\delta_{ab}\delta_{cd} \tag{1.132}$$

$$(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}(\sigma_{-\varsigma},-i\kappa)_{\alpha'_\varsigma cd}=-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}-\varsigma\varepsilon_{abcd}+\delta_{ab}\delta_{cd} \tag{1.133}$$

Reduce a pair of vector indices:

$$(\sigma_\varsigma,i\kappa)^{ac}_{\alpha_\varsigma}(\sigma_\varsigma,-i\kappa)_{\beta_\varsigma cb}=\delta_{\alpha_\varsigma\beta_\varsigma}\delta^a{}_b+2iS_{\alpha_\varsigma\beta_\varsigma}{}^a{}_b[\tfrac{1}{2}\sigma_\varsigma,\kappa] \tag{1.134}$$

$$(\sigma_{-\varsigma},i\kappa)^{ac}_{\alpha'_\varsigma}(\sigma_{-\varsigma},-i\kappa)_{\beta'_\varsigma cb}=\delta_{\alpha'_\varsigma\beta'_\varsigma}\delta^a{}_b+2iS_{\alpha'_\varsigma\beta'_\varsigma}{}^a{}_b[\tfrac{1}{2}\sigma_{-\varsigma},\kappa] \tag{1.135}$$

Reduce two pairs of vector indices:

$$(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}(\sigma_\varsigma,-i\kappa)^{ab}_{\beta_\varsigma}=-4\eta^{\alpha_\varsigma}_{\beta_\varsigma} \qquad (\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}(\sigma_\varsigma,i\kappa)^{ab}_{\beta_\varsigma}=-4\delta^{\alpha_\varsigma}_{\beta_\varsigma} \tag{1.136}$$

$$tr[(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}(\sigma_\varsigma,-i\kappa)^{\beta_\varsigma}]=4\delta^{\alpha_\varsigma\beta_\varsigma} \qquad tr[(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}(\sigma_\varsigma,i\kappa)^{\beta_\varsigma}]=4\eta^{\alpha_\varsigma\beta_\varsigma} \tag{1.137}$$

$$(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}(\sigma_{-\varsigma},-i\kappa)^{ab}_{\beta'_\varsigma}=-4\eta^{\alpha'_\varsigma}_{\beta'_\varsigma} \qquad (\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}(\sigma_{-\varsigma},i\kappa)^{ab}_{\beta'_\varsigma}=-4\delta^{\alpha'_\varsigma}_{\beta'_\varsigma} \tag{1.138}$$

$$tr[(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}(\sigma_{-\varsigma},-i\kappa)^{\beta'_\varsigma}]=4\delta^{\alpha'_\varsigma\beta_\varsigma} \qquad tr[(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}(\sigma_{-\varsigma},i\kappa)^{\beta'_\varsigma}]=4\eta^{\alpha'_\varsigma\beta_\varsigma} \tag{1.139}$$

Reduce all indices:

$$(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}(\sigma_\varsigma,-i\kappa)^{ab}_{\alpha_\varsigma}=-8 \qquad (\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}(\sigma_\varsigma,i\kappa)^{ab}_{\alpha_\varsigma}=-16 \tag{1.140}$$

$$(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}(\sigma_{-\varsigma},-i\kappa)^{ab}_{\alpha'_\varsigma}=-8 \qquad (\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}(\sigma_{-\varsigma},i\kappa)^{ab}_{\alpha'_\varsigma}=-16 \tag{1.141}$$

5.2.2 Identity

$$(\sigma_+,-i)^\alpha|_{ab}=(\sigma_-,-i)_a|_b{}^\alpha \qquad (\sigma_-,i)^{\alpha'}|_{ab}=(\sigma_+,i)_a|_b{}^{\alpha'} \tag{1.142}$$

$$(\sigma_\varsigma,-i\varsigma)^{\alpha_\varsigma}|_{ab}=(\sigma_{-\varsigma},-i\varsigma)_a|_b{}^{\alpha_\varsigma} \qquad (\sigma_{-\varsigma},i\varsigma)^{\alpha'_\varsigma}|_{ab}=(\sigma_\varsigma,i\varsigma)_a|_b{}^{\alpha'_\varsigma} \tag{1.143}$$

5.2.3 Complex conjugation

$$[(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}\partial^a\hat\partial^b]^*=-(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}\partial^a\hat\partial^b \qquad [(\sigma_{-\varsigma},i\kappa)^{\alpha'_\varsigma}_{ab}\partial^a\hat\partial^b]^*=-(\sigma_\varsigma,i\kappa)^{\alpha_\varsigma}_{ab}\partial^a\hat\partial^b \tag{1.144}$$

5.3 Properties of composite spin constant invariant tensors $S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma),S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma)$

5.3.1 Complexity

$$S_{ab}(s,-\varsigma)=-i[\sigma(s),-\tfrac{i\varsigma}{2}]_{[a}[\sigma(s),\tfrac{i\varsigma}{2}]_{b]} \qquad S_{ab}(s,\varsigma)=-i[\sigma(s),\tfrac{i\varsigma}{2}]_{[a}[\sigma(s),-\tfrac{i\varsigma}{2}]_{b]} \tag{1.145}$$

$$S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)=i\sigma^{\alpha'_\varsigma}_{-\varsigma' ab}\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s) \qquad S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma)=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(s) \tag{1.146}$$

$$\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)=\tfrac{i}{4}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma) \qquad \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(s)=\tfrac{i}{4}\sigma^{ab}_{\varsigma\alpha_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma) \tag{1.147}$$

5.3.2 Orthogonality

Reduce two pairs of vector indices:

$$\begin{cases}S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)S^{abm'_\varsigma}{}_{n'_\varsigma}(s,-\varsigma)=4\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s)\sigma^{\alpha'_\varsigma m'_\varsigma}{}_{n'_\varsigma}(s)\\ S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma)S^{ab}{}_{m_\varsigma}{}^{n_\varsigma}(s,\varsigma)=4\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)\sigma_{\alpha_\varsigma m_\varsigma}{}^{n_\varsigma}(s)\end{cases} \tag{1.148}$$

Reduce two pairs of spinor indices:

$$\begin{cases} S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)S_{cd}{}^{l'_\varsigma}{}_{k'_\varsigma}(s,-\varsigma) = -\frac{2}{3}s(s+\frac{1}{2})(s+1)\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha'_\varsigma cd} \\ S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma)S_{cdl_\varsigma}{}^{k_\varsigma}(s,\varsigma) = -\frac{2}{3}s(s+\frac{1}{2})(s+1)\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} \end{cases} \tag{1.149}$$

Reduce a pair of vectors and a pair of spinor indices:

$$\begin{cases} S_{ac}{}^{k'_\varsigma}{}_{m'_\varsigma}(s,-\varsigma)S^{cbm'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma) = -s(s+1)\delta_a{}^b\delta^{k'_\varsigma}{}_{l'_\varsigma} \\ S_{ack_\varsigma}{}^{m_\varsigma}(s,\varsigma)S^{cb}{}_{m_\varsigma}{}^{l_\varsigma}(s,\varsigma) = -s(s+1)\delta_a{}^b\delta_{k_\varsigma}{}^{l_\varsigma} \end{cases} \tag{1.150}$$

Reduce three pairs of indices:

$$\begin{cases} S_{ac}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)S^{cbl'_\varsigma}{}_{k'_\varsigma}(s,-\varsigma) = -2s(s+1)\delta_a{}^b \\ S_{ack_\varsigma}{}^{l_\varsigma}(s,\varsigma)S^{cb}{}_{l_\varsigma}{}^{k_\varsigma}(s,\varsigma) = -2s(s+1)\delta_a{}^b \end{cases} \tag{1.151}$$

$$\begin{cases} S_{ab}{}^{k'_\varsigma}{}_{m'_\varsigma}(s,-\varsigma)S^{abm'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma) = 4s(s+1)\delta^{k'_\varsigma}{}_{l'_\varsigma} \\ S_{abk_\varsigma}{}^{m_\varsigma}(s,\varsigma)S^{ab}{}_{m_\varsigma}{}^{l_\varsigma}(s,\varsigma) = 4s(s+1)\delta_{k_\varsigma}{}^{l_\varsigma} \end{cases} \tag{1.152}$$

Reduce all indices:

$$\begin{cases} S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)S^{abl'_\varsigma}{}_{k'_\varsigma}(s,-\varsigma) = 8s(s+\frac{1}{2})(s+1) \\ S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma)S^{ab}{}_{l_\varsigma}{}^{k_\varsigma}(s,\varsigma) = 8s(s+\frac{1}{2})(s+1) \end{cases} \tag{1.153}$$

5.3.3 Duality

$$S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma) = \varsigma * S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma) \qquad S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma) = -\varsigma * S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma) \tag{1.154}$$

5.3.4 Complex conjugation

$$[S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma)\partial^a\hat{\partial}^b]^* = S_{abl_\varsigma}{}^{k_\varsigma}(s,\varsigma)\partial^a\hat{\partial}^b \qquad [S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma)\partial^a\hat{\partial}^b]^* = S_{ab}{}^{l'_\varsigma}{}_{k'_\varsigma}(s,-\varsigma)\partial^a\hat{\partial}^b \tag{1.155}$$

5.4 Properties of composite spin constant invariant tensors $S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}, S_{abA_\varsigma}{}^{B_\varsigma}$

5.4.1 Complexity

$$S_{ab}(\tfrac{1}{2},-\varsigma) = -\tfrac{i}{4}(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]} \qquad S_{ab}(\tfrac{1}{2},\varsigma) = -\tfrac{i}{4}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]} \tag{1.156}$$

$$S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} \equiv S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}(-\varsigma) \equiv S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}(\tfrac{1}{2},-\varsigma) \qquad S_{abA_\varsigma}{}^{B_\varsigma} \equiv S_{abA_\varsigma}{}^{B_\varsigma}(\varsigma) \equiv S_{abA_\varsigma}{}^{B_\varsigma}(\tfrac{1}{2},\varsigma) \tag{1.157}$$

$$S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} = \tfrac{i}{2}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma} \qquad S_{abA_\varsigma}{}^{B_\varsigma} = \tfrac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma A_\varsigma}{}^{B_\varsigma} \tag{1.158}$$

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma} = \tfrac{i}{2}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} \qquad \sigma_{\alpha_\varsigma A_\varsigma}{}^{B_\varsigma}(s) = \tfrac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}S_{abA_\varsigma}{}^{B_\varsigma} \tag{1.159}$$

5.4.2 Orthogonality

Reduce two pairs of vector indices:

$$S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}S^{abC'_\varsigma}{}_{D'_\varsigma} = \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\sigma^{\alpha'_\varsigma C'_\varsigma}{}_{D'_\varsigma} \qquad S_{abA_\varsigma}{}^{B_\varsigma}S^{ab}{}_{C_\varsigma}{}^{D_\varsigma} = \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\sigma_{\alpha_\varsigma C_\varsigma}{}^{D_\varsigma} \tag{1.160}$$

Reduce two pairs of spinor indices:

$$S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}S_{cd}{}^{B'_\varsigma}{}_{A'_\varsigma} = -\tfrac{1}{2}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha'_\varsigma cd} \qquad S_{abA_\varsigma}{}^{B_\varsigma}S_{cdB_\varsigma}{}^{A_\varsigma} = -\tfrac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} \tag{1.161}$$

Reduce a pair of vectors and a pair of spinor indices:

$$S_{ac}{}^{A'_\varsigma}{}_{C'_\varsigma}S^{cbC'_\varsigma}{}_{B'_\varsigma} = -\tfrac{3}{4}\delta_a{}^b\delta^{A'_\varsigma}{}_{B'_\varsigma} \qquad S_{acA_\varsigma}{}^{C_\varsigma}S^{cb}{}_{C_\varsigma}{}^{B_\varsigma} = -\tfrac{3}{4}\delta_a{}^b\delta_{A_\varsigma}{}^{B_\varsigma} \tag{1.162}$$

Reduce three pairs of indices:

$$S_{ac}{}^{A'_\varsigma}{}_{B'_\varsigma}S^{cbB'_\varsigma}{}_{A'_\varsigma} = -\tfrac{3}{2}\delta_a{}^b \qquad S_{acA_\varsigma}{}^{C_\varsigma}S^{cb}{}_{C_\varsigma}{}^{B_\varsigma} = -\tfrac{3}{2}\delta_a{}^b \tag{1.163}$$

$$S_{ab}{}^{A'_\varsigma}{}_{C'_\varsigma}S^{abC'_\varsigma}{}_{B'_\varsigma} = 3\delta^{A'_\varsigma}{}_{B'_\varsigma} \qquad S_{abA_\varsigma}{}^{C_\varsigma}S^{ab}{}_{C_\varsigma}{}^{B_\varsigma} = 3\delta_{A_\varsigma}{}^{B_\varsigma} \tag{1.164}$$

Reduce all indices:

$$S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}S^{abB'_\varsigma}{}_{A'_\varsigma} = 6 \qquad S_{abA_\varsigma}{}^{B_\varsigma}S^{ab}{}_{B_\varsigma}{}^{A_\varsigma} = 6 \tag{1.165}$$

5.4.3 Duality

$$S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} = \varsigma * S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} \qquad S_{abA_\varsigma}{}^{B_\varsigma} = -\varsigma * S_{abA_\varsigma}{}^{B_\varsigma} \tag{1.166}$$

5.4.4 Complex conjugation

$$[S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}\partial^a\hat{\partial}^b]^* = S_{abB_\varsigma}{}^{A_\varsigma}\partial^a\hat{\partial}^b \qquad [S_{abA_\varsigma}{}^{B_\varsigma}\partial^a\hat{\partial}^b]^* = S_{ab}{}^{B'_\varsigma}{}_{A'_\varsigma}\partial^a\hat{\partial}^b \tag{1.167}$$

5.5 Relations between several basic constant invariant tensors

$$\begin{cases} S_{ab}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma) = i\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s) \\ S_{abk_\varsigma}{}^{l_\varsigma}(s,\varsigma) = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(s) \end{cases} \quad \begin{cases} (\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}(\sigma,i\varsigma)_{bA_\varsigma B'_\varsigma} = \delta_{ab}\delta^{A'_\varsigma}{}_{B'_\varsigma} + 2iS_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} \\ (\sigma,i\varsigma)_{aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_b{}^{A'_\varsigma B_\varsigma} = \delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma} + 2iS_{abA_\varsigma}{}^{B_\varsigma} \end{cases} \tag{1.168}$$

$$\begin{cases} S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} = -\frac{i}{4}(\sigma,-i\varsigma)_{[a}{}^{A'_\varsigma A_\varsigma}(\sigma,i\varsigma)_{b]A_\varsigma B'_\varsigma} \\ \delta_{ab}\delta^{A'_\varsigma}{}_{B'_\varsigma} = \frac{1}{2}(\sigma,-i\varsigma)_{\{a}{}^{A'_\varsigma A_\varsigma}(\sigma,i\varsigma)_{b\}A_\varsigma B'_\varsigma} \end{cases} \quad \begin{cases} S_{abA_\varsigma}{}^{B_\varsigma} = -\frac{i}{4}(\sigma,i\varsigma)_{[aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_{b]}{}^{A'_\varsigma B_\varsigma} \\ \delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma} = \frac{1}{2}(\sigma,i\varsigma)_{\{aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_{b\}}{}^{A'_\varsigma B_\varsigma} \end{cases} \tag{1.169}$$

5.6 Properties of vector spin tensor S_{abcd} and antisymmetric tensor ε_{abcd}

Thm. 5.6.1. $S_{abcd} = -\frac{1}{2}(\sigma^{\alpha'}_{-ab}\sigma_{-\alpha' cd} + \sigma^{\alpha}_{+ab}\sigma_{+\alpha cd}) = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} = \delta_{a[c}\delta_{d]b} = \delta_{c[a}\delta_{b]d}, \vec{S}_{ab} := -iS_{ab|cd}$

Thm. 5.6.2. $\varepsilon_{abcd} = -\frac{1}{2}(\sigma^{\alpha'}_{-ab}\sigma_{-\alpha' cd} - \sigma^{\alpha}_{+ab}\sigma_{+\alpha cd})$

The above two theorems can be proved by expanding them into different cases. The two theorems are the basis and premise for some of the following deductions.
Combine with $(1.250), (1.274), (1.275)$**, then get the following Penrose correspondence notation:**

Cor. 5.6.1. $S_{abcd} \overset{P}{=} \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}$

Cor. 5.6.2. $\varepsilon_{abcd} = -\varepsilon_{acbd} \overset{P}{=} \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'}$

$$\begin{cases} S_{(*ab)(*cd)} = S_{abcd} \\ \varepsilon_{(*ab)(*cd)} = \varepsilon_{abcd} \end{cases} \quad \begin{cases} S_{(*ab)cd} = S_{ab(*cd)} = \varepsilon_{abcd} \\ \varepsilon_{(*ab)cd} = \varepsilon_{ab(*cd)} = S_{abcd} \end{cases} \tag{1.170}$$

$$S_{abcd} = S_{cdab}, S_{abcd} = -S_{bacd}, S_{abcd} = S_{abdc}, S_{abcd} = \frac{1}{2}S_{abef}S^{ef}{}_{cd}, \vartheta_{ab} = \frac{1}{2}S_{abcd}\vartheta^{cd} \tag{1.171}$$

$$\begin{cases} \sigma^{\alpha'}_{-ab}\sigma_{-\alpha' cd} = -(S_{abcd} + \varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \varepsilon_{abcd}) \\ \sigma^{\alpha}_{+ab}\sigma_{+\alpha cd} = -(S_{abcd} - \varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varepsilon_{abcd}) \end{cases} \tag{1.172}$$

$$\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} = -(S_{abcd} - \varsigma\varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varsigma\varepsilon_{abcd}) \tag{1.173}$$

Cor. 5.6.3. $\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma cd} = -\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} + \varsigma\varepsilon_{abc'd'}\eta^{c'}_c\eta^{d'}_d$

Proof: $\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma cd}$
$= \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma c'd'}\eta^{c'}_c\eta^{d'}_d$
$= -(S_{abc'd'} - \varsigma\varepsilon_{abc'd'})\eta^{c'}_c\eta^{d'}_d$
$= (-\delta_{ac'}\delta_{bd'} + \delta_{ad'}\delta_{bc'} + \varsigma\varepsilon_{abc'd'})\eta^{c'}_c\eta^{d'}_d$
$= -\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} + \varsigma\varepsilon_{abc'd'}\eta^{c'}_c\eta^{d'}_d$ □

5.7 Properties of spin tensor $S_{ab}(\varsigma)$

$$S_{ab}(\varsigma) = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma} = -\frac{i}{4}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]} \quad \delta_{ab} = \frac{1}{2}(\sigma,i\varsigma)_{\{a}(\sigma,-i\varsigma)_{b\}} \tag{1.174}$$

$$\begin{cases} i[S_{ab}(\varsigma), S_{cd}(\varsigma)] = \delta_{a[c}S_{d]b}(\varsigma) + S_{a[c}(\varsigma)\delta_{d]b} = -\delta_{c[a}S_{b]d}(\varsigma) - S_{c[a}(\varsigma)\delta_{b]d} \\ \{S_{ab}(\varsigma), S_{cd}(\varsigma)\} = -\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha cd} = \frac{1}{2}(S_{abcd} - \varsigma\varepsilon_{abcd}) \end{cases} \tag{1.175}$$

$$\varepsilon_{abcd} = \varsigma 2tr[S_{ab}(-\varsigma)S_{cd}(-\varsigma) - S_{ab}(\varsigma)S_{cd}(\varsigma)] \qquad S_{ab}(\varsigma) = -\varsigma * S_{ab}(\varsigma) \tag{1.176}$$

$$\begin{cases} 2iS_{ab}(\varsigma)(\sigma,i\varsigma)_c = (\sigma,i\varsigma)_{[a}\delta_{b]c} + \varsigma\varepsilon_{abcd}(\sigma,i\varsigma)^d \\ 2i(\sigma,-i\varsigma)_cS_{ab}(\varsigma) = \delta_{c[a}(\sigma,-i\varsigma)_{b]} - \varsigma\varepsilon_{abcd}(\sigma,-i\varsigma)^d \end{cases} \tag{1.177}$$

5.8 Properties of Dirac spin tensor $S_{ab}(e,\varsigma)$ [20]

$$[\gamma_a(\varsigma), \gamma_5(\varsigma)] = [(\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x), \varsigma I\otimes\sigma_z] \tag{1.178}$$

$$S_{ab}(e,\varsigma) = -\frac{i}{4}[\gamma_a(\varsigma),\gamma_b(\varsigma)] = S_{ab}(\varsigma)\oplus S_{ab}(-\varsigma) \quad \delta_{ab} = \frac{1}{2}\{\gamma_a(\varsigma),\gamma_b(\varsigma)\} \tag{1.179}$$

$$\begin{cases} i[S_{ab}(e,\varsigma), S_{cd}(e,\varsigma)] = \delta_{a[c}S_{d]b}(e,\varsigma) + S_{a[c}(e,\varsigma)\delta_{d]b} = -\delta_{c[a}S_{b]d}(e,\varsigma) - S_{c[a}(e,\varsigma)\delta_{b]d} \\ \{S_{ab}(e,\varsigma), S_{cd}(e,\varsigma)\} = -\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha'_\varsigma cd} \oplus \frac{1}{2}\sigma^{\alpha_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha'_\varsigma cd} = \frac{1}{2}[S_{abcd} - \gamma_5(\varsigma)\varepsilon_{abcd}] \end{cases} \tag{1.180}$$

$$[S_{ab}(e,\varsigma), \gamma_c(\varsigma)] = -i\gamma_{[a}\delta_{b]c} \qquad \{S_{ab}(e,\varsigma), \gamma_c(\varsigma)\} = -i\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma) \tag{1.181}$$

$$S_{ab}(e,\varsigma) = -\gamma_5(\varsigma) * S_{ab}(e,\varsigma) \tag{1.182}$$

5.9 Relations between constant invariant tensors $\varepsilon_{abcd}, \gamma_a(\varsigma)$ [20]

$$\varepsilon_{abcd}\gamma^a(\varsigma)\gamma^b(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma) = 24\gamma_5(\varsigma) \tag{1.183}$$
$$\varepsilon_{abcd}\gamma^b(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma) = -6\gamma_5(\varsigma)\gamma_a(\varsigma) \tag{1.184}$$
$$\varepsilon_{abcd}\gamma^c(\varsigma)\gamma^d(\varsigma) = -4\gamma_5(\varsigma)iS_{ab}(e,\varsigma) \tag{1.185}$$
$$\varepsilon_{abcd}\gamma^d(\varsigma) = \gamma_5(\varsigma)\{\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma) - [\delta_{ab}\gamma_c(\varsigma) + \gamma_{[a}(\varsigma)\delta_{b]c}]\} \tag{1.186}$$
$$\varepsilon_{abcd} = \gamma_5(\varsigma)\{\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma) \tag{1.187}$$
$$-[\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b} + 2i\delta_{ab}S_{cd}(e,\varsigma) + 2iS_{ab}(e,\varsigma)\delta_{cd} + 2i\delta_{a[c}S_{d]b}(e,\varsigma) - 2iS_{a[c}(e,\varsigma)\delta_{d]b}]\} \tag{1.188}$$

5.10 Properties of $tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\cdots]$

$$tr[\gamma_a(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \tag{1.189}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \tag{1.190}$$
$$tr[\gamma_5(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \tag{1.191}$$
$$tr[S_{ab}(e,\varsigma)] = 0 \quad tr[\gamma_c(\varsigma)S_{ab}(e,\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \tag{1.192}$$
$$tr[\gamma_5(\varsigma)S_{ab}(e,\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_c(\varsigma)S_{ab}(e,\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \tag{1.193}$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)] = 4\delta_{ab} \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4[\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b}] \tag{1.194}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4\varepsilon_{abcd} \tag{1.195}$$
$$tr[S_{ab}(e,\varsigma)S_{cd}(e,\varsigma)] = S_{abcd} = \delta_{ac}\delta_{db} - \delta_{ad}\delta_{cb} \quad tr[\gamma_5 S_{ab}(e,\varsigma)S_{cd}(e,\varsigma)] = -\varepsilon_{abcd} \tag{1.196}$$
$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e,\varsigma)] = 2iS_{abcd} \quad tr[\gamma_5\gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e,\varsigma)] = -2i\varepsilon_{abcd} \tag{1.197}$$
$$tr[S_{ab}(e,\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 2iS_{abcd} \quad tr[\gamma_5 S_{ab}(e,\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = -2i\varepsilon_{abcd} \tag{1.198}$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = 2i\{\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef}\} \tag{1.199}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = -2i\{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \tag{1.200}$$
$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{(\delta_{ab}\delta_{cd} - S_{abcd})\delta_{ef} - (\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef})\} \tag{1.201}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{\varepsilon_{abcd}\delta_{ef} + \delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \tag{1.202}$$

$$tr[\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = \delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef} - \delta_{bc}S_{adef} \tag{1.203}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = \delta_{bc}\varepsilon_{adef} - \{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \tag{1.204}$$

$$tr[\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = \delta_{ad}S_{bcef} + \delta_{a[b}S_{c]def} + \delta_{d[b}S_{c]aef} \tag{1.205}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = -\{\delta_{ad}\varepsilon_{bcef} + \delta_{a[b}\varepsilon_{c]def} + \delta_{d[b}\varepsilon_{c]aef}\} \tag{1.206}$$

5.11 Relations between constant invariant tensors $\varepsilon_{abcd}, (\sigma, i\varsigma)_a$

$$\varepsilon_{abcd}(\sigma,i\varsigma)^a(\sigma,-i\varsigma)^b(\sigma,i\varsigma)^c(\sigma,-i\varsigma)^d = 24\varsigma \tag{1.207}$$
$$\varepsilon_{abcd}(\sigma,i\varsigma)^b(\sigma,-i\varsigma)^c(\sigma,i\varsigma)^d = -6\varsigma(\sigma,i\varsigma)^a \tag{1.208}$$
$$\varepsilon_{abcd}(\sigma,i\varsigma)^c(\sigma,-i\varsigma)^d = -4i\varsigma S_{ab}(\varsigma) \tag{1.209}$$
$$\varepsilon_{abcd}(\sigma,i\varsigma)^d = \varsigma\{(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_b(\sigma,i\varsigma)_c - [\delta_{ab}(\sigma,i\varsigma)_c + (\sigma,i\varsigma)_{[a}\delta_{b]c}]\} \tag{1.210}$$
$$\varepsilon_{abcd} = \varsigma\{(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_b(\sigma,i\varsigma)_c(\sigma,-i\varsigma)_d \tag{1.211}$$
$$-[\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b} + 2i\delta_{ab}S_{cd}(\varsigma) + 2iS_{ab}(\varsigma)\delta_{cd} + 2i\delta_{a[c}S_{d]b}(\varsigma) + 2iS_{a[c}(\varsigma)\delta_{d]b}]\} \tag{1.212}$$

5.12 Relations between constant invariant tensors $\varepsilon_{abcd}, \delta_{ab}$

$$\begin{aligned}\varepsilon_{abcd}\varepsilon_{efgh} = &(\delta_{ae}\delta_{bf}\delta_{cg}\delta_{dh} - \delta_{ah}\delta_{be}\delta_{cf}\delta_{dg} + \delta_{ag}\delta_{bh}\delta_{ce}\delta_{df} - \delta_{af}\delta_{bg}\delta_{ch}\delta_{de}\\ &-(\delta_{ae}\delta_{bf}\delta_{ch}\delta_{dg} - \delta_{ag}\delta_{be}\delta_{cf}\delta_{dh} + \delta_{ah}\delta_{bg}\delta_{ce}\delta_{df} - \delta_{af}\delta_{bh}\delta_{cg}\delta_{de})\\ &+(\delta_{ae}\delta_{bg}\delta_{ch}\delta_{df} - \delta_{af}\delta_{be}\delta_{cg}\delta_{dh} + \delta_{ah}\delta_{bf}\delta_{ce}\delta_{dg} - \delta_{ag}\delta_{bh}\delta_{cf}\delta_{de})\\ &-(\delta_{ae}\delta_{bg}\delta_{cf}\delta_{dh} - \delta_{ah}\delta_{be}\delta_{cg}\delta_{df} + \delta_{af}\delta_{bh}\delta_{ce}\delta_{dg} - \delta_{ag}\delta_{bf}\delta_{ch}\delta_{de})\\ &+(\delta_{ae}\delta_{bh}\delta_{cf}\delta_{dg} - \delta_{ag}\delta_{be}\delta_{ch}\delta_{df} + \delta_{af}\delta_{bg}\delta_{ce}\delta_{dh} - \delta_{ah}\delta_{bf}\delta_{cg}\delta_{de})\\ &-(\delta_{ae}\delta_{bh}\delta_{cg}\delta_{df} - \delta_{af}\delta_{be}\delta_{ch}\delta_{dg} + \delta_{ag}\delta_{bf}\delta_{ce}\delta_{dh} - \delta_{ah}\delta_{bg}\delta_{cf}\delta_{de})\end{aligned} \tag{1.213}$$

$$\begin{aligned}\varepsilon_{abcd}\varepsilon_{efgh}\eta^{dh} = &(\delta_{ae}\delta_{bf}\delta_{cg}2 - \eta_{ag}\delta_{be}\delta_{cf} + \delta_{ag}\eta_{bf}\delta_{ce} - \delta_{af}\delta_{bg}\eta_{ce}\\ &- (\delta_{ae}\delta_{bf}\eta_{cg} - \delta_{ag}\delta_{be}\delta_{cf}2 + \eta_{af}\delta_{bg}\delta_{ce} - \delta_{af}\eta_{be}\delta_{cg})\\ &+ (\delta_{ae}\delta_{bg}\eta_{cf} - \delta_{af}\delta_{be}\delta_{cg}2 + \eta_{ag}\delta_{bf}\delta_{ce} - \delta_{ag}\eta_{be}\delta_{cf})\\ &- (\delta_{ae}\delta_{bg}\delta_{cf}2 - \eta_{af}\delta_{be}\delta_{cg} + \delta_{af}\eta_{bg}\delta_{ce} - \delta_{ag}\delta_{bf}\eta_{ce})\\ &+ (\delta_{ae}\eta_{bg}\delta_{cf} - \delta_{ag}\delta_{be}\eta_{cf} + \delta_{af}\delta_{bg}\delta_{ce}2 - \eta_{ae}\delta_{bf}\delta_{cg})\\ &- (\delta_{ae}\eta_{bf}\delta_{cg} - \delta_{af}\delta_{be}\eta_{cg} + \delta_{ag}\delta_{bf}\delta_{ce}2 - \eta_{ae}\delta_{bg}\delta_{cf})\end{aligned} \tag{1.214}$$

$$\begin{aligned}\varepsilon_{abcd}\varepsilon_{a'b'c'd'}\eta^{dd'} = &(\delta_{aa'}\delta_{bb'}\delta_{cc'}2 - \eta_{ac'}\delta_{ba'}\delta_{cb'} + \delta_{ac'}\eta_{bb'}\delta_{ca'} - \delta_{ab'}\delta_{bc'}\eta_{ca'}\\ &- (\delta_{aa'}\delta_{bb'}\eta_{cc'} - \delta_{ac'}\delta_{ba'}\delta_{cb'}2 + \eta_{ab'}\delta_{bc'}\delta_{ca'} - \delta_{ab'}\eta_{ba'}\delta_{cc'})\\ &+ (\delta_{aa'}\delta_{bc'}\eta_{cb'} - \delta_{ab'}\delta_{ba'}\delta_{cc'}2 + \eta_{ac'}\delta_{bb'}\delta_{ca'} - \delta_{ac'}\eta_{ba'}\delta_{cb'})\\ &- (\delta_{aa'}\delta_{bc'}\delta_{cb'}2 - \eta_{ab'}\delta_{ba'}\delta_{cc'} + \delta_{ab'}\eta_{bc'}\delta_{ca'} - \delta_{ac'}\delta_{bb'}\eta_{ca'})\\ &+ (\delta_{aa'}\eta_{bc'}\delta_{cb'} - \delta_{ac'}\delta_{ba'}\eta_{cb'} + \delta_{ab'}\delta_{bc'}\delta_{ca'}2 - \eta_{aa'}\delta_{bb'}\delta_{cc'})\\ &- (\delta_{aa'}\eta_{bb'}\delta_{cc'} - \delta_{ab'}\delta_{ba'}\eta_{cc'} + \delta_{ac'}\delta_{bb'}\delta_{ca'}2 - \eta_{aa'}\delta_{bc'}\delta_{cb'})\end{aligned} \tag{1.215}$$

$$\begin{aligned}\varepsilon_{abcd}\varepsilon_{a'b'c'd'}\eta^{dd'}\partial^{c}\partial^{+c'} = &(\delta_{aa'}\delta_{bb'}\delta_{cc'}2 - \eta_{ac'}\delta_{ba'}\delta_{cb'} + \delta_{ac'}\eta_{bb'}\delta_{ca'} - \delta_{ab'}\delta_{bc'}\eta_{ca'}\\ &- (\delta_{aa'}\delta_{bb'}\eta_{cc'} - \delta_{ac'}\delta_{ba'}\delta_{cb'}2 + \eta_{ab'}\delta_{bc'}\delta_{ca'} - \delta_{ab'}\eta_{ba'}\delta_{cc'})\\ &+ (\delta_{aa'}\delta_{bc'}\eta_{cb'} - \delta_{ab'}\delta_{ba'}\delta_{cc'}2 + \eta_{ac'}\delta_{bb'}\delta_{ca'} - \delta_{ac'}\eta_{ba'}\delta_{cb'})\\ &- (\delta_{aa'}\delta_{bc'}\delta_{cb'}2 - \eta_{ab'}\delta_{ba'}\delta_{cc'} + \delta_{ab'}\eta_{bc'}\delta_{ca'} - \delta_{ac'}\delta_{bb'}\eta_{ca'})\\ &+ (\delta_{aa'}\eta_{bc'}\delta_{cb'} - \delta_{ac'}\delta_{ba'}\eta_{cb'} + \delta_{ab'}\delta_{bc'}\delta_{ca'}2 - \eta_{aa'}\delta_{bb'}\delta_{cc'})\\ &- (\delta_{aa'}\eta_{bb'}\delta_{cc'} - \delta_{ab'}\delta_{ba'}\eta_{cc'} + \delta_{ac'}\delta_{bb'}\delta_{ca'}2 - \eta_{aa'}\delta_{bc'}\delta_{cb'})\end{aligned} \tag{1.216}$$

$$\begin{aligned}\varepsilon_{abcd}\varepsilon_{efgh}\delta^{de} = &(\delta_{ah}\delta_{bf}\delta_{cg} - \delta_{ah}\delta_{bg}\delta_{cf} + \delta_{ag}\delta_{bh}\delta_{cf} - 4\delta_{af}\delta_{bg}\delta_{ch})\\ &- (\delta_{ag}\delta_{bf}\delta_{ch} - \delta_{ag}\delta_{bh}\delta_{cf} + \delta_{ah}\delta_{bg}\delta_{cf} - 4\delta_{af}\delta_{bh}\delta_{cg})\\ &+ (\delta_{af}\delta_{bg}\delta_{ch} - \delta_{af}\delta_{bh}\delta_{cg} + \delta_{ah}\delta_{bf}\delta_{cg} - 4\delta_{ag}\delta_{bh}\delta_{cf})\\ &- (\delta_{ah}\delta_{bg}\delta_{cf} - \delta_{ah}\delta_{bf}\delta_{cg} + \delta_{af}\delta_{bh}\delta_{cg} - 4\delta_{ag}\delta_{bf}\delta_{ch})\\ &+ (\delta_{ag}\delta_{bh}\delta_{cf} - \delta_{ag}\delta_{bf}\delta_{ch} + \delta_{af}\delta_{bg}\delta_{ch} - 4\delta_{ah}\delta_{bf}\delta_{cg})\\ &- (\delta_{af}\delta_{bh}\delta_{cg} - \delta_{af}\delta_{bg}\delta_{ch} + \delta_{ag}\delta_{bf}\delta_{ch} - 4\delta_{ah}\delta_{bg}\delta_{cf})\end{aligned} \tag{1.217}$$

$$\varepsilon_{abcd}\varepsilon_{efgh}\delta^{de} = -(\delta_{af}\delta_{bg}\delta_{ch} - \delta_{af}\delta_{bh}\delta_{cg} + \delta_{ag}\delta_{bh}\delta_{cf} - \delta_{ag}\delta_{bf}\delta_{ch} + \delta_{ah}\delta_{bf}\delta_{cg} - \delta_{ah}\delta_{bg}\delta_{cf}) \tag{1.218}$$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} \tag{1.219}$$

$$\varepsilon_{ijk}\varepsilon^{k}{}_{lm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \varepsilon_{ijk}\varepsilon^{jk}{}_{l} = 2\delta_{il} \tag{1.220}$$

$$\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{C_\varsigma D_\varsigma} = \delta_{A_\varsigma C_\varsigma}\delta_{B_\varsigma D_\varsigma} - \delta_{A_\varsigma D_\varsigma}\delta_{B_\varsigma C_\varsigma} \tag{1.221}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\varepsilon^{\gamma_\varsigma}{}_{\rho_\varsigma\sigma_\varsigma} = \delta_{\alpha_\varsigma\rho_\varsigma}\delta_{\beta_\varsigma\sigma_\varsigma} - \delta_{\alpha_\varsigma\sigma_\varsigma}\delta_{\beta_\varsigma\rho_\varsigma} \qquad \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\varepsilon^{\beta_\varsigma\gamma_\varsigma}{}_{\rho_\varsigma} = 2\delta_{\alpha_\varsigma\rho_\varsigma} \tag{1.222}$$

5.13 Relations between constant invariant tensors $\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \sigma_{\alpha_\varsigma}$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma 4} \tag{1.223}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} = -i(\sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma}\sigma_{\gamma_\varsigma} - \delta_{\beta_\varsigma\gamma_\varsigma}\sigma_{\alpha_\varsigma} + \delta_{\gamma_\varsigma\alpha_\varsigma}\sigma_{\beta_\varsigma} - \delta_{\alpha_\varsigma\beta_\varsigma}\sigma_{\gamma_\varsigma}) \tag{1.224}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\gamma_\varsigma} = -i(\sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma} - \delta_{\alpha_\varsigma\beta_\varsigma}) = -\frac{1}{2}i[\sigma_{\alpha_\varsigma}, \sigma_{\beta_\varsigma}] \tag{1.225}$$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\beta_\varsigma}\sigma^{\gamma_\varsigma} = 2i\sigma_{\alpha_\varsigma} \qquad \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\alpha_\varsigma}\sigma^{\beta_\varsigma}\sigma^{\gamma_\varsigma} = 6i \tag{1.226}$$

$$2S_{\alpha_\varsigma\beta_\varsigma}\sigma_{\gamma_\varsigma} = -i\sigma_{[\alpha_\varsigma}\delta_{\beta_\varsigma]\gamma_\varsigma} + \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \qquad 2\sigma_{\gamma_\varsigma}S_{\alpha_\varsigma\beta_\varsigma} = -i\delta_{\gamma_\varsigma[\alpha_\varsigma}\sigma_{\beta_\varsigma]} + \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \tag{1.227}$$

$$[S_{\alpha_\varsigma\beta_\varsigma}, \sigma_{\gamma_\varsigma}] = -i\sigma_{[\alpha_\varsigma}\delta_{\beta_\varsigma]\gamma_\varsigma} \qquad \{S_{\alpha_\varsigma\beta_\varsigma}, \sigma_{\gamma_\varsigma}\} = \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \tag{1.228}$$

5.14 Relations between constant invariant tensors $\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}, \varepsilon_{abcd}$

$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma 4} \tag{1.229}$$
$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma d}A^d \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}A^4 \tag{1.230}$$
$$\varepsilon_{\alpha_\varsigma\beta_\varsigma cd}F^{cd} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(F^{\gamma_\varsigma 4} - F^{4\gamma_\varsigma}) \tag{1.231}$$
$$\varepsilon_{\alpha_\varsigma bcd}H^{bcd} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(H^{\beta_\varsigma\gamma_\varsigma 4} - H^{\beta_\varsigma 4\gamma_\varsigma} + H^{4\beta_\varsigma\gamma_\varsigma}) \tag{1.232}$$
$$\varepsilon_{abcd}R^{abcd} \equiv \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(R^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma 4} - R^{\alpha_\varsigma\beta_\varsigma 4\gamma_\varsigma} + R^{\alpha_\varsigma 4\beta_\varsigma\gamma_\varsigma} - R^{4\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}) \tag{1.233}$$

5.15 Relations between constant invariant tensors $\varepsilon_{A_\varsigma B_\varsigma}, \varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}$

$$\varepsilon_{A_\varsigma B_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma 3} \tag{1.234}$$
$$\varepsilon_{A_\varsigma B_\varsigma\gamma_\varsigma}A^{\gamma_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma}A^3 \tag{1.235}$$
$$\varepsilon_{A_\varsigma\beta_\varsigma\gamma_\varsigma}F^{\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma}(F^{B_\varsigma 3} - F^{3B_\varsigma}) \tag{1.236}$$
$$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}H^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma} \equiv \varepsilon_{A_\varsigma B_\varsigma}(H^{A_\varsigma B_\varsigma 3} - H^{A_\varsigma 3B_\varsigma} + H^{3A_\varsigma B_\varsigma}) \tag{1.237}$$

6 Properties of several non intuitive composite constant invariant tensors

6.1 Properties of composite constant invariant tensors $\sigma_{\alpha'_\varsigma}^{k'_\varsigma l'_\varsigma}(s), \sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s), \sigma^{\alpha'_\varsigma}_{k'_\varsigma l'_\varsigma}(s), \sigma^{k_\varsigma l_\varsigma}_{\alpha_\varsigma}(s)$

6.1.1 Definition

$$\begin{cases}\sigma_{\alpha'_\varsigma}^{k'_\varsigma l'_\varsigma}(s) := (\varsigma)^{2s}\varepsilon^{l'_\varsigma m'_\varsigma}(s)\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{m'_\varsigma}(s)\\ \sigma^{\alpha'_\varsigma}_{k'_\varsigma l'_\varsigma}(s) := (-\varsigma)^{2s}\varepsilon_{k'_\varsigma m'_\varsigma}(s)\sigma^{\alpha'_\varsigma m'_\varsigma}{}_{l'_\varsigma}(s)\end{cases} \quad \begin{cases}\sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s) := (-\varsigma)^{2s}\varepsilon_{l_\varsigma m_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{m_\varsigma}(s)\\ \sigma^{k_\varsigma l_\varsigma}_{\alpha_\varsigma}(s) := (\varsigma)^{2s}\varepsilon^{k_\varsigma m_\varsigma}(s)\sigma_{\alpha_\varsigma m_\varsigma}{}^{l_\varsigma}(s)\end{cases} \tag{1.238}$$

$$\begin{cases}\sigma_{\alpha'_\varsigma}^{k'_\varsigma l'_\varsigma}(s) := (-\varsigma)^{2s}[\sigma_{\alpha'_\varsigma}(s)\varepsilon(s)]^{k'_\varsigma l'_\varsigma}\\ \sigma^{\alpha'_\varsigma}_{k'_\varsigma l'_\varsigma}(s) := (-\varsigma)^{2s}[\varepsilon(s)\sigma^{\alpha'_\varsigma}(s)]_{k'_\varsigma l'_\varsigma}\end{cases} \quad \begin{cases}\sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s) := (\varsigma)^{2s}[\sigma^{\alpha_\varsigma}(s)\varepsilon(s)]_{k_\varsigma l_\varsigma}\\ \sigma^{k_\varsigma l_\varsigma}_{\alpha_\varsigma}(s) := (\varsigma)^{2s}[\varepsilon(s)\sigma_{\alpha_\varsigma}(s)]^{k_\varsigma l_\varsigma}\end{cases} \tag{1.239}$$

$$\begin{cases}\sigma_{\alpha'_\varsigma}^{k'_\varsigma l'_\varsigma}(s) \simeq (-1)^{2s}\sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s)\\ \sigma^{\alpha'_\varsigma}_{k'_\varsigma l'_\varsigma}(s) \simeq (-1)^{2s}\sigma^{k_\varsigma l_\varsigma}_{\alpha_\varsigma}(s)\end{cases} \quad \begin{cases}\sigma_{\alpha'_\varsigma}^{k'_\varsigma m'_\varsigma}(s)\sigma_{\alpha'_\varsigma m'_\varsigma l'_\varsigma}(s) = (-1)^{2s}\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{m'_\varsigma}(s)\sigma_{\alpha'_\varsigma}{}^{m'_\varsigma}{}_{l'_\varsigma}(s)\\ \sigma_{\alpha_\varsigma k_\varsigma m_\varsigma}(s)\sigma^{m_\varsigma l_\varsigma}_{\alpha_\varsigma}(s) = (-1)^{2s}\sigma_{\alpha_\varsigma k_\varsigma}{}^{m_\varsigma}(s)\sigma_{\alpha_\varsigma m_\varsigma}{}^{l_\varsigma}(s)\end{cases} \tag{1.240}$$

6.1.2 Symmetry and antisymmetry

$$\sigma^*(s) = (-1)^{2s+1}\varepsilon(s)\sigma(s)\varepsilon(s) \Rightarrow \tag{1.241}$$

$$\begin{cases}\sigma_{\alpha'_\varsigma}^{k'_\varsigma l'_\varsigma}(s) = (-1)^{2s+1}\sigma_{\alpha'_\varsigma}^{l'_\varsigma k'_\varsigma}(s)\\ \sigma^{\alpha'_\varsigma}_{k'_\varsigma l'_\varsigma}(s) = (-1)^{2s+1}\sigma^{\alpha'_\varsigma}_{l'_\varsigma k'_\varsigma}(s)\end{cases} \quad \begin{cases}\sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s) = (-1)^{2s+1}\sigma^{\alpha_\varsigma}_{l_\varsigma k_\varsigma}(s)\\ \sigma^{k_\varsigma l_\varsigma}_{\alpha_\varsigma}(s) = (-1)^{2s+1}\sigma^{l_\varsigma k_\varsigma}_{\alpha_\varsigma}(s)\end{cases} \tag{1.242}$$

6.1.3 Complex conjugation

$$\sigma^*(s) = (-1)^{2s+1}\varepsilon(s)\sigma(s)\varepsilon(s) \Rightarrow \tag{1.243}$$

$$[\sigma^{\alpha'_\varsigma}_{k'_\varsigma l'_\varsigma}(s)]^* = (-1)^{2s+1}\sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s) \qquad [\sigma_{\alpha'_\varsigma}^{k'_\varsigma l'_\varsigma}(s)]^* = (-1)^{2s+1}\sigma^{k_\varsigma l_\varsigma}_{\alpha_\varsigma}(s) \tag{1.244}$$

6.2 Properties of composite constant invariant tensors $\sigma_{\alpha'_\varsigma}^{A'_\varsigma B'_\varsigma}, \sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}, \sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}, \sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}$

6.2.1 Definition

$$\begin{cases}\sigma_{\alpha'_\varsigma}^{A'_\varsigma B'_\varsigma} := \varsigma\varepsilon^{B'_\varsigma C'_\varsigma}\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{C'_\varsigma}\\ \sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma} := -\varsigma\varepsilon_{A'_\varsigma C'_\varsigma}\sigma^{\alpha'_\varsigma C'_\varsigma}{}_{B'_\varsigma}\end{cases} \quad \begin{cases}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} := -\varsigma\varepsilon_{B_\varsigma C_\varsigma}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{C_\varsigma}\\ \sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} := \varsigma\varepsilon^{A_\varsigma C_\varsigma}\sigma_{\alpha_\varsigma C_\varsigma}{}^{B_\varsigma}\end{cases} \tag{1.245}$$

$$\begin{cases}\sigma_{\alpha'_\varsigma}^{A'_\varsigma B'_\varsigma} = -\varsigma[\sigma_{\alpha'_\varsigma}\varepsilon]^{A'_\varsigma B'_\varsigma}\\ \sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma} = -\varsigma[\varepsilon\sigma^{\alpha'_\varsigma}]_{A'_\varsigma B'_\varsigma}\end{cases} \quad \begin{cases}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} = \varsigma[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma}\\ \sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} = \varsigma[\varepsilon\sigma_{\alpha_\varsigma}]^{A_\varsigma B_\varsigma}\end{cases} \tag{1.246}$$

$$\begin{cases}\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} = \frac{i}{\sqrt{2}}[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma} = \frac{i}{\sqrt{2}}[-\sigma_z, i, \sigma_x]_{A_\varsigma B_\varsigma}\\ \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} = \frac{i}{\sqrt{2}}[\varepsilon\sigma_{\alpha_\varsigma}]^{A_\varsigma B_\varsigma} = \frac{i}{\sqrt{2}}[\sigma_z, i, -\sigma_x]^{A_\varsigma B_\varsigma}\end{cases} \tag{1.247}$$

$$\begin{cases}\sigma_{\alpha'_\varsigma}^{A'_\varsigma B'_\varsigma} \simeq -\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} \\ \sigma^{\alpha_\varsigma}_{A'_\varsigma B'_\varsigma} \simeq -\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\end{cases} \qquad \begin{cases}\sigma^{A'_\varsigma C'_\varsigma}_{\alpha'_\varsigma}\sigma^{\beta'_\varsigma}{}_{C'_\varsigma B'_\varsigma} = -\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{C'_\varsigma}\sigma^{\beta'_\varsigma C'_\varsigma}{}_{B'_\varsigma} \\ \sigma^{\alpha_\varsigma}_{A_\varsigma C_\varsigma}\sigma_{\beta_\varsigma}{}^{C_\varsigma B_\varsigma} = -\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{C_\varsigma}\sigma_{\beta_\varsigma C_\varsigma}{}^{B_\varsigma}\end{cases} \tag{1.248}$$

6.2.2 Orthogonality

Reduce a pair of complex vector indices:

$$\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{\alpha_\varsigma C_\varsigma D_\varsigma} = \varepsilon^{A_\varsigma D_\varsigma}\varepsilon^{C_\varsigma B_\varsigma} - \varepsilon^{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} \qquad \sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{\alpha'_\varsigma C'_\varsigma D'_\varsigma} = \varepsilon^{A'_\varsigma D'_\varsigma}\varepsilon^{C'_\varsigma B'_\varsigma} - \varepsilon^{A'_\varsigma C'_\varsigma}\varepsilon^{B'_\varsigma D'_\varsigma} \tag{1.249}$$

$$\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma_{\alpha_\varsigma C_\varsigma D_\varsigma} = \varepsilon_{A_\varsigma D_\varsigma}\varepsilon_{C_\varsigma B_\varsigma} - \varepsilon_{A_\varsigma C_\varsigma}\varepsilon_{B_\varsigma D_\varsigma} \qquad \sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\sigma_{\alpha'_\varsigma C'_\varsigma D'_\varsigma} = \varepsilon_{A'_\varsigma D'_\varsigma}\varepsilon_{C'_\varsigma B'_\varsigma} - \varepsilon_{A'_\varsigma C'_\varsigma}\varepsilon_{B'_\varsigma D'_\varsigma} \tag{1.250}$$

$$\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{C_\varsigma D_\varsigma} = -\delta^{(A_\varsigma}_{C_\varsigma}\delta^{B_\varsigma)}_{D_\varsigma} = -\delta^{A_\varsigma}_{(C_\varsigma}\delta^{B_\varsigma}_{D_\varsigma)} \qquad \sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{\alpha'_\varsigma}_{C'_\varsigma D'_\varsigma} = -\delta^{(A'_\varsigma}_{C'_\varsigma}\delta^{B'_\varsigma)}_{D'_\varsigma} = -\delta^{A'_\varsigma}_{(C'_\varsigma}\delta^{B'_\varsigma}_{D'_\varsigma)} \tag{1.251}$$

$$\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\sigma_{\alpha'_\varsigma}{}^{C'_\varsigma}{}_{D'_\varsigma} = -\varsigma(\varepsilon_{A'_\varsigma D'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma} + \delta^{C'_\varsigma}_{A'_\varsigma}\varepsilon_{B'_\varsigma D'_\varsigma}) \qquad \sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{\alpha_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma} = \varsigma(\varepsilon^{A_\varsigma D_\varsigma}\delta^{B_\varsigma}_{C_\varsigma} + \delta^{A_\varsigma}_{C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}) \tag{1.252}$$

$$\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma_{\alpha'_\varsigma}{}^{C'_\varsigma}{}_{D'_\varsigma} = \varsigma(\delta^{A'_\varsigma}_{D'_\varsigma}\varepsilon^{B'_\varsigma C'_\varsigma} + \varepsilon^{A'_\varsigma C'_\varsigma}\delta^{B'_\varsigma}_{D'_\varsigma}) \qquad \sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma} = -\varsigma(\delta^{D_\varsigma}_{A_\varsigma}\varepsilon^{B_\varsigma C_\varsigma} + \varepsilon_{A_\varsigma C_\varsigma}\delta^{D_\varsigma}_{B_\varsigma}) \tag{1.253}$$

$$\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\sigma^{\alpha'_\varsigma C'_\varsigma}{}_{D'_\varsigma} = \delta^{A'_\varsigma}_{D'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma} - \varepsilon^{A'_\varsigma C'_\varsigma}\varepsilon_{B'_\varsigma D'_\varsigma} \qquad \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\sigma_{\alpha_\varsigma C_\varsigma}{}^{D_\varsigma} = \delta^{D_\varsigma}_{A_\varsigma}\delta^{B_\varsigma}_{C_\varsigma} - \varepsilon_{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} \tag{1.254}$$

6.2.3 Symmetry and antisymmetry

$$\begin{cases}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} = \sigma^{B_\varsigma A_\varsigma}_{\alpha_\varsigma} \\ \sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} = \sigma^{\alpha_\varsigma}_{B_\varsigma A_\varsigma}\end{cases} \qquad \begin{cases}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma} = \sigma^{\alpha'_\varsigma}_{B'_\varsigma A'_\varsigma} \\ \sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma} = \sigma^{B'_\varsigma A'_\varsigma}_{\alpha'_\varsigma}\end{cases} \tag{1.255}$$

6.2.4 Complex conjugation

$$\sigma^T = \varepsilon\sigma\varepsilon \tag{1.256}$$

$$[\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}]^* = \sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma} \qquad [\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}]^* = \sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma} \tag{1.257}$$

6.3 Properties of constant invariant tensors $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}, \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}$

6.3.1 Definition

$$\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab} := \tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma} \qquad \sigma^{\alpha'\alpha}_{ab} \overset{P}{=} \tfrac{1}{2}\sigma^{\alpha'}_{A'B'}\sigma^{\alpha}_{AB} \tag{1.258}$$

$$\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} := \tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} \qquad \sigma^{ab}_{\alpha\alpha'} \overset{P}{=} \tfrac{1}{2}\sigma^{AB}_{\alpha}\sigma^{A'B'}_{\alpha'} \tag{1.259}$$

Property:

$$\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab} = \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ba} \qquad \sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} = \sigma^{ba}_{\alpha_\varsigma\alpha'_\varsigma} \qquad \delta^{ab}\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab} = 0 \qquad \delta_{ab}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} = 0 \tag{1.260}$$

$$\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} \simeq \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab} \qquad (\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma})^* = \sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} \qquad (\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab})^* = \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab} \qquad R^{ab} = \sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\psi^{\alpha_\varsigma}\psi^{*\alpha'_\varsigma} \tag{1.261}$$

Cor. 6.3.1. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} = -\frac{1}{2}\sigma^{ac}_{\varsigma\alpha_\varsigma}\delta_{cd}\sigma^{db}_{-\varsigma\alpha'_\varsigma}, \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab} = -\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ac}\delta^{cd}\sigma^{\alpha'_\varsigma}_{-\varsigma db}, \sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} \simeq \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}$

Proof: $\sigma^{\alpha_\varsigma}_{\varsigma a}{}^c\sigma_{-\varsigma\alpha'_\varsigma}{}^c{}_b$

$= \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^c_{B_\varsigma B'_\varsigma}\cdot\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^c{}_{C_\varsigma C'_\varsigma}\sigma_{\alpha'_\varsigma}{}^{C'_\varsigma}{}_{D'_\varsigma}\delta^{C_\varsigma}{}_{D_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{D'_\varsigma D_\varsigma}$

$= \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma}\varepsilon_{B_\varsigma C_\varsigma}\varepsilon_{B'_\varsigma C'_\varsigma}\sigma_{\alpha'_\varsigma}{}^{C'_\varsigma}{}_{D'_\varsigma}\delta^{C_\varsigma}{}_{D_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{D'_\varsigma D_\varsigma}$

$= -\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\varsigma\varepsilon_{B_\varsigma D_\varsigma})(-\varsigma\varepsilon_{A'_\varsigma C'_\varsigma})\sigma_{\alpha'_\varsigma}{}^{C'_\varsigma}{}_{D'_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{D'_\varsigma D_\varsigma}$

$= -\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma D_\varsigma}\sigma_{\alpha'_\varsigma A'_\varsigma D'_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{D'_\varsigma D_\varsigma}$ □

Cor. 6.3.2. $\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{kl} = \frac{1}{2}(\delta^{\alpha_\varsigma}_k\delta^{\alpha'_\varsigma}_l + \delta^{\alpha'_\varsigma}_k\delta^{\alpha_\varsigma}_l - \delta_{kl}\delta^{\alpha_\varsigma\alpha'_\varsigma}), \sigma^{kl}_{\alpha_\varsigma\alpha'_\varsigma} = \frac{1}{2}(\delta^k_{\alpha_\varsigma}\delta^l_{\alpha'_\varsigma} + \delta^k_{\alpha'_\varsigma}\delta^l_{\alpha_\varsigma} - \delta^{kl}\delta_{\alpha_\varsigma\alpha'_\varsigma})$

Proof: $\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{kl}$

$= \frac{i\varsigma}{\sqrt{2}}(\sigma)_k{}^{A'_\varsigma A_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma)_l{}^{B'_\varsigma B_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}$

$= -\frac{1}{4}(\sigma)_k{}^{A'_\varsigma A_\varsigma}(\sigma)_l{}^{B'_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}$

$= -\frac{1}{4}(\sigma)_k{}^{A'_\varsigma A_\varsigma}(\sigma)_l{}^{B'_\varsigma B_\varsigma}\varsigma[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma}\{-\varsigma[\varepsilon\sigma^{\alpha'_\varsigma}]_{A'_\varsigma B'_\varsigma}\}$

$= \frac{1}{4}(\sigma)_k{}^{A'_\varsigma A_\varsigma}[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma}(\sigma^T)_l{}^{B_\varsigma B'_\varsigma}[\sigma^{T\alpha'_\varsigma}\varepsilon^T]_{B'_\varsigma A'_\varsigma}$
$= \frac{1}{4}tr\{\sigma_k\sigma^{\alpha_\varsigma}\varepsilon\sigma_l^T\sigma^{T\alpha'_\varsigma}\varepsilon^T\}$
$= \frac{1}{4}tr\{\sigma_k\sigma^{\alpha_\varsigma}\varepsilon\sigma_l^T\varepsilon^T\varepsilon\sigma^{T\alpha'_\varsigma}\varepsilon^T\}$
$= \frac{1}{4}tr\{\sigma_k\sigma^{\alpha_\varsigma}\sigma_l\sigma^{\alpha'_\varsigma}\}$
$= \frac{1}{4}tr\{(\delta_k^{\alpha_\varsigma}+i\varepsilon_k{}^{\alpha_\varsigma\beta_\varsigma}\sigma_{\beta_\varsigma})(\delta_l^{\alpha'_\varsigma}+i\varepsilon_l{}^{\alpha'_\varsigma\beta'_\varsigma}\sigma_{\beta'_\varsigma})\}$
$= \frac{1}{2}(\delta_k^{\alpha_\varsigma}\delta_l^{\alpha'_\varsigma}-\varepsilon_k{}^{\alpha_\varsigma\beta_\varsigma}\varepsilon_l{}^{\alpha'_\varsigma\beta'_\varsigma}\delta_{\beta_\varsigma\beta'_\varsigma})$
$= \frac{1}{2}(\delta_k^{\alpha_\varsigma}\delta_l^{\alpha'_\varsigma}+\delta_k^{\alpha'_\varsigma}\delta_l^{\alpha_\varsigma}-\delta_{kl}\delta^{\alpha_\varsigma\alpha'_\varsigma})$ □

Cor. 6.3.3. $\sigma_{k\pi}^{\alpha_\varsigma\alpha'_\varsigma} = -\frac{\varsigma}{2}\varepsilon_k{}^{\alpha_\varsigma\alpha'_\varsigma}, \sigma^{k\pi}_{\alpha_\varsigma\alpha'_\varsigma} = -\frac{\varsigma}{2}\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}$

Proof: $\sigma_{k\pi}^{\alpha_\varsigma\alpha'_\varsigma}$
$= \frac{i\varsigma}{\sqrt{2}}(\sigma)_k{}^{A'_\varsigma A_\varsigma}\frac{i\varsigma}{\sqrt{2}}(-i\varsigma)^{B'_\varsigma B_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}$
$= i\varsigma\frac{1}{4}(\sigma)_k{}^{A'_\varsigma A_\varsigma}\delta^{B'_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}$
$= i\varsigma\frac{1}{4}(\sigma)_k{}^{A'_\varsigma A_\varsigma}\delta^{B'_\varsigma B_\varsigma}\varsigma[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma}\{-\varsigma[\varepsilon\sigma^{\alpha'_\varsigma}]_{A'_\varsigma B'_\varsigma}\}$
$= -i\varsigma\frac{1}{4}(\sigma)_k{}^{A'_\varsigma A_\varsigma}[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma}\delta^{B_\varsigma B'_\varsigma}[\sigma^{T\alpha'_\varsigma}\varepsilon^T]_{B'_\varsigma A'_\varsigma}$
$= -i\varsigma\frac{1}{4}tr\{\sigma_k\sigma^{\alpha_\varsigma}\varepsilon I\sigma^{T\alpha'_\varsigma}\varepsilon^T\}$
$= -i\varsigma\frac{1}{4}tr\{\sigma_k\sigma^{\alpha_\varsigma}\sigma^{\alpha'_\varsigma}\}$
$= i\varsigma\frac{1}{4}tr\{(\delta_k^{\alpha_\varsigma}+i\varepsilon_k{}^{\alpha_\varsigma\beta_\varsigma}\sigma_{\beta_\varsigma})\sigma^{\alpha'_\varsigma}\}$
$= -\varsigma\frac{1}{2}\varepsilon_k{}^{\alpha_\varsigma\beta_\varsigma}\delta_{\beta_\varsigma\alpha'_\varsigma}$
$= -\frac{\varsigma}{2}\varepsilon_k{}^{\alpha_\varsigma\alpha'_\varsigma}$ □

Cor. 6.3.4. $\sigma_{\pi\pi}^{\alpha'_\varsigma\alpha_\varsigma} = \frac{1}{2}\delta^{\alpha_\varsigma\alpha'_\varsigma}, \sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma} = \frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}$

Proof: $\sigma_{\pi\pi}^{\alpha'_\varsigma\alpha_\varsigma}$
$= \frac{i\varsigma}{\sqrt{2}}(-i\varsigma)^{A'_\varsigma A_\varsigma}\frac{i\varsigma}{\sqrt{2}}(-i\varsigma)^{B'_\varsigma B_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}$
$= \frac{1}{4}\delta^{A'_\varsigma A_\varsigma}\delta^{B'_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}$
$= \frac{1}{4}\delta^{A'_\varsigma A_\varsigma}\delta^{B'_\varsigma B_\varsigma}\varsigma[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma}\{-\varsigma[\varepsilon\sigma^{\alpha'_\varsigma}]_{A'_\varsigma B'_\varsigma}\}$
$= -\frac{1}{4}\delta^{A'_\varsigma A_\varsigma}[\sigma^{\alpha_\varsigma}\varepsilon]_{A_\varsigma B_\varsigma}\delta^{B_\varsigma B'_\varsigma}[\sigma^{T\alpha'_\varsigma}\varepsilon^T]_{B'_\varsigma A'_\varsigma}$
$= -\frac{1}{4}tr\{I\sigma^{\alpha_\varsigma}\varepsilon I\sigma^{T\alpha'_\varsigma}\varepsilon^T\}$
$= \frac{1}{4}tr\{\sigma^{\alpha_\varsigma}\sigma^{\alpha'_\varsigma}\}$
$= \frac{1}{2}\delta^{\alpha_\varsigma\alpha'_\varsigma}$ □

Cor. 6.3.5. $\begin{cases}\sigma^{kl}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_l = \partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma}-\frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}\nabla^2, \sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\partial_\pi^2 = \frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}\partial_\pi^2\\ \sigma^{k\pi}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_\pi = -\frac{\varsigma}{2}\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_\pi, \sigma^{\pi k}_{\alpha_\varsigma\alpha'_\varsigma}\partial_\pi\partial_k = -\frac{\varsigma}{2}\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_\pi\partial_k\end{cases}$

Cor. 6.3.6. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b = \partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma}-\frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2-\partial_\pi^2)-\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_\pi$

Orthogonality:

Cor. 6.3.7. $\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}\sigma^{ab}_{\beta_\varsigma\beta'_\varsigma} = \delta^{\alpha_\varsigma}{}_{\beta_\varsigma}\delta^{\alpha'_\varsigma}{}_{\beta'_\varsigma}$

Cor. 6.3.8. $\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}\sigma_{\alpha_\varsigma\alpha'_\varsigma cd} = \frac{1}{2}\delta_{ac}\delta_{bd}+\frac{1}{2}\delta_{ad}\delta_{bc}-\frac{1}{4}\delta_{ab}\delta_{cd}$

Proof method 1:

Proof: $\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}\sigma_{\alpha_\varsigma\alpha'_\varsigma cd}$
$= \frac{1}{4}\delta^{ef}\delta^{gh}(\sigma^{\alpha_\varsigma}_{\varsigma ae}\sigma_{\varsigma\alpha_\varsigma cg})(\sigma_{-\varsigma\alpha'_\varsigma hd}\sigma^{\alpha'_\varsigma}_{-\varsigma fb})$
$= \frac{1}{4}\delta^{ef}\delta^{gh}(S_{aecg}-\varsigma\varepsilon_{aecg})(S_{hdfb}+\varsigma\varepsilon_{hdfb})$
$= \frac{1}{4}\delta^{ef}\delta^{gh}(\delta_{ac}\delta_{eg}-\delta_{ag}\delta_{ec}-\varsigma\varepsilon_{aecg})(\delta_{hf}\delta_{db}-\delta_{hb}\delta_{df}+\varsigma\varepsilon_{hdfb})$
$= \frac{1}{4}[(\delta_{ac}\delta_{eg}-\delta_{ag}\delta_{ec})(\delta_{hf}\delta_{db}-\delta_{hb}\delta_{df})+(-\varsigma\varepsilon_{aecg})(\delta_{hf}\delta_{db}-\delta_{hb}\delta_{df})+(\delta_{ac}\delta_{eg}-\delta_{ag}\delta_{ec})(\varsigma\varepsilon_{hdfb})+(-\varsigma\varepsilon_{aecg})(\varsigma\varepsilon_{hdfb})]$
$= \frac{1}{4}[(2\delta_{ac}\delta_{db}+\delta_{ab}\delta_{cd})+(-\varsigma\varepsilon_{abcd})+(\varsigma\varepsilon_{abcd})-(\varepsilon_{aceg})\delta^{ef}\delta^{gh}(\varepsilon_{fhbd})]$
$= \frac{1}{4}[(2\delta_{ac}\delta_{db}+\delta_{ab}\delta_{cd})+(-2S_{acbd})]$
$= \frac{1}{4}[(2\delta_{ac}\delta_{db}+\delta_{ab}\delta_{cd})-2(\delta_{ab}\delta_{cd}-\delta_{ad}\delta_{cb})]$
$= \frac{1}{2}\delta_{ac}\delta_{bd}+\frac{1}{2}\delta_{ad}\delta_{bc}-\frac{1}{4}\delta_{ab}\delta_{cd}$ □

Proof method 2:

Proof:

$$\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}\sigma_{\alpha_\varsigma\alpha'_\varsigma cd}$$

$$= \tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\cdot\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cC_\varsigma C'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dD_\varsigma D'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}\sigma^{C'_\varsigma D'_\varsigma}_{\alpha'_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}$$

$$= \tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cC_\varsigma C'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dD_\varsigma D'_\varsigma}\cdot\tfrac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}\sigma^{C'_\varsigma D'_\varsigma}_{\alpha'_\varsigma}$$

$$= \tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cC_\varsigma C'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dD_\varsigma D'_\varsigma}\cdot\tfrac{1}{2}(\delta^{C_\varsigma}_{A_\varsigma}\delta^{D_\varsigma}_{B_\varsigma}+\delta^{D_\varsigma}_{A_\varsigma}\delta^{C_\varsigma}_{B_\varsigma})\tfrac{1}{2}(\delta^{C'_\varsigma}_{A'_\varsigma}\delta^{D'_\varsigma}_{B'_\varsigma}+\delta^{D'_\varsigma}_{A'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma})$$

$$= \tfrac{1}{4}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cA_\varsigma A'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dB_\varsigma B'_\varsigma}$$

$$+ \tfrac{1}{4}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cB_\varsigma B'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dA_\varsigma A'_\varsigma}$$

$$+ \tfrac{1}{4}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cA_\varsigma B'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dB_\varsigma A'_\varsigma}$$

$$+ \tfrac{1}{4}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cB_\varsigma A'_\varsigma}\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dA_\varsigma B'_\varsigma}$$

$$= \tfrac{1}{4}\delta_{ac}\delta_{bd}+\tfrac{1}{4}\delta_{ad}\delta_{bc}+\tfrac{1}{4}[\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cA_\varsigma B'_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}][\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dB_\varsigma A'_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}]$$

$$+ \tfrac{1}{4}[\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{cB_\varsigma A'_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}][\tfrac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{dA_\varsigma B'_\varsigma}\tfrac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma}]$$

$$= \tfrac{1}{4}\delta_{ac}\delta_{bd}+\tfrac{1}{4}\delta_{ad}\delta_{bc}+\tfrac{1}{4}[\tfrac{1}{2}\delta_{cb}\delta_{A_\varsigma}{}^{B_\varsigma}+iS_{cbA_\varsigma}{}^{B_\varsigma}][\tfrac{1}{2}\delta_{da}\delta_{B_\varsigma}{}^{A_\varsigma}+iS_{daB_\varsigma}{}^{A_\varsigma}]$$

$$+ \tfrac{1}{4}[\tfrac{1}{2}\delta_{ca}\delta_{B_\varsigma}{}^{A_\varsigma}+iS_{caB_\varsigma}{}^{A_\varsigma}][\tfrac{1}{2}\delta_{db}\delta_{A_\varsigma}{}^{B_\varsigma}+iS_{dbA_\varsigma}{}^{B_\varsigma}]$$

$$= \tfrac{3}{8}\delta_{ac}\delta_{bd}+\tfrac{3}{8}\delta_{ad}\delta_{bc}+\tfrac{1}{4}iS_{cbA_\varsigma}{}^{B_\varsigma}iS_{daB_\varsigma}{}^{A_\varsigma}+\tfrac{1}{4}iS_{caB_\varsigma}{}^{A_\varsigma}iS_{dbA_\varsigma}{}^{B_\varsigma}$$

$$= \tfrac{3}{8}\delta_{ac}\delta_{bd}+\tfrac{3}{8}\delta_{ad}\delta_{bc}+\tfrac{1}{8}\sigma^{\alpha_\varsigma}_{\varsigma cb}\sigma_{\varsigma\alpha_\varsigma da}+\tfrac{1}{8}\sigma^{\alpha_\varsigma}_{\varsigma ca}\sigma_{\varsigma\alpha_\varsigma db}$$

$$= \tfrac{3}{8}\delta_{ac}\delta_{bd}+\tfrac{3}{8}\delta_{ad}\delta_{bc}+\tfrac{1}{8}(-\delta_{cd}\delta_{ba}+\delta_{ca}\delta_{bd}+\varsigma\varepsilon_{cbda})+\tfrac{1}{8}(-\delta_{cd}\delta_{ab}+\delta_{cb}\delta_{ad}+\varsigma\varepsilon_{cadb})$$

$$= \tfrac{1}{2}\delta_{ac}\delta_{bd}+\tfrac{1}{2}\delta_{ad}\delta_{bc}-\tfrac{1}{4}\delta_{ab}\delta_{cd}$$ □

6.3.2 Summary

Def. 6.3.1. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} := -\frac{1}{2}(\sigma_{+\varsigma\alpha_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma})^{ab} = \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}, \sigma^{ab}_{\alpha\alpha'} \overset{P}{=} \frac{1}{2}\sigma^{AB}_{\alpha}\sigma^{A'B'}_{\alpha'}$

Cor. 6.3.9. $\sigma^{kl}_{\alpha_\varsigma\alpha'_\varsigma} = \frac{1}{2}(\delta^k_{\alpha_\varsigma}\delta^l_{\alpha'_\varsigma}+\delta^k_{\alpha'_\varsigma}\delta^l_{\alpha_\varsigma}-\delta^{kl}\delta_{\alpha_\varsigma\alpha'_\varsigma}), \sigma^{k\pi}_{\alpha_\varsigma\alpha'_\varsigma} = \sigma^{\pi k}_{\alpha_\varsigma\alpha'_\varsigma} = -\frac{\varsigma}{2}\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}, \sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma} = \frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}$

Cor. 6.3.10. $\sigma^{\alpha_\varsigma\alpha'_\varsigma}_{ab}\sigma^{ab}_{\beta_\varsigma\beta'_\varsigma} = \delta^{\alpha_\varsigma}{}_{\beta_\varsigma}\delta^{\alpha'_\varsigma}{}_{\beta'_\varsigma}, \sigma^{\alpha_\varsigma\alpha'_\varsigma}_{ab}\sigma_{\alpha_\varsigma\alpha'_\varsigma cd} = \frac{1}{2}\delta_{ac}\delta_{bd}+\frac{1}{2}\delta_{ad}\delta_{bc}-\frac{1}{4}\delta_{ab}\delta_{cd}$

Cor. 6.3.11. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b = \partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma}-\frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2-\partial^2_\pi)-\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_\pi = \partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma}-\frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2+\partial^2_t)+i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_t$

6.4 Properties of constant invariant tensors $S_{ab}{}^{k_\varsigma l_\varsigma}(s,\varsigma), S^{ab}{}_{k'_\varsigma l'_\varsigma}(s,-\varsigma), S^{ab}{}_{k_\varsigma l_\varsigma}(s,\varsigma), S_{ab}{}^{k'_\varsigma l'_\varsigma}(s,-\varsigma)$

6.4.1 Definition

$$S_{ab}{}^{k_\varsigma l_\varsigma}(s,\varsigma) = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{k_\varsigma l_\varsigma}_{\alpha_\varsigma}(s) \qquad S^{ab}{}_{k'_\varsigma l'_\varsigma}(s,-\varsigma) = i\sigma^{ab}_{-\varsigma\alpha'_\varsigma}\sigma^{\alpha'_\varsigma}_{k'_\varsigma l'_\varsigma}(s) \tag{1.262}$$

$$S^{ab}{}_{k_\varsigma l_\varsigma}(s,\varsigma) = i\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s) \qquad S_{ab}{}^{k'_\varsigma l'_\varsigma}(s,-\varsigma) = i\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{k'_\varsigma l'_\varsigma}_{\alpha'_\varsigma}(s) \tag{1.263}$$

6.4.2 Symmetry and antisymmetry

$$S_{ab}{}^{k_\varsigma l_\varsigma}(s,\varsigma) = (-1)^{2s+1}S_{ab}{}^{l_\varsigma k_\varsigma}(s,\varsigma) \qquad S^{ab}{}_{k'_\varsigma l'_\varsigma}(s,-\varsigma) = (-1)^{2s+1}S^{ab}{}_{l'_\varsigma k'_\varsigma}(s,-\varsigma) \tag{1.264}$$

$$S^{ab}{}_{k_\varsigma l_\varsigma}(s,\varsigma) = (-1)^{2s+1}S^{ab}{}_{l_\varsigma k_\varsigma}(s,\varsigma) \qquad S_{ab}{}^{k'_\varsigma l'_\varsigma}(s,-\varsigma) = (-1)^{2s+1}S_{ab}{}^{l'_\varsigma k'_\varsigma}(s,-\varsigma) \tag{1.265}$$

$$S_{ab}{}^{k'_\varsigma m'_\varsigma}(s,-\varsigma)S_{cdm'_\varsigma l'_\varsigma}(s,-\varsigma) = -S_{ab}{}^{k'_\varsigma}{}_{m'_\varsigma}(s,-\varsigma)S_{cd}{}^{m'_\varsigma}{}_{l'_\varsigma}(s,-\varsigma) \tag{1.266}$$

$$S_{abk_\varsigma m_\varsigma}(s,\varsigma)S_{cd}{}^{m_\varsigma l_\varsigma}(s,\varsigma) = -S_{abk_\varsigma}{}^{m_\varsigma}(s,\varsigma)S_{cdm_\varsigma}{}^{l_\varsigma}(s,\varsigma) \tag{1.267}$$

6.4.3 Duality

$$S_{ab}{}^{k_\varsigma l_\varsigma}(s,\varsigma) = -\varsigma * S_{ab}{}^{k_\varsigma l_\varsigma}(s,\varsigma) \qquad S^{ab}{}_{k'_\varsigma l'_\varsigma}(s,-\varsigma) = \varsigma * S^{ab}{}_{k'_\varsigma l'_\varsigma}(s,-\varsigma) \tag{1.268}$$

$$S^{ab}{}_{k_\varsigma l_\varsigma}(s,\varsigma) = -\varsigma * S^{ab}{}_{k_\varsigma l_\varsigma}(s,\varsigma) \qquad S_{ab}{}^{k'_\varsigma l'_\varsigma}(s,-\varsigma) = \varsigma * S_{ab}{}^{k'_\varsigma l'_\varsigma}(s,-\varsigma) \tag{1.269}$$

6.4.4 Complex conjugation

$$[S_{ab}{}^{k_\varsigma l_\varsigma}(s,\varsigma)\partial^a\hat{\partial}^b]^* = (-1)^{2s+1}S_{ab}{}^{k'_\varsigma l'_\varsigma}(s,-\varsigma)\partial^a\hat{\partial}^b \qquad [S^{ab}{}_{k_\varsigma l_\varsigma}(s,\varsigma)\partial_a\hat{\partial}_b]^* = (-1)^{2s+1}S^{ab}{}_{k'_\varsigma l'_\varsigma}(s,-\varsigma)\partial_a\hat{\partial}_b \tag{1.270}$$

6.5 Important connections between several basic constant invariant tensors

Connection 1:

$$\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\sigma^{\alpha_\varsigma}_{\varsigma a}{}^b\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b = \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma} \qquad \sigma^{\alpha}_{+a}{}^b \overset{P}{=} \sigma^\alpha{}_A{}^B\delta_{A'}{}^{B'} \tag{1.271}$$

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma}{}^a{}_b\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma} = \sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma}\delta^{A_\varsigma}{}_{B_\varsigma} \qquad \sigma_{-\alpha'}{}^a{}_b \overset{P}{=} \sigma_{\alpha'}{}^{A'}{}_{B'}\delta^A{}_B \tag{1.272}$$

$$\sigma^{\alpha_\varsigma}_{\varsigma a}{}^b = \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)_{bB_\varsigma B'_\varsigma} \tag{1.273}$$

Connection 2:

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{ab}_{\varsigma\alpha_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma} = \varsigma\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma} \qquad \sigma^{ab}_{+\alpha} \overset{P}{=} \sigma^{AB}_{\alpha}\varepsilon^{A'B'} \tag{1.274}$$

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a{}^{A'_\varsigma A_\varsigma}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b{}^{B'_\varsigma B_\varsigma} = -\varsigma\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\varepsilon^{A_\varsigma B_\varsigma} \qquad \sigma^{ab}_{-\alpha'} \overset{P}{=} -\sigma^{A'B'}_{\alpha'}\varepsilon^{AB} \tag{1.275}$$

$$\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\sigma^{\alpha_\varsigma}_{\varsigma ab}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma} = \varsigma\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma} \qquad \sigma^{\alpha}_{+ab} \overset{P}{=} \sigma^{\alpha}_{AB}\varepsilon_{A'B'} \tag{1.276}$$

$$\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma} = -\varsigma\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\varepsilon_{A_\varsigma B_\varsigma} \qquad \sigma^{\alpha'}_{-ab} \overset{P}{=} -\sigma^{\alpha'}_{A'B'}\varepsilon_{AB} \tag{1.277}$$

Connection 3:

$$(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_{[a}(\sigma,i\varsigma)_{b]A_\varsigma B'_\varsigma} = -2\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\alpha'_\varsigma}{}^{A'_\varsigma}{}_{B'_\varsigma} \qquad (\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_{\{a}(\sigma,i\varsigma)_{b\}A_\varsigma B'_\varsigma} = 2\delta_{ab}\delta^{A'_\varsigma}{}_{B'_\varsigma} \tag{1.278}$$

$$(\sigma,i\varsigma)_{[aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)^{A'_\varsigma B_\varsigma}_{b]} = -2\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\alpha'_\varsigma A'_\varsigma}{}^{B'_\varsigma} \qquad (\sigma,i\varsigma)_{\{aA_\varsigma A'_\varsigma}(\sigma,-i\varsigma)^{A'_\varsigma B_\varsigma}_{b\}} = 2\delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma} \tag{1.279}$$

6.6 Properties of spin constant invariant tensors $S_{ab}{}^{A_\varsigma B_\varsigma}, S^{ab}{}_{A'_\varsigma B'_\varsigma}, S^{ab}{}_{A_\varsigma B_\varsigma}, S_{ab}{}^{A'_\varsigma B'_\varsigma}$

6.6.1 Definition

$$S_{ab}{}^{A_\varsigma B_\varsigma} = \tfrac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma} = \tfrac{i}{2}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma} \tag{1.280}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} = \tfrac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma} = \tfrac{i}{2}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma} \tag{1.281}$$

6.6.2 Symmetry and antisymmetry

$$S_{ab}{}^{A_\varsigma B_\varsigma} = S_{ab}{}^{B_\varsigma A_\varsigma} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma} = S^{ab}{}_{B'_\varsigma A'_\varsigma} \tag{1.282}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} = S^{ab}{}_{B_\varsigma A_\varsigma} \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma} = S_{ab}{}^{B'_\varsigma A'_\varsigma} \tag{1.283}$$

$$S_{ab}{}^{A'_\varsigma C'_\varsigma}S_{cdC'_\varsigma B'_\varsigma} = -S_{ab}{}^{A'_\varsigma}{}_{C'_\varsigma}S_{cd}{}^{C'_\varsigma}{}_{B'_\varsigma} \qquad S_{abA_\varsigma C_\varsigma}S_{cd}{}^{C_\varsigma B_\varsigma} = -S_{abA_\varsigma}{}^{C_\varsigma}S_{cdC_\varsigma}{}^{B_\varsigma} \tag{1.284}$$

6.6.3 Duality

$$S_{ab}{}^{A_\varsigma B_\varsigma} = -\varsigma * S_{ab}{}^{A_\varsigma B_\varsigma} \qquad S^{ab}{}_{A'_\varsigma B'_\varsigma} = \varsigma * S^{ab}{}_{A'_\varsigma B'_\varsigma} \tag{1.285}$$

$$S^{ab}{}_{A_\varsigma B_\varsigma} = -\varsigma * S^{ab}{}_{A_\varsigma B_\varsigma} \qquad S_{ab}{}^{A'_\varsigma B'_\varsigma} = \varsigma * S_{ab}{}^{A'_\varsigma B'_\varsigma} \tag{1.286}$$

6.6.4 Complex conjugation

$$[S_{ab}{}^{A_\varsigma B_\varsigma}\partial^a\hat{\partial}^b]^* = S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^a\hat{\partial}^b \qquad [S^{ab}{}_{A_\varsigma B_\varsigma}\partial_a\hat{\partial}_b]^* = S^{ab}{}_{A'_\varsigma B'_\varsigma}\partial_a\hat{\partial}_b \tag{1.287}$$

6.7 Important relations between invariant constant spin tensors

6.7.1 Unified relations between invariant constant spin tensors

$$\begin{cases} S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma} = -\frac{i}{4}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_{[a}\delta_{A_\varsigma}{}^{B_\varsigma}(\sigma,i\varsigma)_{b]B_\varsigma B'_\varsigma} \\ \delta_{ab}\delta^{A'_\varsigma}{}_{B'_\varsigma} = \frac{1}{2}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_{\{a}\delta_{A_\varsigma}{}^{B_\varsigma}(\sigma,i\varsigma)_{b\}B_\varsigma B'_\varsigma} \end{cases} \quad \begin{cases} S_{abA_\varsigma}{}^{B_\varsigma} = -\frac{i}{4}(\sigma,i\varsigma)_{[aA_\varsigma A'_\varsigma}\delta^{A'_\varsigma}{}_{B'_\varsigma}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_{b]} \\ \delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma} = \frac{1}{2}(\sigma,i\varsigma)_{\{aA_\varsigma A'_\varsigma}\delta^{A'_\varsigma}{}_{B'_\varsigma}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_{b\}} \end{cases} \tag{1.288}$$

$$\begin{cases} S_{ab}{}^{A'_\varsigma B'_\varsigma} = -\frac{i\varsigma}{4}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_{[a}\varepsilon_{A_\varsigma B_\varsigma}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_{b]} \\ \delta_{ab}\varepsilon^{A'_\varsigma B'_\varsigma} = -\frac{1}{2}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_{\{a}\varepsilon_{A_\varsigma B_\varsigma}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_{b\}} \end{cases} \quad \begin{cases} S^{ab}{}_{A_\varsigma B_\varsigma} = \frac{i\varsigma}{4}(\sigma,i\varsigma)^{[a}_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma}(\sigma,i\varsigma)^{b]}_{B_\varsigma B'_\varsigma} \\ \delta^{ab}\varepsilon_{A_\varsigma B_\varsigma} = -\frac{1}{2}(\sigma,i\varsigma)^{\{a}_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma}(\sigma,i\varsigma)^{b\}}_{B_\varsigma B'_\varsigma} \end{cases} \tag{1.289}$$

$$\begin{cases} S^{ab}{}_{A'_\varsigma B'_\varsigma} = -\frac{i\varsigma}{4}(\sigma, i\varsigma)^{[a}_{A_\varsigma A'_\varsigma}\varepsilon^{A_\varsigma B_\varsigma}(\sigma, i\varsigma)^{b]}_{B_\varsigma B'_\varsigma} \\ \delta_{ab}\varepsilon_{A'_\varsigma B'_\varsigma} = -\frac{1}{2}(\sigma, i\varsigma)^{\{a}_{A_\varsigma A'_\varsigma}\varepsilon^{A_\varsigma B_\varsigma}(\sigma, i\varsigma)^{b\}}_{B_\varsigma B'_\varsigma} \end{cases} \begin{cases} S_{ab}{}^{A_\varsigma B_\varsigma} = \frac{i\varsigma}{4}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_{[a}\varepsilon_{A'_\varsigma B'_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma B_\varsigma}_{b]} \\ \delta_{ab}\varepsilon^{A_\varsigma B_\varsigma} = -\frac{1}{2}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_{\{a}\varepsilon_{A'_\varsigma B'_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma B_\varsigma}_{b\}} \end{cases} \quad (1.290)$$

6.7.2 Product relation $S^{ac} \otimes S_{bc}$

Cor. 6.7.1. $\begin{cases} S^{ac}{}_{A_\varsigma C_\varsigma}\delta^d_c S_{bd}{}^{B_\varsigma D_\varsigma} = -\frac{1}{8}(\delta_{ab} + 2iS_{ab})_{\{A_\varsigma}{}^{(B_\varsigma}\delta_{C_\varsigma\}}{}^{D_\varsigma)} \\ S_{ac}{}^{A'_\varsigma C'_\varsigma}\delta^c_d S^{bd}{}_{B'_\varsigma D'_\varsigma} = -\frac{1}{8}(\delta_{ab} + 2iS_{ab})^{\{A'_\varsigma}{}_{(B'_\varsigma}\delta^{C'_\varsigma\}}{}_{D'_\varsigma)} \end{cases}$

Proof: $S^{ac}{}_{A_\varsigma C_\varsigma}\delta^d_c S_{bd}{}^{B_\varsigma D_\varsigma}$
$= \frac{i\varsigma}{4}(\sigma, i\varsigma)^{[a}_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\varsigma)^{c]}_{C_\varsigma C'_\varsigma}\delta^d_c \frac{i\varsigma}{4}(\sigma, -i\varsigma)^{B'_\varsigma B_\varsigma}_{[b}\varepsilon_{B'_\varsigma D'_\varsigma}(\sigma, -i\varsigma)^{D'_\varsigma D_\varsigma}_{d]}$
$= -\frac{1}{8}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma B_\varsigma}_b\varepsilon_{B'_\varsigma D'_\varsigma}\delta_{C_\varsigma}{}^{D_\varsigma}\delta_{C'_\varsigma}{}^{D'_\varsigma} + \cdots$
$= -\frac{1}{8}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma, -i\varsigma)^{A'_\varsigma B_\varsigma}_b\delta^{D_\varsigma}_{C_\varsigma} + \cdots$
$= -\frac{1}{8}(\delta_{ab} + 2iS_{ab})_{A_\varsigma}{}^{B_\varsigma}\delta^{D_\varsigma}_{C_\varsigma} - \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{C_\varsigma}{}^{B_\varsigma}\delta^{D_\varsigma}_{A_\varsigma} - \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{A_\varsigma}{}^{D_\varsigma}\delta^{B_\varsigma}_{C_\varsigma} - \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{C_\varsigma}{}^{D_\varsigma}\delta^{B_\varsigma}_{A_\varsigma}$
$= -\frac{1}{8}(\delta_{ab} + 2iS_{ab})_{\{A_\varsigma}{}^{(B_\varsigma}\delta_{C_\varsigma\}}{}^{D_\varsigma)}$ □

Cor. 6.7.2. $\begin{cases} S^{ac}{}_{A_\varsigma C_\varsigma}\delta_{cd}S^{bd}{}_{B'_\varsigma D'_\varsigma} = \frac{1}{8}(\sigma, i\varsigma)^a_{\{A_\varsigma(B'_\varsigma}(\sigma, i\varsigma)^b_{C_\varsigma\}D'_\varsigma)} \\ S_{ac}{}^{A'_\varsigma C'_\varsigma}\delta^{cd}S_{bd}{}^{B_\varsigma D_\varsigma} = \frac{1}{8}(\sigma, i\varsigma)^{\{A'_\varsigma(B_\varsigma}_a(\sigma, i\varsigma)^{C'_\varsigma\}D_\varsigma)}_b \end{cases}$

Proof: $S^{ac}{}_{A_\varsigma C_\varsigma}\delta_{cd}S^{bd}{}_{B'_\varsigma D'_\varsigma}$
$= \frac{i\varsigma}{4}(\sigma, i\varsigma)^{[a}_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\varsigma)^{c]}_{C_\varsigma C'_\varsigma}\delta_{cd}(-\frac{i\varsigma}{4})(\sigma, i\varsigma)^{[b}_{B_\varsigma B'_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}(\sigma, i\varsigma)^{d]}_{D_\varsigma D'_\varsigma}$
$= -\frac{1}{8}\varepsilon_{C_\varsigma D_\varsigma}\varepsilon_{C'_\varsigma D'_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma B'_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} + \frac{1}{8}\varepsilon_{A_\varsigma D_\varsigma}\varepsilon_{A'_\varsigma D'_\varsigma}(\sigma, i\varsigma)^a_{C_\varsigma C'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma B'_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}$
$+ \frac{1}{8}\varepsilon_{C_\varsigma B_\varsigma}\varepsilon_{C'_\varsigma B'_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b_{D_\varsigma D'_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} - \frac{1}{8}\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}(\sigma, i\varsigma)^a_{C_\varsigma C'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b_{D_\varsigma D'_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}$
$= \frac{1}{8}(\sigma, i\varsigma)^a_{A_\varsigma D'_\varsigma}(\sigma, i\varsigma)^b_{C_\varsigma B'_\varsigma} + \frac{1}{8}(\sigma, i\varsigma)^a_{C_\varsigma D'_\varsigma}(\sigma, i\varsigma)^b_{A_\varsigma B'_\varsigma} + \frac{1}{8}(\sigma, i\varsigma)^a_{A_\varsigma B'_\varsigma}(\sigma, i\varsigma)^b_{C_\varsigma D'_\varsigma} + \frac{1}{8}(\sigma, i\varsigma)^a_{C_\varsigma B'_\varsigma}(\sigma, i\varsigma)^b_{A_\varsigma D'_\varsigma}$
$= \frac{1}{8}(\sigma, i\varsigma)^a_{\{A_\varsigma(B'_\varsigma}(\sigma, i\varsigma)^b_{C_\varsigma\}D'_\varsigma)}$ □

Cor. 6.7.3. $S^{ac}{}_{A_\varsigma B_\varsigma}\delta_{cd}S^{bd}{}_{A'_\varsigma B'_\varsigma} = \frac{1}{8}(\sigma, i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma\}B'_\varsigma)}, S_{ac}{}^{A'_\varsigma B'_\varsigma}\delta^{cd}S_{bd}{}^{A_\varsigma B_\varsigma} = \frac{1}{8}(\sigma, -i\varsigma)^{\{A'_\varsigma(A_\varsigma}_a(\sigma, -i\varsigma)^{B'_\varsigma\}B_\varsigma)}_b$

Cor. 6.7.4. $S^{ac}{}_{A_\varsigma C_\varsigma}\delta^d_c S_{bd}{}^{B_\varsigma D_\varsigma}\partial_a\partial^b = \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{\{A_\varsigma}{}^{(B_\varsigma}\delta_{C_\varsigma\}}{}^{D_\varsigma)}\partial^a\partial^b = \frac{1}{8}\delta^{(B_\varsigma}_{\{A_\varsigma}\delta^{D_\varsigma)}_{C_\varsigma\}}\partial^a\partial_a$

Cor. 6.7.5. $S^{ac}{}_{A_\varsigma C_\varsigma}\delta^d_c S_{bd}{}^{B_\varsigma D_\varsigma}\partial_a\partial^b\Delta(x - x') = \frac{1}{8}m^2\delta^{(B_\varsigma}_{\{A_\varsigma}\delta^{D_\varsigma)}_{C_\varsigma\}}\Delta(x - x')$

6.7.3 Product relation $S_{ab}\partial^b \otimes [\sigma_y()]^a, [()\sigma_y]^a \otimes S_{ab}\partial^b$

Cor. 6.7.6.

$$\begin{cases} S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b\delta_{aa'}[\sigma_y(\sigma, -i\varsigma)]^{a'}{}_{B'_\varsigma}{}^{C_\varsigma} = -\frac{\varsigma}{2}\delta^{C_\varsigma}_{\{A_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma\}B'_\varsigma}\partial_b \\ S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\delta^{aa'}[\sigma_y(\sigma, i\varsigma)]_{a'}{}^{B_\varsigma}{}_{C'_\varsigma} = \frac{\varsigma}{2}\delta^{\{A'_\varsigma}_{C'_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma\}B_\varsigma}_b\partial^b \end{cases}$$

$$\begin{cases} S^{ab}{}_{A'_\varsigma B'_\varsigma}\partial_b\delta_{aa'}[(\sigma, -i\varsigma)\sigma_y]^{a'C'_\varsigma}{}_{B_\varsigma} = -\frac{\varsigma}{2}\delta^{C'_\varsigma}_{\{A'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma B'_\varsigma\}}\partial_b \\ S_{ab}{}^{A_\varsigma B_\varsigma}\partial^b\delta^{aa'}[(\sigma, i\varsigma)\sigma_y]_{a'C_\varsigma}{}^{B'_\varsigma} = \frac{\varsigma}{2}\delta^{\{A_\varsigma}_{C_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma B_\varsigma\}}_b\partial^b \end{cases}$$

Proof: $S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b\delta_{aa'}[\sigma_y(\sigma, -i\varsigma)]^{a'}{}_{B'_\varsigma}{}^{C_\varsigma}$
$= \frac{i\varsigma}{4}(\sigma, i\varsigma)^{[a}_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma, i\varsigma)^{b]}_{B_\varsigma C'_\varsigma}\partial_b\delta_{aa'}\sigma^y_{B'_\varsigma D'_\varsigma}(\sigma, -i\varsigma)^{D'_\varsigma C_\varsigma}_{a'}$
$= \frac{\varsigma}{2}\delta^{C_\varsigma}_{A_\varsigma}\delta^{D'_\varsigma}_{A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}\varepsilon_{B'_\varsigma D'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma C'_\varsigma}\partial_b + \cdots$
$= -\frac{\varsigma}{2}\delta^{C_\varsigma}_{A_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b - \frac{\varsigma}{2}\delta^{C_\varsigma}_{B_\varsigma}(\sigma, i\varsigma)^b_{A_\varsigma B'_\varsigma}\partial_b$
$= -\frac{\varsigma}{2}\delta^{C_\varsigma}_{\{A_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma\}B'_\varsigma}\partial_b$ □

Proof: $S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\delta^{aa'}[\sigma_y(\sigma, i\varsigma)]_{a'}{}^{B_\varsigma}{}_{C'_\varsigma}$
$= -\frac{i\varsigma}{4}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_{[a}\varepsilon_{A_\varsigma C_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma C_\varsigma}_{b]}\partial^b\delta^a_{a'}\sigma^{B_\varsigma D_\varsigma}_y(\sigma, i\varsigma)^{a'}_{D_\varsigma C'_\varsigma}$
$= -\frac{\varsigma}{4}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_{[a}\varepsilon_{A_\varsigma C_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma C_\varsigma}_{b]}\partial^b\delta^a_{a'}\varepsilon^{B_\varsigma D_\varsigma}(\sigma, i\varsigma)^{a'}_{D_\varsigma C'_\varsigma}$
$= -\frac{\varsigma}{2}\delta^{A'_\varsigma}_{C'_\varsigma}\delta^{A_\varsigma}_{D_\varsigma}\varepsilon_{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma C_\varsigma}_b\partial^b + \cdots$
$= \frac{\varsigma}{2}\delta^{\{A'_\varsigma}_{C'_\varsigma}(\sigma, -i\varsigma)^{B'_\varsigma\}B_\varsigma}_b\partial^b$ □

Cor. 6.7.7.
$$\begin{cases} S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b\delta^{a'}_a[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{C_\varsigma}{}_{B'_\varsigma} = -\frac{\varsigma}{2}\delta^{C_\varsigma}_{\{A_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma\}B'_\varsigma}\partial_b \\ S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\delta^a_{a'}[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{C'_\varsigma}{}^{B_\varsigma} = \frac{\varsigma}{2}\delta^{\{A'_\varsigma}_{C'_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma\}B_\varsigma}\partial^b \end{cases}$$
$$\begin{cases} S^{ab}{}_{A'_\varsigma B'_\varsigma}\partial_b\delta^{a'}_a[(\sigma,i\varsigma)\sigma_y]_{a'B_\varsigma}{}^{C'_\varsigma} = -\frac{\varsigma}{2}\delta^{C'_\varsigma}_{\{A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma\}}\partial_b \\ S_{ab}{}^{A_\varsigma B_\varsigma}\partial^b\delta^a_{a'}[(\sigma,-i\varsigma)\sigma_y]^{a'B'_\varsigma}{}_{C_\varsigma} = \frac{\varsigma}{2}\delta^{\{A_\varsigma}_{C_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma\}}\partial^b \end{cases}$$

Proof: $S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b\delta^{a'}_a[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{C_\varsigma}{}_{B'_\varsigma}$
$= \frac{i\varsigma}{4}(\sigma,i\varsigma)^{[a}_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma,i\varsigma)^{b]}_{B_\varsigma C'_\varsigma}\partial_b\delta_{aa'}\sigma_y^{C_\varsigma D_\varsigma}(\sigma,i\varsigma)^{a'}_{D_\varsigma B'_\varsigma}$
$= \frac{\varsigma}{4}(\sigma,i\varsigma)^{[a}_{A_\varsigma A'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}(\sigma,i\varsigma)^{b]}_{B_\varsigma C'_\varsigma}\partial_b\delta_{aa'}\varepsilon^{C_\varsigma D_\varsigma}(\sigma,i\varsigma)^{a'}_{D_\varsigma B'_\varsigma}$
$= -\frac{\varsigma}{2}\varepsilon_{A_\varsigma D_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}\varepsilon^{A'_\varsigma C'_\varsigma}\varepsilon^{C_\varsigma D_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma C'_\varsigma}\partial_b + \cdots$
$= -\frac{\varsigma}{2}\delta^{C_\varsigma}_{\{A_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma\}B'_\varsigma}\partial_b$ □

Proof: $S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\delta^a_{a'}[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{C'_\varsigma}{}^{B_\varsigma}$
$= -\frac{i\varsigma}{4}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_{[a}\varepsilon_{A_\varsigma C_\varsigma}(\sigma,-i\varsigma)^{B'_\varsigma C_\varsigma}_{b]}\partial^b\delta^{aa'}\sigma^y_{C'_\varsigma D'_\varsigma}(\sigma,-i\varsigma)^{D'_\varsigma B_\varsigma}_{a'}$
$= \frac{\varsigma}{2}\varepsilon^{A'_\varsigma D'_\varsigma}\varepsilon^{A_\varsigma B_\varsigma}\varepsilon_{A_\varsigma C_\varsigma}\varepsilon_{C'_\varsigma D'_\varsigma}(\sigma,-i\varsigma)^{B'_\varsigma C_\varsigma}_b\partial^b + \cdots$
$= \frac{\varsigma}{2}\delta^{\{A'_\varsigma}_{C'_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma\}B_\varsigma}\partial^b$ □

6.7.4 Product relation $[()\sigma_y]^a \otimes [\sigma_y()]_a$

Cor. 6.7.8. $\begin{cases} [(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\delta_{aa'}[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{A'_\varsigma}{}^{B_\varsigma} = 2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma} \\ [(\sigma,-i\varsigma)\sigma_y]_a{}^{A'_\varsigma}{}_{B_\varsigma}\delta^{aa'}[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{A_\varsigma}{}_{B'_\varsigma} = 2\delta^{A'_\varsigma}_{B'_\varsigma}\delta^{A_\varsigma}_{B_\varsigma} \end{cases}$

Proof: $[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\delta_{aa'}[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{A'_\varsigma}{}^{B_\varsigma}$
$= (\sigma,i\varsigma)^a_{A_\varsigma C'_\varsigma}\sigma_y^{C'_\varsigma B'_\varsigma}\delta^{a'}_a\sigma^y_{A'_\varsigma D'_\varsigma}(\sigma,-i\varsigma)^{D'_\varsigma B_\varsigma}_{a'}$
$= -2\delta^{B_\varsigma}_{A_\varsigma}\delta^{D'_\varsigma}_{C'_\varsigma}\varepsilon^{C'_\varsigma B'_\varsigma}\varepsilon_{A'_\varsigma D'_\varsigma}$
$= 2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}$ □

Cor. 6.7.9. $\begin{cases} [(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\delta^{a'}_a[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{B_\varsigma}{}_{A'_\varsigma} = 2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma} \\ [(\sigma,-i\varsigma)\sigma_y]_a{}^{A'_\varsigma}{}_{B_\varsigma}\delta^a_{a'}[\sigma_y(\sigma,i\varsigma)]^{a'}{}_{B'_\varsigma}{}^{A_\varsigma} = 2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma} \end{cases}$

Proof: $[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\delta^{a'}_a[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{B_\varsigma}{}_{A'_\varsigma}$
$= (\sigma,i\varsigma)^a_{A_\varsigma C'_\varsigma}\sigma_y^{C'_\varsigma B'_\varsigma}\delta_{aa'}\sigma_y^{B_\varsigma D_\varsigma}(\sigma,i\varsigma)^{a'}_{D_\varsigma A'_\varsigma}$
$= 2\varepsilon_{A_\varsigma D_\varsigma}\varepsilon_{C'_\varsigma A'_\varsigma}\varepsilon^{C'_\varsigma B'_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}$
$= 2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}$ □

6.7.5 Detailed proof of important theorems

Lem. 6.7.1. $[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\partial_a[\sigma_y(\sigma,-i\varsigma)]^b{}_{A'_\varsigma}{}^{B_\varsigma}\partial_b + (\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}\partial^b$
$= (i\sigma_z,I,-i\sigma_x,i\varsigma\sigma_y)^a{}_{A_\varsigma}{}^{B'_\varsigma}\partial_a(-i\sigma_z,I,i\sigma_x,-i\varsigma\sigma_y)^b{}_{A'_\varsigma}{}^{B_\varsigma}\partial_b + (\sigma_x,\sigma_y,\sigma_z,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma_x,\sigma_y,\sigma_z,-i\varsigma)_b^{B'_\varsigma B_\varsigma}\partial^b$

Thm. 6.7.1. $[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\partial_a[\sigma_y(\sigma,-i\varsigma)]^b{}_{A'_\varsigma}{}^{B_\varsigma}\partial_b + (\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}\partial^b = \partial^a\partial_a\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}$

Proof: $(i\sigma_z,I,-i\sigma_x,i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z,I,i\sigma_x,-i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x,\sigma_y,\sigma_z,i\varsigma)^a_{1_\varsigma 1'_\varsigma}\partial_a(\sigma_x,\sigma_y,\sigma_z,-i\varsigma)_b^{1'_\varsigma 1_\varsigma}\partial^b$
$= (i\partial_x+\partial_y)(-i\partial_x+\partial_y) + (\partial_z+i\varsigma\partial_\pi)(\partial_z-i\varsigma\partial_\pi)$
$= (\partial_x^2+\partial_y^2) + (\partial_z^2+\partial_\pi^2)$
$= \partial^a\partial_a\delta^{1_\varsigma}_{1_\varsigma}\delta^{1'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z,I,-i\sigma_x,i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z,I,i\sigma_x,-i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x,\sigma_y,\sigma_z,i\varsigma)^a_{1_\varsigma 1'_\varsigma}\partial_a(\sigma_x,\sigma_y,\sigma_z,-i\varsigma)_b^{1'_\varsigma 2_\varsigma}\partial^b$
$= (i\partial_x+\partial_y)(i\partial_z-\varsigma\partial_\pi) + (\partial_z+i\varsigma\partial_\pi)(\partial_x-i\partial_y)$
$= 0$
$= \partial^a\partial_a\delta^{2_\varsigma}_{1_\varsigma}\delta^{1'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{1_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{1'_\varsigma 1_\varsigma}\partial^b$
$= (i\partial_x + \partial_y)(i\partial_z + \varsigma\partial_\pi) + (\partial_x - i\partial_y)(\partial_z - i\varsigma\partial_\pi)$
$= 0$
$= \partial^a\partial_a\delta^{1_\varsigma}_{1_\varsigma}\delta^{2'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{1_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{1'_\varsigma 2_\varsigma}\partial^b$
$= (i\partial_x + \partial_y)(i\partial_x + \partial_y) + (\partial_x - i\partial_y)(\partial_x - i\partial_y)$
$= 0$
$= \partial^a\partial_a\delta^{2_\varsigma}_{1_\varsigma}\delta^{1'_\varsigma}_{2'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{1_\varsigma 1'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 1_\varsigma}\partial^b$
$= (-i\partial_z + \varsigma\partial_\pi)(-i\partial_x + \partial_y) + (\partial_z + i\varsigma\partial_\pi)(\partial_x + i\partial_y)$
$= 0$
$= \partial^a\partial_a\delta^{1_\varsigma}_{1_\varsigma}\delta^{2'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{1_\varsigma 1'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 2_\varsigma}\partial^b$
$= (-i\partial_z + \varsigma\partial_\pi)(i\partial_z - \varsigma\partial_\pi) + (\partial_z + i\varsigma\partial_\pi)(-\partial_z - i\varsigma\partial_\pi)$
$= 0$
$= \partial^a\partial_a\delta^{2_\varsigma}_{1_\varsigma}\delta^{2'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{1_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 1_\varsigma}\partial^b$
$= (-i\partial_z + \varsigma\partial_\pi)(i\partial_z + \varsigma\partial_\pi) + (\partial_x - i\partial_y)(\partial_x - i\partial_y)$
$= (\partial_z^2 + \partial_\pi^2) + (\partial_x^2 + \partial_y^2)$
$= \partial^a\partial_a\delta^{1_\varsigma}_{1_\varsigma}\delta^{2'_\varsigma}_{2'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{1_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{1_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 2_\varsigma}\partial^b$
$= (-i\partial_z + \varsigma\partial_\pi)(i\partial_x + \partial_y) + (\partial_x - i\partial_y)(-\partial_z - i\varsigma\partial_\pi)$
$= 0$
$= \partial^a\partial_a\delta^{2_\varsigma}_{1_\varsigma}\delta^{2'_\varsigma}_{2'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 1'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{1'_\varsigma 1_\varsigma}\partial^b$
$= (-i\partial_z - \varsigma\partial_\pi)(-i\partial_x + \partial_y) + (\partial_x + i\partial_y)(\partial_z - i\varsigma\partial_\pi)$
$= 0$
$= \partial^a\partial_a\delta^{1_\varsigma}_{2_\varsigma}\delta^{1'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 1'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{1'_\varsigma 2_\varsigma}\partial^b$
$= (-i\partial_z - \varsigma\partial_\pi)(i\partial_z - \varsigma\partial_\pi) + (\partial_x + i\partial_y)(\partial_x - i\partial_y)$
$= (\partial_z^2 + \partial_\pi^2) + (\partial_x^2 + \partial_y^2)$
$= \partial^a\partial_a\delta^{2_\varsigma}_{2_\varsigma}\delta^{1'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{1'_\varsigma 1_\varsigma}\partial^b$
$= (-i\partial_z - \varsigma\partial_\pi)(i\partial_z + \varsigma\partial_\pi) + (-\partial_z + i\varsigma\partial_\pi)(\partial_z - i\varsigma\partial_\pi)$
$= 0$
$= \partial^a\partial_a\delta^{1_\varsigma}_{2_\varsigma}\delta^{1'_\varsigma}_{2'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{1'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{1'_\varsigma 2_\varsigma}\partial^b$
$= (-i\partial_z - \varsigma\partial_\pi)(i\partial_x + \partial_y) + (-\partial_z + i\varsigma\partial_\pi)(\partial_x - i\partial_y)$
$= 0$
$= \partial^a\partial_a\delta^{2_\varsigma}_{2_\varsigma}\delta^{1'_\varsigma}_{2'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 1'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 1_\varsigma}\partial^b$
$= (-i\partial_x + \partial_y)(-i\partial_x + \partial_y) + (\partial_x + i\partial_y)(\partial_x + i\varsigma\partial_y)$
$= 0$
$= \partial^a\partial_a\delta^{1_\varsigma}_{2_\varsigma}\delta^{2'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{1'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 1'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 2_\varsigma}\partial^b$
$= (-i\partial_x + \partial_y)(i\partial_z - \varsigma\partial_\pi) + (\partial_x + i\partial_y)(-\partial_z - i\varsigma\partial_\pi)$
$= 0$
$= \partial^a\partial_a\delta^{2_\varsigma}_{2_\varsigma}\delta^{2'_\varsigma}_{1'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{1_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 1_\varsigma}\partial^b$
$= (-i\partial_x + \partial_y)(i\partial_z + \varsigma\partial_\pi) + (-\partial_z + i\varsigma\partial_\pi)(\partial_x + i\partial_y)$
$= 0$
$= \partial^a\partial_a\delta^{1_\varsigma}_{2_\varsigma}\delta^{2'_\varsigma}_{2'_\varsigma}$ □

Proof: $(i\sigma_z, I, -i\sigma_x, i\varsigma\sigma_y)^a{}_{2_\varsigma}{}^{2'_\varsigma}\partial_a(-i\sigma_z, I, i\sigma_x, -i\varsigma\sigma_y)^b{}_{2'_\varsigma}{}^{2_\varsigma}\partial_b + (\sigma_x, \sigma_y, \sigma_z, i\varsigma)^a_{2_\varsigma 2'_\varsigma}\partial_a(\sigma_x, \sigma_y, \sigma_z, -i\varsigma)_b^{2'_\varsigma 2_\varsigma}\partial^b$
$= (-i\partial_x + \partial_y)(i\partial_x + \partial_y) + (-\partial_z + i\varsigma\partial_\pi)(-\partial_z - i\varsigma\partial_\pi)$
$= (\partial_x^2 + \partial_y^2) + (\partial_z^2 + \partial_\pi^2)$
$= \partial^a\partial_a\delta^{2_\varsigma}_{2_\varsigma}\delta^{2'_\varsigma}_{2'_\varsigma}$ □

Thm. 6.7.2. $(\sigma, i\varsigma)^a_{[A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma]B'_\varsigma}\partial_a\partial_b = -\partial^a\partial_a\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}, (\sigma, -i\varsigma)_a^{[A'_\varsigma A_\varsigma}(\sigma, -i\varsigma)_b^{B'_\varsigma]B_\varsigma}\partial^a\partial^b = -\partial^a\partial_a\varepsilon^{A'_\varsigma B'_\varsigma}\varepsilon^{A_\varsigma B_\varsigma}$

Thm. 6.7.3. $(\sigma, i\varsigma)^a_{[A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma]B'_\varsigma}\delta_{ab} = -\delta^a{}_a\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}, (\sigma, -i\varsigma)_a^{[A'_\varsigma A_\varsigma}(\sigma, -i\varsigma)_b^{B'_\varsigma]B_\varsigma}\delta^{ab} = -\delta^a{}_a\varepsilon^{A'_\varsigma B'_\varsigma}\varepsilon^{A_\varsigma B_\varsigma}$

Thm. 6.7.4. $[(\sigma, i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}[\sigma_y(\sigma, -i\varsigma)]^b{}_{A'_\varsigma}{}^{B_\varsigma}\partial_a\partial_b + (\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma, -i\varsigma)_b^{B'_\varsigma B_\varsigma}\partial_a\partial^b = \partial^a\partial_a\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}$

Thm. 6.7.5. $[(\sigma, i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}[\sigma_y(\sigma, -i\varsigma)]^b{}_{A'_\varsigma}{}^{B_\varsigma}\delta^{ab} + (\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma, -i\varsigma)_b^{B'_\varsigma B_\varsigma}\delta_a{}^b = \delta^a{}_a\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}$

6.8 Properties of composite constant invariant tensor $\Sigma^{k_s l_s}_{k_{s'} l_{s'}}(s, s')$

6.8.1 Definition

$$c(s) = [(-1)^{2s}\tfrac{8}{3}s(s + \tfrac{1}{2})(s + 1)]^{-\frac{1}{2}} \tag{1.291}$$

$$\Sigma^{k_s l_s}_{ab}(s) := \begin{cases} c(|s|)S_{ab}{}^{kl}(|s|, +), s > 0 \\ c(|s|)S_{abk'l'}(|s|, -), s < 0 \end{cases} \qquad \Sigma^{ab}_{k_s l_s}(s) := \begin{cases} c(|s|)S^{ab}{}_{kl}(|s|, +), s > 0 \\ c(|s|)S^{abk'l'}(|s|, -), s < 0 \end{cases} \tag{1.292}$$

: $\Sigma^{k_s l_s}_{k_{s'} l_{s'}}(s, s') := \Sigma^{k_s l_s}_{ab}(s)\Sigma^{ab}_{k_{s'} l_{s'}}(s')$ (1.293)

6.8.2 Transitivity

$$\Sigma^{k_s l_s}_{k_{s'} l_{s'}}(s, s')\Sigma^{k_{s'} l_{s'}}_{k_{s''} l_{s''}}(s', s'') = \Sigma^{k_s l_s}_{k_{s''} l_{s''}}(s, s'') \tag{1.294}$$

6.8.3 Symmetry and antisymmetry

$$\Sigma^{k_s l_s}_{k_{s'} l_{s'}}(s, s') = (-1)^{2s+1}\Sigma^{l_s k_s}_{k_{s'} l_{s'}}(s, s') \qquad \Sigma^{l_s k_s}_{k_{s'} l_{s'}}(s, s') = (-1)^{2s'+1}\Sigma^{k_s l_s}_{l_{s'} k_{s'}}(s, s') \tag{1.295}$$

$$\Sigma^{k_s l_s}_{k_{s'} l_{s'}}(s, s') = (-1)^{2(s+s')}\Sigma^{l_s k_s}_{l_{s'} k_{s'}}(s, s') \tag{1.296}$$

7 General theory of constant invariant tensors

7.1 General definition of constant invariant tensors

Define the Lorentz transformation: $\Lambda[L_i] := e^{\frac{i}{2}\vartheta^{ab}S_{ab}[L_i]}$, (1.297)

Define the YM field transformation: $\Lambda[Y_j] := e^{i\theta^\alpha T_\alpha[Y_j]}$ (1.298)

General definition of constant invariant tensors:

$C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n}$ **is constant and equal in any reference system and satisfies the transformation:**

$$C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = \prod_{i=1}^{n} \Lambda^{L_i'}_{L_i}[L_i] \prod_{j=1}^{m} \Lambda^{Y_j}_{Y_j'}[Y_j] C^{Y_1'Y_2'\cdots Y_m'}_{L_1'L_2'\cdots L_n'} \tag{1.299}$$

Then $C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n}$ **is a constant invariant tensor.** $L_i \sim e^{\frac{i}{2}\vartheta^{ab}S_{ab}[L_i]}, Y_j \sim e^{i\theta^\alpha T_\alpha[Y_j]}$

Infinitesimal transformation:

$$0 = \delta C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = \tfrac{1}{2}\vartheta^{ab}\sum_{i=1}^{n} S_{ab}{}^{L_i'}_{L_i}[L_i] C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_i'\cdots L_n} + i\theta^\alpha \sum_{j=1}^{m} T_\alpha{}^{Y_j}_{Y_j'}[L_i] C^{Y_1Y_2\cdots Y_j'\cdots Y_m}_{L_1L_2\cdots L_n}, \forall\vartheta^{ab}, \forall\theta^\alpha \tag{1.300}$$

$$\Leftrightarrow \sum_{i=1}^{n} S_{ab}{}^{L_i'}_{L_i}[L_i] C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_i'\cdots L_n} = 0, \sum_{j=1}^{m} T_\alpha{}^{Y_j}_{Y_j'}[L_i] C^{Y_1Y_2\cdots Y_j'\cdots Y_m}_{L_1L_2\cdots L_n} = 0 \tag{1.301}$$

7.2 Covariant derivatives of constant invariant tensors are zero

$$D_u C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = \partial_u C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} + \tfrac{1}{2}\omega_u{}^{ab}\sum_{i=1}^{n} S_{ab}{}^{L_i'}_{L_i}[L_i] C^{Y_1Y_2\cdots Y_m}_{L_1\cdots L_i'\cdots L_n} + iA_u{}^\alpha \sum_{j=1}^{m} T_\alpha{}^{Y_j}_{Y_j'}[L_i] C^{Y_1\cdots Y_j'\cdots Y_m}_{L_1L_2\cdots L_n} \tag{1.302}$$

$$D_u C^{Y_1Y_2\cdots Y_m}_{L_1L_2\cdots L_n} = 0 + 0 + 0 = 0 \tag{1.303}$$

Therefore, the covariant derivatives of all constant invariant tensors are all zero, which is a very good and convenient property.

Chapter2 Perfect Constant Invariant Tensors

Self comment: The perfect constant invariant tensors created in this chapter have great universality and can be applicable to various situations. It associates a fully symmetric low spin tensor with a high spin tensor. It is a powerful mathematical tool for studying general spin particles. Inspired by the fully antisymmetric tensor, I tried to find a similar fully symmetric tensor. After constant attempts, I finally found such a fully symmetric spin tensor. Then by combining with various basic constant invariant tensors obtained in the previous chapter, several useful special constant invariant tensors have been further developed.

1 Permutation of symmetric indices

Self comment: Unlike the previous sorting starts counting from 1, this section starts counting from 0. This is more natural and naturally matches the w+1 radix.

Def. 1.0.1. $\sigma \otimes I = I \bar{\otimes} \sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$

1.1 Permutation of second-order symmetric indices

1.1.1 Permutation of second-order symmetric indices $A \bar{\otimes} B \bar{\otimes} C \bar{\otimes} D$

Cor. 1.1.1.

$$\begin{cases} \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1 \end{bmatrix} \\ \begin{bmatrix} [0\ 0\ 1\ 0] \\ 0\ 0\ 1\ 1 \end{bmatrix} \end{cases} \quad \begin{cases} \begin{bmatrix} [0\ 1\ 0\ 0] \\ [0\ 1\ 0\ 1] \end{bmatrix} \\ \begin{bmatrix} [0\ 1\ 1\ 0] \\ 0\ 1\ 1\ 1 \end{bmatrix} \end{cases} \quad \begin{cases} \begin{bmatrix} [1\ 0\ 0\ 0] \\ [1\ 0\ 0\ 1] \end{bmatrix} \\ \begin{bmatrix} [1\ 0\ 1\ 0] \\ [1\ 0\ 1\ 1] \end{bmatrix} \end{cases} \quad \begin{cases} \begin{bmatrix} [1\ 1\ 0\ 0] \\ [1\ 1\ 0\ 1] \end{bmatrix} \\ \begin{bmatrix} [1\ 1\ 1\ 0] \\ 1\ 1\ 1\ 1 \end{bmatrix} \end{cases}$$

1.1.2 Deduction of permutation law for second-order symmetric indices $A \leq B \leq C \leq D$

Cor. 1.1.2.

$$\begin{cases} c_3 = 0 \\ c_2 = a_2 = 2 \\ c_1 = a_2 + a_1 = 2 + 1 \\ c_n = C_3^2 - C_n^2 \end{cases} \quad \begin{cases} d_0 = c_3 = 0 \\ d_1 = c_2 = 2 \\ d_2 = c_1 = 3 \\ d_n = C_3^2 - C_{3-n}^2 \end{cases}$$

$$\begin{cases} A_0 d_0 = 0 \\ A_0 d_1 = 2 \\ A_0 d_2 = 3 \\ A_0 d_n = (C_3^2 - C_{3-n}^2) u(C_3^2 - C_{5-3}^2) \end{cases} \quad \begin{cases} A_1 d_0 = 0 + C_3^2 \\ A_1 d_1 = 0 + C_3^2 \\ A_1 d_2 = 1 + C_3^2 \\ A_1 d_n = (C_2^2 - C_{3-n}^2) u(C_2^2 - C_{3-n}^2) + C_3^2 \end{cases}$$

1.1.3 A summary of forward permutation rules for second-order symmetric indices

Cor. 1.1.3.

$$k_{0 \leq 0 \leq \eta \leq \xi} = (C_3^2 - C_{3-\eta}^2) + (\sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0)$$

$$k_{0 \leq \mu \leq \eta \leq \xi} = (\sum_{k=0}^{\mu} C_{3-k}^2 - C_{3-\eta}^2) + (\sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0)$$

$$k_{\lambda \leq \mu \leq \eta \leq \xi} = (C_5^4 - C_{5-\lambda}^4) + (\sum_{k=\lambda}^{\mu} C_{3-k}^2 - C_{3-\eta}^2) + (\sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0)$$

$$k_{d \leq \lambda \leq \mu \leq \eta \leq \xi} = (\sum_{k=0}^{d} C_{5-k}^4 - C_{5-\lambda}^4) + (\sum_{k=\lambda}^{\mu} C_{3-k}^2 - C_{3-\eta}^2) + (\sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0)$$

1.1.4 General conjecture of forward permutation rules for second-order symmetric indices

Ass. 1.1.1.

$$k_{\lambda_{2s} \cdots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+1-k}^{2l} - C_{2l+1-\lambda_{2l}}^{2l}); \lambda_i = 0, i > 2s | i = 0; \lambda_i = (0,1), 1 \leq i \leq 2s$$

1.2 Permutation of fourth-order symmetric indices

1.2.1 Permutation of fourth-order symmetric indices $\lambda\bar{\otimes}\mu\bar{\otimes}\eta\bar{\otimes}\xi$

Def. 1.2.1.

$$\begin{cases}\begin{bmatrix}0\ 0\ 0\ 0\\ 0\ 0\ 0\ 1\\ 0\ 0\ 0\ 2\\ 0\ 0\ 0\ 3\end{bmatrix}\\ \begin{bmatrix}[0\ 0\ 1\ 0]\\ 0\ 0\ 1\ 1\\ 0\ 0\ 1\ 2\\ 0\ 0\ 1\ 3\end{bmatrix}\\ \begin{bmatrix}[0\ 0\ 2\ 0]\\ [0\ 0\ 2\ 1]\\ 0\ 0\ 2\ 2\\ 0\ 0\ 2\ 3\end{bmatrix}\\ \begin{bmatrix}[0\ 0\ 3\ 0]\\ [0\ 0\ 3\ 1]\\ [0\ 0\ 3\ 2]\\ 0\ 0\ 3\ 3\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[0\ 1\ 0\ 0]\\ [0\ 1\ 0\ 1]\\ [0\ 1\ 0\ 2]\\ [0\ 1\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[0\ 1\ 1\ 0]\\ 0\ 1\ 1\ 1\\ 0\ 1\ 1\ 2\\ 0\ 1\ 1\ 3\end{bmatrix}\\ \begin{bmatrix}[0\ 1\ 2\ 0]\\ [0\ 1\ 2\ 1]\\ 0\ 1\ 2\ 2\\ 0\ 1\ 2\ 3\end{bmatrix}\\ \begin{bmatrix}[0\ 1\ 3\ 0]\\ [0\ 1\ 3\ 1]\\ [0\ 1\ 3\ 2]\\ 0\ 1\ 3\ 3\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[0\ 2\ 0\ 0]\\ [0\ 2\ 0\ 1]\\ [0\ 2\ 0\ 2]\\ [0\ 2\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[0\ 2\ 1\ 0]\\ [0\ 2\ 1\ 1]\\ [0\ 2\ 1\ 2]\\ [0\ 2\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[0\ 2\ 2\ 0]\\ [0\ 2\ 2\ 1]\\ 0\ 2\ 2\ 2\\ 0\ 2\ 2\ 3\end{bmatrix}\\ \begin{bmatrix}[0\ 2\ 3\ 0]\\ [0\ 2\ 3\ 1]\\ [0\ 2\ 3\ 2]\\ 0\ 2\ 3\ 3\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[0\ 3\ 0\ 0]\\ [0\ 3\ 0\ 1]\\ [0\ 3\ 0\ 2]\\ [0\ 3\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[0\ 3\ 1\ 0]\\ [0\ 3\ 1\ 1]\\ [0\ 3\ 1\ 2]\\ [0\ 3\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[0\ 3\ 2\ 0]\\ [0\ 3\ 2\ 1]\\ [0\ 3\ 2\ 2]\\ [0\ 3\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[0\ 3\ 3\ 0]\\ [0\ 3\ 3\ 1]\\ [0\ 3\ 3\ 2]\\ 0\ 3\ 3\ 3\end{bmatrix}\end{cases}$$

$$\begin{cases}\begin{bmatrix}[1\ 0\ 0\ 0]\\ [1\ 0\ 0\ 1]\\ [1\ 0\ 0\ 2]\\ [1\ 0\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 0\ 1\ 0]\\ [1\ 0\ 1\ 1]\\ [1\ 0\ 1\ 2]\\ [1\ 0\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 0\ 2\ 0]\\ [1\ 0\ 2\ 1]\\ [1\ 0\ 2\ 2]\\ [1\ 0\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 0\ 3\ 0]\\ [1\ 0\ 3\ 1]\\ [1\ 0\ 3\ 2]\\ [1\ 0\ 3\ 3]\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[1\ 1\ 0\ 0]\\ [1\ 1\ 0\ 1]\\ [1\ 1\ 0\ 2]\\ [1\ 1\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 1\ 1\ 0]\\ 1\ 1\ 1\ 1\\ 1\ 1\ 1\ 2\\ 1\ 1\ 1\ 3\end{bmatrix}\\ \begin{bmatrix}[1\ 1\ 2\ 0]\\ [1\ 1\ 2\ 1]\\ 1\ 1\ 2\ 2\\ 1\ 1\ 2\ 3\end{bmatrix}\\ \begin{bmatrix}[1\ 1\ 3\ 0]\\ [1\ 1\ 3\ 1]\\ [1\ 1\ 3\ 2]\\ 1\ 1\ 3\ 3\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[1\ 2\ 0\ 0]\\ [1\ 2\ 0\ 1]\\ [1\ 2\ 0\ 2]\\ [1\ 2\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 2\ 1\ 0]\\ [1\ 2\ 1\ 1]\\ [1\ 2\ 1\ 2]\\ [1\ 2\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 2\ 2\ 0]\\ [1\ 2\ 2\ 1]\\ 1\ 2\ 2\ 2\\ 1\ 2\ 2\ 3\end{bmatrix}\\ \begin{bmatrix}[1\ 2\ 3\ 0]\\ [1\ 2\ 3\ 1]\\ [1\ 2\ 3\ 2]\\ 1\ 2\ 3\ 3\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[1\ 3\ 0\ 0]\\ [1\ 3\ 0\ 1]\\ [1\ 3\ 0\ 2]\\ [1\ 3\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 3\ 1\ 0]\\ [1\ 3\ 1\ 1]\\ [1\ 3\ 1\ 2]\\ [1\ 3\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 3\ 2\ 0]\\ [1\ 3\ 2\ 1]\\ [1\ 3\ 2\ 2]\\ [1\ 3\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[1\ 3\ 3\ 0]\\ [1\ 3\ 3\ 1]\\ [1\ 3\ 3\ 2]\\ 1\ 3\ 3\ 3\end{bmatrix}\end{cases}$$

$$\begin{cases}\begin{bmatrix}[2\ 0\ 0\ 0]\\ [2\ 0\ 0\ 1]\\ [2\ 0\ 0\ 2]\\ [2\ 0\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 0\ 1\ 0]\\ [2\ 0\ 1\ 1]\\ [2\ 0\ 1\ 2]\\ [2\ 0\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 0\ 2\ 0]\\ [2\ 0\ 2\ 1]\\ [2\ 0\ 2\ 2]\\ [2\ 0\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 0\ 3\ 0]\\ [2\ 0\ 3\ 1]\\ [2\ 0\ 3\ 2]\\ [2\ 0\ 3\ 3]\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[2\ 1\ 0\ 0]\\ [2\ 1\ 0\ 1]\\ [2\ 1\ 0\ 2]\\ [2\ 1\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 1\ 1\ 0]\\ 2\ 1\ 1\ 1\\ 2\ 1\ 1\ 2\\ 2\ 1\ 1\ 3\end{bmatrix}\\ \begin{bmatrix}[2\ 1\ 2\ 0]\\ [2\ 1\ 2\ 1]\\ [2\ 1\ 2\ 2]\\ [2\ 1\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 1\ 3\ 0]\\ [2\ 1\ 3\ 1]\\ [2\ 1\ 3\ 2]\\ [2\ 1\ 3\ 3]\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[2\ 2\ 0\ 0]\\ [2\ 2\ 0\ 1]\\ [2\ 2\ 0\ 2]\\ [2\ 2\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 2\ 1\ 0]\\ [2\ 2\ 1\ 1]\\ [2\ 2\ 1\ 2]\\ [2\ 2\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 2\ 2\ 0]\\ [2\ 2\ 2\ 1]\\ 2\ 2\ 2\ 2\\ 2\ 2\ 2\ 3\end{bmatrix}\\ \begin{bmatrix}[2\ 2\ 3\ 0]\\ [2\ 2\ 3\ 1]\\ [2\ 2\ 3\ 2]\\ 2\ 2\ 3\ 3\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[2\ 3\ 0\ 0]\\ [2\ 3\ 0\ 1]\\ [2\ 3\ 0\ 2]\\ [2\ 3\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 3\ 1\ 0]\\ [2\ 3\ 1\ 1]\\ [2\ 3\ 1\ 2]\\ [2\ 3\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 3\ 2\ 0]\\ [2\ 3\ 2\ 1]\\ [2\ 3\ 2\ 2]\\ [2\ 3\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[2\ 3\ 3\ 0]\\ [2\ 3\ 3\ 1]\\ [2\ 3\ 3\ 2]\\ 2\ 3\ 3\ 3\end{bmatrix}\end{cases}$$

$$\begin{cases}\begin{bmatrix}[3\ 0\ 0\ 0]\\ [3\ 0\ 0\ 1]\\ [3\ 0\ 0\ 2]\\ [3\ 0\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 0\ 1\ 0]\\ [3\ 0\ 1\ 1]\\ [3\ 0\ 1\ 2]\\ [3\ 0\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 0\ 2\ 0]\\ [3\ 0\ 2\ 1]\\ [3\ 0\ 2\ 2]\\ [3\ 0\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 0\ 3\ 0]\\ [3\ 0\ 3\ 1]\\ [3\ 0\ 3\ 2]\\ [3\ 0\ 3\ 3]\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[3\ 1\ 0\ 0]\\ [3\ 1\ 0\ 1]\\ [3\ 1\ 0\ 2]\\ [3\ 1\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 1\ 1\ 0]\\ [3\ 1\ 1\ 1]\\ [3\ 1\ 1\ 2]\\ [3\ 1\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 1\ 2\ 0]\\ [3\ 1\ 2\ 1]\\ [3\ 1\ 2\ 2]\\ [3\ 1\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 1\ 3\ 0]\\ [3\ 1\ 3\ 1]\\ [3\ 1\ 3\ 2]\\ [3\ 1\ 3\ 3]\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[3\ 2\ 0\ 0]\\ [3\ 2\ 0\ 1]\\ [3\ 2\ 0\ 2]\\ [3\ 2\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 2\ 1\ 0]\\ [3\ 2\ 1\ 1]\\ [3\ 2\ 1\ 2]\\ [3\ 2\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 2\ 2\ 0]\\ [3\ 2\ 2\ 1]\\ [3\ 2\ 2\ 2]\\ [3\ 2\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 2\ 3\ 0]\\ [3\ 2\ 3\ 1]\\ [3\ 2\ 3\ 2]\\ [3\ 2\ 3\ 3]\end{bmatrix}\end{cases} \begin{cases}\begin{bmatrix}[3\ 3\ 0\ 0]\\ [3\ 3\ 0\ 1]\\ [3\ 3\ 0\ 2]\\ [3\ 3\ 0\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 3\ 1\ 0]\\ [3\ 3\ 1\ 1]\\ [3\ 3\ 1\ 2]\\ [3\ 3\ 1\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 3\ 2\ 0]\\ [3\ 3\ 2\ 1]\\ [3\ 3\ 2\ 2]\\ [3\ 3\ 2\ 3]\end{bmatrix}\\ \begin{bmatrix}[3\ 3\ 3\ 0]\\ [3\ 3\ 3\ 1]\\ [3\ 3\ 3\ 2]\\ 3\ 3\ 3\ 3\end{bmatrix}\end{cases}$$

Cor. 1.2.1.

$$\begin{cases}number\{[]\}: b_4=\begin{bmatrix}0&4&4&4\\1&1&4&4\\2&2&2&4\\3&3&3&3\end{bmatrix} b_3=\begin{bmatrix}4&4&4&4\\4&1&4&4\\4&2&2&4\\4&3&3&3\end{bmatrix} b_2=\begin{bmatrix}4&4&4&4\\4&4&4&4\\4&4&2&4\\4&4&3&3\end{bmatrix} b_1=\begin{bmatrix}4&4&4&4\\4&4&4&4\\4&4&4&4\\4&4&4&3\end{bmatrix} b_n=4^3-C_{n+2}^3\\ number\{\bar{[]}\}: a_4=\begin{bmatrix}4&0&0&0\\3&3&0&0\\2&2&2&0\\1&1&1&1\end{bmatrix} a_3=\begin{bmatrix}0&0&0&0\\0&3&0&0\\0&2&2&0\\0&1&1&1\end{bmatrix} a_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&2&0\\0&0&1&1\end{bmatrix} a_1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix} a_n=C_{n+2}^3\end{cases}$$

1.2.2 Deduction of permutation law of fourth-order symmetric indices $\lambda\le\mu\le\eta\le\xi$

Cor. 1.2.2.

$\sum\limits_{k=1}^{n}k=\frac{1}{2}n(n+1),\ \sum\limits_{k=1}^{n}k^2=\frac{1}{3}n(n+\frac{1}{2})(n+1),\ \sum\limits_{k=1}^{n}k^3=\frac{1}{4}n^2(n+1)^2$

$$\begin{cases}c_5=0\\c_4=a_4\\c_3=a_4+a_3\\c_2=a_4+a_3+a_2\\c_1=a_4+a_3+a_2+a_1\\c_n=C_7^4-C_{n+2}^4\end{cases}\qquad\begin{cases}d_0=c_5=0\\d_1=c_4=20\\d_2=c_3=30\\d_3=c_2=34\\d_4=c_1=35\\d_n=C_7^4-C_{7-n}^4\end{cases}$$

$$\begin{cases}A_0d_0=A_0c_5=0\\A_0d_1=A_0c_4=4\\A_0d_2=A_0c_3=7\\A_0d_3=A_0c_2=9\\A_0d_4=A_0c_1=10\\A_0d_n=(C_5^2-C_{5-n}^2)u(C_5^2-C_{5-n}^2)\end{cases}\qquad\begin{cases}A_1d_0=A_1c_5=0+C_5^2\\A_1d_1=A_1c_4=0+C_5^2\\A_1d_2=A_1c_3=3+C_5^2\\A_1d_3=A_1c_2=5+C_5^2\\A_1d_4=A_1c_1=6+C_5^2\\A_1d_n=(C_4^2-C_{5-n}^2)u(C_4^2-C_{5-n}^2)+C_5^2\end{cases}$$

$$\begin{cases}A_2d_0=A_2c_5=0+C_5^2+C_4^2\\A_2d_1=A_2c_4=0+C_5^2+C_4^2\\A_2d_2=A_2c_3=0+C_5^2+C_4^2\\A_2d_3=A_2c_2=2+C_5^2+C_4^2\\A_2d_4=A_2c_1=3+C_5^2+C_4^2\\A_2d_n=(C_3^2-C_{5-n}^2)u(C_3^2-C_{5-n}^2)+C_5^2+C_4^2\end{cases}\qquad\begin{cases}A_3d_0=A_3c_5=0+C_5^2+C_4^2+C_3^2\\A_3d_1=A_3c_4=0+C_5^2+C_4^2+C_3^2\\A_3d_2=A_3c_3=0+C_5^2+C_4^2+C_3^2\\A_3d_3=A_3c_2=0+C_5^2+C_4^2+C_3^2\\A_3d_4=A_3c_1=1+C_5^2+C_4^2+C_3^2\\A_3d_n=(C_2^2-C_{5-n}^2)u(C_2^2-C_{5-n}^2)+C_5^2+C_4^2+C_3^2\end{cases}$$

Cor. 1.2.3.

$$\begin{cases} A_0 d_n = (C_5^2 - C_{5-n}^2)u(C_5^2 - C_{5-n}^2) \\ A_1 d_n = (C_4^2 - C_{5-n}^2)u(C_4^2 - C_{5-n}^2) + C_5^2 \\ A_2 d_n = (C_3^2 - C_{5-n}^2)u(C_3^2 - C_{5-n}^2) + C_5^2 + C_4^2 \\ A_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + C_5^2 + C_4^2 + C_3^2 \end{cases}$$

$$\begin{cases} B_0 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 20 \\ B_1 d_n = (C_4^2 - C_{5-n}^2)u(C_4^2 - C_{5-n}^2) + 20 \\ B_2 d_n = (C_3^2 - C_{5-n}^2)u(C_3^2 - C_{5-n}^2) + 20 + C_4^2 \\ B_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + 20 + C_4^2 + C_3^2 \end{cases}$$

$$\begin{cases} C_0 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 30 \\ C_1 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 30 \\ C_2 d_n = (C_3^2 - C_{5-n}^2)u(C_3^2 - C_{5-n}^2) + 30 \\ C_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + 30 + C_3^2 \end{cases}$$

$$\begin{cases} D_0 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 34 \\ D_1 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 34 \\ D_2 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 34 \\ D_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + 34 \end{cases}$$

Cor. 1.2.4.

$$\begin{cases} 00 d_n = (C_5^2 u(0-0) - C_{5-n}^2)u(C_5^2 u(0-0) - C_{5-n}^2) + \sum_{k=0}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4) \\ 01 d_n = (C_4^2 u(1-0) - C_{5-n}^2)u(C_4^2 u(1-0) - C_{5-n}^2) + \sum_{k=0}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4) \\ 02 d_n = (C_3^2 u(2-0) - C_{5-n}^2)u(C_3^2 u(2-0) - C_{5-n}^2) + \sum_{k=0}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4) \\ 03 d_n = (C_2^2 u(3-0) - C_{5-n}^2)u(C_2^2 u(3-0) - C_{5-n}^2) + \sum_{k=0}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4) \end{cases}$$

$$\begin{cases} 10 d_n = (C_5^2 u(0-1) - C_{5-n}^2)u(C_5^2 u(0-1) - C_{5-n}^2) + \sum_{k=1}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \\ 11 d_n = (C_4^2 u(1-1) - C_{5-n}^2)u(C_4^2 u(1-1) - C_{5-n}^2) + \sum_{k=1}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \\ 12 d_n = (C_3^2 u(2-1) - C_{5-n}^2)u(C_3^2 u(2-1) - C_{5-n}^2) + \sum_{k=1}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \\ 13 d_n = (C_2^2 u(3-1) - C_{5-n}^2)u(C_2^2 u(3-1) - C_{5-n}^2) + \sum_{k=1}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \end{cases}$$

$$\begin{cases} 20 d_n = (C_5^2 u(0-2) - C_{5-n}^2)u(C_5^2 u(0-2) - C_{5-n}^2) + \sum_{k=2}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \\ 21 d_n = (C_4^2 u(1-2) - C_{5-n}^2)u(C_4^2 u(1-2) - C_{5-n}^2) + \sum_{k=2}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \\ 22 d_n = (C_3^2 u(2-2) - C_{5-n}^2)u(C_3^2 u(2-2) - C_{5-n}^2) + \sum_{k=2}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \\ 23 d_n = (C_2^2 u(3-2) - C_{5-n}^2)u(C_2^2 u(3-2) - C_{5-n}^2) + \sum_{k=2}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \end{cases}$$

$$\begin{cases} 30 d_n = (C_5^2 u(0-3) - C_{5-n}^2)u(C_5^2 u(0-3) - C_{5-n}^2) + \sum_{k=3}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \\ 31 d_n = (C_4^2 u(1-3) - C_{5-n}^2)u(C_4^2 u(1-3) - C_{5-n}^2) + \sum_{k=3}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \\ 32 d_n = (C_3^2 u(2-3) - C_{5-n}^2)u(C_3^2 u(2-3) - C_{5-n}^2) + \sum_{k=3}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \\ 33 d_n = (C_2^2 u(3-3) - C_{5-n}^2)u(C_2^2 u(3-3) - C_{5-n}^2) + \sum_{k=3}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \end{cases}$$

$$\lambda\mu d_n = (C_{5-\mu}^2 u(\mu-\lambda) - C_{5-n}^2)u(C_{5-\mu}^2 u(\mu-\lambda) - C_{5-n}^2) + \sum_{k=\lambda}^{\mu-1} C_{5-k}^2 + (C_7^4 - C_{7-\lambda}^4)$$

$$k_{\lambda\le\mu\le\eta\le\xi} = (C_{5-\mu}^2 u(\mu-\lambda) - C_{5-\eta}^2)u(C_{5-\mu}^2 u(\mu-\lambda) - C_{5-\eta}^2) + \sum_{k=\lambda}^{\mu-1} C_{5-k}^2 + (C_7^4 - C_{7-\lambda}^4) + (\xi - \eta)$$

$$= (C_7^4 - C_{7-\lambda}^4) + (\sum_{k=\lambda}^{\mu} C_{5-k}^2 - C_{5-\eta}^2) + (\xi - \eta)$$

$k_{0\le\mu\le\eta\le\xi}=(\sum_{k=0}^{\mu}C_{5-k}^{2}-C_{5-\eta}^{2})+(\xi-\eta)$

$k_{0\le 0\le\eta\le\xi}=(C_{5}^{2}-C_{5-\eta}^{2})+(\xi-\eta)$

1.3 Summary of forward permutation rules for fourth-order symmetric indices

Cor. 1.3.1.

$k_{0\le 0\le\eta\le\xi}=(C_{5}^{2}-C_{5-\eta}^{2})+(\sum_{k=\eta}^{\xi}C_{3-k}^{0}-C_{3-\lambda_0}^{0})$

$k_{0\le\mu\le\eta\le\xi}=(\sum_{k=0}^{\mu}C_{5-k}^{2}-C_{5-\eta}^{2})+(\sum_{k=\eta}^{\xi}C_{3-k}^{0}-C_{3-\lambda_0}^{0})$

$k_{\lambda\le\mu\le\eta\le\xi}=(C_{7}^{4}-C_{7-\lambda}^{4})+(\sum_{k=\lambda}^{\mu}C_{5-k}^{2}-C_{5-\eta}^{2})+(\sum_{k=\eta}^{\xi}C_{3-k}^{0}-C_{3-\lambda_0}^{0})$

$k_{d\le\lambda\le\mu\le\eta\le\xi}=(\sum_{k=0}^{d}C_{7-k}^{4}-C_{7-\lambda}^{4})+(\sum_{k=\lambda}^{\mu}C_{5-k}^{2}-C_{5-\eta}^{2})+(\sum_{k=\eta}^{\xi}C_{3-k}^{0}-C_{3-\lambda_0}^{0})$

$k_{c\le d\le\lambda\le\mu\le\eta\le\xi}=(C_{9}^{6}-C_{9-c}^{6})+(\sum_{k=c}^{d}C_{7-k}^{4}-C_{7-\lambda}^{4})+(\sum_{k=\lambda}^{\mu}C_{5-k}^{2}-C_{5-\eta}^{2})+(\sum_{k=\eta}^{\xi}C_{3-k}^{0}-C_{3-\lambda_0}^{0})$

$k_{b\le c\le d\le\lambda\le\mu\le\eta\le\xi}=(\sum_{k=0}^{b}C_{9-k}^{6}-C_{9-c}^{6})+(\sum_{k=c}^{d}C_{7-k}^{4}-C_{7-\lambda}^{4})+(\sum_{k=\lambda}^{\mu}C_{5-k}^{2}-C_{5-\eta}^{2})+(\sum_{k=\eta}^{\xi}C_{3-k}^{0}-C_{3-\lambda_0}^{0})$

$k_{a\le b\le c\le d\le\lambda\le\mu\le\eta\le\xi}=(C_{11}^{8}-C_{11-a}^{8})+(\sum_{k=a}^{b}C_{9-k}^{6}-C_{9-c}^{6})+(\sum_{k=c}^{d}C_{7-k}^{4}-C_{7-\lambda}^{4})+(\sum_{k=\lambda}^{\mu}C_{5-k}^{2}-C_{5-\eta}^{2})+(\sum_{k=\eta}^{\xi}C_{3-k}^{0}-C_{3-\lambda_0}^{0})$

1.3.1 General conjecture of forward permutation rules for fourth-order symmetric indices

Ass. 1.3.1.

$$k_{\lambda_{2s}\cdot\cdot\le\lambda_8\le\lambda_7\le\lambda_6\le\lambda_5\le\lambda_4\le\lambda_3\le\lambda_2\le\lambda_1}$$
$$=\sum_{k=0}^{0}C_{11-k}^{8}+\sum_{k=\lambda_8}^{\lambda_7}C_{9-k}^{6}+\sum_{k=\lambda_6}^{\lambda_5}C_{7-k}^{4}+\sum_{k=\lambda_4}^{\lambda_3}C_{5-k}^{2}+\sum_{k=\lambda_2}^{\lambda_1}C_{3-k}^{0}-(C_{11-\lambda_8}^{8}+C_{9-\lambda_6}^{6}+C_{7-\lambda_4}^{4}+C_{5-\lambda_2}^{2}+C_{3-\lambda_0}^{0})$$
$$=\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C_{2l+3-k}^{2l}-C_{2l+3-\lambda_{2l}}^{2l});\lambda_i=0,i>2s|i=0;\lambda_i=(0,1,2,3),1\le i\le 2s$$

1.4 Permutation of $w+1$-order symmetric indices

1.4.1 General conjecture of forward permutation rules for $w+1$-order symmetric indices

Ass. 1.4.1.

$k_{\lambda_{2s}\cdot\cdot\le\lambda_4\le\lambda_3\le\lambda_2\le\lambda_1}=\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C_{2l-k}^{2l}-C_{2l-\lambda_{2l}}^{2l})=0,\lambda_i=(0)$

$k_{\lambda_{2s}\cdot\cdot\le\lambda_4\le\lambda_3\le\lambda_2\le\lambda_1}=\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C_{2l+1-k}^{2l}-C_{2l+1-\lambda_{2l}}^{2l}),\lambda_i=(0,1)$

$k_{\lambda_{2s}\cdot\cdot\le\lambda_4\le\lambda_3\le\lambda_2\le\lambda_1}=\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+2}}C_{2l+1-k}^{2l}-C_{2l+2-\lambda_{2l}}^{2l}),\lambda_i=(0,1,2)$

$k_{\lambda_{2s}\cdot\cdot\le\lambda_4\le\lambda_3\le\lambda_2\le\lambda_1}=\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C_{2l+3-k}^{2l}-C_{2l+3-\lambda_{2l}}^{2l}),\lambda_i=(0,1,2,3)$

$k_{\lambda_{2s}\cdot\cdot\le\lambda_4\le\lambda_3\le\lambda_2\le\lambda_1}=\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C_{2l+w-k}^{2l}-C_{2l+w-\lambda_{2l}}^{2l}),\lambda_i=(0,1,2,3,\cdot\cdot,w)$

$$k_{\underbrace{w\cdot\cdot w}_{2s}}+1=\sum_{k=\lambda_{2[s]+2}}^{\lambda_{2[s]+1}}C_{2[s]+w-k}^{2[s]}=\begin{cases}C_{2s+w}^{2s},[s]=s\\ \sum_{k=0}^{w}C_{2s-1+w-k}^{2s-1},[s]=s-\frac{1}{2}\end{cases}=C_{2s+w}^{2s},\lambda_i=(0,1,2,3,\cdot\cdot,w)$$

2 Perfect constant invariant tensors $\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}(s;w),\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}(s;w)$

2.1 Introduction of constant invariant tensors $\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}(s),\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}(s)$

Def. 2.1.1.
$$\begin{cases}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s)=\frac{1}{(2s)!}\Gamma^{k_\varsigma}_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdot\cdot)}_{2s}}(s),\Gamma^{k}_{\underbrace{1\cdot\cdot1}_{l}\underbrace{0\cdot\cdot0}_{2s-l}}(s)=\sqrt{C_{2s}^{-k}}\delta_{kl},k,l=0,1,\cdot\cdot,2s\\ \Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s)=\frac{1}{(2s)!}\Gamma_{k_\varsigma}^{\overbrace{(A_\varsigma B_\varsigma C_\varsigma\cdot\cdot)}^{2s}}(s),\Gamma_{k}^{\overbrace{1\cdot\cdot1}^{l}\overbrace{0\cdot\cdot0}^{2s-l}}(s)=\sqrt{C_{2s}^{-k}}\delta_{kl},k,l=0,1,\cdot\cdot,2s\end{cases}$$

Def. 2.1.2. $\psi^{k_\varsigma}(s)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s)\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}=\sqrt{C_{2s}^{k_\varsigma}}\psi^{\overbrace{2\cdot\cdot2}^{k_\varsigma}\overbrace{1\cdot\cdot1}^{2s-k_\varsigma}}$

Def. 2.1.3. $\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma} = (\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix})_{k_\varsigma}$

2.2 Introduction of constant invariant tensors $\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}(s;w), \Gamma^{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{k_\varsigma}(s;w)$

Def. 2.2.1. $\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{2s}}(s;w) = \frac{1}{(2s)!}\Gamma^{k_\varsigma}_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma \cdot\cdot)}_{2s}}(s;w)$

$$\Gamma^{k_\varsigma}_{\underbrace{0_\varsigma \cdot\cdot 0_\varsigma}_{l_0}\underbrace{1_\varsigma \cdot\cdot 1_\varsigma}_{l_1}\cdot\cdot\underbrace{w_\varsigma \cdot\cdot w_\varsigma}_{l_w}}(s;w) = \sqrt{\frac{l_0!l_1!\cdot\cdot l_w!}{(2s)!}}\delta\{k_\varsigma, \sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C^{2l}_{2l+w-k} - C^{2l}_{2l+w-\lambda_{2l}})\}, l_0 + l_1 + \cdot\cdot + l_w = 2s$$

Def. 2.2.2. $\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}^{2s}}_{k_\varsigma}(s;w) = \frac{1}{(2s)!}\Gamma^{\overbrace{(A_\varsigma B_\varsigma C_\varsigma \cdot\cdot)}^{2s}}_{k_\varsigma}(s;w)$

$$\Gamma^{\overbrace{0_\varsigma \cdot\cdot 0_\varsigma}^{l_0}\overbrace{1_\varsigma \cdot\cdot 1_\varsigma}^{l_1}\cdot\cdot\overbrace{w_\varsigma \cdot\cdot w_\varsigma}^{l_w}}_{k_\varsigma}(s;w) = \sqrt{\frac{l_0!l_1!\cdot\cdot l_w!}{(2s)!}}\delta\{k_\varsigma, \sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C^{2l}_{2l+w-k} - C^{2l}_{2l+w-\lambda_{2l}})\}, l_0 + l_1 + \cdot\cdot + l_w = 2s$$

Cor. 2.2.1. $\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{2s}}(s) = \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{2s}}(s;1), \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}^{2s}}_{k_\varsigma}(s) = \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}^{2s}}_{k_\varsigma}(s;1)$

Self comment: The above indicates that $\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}(s;w)$, $\Gamma^{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{k_\varsigma}(s;w)$ are generalization of constant invariant tensors $\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}(s)$, $\Gamma^{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{k_\varsigma}(s)$.

2.3 Introduction of constant matrices $\Gamma(s;w), \bar{\Gamma}(s;w)$

Def. 2.3.1. $\Gamma(s;w) \succ \Gamma_{\underbrace{A_\varsigma \otimes B_\varsigma \otimes C_\varsigma \otimes \cdot\cdot}_{2s}}{}^{k_\varsigma}(s;w), \bar{\Gamma}(s;w) \succ \Gamma_{k_\varsigma}{}^{\overbrace{A_\varsigma \otimes B_\varsigma \otimes C_\varsigma \otimes \cdot\cdot}^{2s}}(s;w) \simeq \Gamma^T(s;w)$

Explicit representation of $\Gamma(s), \bar{\Gamma}(s)$

Cor. 2.3.1. $\Gamma(s = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdot\cdot) = 1, I, \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2} & 0 & 0\\0 & \sqrt{1} & 0\\0 & \sqrt{1} & 0\\0 & 0 & \sqrt{2}\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}\sqrt{3} & 0 & 0 & 0\\0 & \sqrt{1} & 0 & 0\\0 & \sqrt{1} & 0 & 0\\0 & 0 & \sqrt{1} & 0\\0 & \sqrt{1} & 0 & 0\\0 & 0 & \sqrt{1} & 0\\0 & 0 & \sqrt{1} & 0\\0 & 0 & 0 & \sqrt{3}\end{bmatrix}, \cdot\cdot$

Cor. 2.3.2. $\bar{\Gamma}(s = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdot\cdot) = 1, I, \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2} & 0 & 0 & 0\\0 & \sqrt{1} & \sqrt{1} & 0\\0 & 0 & 0 & \sqrt{2}\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & \sqrt{1} & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 & 0\\0 & 0 & 0 & \sqrt{1} & 0 & \sqrt{1} & \sqrt{1} & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3}\end{bmatrix}, \cdot\cdot$

2.4 Basic properties of constant invariant tensors $\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}(s;w), \Gamma^{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{k_\varsigma}(s;w)$

Equality:

Pro. 2.4.1. $\Gamma^{k'_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma \cdot\cdot}_{2s}}(s;w) \simeq \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{2s}}(s;w) \simeq \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}^{2s}}_{k_\varsigma}(s;w) \simeq \Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma \cdot\cdot}^{2s}}_{k'_\varsigma}(s;w)$

Pro. 2.4.2. $[\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{2s}}(s;w)]^* \simeq \Gamma^{k'_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma \cdot\cdot}_{2s}}(s;w), [\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}^{2s}}_{k_\varsigma}(s;w)]^* \simeq \Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma \cdot\cdot}^{2s}}_{k'_\varsigma}(s;w)$

Cor. 2.4.1. $\Gamma(s;w) = \Gamma^*(s;w), \bar{\Gamma}(s;w) = \bar{\Gamma}^*(s;w), \bar{\Gamma}(s;w) = \Gamma^+(s;w), \Gamma(s;w) = \bar{\Gamma}^+(s;w)$

Orthogonality:

Pro. 2.4.3. $\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}^{2s}}_{l_\varsigma}(s;w) = \delta^{k_\varsigma}{}_{l_\varsigma}[\Leftrightarrow]\bar{\Gamma}(s;w)\Gamma(s;w) = I$

Pro. 2.4.4. $\Gamma^{k_\varsigma}_{A_{1\varsigma}A_{2\varsigma}\cdot\cdot A_{2s\varsigma}}(s;w)\Gamma^{B_{1\varsigma}B_{2\varsigma}\cdot\cdot B_{2s\varsigma}}_{k_\varsigma}(s;w) = \frac{1}{(2s)!}\delta^{(B_{1\varsigma}}_{A_{1\varsigma}}\delta^{B_{2\varsigma}}_{A_{2\varsigma}}\cdot\cdot\delta^{B_{2s\varsigma})}_{A_{2s\varsigma}} = \frac{1}{(2s)!}\delta^{B_{1\varsigma}}_{(A_{1\varsigma}}\delta^{B_{2\varsigma}}_{A_{2\varsigma}}\cdot\cdot\delta^{B_{2s\varsigma}}_{A_{2s\varsigma})}$

Comparison:

Pro. 2.4.5. $\varepsilon_{a_1a_2\cdot\cdot a_n}\varepsilon^{b_1b_2\cdot\cdot b_n} = \delta^{[b_1}_{a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_n]}_{a_n} = \delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_n}_{a_n]}$

Other properties:

Pro. 2.4.6. $\Gamma^{k_\varsigma}_{A_\varsigma}(\frac{1}{2};w) = \delta^{k_\varsigma}_{A_\varsigma}, \Gamma^{A_\varsigma}_{k_\varsigma}(\frac{1}{2};w) = \delta^{A_\varsigma}_{k_\varsigma}; \Gamma(0;w) = 1, \bar{\Gamma}(0;w) = 1$

Pro. 2.4.7.

$$\begin{cases}\Gamma^{k_\varsigma}_{\underbrace{0_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s) = \sqrt{\frac{2s-k_\varsigma}{2s}}\Gamma^{k_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2})\\ \Gamma^{k_\varsigma}_{\underbrace{1_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s) = \sqrt{\frac{k_\varsigma}{2s}}\Gamma^{k_\varsigma-1}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2})\end{cases} \quad \begin{cases}\Gamma^{\overbrace{0_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s) = \sqrt{\frac{2s-k_\varsigma}{2s}}\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{k_\varsigma}(s-\frac{1}{2}), k_\varsigma = 0,1,\cdot\cdot,2s-1\\ \Gamma^{\overbrace{1_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s) = \sqrt{\frac{k_\varsigma}{2s}}\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{k_\varsigma-1}(s-\frac{1}{2}), k_\varsigma = 1,2,\cdot\cdot,2s\end{cases}$$

Pro. 2.4.8.

$$\begin{cases}\Gamma^{k_\varsigma}_{\underbrace{1_\varsigma\cdot\cdot 1_\varsigma}_{l}\underbrace{0_\varsigma\cdot\cdot 0_\varsigma}_{n}\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s) = \sqrt{\frac{C^{(k_\varsigma-l)}_{2s-l-n}}{C^{k_\varsigma}_{2s}}}\Gamma^{k_\varsigma-l}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-l-n}}(s-\frac{l+n}{2}), k_\varsigma = l,l+1,\cdot\cdot,2s-n\\ \Gamma^{\overbrace{1_\varsigma\cdot\cdot 1_\varsigma}^{l}\overbrace{0_\varsigma\cdot\cdot 0_\varsigma}^{n}\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-l-n}}_{k_\varsigma}(s) = \sqrt{\frac{C^{(k_\varsigma-l)}_{2s-l-n}}{C^{k_\varsigma}_{2s}}}\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-l-n}}_{k_\varsigma-l}(s-\frac{l+n}{2}), k_\varsigma = l,l+1,\cdot\cdot,2s-n\end{cases}$$

2.5 Introduction and properties of metric constant invariant tensor $\varepsilon_{k_\varsigma l_\varsigma}(s;w)$
(Existing $\varepsilon_{A_\varsigma B_\varsigma}$ is prerequisite.)
Metric definition:

Def. 2.5.1.
$$\begin{cases}\varepsilon_{k_\varsigma l_\varsigma}(s;w) := \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w)\underbrace{\varepsilon_{A_\varsigma E_\varsigma}\varepsilon_{B_\varsigma F_\varsigma}\varepsilon_{C_\varsigma G_\varsigma}\cdot\cdot}_{2s}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s}}_{l_\varsigma}(s;w)\\ \varepsilon^{k_\varsigma l_\varsigma}(s;w) := \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A_\varsigma E_\varsigma}\varepsilon^{B_\varsigma F_\varsigma}\varepsilon^{C_\varsigma G_\varsigma}\cdot\cdot}^{2s}\Gamma^{l_\varsigma}_{\underbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}_{2s}}(s;w)\end{cases}$$

Pro. 2.5.1.
$$\begin{cases}\underbrace{\varepsilon_{A_\varsigma E_\varsigma}\varepsilon_{B_\varsigma F_\varsigma}\varepsilon_{C_\varsigma G_\varsigma}\cdot\cdot}_{2s}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s}}_{l_\varsigma}(s;w)ABC...\\ \overbrace{\varepsilon^{A_\varsigma E_\varsigma}\varepsilon^{B_\varsigma F_\varsigma}\varepsilon^{C_\varsigma G_\varsigma}\cdot\cdot}^{2s}\Gamma^{l_\varsigma}_{\underbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}_{2s}}(s;w)ABC...\end{cases}$$

Cor. 2.5.1. $\varepsilon(s;w) := \bar{\Gamma}(s;w)\underbrace{\varepsilon(\frac{1}{2};w)\otimes\cdot\cdot\otimes\varepsilon(\frac{1}{2};w)}_{2s}\Gamma(s;w)$

Raising and lowering indices:

Pro. 2.5.2.
$$\begin{cases}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w) = \varepsilon^{k_\varsigma l_\varsigma}(s;w)\underbrace{\varepsilon_{A_\varsigma E_\varsigma}\varepsilon_{B_\varsigma F_\varsigma}\varepsilon_{C_\varsigma G_\varsigma}\cdot\cdot}_{2s}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s}}_{l_\varsigma}(s;w)\\ \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w) = \varepsilon_{k_\varsigma l_\varsigma}(s;w)\overbrace{\varepsilon^{A_\varsigma E_\varsigma}\varepsilon^{B_\varsigma F_\varsigma}\varepsilon^{C_\varsigma G_\varsigma}\cdot\cdot}^{2s}\Gamma^{l_\varsigma}_{\underbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}_{2s}}(s;w)\end{cases}$$

Cor. 2.5.2. $\Gamma(s;w)\varepsilon(s;w) = \underbrace{\varepsilon(\frac{1}{2};w)\otimes\cdot\cdot\otimes\varepsilon(\frac{1}{2};w)}_{2s}\Gamma(s;w), \varepsilon(s;w)\bar{\Gamma}(s;w) = \bar{\Gamma}(s;w)\underbrace{\varepsilon(\frac{1}{2};w)\otimes\cdot\cdot\otimes\varepsilon(\frac{1}{2};w)}_{2s}$

Proof: $\varepsilon^{k_\varsigma l_\varsigma}(s;w)\underbrace{\varepsilon_{A_\varsigma E_\varsigma}\varepsilon_{B_\varsigma F_\varsigma}\varepsilon_{C_\varsigma G_\varsigma}\cdot\cdot}_{2s}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s}}_{l_\varsigma}(s;w)$

$$= \Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A'_\varsigma E'_\varsigma}\varepsilon^{B'_\varsigma F'_\varsigma}\varepsilon^{C'_\varsigma G'_\varsigma}\cdot\cdot}^{2s}\Gamma^{l_\varsigma}_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdot\cdot}_{2s}}(s;w)\underbrace{\varepsilon_{A_\varsigma E_\varsigma}\varepsilon_{B_\varsigma F_\varsigma}\varepsilon_{C_\varsigma G_\varsigma}\cdot\cdot}_{2s}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s}}_{l_\varsigma}(s;w)$$

$$= \frac{1}{(2s)!}\Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A'_\varsigma E'_\varsigma}\varepsilon^{B'_\varsigma F'_\varsigma}\varepsilon^{C'_\varsigma G'_\varsigma}\cdot\cdot}^{2s}\delta^{E_\varsigma}_{(E'_\varsigma}\delta^{F_\varsigma}_{F'_\varsigma}\delta^{G_\varsigma}_{G'_\varsigma}\cdot\cdot)\underbrace{\varepsilon_{A_\varsigma E_\varsigma}\varepsilon_{B_\varsigma F_\varsigma}\varepsilon_{C_\varsigma G_\varsigma}\cdot\cdot}_{2s}$$

$= \frac{1}{(2s)!}\Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A'_\varsigma(E_\varsigma}\varepsilon^{B'_\varsigma F_\varsigma}\varepsilon^{C'_\varsigma G_\varsigma)}\cdot\cdot}^{2s}\underbrace{\varepsilon_{A_\varsigma E_\varsigma}\varepsilon_{B_\varsigma F_\varsigma}\varepsilon_{C_\varsigma G_\varsigma}\cdot\cdot}_{2s}$

$= \frac{1}{(2s)!}\Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\delta^{A'_\varsigma}_{(A_\varsigma}\delta^{B'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot{}_{)}$

$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)$ □

Penrose standard raising and lowering rules:

Pro. 2.5.3.
$$\begin{cases}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w) = (-1)^{2s}[\varsigma^{2s}\varepsilon^{k_\varsigma l_\varsigma}(s;w)]\underbrace{(-\varsigma\varepsilon_{A_\varsigma E_\varsigma})(-\varsigma\varepsilon_{B_\varsigma F_\varsigma})(-\varsigma\varepsilon_{C_\varsigma G_\varsigma})\cdot\cdot}_{2s}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s}}_{l_\varsigma}(s;w)\\ \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w) = (-1)^{2s}[(-\varsigma)^{2s}\varepsilon_{k_\varsigma l_\varsigma}(s;w)]\overbrace{(\varsigma\varepsilon^{A_\varsigma E_\varsigma})(\varsigma\varepsilon^{B_\varsigma F_\varsigma})(\varsigma\varepsilon^{C_\varsigma G_\varsigma})\cdot\cdot}^{2s}\Gamma^{l_\varsigma}_{\underbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}_{2s}}(s;w)\end{cases}$$

2.6 Spin constant invariant tensors $\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s;w), S_{ab}(s,\varsigma;w)$

Def. 2.6.1.
$$\begin{cases}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{Z_\varsigma}(\tfrac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{Z_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w) = \frac{1}{2s}\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s;w)[\Leftrightarrow]\bar{\Gamma}(s;w)\sigma^{\alpha_\varsigma}(\tfrac{1}{2};w)\Gamma(s;w) = \frac{1}{2s}\sigma^{\alpha_\varsigma}(s;w)\\ \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w)S_{abA_\varsigma}{}^{Z_\varsigma}(\tfrac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{Z_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w) = \frac{1}{2s}S_{abk_\varsigma}{}^{l_\varsigma}(s;w)[\Leftrightarrow]\bar{\Gamma}(s;w)S_{ab}(\tfrac{1}{2},\varsigma;w)\Gamma(s;w) = \frac{1}{2s}S_{ab}(s,\varsigma;w)\end{cases}$$

2.7 Introduction and properties of constant invariant tensors $\Omega^{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}(s;w)$,$\Omega(s;w)$

Def. 2.7.1.
$$\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w) := \overbrace{\underbrace{\sigma_{A_\varsigma}{}^{A'_\varsigma}(\tfrac{1}{2};w)\delta_{B_\varsigma}{}^{B'_\varsigma}\delta_{C_\varsigma}{}^{C'_\varsigma}\cdot\cdot}_{2s}+\underbrace{\delta_{A_\varsigma}{}^{A'_\varsigma}\sigma_{B_\varsigma}{}^{B'_\varsigma}(\tfrac{1}{2};w)\delta_{C_\varsigma}{}^{C'_\varsigma}\cdot\cdot}_{2s}+\underbrace{\delta_{A_\varsigma}{}^{A'_\varsigma}\delta_{B_\varsigma}{}^{B'_\varsigma}{}_{C_\varsigma}{}^{C'_\varsigma}\sigma(\tfrac{1}{2};w)\cdot\cdot}_{2s}+\cdots}^{2s}$$

$[\Updownarrow]$ $[\Updownarrow]$

Def. 2.7.2. $\Omega(s;w) := \sigma(\frac{1}{2};w)\otimes I_{(w+1)^{2s-1}} + I_{w+1}\otimes\sigma(\frac{1}{2};w)\otimes I_{2^{2s-2}} + \cdot\cdot + I_{(w+1)^{2s-1}}\otimes\sigma(\frac{1}{2};w)$

Cor. 2.7.1. $\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s+1}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w) := \sigma_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\underbrace{\delta_{B_\varsigma}{}^{B'_\varsigma}\delta_{C_\varsigma}{}^{C'_\varsigma}\cdot\cdot}_{2s-1}+\delta_{A_\varsigma}{}^{A'_\varsigma}\Omega^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2};w)$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 2.7.2. $\Omega(s;w) = \sigma(\frac{1}{2};w)\otimes I_{(w+1)^{2s-1}} + I_{w+1}\otimes\Omega(s-\frac{1}{2};w)$

Cor. 2.7.3. $\Omega(s;w) = \Omega(s-s';w)\otimes I_{(w+1)^{2s'}} + I_{(w+1)^{2(s-s')}}\otimes\Omega(s';w)$

Cor. 2.7.4. $\Omega(s;w) = \Omega(s-\frac{1}{2};w)\otimes I_{w+1} + I_{(w+1)^{2s-1}}\otimes\sigma(\frac{1}{2};w)$

Cor. 2.7.5. $\Omega(s;w) = \Omega(s-1;w)\otimes I_{(w+1)^2} + I_{(w+1)^{2s-2}}\otimes\Omega(1;w)$

Lem. 2.7.1. $\Gamma(s;w)\bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w) = \Omega(s;w)\Gamma(s;w)$

Proof: $\Gamma^{k_\varsigma}_{\underbrace{A''_\varsigma B''_\varsigma C''_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w)\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w) = \Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{\underbrace{A''_\varsigma B''_\varsigma C''_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)$

$\Leftrightarrow \Gamma(s;w)\bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w) = \Omega(s;w)\Gamma(s;w)$ □

Thm. 2.7.1. $\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w) = \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)[\Leftrightarrow]\Omega(s;w)\Gamma(s;w) = \Gamma(s;w)\sigma(s;w)$

Proof: $\bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w)=\sigma(s;w)$

$$\Leftrightarrow \Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)=\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$$

$$\Leftrightarrow \Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$$

$\Leftrightarrow \Omega(s;w)\Gamma(s;w)=\Gamma(s;w)\sigma(s;w)$ □

Lem. 2.7.2. $\bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w)\bar{\Gamma}(s;w)=\bar{\Gamma}(s;w)\Omega(s;w)$

Proof: $\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)\Gamma_{l_\varsigma}^{\overbrace{A''_\varsigma B''_\varsigma C''_\varsigma\cdots}^{2s}}(s;w)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Omega^{\overbrace{A''_\varsigma B''_\varsigma C''_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)$

$\Leftrightarrow \bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w)\bar{\Gamma}(s;w)=\bar{\Gamma}(s;w)\Omega(s;w)$ □

Thm. 2.7.2. $\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)=\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)\Gamma_{l_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}(s;w)[\Leftrightarrow]\bar{\Gamma}(s;w)\Omega(s;w)=\sigma(s;w)\bar{\Gamma}(s;w)$

Proof: $\bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w)=\sigma(s;w)$

$$\Leftrightarrow \Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)=\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$$

$$\Leftrightarrow \Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)=\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)\Gamma_{l_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}(s;w)$$

$\Leftrightarrow \bar{\Gamma}(s;w)\Omega(s;w)=\sigma(s;w)\bar{\Gamma}(s;w)$ □

Cor. 2.7.6.
$\bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w)=\sigma(s;w)\Leftrightarrow\Omega(s;w)\Gamma(s;w)=\Gamma(s;w)\sigma(s;w)\Leftrightarrow\bar{\Gamma}(s;w)\Omega(s;w)=\sigma(s;w)\bar{\Gamma}(s;w)$

Cor. 2.7.7. $\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$

$$\Leftrightarrow \sigma(\tfrac{1}{2};w)_{(A_\varsigma}^{A'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots)A'_\varsigma}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$$

Proof: $\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$

$$\Leftrightarrow \sigma(\tfrac{1}{2};w)_{A_\varsigma}^{A'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)+\sigma(\tfrac{1}{2};w)_{B_\varsigma}^{B'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma B'_\varsigma C_\varsigma\cdots}_{2s}}(s;w)+\sigma(\tfrac{1}{2};w)_{C_\varsigma}^{C'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)+\cdots=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$$

$$\Leftrightarrow \sigma(\tfrac{1}{2};w)_{A_\varsigma}^{A'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)+\sigma(\tfrac{1}{2};w)_{B_\varsigma}^{B'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma B'_\varsigma C_\varsigma\cdots}_{2s}}(s;w)+\sigma(\tfrac{1}{2};w)_{C_\varsigma}^{C'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)+\cdots=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$$

$\Leftrightarrow \sigma(\tfrac{1}{2};w)_{(A_\varsigma}^{A'_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots)A'_\varsigma}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$ □

2.8 Several identities of constant matrices $\Gamma(s;w),\bar{\Gamma}(s;w)$

Pro. 2.8.1. $\begin{cases}\bar{\Gamma}(s;w)\Omega(s;w)\Gamma(s;w)=\sigma(s;w),[\Gamma(s;w)\bar{\Gamma}(s;w),\Omega(s;w)]=0\\ \Gamma(s;w)\sigma(s;w)\bar{\Gamma}(s;w)=\Omega(s;w)\Gamma(s;w)\bar{\Gamma}(s;w)=\Gamma(s;w)\bar{\Gamma}(s;w)\Omega(s;w)\end{cases}$

Pro. 2.8.2. $\begin{cases}\bar{\Gamma}(s;w)\Omega_{ab}(s,\varsigma;w)\Gamma(s;w)=S_{ab}(s,\varsigma;w),[\Gamma(s;w)\bar{\Gamma}(s;w),\Omega_{ab}(s,\varsigma;w)]=0\\ \Gamma(s;w)S_{ab}(s,\varsigma;w)\bar{\Gamma}(s;w)=\Omega_{ab}(s,\varsigma;w)\Gamma(s;w)\bar{\Gamma}(s;w)=\Gamma(s;w)\bar{\Gamma}(s;w)\Omega_{ab}(s,\varsigma;w)\end{cases}$

Pro. 2.8.3. $\begin{cases}\bar{\Gamma}(s;w)[\vartheta\cdot\Omega(s;w)]^n\Gamma(s;w)=[\vartheta\cdot\sigma(s;w)]^n,[\Gamma(s;w)\bar{\Gamma}(s;w),[\vartheta\cdot\Omega(s;w)]^n]=0\\ \Gamma(s;w)[\vartheta\cdot\sigma(s;w)]^n\bar{\Gamma}(s;w)=[\vartheta\cdot\Omega(s;w)]^n\Gamma(s;w)\bar{\Gamma}(s;w)=\Gamma(s;w)\bar{\Gamma}(s;w)[\vartheta\cdot\Omega(s;w)]^n\end{cases}$

Pro. 2.8.4. $\begin{cases}\bar{\Gamma}(s;w)[\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n\Gamma(s;w)=[\vartheta^{ab}S_{ab}(s,\varsigma;w)]^n,[\Gamma(s;w)\bar{\Gamma}(s;w),[\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n]=0\\ \Gamma(s;w)[\vartheta^{ab}S_{ab}(s,\varsigma;w)]^n\bar{\Gamma}(s;w)=[\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n\Gamma(s;w)\bar{\Gamma}(s;w)=\Gamma(s;w)\bar{\Gamma}(s;w)[\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n\end{cases}$

Cor. 2.8.1. $\begin{cases}\bar{\Gamma}(s;w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}\Gamma(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)},[\Gamma(s;w)\bar{\Gamma}(s;w),e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}]=0\\ \Gamma(s;w)e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\bar{\Gamma}(s;w)=e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}\Gamma(s;w)\bar{\Gamma}(s;w)=\Gamma(s;w)\bar{\Gamma}(s;w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}\end{cases}$

2.9 Two important corollaries of constant matrices $I_{w+1}\otimes\Gamma(s-\frac{1}{2};w),I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)$

Cor. 2.9.1. $\Omega(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]$

Proof: $\Omega(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]$
$=\Omega(s;w)I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)$
$=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}]+I_{w+1}\otimes[\Gamma(s-\frac{1}{2};w)\sigma(s-\frac{1}{2};w)]$
$=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]$ □

Cor. 2.9.2. $[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)=[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]$

Proof: $[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)$
$=[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)$
$=[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]+I_{w+1}\otimes[\sigma(s-\frac{1}{2};w)\bar{\Gamma}(s-\frac{1}{2};w)]$
$=[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]$ □

2.10 Several identities of constant matrices $\Gamma(s-\frac{1}{2};w),\bar{\Gamma}(s-\frac{1}{2};w)$

Pro. 2.10.1.
$$\begin{cases}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\\ [I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\\ =\Omega(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)\\ [[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)],\Omega(s;w)]=0\end{cases}$$

Pro. 2.10.2.
$$\begin{cases}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega_{ab}(s,\varsigma;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\\ [I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\\ =\Omega_{ab}(s,\varsigma;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega_{ab}(s,\varsigma;w)\\ [[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)],\Omega_{ab}(s,\varsigma;w)]=0\end{cases}$$

Pro. 2.10.3.
$$\begin{cases}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][\vartheta\cdot\Omega(s;w)]^n[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=\{\vartheta\cdot[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\}^n\\ [I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\{\vartheta\cdot[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\}^n[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\\ =[\vartheta\cdot\Omega(s;w)]^n[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][\vartheta\cdot\Omega(s;w)]^n\\ [[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)],[\vartheta\cdot\Omega(s;w)]^n]=0\end{cases}$$

Pro. 2.10.4.
$$\begin{cases}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=\{\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\}^n\\ [I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\{\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\}^n[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\\ =[\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n\\ [[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)],[\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)]^n]=0\end{cases}$$

Cor. 2.10.1.
$$\begin{cases}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]}\\ [I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\\ =e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}\\ [[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)],e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}]=0\end{cases}$$

Cor. 2.10.2.
$$\begin{cases} I_{(w+1)^{2s-1}}\Gamma(s-\frac{1}{2};w)=\Gamma(s-\frac{1}{2};w)I_{C^{2s-1}_{2s-1+w}}, \bar{\Gamma}(s-\frac{1}{2};w)I_{(w+1)^{2s-1}}=I_{C^{2s-1}_{2s-1+w}}\bar{\Gamma}(s-\frac{1}{2};w) \\ [I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\sigma(\frac{1}{2};w)\otimes I_{(w+1)^{2s-1}}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}} \\ [\sigma(\frac{1}{2};w)\otimes I_{(w+1)^{2s-1}}[I_{w+1}]\otimes\Gamma(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}] \\ [I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w)\otimes I_{(w+1)^{2s-1}}]=[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)] \end{cases}$$

Pro. 2.10.5. $(\sigma\otimes I_{(w+1)^{2s-1}},-i\varsigma)_a[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]N(s;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]Z_a(s,\varsigma;w)$

2.11 Permutation properties of constant matrices $\Gamma(s;w),\bar{\Gamma}(s;w)$

Def. 2.11.1. $S_{ex}(s,n)=(\overbrace{I_{w+1}\otimes\cdot\cdot\otimes I_{w+1}\otimes}^{n-1}S_{ex}\overbrace{\otimes I_{w+1}\otimes\cdot\cdot\otimes I}^{2s-n-1})$

Cor. 2.11.1. ??$\Gamma(s;w)=S_{ex}(s,n)\Gamma(s;w),\bar{\Gamma}(s;w)=\bar{\Gamma}(s;w)S_{ex}(s,n)$

Cor. 2.11.2. $S_{ex}(s,n)\Omega(s;w)S_{ex}(s,n)=\Omega(s;w)$

Cor. 2.11.3. $\hat{\psi}(s,\varsigma;w)=S_{ex}(s,n)\hat{\psi}(s,\varsigma;w),\forall n\in\{1,2,\cdot\cdot,2s+1\}$

2.12 Constant invariant tensor properties of matrices $\Gamma(s;w),\bar{\Gamma}(s;w)$

Thm. 2.12.1. $\Gamma(s;w)=e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}\Gamma(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}$

Proof: $\Omega_{ab}(s,\varsigma;w)\Gamma(s;w)=\Gamma(s;w)S_{ab}(s,\varsigma;w)$
$\Leftrightarrow 0=\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)\Gamma(s;w)-\Gamma(s;w)\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)$
$\Leftrightarrow \Gamma(s;w)=e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}\Gamma(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}$ □

Thm. 2.12.2. $\bar{\Gamma}(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\bar{\Gamma}(s;w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}$

Proof: $\bar{\Gamma}(s;w)\Omega_{ab}(s,\varsigma;w)=S_{ab}(s,\varsigma;w)\bar{\Gamma}(s;w)$
$\Leftrightarrow 0=-\bar{\Gamma}(s;w)\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)+\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)\bar{\Gamma}(s;w)$
$\Leftrightarrow \bar{\Gamma}(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\bar{\Gamma}(s;w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}$ □

2.13 Constant invariant tensor properties of matrices $\Gamma(s),\bar{\Gamma}(s)$

Thm. 2.13.1. $\Gamma(s)=e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s)}\Gamma(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$

Proof: $\Omega(s)\Gamma(s)=\Gamma(s)\sigma(s)$
$\Leftrightarrow 0=(i\omega+\varsigma\epsilon)\cdot\Omega(s)\Gamma(s)-(i\omega+\varsigma\epsilon)\cdot\Gamma(s)\sigma(s)$
$\Leftrightarrow \Gamma(s)=e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s)}\Gamma(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$ □

Thm. 2.13.2. $\bar{\Gamma}(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\bar{\Gamma}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s)}$

Proof: $\bar{\Gamma}(s)\Omega(s)=\sigma(s)\bar{\Gamma}(s)$
$\Leftrightarrow 0=-(i\omega+\varsigma\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)+(i\omega+\varsigma\epsilon)\cdot\sigma(s)\bar{\Gamma}(s)$
$\Leftrightarrow \bar{\Gamma}(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\bar{\Gamma}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s)}$ □

2.14 Constant invariant tensor properties of matrices $I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w),I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)$

Thm. 2.14.1. $[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}$

Proof: $\Omega_{ab}(s,\varsigma;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]$
$\Leftrightarrow 0=\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]$
$-\frac{i}{2}\vartheta^{ab}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]$
$\Leftrightarrow [I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]=e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}$ □

Thm. 2.14.2. $[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}$

Proof: $[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega_{ab}(s,\varsigma;w)=[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]$
$\Leftrightarrow 0=-\frac{i}{2}\vartheta^{ab}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega_{ab}(s,\varsigma;w)$
$+\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]$
$\Leftrightarrow [I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;w)}$ □

2.15 Constant invariant tensor properties of matrices $I\otimes\bar\Gamma(s-\frac{1}{2}), I\otimes\Gamma(s-\frac{1}{2})$

Thm. 2.15.1. $[I\otimes\Gamma(s-\frac{1}{2})]=e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s)}[I\otimes\Gamma(s-\frac{1}{2})]e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

Proof: $\Omega(s)[I\otimes\Gamma(s-\frac{1}{2})]=[I\otimes\Gamma(s-\frac{1}{2})][\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})]$
$\Leftrightarrow 0=(i\omega+\varsigma\epsilon)\cdot\Omega(s)[I\otimes\Gamma(s-\frac{1}{2})]$
$-(i\omega+\varsigma\epsilon)\cdot[I\otimes\Gamma(s-\frac{1}{2})][\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})]$
$\Leftrightarrow [I\otimes\Gamma(s-\frac{1}{2})]=e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s)}[I\otimes\Gamma(s-\frac{1}{2})]e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$ □

Thm. 2.15.2. $[I\otimes\bar\Gamma(s-\frac{1}{2})]=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}[I\otimes\bar\Gamma(s-\frac{1}{2})]e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s)}$

Proof: $[I\otimes\bar\Gamma(s-\frac{1}{2})]\Omega(s)=[\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})][I\otimes\bar\Gamma(s-\frac{1}{2})]$
$\Leftrightarrow 0=-(i\omega+\varsigma\epsilon)\cdot[I\otimes\bar\Gamma(s-\frac{1}{2})]\Omega(s)$
$+(i\omega+\varsigma\epsilon)\cdot[\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})][I\otimes\bar\Gamma(s-\frac{1}{2})]$
$\Leftrightarrow [I\otimes\bar\Gamma(s-\frac{1}{2})]=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}[I\otimes\bar\Gamma(s-\frac{1}{2})]e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s)}$ □

3 Perfect constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$

3.1 Introduction of constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$

Def. 3.1.1. $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s):=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s)\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{l_\varsigma}(s-\frac{1}{2}), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s):=\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{k_\varsigma}(s)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2})$

Def. 3.1.2. $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w):=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{l_\varsigma}(s-\frac{1}{2};w), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w):=\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{k_\varsigma}(s;w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)$

Cor. 3.1.1. $N(s;w)=[I_{w+1}\otimes\bar\Gamma(s-\frac{1}{2};w)]\Gamma(s;w), \bar N(s;w)=\bar\Gamma(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]$

Pro. 3.1.1. $\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)=N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w), \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{k_\varsigma}(s;w)=N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{l_\varsigma}(s-\frac{1}{2};w)$

Cor. 3.1.2. $\Gamma(s;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]N(s;w), \bar\Gamma(s;w)=\bar N(s;w)[I_{w+1}\otimes\bar\Gamma(s-\frac{1}{2};w)]$,

Pro. 3.1.2. $\Gamma(s;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar\Gamma(s-\frac{1}{2};w)]\Gamma(s;w)$

Cor. 3.1.3. $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)=N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;1), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)=N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;1)$

Self comment: The above indicates that $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)$,$N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$ are generalization of constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)$,$N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)$.

3.2 Introduction of constant matrices $N_{A_\varsigma}(s;w), N^{A_\varsigma}(s;w); \bar N_{A_\varsigma}(s;w), \bar N^{A_\varsigma}(s;w); N(s;w), \bar N(s;w)$

Def. 3.2.1.

$$\begin{cases} N_{A_\varsigma}(s;w)\prec N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w), N^{A_\varsigma}(s;w)\prec N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)|_{I_{C^{2s}_{2s+w}}\times I_{C^{2s-1}_{2s-1+w}}} \\ \bar N_{A_\varsigma}(s;w):=N^{+}_{A_\varsigma}(s;w)\succ N_{A_\varsigma l_\varsigma}{}^{k_\varsigma}(s;w), \bar N^{A_\varsigma}(s;w):=N^{+A_\varsigma}(s;w)\succ N^{A_\varsigma l_\varsigma}{}_{k_\varsigma}(s;w)|_{I_{C^{2s-1}_{2s-1+w}}\times I_{C^{2s}_{2s+w}}} \\ N(s;w)\prec N_{A_\varsigma\otimes l_\varsigma}{}^{k_\varsigma}(s;w)|_{(w+1)I_{C^{2s-1}_{2s-1+w}}\times I_{C^{2s}_{2s+w}}}, \bar N(s;w)=N^{+}(s;w)\prec N_{k_\varsigma}{}^{A_\varsigma\otimes l_\varsigma}(s;w)|_{I_{C^{2s}_{2s+w}}\times(w+1)I_{C^{2s-1}_{2s-1+w}}} \end{cases}$$

Explicit representation of $N_{A_\varsigma}(s), \bar N_{A_\varsigma}(s)$:

Cor. 3.2.1. $N_{A_\varsigma}(s)\simeq N^{A_\varsigma}(s)=\{\frac{1}{\sqrt{2s}}\begin{bmatrix}\sqrt{2s}&0&0&0&0\\0&\sqrt{2s-1}&0&0&0\\0&0&\cdot\cdot&0&0\\0&0&0&\sqrt{2}&0\\0&0&0&0&\sqrt{1}\\0&0&0&0&0\end{bmatrix}, \frac{1}{\sqrt{2s}}\begin{bmatrix}0&0&0&0&0\\\sqrt{1}&0&0&0&0\\0&\sqrt{2}&0&0&0\\0&0&\cdot\cdot&0&0\\0&0&0&\sqrt{2s-1}&0\\0&0&0&0&\sqrt{2s}\end{bmatrix}\}$

Cor. 3.2.2. $\bar N_{A_\varsigma}(s)\simeq \bar N^{A_\varsigma}(s)=\{\frac{1}{\sqrt{2s}}\begin{bmatrix}\sqrt{2s}&0&0&0&0&0\\0&\sqrt{2s-1}&0&0&0&0\\0&0&\cdot\cdot&0&0&0\\0&0&0&\sqrt{2}&0&0\\0&0&0&0&\sqrt{1}&0\end{bmatrix}, \frac{1}{\sqrt{2s}}\begin{bmatrix}0&\sqrt{1}&0&0&0&0\\0&0&\sqrt{2}&0&0&0\\0&0&0&\cdot\cdot&0&0\\0&0&0&0&\sqrt{2s-1}&0\\0&0&0&0&0&\sqrt{2s}\end{bmatrix}\}$

Explicit representation of $N(s), \bar N(s)$:

Cor. 3.2.3. $N^{A_\varsigma}(s) \leftrightarrow N^{+}(s) \simeq \bar{N}(s) = \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2s-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot\cdot & \cdot\cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s-1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix}$

Cor. 3.2.4. $N^{+}(s) \simeq \bar{N}(s) = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{1} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} \end{bmatrix}, \frac{1}{\sqrt{4}} \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4} \end{bmatrix}, \cdot\cdot$

3.3 Basic properties of constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$

Equality:

Pro. 3.3.1.

$$\begin{cases} N^{k'_\varsigma}_{A'_\varsigma l'_\varsigma}(s;w) \simeq N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w) \simeq N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w) \simeq N^{A'_\varsigma l'_\varsigma}_{k'_\varsigma}(s;w) \\ [N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)]^* \simeq N^{k'_\varsigma}_{A'_\varsigma l'_\varsigma}(s;w), [N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)]^* \simeq N^{A'_\varsigma l'_\varsigma}_{k'_\varsigma}(s;w) \end{cases}$$

Cor. 3.3.1.

$$\begin{cases} N_{A_\varsigma}(s;w) \simeq N^{A_\varsigma}(s;w) \simeq N_{A'_\varsigma}(s;w) \simeq N^{A'_\varsigma}(s;w); \bar{N}_{A_\varsigma}(s;w) \simeq \bar{N}^{A_\varsigma}(s;w) \simeq \bar{N}_{A'_\varsigma}(s;w) \simeq \bar{N}^{A'_\varsigma}(s;w) \\ N_{A_\varsigma}(s;w) = N^*_{A_\varsigma}(s;w), \bar{N}_{A_\varsigma}(s;w) = \bar{N}^*_{A_\varsigma}(s;w); N(s;w) = N^*(s;w), \bar{N}(s;w) = \bar{N}^*(s;w) \end{cases}$$

3.4 Orthogonal properties of constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$

Orthogonality:

Lem. 3.4.1. $\sum\limits_{k=0}^{2s-1} C^{w}_{w+k} = C^{w+1}_{w+2s}$

Lem. 3.4.2. $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w) N^{A'_\varsigma l_\varsigma}_{k'_\varsigma}(s;w) = \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{A'_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s;w)$

Proof: $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w) N^{A'_\varsigma l_\varsigma}_{k'_\varsigma}(s;w)$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s;w)\Gamma^{l_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{(2s-1)!}\delta^{B_\varsigma}_{(B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot{}_{)}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s;w)$$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s;w)\overbrace{\delta^{B_\varsigma}_{B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot}^{2s-1}$$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{A'_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s;w)$$ □

Thm. 3.4.1.

$$\begin{cases} N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{A_\varsigma l_\varsigma}_{m_\varsigma}(s;w) = \delta^{k_\varsigma}_{m_\varsigma}[\Leftrightarrow]N^{A_\varsigma}(s;w)\bar{N}_{A_\varsigma}(s;w) = I_{C^{2s}_{2s+w}}[\Leftrightarrow]\bar{N}(s;w)N(s;w) = I_{C^{2s}_{2s+w}} \\ N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{A_\varsigma m_\varsigma}_{k_\varsigma}(s;w) = (1+\frac{w}{2s})\delta^{m_\varsigma}_{l_\varsigma}[\Leftrightarrow]\bar{N}_{A_\varsigma}(s;w)N^{A_\varsigma}(s;w) = (1+\frac{w}{2s})I_{C^{2s-1}_{2s-1+w}} \\ N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{B_\varsigma l_\varsigma}_{k_\varsigma}(s;w) = \frac{1}{w+1}C^{2s}_{2s+w}\delta^{B_\varsigma}_{A_\varsigma}[\Leftrightarrow]tr[\bar{N}_{A_\varsigma}(s;w)N^{B_\varsigma}(s;w)] = \frac{1}{w+1}C^{2s}_{2s+w}\delta^{B_\varsigma}_{A_\varsigma} \end{cases}$$

Proof: $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{A_\varsigma l_\varsigma}_{m_\varsigma}(s;w)$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{m_\varsigma}(s;w)$$

$$= \delta^{k_\varsigma}_{m_\varsigma}$$ □

Proof: $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{A_\varsigma m_\varsigma}_{k_\varsigma}(s;w)$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{\overbrace{A_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdots}_{2s}}(s;w)\Gamma_{k_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma \cdots}^{2s}}(s;w)\delta^{A_\varsigma}_{A'_\varsigma}\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{(2s)!}\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{B_\varsigma}_{B_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s}\delta^{A_\varsigma}_{A'_\varsigma}\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{(2s)!}\delta^{(A_\varsigma}_{A_\varsigma}\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots)}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{(2s)!}[\delta^{A_\varsigma}_{A_\varsigma}\Gamma_{l_\varsigma}^{\overbrace{(B_\varsigma C_\varsigma \cdots)}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)+(2s-1)\Gamma_{l_\varsigma}^{\overbrace{(A_\varsigma C_\varsigma \cdots)}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{A_\varsigma C_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)]$$

$$= \tfrac{1}{(2s)!}[(2s-1)!\delta^{A_\varsigma}_{A_\varsigma}\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)+(2s-1)(2s-1)!\Gamma_{l_\varsigma}^{\overbrace{A_\varsigma C_\varsigma \cdots}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{A_\varsigma C_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)]$$

$$= \tfrac{1}{(2s)!}[(2s-1)!\delta^{A_\varsigma}_{A_\varsigma}+(2s-1)(2s-1)!]\delta^{m_\varsigma}_{l_\varsigma}$$

$$= (1+\tfrac{w}{2s})\delta^{m_\varsigma}_{l_\varsigma} \qquad \square$$

Proof: $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{A'_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdots}_{2s}}(s;w)\Gamma_{k_\varsigma}^{\overbrace{A'_\varsigma B_\varsigma C_\varsigma \cdots}^{2s}}(s;w)$$

$$= \tfrac{1}{(2s)!}\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{B_\varsigma}_{B_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s}$$

$$= \tfrac{1}{(2s)!}[\overbrace{\delta^{A'_\varsigma}_{A_\varsigma}\delta^{(B_\varsigma}_{B_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s}+\overbrace{\delta^{B_\varsigma}_{A_\varsigma}\delta^{(A'_\varsigma}_{B_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s}+\overbrace{\delta^{C_\varsigma}_{A_\varsigma}\delta^{(B'_\varsigma}_{B_\varsigma}\delta^{A_\varsigma}_{C_\varsigma}\cdots)}^{2s}+\cdots]$$

$$= \tfrac{1}{(2s)!}[\overbrace{\delta^{A'_\varsigma}_{A_\varsigma}\delta^{(B_\varsigma}_{B_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s}+\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s}+\overbrace{\delta^{(B_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{A_\varsigma}\cdots)}^{2s}+\cdots]$$

$$= \tfrac{1}{(2s)!}[\delta^{A'_\varsigma}_{A_\varsigma}\overbrace{\delta^{(B_\varsigma}_{B_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s-1}+(2s-1)\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s-1}]$$

$$= \tfrac{1}{(2s)!}[\delta^{A'_\varsigma}_{A_\varsigma}(2s-1)!\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s-\tfrac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \cdots}_{2s-1}}(s-\tfrac{1}{2};w)+(2s-1)\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s-1}]$$

$$= \tfrac{1}{(2s)!}[\delta^{A'_\varsigma}_{A_\varsigma}(2s-1)!\delta^{l_\varsigma}{}_{l_\varsigma}(s-\tfrac{1}{2};w)+(2s-1)\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s-1}]$$

$$= \tfrac{1}{(2s)!}[\delta^{A'_\varsigma}_{A_\varsigma}(2s-1)!C^{2s-1}_{2s-1+w}+(2s-1)\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s-1}]$$

$$= \tfrac{1}{(2s)!}[\delta^{A'_\varsigma}_{A_\varsigma}\tfrac{(2s-1)!}{(2s-1)!}\tfrac{(2s-1+w)!}{w!}+(2s-1)\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{C_\varsigma}_{C_\varsigma}\cdots)}^{2s-1}]$$

$$= \tfrac{1}{(2s)!}[\tfrac{(2s-1)!}{(2s-1)!}\tfrac{(2s-1+w)!}{w!}+\tfrac{(2s-1)!}{(2s-2)!}\tfrac{(2s-2+w)!}{w!}+\tfrac{(2s-1)!}{(2s-3)!}\tfrac{(2s-3+w)!}{w!}+\cdots+\tfrac{(2s-1)!}{0!}\tfrac{(0+w)!}{w!}]\delta^{A'_\varsigma}_{A_\varsigma}$$

$$= \tfrac{1}{2s}[\tfrac{(2s-1+w)!}{(2s-1)!w!}+\tfrac{(2s-2+w)!}{(2s-2)!w!}+\tfrac{(2s-3+w)!}{(2s-3)!w!}+\cdots+\tfrac{(0+w)!}{0!w!}]\delta^{A'_\varsigma}_{A_\varsigma}$$

$$= \tfrac{1}{2s}\sum_{k=0}^{2s-1}C^{w}_{w+k}\delta^{A'_\varsigma}_{A_\varsigma} = \tfrac{1}{2s}C^{w+1}_{w+2s}\delta^{A'_\varsigma}_{A_\varsigma} = \tfrac{1}{w+1}C^{2s}_{2s+w}\delta^{A'_\varsigma}_{A_\varsigma} \qquad \square$$

Pro. 3.4.1. $\begin{cases} N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{B_\varsigma m_\varsigma}_{k_\varsigma}(s;w)=\frac{1}{2s}[\delta^{B_\varsigma}_{A_\varsigma}\delta^{m_\varsigma}_{l_\varsigma}+(2s-1)N^{m_\varsigma}_{A_\varsigma n_\varsigma}(s-\frac{1}{2};w)N^{B_\varsigma n_\varsigma}_{l_\varsigma}(s-\frac{1}{2};w)] \\ \bar{N}_{A_\varsigma}(s;w)N^{B_\varsigma}(s;w)=\frac{1}{2s}[\delta^{B_\varsigma}_{A_\varsigma}I_{C^{2s-1}_{2s-1+w}}+(2s-1)N^{B_\varsigma}(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s-\frac{1}{2};w)] \end{cases}$

Proof: $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)N^{A'_\varsigma m_\varsigma}_{k_\varsigma}(s;w)$

$$= \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{(2s)!}\overbrace{\delta^{(A'_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot)}^{2s}\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{(2s)!}[\overbrace{\delta^{A'_\varsigma}_{A_\varsigma}\delta^{(B'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot)}^{2s}+\overbrace{\delta^{B'_\varsigma}_{A_\varsigma}\delta^{(A'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot)}^{2s}+\overbrace{\delta^{C'_\varsigma}_{A_\varsigma}\delta^{(B'_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{C_\varsigma}\cdot\cdot)}^{2s}+\cdots]\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{2s}(\overbrace{\delta^{A'_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot}^{2s}+\overbrace{\delta^{B'_\varsigma}_{A_\varsigma}\delta^{A'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot}^{2s}+\overbrace{\delta^{C'_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{C_\varsigma}\cdot\cdot}^{2s}+\cdots)\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{2s}[\overbrace{\delta^{A'_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot}^{2s}+(2s-1)\overbrace{\delta^{B'_\varsigma}_{A_\varsigma}\delta^{A'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot}^{2s}]\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)$$

$$= \tfrac{1}{2s}[\delta^{A'_\varsigma}_{A_\varsigma}\Gamma^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)+(2s-1)\Gamma^{\overbrace{A'_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\Gamma^{m_\varsigma}_{\underbrace{A_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2};w)]$$

$$= \tfrac{1}{2s}[\delta^{B_\varsigma}_{A_\varsigma}\delta^{m_\varsigma}_{l_\varsigma}+(2s-1)N^{m_\varsigma}_{A_\varsigma n_\varsigma}(s-\tfrac{1}{2};w)N^{A'_\varsigma n_\varsigma}_{l_\varsigma}(s-\tfrac{1}{2};w)] \qquad \square$$

3.5 Raising and lowering indices of constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$
(Existing $\varepsilon_{A_\varsigma B_\varsigma}$ is the prerequisite.)
Raising and lowering indices:

Pro. 3.5.1.

$$\begin{cases} N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)=\varepsilon^{k_\varsigma m_\varsigma}(s;w)\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{l_\varsigma n_\varsigma}(s-\tfrac{1}{2};w)N^{B_\varsigma n_\varsigma}_{m_\varsigma}(s;w)\\ N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)=\varepsilon_{k_\varsigma m_\varsigma}(s;w)\varepsilon^{A_\varsigma B_\varsigma}\varepsilon^{l_\varsigma n_\varsigma}(s-\tfrac{1}{2};w)N^{m_\varsigma}_{B_\varsigma n_\varsigma}(s;w)\end{cases}$$

Proof: $\varepsilon^{k_\varsigma m_\varsigma}(s;w)\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)N^{B_\varsigma n_\varsigma}_{m_\varsigma}(s;w)$

$$= \Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A'_\varsigma E'_\varsigma}\varepsilon^{B'_\varsigma F'_\varsigma}\varepsilon^{C'_\varsigma G'_\varsigma}\cdot\cdot}^{2s}\Gamma^{m_\varsigma}_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdot\cdot}_{2s}}(s;w)\varepsilon_{A_\varsigma B_\varsigma}$$

$$\Gamma^{\overbrace{B''_\varsigma C''_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\underbrace{\varepsilon_{B''_\varsigma F''_\varsigma}\varepsilon_{C''_\varsigma G''_\varsigma}\cdot\cdot}_{2s-1}\Gamma^{\overbrace{F''_\varsigma G''_\varsigma\cdot\cdot}^{2s-1}}_{n_\varsigma}(s-\tfrac{1}{2};w)N^{B_\varsigma n_\varsigma}_{m_\varsigma}(s;w)$$

$$= \Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A'_\varsigma E'_\varsigma}\varepsilon^{B'_\varsigma F'_\varsigma}\varepsilon^{C'_\varsigma G'_\varsigma}\cdot\cdot}^{2s}\Gamma^{m_\varsigma}_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdot\cdot}_{2s}}(s;w)\varepsilon_{A_\varsigma B_\varsigma}\Gamma^{\overbrace{B''_\varsigma C''_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)\underbrace{\varepsilon_{B''_\varsigma F''_\varsigma}\varepsilon_{C''_\varsigma G''_\varsigma}\cdot\cdot}_{2s-1}\Gamma^{\overbrace{B_\varsigma F''_\varsigma G''_\varsigma\cdot\cdot}^{2s}}_{m_\varsigma}(s;w)$$

$$= \Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A'_\varsigma E'_\varsigma}\varepsilon^{B'_\varsigma F'_\varsigma}\varepsilon^{C'_\varsigma G'_\varsigma}\cdot\cdot}^{2s}\tfrac{1}{(2s)!}\underbrace{\delta^{B_\varsigma}_{(E'_\varsigma}\delta^{F''_\varsigma}_{F'_\varsigma}\delta^{G''_\varsigma}_{G'_\varsigma}\cdot\cdot)}_{2s}\varepsilon_{A_\varsigma B_\varsigma}\underbrace{\varepsilon_{B''_\varsigma F''_\varsigma}\varepsilon_{C''_\varsigma G''_\varsigma}\cdot\cdot}_{2s-1}\Gamma^{\overbrace{B''_\varsigma C''_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)$$

$$= \Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\overbrace{\varepsilon^{A'_\varsigma B_\varsigma}\varepsilon^{B'_\varsigma F''_\varsigma}\varepsilon^{C'_\varsigma G''_\varsigma}\cdot\cdot}^{2s}\underbrace{\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{B''_\varsigma F''_\varsigma}\varepsilon_{C''_\varsigma G''_\varsigma}\cdot\cdot}_{2s}\Gamma^{\overbrace{B''_\varsigma C''_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)$$

$$= \Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\underbrace{\delta^{A'_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{B''_\varsigma}\delta^{C'_\varsigma}_{C''_\varsigma}\cdot\cdot}_{2s}\Gamma^{\overbrace{B''_\varsigma C''_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)$$

$$= N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w) \qquad \square$$

Cor. 3.5.1.

$$\begin{cases} N_{A_\varsigma}(s;w)\varepsilon(s-\frac{1}{2};w)=\varepsilon_{A_\varsigma B_\varsigma}\varepsilon(s;w)N^{B_\varsigma}(s;w),\varepsilon(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w)=\bar{N}^{B_\varsigma}(s;w)\varepsilon_{B_\varsigma A_\varsigma}\varepsilon(s;w)\\ N^{A_\varsigma}(s;w)\varepsilon(s-\frac{1}{2};w)=\varepsilon^{A_\varsigma B_\varsigma}\varepsilon(s;w)N_{B_\varsigma}(s;w),\varepsilon(s-\frac{1}{2};w)\bar{N}^{A_\varsigma}(s;w)=\bar{N}_{B_\varsigma}(s;w)\varepsilon^{B_\varsigma A_\varsigma}\varepsilon(s;w)\\ N(s;w)\varepsilon(s;w)=[\varepsilon(\frac{1}{2};w)\otimes\varepsilon(s-\frac{1}{2};w)]N(s;w),\varepsilon(s;w)\bar{N}(s;w)=\bar{N}(s;w)[\varepsilon(\frac{1}{2};w)\otimes\varepsilon(s-\frac{1}{2};w)] \end{cases}$$

Proof: $\Gamma(s;w)\varepsilon(s;w)=\underbrace{\varepsilon(\frac{1}{2};w)\otimes\cdot\cdot\otimes\varepsilon(\frac{1}{2};w)}_{2s}\Gamma(s;w)$

$\Leftrightarrow[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Gamma(s;w)\varepsilon(s;w)=[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\underbrace{\varepsilon(\frac{1}{2};w)\otimes\cdot\cdot\otimes\varepsilon(\frac{1}{2};w)}_{2s}\Gamma(s;w)$

$\Leftrightarrow N(s;w)\varepsilon(s;w)=[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\underbrace{\varepsilon(\frac{1}{2};w)\otimes\cdot\cdot\otimes\varepsilon(\frac{1}{2};w)}_{2s}\Gamma(s;w)$

$\Leftrightarrow N(s;w)\varepsilon(s;w)=\{\varepsilon(\frac{1}{2};w)\otimes[\bar{\Gamma}(s-\frac{1}{2};w)\underbrace{\varepsilon(\frac{1}{2};w)\otimes\cdot\cdot\otimes\varepsilon(\frac{1}{2};w)]}_{2s-1}\}\Gamma(s;w)$

$\Leftrightarrow N(s;w)\varepsilon(s;w)=[\varepsilon(\frac{1}{2};w)\otimes\varepsilon(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Gamma(s;w)$
$\Leftrightarrow N(s;w)\varepsilon(s;w)=[\varepsilon(\frac{1}{2};w)\otimes\varepsilon(s-\frac{1}{2};w)]N(s;w)$ □

Penrose standard raising and lowering rules:

Pro. 3.5.2.
$$\begin{cases} N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)=(-1)^{2s}[\varsigma^{2s}\varepsilon^{k_\varsigma m_\varsigma}(s;w)](-\varsigma\varepsilon_{A_\varsigma B_\varsigma})[(-\varsigma)^{2s-1}\varepsilon_{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)]N^{B_\varsigma n_\varsigma}_{m_\varsigma}(s;w)\\ N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)=(-1)^{2s}[(-\varsigma)^{2s}\varepsilon_{k_\varsigma m_\varsigma}(s;w)](\varsigma\varepsilon^{A_\varsigma B_\varsigma})[\varsigma^{2s-1}\varepsilon^{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)]N^{m_\varsigma}_{B_\varsigma n_\varsigma}(s;w) \end{cases}$$

3.6 Spin matrix transformation I of constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w),N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$

Pro. 3.6.1.
$$\begin{cases} N^{A_\varsigma m_\varsigma}_{k_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{l_\varsigma}_{B_\varsigma m_\varsigma}(s;w)=\frac{1}{2s}\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s;w)[\Leftrightarrow]N^{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)=\frac{1}{2s}\sigma^{\alpha_\varsigma}(s;w)\\ [\Leftrightarrow]\bar{N}(s;w)\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}N(s;w)=\frac{1}{2s}\sigma(s;w)\\ N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s;w)=\frac{1}{2s}\sigma^{\alpha_\varsigma}{}_{m_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)[\Leftrightarrow]\bar{N}_{B_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{A_\varsigma}(s;w)=\frac{1}{2s}\sigma^{\alpha_\varsigma}(s-\frac{1}{2};w) \end{cases}$$

Proof: $N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{A'_\varsigma m_\varsigma}(s;w)$

$=\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s;w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2};w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$=\frac{1}{(2s)!}\delta^{A_\varsigma}_{(A'_\varsigma}\delta^{B_\varsigma}_{B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot{}_)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$=\frac{1}{(2s)!}[\delta^{A_\varsigma}_{A'_\varsigma}\delta^{B_\varsigma}_{(B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot{}_)+\delta^{A_\varsigma}_{B'_\varsigma}\delta^{B_\varsigma}_{(A'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot{}_)+\delta^{A_\varsigma}_{C'_\varsigma}\delta^{B_\varsigma}_{(B'_\varsigma}\delta^{C_\varsigma}_{A'_\varsigma}\cdot\cdot{}_)+\cdots]\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$=\frac{1}{(2s)!}[\delta^{A_\varsigma}_{A'_\varsigma}\delta^{B_\varsigma}_{(B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot{}_)+(2s-1)\delta^{A_\varsigma}_{B'_\varsigma}\delta^{B_\varsigma}_{(A'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot{}_)]\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$=\frac{1}{(2s)!}(2s-1)\delta^{A_\varsigma}_{B'_\varsigma}\delta^{B_\varsigma}_{(A'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdot\cdot{}_)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$=\frac{1}{(2s)!}(2s-1)\sigma^{\alpha_\varsigma}{}_{B'_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{(A'_\varsigma C'_\varsigma\cdot\cdot)}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$=\frac{1}{2s}(2s-1)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)\sigma^{\alpha_\varsigma}{}_{B'_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma C'_\varsigma\cdot\cdot}_{2s-1}}(s-\frac{1}{2};w)$

$=\frac{1}{2s}\sigma^{\alpha_\varsigma}{}_{m_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)$ □

Pro. 3.6.2.
$$\begin{cases} N^{A_\varsigma m_\varsigma}_{k_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{l_\varsigma}_{B_\varsigma m_\varsigma}(s;w)=\frac{1}{2s}S_{abk_\varsigma}{}^{l_\varsigma}(s;w)[\Leftrightarrow]N^{A_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)=\frac{1}{2s}S_{ab}(s,\varsigma;w)\\ [\Leftrightarrow]\bar{N}(s;w)S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}N(s;w)=\frac{1}{2s}S_{ab}(s,\varsigma;w)\\ N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s;w)=\frac{1}{2s}S_{abm_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)[\Leftrightarrow]\bar{N}_{B_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{A_\varsigma}(s;w)=\frac{1}{2s}S_{ab}(s-\frac{1}{2},\varsigma;w) \end{cases}$$

Proof: $N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)S_{abA_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{A'_\varsigma m_\varsigma}(s;w)$

$= \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{k_\varsigma}(s;w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)S_{abA_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{k_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$= \frac{1}{(2s)!}\delta^{A_\varsigma}_{(A'_\varsigma}\delta^{B_\varsigma}_{B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdots{}_)S_{abA_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$= \frac{1}{(2s)!}[\delta^{A_\varsigma}_{A'_\varsigma}\delta^{B_\varsigma}_{(B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdots{}_) + \delta^{A_\varsigma}_{B'_\varsigma}\delta^{B_\varsigma}_{(A'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdots{}_) + \delta^{A_\varsigma}_{C'_\varsigma}\delta^{B_\varsigma}_{(B'_\varsigma}\delta^{C_\varsigma}_{A'_\varsigma}\cdots{}_) + \cdots]S_{abA_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$= \frac{1}{(2s)!}[\delta^{A_\varsigma}_{A'_\varsigma}\delta^{B_\varsigma}_{(B'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdots{}_) + (2s-1)\delta^{A_\varsigma}_{B'_\varsigma}\delta^{B_\varsigma}_{(A'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdots{}_)]S_{abA_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$= \frac{1}{(2s)!}(2s-1)\delta^{A_\varsigma}_{B'_\varsigma}\delta^{B_\varsigma}_{(A'_\varsigma}\delta^{C_\varsigma}_{C'_\varsigma}\cdots{}_)S_{abA_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$= \frac{1}{(2s)!}(2s-1)S_{abB'_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{(A'_\varsigma C'_\varsigma\cdots)}_{2s-1}}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)$

$= \frac{1}{2s}(2s-1)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{m_\varsigma}(s-\frac{1}{2};w)S_{abB'_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma C'_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)$

$= \frac{1}{2s}S_{abm_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)$ □

3.7 Spin matrix transformation II of constant invariant tensors $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w), N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)$

Thm. 3.7.1.

$$\begin{cases}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s;w) + \sigma^{\alpha_\varsigma}{}_{l_\varsigma}{}^{m_\varsigma}(s-\frac{1}{2};w)N^{k_\varsigma}_{A_\varsigma m_\varsigma}(s;w) = N^{j_\varsigma}_{A_\varsigma l_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{j_\varsigma}{}^{k_\varsigma}(s;w)\\ N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w) + N^{B_\varsigma m_\varsigma}_{k_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{m_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w) = \sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{j_\varsigma}(s;w)N^{B_\varsigma l_\varsigma}_{j_\varsigma}(s;w)\end{cases}$$

$$\begin{cases}\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w) + \sigma^{\alpha_\varsigma}(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w) = \bar{N}_{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}(s;w)\\ N^{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w) + N^{B_\varsigma}(s;w)\sigma^{\alpha_\varsigma}(s-\frac{1}{2};w) = \sigma^{\alpha_\varsigma}(s;w)N^{B_\varsigma}(s;w)\end{cases}$$

$$\begin{cases}[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}} + I_{w+1}\otimes\sigma^{\alpha_\varsigma}(s-\frac{1}{2};w)]N(s;w) = N(s;w)\sigma^{\alpha_\varsigma}(s;w)\\ \bar{N}(s;w)[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}} + I_{w+1}\otimes\sigma^{\alpha_\varsigma}(s-\frac{1}{2};w)] = \sigma^{\alpha_\varsigma}(s;w)\bar{N}(s;w)\end{cases}$$

Proof: $\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w) = \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$

$\Rightarrow \Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{j_\varsigma}\Omega^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w) = \Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{j_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$

$\Leftrightarrow \Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{j_\varsigma}[\sigma_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\underbrace{\delta^{B'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdots}_{2s-1} + \delta^{A'_\varsigma}_{A_\varsigma}\Omega^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)]\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w) = \Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{j_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$

$\Leftrightarrow [\sigma_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s-1}}_{j_\varsigma} + \delta^{A'_\varsigma}_{A_\varsigma}\sigma_{j_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma\cdots}^{2s}}_{n_\varsigma}(s;w)]\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(s;w) = \Gamma^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}_{j_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$

$\Leftrightarrow \sigma_{A_\varsigma}{}^{A'_\varsigma}(\frac{1}{2};w)N^{l_\varsigma}_{A'_\varsigma j_\varsigma}(s;w) + \sigma_{j_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)N^{l_\varsigma}_{A_\varsigma n_\varsigma}(s;w) = N^{k_\varsigma}_{A_\varsigma j_\varsigma}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$

$\Leftrightarrow \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s;w) + \sigma^{\alpha_\varsigma}{}_{l_\varsigma}{}^{m_\varsigma}(s-\frac{1}{2};w)N^{k_\varsigma}_{A_\varsigma m_\varsigma}(s;w) = N^{j_\varsigma}_{A_\varsigma l_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{j_\varsigma}{}^{k_\varsigma}(s;w)$ □

Thm. 3.7.2.

$$\begin{cases}S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s;w) + S_{abl_\varsigma}{}^{m_\varsigma}(s-\frac{1}{2};w)N^{k_\varsigma}_{A_\varsigma m_\varsigma}(s;w) = N^{j_\varsigma}_{A_\varsigma l_\varsigma}(s;w)S_{abj_\varsigma}{}^{k_\varsigma}(s;w)\\ N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w) + N^{B_\varsigma m_\varsigma}_{k_\varsigma}(s;w)S_{abm_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w) = S_{abk_\varsigma}{}^{j_\varsigma}(s;w)N^{B_\varsigma l_\varsigma}_{j_\varsigma}(s;w)\end{cases}$$

$$\begin{cases}S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)+S_{ab}(s-\frac{1}{2},\varsigma;w)\bar{N}_{A_\varsigma}(s;w)=\bar{N}_{A_\varsigma}(s;w)S_{ab}(s,\varsigma;w)\\ N^{A_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+N^{B_\varsigma}(s;w)S_{ab}(s-\frac{1}{2},\varsigma;w)=S_{ab}(s,\varsigma;w)N^{B_\varsigma}(s;w)\end{cases}$$

$$\begin{cases}[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]N(s;w)=N(s;w)S_{ab}(s,\varsigma;w)\\ \bar{N}(s;w)[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]=S_{ab}(s,\varsigma;w)\bar{N}(s;w)\end{cases}$$

Proof: $\Omega_{ab\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}{}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)S_{abk_\varsigma}{}^{l_\varsigma}(s;w)$

$$\Rightarrow\Gamma_{j_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}\Omega_{ab\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}{}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s;w)\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)=\Gamma_{j_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)S_{abk_\varsigma}{}^{l_\varsigma}(s;w)$$

$$\Leftrightarrow\Gamma_{j_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}[S_{abA_\varsigma}{}^{A'_\varsigma}(\tfrac{1}{2};w)\underbrace{\delta^{B'_\varsigma}_{B_\varsigma}\delta^{C'_\varsigma}_{C_\varsigma}\cdot\cdot}_{2s-1}+\delta^{A'_\varsigma}_{A_\varsigma}\Omega_{ab\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}{}^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}(s-\tfrac{1}{2};w)]\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)=\Gamma_{j_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)S_{abk_\varsigma}{}^{l_\varsigma}(s;w)$$

$$\Leftrightarrow[S_{abA_\varsigma}{}^{A'_\varsigma}(\tfrac{1}{2};w)\Gamma_{j_\varsigma}^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s-1}}+\delta^{A'_\varsigma}_{A_\varsigma}\sigma_{j_\varsigma}{}^{n_\varsigma}(s-\tfrac{1}{2};w)\Gamma_{n_\varsigma}^{\overbrace{B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s;w)]\Gamma^{l_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s;w)=\Gamma_{j_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)S_{abk_\varsigma}{}^{l_\varsigma}(s;w)$$

$$\Leftrightarrow S_{abA_\varsigma}{}^{A'_\varsigma}(\tfrac{1}{2};w)N^{l_\varsigma}_{A'_\varsigma j_\varsigma}(s;w)+S_{abj_\varsigma}{}^{n_\varsigma}(s-\tfrac{1}{2};w)N^{l_\varsigma}_{A_\varsigma n_\varsigma}(s;w)=N^{k_\varsigma}_{A_\varsigma j_\varsigma}(s;w)S_{abk_\varsigma}{}^{l_\varsigma}(s;w)$$
$$\Leftrightarrow S_{abA_\varsigma}{}^{B_\varsigma}(\tfrac{1}{2};w)N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s;w)+S_{abl_\varsigma}{}^{m_\varsigma}(s-\tfrac{1}{2};w)N^{k_\varsigma}_{A_\varsigma m_\varsigma}(s;w)=N^{j_\varsigma}_{A_\varsigma l_\varsigma}(s;w)S_{abj_\varsigma}{}^{k_\varsigma}(s;w)\qquad\square$$

3.8 Introduction and properties of constant invariant tensors $N^{k_\varsigma}_{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_\varsigma}(s;w),N_{k_\varsigma}^{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_\varsigma}(s;w)$

Def. 3.8.1. $\begin{cases}N^{k_\varsigma}_{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w):=\Gamma^{k_\varsigma}_{A_{\varsigma1}\cdot\cdot A_{\varsigma2s}}(s;w)\Gamma_{l_\varsigma}^{A_{\varsigma n+1}\cdot\cdot A_{\varsigma2s}}(s-\frac{n}{2};w)\\ N_{k_\varsigma}^{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w):=\Gamma_{k_\varsigma}^{A_{\varsigma1}\cdot\cdot A_{\varsigma2s}}(s;w)\Gamma^{l_\varsigma}_{A_{\varsigma n+1}\cdot\cdot A_{\varsigma2s}}(s-\frac{n}{2};w)\end{cases}$

Equality:

Pro. 3.8.1. $N^{k'_\varsigma}_{A'_{\varsigma1}\cdot\cdot A'_{\varsigma n}l'_{\varsigma n}}(s;w)\simeq N^{k_\varsigma}_{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w)\simeq N_{k_\varsigma}^{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w)\simeq N_{k'_\varsigma}^{A'_{\varsigma1}\cdot\cdot A'_{\varsigma n}l'_{\varsigma n}}(s;w)$

Pro. 3.8.2. $[N^{k_\varsigma}_{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w)]^*\simeq N^{k'_\varsigma}_{A'_{\varsigma1}\cdot\cdot A'_{\varsigma n}l'_{\varsigma n}}(s;w),[N_{k_\varsigma}^{A_{\varsigma1}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w)]^*\simeq N_{k'_\varsigma}^{A'_{\varsigma1}\cdot\cdot A'_{\varsigma n}l'_{\varsigma n}}(s;w)$

Expansibility:

Pro. 3.8.3.
$$\begin{cases}N^{k_\varsigma}_{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w)=N^{k_\varsigma}_{A_{\varsigma1}l_{\varsigma1}}(s;w)N^{l_{\varsigma1}}_{A_{\varsigma2}l_{\varsigma2}}(s-\frac{1}{2};w)\cdot\cdot N^{l_{\varsigma n-1}}_{A_{\varsigma n}l_{\varsigma n}}(s-\frac{n-1}{2};w)\\ N_{k_\varsigma}^{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma n}l_{\varsigma n}}(s;w)=N_{k_\varsigma}^{A_{\varsigma1}l_{\varsigma1}}(s;w)N_{l_{\varsigma1}}^{A_{\varsigma2}l_{\varsigma2}}(s-\frac{1}{2};w)\cdot\cdot N_{l_{\varsigma n-1}}^{A_{\varsigma n}l_{\varsigma n}}(s-\frac{n-1}{2};w)\end{cases}$$

Pro. 3.8.4.
$$\begin{cases}\Gamma^{k_\varsigma}_{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma2s}}(s;w)=N^{k_\varsigma}_{A_{\varsigma1}l_{\varsigma1}}(s;w)N^{l_{\varsigma1}}_{A_{\varsigma2}l_{\varsigma2}}(s-\frac{1}{2};w)\cdot\cdot N^{l_{\varsigma2s-1}}_{A_{\varsigma2s}l_{\varsigma2s}}(\frac{1}{2};w)\\ \Gamma_{k_\varsigma}^{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma2s}}(s;w)=N_{k_\varsigma}^{A_{\varsigma1}l_{\varsigma1}}(s;w)N_{l_{\varsigma1}}^{A_{\varsigma2}l_{\varsigma2}}(s-\frac{1}{2};w)\cdot\cdot N_{l_{\varsigma2s-1}}^{A_{\varsigma2s}l_{\varsigma2s}}(\frac{1}{2};w)\end{cases}$$
$$\begin{cases}\Gamma^{k_\varsigma}_{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma2s}}(s;w)\succ\Gamma_{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma2s}}(s;w)=N_{A_{\varsigma1}}(s;w)N_{A_{\varsigma2}}(s-\frac{1}{2};w)\cdot\cdot N_{A_{\varsigma2s}}(\frac{1}{2};w)\\ \Gamma_{k_\varsigma}^{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma2s}}(s;w)\succ\Gamma^{A_{\varsigma1}A_{\varsigma2}\cdot\cdot A_{\varsigma2s}}(s;w)=N^{A_{\varsigma1}}(s;w)N^{A_{\varsigma2}}(s-\frac{1}{2};w)\cdot\cdot N^{A_{\varsigma2s}}(\frac{1}{2};w)\end{cases}$$
$$\begin{cases}\bar{\Gamma}(s;w)=\bar{N}(s;w)[I_{w+1}\otimes\bar{N}(s-\frac{1}{2};w)]\cdot\cdot[I_{(w+1)^{2s-2}}\otimes\bar{N}(1)][I_{(w+1)^{2s-1}}\otimes\bar{N}(\frac{1}{2};w)]\\ \Gamma(s;w)=[I_{(w+1)^{2s-1}}\otimes N(\frac{1}{2};w)][I_{(w+1)^{2s-2}}\otimes N(1)]\cdot\cdot[I_{w+1}\otimes N(s-\frac{1}{2};w)]N(s;w)\end{cases}$$

3.9 Several identities of constant matrices $N(s;w),\bar{N}(s;w)$

Pro. 3.9.1.
$$\begin{cases}\bar{N}(s;w)[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]N(s;w)=\sigma(s;w)\\ N(s;w)\sigma(s;w)\bar{N}(s;w)=[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]N(s;w)\bar{N}(s;w)\\ N(s;w)\sigma(s;w)\bar{N}(s;w)=N(s;w)\bar{N}(s;w)\{\vartheta\cdot[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\}^n\\ [N(s;w)\bar{N}(s;w),\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]=0\end{cases}$$

Pro. 3.9.2.
$$\begin{cases}\bar{N}(s;w)[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]N(s;w)=S_{ab}(s,\varsigma;w)\\ N(s;w)S_{ab}(s,\varsigma;w)\bar{N}(s;w)=[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]N(s;w)\bar{N}(s;w)\\ N(s;w)S_{ab}(s,\varsigma;w)\bar{N}(s;w)=N(s;w)\bar{N}(s;w)[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\\ [N(s;w)\bar{N}(s;w),S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]=0\end{cases}$$

Pro. 3.9.3.
$$\begin{cases}\bar{N}(s;w)\{\vartheta\cdot[\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\}^nN(s;w)=[\vartheta\cdot\sigma(s;w)]^n\\ N(s;w)[\vartheta\cdot\sigma(s;w)]^n\bar{N}(s;w)=\{\vartheta\cdot[\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\}^nN(s;w)\bar{N}(s;w)\\ N(s;w)[\vartheta\cdot\sigma(s;w)]^n\bar{N}(s;w)=N(s;w)\bar{N}(s;w)\{\vartheta\cdot[\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\}^n\\ [N(s;w)\bar{N}(s;w),\{\vartheta\cdot[\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]\}^n]=0\end{cases}$$

Pro. 3.9.4.
$$\begin{cases}\bar{N}(s;w)\{\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\}^nN(s;w)=[\vartheta^{ab}S_{ab}(s,\varsigma;w)]^n\\ N(s;w)[\vartheta^{ab}S_{ab}(s,\varsigma;w)]^n\bar{N}(s;w)=\{\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\}^nN(s;w)\bar{N}(s;w)\\ N(s;w)[\vartheta^{ab}S_{ab}(s,\varsigma;w)]^n\bar{N}(s;w)=N(s;w)\bar{N}(s;w)\{\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\}^n\\ [N(s;w)\bar{N}(s;w),\{\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]\}^n]=0\end{cases}$$

Cor. 3.9.1.
$$\begin{cases}\bar{N}(s;w)e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]}N(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\\ N(s;w)e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\bar{N}(s;w)=e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]}N(s;w)\bar{N}(s;w)\\ N(s;w)e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\bar{N}(s;w)=N(s;w)\bar{N}(s;w)e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]}\\ [N(s;w)\bar{N}(s;w),e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]}]=0\end{cases}$$

3.10 Several corollaries of constant matrices $N(s;w),\bar{N}(s;w)$

Cor. 3.10.1.
$$\begin{cases}\bar{N}(s;w)\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}N(s;w)=\frac{1}{2s}\sigma(s;w)\\ \bar{N}(s;w)I_{w+1}\otimes\sigma(s-\frac{1}{2};w)N(s;w)=(1-\frac{1}{2s})\sigma(s;w)\\ N^{A_\varsigma}(s;w)\sigma(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w)=(1-\frac{1}{2s})\sigma(s;w)\\ \bar{N}_{A_\varsigma}(s;w)\sigma(s;w)N^{A_\varsigma}(s;w)=(1+\frac{w+1}{2s})\sigma(s-\frac{1}{2};w)\end{cases}$$

Cor. 3.10.2.
$$\begin{cases}\bar{N}(s;w)S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}N(s;w)=\frac{1}{2s}S_{ab}(s,\varsigma;w)\\ \bar{N}(s;w)I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)N(s;w)=(1-\frac{1}{2s})S_{ab}(s,\varsigma;w)\\ N^{A_\varsigma}(s;w)S_{ab}(s-\frac{1}{2},\varsigma;w)\bar{N}_{A_\varsigma}(s;w)=(1-\frac{1}{2s})S_{ab}(s,\varsigma;w)\\ \bar{N}_{A_\varsigma}(s;w)S_{ab}(s,\varsigma;w)N^{A_\varsigma}(s;w)=(1+\frac{w+1}{2s})S_{ab}(s-\frac{1}{2},\varsigma;w)\end{cases}$$

Cor. 3.10.3.
$$\begin{cases}\bar{N}(1)[\sigma(\frac{1}{2};w)\otimes I_2+I_{w+1}\otimes\sigma(\frac{1}{2};w)]N(1)=\sigma(1)\\ \bar{N}(\frac{3}{2})\{\sigma(\frac{1}{2};w)\otimes I_3+I_{w+1}\otimes\{\bar{N}(1)[\sigma(\frac{1}{2};w)\otimes I_2+I_{w+1}\otimes\sigma(\frac{1}{2};w)]N(1)\}\}N(\frac{3}{2})=\sigma(\frac{3}{2})\\ \bar{N}(s;w)\cdot\cdot\bar{N}(\frac{3}{2})\{\sigma(\frac{1}{2};w)\otimes I_3+I_{w+1}\otimes\{\bar{N}(1)[\sigma(\frac{1}{2};w)\otimes I_2+I_{w+1}\otimes\sigma(\frac{1}{2};w)]N(1)\}\}N(\frac{3}{2})\cdot\cdot N(s;w)=\sigma(s;w)\end{cases}$$

3.11 Constant invariant tensor properties of matrices $N(s;w),\bar{N}(s;w)$

Thm. 3.11.1. $N(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}N(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}$

Proof: $[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]N(s;w)=N(s;w)S_{ab}(s,\varsigma;w)$
$\Leftrightarrow 0=[\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+\frac{i}{2}\vartheta^{ab}I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]N(s;w)-\frac{i}{2}\vartheta^{ab}N(s;w)S_{ab}(s,\varsigma;w)$
$\Leftrightarrow N(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}N(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}$ □

Thm. 3.11.2. $\bar{N}(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\bar{N}(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}$

Proof: $\bar{N}(s;w)[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]=S_{ab}(s,\varsigma;w)\bar{N}(s;w)$
$\Leftrightarrow 0=\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)\bar{N}(s;w)-\bar{N}(s;w)[\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}+\frac{i}{2}\vartheta^{ab}I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]$
$\Leftrightarrow \bar{N}(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\bar{N}(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}$ □

3.12 Constant invariant tensor properties of matrices $N(s),\bar{N}(s)$

Thm. 3.12.1. $N(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}N(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$

Proof: $[\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})]N(s)=N(s)\sigma(s)$
$\Leftrightarrow 0=[(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})\otimes I_{2s}+(i\omega+\varsigma\epsilon)\cdot I\otimes\sigma(s-\frac{1}{2})]N(s)-(i\omega+\varsigma\epsilon)\cdot N(s)\sigma(s)$
$\Leftrightarrow N(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}N(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$ □

Thm. 3.12.2. $\bar{N}(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\bar{N}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

Proof: $\bar{N}(s)[\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})]=\sigma(s)\bar{N}(s)$
$\Leftrightarrow 0=(i\omega+\varsigma\epsilon)\cdot\sigma(s)\bar{N}(s)-\bar{N}(s)[(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})\otimes I_{2s}+(i\omega+\varsigma\epsilon)\cdot I\otimes\sigma(s-\frac{1}{2})]$
$\Leftrightarrow \bar{N}(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\bar{N}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$ □

3.13 Constant invariant tensor properties of matrix $(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a$

Thm. 3.13.1.

$$\begin{cases}(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a\\ =[e^{\vartheta}]_a{}^b[e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},\varsigma;w)}](\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_b[e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},\varsigma;w)}]\\ (\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a\\ =[e^{\vartheta}]_a{}^b[e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},-\varsigma;w)}](\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_b[e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},-\varsigma;w)}]\end{cases}$$

Proof: $(\sigma,-i\varsigma)_a=[e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}(\sigma,-i\varsigma)_b e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}$
$\Leftrightarrow(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a=[e^{\vartheta}]_a{}^b[e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes I_{C^{2s-1}_{2s-1+w}}](\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_b[e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}\otimes I_{C^{2s-1}_{2s-1+w}}]$
$\Leftrightarrow(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a$
$=[e^{\vartheta}]_a{}^b[e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},\varsigma;w)}](\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_b[e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},\varsigma;w)}]$ □

Proof: $(\sigma,-i\varsigma)_a=[e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}(\sigma,-i\varsigma)_b e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}$
$\Leftrightarrow(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a=[e^{\vartheta}]_a{}^b[e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes I_{C^{2s-1}_{2s-1+w}}](\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_b[e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}\otimes I_{C^{2s-1}_{2s-1+w}}]$
$\Leftrightarrow(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a$
$=[e^{\vartheta}]_a{}^b[e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},-\varsigma;w)}](\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_b[e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},-\varsigma;w)}]$ □

3.14 Constant invariant tensor properties of matrix $(\sigma\otimes I_{2s},-i\varsigma)_a$

Thm. 3.14.1.

$$\begin{cases}(\sigma\otimes I_{2s},-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma\otimes I_{2s},-i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}\\ (\sigma\otimes I_{2s},-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma\otimes I_{2s},-i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}\end{cases}$$

Proof: $(\sigma,-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}(\sigma,-i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b[e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes I_{2s}](\sigma\otimes I_{2s},-i\varsigma)_b[e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes I_{2s}]$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma\otimes I_{2s},-i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$ □

Proof: $(\sigma,-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}(\sigma,-i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b[e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes I_{2s}](\sigma\otimes I_{2s},-i\varsigma)_b[e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes I_{2s}]$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_a=[e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}(\sigma\otimes I_{2s},-i\varsigma)_b e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$ □

3.15 Another proof of two theorems

Thm. 3.15.1. $\Omega(s)\Gamma(s)=\Gamma(s)\sigma(s),\bar{\Gamma}(s)\Omega(s)=\sigma(s)\bar{\Gamma}(s)$

Proof: Taking mathematical methods of induction.
1: When $s=\frac{1}{2},\sigma(\frac{1}{2})\Gamma(\frac{1}{2})=\Gamma(\frac{1}{2})\sigma(\frac{1}{2})$ establishes.
2: Assume: when $s=k,\Omega(k)\Gamma(k)=\Gamma(k)\sigma(k)$ establishes.
3: When $s=k+\frac{1}{2}$,
$\Omega(k+\frac{1}{2})\Gamma(k+\frac{1}{2})$
$=[\sigma(\frac{1}{2})\otimes I_{2^{2k}}+I\otimes\Omega(k)][I\otimes\Gamma(k)]N(k+\frac{1}{2})$
$=\{[I\otimes\Gamma(k)][\sigma(\frac{1}{2})\otimes I_{2(k+1)}]+I\otimes[\Gamma(k)\sigma(k)]\}N(k+\frac{1}{2})$
$=[I\otimes\Gamma(k)][\sigma(\frac{1}{2})\otimes I_{2(k+1)}+I\otimes\sigma(k)]N(k+\frac{1}{2})$
$=[I\otimes\Gamma(k)]N(k+\frac{1}{2})\sigma(k+\frac{1}{2})$
$=\Gamma(k+\frac{1}{2})\sigma(k+\frac{1}{2})$
4: So the proposition establishes and has been proved. □

Thm. 3.15.2. $\bar{\Gamma}(s)\Omega(s)=\sigma(s)\bar{\Gamma}(s)$

Proof: Taking mathematical methods of induction.
1: When $s=\frac{1}{2}, \bar{\Gamma}(\frac{1}{2})\sigma(\frac{1}{2})=\sigma(\frac{1}{2})\bar{\Gamma}(\frac{1}{2})$ establishes.
2: Assume: when $s=k, \bar{\Gamma}(k)\Omega(k)=\sigma(k)\bar{\Gamma}(k)$ establishes.
3: When $s=k+\frac{1}{2}$,
$\bar{\Gamma}(k+\frac{1}{2})\Omega(k+\frac{1}{2})$
$=\bar{N}(k+\frac{1}{2})[I\otimes\bar{\Gamma}(k)]\{\sigma(\frac{1}{2})\otimes I_{2^{2k}}+I\otimes\Omega(k)\}$
$=\bar{N}(k+\frac{1}{2})\{[\sigma(\frac{1}{2})\otimes I_{2(k+1)}][I\otimes\bar{\Gamma}(k)]+[I\otimes\sigma(k)][I\otimes\bar{\Gamma}(k)]\}$
$=\bar{N}(k+\frac{1}{2})[\sigma(\frac{1}{2})\otimes I_{2(k+1)}+I\otimes\sigma(k)][I\otimes\bar{\Gamma}(k)]$
$=\sigma(k+\frac{1}{2})\bar{N}(k+\frac{1}{2})[I\otimes\bar{\Gamma}(k)]$
$=\sigma(k+\frac{1}{2})\bar{\Gamma}(k+\frac{1}{2})$
4: So the proposition establishes and has been proved. □

4 Perfect constant invariant tensors $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)$

(Existing $\varepsilon_{A_\varsigma B_\varsigma}$ is the prerequisite.)

4.1 Introduction of perfect constant invariant tensors $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)$

Def. 4.1.1. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w):=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma B_\varsigma}N_{B_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w):=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{A_\varsigma B_\varsigma}N_{l_\varsigma}^{B_\varsigma m_\varsigma}(s-\frac{1}{2};w)$

Pro. 4.1.1. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)\simeq X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)$

4.2 Introduction of constant matrices $X(s;w), \bar{X}(s;w)$

Def. 4.2.1. $\begin{cases} X^{A_\varsigma}(s;w)\prec X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w), X_{A_\varsigma}(s;w)\prec X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w) \\ \bar{X}_{A_\varsigma}(s;w)\prec X_{A_\varsigma l_\varsigma}{}^{m_\varsigma}(s;w), \bar{X}^{A_\varsigma}(s;w)\prec X^{A_\varsigma l_\varsigma}{}_{m_\varsigma}(s;w) \\ X(s;w)\prec X_{A_\varsigma\otimes l_\varsigma}{}^{m_\varsigma}(s;w), \bar{X}(s;w)\prec X_{m_\varsigma}{}^{A_\varsigma\otimes l_\varsigma}(s;w)=X^{+}(s;w) \end{cases}$

Explicit representation of $X(s), \bar{X}(s)$:

Cor. 4.2.1. $\bar{X}(s)=\frac{1}{\sqrt{2s}}\begin{bmatrix} 0 & -\sqrt{2s-1} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2s-2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{2s-1} & 0 \end{bmatrix}$

Cor. 4.2.2. $\bar{X}(s=1,\frac{3}{2},2)=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & -\sqrt{2} & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & \sqrt{2} & 0 \end{bmatrix}, \frac{1}{\sqrt{4}}\begin{bmatrix} 0 & -\sqrt{3} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{3} & 0 \end{bmatrix}$

4.3 Raising and lowering indices of constant invariant tensors $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)$

Pro. 4.3.1.
$\begin{cases} X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)=\varepsilon^{A_\varsigma B_\varsigma}\varepsilon^{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)\varepsilon_{m_\varsigma r_\varsigma}(s-1;w)X_{B_\varsigma n_\varsigma}^{r_\varsigma}(s-\frac{1}{2};w) \\ X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)=\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)\varepsilon^{m_\varsigma r_\varsigma}(s-1;w)X_{r_\varsigma}^{B_\varsigma n_\varsigma}(s-\frac{1}{2};w) \end{cases}$

Proof: $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s-\frac{1}{2};w)=\varepsilon^{k_\varsigma m_\varsigma}(s-\frac{1}{2};w)\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{l_\varsigma n_\varsigma}(s-1;w)N_{m_\varsigma}^{B_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$\Leftrightarrow \varepsilon^{C_\varsigma A_\varsigma}N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s-\frac{1}{2};w)=\varepsilon^{C_\varsigma A_\varsigma}\varepsilon^{k_\varsigma m_\varsigma}(s-\frac{1}{2};w)\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{l_\varsigma n_\varsigma}(s-1;w)N_{m_\varsigma}^{B_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$\Leftrightarrow X_{l_\varsigma}^{C_\varsigma k_\varsigma}(s;w)=\varepsilon^{C_\varsigma A_\varsigma}\varepsilon^{k_\varsigma m_\varsigma}(s-\frac{1}{2};w)\varepsilon_{l_\varsigma n_\varsigma}(s-1;w)X_{A_\varsigma m_\varsigma}^{n_\varsigma}(s-\frac{1}{2};w)$
$\Leftrightarrow X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)=\varepsilon^{A_\varsigma B_\varsigma}\varepsilon^{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)\varepsilon_{m_\varsigma r_\varsigma}(s-1;w)X_{B_\varsigma n_\varsigma}^{r_\varsigma}(s-\frac{1}{2};w)$ □

4.4 Orthogonality of constant invariant tensors $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)$

Pro. 4.4.1. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)X_{A_\varsigma l_\varsigma}^{n_\varsigma}(s;w)=\delta_{m_\varsigma}{}^{n_\varsigma}[\Leftrightarrow]X^{A_\varsigma}(s;w)\bar{X}_{A_\varsigma}(s;w)=I_{C_{2s-2+w}^{2s-2}}[\Leftrightarrow]\bar{X}(s;w)X(s;w)=I_{C_{2s-2+w}^{2s-2}}$

Proof: $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)X_{A_\varsigma l_\varsigma}^{n_\varsigma}(s;w)$
$=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{A_\varsigma D_\varsigma}N_{l_\varsigma}^{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$=\frac{2s-1}{2s-1+w}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)\delta_{D_\varsigma}{}^{C_\varsigma}N_{l_\varsigma}^{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$=\frac{2s-1}{2s-1+w}(1+\frac{w}{2s-1})\delta_{m_\varsigma}{}^{n_\varsigma}$
$=\delta_{m_\varsigma}{}^{n_\varsigma}$ □

Pro. 4.4.2. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s;w)=0$
$[\Leftrightarrow]X^{A_\varsigma}(s;w)\bar{N}_{A_\varsigma}(s;w)=0, N_{A_\varsigma}(s;w)\bar{X}^{A_\varsigma}(s;w)=0[\Leftrightarrow]\bar{X}(s;w)N(s;w)=0, \bar{N}(s;w)X(s;w)=0$

Proof: $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s;w)$
$=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s;w)$

$$= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}\Gamma^{l_\varsigma}_{\underbrace{C_\varsigma C''_\varsigma D''_\varsigma\cdot\cdot}_{2s}}(s-\tfrac{1}{2};w)\Gamma^{\overbrace{C''_\varsigma D''_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-1;w)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{B'_\varsigma C'_\varsigma D'_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2};w)$$

$$= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}\frac{1}{(2s-1)!}\delta^{B'_\varsigma}_{(C_\varsigma}\delta^{C'_\varsigma}_{C''_\varsigma}\delta^{D'_\varsigma}_{D''_\varsigma}\cdot\cdot{}_)\Gamma^{\overbrace{C''_\varsigma D''_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-1;w)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma\cdot\cdot}_{2s}}(s;w)$$

$$= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma B_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C''_\varsigma D''_\varsigma\cdot\cdot}_{2s}}(s;w)\Gamma^{\overbrace{C''_\varsigma D''_\varsigma\cdot\cdot}^{2s-1}}_{m_\varsigma}(s-1;w)$$

$$= 0 \qquad \square$$

Pro. 4.4.3. $X^{m_\varsigma}_{A_\varsigma l_\varsigma}(s;w)X^{A_\varsigma k_\varsigma}_{m_\varsigma}(s;w) = \frac{2s-1}{2s-1+w}\delta_{l_\varsigma}{}^{k_\varsigma}[\Leftrightarrow]\bar{X}_{A_\varsigma}(s;w)X^{A_\varsigma}(s;w) = \frac{2s-1}{2s-1+w}I_{C^{2s-1}_{2s-1+w}}$

Proof: $X^{m_\varsigma}_{A_\varsigma l_\varsigma}(s;w)X^{A_\varsigma k_\varsigma}_{m_\varsigma}(s;w)$
$= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{A_\varsigma B_\varsigma}N^{B_\varsigma m_\varsigma}_{l_\varsigma}(s-\frac{1}{2};w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N^{k_\varsigma}_{C_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$= \frac{2s-1}{2s-1+w}N^{B_\varsigma m_\varsigma}_{l_\varsigma}(s-\frac{1}{2};w)N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$= \frac{2s-1}{2s-1+w}\delta_{l_\varsigma}{}^{k_\varsigma}$ $\square$

Pro. 4.4.4. $X^{m_\varsigma}_{A_\varsigma l_\varsigma}(s;w)X^{B_\varsigma l_\varsigma}_{m_\varsigma}(s;w) = \frac{1}{w+1}C^{2s-2}_{2s-2+w}\delta^{B_\varsigma}_{A_\varsigma}[\Leftrightarrow]tr[\bar{X}_{A_\varsigma}(s;w)X^{B_\varsigma}(s;w)] = \frac{1}{w+1}C^{2s-2}_{2s-2+w}\delta^{B_\varsigma}_{A_\varsigma}$

Proof: $X^{m_\varsigma}_{A_\varsigma l_\varsigma}(s;w)X^{B_\varsigma l_\varsigma}_{m_\varsigma}(s;w)$
$= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{A_\varsigma C_\varsigma}N^{C_\varsigma m_\varsigma}_{l_\varsigma}(s-\frac{1}{2};w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{B_\varsigma D_\varsigma}N^{l_\varsigma}_{D_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$= \frac{2s-1}{2s-1+w}\varepsilon_{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}N^{C_\varsigma m_\varsigma}_{l_\varsigma}(s-\frac{1}{2};w)N^{l_\varsigma}_{D_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$= \frac{2s-1}{2s-1+w}\varepsilon_{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma}\frac{1}{w+1}C^{2s-1}_{2s-1+w}\delta^{D_\varsigma}_{C_\varsigma}$
$= \frac{1}{w+1}C^{2s-2}_{2s-2+w}\delta^{B_\varsigma}_{A_\varsigma}$ $\square$

Cor. 4.4.1. $\bar{N}(s;w)N(s;w) = I_{C^{2s}_{2s+w}}, \bar{X}(s;w)X(s;w) = I_{C^{2s-2}_{2s-2+w}}, \bar{N}(s;w)X(s;w) = 0, \bar{X}(s;w)N(s;w) = 0$

4.5 The joint orthogonal properties of maxtices $N(s), \bar{N}(s), X(s), \bar{X}(s)$

Pro. 4.5.1. $\begin{cases} X^{n_\varsigma}_{A_\varsigma l_\varsigma}(s)X^{B_\varsigma m_\varsigma}_{n_\varsigma}(s) = \frac{1}{2s}[(2s-1)\delta^{B_\varsigma}_{A_\varsigma}\delta^{m_\varsigma}_{l_\varsigma} - (2s-1)N^{m_\varsigma}_{A_\varsigma n_\varsigma}(s-\frac{1}{2})N^{B_\varsigma n_\varsigma}_{l_\varsigma}(s-\frac{1}{2})] \\ \bar{X}_{A_\varsigma}(s)X^{B_\varsigma}(s) = \frac{1}{2s}[(2s-1)\delta^{B_\varsigma}_{A_\varsigma}I_{2s} - (2s-1)N^{B_\varsigma}(s-\frac{1}{2})\bar{N}_{A_\varsigma}(s-\frac{1}{2};w)] \end{cases}$

Proof: $X^{n_\varsigma}_{A_\varsigma l_\varsigma}(s)X^{A'_\varsigma m_\varsigma}_{n_\varsigma}(s) = \frac{2s-1}{2s}\varepsilon_{A_\varsigma E_\varsigma}\varepsilon^{A'_\varsigma E'_\varsigma}N^{E_\varsigma n_\varsigma}_{l_\varsigma}(s-\frac{1}{2})N^{m_\varsigma}_{E'_\varsigma n_\varsigma}(s-\frac{1}{2})$

$$= \frac{2s-1}{2s}\varepsilon_{A_\varsigma E_\varsigma}\varepsilon^{A'_\varsigma E'_\varsigma}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2})\Gamma^{n_\varsigma}_{\underbrace{F_\varsigma G_\varsigma\cdot\cdot}_{2s-2}}(s-1)\Gamma^{m_\varsigma}_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2})\Gamma^{\overbrace{F'_\varsigma G'_\varsigma\cdot\cdot}^{2s-2}}_{n_\varsigma}(s-1)$$

$$= \frac{2s-1}{2s}\varepsilon_{A_\varsigma E_\varsigma}\varepsilon^{A'_\varsigma E'_\varsigma}\frac{1}{(2s-2)!}\overbrace{\delta^{(F'_\varsigma}_{F_\varsigma}\delta^{G'_\varsigma}_{G_\varsigma}\cdot\cdot{}^)}^{2s-2}\Gamma^{\overbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2})\Gamma^{m_\varsigma}_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2})$$

$$= \frac{2s-1}{2s}\varepsilon_{A_\varsigma E_\varsigma}\varepsilon^{A'_\varsigma E'_\varsigma}\Gamma^{\overbrace{E_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2})\Gamma^{m_\varsigma}_{\underbrace{E'_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2})$$

$$= \frac{2s-1}{2s}\delta^{[A'_\varsigma}_{A_\varsigma}\delta^{E'_\varsigma]}_{E_\varsigma}\Gamma^{\overbrace{E_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2})\Gamma^{m_\varsigma}_{\underbrace{E'_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2})$$

$$= \frac{2s-1}{2s}\delta^{A'_\varsigma}_{A_\varsigma}\Gamma^{\overbrace{E_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2})\Gamma^{m_\varsigma}_{\underbrace{E_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2}) - \frac{2s-1}{2s}\Gamma^{\overbrace{A'_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s-1}}_{l_\varsigma}(s-\tfrac{1}{2})\Gamma^{m_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(s-\tfrac{1}{2})$$

$= \frac{1}{2s}[(2s-1)\delta^{A'_\varsigma}_{A_\varsigma}\delta^{m_\varsigma}_{l_\varsigma} - (2s-1)N^{m_\varsigma}_{A_\varsigma n_\varsigma}(s-\frac{1}{2})N^{A'_\varsigma n_\varsigma}_{l_\varsigma}(s-\frac{1}{2})]$ $\square$

Cor. 4.5.1. $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)N^{B_\varsigma m_\varsigma}_{k_\varsigma}(s) + X^{n_\varsigma}_{A_\varsigma l_\varsigma}(s)X^{B_\varsigma m_\varsigma}_{n_\varsigma}(s) = \delta^{B_\varsigma}_{A_\varsigma}\delta^{m_\varsigma}_{l_\varsigma}$
$[\Leftrightarrow]\bar{N}_{A_\varsigma}(s)N^{B_\varsigma}(s) + \bar{X}_{A_\varsigma}(s)X^{B_\varsigma}(s) = \delta^{B_\varsigma}_{A_\varsigma}I_{2s}[\Leftrightarrow]N(s)\bar{N}(s) + X(s)\bar{X}(s) = I_{4s}$

Cor. 4.5.2. $\begin{bmatrix}\bar{N}(s)\\ \bar{X}(s)\end{bmatrix}[N(s),X(s)]=[N(s),X(s)]\begin{bmatrix}\bar{N}(s)\\ \bar{X}(s)\end{bmatrix}=I_{4s}[\Leftrightarrow]\begin{cases}N(s)\bar{N}(s)+X(s)\bar{X}(s)=I_{4s}\\ \bar{N}(s)N(s)=I_{2s+1},\bar{X}(s)X(s)=I_{2s-1}\\ \bar{N}(s)X(s)=0,\bar{X}(s)N(s)=0\end{cases}$

4.6 Spin transformation I of constant invariant tensors $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)$

Cor. 4.6.1. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma l_\varsigma}^{n_\varsigma}(s;w)=-\frac{1}{2s-1+w}\sigma^{\alpha_\varsigma}{}_{m_\varsigma}{}^{n_\varsigma}(s-1;w)$
$[\Leftrightarrow]X^{A_\varsigma}(s;w)\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{X}_{B_\varsigma}(s;w)=-\frac{1}{2s-1+w}\sigma(s-1;w)$
$[\Leftrightarrow]\bar{X}(s;w)\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}X(s;w)=-\frac{1}{2s-1+w}\sigma(s-1;w)$

Proof: $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma l_\varsigma}^{n_\varsigma}(s;w)$
$=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{B_\varsigma D_\varsigma}N_{l_\varsigma}^{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{2s-1}{2s-1+w}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)\sigma^{\alpha_\varsigma}{}_{D_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)N_{l_\varsigma}^{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{1}{2s-1+w}\sigma^{\alpha_\varsigma}{}_{m_\varsigma}{}^{n_\varsigma}(s-1;w)$ □

Cor. 4.6.2. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma k_\varsigma}^{m_\varsigma}(s;w)=-\frac{1}{2s-1+w}\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)$
$[\Leftrightarrow]\bar{X}^{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma}(s;w)=-\frac{1}{2s-1+w}\sigma^{\alpha_\varsigma}(s-\frac{1}{2};w)$

Proof: $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma k_\varsigma}^{m_\varsigma}(s;w)$
$=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{B_\varsigma D_\varsigma}N_{k_\varsigma}^{D_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{2s-1}{2s-1+w}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)\sigma^{\alpha_\varsigma}{}_{D_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)N_{k_\varsigma}^{D_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{1}{2s-1+w}\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)$ □

Cor. 4.6.3. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma l_\varsigma}^{n_\varsigma}(s;w)=-\frac{1}{2s-1+w}S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)$
$[\Leftrightarrow]X^{A_\varsigma}(s;w)S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-2+w}^{2s-2}}\bar{X}_{A_\varsigma}(s;w)=-\frac{1}{2s-1+w}S_{ab}(s-1,\varsigma;w)$
$[\Leftrightarrow]\bar{X}(s;w)S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C_{2s-1+w}^{2s-1}}X(s;w)=-\frac{1}{2s-1+w}S_{ab}(s-1,\varsigma;w)$

Proof: $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma l_\varsigma}^{n_\varsigma}(s;w)$
$=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{B_\varsigma D_\varsigma}N_{l_\varsigma}^{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{2s-1}{2s-1+w}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)S_{abD_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)N_{l_\varsigma}^{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{1}{2s-1+w}S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)$ □

Cor. 4.6.4. $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma k_\varsigma}^{m_\varsigma}(s;w)=-\frac{1}{2s}S_{abk_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)$
$[\Leftrightarrow]\bar{X}^{A_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma}(s;w)=-\frac{1}{2s-1+w}S_{ab}(s-\frac{1}{2},\varsigma;w)$

Proof: $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)X_{B_\varsigma k_\varsigma}^{m_\varsigma}(s;w)$
$=\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon_{B_\varsigma D_\varsigma}N_{k_\varsigma}^{D_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{2s-1}{2s-1+w}N_{C_\varsigma m_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)S_{abD_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)N_{k_\varsigma}^{D_\varsigma m_\varsigma}(s-\frac{1}{2};w)$
$=-\frac{1}{2s-1+w}S_{abk_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)$ □

4.7 Spin transformation II of constant invariant tensors $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)$

Thm. 4.7.1.
$$\begin{cases}X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s;w)[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{l_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]=\sigma_{m_\varsigma}{}^{n_\varsigma}(s-1;w)X_{n_\varsigma}^{B_\varsigma k_\varsigma}(s;w)\\ [\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{l_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]X_{B_\varsigma k_\varsigma}^{n_\varsigma}(s;w)=X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s;w)\sigma_{m_\varsigma}{}^{n_\varsigma}(s-1;w)\end{cases}$$
$$\begin{cases}X^{A_\varsigma}(s;w)[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma(s-\frac{1}{2};w)]=\sigma(s-1;w)X^{B_\varsigma}(s;w)\\ [\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma(s-\frac{1}{2};w)]\bar{X}_{B_\varsigma}(s;w)=\bar{X}_{A_\varsigma}(s;w)\sigma(s-1;w)\end{cases}$$
$$\begin{cases}\bar{X}(s;w)[\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]=\sigma(s-1;w)\bar{X}(s;w)\\ [\sigma(\frac{1}{2};w)\otimes I_{C_{2s-1+w}^{2s-1}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]X(s;w)=X(s;w)\sigma(s-1;w)\end{cases}$$

Proof: $X_{m_\varsigma}^{A_\varsigma k_\varsigma}(s;w)[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{k_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)]=\sigma_{m_\varsigma}{}^{n_\varsigma}(s-1;w)X_{n_\varsigma}^{B_\varsigma l_\varsigma}(s;w)$
$\Leftrightarrow\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{k_\varsigma}(s-\frac{1}{2};w)[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{k_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)]=\sigma_{m_\varsigma}{}^{n_\varsigma}(s-1;w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{B_\varsigma D_\varsigma}N_{D_\varsigma n_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)$
$\Leftrightarrow\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{k_\varsigma}(s-\frac{1}{2};w)[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{k_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)]=\sigma_{m_\varsigma}{}^{n_\varsigma}(s-1;w)\varepsilon^{B_\varsigma D_\varsigma}N_{D_\varsigma n_\varsigma}^{l_\varsigma}(s-\frac{1}{2};w)$
$\Leftrightarrow\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{k_\varsigma}(s;w)[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)]=\sigma_{m_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)\varepsilon^{B_\varsigma D_\varsigma}N_{D_\varsigma n_\varsigma}^{l_\varsigma}(s;w)$
$\Leftrightarrow\varepsilon_{E_\varsigma B_\varsigma}\varepsilon^{A_\varsigma C_\varsigma}N_{C_\varsigma m_\varsigma}^{k_\varsigma}(s;w)[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)]=\varepsilon_{E_\varsigma B_\varsigma}\sigma_{m_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)\varepsilon^{B_\varsigma D_\varsigma}N_{D_\varsigma n_\varsigma}^{l_\varsigma}(s;w)$

$\Leftrightarrow N^{k_\varsigma}_{C_\varsigma m_\varsigma}(s;w)[\sigma(\frac{1}{2};w)_{E_\varsigma}{}^{C_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}-\delta_{E_\varsigma}{}^{C_\varsigma}\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)]=-\sigma_{m_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)N^{l_\varsigma}_{E_\varsigma n_\varsigma}(s;w)$
$\Leftrightarrow [\sigma(\frac{1}{2};w)_{E_\varsigma}{}^{C_\varsigma}N^{l_\varsigma}_{C_\varsigma m_\varsigma}(s;w)+\sigma_{m_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)N^{l_\varsigma}_{E_\varsigma n_\varsigma}(s;w)=N^{k_\varsigma}_{E_\varsigma m_\varsigma}(s;w)\sigma_{k_\varsigma}{}^{l_\varsigma}(s;w)$
$\Leftrightarrow \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s;w)+\sigma^{\alpha_\varsigma}{}_{l_\varsigma}{}^{m_\varsigma}(s-\frac{1}{2};w)N^{k_\varsigma}_{A_\varsigma m_\varsigma}(s;w)=N^{j_\varsigma}_{A_\varsigma l_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{j_\varsigma}{}^{k_\varsigma}(s;w)$ □

Thm. 4.7.2.

$$\begin{cases}X^{A_\varsigma l_\varsigma}_{m_\varsigma}(s;w)[S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abl_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]=S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)X^{B_\varsigma\otimes k_\varsigma}_{n_\varsigma}(s;w)\\ [S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abl_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]X^{n_\varsigma}_{B_\varsigma k_\varsigma}(s;w)=X^{m_\varsigma}_{A_\varsigma l_\varsigma}(s;w)S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)\end{cases}$$

$$\begin{cases}X^{A_\varsigma}(s;w)[S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}S_{ab}(s-\frac{1}{2},\varsigma;w)]=S_{ab}(s-1,\varsigma;w)X^{B_\varsigma}(s;w)\\ [S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}S_{ab}(s-\frac{1}{2},\varsigma;w)]\bar{X}_{B_\varsigma}(s;w)=\bar{X}_{A_\varsigma}(s;w)S_{ab}(s-1,\varsigma;w)\end{cases}$$

$$\begin{cases}\bar{X}(s;w)[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]=S_{ab}(s-1,\varsigma;w)\bar{X}(s;w)\\ [S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]X(s;w)=X(s;w)S_{ab}(s-1,\varsigma;w)\end{cases}$$

Proof: $X^{A_\varsigma k_\varsigma}_{m_\varsigma}(s;w)[S_{abA_\varsigma}{}^{B_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)]=S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)X^{B_\varsigma l_\varsigma}_{n_\varsigma}(s;w)$
$\Leftrightarrow \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{A_\varsigma C_\varsigma}N^{k_\varsigma}_{C_\varsigma m_\varsigma}(s-\frac{1}{2};w)[S_{abA_\varsigma}{}^{B_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)]$
$=S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)\frac{\sqrt{2s-1}}{\sqrt{2s-1+w}}\varepsilon^{B_\varsigma D_\varsigma}N^{l_\varsigma}_{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$\Leftrightarrow \varepsilon^{A_\varsigma C_\varsigma}N^{k_\varsigma}_{C_\varsigma m_\varsigma}(s-\frac{1}{2};w)[S_{abA_\varsigma}{}^{B_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)]=S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)\varepsilon^{B_\varsigma D_\varsigma}N^{l_\varsigma}_{D_\varsigma n_\varsigma}(s-\frac{1}{2};w)$
$\Leftrightarrow \varepsilon^{A_\varsigma C_\varsigma}N^{k_\varsigma}_{C_\varsigma m_\varsigma}(s;w)[S_{abA_\varsigma}{}^{B_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(s;w)]=S_{abm_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)\varepsilon^{B_\varsigma D_\varsigma}N^{l_\varsigma}_{D_\varsigma n_\varsigma}(s;w)$
$\Leftrightarrow \varepsilon_{E_\varsigma B_\varsigma}\varepsilon^{A_\varsigma C_\varsigma}N^{k_\varsigma}_{C_\varsigma m_\varsigma}(s;w)[S_{abA_\varsigma}{}^{B_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(s;w)]=\varepsilon_{E_\varsigma B_\varsigma}S_{abm_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)\varepsilon^{B_\varsigma D_\varsigma}N^{l_\varsigma}_{D_\varsigma n_\varsigma}(s;w)$
$\Leftrightarrow N^{k_\varsigma}_{C_\varsigma m_\varsigma}(s;w)[S_{abE_\varsigma}{}^{C_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}-\delta_{E_\varsigma}{}^{C_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(s;w)]=-S_{abm_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)N^{l_\varsigma}_{E_\varsigma n_\varsigma}(s;w)$
$\Leftrightarrow [S_{abE_\varsigma}{}^{C_\varsigma}N^{l_\varsigma}_{C_\varsigma m_\varsigma}(s;w)+S_{abm_\varsigma}{}^{n_\varsigma}(s-\frac{1}{2};w)N^{l_\varsigma}_{E_\varsigma n_\varsigma}(s;w)=N^{k_\varsigma}_{E_\varsigma m_\varsigma}(s;w)S_{abk_\varsigma}{}^{l_\varsigma}(s;w)$
$\Leftrightarrow S_{abA_\varsigma}{}^{B_\varsigma}N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s;w)+S_{abl_\varsigma}{}^{m_\varsigma}(s-\frac{1}{2};w)N^{k_\varsigma}_{A_\varsigma m_\varsigma}(s;w)=N^{j_\varsigma}_{A_\varsigma l_\varsigma}(s;w)S_{abj_\varsigma}{}^{k_\varsigma}(s;w)$ □

Cor. 4.7.1.

$$\begin{cases}N_{C_\varsigma m_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)\varepsilon^{C_\varsigma A_\varsigma}[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{l_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]=\sigma_{m_\varsigma}{}^{n_\varsigma}(s-1;w)N_{D_\varsigma n_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)\varepsilon^{D_\varsigma B_\varsigma}\\ [\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma_{l_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]\varepsilon_{B_\varsigma C_\varsigma}N^{C_\varsigma n_\varsigma}_{k_\varsigma}(s-\frac{1}{2};w)=\varepsilon_{A_\varsigma D_\varsigma}N^{D_\varsigma m_\varsigma}_{l_\varsigma}(s-\frac{1}{2};w)\sigma_{m_\varsigma}{}^{n_\varsigma}(s-1;w)\end{cases}$$

$$\begin{cases}N_{C_\varsigma}(s-\frac{1}{2};w)\varepsilon^{C_\varsigma A_\varsigma}[\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma(s-\frac{1}{2};w)]=\sigma(s-1;w)N_{D_\varsigma}(s-\frac{1}{2};w)\varepsilon^{D_\varsigma B_\varsigma}\\ [\sigma_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}\sigma(s-\frac{1}{2};w)]\varepsilon_{B_\varsigma C_\varsigma}N^{C_\varsigma}(s-\frac{1}{2};w)=\varepsilon_{A_\varsigma D_\varsigma}N^{D_\varsigma}(s-\frac{1}{2};w)\sigma(s-1;w)\end{cases}$$

Cor. 4.7.2.

$$\begin{cases}N_{C_\varsigma m_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2};w)\varepsilon^{C_\varsigma A_\varsigma}[S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abl_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]=S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)N_{D_\varsigma n_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)\varepsilon^{D_\varsigma B_\varsigma}\\ [S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\delta_{l_\varsigma}{}^{k_\varsigma}+\delta_{A_\varsigma}{}^{B_\varsigma}S_{abl_\varsigma}{}^{k_\varsigma}(s-\frac{1}{2};w)]\varepsilon_{B_\varsigma C_\varsigma}N^{C_\varsigma n_\varsigma}_{k_\varsigma}(s-\frac{1}{2};w)=\varepsilon_{A_\varsigma D_\varsigma}N^{D_\varsigma m_\varsigma}_{l_\varsigma}(s-\frac{1}{2};w)S_{abm_\varsigma}{}^{n_\varsigma}(s-1;w)\end{cases}$$

$$\begin{cases}N_{C_\varsigma}(s-\frac{1}{2};w)\varepsilon^{C_\varsigma A_\varsigma}[S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}S_{ab}(s-\frac{1}{2},\varsigma;w)]=S_{ab}(s-1,\varsigma;w)N_{D_\varsigma}(s-\frac{1}{2};w)\varepsilon^{D_\varsigma B_\varsigma}\\ [S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)+\delta_{A_\varsigma}{}^{B_\varsigma}S_{ab}(s-\frac{1}{2},\varsigma;w)]\varepsilon_{B_\varsigma C_\varsigma}N^{C_\varsigma}(s-\frac{1}{2};w)=\varepsilon_{A_\varsigma D_\varsigma}N^{D_\varsigma}(s-\frac{1}{2};w)S_{ab}(s-1,\varsigma;w)\end{cases}$$

4.8 Important corollaries of constant matrices $X(s;w),\bar{X}(s;w)$

Cor. 4.8.1.

$$\begin{cases}\bar{X}(s;w)[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]X(s;w)=\sigma(s-1;w)\\ X(s;w)\sigma(s-1;w)\bar{X}(s;w)=[\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]X(s;w)\bar{X}(s;w)\\ [X(s;w)\bar{X}(s;w),\sigma(\frac{1}{2};w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes\sigma(s-\frac{1}{2};w)]=0\end{cases}$$

Cor. 4.8.2.

$$\begin{cases}\bar{X}(s;w)[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]X(s;w)=S_{ab}(s-1,\varsigma;w)\\ X(s;w)S_{ab}(s,\varsigma-1,\varsigma;w)\bar{X}(s;w)=[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]X(s;w)\bar{X}(s;w)\\ [X(s;w)\bar{X}(s;w),S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]=0\end{cases}$$

Cor. 4.8.3. $X^{A_\varsigma}(s;w)\sigma(s-\frac{1}{2};w)\bar{X}_{A_\varsigma}(s;w)=\frac{2s+w}{2s-1+w}\sigma(s-1;w)$
$[\Leftrightarrow]\bar{X}(s;w)I_{w+1}\otimes\sigma(s-\frac{1}{2};w)X(s;w)=\frac{2s+w}{2s-1+w}\sigma(s-1;w)$

Cor. 4.8.4. $X^{A_\varsigma}(s;w)I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)\bar{X}_{A_\varsigma}(s;w)=\frac{2s+w}{2s-1+w}S_{ab}(s-1,\varsigma;w)$
$[\Leftrightarrow]\bar{X}(s;w)I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)X(s;w)=\frac{2s+w}{2s-1+w}S_{ab}(s-1,\varsigma;w)$

4.9 Constant invariant tensor properties of matrices $X(s;w),\bar{X}(s;w)$

Thm. 4.9.1. $X(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}X(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-1,\varsigma;w)}$

Proof: $[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]X(s;w)=X(s;w)S_{ab}(s-1,\varsigma;w)$
$\Leftrightarrow 0=[\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+\frac{i}{2}\vartheta^{ab}I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]X(s;w)-\frac{i}{2}\vartheta^{ab}X(s;w)S_{ab}(s-1,\varsigma;w)$
$\Leftrightarrow X(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}X(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-1,\varsigma;w)}$ □

Thm. 4.9.2. $\bar{X}(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-1,\varsigma;w)}\bar{X}(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}$

Proof: $\bar{X}(s;w)[S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]=S_{ab}(s-1,\varsigma;w)\bar{X}(s;w)$
$\Leftrightarrow 0=\frac{i}{2}\vartheta^{ab}S_{ab}(s-1,\varsigma;w)\bar{X}(s;w)-\bar{X}(s;w)[\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)\otimes I_{C^{2s-1}_{2s-1+w}}+\frac{i}{2}\vartheta^{ab}I_{w+1}\otimes S_{ab}(s-\frac{1}{2},\varsigma;w)]$
$\Leftrightarrow \bar{X}(s;w)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-1,\varsigma;w)}\bar{X}(s;w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\varsigma;w)}$ □

4.10 Constant invariant tensor properties of matrices $X(s),\bar{X}(s)$

Thm. 4.10.1. $X(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}X(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-1)}$

Proof: $[\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})]X(s)=X(s)\sigma(s-1)$
$\Leftrightarrow 0=[(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})\otimes I_{2s}+(i\omega+\varsigma\epsilon)\cdot I\otimes\sigma(s-\frac{1}{2})]X(s)-(i\omega+\varsigma\epsilon)\cdot X(s)\sigma(s-1)$
$\Leftrightarrow X(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}X(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-1)}$ □

Thm. 4.10.2. $\bar{X}(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-1)}\bar{X}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

Proof: $\bar{X}(s)[\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})]=\sigma(s-1)\bar{X}(s)$
$\Leftrightarrow 0=(i\omega+\varsigma\epsilon)\cdot\sigma(s-1)\bar{X}(s)-\bar{X}(s)[(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})\otimes I_{2s}+(i\omega+\varsigma\epsilon)\cdot I\otimes\sigma(s-\frac{1}{2})]$
$\Leftrightarrow \bar{X}(s)=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-1)}\bar{X}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$ □

Cor. 4.10.1. $[N(s),X(s)]=e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}[N(s),X(s)][e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\oplus e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-1)}]$

Cor. 4.10.2. $\begin{bmatrix}\bar{N}(s)\\ \bar{X}(s)\end{bmatrix}=[e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\oplus e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-1)}]\begin{bmatrix}\bar{N}(s)\\ \bar{X}(s)\end{bmatrix}e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

4.11 Commutative properties of constant matrices $\Omega(s;w),\sigma(s-1;w)$

Cor. 4.11.1. $\begin{cases}\Omega(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]X(s;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]X(s;w)\sigma(s-1;w)\\ \bar{X}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)=\sigma(s-1;w)\bar{X}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\end{cases}$

Cor. 4.11.2. $\begin{cases}\sigma(s;w)=\bar{N}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]N(s;w)\\ \sigma(s-1;w)=\bar{X}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega(s;w)[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]X(s;w)\end{cases}$

Cor. 4.11.3. $\begin{cases}[\vec{\vartheta}\cdot\sigma(s;w)]^n=\bar{N}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][\vec{\vartheta}\cdot\Omega(s;w)]^n[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]N(s;w)\\ [\vec{\vartheta}\cdot\sigma(s-1;w)]^n=\bar{X}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][\vec{\vartheta}\cdot\Omega(s;w)]^n[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]X(s;w)\end{cases}$

Cor. 4.11.4. $\begin{cases}e^{\vec{\vartheta}\cdot\sigma(s;w)}=\bar{N}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{\vec{\vartheta}\cdot\Omega(s;w)}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]N(s;w)\\ e^{\vec{\vartheta}\cdot\sigma(s-1;w)}=\bar{X}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{\vec{\vartheta}\cdot\Omega(s;w)}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]X(s;w)\end{cases}$

4.12 Isomorphic representation of constant matrices $\Omega(s-l;w),[\vec{\vartheta}\cdot\Omega(s-l;w)]^n,e^{\vec{\vartheta}\cdot\Omega(s-l;w)}$

Cor. 4.12.1. $\Omega(s;w)=\Omega(s-1;w)\otimes I_{(w+1)^2}+I_{(w+1)^{2s-2}}\otimes\Omega(1;w)$

Cor. 4.12.2.
$\begin{cases}\Omega(s;w)I_{(w+1)^{2s-2}}\otimes\{[I_{w+1}\otimes\Gamma(\frac{1}{2};w)]X(1;w)\}=I_{(w+1)^{2s-2}}\otimes\{[I_{w+1}\otimes\Gamma(\frac{1}{2};w)]X(1;w)\}\Omega(s-1;w)\\ I_{(w+1)^{2s-2}}\otimes\{\bar{X}(1;w)[I_{w+1}\otimes\bar{\Gamma}(\frac{1}{2};w)]\}\Omega(s;w)=\Omega(s-1;w)I_{(w+1)^{2s-2}}\otimes\{\bar{X}(1;w)[I_{w+1}\otimes\bar{\Gamma}(\frac{1}{2};w)]\}\end{cases}$

Cor. 4.12.3.
$\begin{cases}\Omega(s-1;w)=I_{(w+1)^{2s-2}}\otimes\{\bar{X}(1;w)[I_{w+1}\otimes\bar{\Gamma}(\frac{1}{2};w)]\}\Omega(s;w)I_{(w+1)^{2s-2}}\otimes\{[I_{w+1}\otimes\Gamma(\frac{1}{2};w)]X(1;w)\}\\ [\vec{\vartheta}\cdot\Omega(s-1;w)]^n=I_{(w+1)^{2s-2}}\otimes\{\bar{X}(1;w)[I_{w+1}\otimes\bar{\Gamma}(\frac{1}{2};w)]\}[\vec{\vartheta}\cdot\Omega(s;w)]^nI_{(w+1)^{2s-2}}\otimes\{[I_{w+1}\otimes\Gamma(\frac{1}{2};w)]X(1;w)\}\\ e^{\vec{\vartheta}\cdot\Omega(s-1;w)}=I_{(w+1)^{2s-2}}\otimes\{\bar{X}(1;w)[I_{w+1}\otimes\bar{\Gamma}(\frac{1}{2};w)]\}e^{\vec{\vartheta}\cdot\Omega(s;w)}I_{(w+1)^{2s-2}}\otimes\{[I_{w+1}\otimes\Gamma(\frac{1}{2};w)]X(1;w)\}\end{cases}$

Def. 4.12.1.
$\begin{cases}T(s;w):=I_{(w+1)^{2s-2}}\otimes\{[I_{w+1}\otimes\Gamma(\frac{1}{2};w)]X(1;w)\}\\ \bar{T}(s;w):=I_{(w+1)^{2s-2}}\otimes\{\bar{X}(1;w)[I_{w+1}\otimes\bar{\Gamma}(\frac{1}{2};w)]\}=T^+(s;w)\end{cases}$

Cor. 4.12.4.

$$\begin{cases}\Omega(s-l;w)=\bar{T}(s-l+1;w)\cdots\bar{T}(s-1;w)\bar{T}(s;w)\Omega(s;w)T(s;w)T(s-1;w)\cdots T(s-l+1;w)\\ [\vec{\vartheta}\cdot\Omega(s-l;w)]^n=\bar{T}(s-l+1;w)\cdots\bar{T}(s-1;w)\bar{T}(s;w)[\vec{\vartheta}\cdot\Omega(s;w)]^nT(s;w)T(s-1;w)\cdots T(s-l+1;w)\\ e^{\vec{\vartheta}\cdot\Omega(s-l;w)}=\bar{T}(s-l+1;w)\cdots\bar{T}(s-1;w)\bar{T}(s;w)e^{\vec{\vartheta}\cdot\Omega(s;w)}T(s;w)T(s-1;w)\cdots T(s-l+1;w)\end{cases}$$

Cor. 4.12.5.

$$\begin{cases}\sigma(s-l;w)=\bar{\Gamma}(s-l;w)\bar{T}(s-l+1;w)\cdots\bar{T}(s;w)\Omega(s;w)T(s;w)\cdots T(s-l+1;w)\Gamma(s-l;w)\\ [\vec{\vartheta}\cdot\sigma(s-l;w)]^n=\bar{\Gamma}(s-l;w)\bar{T}(s-l+1;w)\cdots\bar{T}(s;w)[\vec{\vartheta}\cdot\Omega(s;w)]^nT(s;w)\cdots T(s-l+1;w)\Gamma(s-l;w)\\ e^{\vec{\vartheta}\cdot\sigma(s-l;w)}=\bar{\Gamma}(s-l;w)\bar{T}(s-l+1;w)\cdots\bar{T}(s;w)e^{\vec{\vartheta}\cdot\Omega(s;w)}T(s;w)\cdots T(s-l+1;w)\Gamma(s-l;w)\end{cases}$$

Chapter3 Important Composite Constant Invariant Tensors

Self comment: This chapter conducts an in-depth analysis of multiple complex constant invariant tensors and obtains some very useful conclusions. It is a powerful mathematical tool for studying high spin particles.

1 Composite constant invariant tensor $X^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}}(s,n,l;w)$

1.1 Introduction of composite constant invariant tensor $X^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}}(s,n,l;w)$

1.1.1 Definition of composite constant invariant tensor $X^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}}(s,n,l;w)$

Def. 1.1.1.

$$\begin{cases}X^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma};\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}}(s,n,l;w):=N^{A_\varsigma}(s;w)[\sigma^{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{l\varsigma}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)\\ X(s,0,0):=1,X^{\alpha_{1\varsigma}}(s,1,0;w)=\frac{1}{2s}\sigma^{\alpha_{1\varsigma}}(s;w),X^{\beta_{1\varsigma}}(s,0,1;w)=(1-\frac{1}{2s})\sigma^{\beta_{1\varsigma}}(s;w)\end{cases}$$

Cor. 1.1.1.

$$\begin{cases}X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n,l;w):=N^{A_\varsigma}(s;w)[\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{l\varsigma}\}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)\\ X^{\{\}}(s,0,0;w):=1,X^{\{\alpha_{1\varsigma}\}}(s,1,0;w)=\frac{1}{2s}\sigma^{\alpha_{1\varsigma}}(s;w),X^{\{\beta_{1\varsigma}\}}(s,0,1;w)=(1-\frac{1}{2s})\sigma^{\beta_{1\varsigma}}(s;w)\end{cases}$$

1.1.2 Recursive relations of composite constant invariant tensor $X^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}}(s,n,l;w)$

Thm. 1.1.1. $X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n,l;w)=\frac{1}{4}\frac{1}{(n+l-2)!}X^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n-2,l;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$

Proof: $X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n,l;w)=N^{A_\varsigma}(s;w)[\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{l\varsigma}\}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)$
$=N^{A_\varsigma}(s;w)[\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{(n-2)\varsigma}}(\frac{1}{2};w)\sigma^{\alpha_{(n-1)\varsigma}}(\frac{1}{2};w)\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{l\varsigma}\}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)$
$=N^{A_\varsigma}(s;w)\{\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{(n-2)\varsigma}}(\frac{1}{2};w)[\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}}]\}_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{l\varsigma}\}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)$
$=\frac{1}{4}\frac{1}{(n+l-2)!}X^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n-2,l;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$ □

Cor. 1.1.2. $X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n,l;w)=\begin{cases}\frac{1}{2^n}\frac{1}{l!}\delta^{\{\alpha_{1\varsigma}\alpha_{2\varsigma}}\cdot\cdot\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}}X^{\{\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}\}}(s,0,l;w),n=2k\\ \frac{1}{2^{(n-1)}}\frac{1}{(l+1)!}\delta^{\{\alpha_{1\varsigma}\alpha_{2\varsigma}}\cdot\cdot\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}X^{\{\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}\}}(s,1,l;w),n=2k+1\end{cases}$

Thm. 1.1.2. $X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n,l;w)$
$=\frac{1}{(n+l-1)!}X^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{(l-1)\varsigma}\}}(s,n,l-1;w)\sigma^{\beta_{l\varsigma}\}}(s;w)-\frac{1}{4}\frac{1}{(n+l-2)!}X^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-1)\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{(l-1)\varsigma}\}}(s,n-1,l-1;w)\delta^{\alpha_{n\varsigma}\beta_{l\varsigma}\}}$

Proof: $X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n,l;w)=N^{A_\varsigma}(s;w)[\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{l\varsigma}\}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)$
$=N^{A_\varsigma}(s;w)[\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{(l-1)\varsigma}}(s-\frac{1}{2};w)][\bar{N}_{B_\varsigma}(s;w)\sigma^{\beta_{l\varsigma}\}}(s;w)-\sigma^{\beta_{l\varsigma}\}}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)]$
$=N^{A_\varsigma}(s;w)[\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{(l-1)\varsigma}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)\sigma^{\beta_{l\varsigma}\}}(s;w)$
$-N^{A_\varsigma}(s;w)[\sigma^{\{\alpha_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(\frac{1}{2};w)\sigma^{\beta_{l\varsigma}}(\frac{1}{2};w)]_{A_\varsigma}{}^{B_\varsigma}[\sigma^{\beta_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\beta_{(l-1)\varsigma}}(s-\frac{1}{2};w)]\bar{N}_{B_\varsigma}(s;w)$
$=\frac{1}{(n+l-1)!}X^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{(l-1)\varsigma}\}}(s,n,l-1;w)\sigma^{\beta_{l\varsigma}\}}(s;w)-X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{l\varsigma}\}}(s,n+1,l-1;w)$
$=\frac{1}{(n+l-1)!}X^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{(l-1)\varsigma}\}}(s,n,l-1;w)\sigma^{\beta_{l\varsigma}\}}(s;w)-\frac{1}{4}\frac{1}{(n+l-2)!}X^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-1)\varsigma}\beta_{1\varsigma}\cdot\cdot\beta_{(l-1)\varsigma}\}}(s,n-1,l-1;w)\delta^{\alpha_{n\varsigma}\beta_{l\varsigma}\}}$
□

1.2 Composite constant invariant tensors $M^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$ and $N^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$

1.2.1 Introduction of $M^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$ and $N^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$

Def. 1.2.1.

$$\begin{cases}M^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w):=N^{A_\varsigma}(s;w)\sigma^{\alpha_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w)=X^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,0,n;w)\\ M(s,0;w)=1,M^{\alpha_{1\varsigma}}(s,1;w)=(1-\frac{1}{2s})\sigma^{\alpha_{1\varsigma}}(s;w)\end{cases}$$

Cor. 1.2.1.

$$\begin{cases}M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w):=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}\}}(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w)=X^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,0,n;w)\\ M^{\{\}}(s,0;w)=1,M^{\{\alpha_{1\varsigma}\}}(s,1;w)=(1-\frac{1}{2s})\sigma^{\alpha_{1\varsigma}}(s;w)\end{cases}$$

Def. 1.2.2.

$$\begin{cases}N^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w):=N^{A_\varsigma}(s;w)\sigma^{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\alpha_{2\varsigma}}(s-\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)=X^{\alpha_{1\varsigma};\alpha_{2\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,1,n-1;w)\\ N(s,0;w)=1,N^{\alpha_{1\varsigma}}(s,1;w)=\frac{1}{2s}\sigma^{\alpha_{1\varsigma}}(s;w)\end{cases}$$

Cor. 1.2.2.
$$\begin{cases} N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w):=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\tfrac{1}{2};w)\sigma^{\alpha_{2\varsigma}}(s-\tfrac{1}{2};w)\cdots\sigma^{\alpha_{n\varsigma}\}}(s-\tfrac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)=X^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,1,n-1;w) \\ N^{\{\}}(s,0;w)=1, N^{\{\alpha_{1\varsigma}\}}(s,1;w)=\frac{1}{2s}\sigma^{\alpha_{1\varsigma}}(s;w) \end{cases}$$

1.2.2 Recursive relations of $M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$ and $N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$

Thm. 1.2.1.
$$N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)=\frac{1}{(n-1)!}N^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$$

Proof: $N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\alpha_{2\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{n\varsigma}\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$
$= N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\alpha_{2\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{(n-1)\varsigma}}(s-\frac{1}{2};w)[\bar{N}_{B_\varsigma}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\sigma^{\alpha_{n\varsigma}\}}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)]$
$= N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\alpha_{2\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{(n-1)\varsigma}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$- N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\alpha_{2\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{(n-1)\varsigma}}(s-\frac{1}{2};w)\sigma^{\alpha_{n\varsigma}\}}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)$
$= \frac{1}{(n-1)!}N^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$ □

Thm. 1.2.2. $N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)=\frac{1}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$

Proof: $M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{n\varsigma}\}}(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w)$
$= N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{(n-1)\varsigma}}(s-\frac{1}{2};w)[\bar{N}_{A_\varsigma}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\sigma^{\alpha_{n\varsigma}\}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)]$
$= N^{A_\varsigma}(s;w)\frac{1}{(n-1)!}\sigma^{\{\{\alpha_{1\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{(n-1)\varsigma}\}}(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$- N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}(s-\frac{1}{2};w)\cdots\sigma^{\alpha_{(n-1)\varsigma}}(s-\frac{1}{2};w)\sigma^{\alpha_{n\varsigma}\}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$
$= \frac{1}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$ □

Cor. 1.2.3.
$M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= \frac{2}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+ \frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}, n\geq 2$

Proof: $N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= \frac{1}{(n-1)!}N^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$
$\Leftrightarrow \frac{1}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= \frac{1}{(n-1)!}[\frac{1}{(n-2)!}M^{\{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}\}}(s;w)-M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)]\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$- \frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$
$\Leftrightarrow \frac{1}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= \frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$- \frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$
$\Leftrightarrow M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= \frac{2}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+ \frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}, n\geq 2$ □

Cor. 1.2.4. $M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)=\frac{2}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+ \frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)[\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}-\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$

1.2.3 Recursive relations deepening

Cor. 1.2.5.
$M^{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)$
$= \frac{2}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}\}}(s;w)-\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}\}}(s;w)$
$+ \frac{1}{4}\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}\}}, n\geq 2$

Cor. 1.2.6. $M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= \frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)[\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}+3\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$
$+ \frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-3)\varsigma}\}}(s,n-3;w)[\frac{1}{2}\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}-2\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)]\sigma^{\alpha_{n\varsigma}\}}(s;w)$

Proof: $M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$
$= \frac{2}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+ \frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}, n\geq 2$
$= \frac{2}{(n-1)!}[\frac{2}{(n-2)!}M^{\{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}\}}(s;w)$

$-\frac{1}{(n-3)!}M^{\{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}\}}(s;w)$
$+\frac{1}{4}\frac{1}{(n-3)!}M^{\{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}\}}]\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$-\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+\frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}, n\geq 2$
$=2[\frac{2}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$-\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+\frac{1}{4}\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}\sigma^{\alpha_{n\varsigma}\}}(s;w)]$
$-\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+\frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}, n\geq 2$
$=\frac{3}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+\frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$
$-\frac{2}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+\frac{1}{4}\frac{2}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$=\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)[\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}+3\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$
$+\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)[\frac{1}{2}\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}-2\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)]\sigma^{\alpha_{n\varsigma}\}}(s;w)$ □

Cor. 1.2.7. $M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n-2\varsigma}\}}(s,n;w)$
$=\frac{2}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\sigma^{\alpha_{(n-2)\varsigma}\}}(s;w)+\frac{1}{(n-4)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-4)\varsigma}\}}(s,n-4;w)[\frac{1}{4}\delta^{\alpha_{(n-3)\varsigma}\alpha_{(n-2)\varsigma}\}}-\sigma^{\alpha_{(n-3)\varsigma}}(s;w)\sigma^{\alpha_{(n-2)\varsigma}\}}(s;$

Cor. 1.2.8. $M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w)$
$=\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-2)\varsigma}\}}(s,n-2;w)[\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}+3\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$
$+\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)[\frac{1}{2}\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}-2\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)]\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$=\frac{1}{(n-2)!}\{\frac{2}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\sigma^{\alpha_{(n-2)\varsigma}\}}(s;w)$
$+\frac{1}{(n-4)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-4)\varsigma}\}}(s,n-4;w)[\frac{1}{4}\delta^{\alpha_{(n-3)\varsigma}\alpha_{(n-2)\varsigma}\}}-\sigma^{\alpha_{(n-3)\varsigma}}(s;w)\sigma^{\alpha_{(n-2)\varsigma}\}}(s;w)]\}[\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}+3\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$
$+\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)[\frac{1}{2}\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}-2\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)]\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$=\frac{2}{(n-3)!}M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)\sigma^{\alpha_{(n-2)\varsigma}}(s;w)[\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}+3\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$
$+\frac{1}{(n-4)!}M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-4)\varsigma}\}}(s,n-4;w)[\frac{1}{4}\delta^{\alpha_{(n-3)\varsigma}\alpha_{(n-2)\varsigma}}-\sigma^{\alpha_{(n-3)\varsigma}}(s;w)\sigma^{\alpha_{(n-2)\varsigma}}(s;w)][\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}+3\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$
$+\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)[\frac{1}{2}\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}-2\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)]\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$=\frac{1}{(n-3)!}M^{\{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-3)\varsigma}\}}(s,n-3;w)[\delta^{\alpha_{(n-2)\varsigma}\alpha_{(n-1)\varsigma}}+4\sigma^{\alpha_{(n-2)\varsigma}}(s;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)]\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+\frac{1}{(n-4)!}M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{(n-4)\varsigma}\}}(s,n-4;w)[\frac{1}{4}\delta^{\alpha_{(n-3)\varsigma}\alpha_{(n-2)\varsigma}}-\sigma^{\alpha_{(n-3)\varsigma}}(s;w)\sigma^{\alpha_{(n-2)\varsigma}}(s;w)][\frac{1}{4}\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}+3\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)]$

1.2.4 Direct calculation of $N^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$ by using full symmetry technique

Pro. 1.2.1. $N^{\{\alpha_\varsigma\}}(s,1)=N^{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)=\frac{1}{2s}\sigma^{\alpha_\varsigma}(s;w)$

Pro. 1.2.2. $N^{\{\alpha_\varsigma\beta_\varsigma\}}(s,2)=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$
$=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-\frac{s}{2}\delta^{\{\alpha_\varsigma\beta_\varsigma\}}]$

Proof: $N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$
$=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)[\bar{N}_{B_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-\sigma^{\beta_\varsigma\}}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)]$
$=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma\}}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)$
$=\frac{1}{2s}\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-N^{A_\varsigma}(s;w)\frac{1}{2}\delta_{A_\varsigma}{}^{C_\varsigma}\delta^{\alpha_\varsigma\beta_\varsigma}\bar{N}_{C_\varsigma}(s;w)$
$=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-\frac{s}{2}\delta^{\{\alpha_\varsigma\beta_\varsigma\}}]$ □

Pro. 1.2.3. $N^{\{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\}}(s,3)=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}(s-\frac{1}{2};w)\sigma^{\gamma_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$
$=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)+\frac{1-4s}{4}\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}]$

Proof: $N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}(s-\frac{1}{2};w)\sigma^{\gamma_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$
$=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}(s-\frac{1}{2};w)[\bar{N}_{B_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)-\sigma^{\gamma_\varsigma\}}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)]$
$=N^{A_\varsigma}(s;w)\frac{1}{2!}\sigma^{\{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)$
$-N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\gamma_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)$
$=\frac{1}{2s}\frac{1}{2!}[\sigma^{\{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-\frac{s}{2}\delta^{\{\{\alpha_\varsigma\beta_\varsigma\}}]\sigma^{\gamma_\varsigma\}}(s;w)$
$-N^{A_\varsigma}(s;w)\frac{1}{2!}\sigma^{\{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\gamma_\varsigma\}}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)$
$=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)-\frac{s}{2}\delta^{\{\alpha_\varsigma\beta_\varsigma}]\sigma^{\gamma_\varsigma\}}(s;w)-N^{A_\varsigma}(s;w)\frac{1}{2!}\frac{1}{2}\delta_{A_\varsigma}{}^{C_\varsigma}\delta^{\{\alpha_\varsigma\gamma_\varsigma}\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)$
$=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)-\frac{s}{2}\delta^{\{\alpha_\varsigma\beta_\varsigma}]\sigma^{\gamma_\varsigma\}}(s;w)-\frac{1}{2!}\frac{1}{2}\delta^{\{\alpha_\varsigma\gamma_\varsigma}N^{A_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{A_\varsigma}(s;w)$
$=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)+\frac{1-4s}{4}\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}]$ □

Pro. 1.2.4. $N^{\{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\eta_\varsigma\}}(s,4;w)=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}(s-\frac{1}{2};w)\sigma^{\gamma_\varsigma}(s-\frac{1}{2};w)\sigma^{\eta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$
$=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)+\frac{3}{4}(1-2s)\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\delta^{\gamma_\varsigma\eta_\varsigma\}}-\frac{s}{8}\delta^{\{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma\}}]$

Cor. 1.2.9.

$$\begin{cases}N^{\{\alpha_\varsigma\}}(s,1;w)=\frac{1}{2s}\sigma^{\alpha_\varsigma}(s;w)\\ N^{\{\alpha_\varsigma\beta_\varsigma\}}(s,2;w)=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-\frac{s}{2}\delta^{\{\alpha_\varsigma\beta_\varsigma\}}]\\ N^{\{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\}}(s,3;w)=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)-\frac{1+2s}{4}\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}]\\ N^{\{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\eta_\varsigma\}}(s,4;w)=\frac{1}{2s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)+\frac{3}{4}(1-2s)\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\delta^{\gamma_\varsigma\eta_\varsigma\}}-\frac{s}{8}\delta^{\{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma\}}]\end{cases}$$

1.2.5 Direct calculation of first few items for constant invariant tensors $N^{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}}(s,n)$

Pro. 1.2.5. $N^{\alpha_\varsigma}(s,1)=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\bar{N}_{B_\varsigma}(s)=\frac{1}{2s}\sigma^{\alpha_\varsigma}(s)$

Pro. 1.2.6. $N^{\alpha_\varsigma\beta_\varsigma}(s,2)=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)=\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma\}}(s)-s\delta^{\alpha_\varsigma\beta_\varsigma}]$

Proof: $N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)$
$=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})[-\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\bar{N}_{C_\varsigma}(s)+\bar{N}_{B_\varsigma}(s)\sigma^{\beta_\varsigma}(s)]$
$=-\frac{1}{4s}[s\delta^{\alpha_\varsigma\beta_\varsigma}+\sigma^{[\alpha_\varsigma}(s)\sigma^{\beta_\varsigma]}(s)]+\frac{1}{2s}\sigma^{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma}(s)$
$=\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma\}}(s)-s\delta^{\alpha_\varsigma\beta_\varsigma}]$ □

Pro. 1.2.7. $N^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}(s,3)=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)$
$=\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}-\delta^{\alpha_\varsigma[\beta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]-\frac{1}{4}\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s)+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\sigma^{[\beta_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)+\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma\}}(s)]$

Proof: $N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)$
$=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})[-\sigma^{\gamma_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\bar{N}_{C_\varsigma}(s)+\bar{N}_{B_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)]$
$=-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\gamma_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\bar{N}_{C_\varsigma}(s)+\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma\}}(s)-s\delta^{\alpha_\varsigma\beta_\varsigma}]\sigma^{\gamma_\varsigma}(s)$
$=-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\gamma_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})[-\sigma^{\beta_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}(\frac{1}{2})\bar{N}_{D_\varsigma}(s)+\bar{N}_{C_\varsigma}(s)\sigma^{\beta_\varsigma}(s)]+\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma\}}(s)-s\delta^{\alpha_\varsigma\beta_\varsigma}]\sigma^{\gamma_\varsigma}(s)$
$=\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\gamma_\varsigma\beta_\varsigma}-\delta^{\alpha_\varsigma[\beta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]-\frac{1}{4s}[s\delta^{\alpha_\varsigma\gamma_\varsigma}+\sigma^{[\alpha_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)]\sigma^{\beta_\varsigma}(s)+\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma\}}(s)-s\delta^{\alpha_\varsigma\beta_\varsigma}]\sigma^{\gamma_\varsigma}(s)$
$=\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\gamma_\varsigma\beta_\varsigma}-\delta^{\alpha_\varsigma[\beta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]-\frac{1}{4}\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s)+\frac{1}{4s}\{\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma\}}(s)\sigma^{\gamma_\varsigma}(s)-\sigma^{[\alpha_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)\sigma^{\beta_\varsigma}(s)\}$
$=\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}-\delta^{\alpha_\varsigma[\beta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]-\frac{1}{4}\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s)+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\sigma^{[\beta_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)+\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma\}}(s)]$ □

Cor. 1.2.10.
$N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma\}}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)=\frac{1}{4s}\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}-\frac{1}{2}\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s)+\frac{1}{2s}\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma\}}(s)$

Pro. 1.2.8. $N^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\eta_\varsigma}(s,4)=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\sigma^{\eta_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)$
$=-\frac{1}{16}(\delta^{\alpha_\varsigma\eta_\varsigma}\delta^{\gamma_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma}-\delta^{\alpha_\varsigma\gamma_\varsigma}\delta^{\beta_\varsigma\eta_\varsigma})+\frac{i}{16s}[\varepsilon^{\gamma_\varsigma\beta_\varsigma\alpha_\varsigma}\sigma^{\eta_\varsigma}(s)-\varepsilon^{\gamma_\varsigma\beta_\varsigma\eta_\varsigma}\sigma^{\alpha_\varsigma}(s)]$
$-\frac{1}{16s}[\delta^{\alpha_\varsigma\eta_\varsigma}\sigma^{[\gamma_\varsigma}(s)\sigma^{\beta_\varsigma]}(s)+\sigma^{[\alpha_\varsigma}(s)\sigma^{\eta_\varsigma]}(s)\delta^{\gamma_\varsigma\beta_\varsigma}]$
$+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\gamma_\varsigma}+\delta^{\alpha_\varsigma[\eta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]\sigma^{\beta_\varsigma}(s)+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma[\eta_\varsigma}\sigma^{\beta_\varsigma]}(s)]\sigma^{\gamma_\varsigma}(s)+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}-\delta^{\alpha_\varsigma[\beta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]\sigma^{\eta_\varsigma}(s)$
$-\frac{1}{4}[\delta^{\alpha_\varsigma\eta_\varsigma}\sigma^{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)+\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s)\sigma^{\eta_\varsigma}(s)]$
$+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\sigma^{[\beta_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)\sigma^{\eta_\varsigma}(s)+\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma\}}(s)\sigma^{\eta_\varsigma}(s)-\sigma^{[\alpha_\varsigma}(s)\sigma^{\eta_\varsigma]}(s)\sigma^{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)]$

Proof: $N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\sigma^{\eta_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)$
$=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})[-\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\bar{N}_{C_\varsigma}(s)+\bar{N}_{B_\varsigma}(s)\sigma^{\eta_\varsigma}(s)]$
$=-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{C_\varsigma}(s)$
$+N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)\sigma^{\eta_\varsigma}(s)$
$=-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})[-\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}(\frac{1}{2})\bar{N}_{D_\varsigma}(s)+\bar{N}_{C_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)]$
$+N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)\sigma^{\eta_\varsigma}(s)$
$=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\bar{N}_{D_\varsigma}(s)$
$-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\bar{N}_{C_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)$
$+N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)\sigma^{\eta_\varsigma}(s)$
$=N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}(\frac{1}{2})[-\sigma^{\beta_\varsigma}{}_{D_\varsigma}{}^{E_\varsigma}(\frac{1}{2})\bar{N}_{E_\varsigma}(s)+\bar{N}_{D_\varsigma}(s)\sigma^{\beta_\varsigma}(s)]$
$-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\bar{N}_{C_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)$
$+N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)\sigma^{\eta_\varsigma}(s)$
$=-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}{}_{D_\varsigma}{}^{E_\varsigma}(\frac{1}{2})\bar{N}_{E_\varsigma}(s)$
$+N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}(\frac{1}{2})\bar{N}_{D_\varsigma}(s)\sigma^{\beta_\varsigma}(s)$
$-N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\eta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\bar{N}_{C_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)$
$+N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\sigma^{\gamma_\varsigma}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)\sigma^{\eta_\varsigma}(s)$
$=-\frac{1}{16}(\delta^{\alpha_\varsigma\eta_\varsigma}\delta^{\gamma_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma}-\delta^{\alpha_\varsigma\gamma_\varsigma}\delta^{\beta_\varsigma\eta_\varsigma})$
$-\frac{1}{16s}[\delta^{\alpha_\varsigma\eta_\varsigma}\sigma^{[\gamma_\varsigma}(s)\sigma^{\beta_\varsigma]}(s)+\sigma^{[\alpha_\varsigma}(s)\sigma^{\eta_\varsigma]}(s)\delta^{\gamma_\varsigma\beta_\varsigma}]+\frac{i}{16s}[\varepsilon^{\gamma_\varsigma\beta_\varsigma\alpha_\varsigma}\sigma^{\eta_\varsigma}(s)-\varepsilon^{\gamma_\varsigma\beta_\varsigma\eta_\varsigma}\sigma^{\alpha_\varsigma}(s)]$
$+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\gamma_\varsigma}+\delta^{\alpha_\varsigma[\eta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]\sigma^{\beta_\varsigma}(s)$
$+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma[\eta_\varsigma}\sigma^{\beta_\varsigma]}(s)]\sigma^{\gamma_\varsigma}(s)-\frac{1}{4s}[s\delta^{\alpha_\varsigma\eta_\varsigma}+\sigma^{[\alpha_\varsigma}(s)\sigma^{\eta_\varsigma]}(s)]\sigma^{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)$
$+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}-\delta^{\alpha_\varsigma[\beta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]\sigma^{\eta_\varsigma}(s)-\frac{1}{4}\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s)\sigma^{\eta_\varsigma}(s)$

$+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\sigma^{[\beta_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)+\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma\}}(s)]\sigma^{\eta_\varsigma}(s)$
$=-\frac{1}{16}(\delta^{\alpha_\varsigma\eta_\varsigma}\delta^{\gamma_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma}-\delta^{\alpha_\varsigma\gamma_\varsigma}\delta^{\beta_\varsigma\eta_\varsigma})+\frac{i}{16s}[\varepsilon^{\gamma_\varsigma\beta_\varsigma\alpha_\varsigma}\sigma^{\eta_\varsigma}(s)-\varepsilon^{\gamma_\varsigma\beta_\varsigma\eta_\varsigma}\sigma^{\alpha_\varsigma}(s)]$
$-\frac{1}{16s}[\delta^{\alpha_\varsigma\eta_\varsigma}\sigma^{[\gamma_\varsigma}(s)\sigma^{\beta_\varsigma]}(s)+\sigma^{[\alpha_\varsigma}(s)\sigma^{\eta_\varsigma]}(s)\delta^{\gamma_\varsigma\beta_\varsigma}]$
$+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\gamma_\varsigma}+\delta^{\alpha_\varsigma[\eta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]\sigma^{\beta_\varsigma}(s)+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma[\eta_\varsigma}\sigma^{\beta_\varsigma]}(s)]\sigma^{\gamma_\varsigma}(s)+\frac{1}{8s}[\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}-\delta^{\alpha_\varsigma[\beta_\varsigma}\sigma^{\gamma_\varsigma]}(s)]\sigma^{\eta_\varsigma}(s)$
$-\frac{1}{4}[\delta^{\alpha_\varsigma\eta_\varsigma}\sigma^{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)+\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s)\sigma^{\eta_\varsigma}(s)]$
$+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\sigma^{[\beta_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)\sigma^{\eta_\varsigma}(s)+\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma\}}(s)\sigma^{\eta_\varsigma}(s)-\sigma^{[\alpha_\varsigma}(s)\sigma^{\eta_\varsigma]}(s)\sigma^{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)]$ □

Cor. 1.2.11. $N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{\{\gamma_\varsigma}(s-\frac{1}{2})[\sigma^{\beta_\varsigma}(s-\frac{1}{2})]\sigma^{\eta_\varsigma\}}(s-\frac{1}{2})\bar{N}_{B_\varsigma}(s)$
$=\frac{1}{8}(-\delta^{\alpha_\varsigma\eta_\varsigma}\delta^{\gamma_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma}-\delta^{\alpha_\varsigma\gamma_\varsigma}\delta^{\beta_\varsigma\eta_\varsigma})-\frac{i}{16s}[\varepsilon^{\gamma_\varsigma\beta_\varsigma\alpha_\varsigma}\sigma^{\eta_\varsigma}(s)+\varepsilon^{\eta_\varsigma\beta_\varsigma\alpha_\varsigma}\sigma^{\gamma_\varsigma}(s)]$
$\frac{1}{16s}[\delta^{\alpha_\varsigma\eta_\varsigma}\sigma^{[\gamma_\varsigma}(s)\sigma^{\beta_\varsigma]}(s)+\delta^{\alpha_\varsigma\gamma_\varsigma}\sigma^{[\eta_\varsigma}(s)\sigma^{\beta_\varsigma]}(s)-\sigma^{[\alpha_\varsigma}(s)\sigma^{\eta_\varsigma]}(s)\delta^{\gamma_\varsigma\beta_\varsigma}-\sigma^{[\alpha_\varsigma}(s)\sigma^{\gamma_\varsigma]}(s)\delta^{\eta_\varsigma\beta_\varsigma}]$
$+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\gamma_\varsigma}]\sigma^{\beta_\varsigma}(s)+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\delta^{\eta_\varsigma\beta_\varsigma}]\sigma^{\gamma_\varsigma}(s)+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}]\sigma^{\eta_\varsigma}(s)$
$-\frac{1}{4}[\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma}(s)\sigma^{\eta_\varsigma\}}(s)]$
$+\frac{1}{4s}[\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma}(s)\sigma^{\eta_\varsigma\}}(s)-\sigma^{\alpha_\varsigma}(s)\sigma^{\{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma}(s)\sigma^{\eta_\varsigma\}}(s)+2\sigma^{\alpha_\varsigma}(s)\sigma^{\{\gamma_\varsigma}(s)[\sigma^{\beta_\varsigma}(s)]\sigma^{\eta_\varsigma\}}(s)]$

Cor. 1.2.12. $N^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})[(s-\frac{3}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2})\delta^{\gamma_\varsigma\eta_\varsigma}+(s-\frac{1}{2})\delta^{\beta_\varsigma\{\gamma_\varsigma}\sigma^{\eta_\varsigma\}}(s-\frac{1}{2})]\bar{N}_{B_\varsigma}(s)$
$=(s-\frac{3}{2})\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma\}}(s)-s\delta^{\alpha_\varsigma\beta_\varsigma}]\delta^{\gamma_\varsigma\eta_\varsigma}+(s-\frac{1}{2})\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s)\sigma^{\gamma_\varsigma\}}(s)\delta^{\eta_\varsigma\beta_\varsigma}+\sigma^{\{\alpha_\varsigma}(s)\sigma^{\eta_\varsigma\}}(s)\delta^{\gamma_\varsigma\beta_\varsigma}-s\delta^{\alpha_\varsigma\{\gamma_\varsigma}\delta^{\eta_\varsigma\}\beta_\varsigma}]\}$

Pro. 1.2.9. $\sigma^{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma}+\sigma^{\{\beta_\varsigma}(s)[\sigma^{\alpha_\varsigma}(s)]\sigma^{\gamma_\varsigma\}}(s)=\frac{1}{3!}[\sigma^{\{\alpha_\varsigma}(s)\delta^{\beta_\varsigma\gamma_\varsigma\}}+2\sigma^{\{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma}(s)\sigma^{\gamma_\varsigma\}}(s)]$

1.3 Introduction of composite constant invariant tensor $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s;w)$

1.3.1 Definition of composite constant invariant tensor $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n;w)$

Def. 1.3.1.
$$\begin{cases}\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n;w):=\Gamma^{A_{1\varsigma}\cdot\cdot A_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}_{k_\varsigma}(s;w)\prod_{i=1}^{n}\sigma^{\alpha_{i\varsigma}}{}_{A_{i\varsigma}}{}^{B_{i\varsigma}}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{B_{1\varsigma}\cdot\cdot B_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}(s;w)\\ \Gamma_{\alpha'_{1\varsigma}\cdot\cdot\alpha'_{n\varsigma}}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,n;w):=\Gamma^{k'_\varsigma}_{A'_{1\varsigma}\cdot\cdot A'_{n\varsigma}A'_{(n+1)\varsigma}\cdot\cdot A'_{(2s)\varsigma}}(s;w)\prod_{i=1}^{n}\sigma_{\alpha'_{i\varsigma}}{}^{A'_{i\varsigma}}{}_{B'_{i\varsigma}}(\frac{1}{2};w)\Gamma^{B'_{1\varsigma}\cdot\cdot B'_{n\varsigma}A'_{(n+1)\varsigma}\cdot\cdot A'_{(2s)\varsigma}}_{l'_\varsigma}(s;w)\end{cases}$$

Def. 1.3.2. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w):\prec\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n;w),\Gamma_{\alpha'_{1\varsigma}\cdot\cdot\alpha'_{n\varsigma}}(s,n;w):\prec\Gamma_{\alpha'_{1\varsigma}\cdot\cdot\alpha'_{n\varsigma}}{}^{k'_\varsigma}{}_{l'_\varsigma}(s,n;w)$

Cor. 1.3.1. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s;w)\simeq\Gamma_{\alpha'_{1\varsigma}\cdot\cdot\alpha'_{n\varsigma}}{}^{k'_\varsigma}{}_{l'_\varsigma}(s;w),\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s;w)\simeq\Gamma_{\alpha'_{1\varsigma}\cdot\cdot\alpha'_{n\varsigma}}(s;w)$

1.3.2 Recursive formula of composite constant invariant tensor $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n;w)$

Thm. 1.3.1. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)=N^{A_{1\varsigma}}(s;w)\sigma^{\alpha_{1\varsigma}}{}_{A_{1\varsigma}}{}^{B_{1\varsigma}}(\frac{1}{2};w)\Gamma^{\alpha_{2\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n-1;w)\bar{N}_{B_{1\varsigma}}(s;w)$

Proof: $\Gamma^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w)$
$=\Gamma^{A_{1\varsigma}\cdot\cdot A_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}_{k_{1\varsigma}}(s;w)\prod_{i=1}^{n}\sigma^{\alpha_{i\varsigma}}{}_{A_{i\varsigma}}{}^{B_{i\varsigma}}(\frac{1}{2};w)\Gamma^{l_{1\varsigma}}_{B_{1\varsigma}\cdot\cdot B_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}(s;w)$
$=N^{A_{1\varsigma}k_{2\varsigma}}_{k_{1\varsigma}}(s;w)\Gamma^{A_{2\varsigma}\cdot\cdot A_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}_{k_{2\varsigma}}(s-\frac{1}{2};w)\prod_{i=1}^{n}\sigma^{\alpha_{i\varsigma}}{}_{A_{i\varsigma}}{}^{B_{i\varsigma}}(\frac{1}{2};w)N^{l_{1\varsigma}}_{B_{2\varsigma}l_{2\varsigma}}\Gamma^{l_{2\varsigma}}_{B_{2\varsigma}\cdot\cdot B_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}(s;w)$
$=N^{A_{1\varsigma}k_{2\varsigma}}_{k_{1\varsigma}}(s;w)\sigma^{\alpha_{1\varsigma}}{}_{A_{1\varsigma}}{}^{B_{1\varsigma}}(\frac{1}{2};w)$
$[\Gamma^{A_{2\varsigma}\cdot\cdot A_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}_{k_{2\varsigma}}(s-\frac{1}{2};w)\prod_{i=2}^{n}\sigma^{\alpha_{i\varsigma}}{}_{A_{i\varsigma}}{}^{B_{i\varsigma}}(\frac{1}{2};w)\Gamma^{l_{2\varsigma}}_{B_{2\varsigma}\cdot\cdot B_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}(s-\frac{1}{2};w)]N^{l_{1\varsigma}}_{B_{1\varsigma}l_{1\varsigma}}(s;w)$
$=N^{A_{1\varsigma}k_{2\varsigma}}_{k_{1\varsigma}}(s;w)\sigma^{\alpha_{1\varsigma}}{}_{A_{1\varsigma}}{}^{B_{1\varsigma}}(\frac{1}{2};w)\Gamma^{\alpha_{2\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_{2\varsigma}}{}^{l_{2\varsigma}}(s,n-1;w)N^{l_{1\varsigma}}_{B_{1\varsigma}l_{1\varsigma}}(s;w)$
$=N^{A_{1\varsigma}}(s;w)\sigma^{\alpha_{1\varsigma}}{}_{A_{1\varsigma}}{}^{B_{1\varsigma}}(\frac{1}{2};w)\Gamma^{\alpha_{2\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n-1;w)\bar{N}_{B_{1\varsigma}}(s;w)$ □

Cor. 1.3.2. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$
$=N^{A_{1\varsigma}}(s;w)\cdot\cdot N^{A_{n\varsigma}}(s-\frac{n-1}{2};w)\sigma^{\alpha_{1\varsigma}}{}_{A_{1\varsigma}}{}^{B_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}{}_{A_{n\varsigma}}{}^{B_{n\varsigma}}(\frac{1}{2};w)\bar{N}_{B_{n\varsigma}}(s-\frac{n-1}{2};w)\cdot\cdot\bar{N}_{B_{1\varsigma}}(s;w)$

Cor. 1.3.3. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)=N^{A_{\varsigma1}\cdot\cdot A_{\varsigma n}}(s,n;w)\sigma^{\alpha_{1\varsigma}}{}_{A_{1\varsigma}}{}^{B_{1\varsigma}}(\frac{1}{2};w)\cdot\cdot\sigma^{\alpha_{n\varsigma}}{}_{A_{n\varsigma}}{}^{B_{n\varsigma}}(\frac{1}{2};w)\bar{N}_{B_{\varsigma1}\cdot\cdot B_{\varsigma n}}(s,n;w)$

1.3.3 Direct calculation of first few items for constant invariant tensor $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n;w)$

Pro. 1.3.1. $N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{C_\varsigma}(\frac{1}{2};w)\sigma^{\gamma_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{C_\varsigma}(s;w)=\frac{1}{4}(1-\frac{1}{2s})\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}$

Pro. 1.3.2. $\Gamma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s,1)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)=\frac{1}{2s}\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s;w)$
$=\frac{1}{(1!)^2}C_{2s}^{-1}\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s;w)$

Pro. 1.3.3. $\Gamma^{\alpha_\varsigma\beta_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s,2)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{J_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma J_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)$
$=\frac{1}{(2!)^2}C_{2s}^{-2}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-\frac{s}{2}\delta^{\{\alpha_\varsigma\beta_\varsigma\}}]_{k_\varsigma}{}^{l_\varsigma}$

Proof: $\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{J_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma J_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)$

$= \frac{1}{2!}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma\}}{}_{B_\varsigma}{}^{J_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma J_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s;w)$
$= \frac{1}{2!}[N^{A_\varsigma}(s;w)\sigma^{\{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\frac{1}{2s-1}\sigma^{\beta_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)]_{k_\varsigma}{}^{l_\varsigma}$
$= \frac{1}{2s-1}\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-s\delta^{\alpha_\varsigma\beta_\varsigma}]_{k_\varsigma}{}^{l_\varsigma}$
$= \frac{1}{(2!)^2}C_{2s}^{-2}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-\frac{s}{2}\delta^{\{\alpha_\varsigma\beta_\varsigma\}}]_{k_\varsigma}{}^{l_\varsigma}$ □

Pro. 1.3.4. $\Gamma^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s,3) = \Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{J_\varsigma}(\frac{1}{2};w)\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{K_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma J_\varsigma K_\varsigma D_\varsigma\cdot\cdot}_{2s}}(s;w)$

$= \frac{1}{(3!)^2}C_{2s}^{-3}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)+\frac{1-3s}{2}\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}]$

Proof: $\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{J_\varsigma}(\frac{1}{2};w)\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{K_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma J_\varsigma K_\varsigma D_\varsigma\cdot\cdot}_{2s}}(s;w)$

$= N^{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\frac{1}{2s-2}[\frac{1}{4s-2}\sigma^{\{\beta_\varsigma}(s-\frac{1}{2};w)\sigma^{\gamma_\varsigma\}}(s-\frac{1}{2};w)-\frac{1}{4}\delta^{\beta_\varsigma\gamma_\varsigma}]\bar{N}_{B_\varsigma}(s;w)$
$= \frac{1}{4(s-1)(2s-1)}N^{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sigma^{\{\beta_\varsigma}(s-\frac{1}{2};w)\sigma^{\gamma_\varsigma\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)-\frac{1}{16s(s-1)}\sigma^{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma}$
$= \frac{1}{4(s-1)(2s-1)}\{\frac{1}{2s}[\sigma^{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma}+\sigma^{\beta_\varsigma}(s;w)\sigma^{\alpha_\varsigma}(s;w)\sigma^{\gamma_\varsigma}(s;w)+\sigma^{\gamma_\varsigma}(s;w)\sigma^{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)]-\frac{1}{4}\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}\}$
$= \frac{1}{4(s-1)(2s-1)}\{\frac{1}{2s}\frac{1}{3!}[\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}+2\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)]-\frac{1}{4}\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}\}$
$= \frac{1}{48s(s-1)(2s-1)}[2\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)+(1-3s)\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}]$
$= \frac{1}{16s(s-\frac{1}{2};w)(s-1)}\{\sigma^{\{\beta_\varsigma}(s;w)[\sigma^{\alpha_\varsigma}(s;w)]\sigma^{\gamma_\varsigma\}}(s;w)-[(s-1)\sigma^{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma}+s\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma\}}(s;w)]\}$
$= \frac{1}{(3!)^2}C_{2s}^{-3}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)+\frac{1-3s}{2}\sigma^{\{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma\}}]$ □

Pro. 1.3.5. $\Gamma^{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma\eta_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s,4)$
$= \Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{J_\varsigma}(\frac{1}{2};w)\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{K_\varsigma}(\frac{1}{2};w)\sigma^{\eta_\varsigma}{}_{D_\varsigma}{}^{L_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma J_\varsigma K_\varsigma L_\varsigma\cdot\cdot}_{2s}}(s;w)$
$= \frac{1}{(4!)^2}C_{2s}^{-4}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)+(2-3s)\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\delta^{\gamma_\varsigma\eta_\varsigma\}}+\frac{3}{4}s(s-1)\delta^{\{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma\}}]$

Proof: $\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2s}}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{I_\varsigma}(\frac{1}{2};w)\sigma^{\beta_\varsigma}{}_{B_\varsigma}{}^{J_\varsigma}(\frac{1}{2};w)\sigma^{\gamma_\varsigma}{}_{C_\varsigma}{}^{K_\varsigma}(\frac{1}{2};w)\sigma^{\eta_\varsigma}{}_{D_\varsigma}{}^{L_\varsigma}(\frac{1}{2};w)\Gamma^{l_\varsigma}_{\underbrace{I_\varsigma J_\varsigma K_\varsigma L_\varsigma\cdot\cdot}_{2s}}(s;w)$

$= \frac{1}{16(s-\frac{1}{2};w)(s-1)(s-\frac{3}{2})}N^{A_\varsigma}(s;w)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)$
$\{\sigma^{\{\gamma_\varsigma}(s-\frac{1}{2};w)[\sigma^{\beta_\varsigma}(s-\frac{1}{2};w)]\sigma^{\eta_\varsigma\}}(s-\frac{1}{2};w)-[(s-\frac{3}{2})\sigma^{\beta_\varsigma}(s-\frac{1}{2};w)\delta^{\gamma_\varsigma\eta_\varsigma}+(s-\frac{1}{2})\delta^{\beta_\varsigma\{\gamma_\varsigma}\sigma^{\eta_\varsigma\}}(s-\frac{1}{2};w)]\}\bar{N}_{B_\varsigma}(s;w)$
$= \frac{1}{16(s-\frac{1}{2};w)(s-1)(s-\frac{3}{2})}$
$\{\frac{1}{8}(-\delta^{\alpha_\varsigma\eta_\varsigma}\delta^{\gamma_\varsigma\beta_\varsigma}+\delta^{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma}-\delta^{\alpha_\varsigma\gamma_\varsigma}\delta^{\beta_\varsigma\eta_\varsigma})-\frac{i}{16s}[\varepsilon^{\gamma_\varsigma\beta_\varsigma\alpha_\varsigma}\sigma^{\eta_\varsigma}(s;w)+\varepsilon^{\eta_\varsigma\beta_\varsigma\alpha_\varsigma}\sigma^{\gamma_\varsigma}(s;w)]$
$\frac{1}{16s}[\delta^{\alpha_\varsigma\eta_\varsigma}\sigma^{[\gamma_\varsigma}(s;w)\sigma^{\beta_\varsigma]}(s;w)+\delta^{\alpha_\varsigma\gamma_\varsigma}\sigma^{[\eta_\varsigma}(s;w)\sigma^{\beta_\varsigma]}(s;w)-\sigma^{[\alpha_\varsigma}(s;w)\sigma^{\eta_\varsigma]}(s;w)\delta^{\gamma_\varsigma\beta_\varsigma}-\sigma^{[\alpha_\varsigma}(s;w)\sigma^{\gamma_\varsigma]}(s;w)\delta^{\eta_\varsigma\beta_\varsigma}]$
$+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s;w)\delta^{\eta_\varsigma\gamma_\varsigma}]\sigma^{\beta_\varsigma}(s;w)+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s;w)\delta^{\eta_\varsigma\beta_\varsigma}]\sigma^{\gamma_\varsigma}(s;w)+\frac{1}{4s}[\sigma^{\alpha_\varsigma}(s;w)\delta^{\beta_\varsigma\gamma_\varsigma}]\sigma^{\eta_\varsigma}(s;w)-\frac{1}{4}[\delta^{\alpha_\varsigma\{\beta_\varsigma}\sigma^{\gamma_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)]$
$+\frac{1}{4s}[\sigma^{\{\beta_\varsigma}(s;w)[\sigma^{\alpha_\varsigma}(s;w)]\sigma^{\gamma_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)-\sigma^{\alpha_\varsigma}(s;w)\sigma^{\{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)+2\sigma^{\alpha_\varsigma}(s;w)\sigma^{\{\gamma_\varsigma}(s;w)[\sigma^{\beta_\varsigma}(s;w)]\sigma^{\eta_\varsigma\}}(s;w)]$
$-(s-\frac{3}{2})\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma\}}(s;w)-s\delta^{\alpha_\varsigma\beta_\varsigma}]\delta^{\gamma_\varsigma\eta_\varsigma}-(s-\frac{1}{2};w)\frac{1}{4s}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\gamma_\varsigma\}}(s;w)\delta^{\eta_\varsigma\beta_\varsigma}+\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)\delta^{\gamma_\varsigma\beta_\varsigma}-$
$s\delta^{\alpha_\varsigma\{\gamma_\varsigma}\delta^{\eta_\varsigma\}\beta_\varsigma}]\}$
$= \frac{1}{(4!)^2}C_{2s}^{-4}[\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\sigma^{\gamma_\varsigma}(s;w)\sigma^{\eta_\varsigma\}}(s;w)+(2-3s)\sigma^{\{\alpha_\varsigma}(s;w)\sigma^{\beta_\varsigma}(s;w)\delta^{\gamma_\varsigma\eta_\varsigma\}}+\frac{3}{4}s(s-1)\delta^{\{\alpha_\varsigma\beta_\varsigma}\delta^{\gamma_\varsigma\eta_\varsigma\}}]$ □

1.4 Expansion of $M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w), N^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w), \Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$

1.4.1 Expansion and its recurrence relations of $M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w)$

Def. 1.4.1. $\Omega^i(s,n;w) := \sigma^{\{\alpha_{1\varsigma}}(s;w)\cdot\cdot\sigma^{\alpha_{i\varsigma}}(s;w)\delta^{\alpha_{(i+1)\varsigma}\alpha_{(i+1)\varsigma}}\cdot\cdot\delta^{[\alpha_{(n-1)\varsigma}]\alpha_{n\varsigma}\}}$

Def. 1.4.2. $\Omega^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n,n-2k;w) := \delta^{\{\alpha_{1\varsigma}\alpha_{2\varsigma}}\cdot\cdot\delta^{\alpha_{(2k-1)\varsigma}\alpha_{(2k)\varsigma}}\sigma^{\alpha_{(2k+1)\varsigma}}(s;w)\cdot\cdot\sigma^{\alpha_{n\varsigma}\}}(s;w), 1\le k\le[n/2]$

Pro. 1.4.1. $\sum_{k=0}^{[n/2]}c_{n-2k}\Omega^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n,n-2k;w)=0 \Leftrightarrow c_{n-2k}=0, 1\le k\le[n/2]; \forall\alpha_{i\varsigma}$

Proof: $\sum\limits_{k=0}^{[n/2]} c_{n-2k}\Omega^{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}}(s,n,n-2k;w)=0$

$\Leftrightarrow c_n\sigma^{\{\alpha_{1\varsigma}}(s;w)\cdot\cdot\sigma^{\alpha_{n\varsigma}\}}(s;w)+\delta^{\{\alpha_{1\varsigma}\alpha_{2\varsigma}}\sum\limits_{k=0}^{[n/2]-1} c_{n-2-2k}\Omega^{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}}(s,n-2,n-2-2k;w)=0$ □

Thm. 1.4.1. $M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)=\sum\limits_{i=0}^{n} m(s,n;i)\Omega^i(s,n;w)=\sum\limits_{k=0}^{[n/2]} m(s,n;n-2k)\Omega^{n-2k}(s,n;n-2k;w)$

Thm. 1.4.2.

$$\begin{cases} m(s,n;0)=\frac{1}{4}m(s,n-2;0)\\ m(s,n;1)=2m(s,n-1;0)+\frac{1}{4}m(s,n-2;1)\\ m(s,n;i)=2m(s,n-1;i-1)-m(s,n-2;i-2)+\frac{1}{4}m(s,n-2;i), 2\le i\le n-2\\ m(s,n;i)=2m(s,n-1;i-1)-m(s,n-2;i-2), 2<n-1\le i\le n\end{cases}\qquad \begin{cases} m(s,0;0)=1\\ m(s,1;0)=0\\ m(s,1;1)=1-\frac{1}{2s}\end{cases}$$

Proof:

$M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)=\frac{2}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\sigma^{\alpha_{(n-1)\varsigma}}(s;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)$
$+\frac{1}{4}\frac{1}{(n-2)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-2)\varsigma}\}}(s,n-2;w)\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}, n\ge 2$

$\Leftrightarrow$

$\sum\limits_{i=0}^{n} m(s,n;i)\Omega^i(s;w)=2\sum\limits_{i=0}^{n-1} m(s,n-1;i)\Omega^{i+1}(s;w)-\sum\limits_{i=0}^{n-2} m(s,n-2;i)\Omega^{i+2}(s;w)+\frac{1}{4}\sum\limits_{i=0}^{n-2} m(s,n-2;i)\Omega^i(s;w), n\ge 2$

$\Leftrightarrow$

$\sum\limits_{i=0}^{n} m(s,n;i)\Omega^i(s;w)=2\sum\limits_{i=1}^{n} m(s,n-1;i-1)\Omega^{i}(s;w)-\sum\limits_{i=2}^{n} m(s,n-2;i-2)\Omega^{i}(s;w)+\frac{1}{4}\sum\limits_{i=0}^{n-2} m(s,n-2;i)\Omega^i(s;w), n\ge 2$

$\Leftrightarrow$

$$\begin{cases} m(s,n;0)=\frac{1}{4}m(s,n-2;0)\\ m(s,n;1)=2m(s,n-1;0)+\frac{1}{4}m(s,n-2;1)\\ m(s,n;i)=2m(s,n-1;i-1)-m(s,n-2;i-2)+\frac{1}{4}m(s,n-2;i), 2\le i\le n-2\\ m(s,n;i)=2m(s,n-1;i-1)-m(s,n-2;i-2), 2<n-1\le i\le n\end{cases}$$ □

The above expansion coefficient recurrence relation and initial conditions are independent of w. Therefore, the general term expansion coefficients are also independent of w. For all w, they are algebraic isomorphism and have the same expansion coefficients. Just need to figure out the expansion coefficients of any w-algebra, whichever is convenient to use.

1.4.2 Expansion and its recurrence relations of $N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$

Ass. 1.4.1. $N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)=\sum\limits_{i=0}^{n} n(s,n;i)\Omega^i(s,n;w)$

Thm. 1.4.3. $\begin{cases} n(s,n;0)=-m(s,n;0)\\ n(s,n;i)=m(s,n-1;i-1)-m(s,n;i), 1\le i\le n\end{cases}$

Proof: $N^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)=\frac{1}{(n-1)!}M^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(n-1)\varsigma}\}}(s,n-1;w)\sigma^{\alpha_{n\varsigma}\}}(s;w)-M^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s,n;w)$

$\Leftrightarrow$

$\sum\limits_{i=0}^{n} n(s,n;i)\Omega^i(s;w)=\sum\limits_{i=0}^{n-1} m(s,n-1;i)\Omega^{i+1}(s;w)-\sum\limits_{i=0}^{n} m(s,n;i)\Omega^i(s;w)$

$\Leftrightarrow$

$\sum\limits_{i=0}^{n} n(s,n;i)\Omega^i(s;w)=\sum\limits_{i=1}^{n} m(s,n-1;i-1)\Omega^{i}(s;w)-\sum\limits_{i=0}^{n} m(s,n;i)\Omega^i(s;w)$

$\Leftrightarrow$

$$\begin{cases} n(s,n;0)=-m(s,n;0)\\ n(s,n;i)=m(s,n-1;i-1)-m(s,n;i), 1\le i\le n\end{cases}$$ □

Because $m(s,n;i)$ is independent of w, they are isomorphic for all w-algebras. And they have the same expansion coefficients. Just need to figure out the expansion coefficients of any w-algebra, whichever is convenient to use.

1.4.3 Expansion and its recurrence relations of $\Gamma^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s;w)$

Ass. 1.4.2. $\Gamma^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s;w)=\sum\limits_{i=0}^{n} c(s,n;i)\Omega^i(s,n;w)$

Cor. 1.4.1. $\Gamma^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s;w)=N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\Gamma^{\alpha_{2\varsigma}\cdots\alpha_{n\varsigma}\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$

Thm. 1.4.4. $c(s,n;i)=\sum\limits_{j=i-1}^{n-1} c(s-\frac{1}{2},n-1;j)n(s,j+1;i)$

Proof: $\Gamma^{\{\alpha_{1\varsigma}\cdots\alpha_{n\varsigma}\}}(s;w) = N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\Gamma^{\alpha_{2\varsigma}\cdots\alpha_{n\varsigma}\}}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$

$\Leftrightarrow \sum_{i=0}^{n} c(s,n;i)\Omega^i(s;w) = N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\sum_{i=0}^{n-1} c(s-\frac{1}{2},n-1;i)\Omega^i\}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$

$\Leftrightarrow \sum_{i=0}^{n} c(s,n;i)\Omega^i(s;w) = \sum_{i=0}^{n-1} c(s-\frac{1}{2},n-1;i)N^{A_\varsigma}(s;w)\sigma^{\{\alpha_{1\varsigma}}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2};w)\Omega^i\}(s-\frac{1}{2};w)\bar{N}_{B_\varsigma}(s;w)$

$\Leftrightarrow \sum_{i=0}^{n} c(s,n;i)\Omega^i(s;w) = \sum_{i=0}^{n-1} c(s-\frac{1}{2},n-1;i)\frac{1}{(i+1)!}N^{\{\{\alpha_{1\varsigma}\cdots\alpha_{(i+1)\varsigma}\}}(s,i+1)\delta^{\alpha_{(i+2)\varsigma}\alpha_{(i+3)\varsigma}}\cdots\delta^{\alpha_{(n-1)\varsigma}\alpha_{n\varsigma}\}}$

$\Leftrightarrow \sum_{i=0}^{n} c(s,n;i)\Omega^i(s;w) = \sum_{i=0}^{n-1} c(s-\frac{1}{2},n-1;i)\sum_{j=0}^{i+1} n(s,i+1;j)\Omega^j(s;w)$

$\Leftrightarrow \sum_{i=0}^{n} c(s,n;i)\Omega^i(s;w) = \sum_{j=0}^{n-1}\sum_{i=0}^{j+1} c(s-\frac{1}{2},n-1;j)n(s,j+1;i)\Omega^i(s;w)$

$\Leftrightarrow c(s,n;i) = \sum_{j=i-1}^{n-1} c(s-\frac{1}{2},n-1;j)n(s,j+1;i)$ □

Because $n(s,n;i)$ is independent of w, they are isomorphic for all w-algebras. And they have the same expansion coefficients. Then $c(s,n;i)$ is also independent of w and they are isomorphic for all w-algebras. They have the same expansion coefficients. Just need to figure out the expansion coefficients of any w-algebra, whichever is convenient to use.

1.5 Apply iterative method to calculate expansion coefficients of first few items

Cor. 1.5.1.

$$\begin{cases} m(s,n;0) = \frac{1}{4}m(s,n-2;0) \\ m(s,n;1) = 2m(s,n-1;0) + \frac{1}{4}m(s,n-2;1) \\ m(s,n;i) = 2m(s,n-1;i-1) - m(s,n-2;i-2) + \frac{1}{4}m(s,n-2;i), 2 \le i \le n-2 \\ m(s,n;i) = 2m(s,n-1;i-1) - m(s,n-2;i-2), 2 < n-1 \le i \le n \\ m(s,0;0) = 1, m(s,1;0) = 0, m(s,1;1) = 1 - \frac{1}{2s} \end{cases}$$

Cor. 1.5.2.

$$\begin{cases} m(s,0;0) = 1 \\ m(s,1;0) = 0, m(s,1;1) = 1 - \frac{1}{2s} \\ m(s,2;0) = \frac{1}{4}, m(s,2;1) = 0, m(s,2;2) = 1 - \frac{2}{2s} \\ m(s,3;0) = 0, m(s,3;1) = \frac{3}{4} - \frac{1}{8s}, m(s,3;2) = 0, m(s,3;3) = 1 - \frac{3}{2s} \\ m(s,4;0) = \frac{1}{16}, m(s,4;1) = 0, m(s,4;2) = \frac{3}{2} - \frac{1}{2s}, m(s,4;3) = 0, m(s,4;4) = 1 - \frac{4}{2s} \end{cases}$$

Cor. 1.5.3.

$$\begin{cases} n(s,0;0) = 1 \\ n(s,1;0) = 0, n(s,1;1) = \frac{1}{2s} \\ n(s,2;0) = -\frac{1}{4}, n(s,2;1) = 0, n(s,2;2) = \frac{1}{2s} \\ n(s,3;0) = 0, n(s,3;1) = -\frac{1}{2} + \frac{1}{8s}, n(s,3;2) = 0, n(s,3;3) = \frac{1}{2s} \\ n(s,4;0) = -\frac{1}{16}, n(s,4;1) = 0, n(s,4;2) = -\frac{3}{4} + \frac{3}{8s}, n(s,4;3) = 0, n(s,4;4) = \frac{1}{2s} \end{cases}$$

Cor. 1.5.4. $c(s,n;i) = \sum_{j=i-1}^{n-1} c(s-\frac{1}{2},n-1;j)n(s,j+1;i); c(s,0;0) = 1, c(s,1;0) = 0, c(s,1;1) = \frac{1}{2s}$

$$\Rightarrow \begin{cases} c(s,0;0) = 1 \\ c(s,1;0) = 0, c(s,1;1) = \frac{(2s-1)!}{(2s)!} \\ c(s,2;0) = \frac{(2s-2)!}{(2s)!}\frac{-s}{2}, c(s,2;1) = 0, c(s,2;2) = \frac{(2s-2)!}{(2s)!} \\ c(s,3;0) = 0, c(s,3;1) = \frac{(2s-3)!}{(2s)!}\frac{1-3s}{2}, c(s,3;2) = 0, c(s,3;3) = \frac{(2s-3)!}{(2s)!} \\ c(s,4;0) = \frac{(2s-4)!}{(2s)!}\frac{3s(s-1)}{4}, c(s,4;1) = 0, c(s,4;2) = \frac{(2s-4)!}{(2s)!}(2-3s), c(s,4;3) = 0, c(s,4;4) = \frac{(2s-4)!}{(2s)!} \end{cases}$$

The above calculation results can be mutually verified with the direct calculation methods in the previous sections, and the results are identical, indicating that this analytical method is correct and effective.

1.6 Solving general expansion coefficients by iterative method

Cor. 1.6.1.

$$\begin{cases} m(s,2k;0) = \frac{1}{2^{2k}}, m(s,2k+1;0) = 0 \\ m(s,2k;1) = 0, m(s,2k+1;1) = \frac{2}{2^{2k+1}}(2k+1-\frac{1}{2s}) \\ m(s,n;i) = 2m(s,n-1;i-1) - m(s,n-2;i-2) + \frac{1}{4}m(s,n-2;i), 2 \le i \le n-2 \\ m(s,n;n-1) = 0, m(s,n;n) = 1 - \frac{n}{2s} \end{cases}$$

Cor. 1.6.2.
$$\begin{cases} m(s,n;2)-\frac{1}{4}m(s,n-2;2)=2m(s,n-1;1)-m(s,n-2;0), 2\le 2\le n-2 \\ m(s,n;3)-\frac{1}{4}m(s,n-2;3)=2m(s,n-1;2)-m(s,n-2;1), 2\le 3\le n-2 \\ m(s,n;4)-\frac{1}{4}m(s,n-2;4)=2m(s,n-1;3)-m(s,n-2;2), 2\le 4\le n-2 \end{cases}$$

Cor. 1.6.3. $m(s,2k;2)-\frac{1}{4}m(s,2k-2;2)=\frac{k}{4^{k-2}}-\frac{1}{4^{k-1}}(3+\frac{1}{s})$

Proof: $m(s,2k;2)-\frac{1}{4}m(s,2k-2;2)$
$=2m(s,2k-1;1)-m(s,2k-2;0), 2\le 2\le 2k-2$
$=2\frac{2}{2^{2k-1}}(2k-1-\frac{1}{2s})-\frac{1}{2^{2k-2}}$
$=\frac{1}{2^{2k-2}}(4k-3-\frac{1}{s})$
$=\frac{k}{4^{k-2}}-\frac{1}{4^{k-1}}(3+\frac{1}{s})$ □

Cor. 1.6.4. $m(s,2k;2)=\frac{1}{4^{k-1}}[2k^2-(1+\frac{1}{s})k], k\ge 1$

Proof:
$$\begin{cases} m(s,2k;2)-\frac{1}{4}m(s,2(k-1);2)=\frac{k}{4^{k-2}}-\frac{1}{4^{k-1}}(3+\frac{1}{s}) \\ \frac{1}{4}m(s,2(k-1);2)-(\frac{1}{4})^2m(s,2(k-2);2)=\frac{k-1}{4^{k-2}}-\frac{1}{4^{k-1}}(3+\frac{1}{s}) \\ .. \\ (\frac{1}{4})^{k-2}m(s,4;2)-(\frac{1}{4})^{k-1}m(s,2;2)=\frac{2}{4^{k-2}}-\frac{1}{4^{k-1}}(3+\frac{1}{s}) \end{cases}$$
$\Rightarrow m(s,2k;2)-(\frac{1}{4})^{k-1}m(s,2;2)=\frac{k(k+1)-2}{2\cdot 4^{k-2}}-\frac{k-1}{4^{k-1}}(3+\frac{1}{s})$
$\Rightarrow m(s,2k;2)=\frac{k(k+1)-2}{2\cdot 4^{k-2}}-\frac{k-1}{4^{k-1}}(3+\frac{1}{s})+\frac{1}{4^{k-1}}(1-\frac{1}{s})=\frac{1}{4^{k-1}}[2k^2-(1+\frac{1}{s})k]$ □

Cor. 1.6.5. $m(s,2k+1;3)=\frac{1}{4^{k-1}}[\frac{4}{3}k^3-\frac{2}{2s}k^2-(\frac{1}{3}+\frac{1}{2s})k^1], k\ge 1$

Proof: $m(s,2k+1;3)-\frac{1}{4}m(s,2k-1;3)$
$=2m(s,2k;2)-m(s,2k-1;1), k\ge 2$
$=2\frac{1}{4^{k-1}}[2k^2-(1+\frac{2}{2s})k]-\frac{1}{4^{k-1}}[2k^1-(1+\frac{1}{2s})]$
$\Rightarrow m(s,2k+1;3)-\frac{1}{4^{k-1}}m(s,3;3)=\sum\limits_{i=2}^{k}2\frac{1}{4^{i-1}}[2i^2-(1+\frac{2}{2s})i]-\frac{1}{4^{i-1}}[2i^1-(1+\frac{1}{2s})]$
$\Rightarrow m(s,2k+1;3)=\frac{1}{4^{k-1}}[\frac{4}{3}k^3-\frac{2}{2s}k^2-(\frac{1}{3}+\frac{1}{2s})k^1]$ □

Cor. 1.6.6.
$m(s,n;n)=1-\frac{n}{2s}, n\ge 0$
$$\begin{cases} m(s,2k-2;0)=\frac{1}{4^{k-1}}, k\ge 1 \\ \sum\limits_{i=2}^{k}4^{i-1}m(s,2i-2;0)=k-1 \end{cases}$$
$$\begin{cases} m(s,2k-1;1)=\frac{1}{4^{k-1}}[2k^1-(1+\frac{1}{2s})], k\ge 1 \\ \sum\limits_{i=2}^{k}4^{i-1}m(s,2i-1;1)=k^2-\frac{1}{2s}k-(1-\frac{1}{2s}) \end{cases}$$
$$\begin{cases} m(s,2k-0;2)=\frac{1}{4^{k-1}}[2k^2-(1+\frac{2}{2s})k], k\ge 1 \\ \sum\limits_{i=2}^{k}4^{i-1}m(s,2i-0;2)=\frac{2}{3}k^3+\frac{1}{2}(1-\frac{2}{2s})k^2-(\frac{1}{6}+\frac{1}{2s})k-(1-\frac{2}{2s}) \end{cases}$$
$$\begin{cases} m(s,2k+1;3)=\frac{1}{4^{k-1}}[\frac{4}{3}k^3-\frac{2}{2s}k^2-\frac{1}{3}(1+\frac{3}{2s})k^1], k\ge 1 \end{cases}$$

Cor. 1.6.7.
$$\begin{cases} n(s,2k;0)=-\frac{1}{2^{2k}}, n(s,2k+1;0)=0 \\ n(s,2k;1)=0, n(s,2k+1;1)=-\frac{2}{2^{2k+1}}(2k+1-1-\frac{1}{2s}) \\ n(s,n;i)=n(s,n-1;i-1)-\frac{1}{4}m(s,n-2;i), 2\le i\le n-2 \\ n(s,n;n-1)=0, n(s,n;n)=\frac{1}{2s} \end{cases}$$

According to the above method, we can recursively figure out all $m(s,n;i)$ and $n(s,n;i)$. Where there must appear the Bernoulli number B_k

Pro. 1.6.1.
$$\begin{cases} \sum\limits_{i=0}^{n}i^p=\frac{1}{p+1}\sum\limits_{k=0}^{p}(-1)^kC_{p+1}^kB_kn^{p+1-k}, B_k=\delta_{k0}-\frac{1}{k+1}\sum\limits_{j=0}^{k-1}C_{k+1}^jB_j, \frac{z}{e^z-1}=\sum\limits_{k=0}^{\infty}B_k\frac{z^k}{k!} \\ B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, B_8=-\frac{1}{30}, B_{2k+1}=0(k\ge 1) \end{cases}$$

1.7 Linear algebraic method for solving expansion coefficients

The linear algebraic solution in this section embodies the holographic principle of mathematics. By solving only one projection direction, the solution of the entire space can be obtained. That is, one projection direction contains the information of the entire space, reflecting the holographic principle.

1.7.1 Linear algebraic method for solving expansion coefficients of $M^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w)$

Because $m(s,n;i)$ is independent of w, they are isomorphic for all w-algebras. And they have the same expansion coefficients. Just need to figure out the expansion coefficients of any w-algebra, whichever is convenient to use. Here take $w=1$

Thm. 1.7.1. $$2s\begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdot\cdot & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdot\cdot & (-s)^{n-2[n/2]} \end{bmatrix}\begin{bmatrix} m(s,n;n) \\ m(s,n;n-2) \\ m(s,n;n-4) \\ \cdot\cdot \\ m(s,n;n-2[n/2]) \end{bmatrix}=\begin{bmatrix} 2s(s-1/2)^n \\ (2s-1)(s-3/2)^n+1(s-1/2)^n \\ (2s-2)(s-5/2)^n+2(s-3/2)^n \\ \cdot\cdot \\ 2s(1/2-s)^n \end{bmatrix}$$

Proof: $\frac{1}{n!}M^{\{z_{1\varsigma}\cdot\cdot z_{n\varsigma}\}}(s,n)=N^{A_\varsigma}(s)\sigma^n_{z_\varsigma}(s-\frac{1}{2})\bar{N}_{A_\varsigma}(s)=\sum\limits_{k=0}^{[n/2]}m(s,n;n-2k)\sigma^{n-2k}_{z_\varsigma}(s)$

$$\Leftrightarrow N^{A_\varsigma}(s)\begin{bmatrix} (s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^n \end{bmatrix}\bar{N}_{A_\varsigma}(s)=\sum_{k=0}^{[n/2]}m(s,n;n-2k)\begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix}$$

$$\Leftrightarrow \frac{1}{2s}\begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^n \end{bmatrix}\begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} & 0 \end{bmatrix}$$

$$+\frac{1}{2s}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix}\begin{bmatrix} (s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^n \end{bmatrix}\begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix}$$

$$=\sum_{k=0}^{[n/2]}m(s,n;n-2k)\begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix}$$

$$\Leftrightarrow \frac{1}{2s}\begin{bmatrix} 2s(s-1/2)^n & 0 & 0 & 0 & 0 & 0 \\ 0 & (2s-1)(s-3/2)^n & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(3/2-s)^n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1(1/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}+\frac{1}{2s}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1(s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & 0 & 2(s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & (2s-1)(3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & 0 & 2s(1/2-s)^n \end{bmatrix}$$

$$=\sum_{k=0}^{[n/2]}m(s,n;n-2k)\begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix}$$

$$\Leftrightarrow\begin{cases} 2s\sum\limits_{k=0}^{[n/2]}m(s,n;n-2k)s^{n-2k}=2s(s-1/2)^n \\ 2s\sum\limits_{k=0}^{[n/2]}m(s,n;n-2k)(s-1)^{n-2k}=(2s-1)(s-3/2)^n+1(s-1/2)^n \\ \cdots\cdots \\ 2s\sum\limits_{k=0}^{[n/2]}m(s,n;n-2k)(1-s)^{n-2k}=1(1/2-s)^n+(2s-1)(3/2-s)^n \\ 2s\sum\limits_{k=0}^{[n/2]}m(s,n;n-2k)(-s)^{n-2k}=2s(1/2-s)^n \end{cases}$$

$$\Leftrightarrow 2s\begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdot\cdot & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdot\cdot & (-s)^{n-2[n/2]} \end{bmatrix}\begin{bmatrix} m(s,n;n) \\ m(s,n;n-2) \\ m(s,n;n-4) \\ \cdot\cdot \\ m(s,n;n-2[n/2]) \end{bmatrix}=\begin{bmatrix} 2s(s-1/2)^n \\ (2s-1)(s-3/2)^n+1(s-1/2)^n \\ (2s-2)(s-5/2)^n+2(s-3/2)^n \\ \cdot\cdot \\ 2s(1/2-s)^n \end{bmatrix}\qquad\square$$

Cor. 1.7.1. $$\begin{bmatrix} m(s,n;n) \\ m(s,n;n-2) \\ m(s,n;n-4) \\ \cdot\cdot \\ m(s,n;n-2[n/2]) \end{bmatrix}$$

$$=\frac{1}{2s}\begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ (s-2)^n & (s-2)^{n-2} & (s-2)^{n-4} & \cdot\cdot & (s-2)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (s-[n/2])^n & (s-[n/2])^{n-2} & (s-[n/2])^{n-4} & \cdot\cdot & (s-[n/2])^{n-2[n/2]} \end{bmatrix}^{-1}\begin{bmatrix} 2s(s-1/2)^n \\ (2s-1)(s-3/2)^n+1(s-1/2)^n \\ (2s-2)(s-5/2)^n+2(s-3/2)^n \\ \cdot\cdot \\ (2s-[n/2])(s-1/2-[n/2])^n+[n/2](s+1/2-[n/2])^n \end{bmatrix}$$

1.7.2 Linear algebraic method for solving $N^{\{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}\}}(s,n;w)$ expansion coefficients

Because $n(s,n;i)$ is independent of w, they are isomorphic for all w-algebras. And they have the same expansion coefficients. Just need to figure out the expansion coefficients of any w-algebra, whichever is convenient to use. Here take $w=1$

Thm. 1.7.2. $$4s\begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdot\cdot & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdot\cdot & (-s)^{n-2[n/2]} \end{bmatrix}\begin{bmatrix} n(s,n;n) \\ n(s,n;n-2) \\ n(s,n;n-4) \\ \cdot\cdot \\ n(s,n;n-2[n/2]) \end{bmatrix}=\begin{bmatrix} 2s(s-1/2)^{n-1} \\ (2s-1)(s-3/2)^{n-1}-1(s-1/2)^{n-1} \\ (2s-2)(s-5/2)^{n-1}-2(s-3/2)^{n-1} \\ \cdot\cdot \\ -2s(1/2-s)^{n-1} \end{bmatrix}$$

Proof: $\frac{1}{n!}N^{\{z_{1\varsigma}\cdot\cdot z_{n\varsigma}\}}(s,n)=N^{A_\varsigma}(s)\sigma^{z_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\sigma^{n-1}_{z_\varsigma}(s-\frac{1}{2})\bar{N}_{A_\varsigma}(s)=\sum_{k=0}^{[n/2]}n(s,n;n-2k)\sigma^{n-2k}_{z_\varsigma}(s),n\geq 1$

$$\Leftrightarrow N^{A_\varsigma}(s)\sigma^{z_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}\begin{bmatrix} (s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^{n-1} \end{bmatrix}\bar{N}_{A_\varsigma}(s)$$

$$=\sum_{k=0}^{[n/2]}n(s,n;n-2k)\begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix}$$

$$\Leftrightarrow \frac{1}{4s}\begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} (s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^{n-1} \end{bmatrix}\begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} & 0 \end{bmatrix}$$

$$-\frac{1}{4s}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix}\begin{bmatrix} (s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^{n-1} \end{bmatrix}\begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix}$$

$$=\sum_{k=0}^{[n/2]}n(s,n;n-2k)\begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix}$$

$$\Leftrightarrow \frac{1}{4s}\begin{bmatrix} 2s(s-1/2)^{n-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (2s-1)(s-3/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(3/2-s)^{n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1(1/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{1}{4s}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1(s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2(s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & (2s-1)(3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2s(1/2-s)^{n-1} \end{bmatrix}$$

$$=\sum_{k=0}^{[n/2]}n(s,n;n-2k)\begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix}$$

$$\Leftrightarrow\begin{cases} 2s\sum_{k=0}^{[n/2]}n(s,n;n-2k)s^{n-2k}=2s(s-1/2)^{n-1} \\ 2s\sum_{k=0}^{[n/2]}n(s,n;n-2k)(s-1)^{n-2k}=(2s-1)(s-3/2)^{n-1}-1(s-1/2)^{n-1} \\ \cdots\cdots \\ 2s\sum_{k=0}^{[n/2]}n(s,n;n-2k)(1-s)^{n-2k}=1(1/2-s)^{n-1}-(2s-1)(3/2-s)^{n-1} \\ 2s\sum_{k=0}^{[n/2]}n(s,n;n-2k)(-s)^{n-2k}=-2s(1/2-s)^{n-1} \end{cases}$$

$$\Leftrightarrow 4s\begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdot\cdot & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdot\cdot & (-s)^{n-2[n/2]} \end{bmatrix}\begin{bmatrix} n(s,n;n) \\ n(s,n;n-2) \\ n(s,n;n-4) \\ \cdot\cdot \\ n(s,n;n-2[n/2]) \end{bmatrix}=\begin{bmatrix} 2s(s-1/2)^{n-1} \\ (2s-1)(s-3/2)^{n-1}-1(s-1/2)^{n-1} \\ (2s-2)(s-5/2)^{n-1}-2(s-3/2)^{n-1} \\ \cdot\cdot \\ -2s(1/2-s)^{n-1} \end{bmatrix}$$ □

Cor. 1.7.2. $$\begin{bmatrix} n(s,n;n) \\ n(s,n;n-2) \\ n(s,n;n-4) \\ \cdot\cdot \\ n(s,n;n-2[n/2]) \end{bmatrix}$$

$$= \frac{1}{4s} \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ (s-2)^n & (s-2)^{n-2} & (s-2)^{n-4} & \cdot\cdot & (s-2)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (s-[n/2])^n & (s-[n/2])^{n-2} & (s-[n/2])^{n-4} & \cdot\cdot & (s-[n/2])^{n-2[n/2]} \end{bmatrix}^{-1} \begin{bmatrix} 2s(s-1/2)^{n-1} \\ (2s-1)(s-3/2)^{n-1}-1(s-1/2)^{n-1} \\ (2s-2)(s-5/2)^{n-1}-2(s-3/2)^{n-1} \\ \cdot\cdot \\ (2s-[n/2])(s-1/2-[n/2])^n-[n/2](s+1/2-[n/2])^n \end{bmatrix}$$

1.7.3 Properties of $\Gamma^{z_{1\varsigma}\cdot\cdot z_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n)$

Cor. 1.7.3. $\Gamma^{z_{1\varsigma}\cdot\cdot z_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n) := \frac{1}{2^n}\Gamma_{k_\varsigma}^{A_{1\varsigma}\cdot\cdot A_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}(s)\prod_{i=1}^{n}\sigma^z{}_{A_{i\varsigma}}{}^{B_{i\varsigma}}\Gamma^{l_\varsigma}_{B_{1\varsigma}\cdot\cdot B_{n\varsigma}A_{(n+1)\varsigma}\cdot\cdot A_{(2s)\varsigma}}(s)$

Lem. 1.7.1.

$$\Gamma^{z_{1\varsigma}\cdot\cdot z_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n) = \frac{1}{2^n} \begin{bmatrix} C_{2s}^{-0}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{0-i} & 0 & \cdot\cdot & 0 & 0 \\ 0 & C_{2s}^{-1}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{1-i} & \cdot\cdot & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & \cdot\cdot\ C_{2s}^{1-2s}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{2s-1-i} & & 0 \\ 0 & 0 & \cdot\cdot & 0 & C_{2s}^{-2s}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{2s-i} \end{bmatrix}$$

1.7.4 Linear algebraic method for solving $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)$ expansion coefficients

Ass. 1.7.1. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w) = \frac{1}{n!}\sum_{k}^{[n/2]}c(s,n;n-2k)\Omega^{n-2k}(s;w), c(s,n;n-2k-1) = 0$

Because $c(s,n;i)$ is independent of w, they are isomorphic for all w-algebras. And they have the same expansion coefficients. Just need to figure out the expansion coefficients of any w-algebra, whichever is convenient to use. Here take $w=1$. How to push it out still needs to be strictly and carefully written out, and I can't remember it for a long time.

Cor. 1.7.4. $\Gamma^{z_{1\varsigma}\cdot\cdot z_{n\varsigma}}(s,n;w=1) = \sum_{k}^{[n/2]}c(s,n;n-2k)\sigma_z^{n-2k}(s;w=1) = \frac{1}{2^n}$

$$\begin{bmatrix} C_{2s}^{-0}(-1)^0C_n^0C_{2s-n}^0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & C_{2s}^{-1}[(-1)^0C_n^0C_{2s-n}^1+(-1)^1C_n^1C_{2s-n}^0] & \cdot\cdot & 0 & 0 \\ 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & \cdot\cdot & C_{2s}^{-(2s-1)}[(-1)^{n-1}C_n^{n-1}C_{2s-n}^{2s-n}+(-1)^nC_n^nC_{2s-n}^{2s-n-1}] & 0 \\ 0 & 0 & \cdot\cdot & 0 & C_{2s}^{-2s}(-1)^nC_n^nC_{2s-n}^{2s-n} \end{bmatrix}$$

$$\Rightarrow \frac{1}{2^n}C_{2s}^{-(s-h)}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{s-h-i} = \sum_{k}^{[n/2]}c(s,n;n-2k)h^{n-2k}, h = s, s-1, \cdot\cdot, -(s-1), -s$$

Cor. 1.7.5.

$$2^n \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdot\cdot & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdot\cdot & (-s)^{n-2[n/2]} \end{bmatrix} \begin{bmatrix} c(s,n;n) \\ c(s,n;n-2) \\ c(s,n;n-4) \\ \cdot\cdot \\ c(s,n;n-2[n/2]) \end{bmatrix} = \begin{bmatrix} C_{2s}^{-0}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{0-i} \\ C_{2s}^{-1}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{1-i} \\ \cdot\cdot \\ C_{2s}^{-(2s-1)}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{2s-1-i} \\ C_{2s}^{-2s}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{2s-i} \end{bmatrix} = \begin{bmatrix} 1 \\ 1-\frac{n}{s} \\ 1-\frac{n(2s-n)}{s(s-1/2)} \\ 1-\frac{n(6s^2-6ns-3s+2n^2+1)}{2s(s-1/2)(s-1)} \\ \cdot\cdot \end{bmatrix}$$

Cor. 1.7.6.

$$\begin{bmatrix} c(s,n;n) \\ c(s,n;n-2) \\ c(s,n;n-4) \\ \cdot\cdot \\ c(s,n;n-2[n/2]) \end{bmatrix} = \frac{1}{2^n} \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdot\cdot & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdot\cdot & (s-1)^{n-2[n/2]} \\ (s-2)^n & (s-2)^{n-2} & (s-2)^{n-4} & \cdot\cdot & (s-2)^{n-2[n/2]} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (s-[n/2])^n & (s-[n/2])^{n-2} & (s-[n/2])^{n-4} & \cdot\cdot & (s-[n/2])^{n-2[n/2]} \end{bmatrix}^{-1} \begin{bmatrix} C_{2s}^{-0}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{0-i} \\ C_{2s}^{-1}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{1-i} \\ C_{2s}^{-2}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{2-i} \\ \cdot\cdot \\ C_{2s}^{-[n/2]}\sum_{i=0}^{n}(-1)^iC_n^iC_{2s-n}^{[n/2]-i} \end{bmatrix}$$

1.7.5 Verification of linear algebra solution for first several items of $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}{}_{k_\varsigma}{}^{l_\varsigma}(s,n;w)$

Cor. 1.7.7.

$$\begin{bmatrix} s^0 \\ (s-1)^0 \\ \cdot\cdot \\ (1-s)^0 \\ (-s)^0 \end{bmatrix} \begin{bmatrix} c(s,0;0) \end{bmatrix} = \begin{bmatrix} C_{2s}^{-0}(-1)^0C_0^0C_{2s-0}^{0-0} \\ C_{2s}^{-1}(-1)^0C_0^0C_{2s-0}^{1-0} \\ \cdot\cdot \\ C_{2s}^{-(2s-1)}(-1)^0C_0^0C_{2s-0}^{2s-1-0} \\ C_{2s}^{-2s}(-1)^0C_0^0C_{2s-n}^{2s-0} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \cdot\cdot \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow c(s,0;0) = 1$$

$s^0c(s,0;0) = 1 \Leftrightarrow c(s,0;0) = 1, s \geq \frac{1}{2}$

Cor. 1.7.8.

$$\begin{bmatrix} s^1 \\ (s-1)^1 \\ \cdot\cdot \\ (1-s)^1 \\ (-s)^1 \end{bmatrix} \begin{bmatrix} c(s,1;1) \end{bmatrix} = \frac{1}{2^1} \begin{bmatrix} C_{2s}^{-0}[(-1)^0C_1^0C_{2s-1}^{0-0}+(-1)^1C_1^1C_{2s-1}^{0-1}] \\ C_{2s}^{-1}[(-1)^0C_1^0C_{2s-1}^{1-0}+(-1)^1C_1^1C_{2s-1}^{1-1}] \\ \cdot\cdot \\ C_{2s}^{-(2s-1)}[(-1)^0C_1^0C_{2s-1}^{2s-1-0}+(-1)^1C_1^1C_{2s-1}^{2s-1-1}] \\ C_{2s}^{-2s}[(-1)^0C_1^0C_{2s-1}^{2s-0}+(-1)^1C_1^1C_{2s-1}^{2s-1}] \end{bmatrix} \Leftrightarrow c(s,1;1) = \frac{1}{2s}$$

$s^1c(s,1;1) = \frac{1}{2} \Leftrightarrow c(s,1;1) = \frac{1}{2s}, s \geq \frac{1}{2}$

Cor. 1.7.9.

$$\begin{bmatrix} s^2 & s^0 \\ (s-1)^2 & (s-1)^0 \end{bmatrix}\begin{bmatrix} c(s,2;2)\\ c(s,2;0)\end{bmatrix} = \frac{1}{2^2}\begin{bmatrix} C_{2s}^{-0}\sum\limits_{i=0}^{2}(-1)^i C_2^i C_{2s-2}^{0-i} \\ C_{2s}^{-1}\sum\limits_{i=0}^{2}(-1)^iC_2^iC_{2s-2}^{1-i}\end{bmatrix} = \frac{1}{2^2}\begin{bmatrix} C_{2s}^{-0}(-1)^0C_2^0C_{2s-2}^{0-0} \\ C_{2s}^{-1}[(-1)^0C_2^0C_{2s-2}^{1-0}+(-1)^1C_2^1C_{2s-2}^{1-1}]\end{bmatrix} = \frac{1}{2^2}\begin{bmatrix}1\\1-\frac{2}{s}\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c(s,2;2)\\ c(s,2;0)\end{bmatrix} = \frac{1}{2^2}\begin{bmatrix} \frac{1}{s(s-1/2)} \\ 1-\frac{s^2}{s(s-1/2)}\end{bmatrix} = \frac{(2s-2)!}{(2s)!}\begin{bmatrix}1\\ -\frac{1}{2}s\end{bmatrix} = \frac{(2s-2)!}{(2s)!}\begin{bmatrix}1\\ \frac{1}{2}(C_2^3-C_2^2s)\end{bmatrix}, s\geq 1$$

Cor. 1.7.10.

$$\begin{bmatrix} s^3 & s^1 \\ (s-1)^3 & (s-1)^1 \end{bmatrix}\begin{bmatrix} c(s,3;3)\\ c(s,3;1)\end{bmatrix} = \frac{1}{2^3}\begin{bmatrix} C_{2s}^{-0}\sum\limits_{i=0}^{3}(-1)^i C_3^i C_{2s-3}^{0-i} \\ C_{2s}^{-1}\sum\limits_{i=0}^{3}(-1)^iC_3^iC_{2s-3}^{1-i}\end{bmatrix} = \frac{1}{2^3}\begin{bmatrix} C_{2s}^{-0}(-1)^0C_3^0C_{2s-3}^{0-0} \\ C_{2s}^{-1}[(-1)^0C_3^0C_{2s-3}^{1-0}+(-1)^1C_3^1C_{2s-3}^{1-1}]\end{bmatrix} = \frac{1}{2^3}\begin{bmatrix}1\\1-\frac{3}{s}\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c(s,3;3)\\ c(s,3;1)\end{bmatrix} = \frac{1}{2^3}\begin{bmatrix} \frac{1}{s(s-1/2)(s-1)} \\ \frac{1}{s}-\frac{s^2}{s(s-1/2)(s-1)}\end{bmatrix} = \frac{(2s-3)!}{(2s)!}\begin{bmatrix}1\\ \frac{1}{2}(1-3s)\end{bmatrix} = \frac{(2s-3)!}{(2s)!}\begin{bmatrix}1\\ \frac{1}{2}(C_3^3-C_3^2s)\end{bmatrix}, s\geq \frac{3}{2}$$

Cor. 1.7.11.

$$\begin{bmatrix} s^4 & s^2 & s^0 \\ (s-1)^4 & (s-1)^2 & (s-1)^0 \\ (s-2)^4 & (s-2)^2 & (s-2)^0 \end{bmatrix}\begin{bmatrix} c(s,4;4)\\ c(s,4;2)\\ c(s,4;0)\end{bmatrix} = \frac{1}{2^4}\begin{bmatrix} C_{2s}^{-0}\sum\limits_{i=0}^{4}(-1)^i C_4^i C_{2s-4}^{0-i} \\ C_{2s}^{-1}\sum\limits_{i=0}^{4}(-1)^iC_4^iC_{2s-4}^{1-i} \\ C_{2s}^{-2}\sum\limits_{i=0}^{4}(-1)^iC_4^iC_{2s-4}^{2-i}\end{bmatrix} = \frac{1}{2^4}\begin{bmatrix}1\\1-\frac{4}{s}\\ 1-\frac{4(2s-4)}{s(s-1/2)}\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c(s,4;4)\\ c(s,4;2)\\ c(s,4;0)\end{bmatrix} = \frac{1}{2^4}\begin{bmatrix} \frac{1}{s(s-1/2)(s-1)(s-3/2)} \\ \frac{2}{s(s-1/2)}-\frac{s^2+(s-1)^2}{s(s-1/2)(s-1)(s-3/2)} \\ 1-\frac{2s^2}{s(s-1/2)}+\frac{s^2(s-1)^2}{s(s-1/2)(s-1)(s-3/2)}\end{bmatrix} = \frac{(2s-4)!}{(2s)!}\begin{bmatrix}1\\ 2-3s\\ \frac{3}{4}s(s-1)\end{bmatrix} = \frac{(2s-4)!}{(2s)!}\begin{bmatrix}1\\ \frac{1}{2}(C_4^3-C_4^2s)\\ \frac{3}{4}s(s-1)\end{bmatrix}, s\geq 2$$

Cor. 1.7.12.

$$\begin{bmatrix} s^5 & s^3 & s^1 \\ (s-1)^5 & (s-1)^3 & (s-1)^1 \\ (s-2)^5 & (s-2)^3 & (s-2)^1 \end{bmatrix}\begin{bmatrix} c(s,5;5)\\ c(s,5;3)\\ c(s,5;1)\end{bmatrix} = \frac{1}{2^5}\begin{bmatrix} C_{2s}^{-0}\sum\limits_{i=0}^{5}(-1)^i C_5^i C_{2s-5}^{0-i} \\ C_{2s}^{-1}\sum\limits_{i=0}^{5}(-1)^iC_5^iC_{2s-5}^{1-i} \\ C_{2s}^{-2}\sum\limits_{i=0}^{5}(-1)^iC_5^iC_{2s-5}^{2-i}\end{bmatrix} = \frac{1}{2^5}\begin{bmatrix}1\\1-\frac{5}{s}\\ 1-\frac{5(2s-5)}{s(s-1/2)}\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c(s,5;5)\\ c(s,5;3)\\ c(s,5;1)\end{bmatrix} = \frac{1}{2^5}\begin{bmatrix} \frac{1}{s(s-1/2)(s-1)(s-3/2)(s-2)} \\ \frac{2}{s(s-1/2)(s-1)}-\frac{s^2+(s-1)^2}{s(s-1/2)(s-1)(s-3/2)(s-2)} \\ \frac{1}{s}-\frac{2s^2}{s(s-1/2)(s-1)}+\frac{s^2(s-1)^2}{s(s-1/2)(s-1)(s-3/2)(s-2)}\end{bmatrix} = \frac{(2s-5)!}{(2s)!}\begin{bmatrix}1\\ \frac{1}{2}(C_5^3-C_5^2s)\\ \frac{1}{4}(15s^2-25s+6)\end{bmatrix}, s\geq \frac{5}{2}$$

Cor. 1.7.13.

$$\begin{bmatrix} s^6 & s^4 & s^2 & s^0 \\ (s-1)^6 & (s-1)^4 & (s-1)^2 & (s-1)^0 \\ (s-2)^6 & (s-2)^4 & (s-2)^2 & (s-2)^0 \\ (s-3)^6 & (s-3)^4 & (s-2)^3 & (s-3)^0 \end{bmatrix}\begin{bmatrix} c(s,6;6)\\ c(s,6;4)\\ c(s,6;2)\\ c(s,6;0)\end{bmatrix} = \frac{1}{2^6}\begin{bmatrix} C_{2s}^{-0}\sum\limits_{i=0}^{6}(-1)^i C_6^i C_{2s-6}^{0-i} \\ C_{2s}^{-1}\sum\limits_{i=0}^{6}(-1)^iC_6^iC_{2s-6}^{1-i} \\ C_{2s}^{-2}\sum\limits_{i=0}^{6}(-1)^iC_6^iC_{2s-6}^{2-i} \\ C_{2s}^{-3}\sum\limits_{i=0}^{6}(-1)^iC_6^iC_{2s-6}^{3-i}\end{bmatrix} = \frac{1}{2^6}\begin{bmatrix}1\\1-\frac{6}{s}\\ 1-\frac{6(2s-6)}{s(s-1/2)} \\ 1-\frac{3(6s^2-39s+73)}{s(s-1/2)(s-1)}\end{bmatrix}, s\geq 3$$

Cor. 1.7.14.

$$\begin{bmatrix} s^7 & s^5 & s^3 & s^1 \\ (s-1)^7 & (s-1)^5 & (s-1)^3 & (s-1)^1 \\ (s-2)^7 & (s-2)^5 & (s-2)^3 & (s-2)^1 \\ (s-3)^7 & (s-3)^5 & (s-2)^3 & (s-3)^1 \end{bmatrix}\begin{bmatrix} c(s,7;7)\\ c(s,7;5)\\ c(s,7;3)\\ c(s,7;1)\end{bmatrix} = \frac{1}{2^7}\begin{bmatrix} C_{2s}^{-0}\sum\limits_{i=0}^{7}(-1)^i C_7^i C_{2s-7}^{0-i} \\ C_{2s}^{-1}\sum\limits_{i=0}^{7}(-1)^iC_7^iC_{2s-7}^{1-i} \\ C_{2s}^{-2}\sum\limits_{i=0}^{7}(-1)^iC_7^iC_{2s-7}^{2-i} \\ C_{2s}^{-3}\sum\limits_{i=0}^{7}(-1)^iC_7^iC_{2s-7}^{3-i}\end{bmatrix} = \frac{1}{2^7}\begin{bmatrix}1\\1-\frac{7}{s}\\ 1-\frac{7(2s-7)}{s(s-1/2)} \\ 1-\frac{7(6s^2-45s+99)}{2s(s-1/2)(s-1)}\end{bmatrix}, s\geq \frac{7}{2}$$

1.7.6 Expansion of composite constant invariant tensors $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)\hat{p}_{\alpha_{1\varsigma}}\cdot\cdot\hat{p}_{\alpha_{n\varsigma}}$

Cor. 1.7.15. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)\hat{p}_{\alpha_{1\varsigma}}\cdot\cdot\hat{p}_{\alpha_{n\varsigma}} = \sum\limits_{k}^{[n/2]} c(s,n;n-2k)[\sigma(s;w)\cdot\hat{p}]^{n-2k}$

Cor. 1.7.16. $\Gamma^{\alpha_{1\varsigma}\cdot\cdot\alpha_{n\varsigma}}(s,n;w)\hat{\partial}_{\alpha_{1\varsigma}}\cdot\cdot\hat{\partial}_{\alpha_{n\varsigma}} = \sum\limits_{k}^{[n/2]} c(s,n;n-2k)[\sigma(s;w)\cdot\hat{\nabla}]^{n-2k}$

2 Constant invariant tensors $Z_{al_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma;w), Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)$

2.1 Introduction of constant invariant tensors $Z_{al_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma;w), Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)$

Def. 2.1.1. $Z_{al_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma;w) := \frac{i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,-i\varsigma)_a^{A'_\varsigma A_\varsigma}N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s;w), Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w) := \frac{-i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s;w)$

Pro. 2.1.1. $\begin{cases} Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)=\delta^{ab}\varepsilon_{k_\varsigma m_\varsigma}(s;w)\varepsilon^{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)\varepsilon_{A'_\varsigma B'_\varsigma}Z^{B'_\varsigma m_\varsigma}_{bn_\varsigma}(s,\varsigma;w) \\ Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma;w)=\delta_{ab}\varepsilon^{k_\varsigma m_\varsigma}(s;w)\varepsilon_{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)\varepsilon^{A'_\varsigma B'_\varsigma}Z^{bn_\varsigma}_{B'_\varsigma m_\varsigma}(s,\varsigma;w) \end{cases}$

Pro. 2.1.2. $\begin{cases} Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)=(-1)^{2s+1}\delta^{ab}[(-\varsigma)^{2s}\varepsilon_{k_\varsigma m_\varsigma}(s;w)][\varsigma^{2s-1}\varepsilon^{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)][-\varsigma\varepsilon_{A'_\varsigma B'_\varsigma}]Z^{B'_\varsigma m_\varsigma}_{bn_\varsigma}(s,\varsigma;w) \\ Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma;w)=(-1)^{2s+1}\delta_{ab}[(\varsigma)^{2s}\varepsilon^{k_\varsigma m_\varsigma}(s;w)][(-\varsigma)^{2s-1}\varepsilon_{l_\varsigma n_\varsigma}(s-\frac{1}{2};w)][\varsigma\varepsilon^{A'_\varsigma B'_\varsigma}]Z^{bn_\varsigma}_{B'_\varsigma m_\varsigma}(s,\varsigma;w) \end{cases}$

2.2 Introduction of constant matrices $Z^{A'_\varsigma}_a(s,\varsigma;w), Z^a_{A'_\varsigma}(s,\varsigma;w)$

Def. 2.2.1. $\begin{cases} Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma;w)\succ Z^{A'_\varsigma}_a(s,\varsigma;w):=\frac{i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\bar{N}_{A_\varsigma}(s;w) \\ Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)\succ Z^a_{A'_\varsigma}(s,\varsigma;w):=\frac{-i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}\bar{N}^{A_\varsigma}(s;w) \end{cases}$

Def. 2.2.2. $\begin{cases} \bar{Z}^{A'_\varsigma}_a(s,\varsigma;w):=Z^{TA'_\varsigma}_a(s,\varsigma;w)=\frac{i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,-i\varsigma)^{A'_\varsigma A_\varsigma}_a N_{A_\varsigma}(s;w) \\ \bar{Z}^a_{A'_\varsigma}(s,\varsigma;w):=Z^{Ta}_{A'_\varsigma}(s,\varsigma;w)=\frac{-i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma}(s;w) \end{cases}$

2.3 Introduction of constant invariant tensor matrices $Z_a(s,\varsigma;w),\bar{Z}_a(s,\varsigma;w)$

Def. 2.3.1. $\begin{cases} Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma;w)|^{A'_\varsigma}{}_{\otimes l_\varsigma}{}^{k_\varsigma}\succ Z_a(s,\varsigma;w):=\frac{i\varsigma}{\sqrt{2}}(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)_a N(s;w) \\ Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)|_{k_\varsigma A'_\varsigma}{}^{\otimes l_\varsigma}\succ\bar{Z}_a(s,\varsigma;w):=\frac{-i\varsigma}{\sqrt{2}}\bar{N}(s;w)(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},i\varsigma)_a\simeq Z^+_a(s,\varsigma;w) \end{cases}$

2.4 Constant invariant tensor properties of matrices $Z_a(s,\varsigma;w),\bar{Z}_a(s,\varsigma;w)$

Pro. 2.4.1. $Z_a(s,\varsigma;w)=[e^\vartheta]_a{}^b[e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes e^{\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},\varsigma;w)}]Z_b(s,\varsigma;w)e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(s,\varsigma;w)}$

Pro. 2.4.2. $\bar{Z}_a(s,\varsigma;w)=[e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}(s,\varsigma;w)}\bar{Z}_b(s,\varsigma;w)[e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2},-\varsigma;w)}\otimes e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2},\varsigma;w)}]$

2.5 Properties of constant invariant tensors $Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma;w), Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)$

1. Reduce two pairs of indices $A'_\varsigma, l_\varsigma$

Pro. 2.5.1. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma m_\varsigma}_{bl_\varsigma}(s,\varsigma;w)=\frac{1}{2s}[s\delta^a{}_b\delta_{k_\varsigma}{}^{m_\varsigma}+iS^a{}_{bk_\varsigma}{}^{m_\varsigma}(s,\varsigma;w)]$

$[\Leftrightarrow]\bar{Z}^a_{A'_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma}_b(s,\varsigma;w)=\frac{1}{2s}[s\delta^a{}_b+iS^a{}_b(s,\varsigma;w)][\Leftrightarrow]\bar{Z}_a(s,\varsigma;w)Z_b(s,\varsigma;w)=\frac{1}{2s}[s\delta_{ab}+iS_{ab}(s,\varsigma;w)]$

Proof: $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma m_\varsigma}_{bl_\varsigma}(s,\varsigma;w)$
$=\frac{-i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\frac{i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,-i\varsigma)^{A'_\varsigma B_\varsigma}_b N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s;w)$
$=\frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma\langle w\rangle,-i\varsigma)^{A'_\varsigma B_\varsigma}_b N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s;w)$
$=\frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)(\delta^a{}_b\delta_{A_\varsigma}{}^{B_\varsigma}+2iS^a{}_{bA_\varsigma}{}^{B_\varsigma})N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s;w)$
$=\frac{1}{2s}[s\delta^a{}_b\delta_{k_\varsigma}{}^{m_\varsigma}+iS^a{}_{bk_\varsigma}{}^{m_\varsigma}(s,\varsigma;w)]$ □

Cor. 2.5.1. $Z_a{}^{l_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma m_\varsigma}_{al_\varsigma}(s,\varsigma;w)=\frac{1}{2}\delta_{k_\varsigma}{}^{m_\varsigma}$

$[\Leftrightarrow]\bar{Z}_{aA'_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma}_a(s,\varsigma;w)=\frac{1}{2}I_{C^{2s}_{2s+w}}[\Leftrightarrow]\bar{Z}_a(s,\varsigma;w)Z_a(s,\varsigma;w)=\frac{1}{2}I_{C^{2s}_{2s+w}}$

2. Reduce two pairs of indices $A'_\varsigma, k_\varsigma$

Pro. 2.5.2. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma k_\varsigma}_{bm_\varsigma}(s,\varsigma;w)=\frac{1}{2s}[(s+\frac{w}{2})\delta^a{}_b\delta_{m_\varsigma}{}^{l_\varsigma}+iS^a{}_{bm_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2},\varsigma;w)]$

$[\Leftrightarrow]Z^{A'_\varsigma}_b(s,\varsigma;w)\bar{Z}^a_{A'_\varsigma}(s,\varsigma;w)=\frac{1}{2s}[(s+\frac{w}{2})\delta^a{}_b+iS^a{}_b(s-\frac{1}{2},\varsigma;w)]$

Proof: $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma k_\varsigma}_{bm_\varsigma}(s,\varsigma;w)$
$=\frac{-i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\frac{i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,-i\varsigma)^{A'_\varsigma B_\varsigma}_b N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s;w)$
$=\frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma\langle w\rangle,-i\varsigma)^{A'_\varsigma B_\varsigma}_b N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s;w)$
$=\frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)(\delta^a{}_b\delta_{A_\varsigma}{}^{B_\varsigma}+2iS^a{}_{bA_\varsigma}{}^{B_\varsigma})N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s;w)$
$=\frac{1}{2s}[(s+\frac{w}{2})\delta^a{}_b\delta_{m_\varsigma}{}^{l_\varsigma}+iS^a{}_{bm_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2},\varsigma;w)]$ □

Cor. 2.5.2. $Z_a{}^{l_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{A'_\varsigma k_\varsigma}_{am_\varsigma}(s,\varsigma;w)=\frac{1}{2}(1+\frac{w}{2s})\delta_{m_\varsigma}{}^{l_\varsigma}[\Leftrightarrow]Z^{A'_\varsigma}_a(s,\varsigma;w)\bar{Z}_{aA'_\varsigma}(s,\varsigma;w)=\frac{1}{2}(1+\frac{w}{2s})$

3. Reduce two pairs of indices k_ς, l_ς

Pro. 2.5.3. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{B'_\varsigma k_\varsigma}_{bl_\varsigma}(s,\varsigma;w)=\frac{1}{w+1}C^{2s}_{2s+w}(\frac{1}{2}\delta_b{}^a\delta^{B'_\varsigma}{}_{A'_\varsigma}+iS_b{}^{aB'_\varsigma}{}_{A'_\varsigma})$
$[\Leftrightarrow]tr[\bar{Z}^a_{A'_\varsigma}(s,\varsigma;w)Z^{B'_\varsigma}_b(s,\varsigma;w)]=\frac{1}{w+1}C^{2s}_{2s+w}(\frac{1}{2}\delta_b{}^a\delta^{B'_\varsigma}{}_{A'_\varsigma}+iS_b{}^{aB'_\varsigma}{}_{A'_\varsigma})$

Proof: $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{B'_\varsigma k_\varsigma}_{bl_\varsigma}(s,\varsigma;w)$
$=\frac{-i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\frac{i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,-i\varsigma)^{B'_\varsigma B_\varsigma}_b N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s;w)$
$=\frac{1}{2}\frac{1}{w+1}C^{2s}_{2s+w}(\sigma\langle w\rangle,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma\langle w\rangle,-i\varsigma)^{B'_\varsigma A_\varsigma}_b$
$=\frac{1}{w+1}C^{2s}_{2s+w}(\frac{1}{2}\delta_b{}^a\delta^{B'_\varsigma}{}_{A'_\varsigma}+iS_b{}^{aB'_\varsigma}{}_{A'_\varsigma})$ □

Cor. 2.5.3. $Z_a{}^{l_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{B'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma;w)=\frac{1}{2(w+1)}C^{2s}_{2s+w}\delta^{B'_\varsigma}_{A'_\varsigma}[\Leftrightarrow]tr[\bar{Z}^a_{A'_\varsigma}(s,\varsigma;w)Z^{B'_\varsigma}_b(s,\varsigma;w)]=\frac{1}{2(w+1)}C^{2s}_{2s+w}\delta^{B'_\varsigma}_{A'_\varsigma}$

2.6 Conjecture(not general)

Ass. 2.6.1. $\frac{i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\frac{-i\varsigma}{\sqrt{2}}(\sigma\langle w\rangle,i\varsigma)^a_{B_\varsigma B'_\varsigma}=\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}$

4. Reduce two pairs of indices a,l_ς

Pro. 2.6.1. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma;w)Z^{B'_\varsigma m_\varsigma}_{al_\varsigma}(s,\varsigma;w)=\delta^{B'_\varsigma}_{A'_\varsigma}\delta^{m_\varsigma}_{k_\varsigma}[\Leftrightarrow]\bar{Z}^a_{A'_\varsigma}(s,\varsigma;w)Z^{B'_\varsigma}_a(s,\varsigma;w)=\delta^{B'_\varsigma}_{A'_\varsigma}I_{C^{2s}_{2s+w}}$

5. Reduce two pairs of indices a,k_ς

Pro. 2.6.2. $Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma;w)Z^{am_\varsigma}_{B'_\varsigma k_\varsigma}(s,\varsigma;w)=(1+\frac{w}{2s})\delta^{A'_\varsigma}_{B'_\varsigma}\delta^{m_\varsigma}_{l_\varsigma}$
$[\Leftrightarrow]Z^{A'_\varsigma}_a(s,\varsigma;w)\bar{Z}^a_{B'_\varsigma}(s,\varsigma;w)=(1+\frac{w}{2s})\delta^{A'_\varsigma}_{B'_\varsigma}I_{C^{2s-1}_{2s-1+w}}[\Leftrightarrow]Z_a(s,\varsigma;w)\bar{Z}^a(s,\varsigma;w)=(1+\frac{w}{2s})I_{(w+1)C^{2s-1}_{2s-1+w}}$

2.7 Properties (not general) of constant invariant tensor matrices $Z_a(s,\varsigma;w),\bar{Z}_a(s,\varsigma;w)$

Pro. 2.7.1. $\begin{cases}\bar{Z}_a(s,\varsigma;w)Z_b(s,\varsigma;w)=\frac{1}{2s}[s\delta_{ab}+iS_{ab}(s,\varsigma;w)]\\ Z^a(s,\varsigma;w)\bar{Z}_a(s,\varsigma;w)=(1+\frac{w}{2s})I_{(w+1)C^{2s-1}_{2s-1+w}}\end{cases}$

Pro. 2.7.2. $\begin{cases}(s+w)Z_b(s,\varsigma;w)=Z^a(s,\varsigma;w)iS_{ab}(s,\varsigma;w),(s+w)\bar{Z}_a(s,\varsigma;w)=iS_{ab}(s,\varsigma;w)\bar{Z}^b(s,\varsigma;w)\\ Z^a(s,\varsigma;w)iS_{ab}(s,\varsigma;w)\bar{Z}^b(s,\varsigma;w)=(s+w)(1+\frac{w}{2s}),Z_a(s,\varsigma;w)\bar{Z}_a(s,\varsigma;w)\neq kI_{(w+1)C^{2s-1}_{2s-1+w}}\end{cases}$

Pro. 2.7.3. $\begin{cases}-S_{ac}(s,\varsigma;w)S^c{}_b(s,\varsigma;w)=s(s+w)\delta_{ab}+iwS_{ab}(s,\varsigma;w)\\ \bar{Z}_a(s,\varsigma;w)Z_b(s,\varsigma;w)=-\frac{1}{2sw}[s^2\delta_{ab}+S_{ac}(s,\varsigma;w)S^c{}_b(s,\varsigma;w)]\end{cases}$

Pro. 2.7.4. $[\sigma(s;w),i\varsigma(s+w)]^a\bar{Z}_a(s,\varsigma;w)=0,Z_a(s,\varsigma;w)[\sigma(s;w),-i\varsigma(s+w)]^a=0$

Proof: $[\sigma(s;w),i\varsigma(s+w)]^a\bar{Z}_a(s,\varsigma;w)$
$=\frac{-i\varsigma}{\sqrt{2}}[\sigma(s;w),i\varsigma(s+w)]^a\bar{N}(s;w)(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},i\varsigma)_a$
$=\frac{-i\varsigma}{\sqrt{2}}\bar{N}(s;w)[s\sigma\otimes I_{C^{2s-1}_{2s-1+w}},i\varsigma(s+w)]^aN(s;w)\bar{N}(s;w)(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},i\varsigma)_a$
$=\frac{-i\varsigma}{\sqrt{2}}[\bar{N}(s;w)(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},-i\varsigma)^aN(s;w)s\bar{N}(s;w)(\sigma\otimes I_{C^{2s-1}_{2s-1+w}},i\varsigma)_a-(2s+w)\bar{N}(s;w)]$
$=\frac{-i\varsigma}{\sqrt{2}}[\bar{N}(s;w)Z^a(s,\varsigma;w)2s\bar{Z}_a(s,\varsigma;w)-(2s+w)\bar{N}(s;w)]$
$=\frac{-i\varsigma}{\sqrt{2}}[\bar{N}(s;w)(2s+w)-(2s+w)\bar{N}(s;w)]$
$=0$ □

3 Constant invariant tensors $Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma),Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)$

3.1 Introduction of constant invariant tensors $Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma),Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)$

Def. 3.1.1. $Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma):=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_aN^{k_\varsigma}_{A_\varsigma l_\varsigma}(s),Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma):=\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)$

Pro. 3.1.1. $\begin{cases}Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)=\delta^{ab}\varepsilon_{k_\varsigma m_\varsigma}(s)\varepsilon^{l_\varsigma n_\varsigma}(s-\frac{1}{2})\varepsilon_{A'_\varsigma B'_\varsigma}Z^{B'_\varsigma m_\varsigma}_{bn_\varsigma}(s,\varsigma)\\ Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma)=\delta_{ab}\varepsilon^{k_\varsigma m_\varsigma}(s)\varepsilon_{l_\varsigma n_\varsigma}(s-\frac{1}{2})\varepsilon^{A'_\varsigma B'_\varsigma}Z^{bn_\varsigma}_{B'_\varsigma m_\varsigma}(s,\varsigma)\end{cases}$

Pro. 3.1.2. $\begin{cases}Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)=(-1)^{2s+1}\delta^{ab}[(-\varsigma)^{2s}\varepsilon_{k_\varsigma m_\varsigma}(s)][\varsigma^{2s-1}\varepsilon^{l_\varsigma n_\varsigma}(s-\frac{1}{2})][-\varsigma\varepsilon_{A'_\varsigma B'_\varsigma}]Z^{B'_\varsigma m_\varsigma}_{bn_\varsigma}(s,\varsigma)\\ Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma)=(-1)^{2s+1}\delta_{ab}[(\varsigma)^{2s}\varepsilon^{k_\varsigma m_\varsigma}(s)][(-\varsigma)^{2s-1}\varepsilon_{l_\varsigma n_\varsigma}(s-\frac{1}{2})][\varsigma\varepsilon^{A'_\varsigma B'_\varsigma}]Z^{bn_\varsigma}_{B'_\varsigma m_\varsigma}(s,\varsigma)\end{cases}$

3.2 Introduction of constant matrices $Z_a^{A'_\varsigma}(s,\varsigma), Z^a_{A'_\varsigma}(s,\varsigma)$

Def. 3.2.1. $\begin{cases} Z_{al_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma) \succ Z_a^{A'_\varsigma}(s,\varsigma) := \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\bar{N}_{A_\varsigma}(s) \\ Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma) \succ Z^a_{A'_\varsigma}(s,\varsigma) := \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\bar{N}^{A_\varsigma}(s) \end{cases}$

Def. 3.2.2. $\begin{cases} \bar{Z}_a^{A'_\varsigma}(s,\varsigma) := Z_a^{TA'_\varsigma}(s,\varsigma) = \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}N_{A_\varsigma}(s) \\ \bar{Z}^a_{A'_\varsigma}(s,\varsigma) := Z^{Ta}_{A'_\varsigma}(s,\varsigma) = \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma}(s) \end{cases}$

3.3 Introduction of constant invariant tensor matrices $Z_a(s,\varsigma), \bar{Z}_a(s,\varsigma)$

Def. 3.3.1. $\begin{cases} Z_{al_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma)|^{A'_\varsigma}{}_{\otimes l_\varsigma}{}^{k_\varsigma} \succ Z_a(s,\varsigma) := \frac{i\varsigma}{\sqrt{2}}(\sigma\otimes I_{2s},-i\varsigma)_a N(s) \\ Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)|_{k_\varsigma A'_\varsigma}{}^{\otimes l_\varsigma} \succ \bar{Z}_a(s,\varsigma) := \frac{-i\varsigma}{\sqrt{2}}\bar{N}(s)(\sigma\otimes I_{2s},i\varsigma)_a \simeq Z_a^+(s,\varsigma) \end{cases}$

3.4 Constant invariant tensor properties of matrices $Z_a(s,\varsigma), \bar{Z}_a(s,\varsigma)$

Pro. 3.4.1. $Z_a(s,\varsigma) = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})} Z_b(s,\varsigma) e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$

Pro. 3.4.2. $\bar{Z}_a(s,\varsigma) = [e^{(i\omega\cdot R+\epsilon\cdot L)}]_a{}^b e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)} \bar{Z}_b(s,\varsigma) e^{-(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2})}$

3.5 Properties I of constant invariant tensors $Z_{al_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma), Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)$

1. Reduce two pairs of indices $A'_\varsigma, l_\varsigma$

Pro. 3.5.1. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{bl_\varsigma}^{A'_\varsigma m_\varsigma}(s,\varsigma) = \frac{1}{2s}[s\delta^a{}_b\delta_{k_\varsigma}{}^{m_\varsigma} + iS^a{}_{bk_\varsigma}{}^{m_\varsigma}(s,\varsigma)]$

$[\Leftrightarrow]\bar{Z}^a_{A'_\varsigma}(s,\varsigma)Z_b^{A'_\varsigma}(s,\varsigma) = \frac{1}{2s}[s\delta^a{}_b + iS^a{}_b(s,\varsigma)][\Leftrightarrow]\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma) = \frac{1}{2s}[s\delta_{ab} + iS_{ab}(s,\varsigma)]$

Proof: $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{bl_\varsigma}^{A'_\varsigma m_\varsigma}(s,\varsigma)$

$= \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{A'_\varsigma B_\varsigma}N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s)$

$= \frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_b^{A'_\varsigma B_\varsigma}N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s)$

$= \frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)(\delta^a{}_b\delta_{A_\varsigma}{}^{B_\varsigma} + 2iS^a{}_{bA_\varsigma}{}^{B_\varsigma})N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s)$

$= \frac{1}{2s}[s\delta^a{}_b\delta_{k_\varsigma}{}^{m_\varsigma} + iS^a{}_{bk_\varsigma}{}^{m_\varsigma}(s,\varsigma)]$ □

Cor. 3.5.1. $Z_a{}^{l_\varsigma}{}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{al_\varsigma}^{A'_\varsigma m_\varsigma}(s,\varsigma) = \frac{1}{2}\delta_{k_\varsigma}{}^{m_\varsigma}$

$[\Leftrightarrow]\bar{Z}_{aA'_\varsigma}(s,\varsigma)Z_a^{A'_\varsigma}(s,\varsigma) = \frac{1}{2}I_{2s+1}[\Leftrightarrow]\bar{Z}_a(s,\varsigma)Z_a(s,\varsigma) = \frac{1}{2}I_{2s+1}$

2. Reduce two pairs of indices $A'_\varsigma, k_\varsigma$

Pro. 3.5.2. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{bm_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma) = \frac{1}{2s}[(s+\frac{1}{2})\delta^a{}_b\delta_{m_\varsigma}{}^{l_\varsigma} + iS^a{}_{bm_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2},\varsigma)]$

$[\Leftrightarrow]Z_b^{A'_\varsigma}(s,\varsigma)\bar{Z}^a_{A'_\varsigma}(s,\varsigma) = \frac{1}{2s}[(s+\frac{1}{2})\delta^a{}_b + iS^a{}_b(s-\frac{1}{2},\varsigma)]$

Proof: $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{bm_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma)$

$= \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{A'_\varsigma B_\varsigma}N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s)$

$= \frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_b^{A'_\varsigma B_\varsigma}N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s)$

$= \frac{1}{2}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)(\delta^a{}_b\delta_{A_\varsigma}{}^{B_\varsigma} + 2iS^a{}_{bA_\varsigma}{}^{B_\varsigma})N^{k_\varsigma}_{B_\varsigma m_\varsigma}(s)$

$= \frac{1}{2s}[(s+\frac{1}{2})\delta^a{}_b\delta_{m_\varsigma}{}^{l_\varsigma} + iS^a{}_{bm_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2},\varsigma)]$ □

Cor. 3.5.2. $Z_a{}^{l_\varsigma}{}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{am_\varsigma}^{A'_\varsigma k_\varsigma}(s,\varsigma) = \frac{1}{2}(1+\frac{1}{2s})\delta_{m_\varsigma}{}^{l_\varsigma}[\Leftrightarrow]Z_a^{A'_\varsigma}(s,\varsigma)\bar{Z}_{aA'_\varsigma}(s,\varsigma) = \frac{1}{2}(1+\frac{1}{2s})I_{2s}$

3. Reduce two pairs of indices k_ς, l_ς

Pro. 3.5.3. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{bl_\varsigma}^{B'_\varsigma k_\varsigma}(s,\varsigma) = (s+\frac{1}{2})(\frac{1}{2}\delta_b{}^a\delta^{B'_\varsigma}{}_{A'_\varsigma} + iS_b{}^{aB'_\varsigma}{}_{A'_\varsigma})$

$[\Leftrightarrow]tr[\bar{Z}^a_{A'_\varsigma}(s,\varsigma)Z_b^{B'_\varsigma}(s,\varsigma)] = (s+\frac{1}{2})(\frac{1}{2}\delta_b{}^a\delta^{B'_\varsigma}{}_{A'_\varsigma} + iS_b{}^{aB'_\varsigma}{}_{A'_\varsigma})$

Proof: $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{bl_\varsigma}^{B'_\varsigma k_\varsigma}(s,\varsigma)$

$= \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}N^{k_\varsigma}_{B_\varsigma l_\varsigma}(s)$

$= \frac{1}{2}(s+\frac{1}{2})(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma A_\varsigma}$

$= (s+\frac{1}{2})(\frac{1}{2}\delta_b{}^a\delta^{B'_\varsigma}{}_{A'_\varsigma} + iS_b{}^{aB'_\varsigma}{}_{A'_\varsigma})$ □

Cor. 3.5.3. $Z_a{}^{l_\varsigma}{}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z_{al_\varsigma}^{B'_\varsigma k_\varsigma}(s,\varsigma) = \frac{1}{2}(s+\frac{1}{2})\delta^{B'_\varsigma}_{A'_\varsigma}[\Leftrightarrow]tr[\bar{Z}^a_{A'_\varsigma}(s,\varsigma)Z_b^{B'_\varsigma}(s,\varsigma)] = \frac{1}{2}(s+\frac{1}{2})\delta^{B'_\varsigma}_{A'_\varsigma}$

3.6 Properties II of constant invariant tensors $Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma), Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)$

Pro. 3.6.1. $\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{B_\varsigma B'_\varsigma}=\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}$

4. Reduce two pairs of indices a, l_ς

Pro. 3.6.2. $Z^{al_\varsigma}_{A'_\varsigma k_\varsigma}(s,\varsigma)Z^{B'_\varsigma m_\varsigma}_{al_\varsigma}(s,\varsigma)=\delta^{B'_\varsigma}_{A'_\varsigma}\delta^{m_\varsigma}_{k_\varsigma}[\Leftrightarrow]\bar{Z}^a_{A'_\varsigma}(s,\varsigma)Z^{B'_\varsigma}_a(s,\varsigma)=\delta^{B'_\varsigma}_{A'_\varsigma}I_{2s+1}$

5. Reduce two pairs of indices a, k_ς

Pro. 3.6.3. $Z^{A'_\varsigma k_\varsigma}_{al_\varsigma}(s,\varsigma)Z^{am_\varsigma}_{B'_\varsigma k_\varsigma}(s,\varsigma)=(1+\frac{1}{2s})\delta^{A'_\varsigma}_{B'_\varsigma}\delta^{m_\varsigma}_{l_\varsigma}$
$[\Leftrightarrow]Z^{A'_\varsigma}_a(s,\varsigma)\bar{Z}^a_{B'_\varsigma}(s,\varsigma)=(1+\frac{1}{2s})\delta^{A'_\varsigma}_{B'_\varsigma}I_{2s}[\Leftrightarrow]Z_a(s,\varsigma)\bar{Z}^a(s,\varsigma)=(1+\frac{1}{2s})I_{4s}$

3.7 Properties of constant invariant tensor matrices $Z_a(s,\varsigma), \bar{Z}_a(s,\varsigma)$

Pro. 3.7.1. $\begin{cases}\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)=\frac{1}{2s}[s\delta_{ab}+iS_{ab}(s,\varsigma)]\\ Z^a(s,\varsigma)\bar{Z}_a(s,\varsigma)=(1+\frac{1}{2s})I_{4s}\end{cases}$

Pro. 3.7.2. $\begin{cases}(s+1)Z_b(s,\varsigma)=Z^a(s,\varsigma)iS_{ab}(s,\varsigma),(s+1)\bar{Z}_a(s,\varsigma)=iS_{ab}(s,\varsigma)\bar{Z}^b(s,\varsigma)\\ Z^a(s,\varsigma)iS_{ab}(s,\varsigma)\bar{Z}^b(s,\varsigma)=(s+1)(1+\frac{1}{2s}),Z_a(s,\varsigma)\bar{Z}_a(s,\varsigma)\neq kI_{4s}\end{cases}$

Pro. 3.7.3. $\begin{cases}-S_{ac}(s,\varsigma)S^c{}_b(s,\varsigma)=s(s+1)\delta_{ab}+iS_{ab}(s,\varsigma)\\ \bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)=-\frac{1}{2s}[s^2\delta_{ab}+S_{ac}(s,\varsigma)S^c{}_b(s,\varsigma)]\end{cases}$

Pro. 3.7.4. $[\sigma(s),i\varsigma(s+1)]^a\bar{Z}_a(s,\varsigma)=0, Z_a(s,\varsigma)[\sigma(s),-i\varsigma(s+1)]^a=0$

Proof: $[\sigma(s),i\varsigma(s+1)]^a\bar{Z}_a(s,\varsigma)$
$=\frac{-i\varsigma}{\sqrt{2}}[\sigma(s),i\varsigma(s+1)]^a\bar{N}(s)(\sigma\otimes I_{2s},i\varsigma)_a$
$=\frac{-i\varsigma}{\sqrt{2}}\bar{N}(s)[s\sigma\otimes I_{2s},i\varsigma(s+1)]^aN(s)\bar{N}(s)(\sigma\otimes I_{2s},i\varsigma)_a$
$=\frac{-i\varsigma}{\sqrt{2}}[\bar{N}(s)(\sigma\otimes I_{2s},-i\varsigma)^aN(s)s\bar{N}(s)(\sigma\otimes I_{2s},i\varsigma)_a-(2s+1)\bar{N}(s)]$
$=\frac{-i\varsigma}{\sqrt{2}}[\bar{N}(s)Z^a(s,\varsigma)2s\bar{Z}_a(s,\varsigma)-(2s+1)\bar{N}(s)]$
$=\frac{-i\varsigma}{\sqrt{2}}[\bar{N}(s)(2s+1)-(2s+1)\bar{N}(s)]$
$=0$ □

4 Several important composite constant invariant tensors

4.1 Composite constant invariant tensors $\Gamma^{k_\varsigma}_{\alpha_\varsigma\beta_\varsigma\cdots}(n),\Gamma^{\alpha_\varsigma\beta_\varsigma\cdots}_{k_\varsigma}(n)$

Def. 4.1.1.

$$\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{2n}}(n):=(\frac{i\varsigma}{\sqrt{2}})^n\underbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots}_{n}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n),\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{2n}}_{k_\varsigma}(n):=(\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)$$

Equivalence:

Cor. 4.1.1.

$$\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)=(\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)\Leftrightarrow\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)=(\frac{i\varsigma}{\sqrt{2}})^n\underbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots}_{n}\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)$$

Cor. 4.1.2.

$$\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)=(\frac{i\varsigma}{\sqrt{2}})^n\underbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots}_{n}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n)\Leftrightarrow\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n)=(\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)$$

Equality:

Cor. 4.1.3. $\Gamma^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdots}^{n}}_{k'_\varsigma}(n)=[\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)]^*\simeq\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n),\Gamma^{k'_\varsigma}_{\underbrace{\alpha'_\varsigma\beta'_\varsigma\cdots}_{n}}(n)=[\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)]^*\simeq\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)$

Full symmetry:

Cor. 4.1.4. $\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)=\frac{1}{n!}\Gamma^{\overbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}^{n}}_{k_\varsigma}(n),\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)=\frac{1}{n!}\Gamma^{k_\varsigma}_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}}(n)$

Tracelessness:

Cor. 4.1.5. $\delta_{\alpha_\varsigma\beta_\varsigma}\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)=0,\delta^{\alpha_\varsigma\beta_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)=0$

Similar Penrose correspondence:

$$\Gamma^{\overbrace{\alpha\beta\cdots}^{n}}_{k}(n)\overset{P}{=}\Gamma^{\overbrace{ABCD\cdots}^{2n}}_{k}(n) \qquad \Gamma^{\overbrace{\alpha'\beta'\cdots}^{n}}_{k'}(n)\overset{P}{=}\Gamma^{\overbrace{A'B'C'D'\cdots}^{2n}}_{k'}(n) \tag{3.1}$$

$$\Gamma^{k}_{\underbrace{\alpha\beta\cdots}_{n}}(n)\overset{P}{=}\Gamma^{k}_{\underbrace{ABCD\cdots}_{2n}}(n) \qquad \Gamma^{k'}_{\underbrace{\alpha'\beta'\cdots}_{n}}(n)\overset{P}{=}\Gamma^{k'}_{\underbrace{A'B'C'D'\cdots}_{2n}}(n) \tag{3.2}$$

Orthogonality:

Cor. 4.1.6. $\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{l_\varsigma}(n)=\delta^{k_\varsigma}{}_{l_\varsigma}$

Cor. 4.1.7. $\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{\alpha_\varsigma}_{l_\varsigma}(1)=\delta^{k_\varsigma}_{l_\varsigma},\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{\beta_\varsigma}_{k_\varsigma}(1)=\delta^{\beta_\varsigma}_{\alpha_\varsigma}$

Cor. 4.1.8. $\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)=\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Leftrightarrow\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)$

Cor. 4.1.9. $\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Leftrightarrow\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)$

Cor. 4.1.10. $\Gamma^{k_\varsigma}_{\alpha_{1\varsigma}\alpha_{2\varsigma}}(2)\Gamma^{\beta_{1\varsigma}\beta_{2\varsigma}}_{k_\varsigma}(2)=\frac{1}{2!}\delta^{(\beta_{1\varsigma}}_{\alpha_{1\varsigma}}\delta^{\beta_{2\varsigma})}_{\alpha_{2\varsigma}}-\frac{1}{3!}\delta^{(\beta_{1\varsigma}\beta_{2\varsigma})}\delta_{\alpha_{1\varsigma}\alpha_{2\varsigma}}$

Proof: $\Gamma^{k_\varsigma}_{\alpha_{1\varsigma}\alpha_{2\varsigma}}(2)\Gamma^{\beta_{1\varsigma}\beta_{2\varsigma}}_{k_\varsigma}(2)$
$=(\frac{i\varsigma}{\sqrt{2}})^2\sigma^{A_{1\varsigma}A_{2\varsigma}}_{\alpha_{1\varsigma}}\sigma^{A_{3\varsigma}A_{4\varsigma}}_{\alpha_{2\varsigma}}(\frac{i\varsigma}{\sqrt{2}})^2\sigma^{\beta_{1\varsigma}}_{B_{1\varsigma}B_{2\varsigma}}\sigma^{\beta_{2\varsigma}}_{B_{3\varsigma}B_{4\varsigma}}\Gamma^{k_\varsigma}_{A_{1\varsigma}A_{2\varsigma}\cdots A_{2s\varsigma}}(s)\Gamma^{B_{1\varsigma}B_{2\varsigma}\cdots B_{2s\varsigma}}_{k_\varsigma}(s)$
$=(\frac{i\varsigma}{\sqrt{2}})^2\sigma^{A_{1\varsigma}A_{2\varsigma}}_{\alpha_{1\varsigma}}\sigma^{A_{3\varsigma}A_{4\varsigma}}_{\alpha_{2\varsigma}}(\frac{i\varsigma}{\sqrt{2}})^2\sigma^{\beta_{1\varsigma}}_{B_{1\varsigma}B_{2\varsigma}}\sigma^{\beta_{2\varsigma}}_{B_{3\varsigma}B_{4\varsigma}}\frac{1}{4!}\delta^{B_{1\varsigma}}_{(A_{1\varsigma}}\delta^{B_{2\varsigma}}_{A_{2\varsigma}}\delta^{B_{3\varsigma}}_{A_{3\varsigma}}\delta^{B_{4\varsigma}}_{A_{4\varsigma})}$
$=\frac{1}{4!}[12\delta^{(\beta_{1\varsigma}}_{\alpha_{1\varsigma}}\delta^{\beta_{2\varsigma})}_{\alpha_{2\varsigma}}-8\delta^{\beta_{1\varsigma}\beta_{2\varsigma}}\delta_{\alpha_{1\varsigma}\alpha_{2\varsigma}}]$
$=\frac{1}{2!}\delta^{(\beta_{1\varsigma}}_{\alpha_{1\varsigma}}\delta^{\beta_{2\varsigma})}_{\alpha_{2\varsigma}}-\frac{1}{3!}\delta^{(\beta_{1\varsigma}\beta_{2\varsigma})}\delta_{\alpha_{1\varsigma}\alpha_{2\varsigma}}$ □

4.2 Composite constant invariant tensors $\Gamma^{k_\varsigma}_{abcd\cdots}(n),\Gamma^{abcd\cdots}_{k_\varsigma}(n)$

Def. 4.2.1. $\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n):=(\frac{i}{2})^n\underbrace{\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}\cdots}_{n}\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n),\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n):=(\frac{i}{2})^n\overbrace{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{\varsigma cd}\cdots}^{n}\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)$

Cor. 4.2.1.

$$\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)=(\frac{i}{2})^n\underbrace{\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}\cdots}_{n}\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)\Rightarrow\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)=(\frac{i}{2})^n\overbrace{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{\varsigma cd}\cdots}^{n}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)$$

$$\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)=(\frac{i}{2})^n\overbrace{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{\varsigma cd}\cdots}^{n}\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\Rightarrow\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)=(\frac{i}{2})^n\underbrace{\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}\cdots}_{n}\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)$$

Cor. 4.2.2. *The following two equations can be deduced from each other and are equivalent.*

$$\begin{cases}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)=(\frac{-i\varsigma}{\sqrt{2}})^{2n}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^d_{D_\varsigma D'_\varsigma}\cdots}^{2n}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)\overbrace{\varepsilon^{A'_\varsigma B'_\varsigma}\varepsilon^{C'_\varsigma D'_\varsigma}\cdots}^{n}\\ \Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)\overbrace{\varepsilon^{A'_\varsigma B'_\varsigma}\varepsilon^{C'_\varsigma D'_\varsigma}\cdots}^{n}=(\frac{i\varsigma}{\sqrt{2}})^{2n}\overbrace{(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b(\sigma,-i\varsigma)^{C'_\varsigma C_\varsigma}_c(\sigma,-i\varsigma)^{D'_\varsigma D_\varsigma}_d\cdots}^{2n}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)\end{cases}$$

Cor. 4.2.3. *The following two equations can be deduced from each other and are equivalent.*

$$\begin{cases}\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)=(\frac{i\varsigma}{\sqrt{2}})^{2n}\overbrace{(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b(\sigma,-i\varsigma)^{C'_\varsigma C_\varsigma}_c(\sigma,-i\varsigma)^{D'_\varsigma D_\varsigma}_d\cdots}^{2n}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n)\overbrace{\varepsilon_{A'_\varsigma B'_\varsigma}\varepsilon_{C'_\varsigma D'_\varsigma}\cdots}^{n}\\ \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n)\overbrace{\varepsilon_{A'_\varsigma B'_\varsigma}\varepsilon_{C'_\varsigma D'_\varsigma}\cdots}^{n}=(\frac{-i\varsigma}{\sqrt{2}})^{2n}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^d_{D_\varsigma D'_\varsigma}\cdots}^{2n}\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)\end{cases}$$

Symmetry:

Cor. 4.2.4. $\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)=\frac{1}{n!}\frac{1}{2^n}\Gamma^{\overbrace{([ab][cd]\cdots)}^{2n}}_{k_\varsigma}(n),\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)=\frac{1}{n!}\frac{1}{2^n}\Gamma^{k_\varsigma}_{\underbrace{([ab][cd]\cdots)}_{2n}}(n)$

Duality:

Cor. 4.2.5. $\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)=[-\varsigma]^n\Gamma^{\overbrace{*ab*cd\cdots}^{2n}}_{k_\varsigma}(n),\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)=[-\varsigma]^n\Gamma^{k_\varsigma}_{\underbrace{*ab*cd\cdots}_{2n}}(n)$

Tracelessness:

Cor. 4.2.6. $\delta_{ab}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)=0,\delta_{ac}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k_\varsigma}(n)=0,\delta^{ab}\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)=0,\delta^{ac}\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)=0$

Penrose correspondence:

$$\begin{cases}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{\overbrace{ABCD\cdots}^{2n}}_{k}(n)\overbrace{\varepsilon^{A'B'}\varepsilon^{C'D'}\cdots}^{n},\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k'}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{\overbrace{A'B'C'D'\cdots}^{2n}}_{k'}(n)\overbrace{\varepsilon^{AB}\varepsilon^{CD}\cdots}^{n}\\ \Gamma^{k}_{\underbrace{abcd\cdots}_{2n}}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{k}_{\underbrace{ABCD\cdots}_{2n}}(n)\overbrace{\varepsilon_{A'B'}\varepsilon_{C'D'}\cdots}^{n},\Gamma^{k'}_{\underbrace{abcd\cdots}_{2n}}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{k'}_{\underbrace{A'B'C'D'\cdots}_{2n}}(n)\overbrace{\varepsilon_{AB}\varepsilon_{CD}\cdots}^{n}\end{cases}\tag{3.3}$$

Similar Penrose correspondence:

$$\begin{cases}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{\overbrace{\alpha\beta\cdots}^{n}}_{k}(n)\overbrace{\varepsilon^{A'B'}\varepsilon^{C'D'}\cdots}^{n},\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k'}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{\overbrace{\alpha'\beta'\cdots}^{n}}_{k'}(n)\overbrace{\varepsilon^{AB}\varepsilon^{CD}\cdots}^{n}\\ \Gamma^{k}_{\underbrace{abcd\cdots}_{2n}}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{k}_{\underbrace{\alpha\beta\cdots}_{n}}(n)\overbrace{\varepsilon_{A'B'}\varepsilon_{C'D'}\cdots}^{n},\Gamma^{k'}_{\underbrace{abcd\cdots}_{2n}}(n)\overset{P}{=}(\frac{1}{\sqrt{2}})^n\Gamma^{k'}_{\underbrace{\alpha'\beta'\cdots}_{n}}(n)\overbrace{\varepsilon_{AB}\varepsilon_{CD}\cdots}^{n}\end{cases}\tag{3.4}$$

One to one correspondence:

$$\begin{cases}\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k}(n)\leftrightarrow\Gamma^{\overbrace{\alpha\beta\cdots}^{n}}_{k}(n)\leftrightarrow\Gamma^{\overbrace{ABCD\cdots}^{2n}}_{k}(n),\Gamma^{\overbrace{abcd\cdots}^{2n}}_{k'}(n)\leftrightarrow\Gamma^{\overbrace{\alpha'\beta'\cdots}^{n}}_{k'}(n)\leftrightarrow\Gamma^{\overbrace{A'B'C'D'\cdots}^{2n}}_{k'}(n)\\ \Gamma^{k}_{\underbrace{abcd\cdots}_{2n}}(n)\leftrightarrow\Gamma^{k}_{\underbrace{\alpha\beta\cdots}_{n}}(n)\leftrightarrow\Gamma^{k}_{\underbrace{ABCD\cdots}_{2n}}(n),\Gamma^{k'}_{\underbrace{abcd\cdots}_{2n}}(n)\leftrightarrow\Gamma^{k'}_{\underbrace{\alpha'\beta'\cdots}_{n}}(n)\leftrightarrow\Gamma^{k'}_{\underbrace{A'B'C'D'\cdots}_{2n}}(n)\end{cases}\tag{3.5}$$

Orthogonality:

Cor. 4.2.7. $\Gamma^{k_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)\Gamma^{\overbrace{abcd\cdots}^{2n}}_{l_\varsigma}(n)=2^n\delta^{k_\varsigma}{}_{l_\varsigma}$

4.3 Introduction of composite constant invariant tensors $N^{k_\varsigma k'_\varsigma}_{l_\varsigma l'_\varsigma a}(s),N^{l'_\varsigma l_\varsigma a}_{k'_\varsigma k_\varsigma}(s)$

Def. 4.3.1. $N^{k_\varsigma k'_\varsigma}_{l_\varsigma l'_\varsigma a}(s):=\frac{i\varsigma}{\sqrt{2}}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)N^{k'_\varsigma}_{A'_\varsigma l'_\varsigma}(s)(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a,N^{l'_\varsigma l_\varsigma a}_{k'_\varsigma k_\varsigma}(s):=\frac{-i\varsigma}{\sqrt{2}}N^{A'_\varsigma l'_\varsigma}_{k'_\varsigma}(s)N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}$

Cor. 4.3.1. $N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)N^{k'_\varsigma}_{A'_\varsigma l'_\varsigma}(s)=\frac{-i\varsigma}{\sqrt{2}}N^{k_\varsigma k'_\varsigma}_{l_\varsigma l'_\varsigma a}(s)(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma},N^{A'_\varsigma l'_\varsigma}_{k'_\varsigma}(s)N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)=\frac{i\varsigma}{\sqrt{2}}N^{l'_\varsigma l_\varsigma a}_{k'_\varsigma k_\varsigma}(s)(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a$

Penrose notation:

$$N^{kk'}_{ll'a}(s)\overset{P}{=}N^{k}_{Al}(s)N^{k'}_{A'l'}(s)\qquad N^{l'la}_{k'k}(s)\overset{P}{=}N^{A'l'}_{k'}(s)N^{Al}_{k}(s)\tag{3.6}$$

4.4 Introduction of composite constant invariant tensors $\Gamma^{a_1b_1c_1d_1\cdots a_2b_2c_2d_2\cdots}_{abcd\cdots}(n)$

Def. 4.4.1. $\Gamma^{\overbrace{a_1b_1c_1d_1\cdots}^{2n}\overbrace{a_2b_2c_2d_2\cdots}^{2n}}_{\underbrace{abcd\cdots}_{2n}}(n):=\Gamma^{k_\varsigma k'_\varsigma}_{\underbrace{abcd\cdots}_{2n}}(n)\Gamma^{\overbrace{a_1b_1c_1d_1\cdots}^{2n}}_{k_\varsigma}(n)\Gamma^{\overbrace{a_2b_2c_2d_2\cdots}^{2n}}_{k'_\varsigma}(n)$

4.5 Introduction of composite constant invariant tensors $\Gamma^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{\alpha_\varsigma\beta_\varsigma\cdots}(n),\Gamma^{\alpha_\varsigma\beta_\varsigma\cdots}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}(n)$

Def. 4.5.1.

$$\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{2n}}(n):=\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{2n}}(n)\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)=(\frac{i\varsigma}{\sqrt{2}})^n\frac{1}{(2n)!}\underbrace{\sigma^{(A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots)}_{n}$$

$$\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{2n}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n):=\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n)=(\frac{i\varsigma}{\sqrt{2}})^n\frac{1}{(2n)!}\overbrace{\sigma^{\alpha_\varsigma}_{(A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots)}^{n}$$

5 Composite constant invariant tensors $\Gamma^{k_\varsigma k'_\varsigma}_{abc\cdot\cdot}(s),\Gamma^{abc\cdot\cdot}_{k_\varsigma k'_\varsigma}(s)$

5.1 Introduction of composite constant invariant tensors $\Gamma^{k_\varsigma k'_\varsigma}_{abc\cdot\cdot}(s),\Gamma^{abc\cdot\cdot}_{k_\varsigma k'_\varsigma}(s)$

Def. 5.1.1.

$$\begin{cases}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s):=(\frac{i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}(\sigma,-i\varsigma)_c^{C'_\varsigma C_\varsigma}\cdot\cdot}^{2s}\Gamma^{k'_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s)\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s):=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s)\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s)\end{cases}$$

$\Leftrightarrow$

Cor. 5.1.1.

$$\begin{cases}\Gamma^{k'_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s)=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)\\ \Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s)\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s)=(\frac{i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}(\sigma,-i\varsigma)_c^{C'_\varsigma C_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\end{cases}$$

Non covariant relation:

Cor. 5.1.2.

$$\begin{cases}\Gamma^{k_\varsigma k'_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)\overbrace{\partial^a\partial^b\partial^c\cdot\cdot}^{2s}\simeq(-1)^{2s}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial^+_a\partial^+_b\partial^+_c\cdot\cdot}^{2s}\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k'_\varsigma k_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\simeq(-1)^{2s}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)\overbrace{\partial^{+a}\partial^{+b}\partial^{+c}\cdot\cdot}^{2s}\end{cases}$$

Full symmetry:

Cor. 5.1.3. $\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)=\frac{1}{(2s)!}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{(abc\cdot\cdot)}_{2s}}(s),\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)=\frac{1}{(2s)!}\Gamma^{\overbrace{(abc\cdot\cdot)}^{2s}}_{k_\varsigma k'_\varsigma}(s)$

Tracelessness:

Cor. 5.1.4. $\delta^{ab}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)=0,\delta_{ab}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)=0$

Penrose notation:

$$\Gamma^{k'k}_{\underbrace{abc\cdot\cdot}_{2s}}(s)\overset{P}{=}\Gamma^{k'}_{\underbrace{A'B'C'\cdot\cdot}_{2s}}(s)\Gamma^{k}_{\underbrace{ABC\cdot\cdot}_{2s}}(s)\qquad\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{kk'}(s)\overset{P}{=}\Gamma_k^{\overbrace{ABC\cdot\cdot}^{2s}}(s)\Gamma_{k'}^{\overbrace{A'B'C'\cdot\cdot}^{2s}}(s)\tag{3.7}$$

Orthogonality:

Cor. 5.1.5. $\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{l_\varsigma l'_\varsigma}(s)=\delta^{k_\varsigma}{}_{l_\varsigma}\delta^{k'_\varsigma}{}_{l'_\varsigma}$

5.2 Introduction of composite constant invariant tensors $\Gamma^{k_\varsigma k'_\varsigma}_{abc\cdot\cdot}(s,w),\Gamma^{abc\cdot\cdot}_{k_\varsigma k'_\varsigma}(s,w)$

Def. 5.2.1.

$$\begin{cases}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s,w):=(\frac{i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}(\sigma,-i\varsigma)_c^{C'_\varsigma C_\varsigma}\cdot\cdot}^{2s,w}\Gamma^{k'_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s,w)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s,w)\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s,w):=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s,w}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s,w)\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s,w)\end{cases}$$

5.3 Introduction of composite constant invariant tensors $\Gamma^{\alpha_\varsigma\alpha'_\varsigma\beta_\varsigma\beta'_\varsigma\cdot\cdot}_{abcd\cdot\cdot}(n), \Gamma^{abcd\cdot\cdot}_{\alpha_\varsigma\alpha'_\varsigma\beta_\varsigma\beta'_\varsigma\cdot\cdot}(n)$

Def. 5.3.1.

$$\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}_{2n}}(n) := (\tfrac{i\varsigma}{\sqrt2})^n \underbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdot\cdot}_{n}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}_{2n}}(n), \Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{2n}}_{k_\varsigma}(n) := (\tfrac{i\varsigma}{\sqrt2})^n \overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdot\cdot}^{n}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2n}}_{k_\varsigma}(n)$$

Def. 5.3.2.

$$\begin{cases}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s) := (\tfrac{i\varsigma}{\sqrt2})^{2s}\overbrace{(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b(\sigma,-i\varsigma)^{C'_\varsigma C_\varsigma}_c\cdot\cdot}^{2s}\Gamma^{k'_\varsigma}_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(s)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s)\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s) := (\tfrac{-i\varsigma}{\sqrt2})^{2s}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\end{cases}$$

Pro. 5.3.1. $\Gamma^{k_\varsigma}_{A_{1\varsigma}A_{2\varsigma}\cdot\cdot A_{2s\varsigma}}(s)\Gamma^{B_{1\varsigma}B_{2\varsigma}\cdot\cdot B_{2s\varsigma}}_{k_\varsigma}(s) = \frac{1}{(2s)!}\delta^{(B_{1\varsigma}}_{A_{1\varsigma}}\delta^{B_{2\varsigma}}_{A_{2\varsigma}}\cdot\cdot\delta^{B_{2s\varsigma})}_{A_{2s\varsigma}} = \frac{1}{(2s)!}\delta^{B_{1\varsigma}}_{(A_{1\varsigma}}\delta^{B_{2\varsigma}}_{A_{2\varsigma}}\cdot\cdot\delta^{B_{2s\varsigma}}_{A_{2s\varsigma})}$

Def. 5.3.3.

$$\begin{cases}\Gamma^{\overbrace{\alpha'_\varsigma\alpha_\varsigma\beta'_\varsigma\beta_\varsigma\cdot\cdot}^{2n}}_{\underbrace{abc\cdot\cdot}_{2s}}(n) := \Gamma^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}_{k'_\varsigma}(n)\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}_{k_\varsigma}(n)\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2n}}(n)\\ = (-\tfrac14)^n\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdot\cdot}^{n}\overbrace{\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\sigma^{\beta'_\varsigma}_{C'_\varsigma D'_\varsigma}\cdot\cdot}^{n}\overbrace{(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b(\sigma,-i\varsigma)^{C'_\varsigma C_\varsigma}_c\cdot\cdot}^{2n}\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2n}}_{\underbrace{\alpha_\varsigma\alpha'_\varsigma\beta_\varsigma\beta'_\varsigma\cdot\cdot}_{2n}}(n) := \Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}_{2n}}(n)\Gamma^{k'_\varsigma}_{\underbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}_{2n}}(n)\Gamma^{\overbrace{abc\cdot\cdot}^{2n}}_{k_\varsigma k'_\varsigma}(n)\\ = (-\tfrac14)^n\underbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdot\cdot}_{n}\underbrace{\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\beta'_\varsigma}\cdot\cdot}_{n}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2n}\end{cases}$$

$\Rightarrow$

Cor. 5.3.1.

$$\begin{cases}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2n}}(n) := \Gamma^{k'_\varsigma}_{\underbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}_{2n}}(n)\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}_{2n}}(n)\Gamma^{\overbrace{\alpha'_\varsigma\alpha_\varsigma\beta'_\varsigma\beta_\varsigma\cdot\cdot}^{2n}}_{\underbrace{abc\cdot\cdot}_{2s}}(n)\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2n}}_{k_\varsigma k'_\varsigma}(n) := \Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}_{k_\varsigma}(n)\Gamma^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}_{k'_\varsigma}(n)\Gamma^{\overbrace{abc\cdot\cdot}^{2n}}_{\underbrace{\alpha_\varsigma\alpha'_\varsigma\beta_\varsigma\beta'_\varsigma\cdot\cdot}_{2n}}(n)\end{cases}$$

Cor. 5.3.2.

$$\begin{cases}\Gamma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}(1) := \Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{k'_\varsigma k_\varsigma}_{ab}(1)\\ \Gamma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}(1) := \Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)\Gamma^{ab}_{k_\varsigma k'_\varsigma}(1)\end{cases} [\Leftrightarrow] \begin{cases}\Gamma^{k'_\varsigma k_\varsigma}_{ab}(1) := \Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}(1)\\ \Gamma^{ab}_{k_\varsigma k'_\varsigma}(1) := \Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)\Gamma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}(1)\end{cases} \begin{cases}\Gamma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}(1) = \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}\\ \Gamma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}(1) = \sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\end{cases}$$

5.4 Concrete expansion of first few items for $\Gamma^{k_\varsigma k'_\varsigma}_{abc\cdot\cdot}(s), \Gamma^{abc\cdot\cdot}_{k_\varsigma k'_\varsigma}(s)$

Proof: $\Gamma^{\overbrace{\pi\pi\pi\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)$

$$= (\tfrac{-i\varsigma}{\sqrt2})^{2s}\overbrace{(i\varsigma)_{A_\varsigma A'_\varsigma}(i\varsigma)_{B_\varsigma B'_\varsigma}(i\varsigma)_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$$
$$= (\tfrac{1}{\sqrt2})^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\overbrace{\delta_{A_\varsigma A'_\varsigma}\delta_{B_\varsigma B'_\varsigma}\delta_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$$
$$= (\tfrac{1}{\sqrt2})^{2s}\delta_{k_\varsigma k'_\varsigma}$$
□

Proof: $\Gamma^{\overbrace{i\pi\pi\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)$

$$= (\tfrac{-i\varsigma}{\sqrt2})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(i\varsigma)_{B_\varsigma B'_\varsigma}(i\varsigma)_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$$
$$= -i\varsigma(\tfrac{1}{\sqrt2})^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}\delta_{B_\varsigma B'_\varsigma}\delta_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$$
$$= -i\varsigma(\tfrac{1}{\sqrt2})^{2s}\tfrac{1}{s}\sigma^i(s)_{k_\varsigma k'_\varsigma}$$
□

Proof: $\Gamma^{\overbrace{ij\pi\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)$

$= (\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}(i\varsigma)_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$
$= -(\frac{1}{\sqrt{2}})^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}\delta_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$
$= -(\frac{1}{\sqrt{2}})^{2s}\frac{1}{2s(s-\frac{1}{2})}[\{\sigma^i(s),\sigma^j(s)\} - s\delta^{ij}]_{k_\varsigma k'_\varsigma}$ □

Proof: $\Gamma^{\overbrace{ijk\pi\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)$

$= (\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}(\sigma)^k_{C_\varsigma C'_\varsigma}(i\varsigma)_{D_\varsigma D'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$
$= (\frac{1}{\sqrt{2}})^{2s}\frac{i\varsigma}{2s(s-\frac{1}{2})(s-1)}\{\sigma^{\{j}(s)[\sigma^i(s)]\sigma^{k\}}(s) - [(s-1)\sigma^i(s)\delta^{jk} + s\delta^{i\{j}\sigma^{k\}}(s)]\}_{k_\varsigma k'_\varsigma}$ □

Proof: $\Gamma^{\overbrace{ijkl\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\partial_i\partial_j\partial_k\partial_l$

$= (\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}(\sigma)^k_{C_\varsigma C'_\varsigma}(\sigma)^l_{D_\varsigma D'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$
$= (\frac{1}{\sqrt{2}})^{2s}\frac{1}{s(s-\frac{1}{2})(s-1)(s-\frac{3}{2})}[\sigma^i(s)\sigma^j(s)\sigma^k(s)\sigma^l(s) + (2-3s)\sigma^i(s)\sigma^j(s)\delta^{kl} + \frac{3s(s-1)}{4}\delta^{ij}\delta^{kl}]_{k_\varsigma k'_\varsigma}\partial_i\partial_j\partial_k\partial_l$ □

5.5 Definition of $\Gamma^{abc\cdot\cdot}_+(s)$ and $\Gamma^{abc\cdot\cdot}_-(s)$

Def. 5.5.1. $odd := -, even := +$

Def. 5.5.2.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s) = 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s), 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s), l=0,\cdot\cdot,2s\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_+(s) := 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s), 0\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s), l=0,\cdot\cdot,2s\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_-(s) := 0\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s), 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s), l=0,\cdot\cdot,2s\end{cases}$$

Cor. 5.5.1. $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s) = \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_+(s) + \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_-(s)$

5.6 Basic properties of operators $\Gamma^{abc\cdot\cdot}_\pm(s)p_ap_bp_c\cdot\cdot$ and $\Gamma^{abc\cdot\cdot}_\pm(s)\partial_a\partial_b\partial_c\cdot\cdot$

Pro. 5.6.1.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{p_ap_bp_c\cdot\cdot}^{2s} = \sum_{n=0}^{2s}C^n_{2s}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}(s)\overbrace{p_ip_j\cdot\cdot}^{2s-n}p^n_\pi\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s} = \sum_{n=0}^{2s}C^n_{2s}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}(s)\overbrace{\partial_i\partial_j\cdot\cdot}^{2s-n}\partial^n_\pi\end{cases}$$

Pro. 5.6.2.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_+(s)\overbrace{p_ap_bp_c\cdot\cdot}^{2s} := \sum_{l=0}^{[s]}C^{2l}_{2s}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s)\overbrace{p_ip_j\cdot\cdot}^{2s-2l}p^{2l}_\pi\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_+(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s} := \sum_{l=0}^{[s]}C^{2l}_{2s}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s)\overbrace{\partial_i\partial_j\cdot\cdot}^{2s-2l}\partial^{2l}_\pi\end{cases}$$

Pro. 5.6.3.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_-(s)\overbrace{p_ap_bp_c\cdot\cdot}^{2s} := \sum_{l=0}^{[s-\frac{1}{2}]}C^{2l+1}_{2s}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\overbrace{p_ip_j\cdot\cdot}^{2s-2l-1}p^{2l+1}_\pi\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_-(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s} := \sum_{l=0}^{[s-\frac{1}{2}]}C^{2l+1}_{2s}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\overbrace{\partial_i\partial_j\cdot\cdot}^{2s-2l-1}\partial^{2l+1}_\pi\end{cases}$$

Pro. 5.6.4.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{p_ap_bp_c\cdot\cdot}^{2s} = \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_+(s)\overbrace{p_ap_bp_c\cdot\cdot}^{2s} + \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_-(s)\overbrace{p_ap_bp_c\cdot\cdot}^{2s}\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s} = \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_+(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s} + \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_-(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\end{cases}$$

5.7 Definition of operators $\hat{p}_a$ and $\hat{\partial}_a$

Def. 5.7.1. $\hat{p}_a := \frac{p_a}{|\vec{p}|} = (\hat{p}, i); \hat{p} = \frac{\vec{p}}{|\vec{p}|}, \hat{p}_\pi = \frac{p_\pi}{|\vec{p}|} = i; \hat{p}^2 = 1, \hat{p}_\pi^2 = i^2$

Def. 5.7.2. $\hat{\partial}_a := \frac{\partial_a}{i\sqrt{-\nabla^2}} = \frac{-i\partial_a}{\sqrt{-\nabla^2}} = \frac{(-i\nabla, -\partial_t)}{\sqrt{-\nabla^2}}; \hat{\nabla} = \frac{\nabla}{i\sqrt{-\nabla^2}} = \frac{-i\nabla}{\sqrt{-\nabla^2}}; \hat{\nabla}^2 = 1, \hat{\nabla}_\pi^2 = i^2$

Cor. 5.7.1. $p_a \simeq -i\partial_a, |\vec{p}| \simeq \sqrt{-\nabla^2}, \hat{p}_a \simeq \hat{\partial}_a, p_a = |\vec{p}|\hat{p}_a, \partial_a = (i\sqrt{-\nabla^2})\hat{\partial}_a$

5.8 Basic properties of operators $\Gamma^{abc\cdot\cdot}(s)\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot$ and $\Gamma^{abc\cdot\cdot}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot$

Cor. 5.8.1.
$$\begin{cases} \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{p_a p_b p_c\cdot\cdot}^{2s} = |\vec{p}|^{2s}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s} = (i\sqrt{-\nabla^2})^{2s}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s} \end{cases}$$

Pro. 5.8.1.
$$\begin{cases} \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} = \sum_{n=0}^{2s} i^n C_{2s}^n \Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{2s-n} = \sum_{n=0}^{2s} i^{2s-n} C_{2s}^n \Gamma^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{n} \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i} = \sum_{n=0}^{2s} i^n C_{2s}^n \Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{2s-n} = \sum_{n=0}^{2s} i^{2s-n} C_{2s}^n \Gamma^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{n} \end{cases}$$

Pro. 5.8.2.
$$\begin{cases} \Gamma_+^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} = \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{2s-2l} \\ \Gamma_+^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i} = \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{2s-2l} \end{cases}$$

Pro. 5.8.3.
$$\begin{cases} \Gamma_-^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} = i\sum_{l=0}^{[s-\frac{1}{2}]} (-1)^l C_{2s}^{2l+1} \Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{2s-2l-1} \\ \Gamma_-^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i} = i\sum_{l=0}^{[s-\frac{1}{2}]} (-1)^l C_{2s}^{2l+1} \Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{2s-2l-1} \end{cases}$$

5.9 Expansion of operators $\Gamma_{k_\varsigma k'_\varsigma}^{ij\cdot\cdot\pi\cdot\cdot\pi}(s)\hat{p}_i\hat{p}_j\cdot\cdot$ and $\Gamma_{k_\varsigma k'_\varsigma}^{ij\cdot\cdot\pi\cdot\cdot\pi}(s)\hat{\partial}_i\hat{\partial}_j\cdot\cdot$

Cor. 5.9.1. $\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}(s) = (-i\varsigma)^n 2^{n-s}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{n}}(s,n) = \frac{(-i\varsigma)^n}{2^{s-n}}\frac{1}{n!}\sum_{k}^{[n/2]} c(s,n;n-2k)\Omega^{n-2k}(s)$

Proof: $\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}(s)$

$$= (\tfrac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)_{A_\varsigma A'_\varsigma}^i(\sigma)_{B_\varsigma B'_\varsigma}^j\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P_\varsigma P'_\varsigma}(i\varsigma)_{Q_\varsigma Q'_\varsigma}\cdot\cdot}^{2s-n}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}(s)\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}(s)$$
$$= (\tfrac{-i\varsigma}{\sqrt{2}})^{2s}2^n(i\varsigma)^{2s-n}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}(s)\overbrace{\sigma(\tfrac{1}{2})_{A_\varsigma A'_\varsigma}^i\sigma(\tfrac{1}{2})_{B_\varsigma B'_\varsigma}^j\cdot\cdot}^{n}\overbrace{\delta_{P_\varsigma P'_\varsigma}\delta_{Q_\varsigma Q'_\varsigma}\cdot\cdot}^{2s-n}\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}(s)$$
$$= (-i\varsigma)^n 2^{n-s}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{n}}(s,n) = \frac{(-i\varsigma)^n}{2^{s-n}}\frac{1}{n!}\sum_{k=0}^{[n/2]} c(s,n;n-2k)\Omega^{n-2k}(s) \qquad \square$$

Cor. 5.9.2.
$$\begin{cases} \Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{n} = \frac{(-i\varsigma)^n}{2^{s-n}}\sum_{k=0}^{[n/2]} c(s,n;n-2k)[\sigma(s)\cdot\hat{p}]^{n-2k} \\ \Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{n} = \frac{(-i\varsigma)^n}{2^{s-n}}\sum_{k=0}^{[n/2]} c(s,n;n-2k)[\sigma(s)\cdot\hat{\nabla}]^{n-2k} \end{cases}$$

Cor. 5.9.3.
$$\begin{cases} \Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{2s-n} = \frac{(-i\varsigma)^{2s-n}}{2^{n-s}}\sum_{k=0}^{[(2s-n)/2]} c(s,2s-n;2s-n-2k)[\sigma(s)\cdot\hat{p}]^{2s-n-2k} \\ \Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{n} = \frac{(-i\varsigma)^{2s-n}}{2^{n-s}}\sum_{k=0}^{[(2s-n)/2]} c(s,2s-n;2s-n-2k)[\sigma(s)\cdot\hat{\nabla}]^{2s-n-2k} \end{cases}$$

5.10 Expansion of operators $\Gamma^{abc\cdot\cdot}(s)\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot$ and $\Gamma^{abc\cdot\cdot}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot$

Pro. 5.10.1.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=\frac{i^{2s}}{2^s}\sum\limits_{n=0}^{2s}\sum\limits_{k=0}^{[n/2]}C_{2s}^n(-2\varsigma)^n c(s,n;n-2k)[\sigma(s)\cdot\hat{p}]^{n-2k}(s)\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}=\frac{i^{2s}}{2^s}\sum\limits_{n=0}^{2s}\sum\limits_{k=0}^{[n/2]}C_{2s}^n(-2\varsigma)^n c(s,n;n-2k)[\sigma(s)\cdot\hat{\nabla}]^{n-2k}(s)\end{cases}$$

Pro. 5.10.2.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=\frac{i^{2s}}{2^s}\sum\limits_{n=0}^{2s}\sum\limits_{k=0}^{[(2s-n)/2]}C_{2s}^n(-2\varsigma)^{2s-n}c(s,2s-n;2s-n-2k)[\sigma(s)\cdot\hat{p}]^{2s-n-2k}(s)\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}=\frac{i^{2s}}{2^s}\sum\limits_{n=0}^{2s}\sum\limits_{k=0}^{[(2s-n)/2]}C_{2s}^n(-2\varsigma)^{2s-n}c(s,2s-n;2s-n-2k)[\sigma(s)\cdot\hat{\nabla}]^{2s-n-2k}(s)\end{cases}$$

Pro. 5.10.3.
$$\begin{cases}\Gamma_+^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=\frac{(-i\varsigma)^{2s}}{2^{-s}}\sum\limits_{l=0}^{[s]}\sum\limits_{k=0}^{[s-l]}C_{2s}^{2l}2^{-2l}c(s,2s-2l;2s-2l-2k)[\sigma(s)\cdot\hat{p}]^{2s-2l-2k}\\ \Gamma_+^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}=\frac{(-i\varsigma)^{2s}}{2^{-s}}\sum\limits_{l=0}^{[s]}\sum\limits_{k=0}^{[s-l]}C_{2s}^{2l}2^{-2l}c(s,2s-2l;2s-2l-2k)[\sigma(s)\cdot\hat{\nabla}]^{2s-2l-2k}\end{cases}$$

Pro. 5.10.4.
$$\begin{cases}\Gamma_-^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=i\frac{(-i\varsigma)^{2s-1}}{2^{1-s}}\sum\limits_{l=0}^{[s-\frac{1}{2}]}\sum\limits_{k=0}^{[s-\frac{1}{2}-l]}2^{-2l}C_{2s}^{2l+1}c(s,2s-2l-1;2s-2l-1-2k)[\sigma(s)\cdot\hat{p}]^{2s-2l-1-2k}\\ \Gamma_-^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}=i\frac{(-i\varsigma)^{2s-1}}{2^{1-s}}\sum\limits_{l=0}^{[s-\frac{1}{2}]}\sum\limits_{k=0}^{[s-\frac{1}{2}-l]}2^{-2l}C_{2s}^{2l+1}c(s,2s-2l-1;2s-2l-1-2k)[\sigma(s)\cdot\hat{\nabla}]^{2s-2l-1-2k}\end{cases}$$

5.11 Linear algebraic method on expansion coefficients of $\Gamma^{abc\cdot\cdot}(s)\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot,\Gamma^{abc\cdot\cdot}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot$

Pro. 5.11.1. $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=\sum\limits_{n=0}^{2s}C_n[\sigma(s)\cdot\hat{p}]^n,\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}=\sum\limits_{n=0}^{2s}C_n[\sigma(s)\cdot\hat{\nabla}]^n$

Proof: $\lambda^+(\hat{p},h;s)\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}\lambda(\hat{p},h;s)=\lambda^+(\hat{p},h;s)\sum\limits_{n=0}^{2s}C_n(\varsigma)[\sigma(s)\cdot\hat{p}]^n(s)\lambda(\hat{p},h;s)$

$\Leftrightarrow\lambda^+(\hat{p},h;s)(i\sqrt{2})^{2s}\lambda(\hat{p},-s\varsigma)\lambda^+(\hat{p},-s\varsigma)\lambda(\hat{p},h;s)=\sum\limits_{n=0}^{2s}C_n(\varsigma)\lambda^+(\hat{p},h;s)[\sigma(s)\cdot\hat{p}]^n(s)\lambda(\hat{p},h;s)$

$\Leftrightarrow(i\sqrt{2})^{2s}\delta(-s\varsigma,h)=\sum\limits_{n=0}^{2s}C_n(\varsigma)h^n,h=-s\varsigma,\cdot\cdot,s\varsigma$

$\Leftrightarrow$

$$\begin{bmatrix}s^0&s^1&\cdots&s^{2s-1}&s^{2s}\\(s-1)^0&(s-1)^1&\cdots&(s-1)^{2s-1}&(s-1)^{2s}\\\cdots&\cdots&\cdots&\cdots&\cdots\\(1-s)^0&(1-s)^1&\cdots&(1-s)^{2s-1}&(1-s)^{2s}\\(-s)^0&(-s)^1&\cdots&(-s)^{2s-1}&(-s)^{2s}\end{bmatrix}\begin{bmatrix}C_0(-1)\\C_1(-1)\\\cdots\\C_{2s-1}(-1)\\C_{2s}(-1)\end{bmatrix}=(i\sqrt{2})^{2s}\begin{bmatrix}1\\0\\\cdots\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}s^0&s^1&\cdots&s^{2s-1}&s^{2s}\\(s-1)^0&(s-1)^1&\cdots&(s-1)^{2s-1}&(s-1)^{2s}\\\cdots&\cdots&\cdots&\cdots&\cdots\\(1-s)^0&(1-s)^1&\cdots&(1-s)^{2s-1}&(1-s)^{2s}\\(-s)^0&(-s)^1&\cdots&(-s)^{2s-1}&(-s)^{2s}\end{bmatrix}\begin{bmatrix}C_0(1)\\C_1(1)\\\cdots\\C_{2s-1}(1)\\C_{2s}(1)\end{bmatrix}=(i\sqrt{2})^{2s}\begin{bmatrix}0\\0\\\cdots\\0\\1\end{bmatrix}$$

$\Leftrightarrow$

$$\begin{bmatrix}C_0(-1)\\C_1(-1)\\\cdots\\C_{2s-1}(-1)\\C_{2s}(-1)\end{bmatrix}=(i\sqrt{2})^{2s}\begin{bmatrix}s^0&s^1&\cdots&s^{2s-1}&s^{2s}\\(s-1)^0&(s-1)^1&\cdots&(s-1)^{2s-1}&(s-1)^{2s}\\\cdots&\cdots&\cdots&\cdots&\cdots\\(1-s)^0&(1-s)^1&\cdots&(1-s)^{2s-1}&(1-s)^{2s}\\(-s)^0&(-s)^1&\cdots&(-s)^{2s-1}&(-s)^{2s}\end{bmatrix}^{-1}\begin{bmatrix}1\\0\\\cdots\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}C_0(1)\\C_1(1)\\\cdots\\C_{2s-1}(1)\\C_{2s}(1)\end{bmatrix}=(i\sqrt{2})^{2s}\begin{bmatrix}s^0&s^1&\cdots&s^{2s-1}&s^{2s}\\(s-1)^0&(s-1)^1&\cdots&(s-1)^{2s-1}&(s-1)^{2s}\\\cdots&\cdots&\cdots&\cdots&\cdots\\(1-s)^0&(1-s)^1&\cdots&(1-s)^{2s-1}&(1-s)^{2s}\\(-s)^0&(-s)^1&\cdots&(-s)^{2s-1}&(-s)^{2s}\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\\cdots\\0\\1\end{bmatrix}\qquad\square$$

Pro. 5.11.2.
$$\begin{cases}\Gamma_+^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=\sum\limits_{l=0}^{[s]}C_{2s-2l}[\sigma(s)\cdot\hat{p}]^{2s-2l}\\ \Gamma_+^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}=\sum\limits_{l=0}^{[s]}C_{2s-2l}[\sigma(s)\cdot\hat{\nabla}]^{2s-2l}\end{cases}$$

Pro. 5.11.3. $$\begin{cases}\Gamma_{-}^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=\sum\limits_{l=0}^{[s-\frac{1}{2}]}C_{2s-2l-1}[\sigma(s)\cdot\hat{p}]^{2s-2l-1}\\ \Gamma_{-}^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}=\sum\limits_{l=0}^{[s-\frac{1}{2}]}C_{2s-2l-1}[\sigma(s)\cdot\hat{\nabla}]^{2s-2l-1}\end{cases}$$

Thm. 5.11.1. $\sum\limits_{n=0}^{2s}k_n[\sigma(s)\cdot\hat{p}]^n=0\Leftrightarrow k_n=0$

Proof: $\sum\limits_{n=0}^{2s}k_n[\sigma(s)\cdot\hat{p}]^n=0(\Rightarrow\sum\limits_{n=0}^{2s}k_n[\sigma_z(s)]^n=0)$

$\Rightarrow\lambda^+(\hat{p},h;s)\sum\limits_{n=0}^{2s}k_n[\sigma(s)\cdot\hat{p}]^n\lambda(\hat{p},h;s)=0$

$\Leftrightarrow\sum\limits_{n=0}^{2s}k_nh^n=0,h=-s,\cdot\cdot,s$

$$\Leftrightarrow\begin{bmatrix}s^0 & s^1 & \cdots & s^{2s-1} & s^{2s}\\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s}\\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s}\end{bmatrix}\begin{bmatrix}k_0\\ k_1\\ \cdots\\ k_{2s-1}\\ k_{2s}\end{bmatrix}=0$$

$\Leftrightarrow k_n=0$ □

5.12 Relations between special composite constant invariant tensors

Cor. 5.12.1. $$\begin{cases}\Gamma_{k_\varsigma}^{\overbrace{a\tilde{a}c\tilde{c}\cdot\cdot}^{2n}}(n)\overbrace{\delta_{\tilde{a}\tilde{b}}\delta_{\tilde{c}\tilde{d}}\cdot\cdot}^{n}\Gamma_{k'_\varsigma}^{\overbrace{b\tilde{b}d\tilde{d}\cdot\cdot}^{2n}}(n)=(\frac{1}{2})^n\overbrace{\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab}\sigma_{\beta_\varsigma\beta'_\varsigma}^{cd}\cdot\cdot}^{n}\Gamma_{k_\varsigma}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}(n)\Gamma_{k'_\varsigma}^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}(n)\\ \Gamma^{k'_\varsigma}_{\underbrace{a\tilde{a}c\tilde{c}\cdot\cdot}_{2n}}(n)\overbrace{\delta^{\tilde{a}\tilde{b}}\delta^{\tilde{c}\tilde{d}}\cdot\cdot}^{n}\Gamma^{k_\varsigma}_{\underbrace{b\tilde{b}d\tilde{d}\cdot\cdot}_{2n}}(n)=(\frac{1}{2})^n\overbrace{\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}\sigma^{\beta'_\varsigma\beta_\varsigma}_{cd}\cdot\cdot}^{n}\Gamma^{k'_\varsigma}_{\underbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}_{n}}(n)\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}_{n}}(n)\end{cases}$$

Proof: $\Gamma_{k_\varsigma}^{\overbrace{a\tilde{a}c\tilde{c}\cdot\cdot}^{2n}}(n)\overbrace{\delta_{\tilde{a}\tilde{b}}\delta_{\tilde{c}\tilde{d}}\cdot\cdot}^{n}\Gamma_{k'_\varsigma}^{\overbrace{b\tilde{b}d\tilde{d}\cdot\cdot}^{2n}}(n)$

$=(\frac{i}{2})^n\underbrace{\sigma^{a\tilde{a}}_{\varsigma\alpha_\varsigma}\sigma^{c\tilde{c}}_{\varsigma\beta_\varsigma}\cdot\cdot}_{n}\Gamma_{k_\varsigma}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}(n)\overbrace{\delta_{\tilde{a}\tilde{b}}\delta_{\tilde{c}\tilde{d}}\cdot\cdot}^{n}(\frac{i}{2})^n\underbrace{\sigma^{b\tilde{b}}_{-\varsigma\alpha'_\varsigma}\sigma^{d\tilde{d}}_{-\varsigma\beta'_\varsigma}\cdot\cdot}_{n}\Gamma_{k'_\varsigma}^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}(n)$

$=(\frac{1}{2})^n\underbrace{\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\cdot\cdot}_{n}\Gamma_{k_\varsigma}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}(n)\Gamma_{k'_\varsigma}^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}(n)$ □

Cor. 5.12.2. $$\begin{cases}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdot\cdot}^{2n}}(n)=\underbrace{\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\cdot\cdot}_{n}\Gamma_{k_\varsigma}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}(n)\Gamma_{k'_\varsigma}^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}(n)=2^n\Gamma_{k_\varsigma}^{\overbrace{a\tilde{a}c\tilde{c}\cdot\cdot}^{2n}}(n)\overbrace{\delta_{\tilde{a}\tilde{b}}\delta_{\tilde{c}\tilde{d}}\cdot\cdot}^{n}\Gamma_{k'_\varsigma}^{\overbrace{b\tilde{b}d\tilde{d}\cdot\cdot}^{2n}}(n)\\ \Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2n}}(n)=\overbrace{\sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab}\sigma^{\beta'_\varsigma\beta_\varsigma}_{cd}\cdot\cdot}^{n}\Gamma^{k'_\varsigma}_{\underbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}_{n}}(n)\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}_{n}}(n)=2^n\Gamma^{k'_\varsigma}_{\underbrace{a\tilde{a}c\tilde{c}\cdot\cdot}_{2n}}(n)\overbrace{\delta^{\tilde{a}\tilde{b}}\delta^{\tilde{c}\tilde{d}}\cdot\cdot}^{n}\Gamma^{k_\varsigma}_{\underbrace{b\tilde{b}d\tilde{d}\cdot\cdot}_{2n}}(n)\end{cases}$$

Proof: $\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdot\cdot}^{2n}}(n)=(\frac{-i\varsigma}{\sqrt{2}})^{2n}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^d_{D_\varsigma D'_\varsigma}\cdot\cdot}^{2n}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s)\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2n}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2n}(\frac{i\varsigma}{\sqrt{2}})^n\underbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdot\cdot}_{n}\Gamma_{k_\varsigma}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}(n)(\frac{-i\varsigma}{\sqrt{2}})^n\underbrace{\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\beta'_\varsigma}\cdot\cdot}_{n}\Gamma_{k'_\varsigma}^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}(n)$

$=\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^c{}_{C_\varsigma C'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^d{}_{D_\varsigma D'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{C'_\varsigma D'_\varsigma}_{\alpha'_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}\cdot\cdot$

$\Gamma_{k_\varsigma}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}(n)\Gamma_{k'_\varsigma}^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}(n)$

$=\underbrace{\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\cdot\cdot}_{n}\Gamma_{k_\varsigma}^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}^{n}}(n)\Gamma_{k'_\varsigma}^{\overbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}^{n}}(n)$

$=2^n\Gamma_{k_\varsigma}^{\overbrace{a\tilde{a}c\tilde{c}\cdot\cdot}^{2n}}(n)\overbrace{\delta_{\tilde{a}\tilde{b}}\delta_{\tilde{c}\tilde{d}}\cdot\cdot}^{n}\Gamma_{k'_\varsigma}^{\overbrace{b\tilde{b}d\tilde{d}\cdot\cdot}^{2n}}(n)$ □

6 The most basic constant invariant tensors

The most basic constant invariant tensors:

$$\varepsilon_{AB},\varepsilon^{AB};\delta_{ab},\delta^{ab};\delta_{\alpha\beta},\delta^{\alpha\beta};\varepsilon_{\alpha\beta\gamma};(\sigma,-i)_a^{A'A};\sigma^{\alpha}{}_{A}{}^{B};N_{Al}^{k}(s),N_k^{Al}(s) \tag{3.8}$$

The above are the most basic constant invariant tensors. All constant invariant tensors in this chapter and the previous chapters can be derived from them. Therefore, as long as the covariance of the above constant invariant tensors is proved, the covariance of the constant invariant tensors derived from them naturally holds.

7 Generalized constant invariant tensors in 4 dimensional space-time

7.1 Generalized constant invariant tensors in 4 dimensional space-time

7.1.1 Introduction of Dirac type constant invariant tensor $\Gamma^{k_\varsigma}_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdot\cdot}(s;3),\Gamma_{k_\varsigma}^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdot\cdot}(s;3)$ in 4D

Def. 7.1.1.

$$\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdot\cdot}_{2s}}(s;3)=\frac{1}{(2s)!}\Gamma^{k_\varsigma}_{\underbrace{(\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdot\cdot)}_{2s}}(s;3)$$

$$\Gamma^{k_\varsigma}_{\underbrace{0_\varsigma\cdot\cdot0_\varsigma}_{l_0}\underbrace{1_\varsigma\cdot\cdot1_\varsigma}_{l_1}\underbrace{2_\varsigma\cdot\cdot2_\varsigma}_{l_2}\underbrace{3_\varsigma\cdot\cdot3_\varsigma}_{l_3}}(s;3)=\sqrt{\frac{l_0!l_1!l_2!l_3!}{(2s)!}}\delta\{k_\varsigma,\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C^{2l}_{2l+3-k}-C^{2l}_{2l+3-\lambda_{2l}})\},l_0+l_1+l_2+l_3=2s$$

Def. 7.1.2.

$$\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdot\cdot}^{2s}}(s;3)=\frac{1}{(2s)!}\Gamma_{k_\varsigma}^{\overbrace{(\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdot\cdot)}^{2s}}(s;3)$$

$$\Gamma_{k_\varsigma}^{\overbrace{0_\varsigma\cdot\cdot0_\varsigma}^{l_0}\overbrace{1_\varsigma\cdot\cdot1_\varsigma}^{l_1}\overbrace{2_\varsigma\cdot\cdot2_\varsigma}^{l_2}\overbrace{3_\varsigma\cdot\cdot3_\varsigma}^{l_3}}(s;3)=\sqrt{\frac{l_0!l_1!l_2!l_3!}{(2s)!}}\delta\{k_\varsigma,\sum_{l=0}^{[s]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C^{2l}_{2l+3-k}-C^{2l}_{2l+3-\lambda_{2l}})\},l_0+l_1+l_2+l_3=2s$$

7.1.2 Properties of Dirac type constant invariant tensors $\Gamma^{k_\varsigma}_{\lambda_\varsigma\mu_\varsigma}(1;3),\Gamma_{k_\varsigma}^{\lambda_\varsigma\mu_\varsigma}(1;3)$ in 4D

Def. 7.1.3.

$$\Gamma^{k_\varsigma}_{\underbrace{0_\varsigma\cdot\cdot0_\varsigma}_{l_0}\underbrace{1_\varsigma\cdot\cdot1_\varsigma}_{l_1}\underbrace{2_\varsigma\cdot\cdot2_\varsigma}_{l_2}\underbrace{3_\varsigma\cdot\cdot3_\varsigma}_{l_3}}(\tfrac{1}{2};3)=\delta\{k_\varsigma,\sum_{l=0}^{0}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C^{2l}_{2l+3-k}-C^{2l}_{2l+3-\lambda_{2l}})\},l_0+l_1+l_2+l_3=1$$
$$=\delta\{k_\varsigma,\lambda_1\},l_0+l_1+l_2+l_3=1$$

Def. 7.1.4.

$$\Gamma^{k_\varsigma}_{\lambda_\varsigma\mu_\varsigma}(1;3)=\Gamma^{k_\varsigma}_{\mu_\varsigma\lambda_\varsigma}(1;3)$$

$$\Gamma^{k_\varsigma}_{\underbrace{0_\varsigma\cdot\cdot0_\varsigma}_{l_0}\underbrace{1_\varsigma\cdot\cdot1_\varsigma}_{l_1}\underbrace{2_\varsigma\cdot\cdot2_\varsigma}_{l_2}\underbrace{3_\varsigma\cdot\cdot3_\varsigma}_{l_3}}(1;3)=\sqrt{\frac{l_0!l_1!l_2!l_3!}{2!}}\delta\{k_\varsigma,\sum_{l=0}^{[1]}(\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}}C^{2l}_{2l+3-k}-C^{2l}_{2l+3-\lambda_{2l}})\},l_0+l_1+l_2+l_3=2$$
$$=\sqrt{\frac{l_0!l_1!l_2!l_3!}{2!}}\delta\{k_\varsigma,(\lambda_1-\lambda_2)+(C^2_5-C^2_{5-\lambda_2})\},l_0+l_1+l_2+l_3=2$$

Cor. 7.1.1.

$$\Gamma^{k_\varsigma}_{0_\varsigma0_\varsigma}(1;3)=\sqrt{\tfrac{2!}{2!}}\delta\{k_\varsigma,0\},\Gamma^{k_\varsigma}_{0_\varsigma1_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,1\},\Gamma^{k_\varsigma}_{0_\varsigma2_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,2\},\Gamma^{k_\varsigma}_{0_\varsigma3_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,3\}$$
$$\Gamma^{k_\varsigma}_{1_\varsigma0_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,1\},\Gamma^{k_\varsigma}_{1_\varsigma1_\varsigma}(1;3)=\sqrt{\tfrac{2!}{2!}}\delta\{k_\varsigma,4\},\Gamma^{k_\varsigma}_{1_\varsigma2_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,5\},\Gamma^{k_\varsigma}_{1_\varsigma3_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,6\}$$
$$\Gamma^{k_\varsigma}_{2_\varsigma0_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,2\},\Gamma^{k_\varsigma}_{2_\varsigma1_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,5\},\Gamma^{k_\varsigma}_{2_\varsigma2_\varsigma}(1;3)=\sqrt{\tfrac{2!}{2!}}\delta\{k_\varsigma,7\},\Gamma^{k_\varsigma}_{2_\varsigma3_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,8\}$$
$$\Gamma^{k_\varsigma}_{3_\varsigma0_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,3\},\Gamma^{k_\varsigma}_{3_\varsigma1_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,6\},\Gamma^{k_\varsigma}_{3_\varsigma2_\varsigma}(1;3)=\sqrt{\tfrac{0!1!}{2!}}\delta\{k_\varsigma,8\},\Gamma^{k_\varsigma}_{3_\varsigma3_\varsigma}(1;3)=\sqrt{\tfrac{2!}{2!}}\delta\{k_\varsigma,9\}$$

Cor. 7.1.2.

$$\Gamma^{0_\varsigma}(1;3)=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\Gamma^{1_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\Gamma^{2_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix},\Gamma^{3_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}$$
$$\Gamma^{4_\varsigma}(1;3)=\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\Gamma^{5_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{bmatrix},\Gamma^{6_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{bmatrix}$$
$$\Gamma^{7_\varsigma}(1;3)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\Gamma^{8_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$
$$\Gamma^{9_\varsigma}(1;3)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

7.2 Properties I of Dirac type constant invariant tensors $N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(1;3), N^{\lambda_\varsigma l_\varsigma}_{k_\varsigma}(1;3)$ in 4D

Cor. 7.2.1.

$$N_{0_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, N_{1_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, N_{2_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, N_{3_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

Cor. 7.2.2.

$$\bar{N}_{0_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{N}_{1_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{N}_{2_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \bar{N}_{3_\varsigma}(1;3) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

7.3 Properties II of Dirac type constant invariant tensors $N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(1;3), N^{\lambda_\varsigma l_\varsigma}_{k_\varsigma}(1;3)$ in 4D

Lem. 7.3.1.

$$\begin{cases} \bar{N}_{0_\varsigma}(1;3)N_{0_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \bar{N}_{0_\varsigma}(1;3)N_{1_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \\ \bar{N}_{0_\varsigma}(1;3)N_{2_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}, \bar{N}_{0_\varsigma}(1;3)N_{3_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \bar{N}_{1_\varsigma}(1;3)N_{0_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \bar{N}_{1_\varsigma}(1;3)N_{1_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} \\ \bar{N}_{1_\varsigma}(1;3)N_{2_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}, \bar{N}_{1_\varsigma}(1;3)N_{3_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \bar{N}_{2_\varsigma}(1;3)N_{0_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \bar{N}_{2_\varsigma}(1;3)N_{1_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \\ \bar{N}_{2_\varsigma}(1;3)N_{2_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&1 \end{bmatrix}, \bar{N}_{2_\varsigma}(1;3)N_{3_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \bar{N}_{3_\varsigma}(1;3)N_{0_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \bar{N}_{3_\varsigma}(1;3)N_{1_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} \\ \bar{N}_{3_\varsigma}(1;3)N_{2_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}, \bar{N}_{3_\varsigma}(1;3)N_{3_\varsigma}(1;3) = \frac{1}{2}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&2 \end{bmatrix} \end{cases}$$

Thm. 7.3.1. $\bar{N}^{\lambda_\varsigma}(1;3)\gamma^a{}_{\lambda_\varsigma}{}^{\mu_\varsigma}N_{\mu_\varsigma}(1;3) = \frac{1}{2}\gamma^{Ta}, \bar{N}_{\mu_\varsigma}(1;3)\gamma^a{}_{\lambda_\varsigma}{}^{\mu_\varsigma}N^{\lambda_\varsigma}(1;3) = \frac{1}{2}\gamma^a$

Proof: $\bar{N}^{\lambda_\varsigma}(1;3)\gamma^x{}_{\lambda_\varsigma}{}^{\mu_\varsigma}N_{\mu_\varsigma}(1;3)$
$= -i\bar{N}^{0_\varsigma}(1;3)N_{3_\varsigma}(1;3) - i\bar{N}^{1_\varsigma}(1;3)N_{2_\varsigma}(1;3) + i\bar{N}^{2_\varsigma}(1;3)N_{1_\varsigma}(1;3) + i\bar{N}^{3_\varsigma}(1;3)N_{0_\varsigma}(1;3)$
$$= -\frac{i}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} - \frac{i}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} + \frac{i}{2}\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} + \frac{i}{2}\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0 \end{bmatrix}$$ □

Proof: $\bar{N}^{\lambda_\varsigma}(1;3)\gamma^y{}_{\lambda_\varsigma}{}^{\mu_\varsigma}N_{\mu_\varsigma}(1;3)$
$= -\bar{N}^{0_\varsigma}(1;3)N_{3_\varsigma}(1;3) + \bar{N}^{1_\varsigma}(1;3)N_{2_\varsigma}(1;3) + \bar{N}^{2_\varsigma}(1;3)N_{1_\varsigma}(1;3) - \bar{N}^{3_\varsigma}(1;3)N_{0_\varsigma}(1;3)$
$$= -\frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0 \end{bmatrix}$$ □

Proof: $\bar{N}^{\lambda_\varsigma}(1;3)\gamma^z{}_{\lambda_\varsigma}{}^{\mu_\varsigma}N_{\mu_\varsigma}(1;3)$
$= -i\bar{N}^{0_\varsigma}(1;3)N_{2_\varsigma}(1;3) + i\bar{N}^{1_\varsigma}(1;3)N_{3_\varsigma}(1;3) + i\bar{N}^{2_\varsigma}(1;3)N_{0_\varsigma}(1;3) - i\bar{N}^{3_\varsigma}(1;3)N_{1_\varsigma}(1;3)$
$$= -\frac{i}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix} + \frac{i}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix} + \frac{i}{2}\begin{bmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \frac{i}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0 \end{bmatrix}$$ □

Proof: $\bar{N}^{\lambda_\varsigma}(1;3)\gamma^4{}_{\lambda_\varsigma}{}^{\mu_\varsigma}N_{\mu_\varsigma}(1;3)$
$= \varsigma\bar{N}^{0_\varsigma}(1;3)N_{2_\varsigma}(1;3) + \varsigma\bar{N}^{1_\varsigma}(1;3)N_{3_\varsigma}(1;3) + \varsigma\bar{N}^{2_\varsigma}(1;3)N_{0_\varsigma}(1;3) + \varsigma\bar{N}^{3_\varsigma}(1;3)N_{1_\varsigma}(1;3)$
$$= \frac{\varsigma}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix} + \frac{\varsigma}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix} + \frac{\varsigma}{2}\begin{bmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} + \frac{\varsigma}{2}\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} = \frac{\varsigma}{2}\begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}$$ □

Proof: $\bar{N}^{\lambda_\varsigma}(1;3)\gamma^5{}_{\lambda_\varsigma}{}^{\mu_\varsigma}N_{\mu_\varsigma}(1;3)$
$= \varsigma\bar{N}^{0_\varsigma}(1;3)N_{0_\varsigma}(1;3) + \varsigma\bar{N}^{1_\varsigma}(1;3)N_{1_\varsigma}(1;3) + \varsigma\bar{N}^{2_\varsigma}(1;3)N_{2_\varsigma}(1;3) + \varsigma\bar{N}^{3_\varsigma}(1;3)N_{3_\varsigma}(1;3)$
$$= \frac{\varsigma}{2}\begin{bmatrix} 2&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} + \frac{\varsigma}{2}\begin{bmatrix} 1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} - \frac{\varsigma}{2}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&1 \end{bmatrix} - \frac{\varsigma}{2}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&2 \end{bmatrix} = \frac{\varsigma}{2}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1 \end{bmatrix}$$ □

7.4 Properties III of Dirac type constant invariant tensors $N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(1;3), N_{k_\varsigma}^{\lambda_\varsigma l_\varsigma}(1;3)$ in 4D

Cor. 7.4.1.

$$N_{3_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\\sqrt{2}&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&\sqrt{2}&0&0&0&0&0&0\end{bmatrix},N_{3_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&\sqrt{2}&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&\sqrt{2}&0&0&0\end{bmatrix}$$

$$N_{3_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&\sqrt{2}&0&0\\0&0&0&0&0&0&0&0&\sqrt{2}&0\end{bmatrix},N_{3_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&2\end{bmatrix}$$

Cor. 7.4.2.

$$N_{2_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\\sqrt{2}&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&\sqrt{2}&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},N_{2_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&\sqrt{2}&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\sqrt{2}&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

$$N_{2_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&2&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},N_{2_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&\sqrt{2}&0\\0&0&0&0&0&0&0&0&0&\sqrt{2}\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

Cor. 7.4.3.

$$N_{1_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\\sqrt{2}&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&\sqrt{2}&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},N_{1_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&2&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

$$N_{1_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\sqrt{2}&0&0&0&0\\0&0&0&0&0&0&0&\sqrt{2}&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},N_{1_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&\sqrt{2}&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&\sqrt{2}\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

Cor. 7.4.4.

$$N_{0_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}2&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},N_{0_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&\sqrt{2}&0&0&0&0&0&0&0&0\\0&0&0&0&\sqrt{2}&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

$$N_{0_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&\sqrt{2}&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&\sqrt{2}&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},N_{0_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3)=\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0&\sqrt{2}&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&\sqrt{2}\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

Cor. 7.4.5. $[\gamma_a(\varsigma),\gamma_5(\varsigma)]=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z]$

$$=\begin{bmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix},\begin{bmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{bmatrix},\varsigma\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix},\varsigma\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

Pro. 7.4.1. $N^{\lambda_\varsigma}(1;3)\gamma^x{}_{\lambda_\varsigma}{}^{\mu_\varsigma}\bar{N}_{\mu_\varsigma}(1;3)$

$=-iN^{0_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3)-iN^{1_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3)+iN^{2_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3)+iN^{3_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3)$

$$=-\frac{i}{\sqrt{2}}\begin{bmatrix}0&0&0&\sqrt{2}&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&\sqrt{2}\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}-\frac{i}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\sqrt{2}&0&0&0&0\\0&0&0&0&0&0&0&\sqrt{2}&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}+\frac{i}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&\sqrt{2}&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\sqrt{2}&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}+\frac{i}{\sqrt{2}}\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\\sqrt{2}&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&\sqrt{2}&0&0&0&0&0&0\end{bmatrix}$$

$$
= \frac{i}{\sqrt{2}}\begin{bmatrix}
0&0&0&-\sqrt{2}&0&0&0&0&0&0\\
0&0&-1&0&0&0&-1&0&0&0\\
0&1&0&0&0&0&0&0&-1&0\\
\sqrt{2}&0&0&0&0&0&0&0&0&-\sqrt{2}\\
0&0&0&0&0&-\sqrt{2}&0&0&0&0\\
0&0&0&0&\sqrt{2}&0&0&-\sqrt{2}&0&0\\
0&1&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&\sqrt{2}&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0\\
0&0&0&\sqrt{2}&0&0&0&0&0&0
\end{bmatrix}^2
= \frac{1}{2}\begin{bmatrix}
2&0&0&0&0&0&0&0&0&-2\\
0&2&0&0&0&0&0&0&-2&0\\
0&0&2&0&0&0&2&0&0&0\\
0&0&0&4&0&0&0&0&0&0\\
0&0&0&0&2&0&0&-2&0&0\\
0&0&0&0&0&4&0&0&0&0\\
0&0&2&0&0&0&2&0&0&0\\
0&0&0&0&-2&0&0&2&0&0\\
0&-2&0&0&0&0&0&0&2&0\\
-2&0&0&0&0&0&0&0&0&2
\end{bmatrix}
$$

Pro. 7.4.2. $N^{\lambda_\varsigma}(1;3)\gamma^{y}{}_{\lambda_\varsigma}{}^{\mu_\varsigma}\bar{N}_{\mu_\varsigma}(1;3)$
$= -N^{0_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3) + N^{1_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3) + N^{2_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3) - N^{3_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3)$

$$
= -\frac{1}{\sqrt{2}}\begin{bmatrix}
0&0&0&\sqrt{2}&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&\sqrt{2}\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{1}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\sqrt{2}&0&0&0&0\\
0&0&0&0&0&0&0&\sqrt{2}&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{1}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\sqrt{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\sqrt{2}&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
- \frac{1}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
\sqrt{2}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&\sqrt{2}&0&0&0&0&0&0
\end{bmatrix}
$$

$$
= \frac{1}{\sqrt{2}}\begin{bmatrix}
0&0&0&-\sqrt{2}&0&0&0&0&0&0\\
0&0&1&0&0&0&-1&0&0&0\\
0&1&0&0&0&0&0&0&-1&0\\
-\sqrt{2}&0&0&0&0&0&0&0&0&-\sqrt{2}\\
0&0&0&0&0&\sqrt{2}&0&0&0&0\\
0&0&0&0&\sqrt{2}&0&0&\sqrt{2}&0&0\\
0&-1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&\sqrt{2}&0&0&0&0\\
0&0&-1&0&0&0&1&0&0&0\\
0&0&0&-\sqrt{2}&0&0&0&0&0&0
\end{bmatrix}^2
= \frac{1}{2}\begin{bmatrix}
2&0&0&0&0&0&0&0&0&2\\
0&2&0&0&0&0&0&0&-2&0\\
0&0&2&0&0&0&-2&0&0&0\\
0&0&0&4&0&0&0&0&0&0\\
0&0&0&0&2&0&0&2&0&0\\
0&0&0&0&0&4&0&0&0&0\\
0&0&-2&0&0&0&2&0&0&0\\
0&0&0&0&2&0&0&2&0&0\\
0&-2&0&0&0&0&0&0&2&0\\
2&0&0&0&0&0&0&0&0&2
\end{bmatrix}
$$

Pro. 7.4.3. $N^{\lambda_\varsigma}(1;3)\gamma^{z}{}_{\lambda_\varsigma}{}^{\mu_\varsigma}\bar{N}_{\mu_\varsigma}(1;3)$
$= -iN^{0_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3) + iN^{1_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3) + iN^{2_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3) - iN^{3_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3)$

$$
= -\frac{i}{\sqrt{2}}\begin{bmatrix}
0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&\sqrt{2}&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{i}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\sqrt{2}&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&\sqrt{2}\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{i}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
\sqrt{2}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
- \frac{i}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\sqrt{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&\sqrt{2}&0&0&0
\end{bmatrix}
$$

$$
= \frac{i}{\sqrt{2}}\begin{bmatrix}
0&0&-\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&1&0&-1&0&0&0&0\\
\sqrt{2}&0&0&0&0&0&0&-\sqrt{2}&0&0\\
0&-1&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&\sqrt{2}&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&-\sqrt{2}&0&0&0&0&\sqrt{2}\\
0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&1&0&-1&0&0&0&0\\
0&0&0&0&0&0&-\sqrt{2}&0&0&0
\end{bmatrix}^2
= \frac{1}{2}\begin{bmatrix}
2&0&0&0&0&0&0&-2&0&0\\
0&2&0&0&0&0&0&0&2&0\\
0&0&4&0&0&0&0&0&0&0\\
0&0&0&2&0&-2&0&0&0&0\\
0&0&0&0&2&0&0&0&0&-2\\
0&0&0&-2&0&2&0&0&0&0\\
0&0&0&0&0&0&4&0&0&0\\
-2&0&0&0&0&0&0&2&0&0\\
0&2&0&0&0&0&0&0&2&0\\
0&0&0&0&-2&0&0&0&0&2
\end{bmatrix}
$$

Pro. 7.4.4. $N^{\lambda_\varsigma}(1;3)\gamma^{4}{}_{\lambda_\varsigma}{}^{\mu_\varsigma}\bar{N}_{\mu_\varsigma}(1;3)$
$= \varsigma N^{0_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3) + \varsigma N^{1_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3) + \varsigma N^{2_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3) + \varsigma N^{3_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3)$

$$
= \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&\sqrt{2}&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\sqrt{2}&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&\sqrt{2}\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
\sqrt{2}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\sqrt{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&\sqrt{2}&0&0&0
\end{bmatrix}
$$

$$
= \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
\sqrt{2}&0&0&0&0&0&0&\sqrt{2}&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&\sqrt{2}&0&0&0\\
0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&\sqrt{2}&0&0&0&0&\sqrt{2}\\
0&0&\sqrt{2}&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&\sqrt{2}&0&0&0
\end{bmatrix}^2
= \frac{1}{2}\begin{bmatrix}
2&0&0&0&0&0&0&2&0&0\\
0&2&0&0&0&0&0&0&2&0\\
0&0&4&0&0&0&0&0&0&0\\
0&0&0&2&0&2&0&0&0&0\\
0&0&0&0&2&0&0&0&0&2\\
0&0&0&2&0&2&0&0&0&0\\
0&0&0&0&0&0&4&0&0&0\\
2&0&0&0&0&0&0&2&0&0\\
0&2&0&0&0&0&0&0&2&0\\
0&0&0&0&2&0&0&0&0&2
\end{bmatrix}
$$

Pro. 7.4.5. $N^{\lambda_\varsigma}(1;3)\gamma^{5}{}_{\lambda_\varsigma}{}^{\mu_\varsigma}\bar{N}_{\mu_\varsigma}(1;3)$
$= \varsigma N^{0_\varsigma}(1;3)\bar{N}_{0_\varsigma}(1;3) + \varsigma N^{1_\varsigma}(1;3)\bar{N}_{1_\varsigma}(1;3) - \varsigma N^{2_\varsigma}(1;3)\bar{N}_{2_\varsigma}(1;3) - \varsigma N^{3_\varsigma}(1;3)\bar{N}_{3_\varsigma}(1;3)$

$$
= \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
2&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
+ \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
- \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
- \frac{\varsigma}{\sqrt{2}}\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&2
\end{bmatrix}
$$

$$= \frac{\varsigma}{\sqrt{2}} \begin{bmatrix} 2&0&0&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-2&0&0\\ 0&0&0&0&0&0&0&0&-2&0\\ 0&0&0&0&0&0&0&0&0&-2 \end{bmatrix}^2 = \frac{1}{2} \begin{bmatrix} 4&0&0&0&0&0&0&0&0&0\\ 0&4&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&4&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&4&0&0\\ 0&0&0&0&0&0&0&0&4&0\\ 0&0&0&0&0&0&0&0&0&4 \end{bmatrix}$$

Pro. 7.4.6.

$$\frac{1}{2} \begin{bmatrix} 2&0&0&0&0&0&0&0&0&-2\\ 0&2&0&0&0&0&0&0&-2&0\\ 0&0&2&0&0&0&2&0&0&0\\ 0&0&0&4&0&0&0&0&0&0\\ 0&0&0&0&2&0&0&-2&0&0\\ 0&0&0&0&0&4&0&0&0&0\\ 0&0&2&0&0&0&2&0&0&0\\ 0&0&0&0&-2&0&0&2&0&0\\ 0&-2&0&0&0&0&0&0&2&0\\ -2&0&0&0&0&0&0&0&0&2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2&0&0&0&0&0&0&0&0&2\\ 0&2&0&0&0&0&0&0&-2&0\\ 0&0&2&0&0&0&-2&0&0&0\\ 0&0&0&4&0&0&0&0&0&0\\ 0&0&0&0&2&0&0&2&0&0\\ 0&0&0&0&0&4&0&0&0&0\\ 0&0&-2&0&0&0&2&0&0&0\\ 0&0&0&0&2&0&0&2&0&0\\ 0&-2&0&0&0&0&0&0&2&0\\ 2&0&0&0&0&0&0&0&0&2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2&0&0&0&0&0&0&-2&0&0\\ 0&2&0&0&0&0&0&0&2&0\\ 0&0&4&0&0&0&0&0&0&0\\ 0&0&0&2&0&-2&0&0&0&0\\ 0&0&0&0&2&0&0&0&0&-2\\ 0&0&0&-2&0&2&0&0&0&0\\ 0&0&0&0&0&0&4&0&0&0\\ -2&0&0&0&0&0&0&2&0&0\\ 0&2&0&0&0&0&0&0&2&0\\ 0&0&0&0&-2&0&0&0&0&2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2&0&0&0&0&0&0&2&0&0\\ 0&2&0&0&0&0&0&0&2&0\\ 0&0&4&0&0&0&0&0&0&0\\ 0&0&0&2&0&2&0&0&0&0\\ 0&0&0&0&2&0&0&0&0&2\\ 0&0&0&2&0&2&0&0&0&0\\ 0&0&0&0&0&0&4&0&0&0\\ 2&0&0&0&0&0&0&2&0&0\\ 0&2&0&0&0&0&0&0&2&0\\ 0&0&0&0&2&0&0&0&0&2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 8&0&0&0&0&0&0&0&0&0\\ 0&8&0&0&0&0&0&0&0&0\\ 0&0&12&0&0&0&0&0&0&0\\ 0&0&0&12&0&0&0&0&0&0\\ 0&0&0&0&8&0&0&0&0&0\\ 0&0&0&0&0&12&0&0&0&0\\ 0&0&0&0&0&0&12&0&0&0\\ 0&0&0&0&0&0&0&8&0&0\\ 0&0&0&0&0&0&0&0&8&0\\ 0&0&0&0&0&0&0&0&0&8 \end{bmatrix}$$

Cor. 7.4.6. $[\gamma_a(\varsigma)(1;3), \gamma_5(\varsigma)(1;3)]^2 = 6$

Cor. 7.4.7.

$$\frac{i}{\sqrt{2}} \begin{bmatrix} 0&0&0&-\sqrt{2}&0&0&0&0&0&0\\ 0&0&-1&0&0&0&-1&0&0&0\\ 0&1&0&0&0&0&0&0&-1&0\\ \sqrt{2}&0&0&0&0&0&0&0&0&-\sqrt{2}\\ 0&0&0&0&0&-\sqrt{2}&0&0&0&0\\ 0&0&0&0&\sqrt{2}&0&0&-\sqrt{2}&0&0\\ 0&1&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&\sqrt{2}&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0\\ 0&0&0&\sqrt{2}&0&0&0&0&0&0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0&0&0&-\sqrt{2}&0&0&0&0&0&0\\ 0&0&1&0&0&0&-1&0&0&0\\ 0&1&0&0&0&0&0&0&-1&0\\ -\sqrt{2}&0&0&0&0&0&0&0&0&-\sqrt{2}\\ 0&0&0&0&0&\sqrt{2}&0&0&0&0\\ 0&0&0&0&\sqrt{2}&0&0&\sqrt{2}&0&0\\ 0&-1&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&\sqrt{2}&0&0&0&0\\ 0&0&-1&0&0&0&1&0&0&0\\ 0&0&0&-\sqrt{2}&0&0&0&0&0&0 \end{bmatrix}$$

$$= \frac{i}{2} \begin{bmatrix} 2&0&0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&-2&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-2&0&0&-2&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&-2&0&0&0\\ 0&0&0&0&2&0&0&2&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ -2&0&0&0&0&0&0&0&0&-2 \end{bmatrix}$$

Cor. 7.4.8. $[\gamma_x(1;3), \gamma_y(1;3)] = i \begin{bmatrix} 2&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-2&0&0&0\\ 0&0&0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-2 \end{bmatrix}$, $\{\gamma_x(1;3), \gamma_y(1;3)\} = i \begin{bmatrix} 0&0&0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-2&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-2&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ -2&0&0&0&0&0&0&0&0&0 \end{bmatrix}$

Chapter4 Preliminary Study on High Dimensional Constant Invariant Tensor

Self comment: Based on the previous chapters, this chapter comprehensively and systematically develops constant invariant tensors towards higher order and even infinite order, high and low dimensional space-time. Various constant invariant tensors in high and low dimensional space-time are obtained, which provides a powerful mathematical tool for studying particle physics in high and low dimensional space-time.

1 Lorentz group representation in n=N+1 dimensional space-time

1.1 Recursive representation Dirac matrices in n=N+1-dimensional space-time

Def. 1.1.1. $\begin{cases}\gamma_a(1)=(1)\\ \gamma_1(1)=1\end{cases}$

Def. 1.1.2. $\begin{cases}\gamma_a(2):=(\gamma_a(1)\otimes\sigma_x,1\otimes\sigma_y)=(\sigma_x,\sigma_y)\\ \Gamma^a(2):=[\gamma_a(1),i\varsigma]=(1,i\varsigma)\\ \gamma_1(2)\gamma_2(2)=i\sigma_z\end{cases}$ $\begin{cases}C_1(2):=\gamma_1(2)=\sigma_x,C_2(2):=\gamma_2(2)=\sigma_y\\ C_1^+(2)\gamma_a(2)C_1(2)=\gamma_a^T(2),C_1^T(2)=C_1(2)\\ C_2^+(2)\gamma_a(2)C_2(2)=-\gamma_a^T(2),C_2^T(2)=-C_2(2)\end{cases}$

Def. 1.1.3. $\begin{cases}\gamma_a(3)=[\gamma_a(2),1\otimes\sigma_z]=(\sigma_x,\sigma_y,\sigma_z)\\ \gamma_1(3)\cdots\gamma_3(3)=i\end{cases}$ $\begin{cases}C(3):=\gamma_2(3)=\sigma_y,C(3)=C_2(2)\\ C^+(3)\gamma_a(3)C(3)=-\gamma_a^T(3),C^T(3)=-C(3)\end{cases}$

Def. 1.1.4. $\begin{cases}\gamma_a(4)=[\gamma_a(3)\otimes\sigma_x,I\otimes\sigma_y]=(\sigma\otimes\sigma_x,I\otimes\sigma_y)\\ \Gamma^a(4)=[\gamma_a(3),i\varsigma]\\ \gamma_1(4)\cdots\gamma_4(4)=-I\otimes\sigma_z\end{cases}$ $\begin{cases}C_1(4):=\gamma_1(4)\gamma_3(4)=-i\sigma_y\otimes I\\ C_2(4):=\gamma_2(4)\gamma_4(4)=i\sigma_y\otimes\sigma_z\\ C_1^+(4)\gamma_a(4)C_1(4)=-\gamma_a^T(4),C_1^T(4)=-C_1(4)\\ C_2^+(4)\gamma_a(4)C_2(4)=\gamma_a^T(4),C_2^T(4)=-C_2(4)\end{cases}$

Def. 1.1.5. $\begin{cases}\gamma_a(5)=[\gamma_a(4),I\otimes\sigma_z]\\ \gamma_1(5)\cdots\gamma_5(5)=-1\end{cases}$ $\begin{cases}C(5):=\gamma_2(4)\gamma_4(4)=i\sigma_y\otimes\sigma_z,C(5)=C_2(4)\\ C^+(5)\gamma_a(5)C(5)=\gamma_a^T(5),C^T(5)=-C(5)\end{cases}$

Def. 1.1.6. $\begin{cases}\gamma_a(6)=[\gamma_a(5)\otimes\sigma_x,I_4\otimes\sigma_y]\\ \Gamma^a(6)=[\gamma_a(5),i\varsigma]\\ \gamma_1(6)\cdots\gamma_6(6)=-iI_4\otimes\sigma_z\end{cases}$ $\begin{cases}C_1(6):=\gamma_1(6)\gamma_3(6)\gamma_5(6)=-i\sigma_y\otimes\sigma_z\otimes\sigma_x\\ C_2(6):=\gamma_2(6)\gamma_4(6)\gamma_6(6)=i\sigma_y\otimes\sigma_z\otimes\sigma_y\\ C_1^+(6)\gamma_a(6)C_1(6)=\gamma_a^T(6),C_1^T(6)=-C_1(6)\\ C_2^+(6)\gamma_a(6)C_2(6)=-\gamma_a^T(6),C_2^T(6)=C_2(6)\end{cases}$

Def. 1.1.7. $\begin{cases}\gamma_a(7)=[\gamma_a(6),I_4\otimes\sigma_z]\\ \gamma_1(7)\cdots\gamma_7(7)=-i\end{cases}$ $\begin{cases}C(7):=\gamma_2(7)\gamma_4(7)\gamma_6(7)=i\sigma_y\otimes\sigma_z\otimes\sigma_y,C(7)=C_2(6)\\ C^+(7)\gamma_a(7)C(7)=-\gamma_a^T(7),C^T(7)=C(7)\end{cases}$

Def. 1.1.8. $\begin{cases}\gamma_a(8)=[\gamma_a(7)\otimes\sigma_x,I_8\otimes\sigma_y]\\ \Gamma^a(8)=[\gamma_a(7),i\varsigma]\\ \gamma_1(8)\cdots\gamma_8(8)=I_8\otimes\sigma_z\end{cases}$ $\begin{cases}C_1(8):=\gamma_1(8)\gamma_3(8)\gamma_5(8)\gamma_7(8)=-\sigma_y\otimes\sigma_z\otimes\sigma_y\otimes I\\ C_2(8):=\gamma_2(8)\gamma_4(8)\gamma_6(8)\gamma_8(8)=-\sigma_y\otimes\sigma_z\otimes\sigma_y\otimes\sigma_z\\ C_1^+(8)\gamma_a(8)C_1(8)=-\gamma_a^T(8),C_1^T(8)=C_1(8)\\ C_2^+(8)\gamma_a(8)C_2(8)=\gamma_a^T(8),C_2^T(8)=C_2(8)\end{cases}$

Def. 1.1.9. $\begin{cases}\gamma_a(9)=[\gamma_a(8),I_8\otimes\sigma_z]\\ \gamma_1(9)\cdots\gamma_9(9)=1\end{cases}$ $\begin{cases}C(9):=\gamma_2(8)\gamma_4(8)\gamma_6(8)\gamma_8(8)=-\sigma_y\otimes\sigma_z\otimes\sigma_y\otimes\sigma_z,C(9)=C_2(8)\\ C^+(9)\gamma_a(9)C(9)=\gamma_a^T(9),C^T(9)=C(9)\end{cases}$

Def. 1.1.10. $\begin{cases}\gamma_a(10)=[\gamma_a(9)\otimes\sigma_x,I_{16}\otimes\sigma_y]\\ \Gamma^a(10)=[\gamma_a(9),i\varsigma]\\ \gamma_1(10)\cdots\gamma_{10}(10)=iI_{16}\otimes\sigma_z\end{cases}$ $\begin{cases}C_1(10):=\gamma_1(10)\gamma_3(10)\gamma_5(10)\gamma_7(10)\gamma_9(10)\\ =-\sigma_y\otimes\sigma_z\otimes\sigma_y\otimes\sigma_z\otimes\sigma_x\\ C_2(10):=\gamma_2(10)\gamma_4(10)\gamma_6(10)\gamma_8(10)\gamma_{10}(10)\\ =-\sigma_y\otimes\sigma_z\otimes\sigma_y\otimes\sigma_z\otimes\sigma_y\\ C_2^+(10)\gamma_a(10)C_2(10)=-\gamma_a^T(10),C_2^T(10)=-C_2(10)\\ C_1^+(10)\gamma_a(10)C_1(10)=\gamma_a^T(10),C_1^T(10)=C_1(10)\end{cases}$

Def. 1.1.11. $\begin{cases}\gamma_a(11)=[\gamma_a(10),I_{16}\otimes\sigma_z]\\ \gamma_1(11)\cdots\gamma_{11}(11)=i\end{cases}$ $\begin{cases}C(11):=\gamma_2(11)\gamma_4(11)\gamma_6(11)\gamma_8(11)\gamma_{10}(11)\\ =-\sigma_y\otimes\sigma_z\otimes\sigma_y\otimes\sigma_z\otimes\sigma_y, C(11)=C_2(10)\\ C^+(11)\gamma_a(10)C(11)=-\gamma_a^T(11), C^T(11)=-C(11)\end{cases}$

1.2 Dirac matrix, minimum spinor and real form in n=N+1 dimensional space-time

Def. 1.2.1. $\begin{cases}\gamma_a(2)=(\sigma_x,\sigma_y), 2^1\times 2^1\\ \Gamma^a(2)=(1,i\varsigma), 2^0\times 2^0\\ \gamma_1(2)\gamma_2(2)=i\sigma_z\end{cases}$

Def. 1.2.2. $\begin{cases}\gamma_a(3)=(\sigma_x,\sigma_y,\sigma_z)\to(\sigma_z,\sigma_x,\sigma_y), 2^1\times 2^1\\ \gamma_1(3)\cdots\gamma_3(3)=i\end{cases}$

Def. 1.2.3. $\begin{cases}\gamma_a(4)=(\sigma\otimes\sigma_x, I\otimes\sigma_y)\to(\sigma_+\sigma_{-x},\sigma_{-y}), 2^2\times 2^2\\ \Gamma^a(4)=[\gamma_a(3),i\varsigma]\to Null, 2^1\times 2^1\\ \gamma_1(4)\cdots\gamma_4(4)=-I\otimes\sigma_z\end{cases}$

Def. 1.2.4. $\begin{cases}\gamma_a(5)=[\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z]\to Null, 2^2\times 2^2\\ \gamma_1(5)\cdots\gamma_5(5)=-1\end{cases}$

Def. 1.2.5. $\begin{cases}\gamma_a(6)=[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y], 2^3\times 2^3\\ \Gamma^a(6)=[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z), i\varsigma], 2^2\times 2^2\\ \gamma_1(6)\cdots\gamma_6(6)=-iI_4\otimes\sigma_z\end{cases}$

Def. 1.2.6. $\begin{cases}\gamma_a(7)=[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y, I_4\otimes\sigma_z], 2^3\times 2^3\\ \gamma_1(7)\cdots\gamma_7(7)=-i\end{cases}$

Def. 1.2.7. $\begin{cases}\gamma_a(8)=[[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y, I_4\otimes\sigma_z]\otimes\sigma_x, I_8\otimes\sigma_y], 2^4\times 2^4\\ \Gamma^a(8)=[[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y, I_4\otimes\sigma_z], i\varsigma]\to Null, 2^3\times 2^3\\ \gamma_1(8)\cdots\gamma_8(8)=I_8\otimes\sigma_z\end{cases}$

Def. 1.2.8.
$$\begin{cases}\gamma_a(9)=[[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y, I_4\otimes\sigma_z]\otimes\sigma_x, I_8\otimes\sigma_y, I_8\otimes\sigma_z]\to Null, 2^4\times 2^4\\ \gamma_1(9)\cdots\gamma_9(9)=1\\ \gamma_s^a(9)=S\gamma^a(9)S^+, S=[\sigma_z\otimes\sigma_y\otimes I\otimes\sigma_y][S_{ex}S_{em}(-1)\otimes I_4][I\otimes S_{em}(-1)\otimes S_c(\frac{1}{2})]\\ =-\{[(\sigma_x\otimes I,\sigma_y\otimes\sigma_y,\sigma_z\otimes I)\otimes\sigma_y,\sigma_y\otimes\sigma_x\otimes I, I\otimes\sigma_y\otimes\sigma_z, I\otimes\sigma_y\otimes\sigma_x,\sigma_y\otimes\sigma_z\otimes I]\otimes\sigma_y\\ , I\otimes I\otimes I\otimes\sigma_z, I\otimes I\otimes I\otimes\sigma_x\}\in R\end{cases}$$

Def. 1.2.9.
$$\begin{cases}\gamma_a(10)=[[[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y, I_4\otimes\sigma_z]\otimes\sigma_x, I_8\otimes\sigma_y, I_8\otimes\sigma_z]\otimes\sigma_x, I_{16}\otimes\sigma_y], 2^5\times 2^5\\ \Gamma^a(10)=[[[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y, I_4\otimes\sigma_z]\otimes\sigma_x, I_8\otimes\sigma_y, I_8\otimes\sigma_z], i\varsigma], 2^4\times 2^4\\ \gamma_1(10)\cdots\gamma_{10}(10)=iI_{16}\otimes\sigma_z\\ \Gamma_s^a(10)=[\gamma_s(9), i\varsigma], \gamma_s^a(10)=[\gamma_s(9)\otimes\sigma_x, I_{16}\otimes\sigma_y]\end{cases}$$

Def. 1.2.10.
$$\begin{cases}\gamma_a(11)=[[[(\sigma\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z)\otimes\sigma_x, I_4\otimes\sigma_y, I_4\otimes\sigma_z]\otimes\sigma_x, I_8\otimes\sigma_y, I_8\otimes\sigma_z]\otimes\sigma_x, I_{16}\otimes\sigma_y, I_{16}\otimes\sigma_z]\\ \gamma_1(11)\cdots\gamma_{11}(11)=i\\ \gamma_s^a(11)=[\Gamma_s^i(10)\otimes\sigma_x, I_{16}\otimes\sigma_z, I_{16}\otimes\sigma_y]\end{cases}$$

1.3 Concise construction of Dirac matrix in n=N+1 dimensional space-time

Def. 1.3.1.
$$\begin{cases}\gamma_{n+1}(n):=i^{-[n/2]}\gamma_1(n)\cdots\gamma_n(n)\\ \gamma_a(4)=[\gamma_a(2)\otimes I,\gamma_3(2)\otimes\gamma_a(2)]\\ \gamma_a(5)=[\gamma_a(2)\otimes I,\gamma_3(2)\otimes\gamma_a(3)]=[\gamma_a(2)\otimes I,\gamma_3(2)\otimes\gamma_a(2),\gamma_3(2)\otimes\gamma_3(2)]\\ \gamma_a(6)=[\gamma_a(2)\otimes I_4,\gamma_3(2)\otimes\gamma_a(4)]\\ \gamma_a(7)=[\gamma_a(2)\otimes I_4,\gamma_3(2)\otimes\gamma_a(5)]=[\gamma_a(2)\otimes I_4,\gamma_3(2)\otimes\gamma_a(4),\gamma_3(2)\otimes\gamma_5(4)]\\ \gamma_a(8)=[\gamma_a(4)\otimes I_4,\gamma_5(4)\otimes\gamma_a(4)]\\ \gamma_a(9)=[\gamma_a(4)\otimes I_4,\gamma_5(4)\otimes\gamma_a(5)]=[\gamma_a(4)\otimes I_4,\gamma_5(4)\otimes\gamma_a(4),\gamma_5(4)\otimes\gamma_5(4)]\\ \gamma_a(10)=[\gamma_a(4)\otimes I_8,\gamma_5(4)\otimes\gamma_a(6)]\\ \gamma_a(11)=[\gamma_a(4)\otimes I_8,\gamma_5(4)\otimes\gamma_a(6),\gamma_5(4)\otimes\gamma_7(6)]\end{cases}$$

Def. 1.3.2. $\begin{cases}\gamma_s^a(10)=[\gamma_s^a(9)\otimes\sigma_x, I_{16}\otimes\sigma_y]\\ \gamma_s^a(11)=[\gamma_s^a(9)\otimes\sigma_x, I_{16}\otimes\sigma_z, I_{16}\otimes\sigma_y]\end{cases}$

1.4 Recursive relations of Dirac matrix in n=N+1 dimensional space-time

Def. 1.4.1. $\begin{cases}\gamma_a(2n)=[\gamma_a(2n-1)\otimes\sigma_x, I_{2^n}\otimes\sigma_y]\\ \Gamma^a(2n)=[\gamma_a(2n-1), i\varsigma]\\ \gamma_1(2n)\cdots\gamma_{2n}(2n)=i^n I_{2^n}\otimes\sigma_z=i^n\gamma_{2n+1}(2n+1)\end{cases}$ $\begin{cases}C_1(2n):=\gamma_1(2n)\gamma_3(2n)\cdots\gamma_{2n-1}(2n)\\ C_2(2n):=\gamma_2(2n)\gamma_4(2n)\cdots\gamma_{2n}(2n)\\ C_r^+(2n)\gamma_a(2n)C_r(2n)=(-1)^{n+r}\gamma_a^*(2n)\end{cases}$

Def. 1.4.2. $\begin{cases}\gamma_a(2n+1)=[\gamma_a(2n), I_{2^n}\otimes\sigma_z]\\ \gamma_1(2n+1)\cdots\gamma_{2n+1}(2n+1)=i^n\end{cases}$ $\begin{cases}C(2n+1)=C_2(2n)\\ C^+(2n+1)\gamma_a(2n+1)C(2n+1)=(-1)^n\gamma_a^*(2n+1)\end{cases}$

1.5 Spin tensors in n=N+1 dimensional space-time

Def. 1.5.1. $\begin{cases}S_{ab}(\nu;2n):=\begin{bmatrix}S_{ij}(e;2n-1) & -\frac{1}{2}\vec{\gamma}(2n-1)\\ \frac{1}{2}\vec{\gamma}(2n-1) & 0\end{bmatrix}=-\frac{i}{4}[\gamma(2n-1),i]_{[a}[\gamma(2n-1),-i]_{b]}\\ S_{ab}(\bar{\nu};2n):=\begin{bmatrix}S_{ij}(e;2n-1) & \frac{1}{2}\vec{\gamma}(2n-1)\\ -\frac{1}{2}\vec{\gamma}(2n-1) & 0\end{bmatrix}=-\frac{i}{4}[\gamma(2n-1),-i]_{[a}[\gamma(2n-1),i]_{b]}\end{cases}$

Cor. 1.5.1. $S_{ab}(e;2n)=S_{ab}(\nu;2n)\oplus S_{ab}(\bar{\nu};2n)=-\frac{i}{4}[\gamma_a(2n),\gamma_b(2n)]$

Cor. 1.5.2. $S_{ab}(\varsigma;2n)=-\frac{i}{4}[\gamma(2n-1),i\varsigma]_{[a}[\gamma(2n-1),-i\varsigma]_{b]}\Leftrightarrow S_{ab}(\varsigma;2n):=\begin{bmatrix}S_{ij}(e;2n-1) & -\frac{\varsigma}{2}\vec{\gamma}(2n-1)\\ \frac{\varsigma}{2}\vec{\gamma}(2n-1) & 0\end{bmatrix}$

Cor. 1.5.3. $S_{ab}(e;2n+1)=-\frac{i}{4}[\gamma_a(2n+1),\gamma_b(2n+1)]=-\frac{i}{4}[i\varsigma\gamma(2n)\gamma_0(2n),-i\varsigma]_{[a}[i\varsigma\gamma(2n)\gamma_0(2n),i\varsigma]_{b]}$

Cor. 1.5.4. $\frac{i}{2}\vartheta^{ab}S_{ab}(e;2n+1)=\frac{i}{2}\vartheta^{ab}[\gamma_a(2n+1),\gamma_b(2n+1)]$

Cor. 1.5.5. $\frac{i}{2}\vartheta^{ab}S_{ab}(e;2n)=\frac{i}{2}\vartheta^{ab}S_{ab}(\nu;2n)\oplus\frac{i}{2}\vartheta^{ab}S_{ab}(\bar{\nu};2n)$

Proof: $\frac{i}{2}\vartheta^{ab}S_{ab}(e;2n)=\frac{1}{8}\vartheta^{ab}[\gamma_a(2n),\gamma_b(2n)]=\frac{1}{4}\vartheta^{i<j}[\gamma_i(2n),\gamma_j(2n)]+\frac{1}{4}\vartheta^{i\pi}[\gamma_i(2n),\gamma_\pi(2n)]$
$=\frac{1}{4}\vartheta^{i<j}[\gamma_i(2n-1),\gamma_j(2n-1)]\otimes I-\frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1)\otimes\sigma_z$
$=i\vartheta^{i<j}S_{ij}(e;2n-1)\otimes I-\frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1)\otimes\sigma_z$
$=[i\vartheta^{i<j}S_{ij}(e;2n-1)-\frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1)]\oplus[i\vartheta^{i<j}S_{ij}(e;2n-1)+\frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1)]$
$=[i\vartheta^{i<j}S_{ij}(e;2n-1)+\epsilon\cdot\frac{1}{2}\gamma(2n-1)]\oplus[i\vartheta^{i<j}S_{ij}(e;2n-1)-\epsilon\cdot\frac{1}{2}\gamma(2n-1)]$
$=\frac{i}{2}\vartheta^{ab}S_{ab}(\nu;2n)\oplus\frac{i}{2}\vartheta^{ab}S_{ab}(\bar{\nu};2n)$ □

1.6 Lorentz group representation in n=N+1 dimensional space-time

Cor. 1.6.1. $\{\gamma_a,\gamma_b\}=2g_{ab}\Rightarrow i[S_{ab},S_{cd}]=g_{ad}S_{bc}-g_{ac}S_{bd}+g_{bc}S_{ad}-g_{bd}S_{ac}; S_{ab}:=-\frac{i}{4}[\gamma_a,\gamma_b]$

Proof: $i[S_{ab},S_{cd}]$
$=-\frac{i}{16}[[\gamma_a,\gamma_b],[\gamma_c,\gamma_d]]$
$=-\frac{i}{16}\{[[\gamma_a,\gamma_b],\gamma_c\gamma_d]-[[\gamma_a,\gamma_b],\gamma_d\gamma_c]\}$
$=-\frac{i}{16}\{[[\gamma_a,\gamma_b],\gamma_c]\gamma_d+\gamma_c[[\gamma_a,\gamma_b],\gamma_d]-[[\gamma_a,\gamma_b],\gamma_d]\gamma_c-\gamma_d[[\gamma_a,\gamma_b],\gamma_c]\}$
$=-\frac{i}{16}\{-\{\{\gamma_c,\gamma_a\},\gamma_b\}\gamma_d+\{\gamma_a,\{\gamma_c,\gamma_b\}\}\gamma_d-\gamma_c\{\{\gamma_d,\gamma_a\},\gamma_b\}+\gamma_c\{\gamma_a,\{\gamma_d,\gamma_b\}\}$
$\quad+\{\{\gamma_d,\gamma_a\},\gamma_b\}\gamma_c-\{\gamma_a,\{\gamma_d,\gamma_b\}\}\gamma_c+\gamma_d\{\{\gamma_c,\gamma_a\},\gamma_b\}-\gamma_d\{\gamma_a,\{\gamma_c,\gamma_b\}\}\}$
$=-\frac{i}{16}\{-4\delta_{ca}\gamma_b\gamma_d+4\delta_{cb}\gamma_a\gamma_d-4\delta_{da}\gamma_c\gamma_b+4\delta_{db}\gamma_c\gamma_a+4\delta_{da}\gamma_b\gamma_c-4\delta_{db}\gamma_a\gamma_c+4\delta_{ca}\gamma_d\gamma_b-4\delta_{cb}\gamma_d\gamma_a\}$
$=-\frac{i}{4}\{\delta_{da}[\gamma_b,\gamma_c]-\delta_{db}[\gamma_a,\gamma_c]-\delta_{ca}[\gamma_b,\gamma_d]+\delta_{cb}[\gamma_a,\gamma_d]\}$
$=g_{ad}S_{bc}-g_{ac}S_{bd}+g_{bc}S_{ad}-g_{bd}S_{ac}$ □

Cor. 1.6.2. $\begin{cases}i[S_{ab}(\nu;2n),S_{cd}(\nu;2n)]=g_{ad}S_{bc}(\nu;2n)-g_{ac}S_{bd}(\nu;2n)+g_{bc}S_{ad}(\nu;2n)-g_{bd}S_{ac}(\nu;2n)\\ i[S_{ab}(\bar{\nu};2n),S_{cd}(\bar{\nu};2n)]=g_{ad}S_{bc}(\bar{\nu};2n)-g_{ac}S_{bd}(\bar{\nu};2n)+g_{bc}S_{ad}(\bar{\nu};2n)-g_{bd}S_{ac}(\bar{\nu};2n)\\ i[S_{ab}(\varsigma;2n),S_{cd}(\varsigma;2n)]=g_{ad}S_{bc}(\varsigma;2n)-g_{ac}S_{bd}(\varsigma;2n)+g_{bc}S_{ad}(\varsigma;2n)-g_{bd}S_{ac}(\varsigma;2n)\end{cases}$

Cor. 1.6.3. $\vec{S}_{ab}:=-iS_{ab|cd}=-i(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})\Rightarrow i[\vec{S}_{ab},\vec{S}_{cd}]=g_{ad}\vec{S}_{bc}-g_{ac}\vec{S}_{bd}+g_{bc}\vec{S}_{ad}-g_{bd}\vec{S}_{ac}$

Cor. 1.6.4. $e^{\frac{i}{2}\theta^{ab}\vec{S}_{ab}}=e^{\theta}$

1.7 Metric tensor and charge conjugate matrix in n=N+1 dimensional space-time

Def. 1.7.1. $C^{+}\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = -C, C^{+}C = CC^{+} = I$

Cor. 1.7.1.
$$\begin{cases} C(n)S_{ab}(e;n) = [C(n)S_{ab}(e;n)]^T, C(2n-1)S_{ab}(\varsigma;2n) = [C(2n-1)S_{ab}(\varsigma;2n)]^T \\ C(n)\gamma_a(e;n) = [C(n)\gamma_a(e;n)]^T, C(2n-1)\gamma_a(2n-1) = [C(2n-1)\gamma_a(2n-1)]^T \end{cases}$$

Cor. 1.7.2.
$$\begin{cases} \varepsilon(2n)S_{ab}(e;2n) = -S_{ab}^T(e;2n)\varepsilon(2n), \varepsilon(2n-1)S_{ab}(\varsigma;2n) = -S_{ab}^T(\varsigma;2n)\varepsilon(2n-1) \\ \varepsilon(2n-1)S_{ab}(e;2n-1) = -S_{ab}^T(e;2n-1)\varepsilon(2n-1), \varepsilon(2n-1)\gamma_a(2n-1) = -\gamma_a^T(2n-1)\varepsilon(2n-1) \end{cases}$$

1.8 Constant invariant tensors in n=N+1 dimensional space-time
(Finally successfully promoted.)
The following theorem exists in any n=N+1 dimensional space-time.

Thm. 1.8.1. $[\Gamma(N), i\varsigma]^a = [e^{\vartheta}]^a{}_b e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i(N),\Gamma_j(N)]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma(N)}[\Gamma(N), i\varsigma]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i(N),\Gamma_j(N)]+\varsigma\epsilon\cdot\frac{1}{2}\Gamma(N)}$

Self comment: Therefore, $[\Gamma(N), i\varsigma]^a_{A_\varsigma A'_\varsigma}$ **and** $[\Gamma(N), -i\varsigma]_a^{A'_\varsigma A_\varsigma}$ **are constant invariant tensors. This is a generalization of Penrose spinors in high and low dimensional space-time.**
The following theorem exists in any n=N+1 dimensional space-time.

Thm. 1.8.2. $\Gamma_{ab} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}\Gamma_{cd}e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}} \Leftrightarrow i[\Gamma_{ab},\Gamma_{cd}] = \delta_{ad}\Gamma_{bc} - \delta_{ac}\Gamma_{bd} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}$

Therefore, $S_{ab\lambda_\varsigma}{}^{\mu_\varsigma}(e,\varsigma;n), S_{ab}{}^{\lambda'_\varsigma}{}_{\mu'_\varsigma}(e,-\varsigma;n)$**,** $S_{abA_\varsigma}{}^{B_\varsigma}(\frac{1}{2},\varsigma;2n), S_{ab}{}^{A'_\varsigma}{}_{B'_\varsigma}(\frac{1}{2},-\varsigma;2n)$ **are constant invariant tensors.**

Thm. 1.8.3. $\Gamma_a = [e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}\Gamma_{cd}}\Gamma_b e^{-\frac{i}{2}\vartheta^{cd}\Gamma_{cd}} \Leftrightarrow i[\Gamma_a,\Gamma_{cd}] = \delta_{a[c}\Gamma_{d]}$

Thm. 1.8.4.
$$\begin{cases} \Gamma_0 = e^{\frac{i}{2}\vartheta^{ab}S_{ab}}\Gamma_0 e^{-\frac{i}{2}\vartheta^{ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a,\Gamma_b], \Gamma_0 = \Gamma_1\cdots\Gamma_{N+1} \\ \Gamma_0 = e^{\frac{i}{2}\vartheta^{*ab}S_{ab}}\Gamma_0 e^{-\frac{i}{2}\vartheta^{*ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a,\Gamma_b], \Gamma_0 = \Gamma_1\cdots\Gamma_{N+1} \end{cases}$$

Therefore, $\gamma^a{}_{\lambda_\varsigma}{}^{\mu_\varsigma}(n), \gamma_{a'}{}^{\lambda'_\varsigma}{}_{\mu'_\varsigma}(n)\ \gamma^0{}_{\lambda_\varsigma}{}^{\mu_\varsigma}(n), \gamma_0{}^{\lambda'_\varsigma}{}_{\mu'_\varsigma}(n)$ **are constant invariant tensors.**

Thm. 1.8.5.
$$\begin{cases} \Gamma_{N+1} = e^{\frac{i}{2}\vartheta^{*ab}S_{ab}}\Gamma_{N+1}e^{-\frac{i}{2}\vartheta^{ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a,\Gamma_b] \\ \Gamma_{N+1} = e^{\frac{i}{2}\vartheta^{ab}S_{ab}}\Gamma_{N+1}e^{-\frac{i}{2}\vartheta^{*ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a,\Gamma_b] \end{cases}$$

Therefore, $\gamma_n^{\lambda'_\varsigma\lambda_\varsigma}(n), \gamma^n_{\lambda_\varsigma\lambda'_\varsigma}(n)$**,** $(\gamma_n\gamma_a)^{\lambda'_\varsigma\lambda_\varsigma}(n), (\gamma^a\gamma^n)_{\lambda_\varsigma\lambda'_\varsigma}(n)$**,** $(\gamma^n\gamma^{a'})_{\lambda_\varsigma\lambda'_\varsigma}(n), (\gamma_{a'}\gamma_n)^{\lambda'_\varsigma\lambda_\varsigma}(n)$ **are constant invariant tensors.**

1.9 Properties of constant invariant tensors in n=N+1 dimensional space-time

Thm. 1.9.1.
$$\begin{cases} S_{ab}(e;n) = -\frac{i}{4}[\gamma_a(n),\gamma_b(n)] = -\frac{i}{4}[i\varsigma\gamma(N)\gamma_0(N),-i\varsigma]_{[a}[i\varsigma\gamma(N)\gamma_0(N),i\varsigma]_{b]} \\ 2\delta_{ab} = \{\gamma_a(n),\gamma_b(n)\} = [i\varsigma\gamma(N)\gamma_0(N),-i\varsigma]_{\{a}[i\varsigma\gamma(N)\gamma_0(N),i\varsigma]_{b\}} \\ \gamma_a(n)\gamma_b(n) = [i\varsigma\gamma(N)\gamma_0(N),-i\varsigma]_a[i\varsigma\gamma(N)\gamma_0(N),i\varsigma]_b = \delta_{ab} + 2iS_{ab}(e,\varsigma;n) \end{cases}$$

Thm. 1.9.2.
$$\begin{cases} S_{ab}(\frac{1}{2},\varsigma;n) = -\frac{i}{4}[\Gamma(N),i\varsigma]_{[a}[\Gamma(N),-i\varsigma]_{b]} \\ 2\delta_{ab} = [\Gamma(N),i\varsigma]_{\{a}[\Gamma(N),-i\varsigma]_{b\}} \\ [\Gamma(N),i\varsigma]_a[\Gamma(N),-i\varsigma]_b = \delta_{ab} + 2iS_{ab}(\frac{1}{2},\varsigma;n) \end{cases}$$

1.10 Penrose transform in n=N+1 dimensional space-time

Thm. 1.10.1. $x^{A'_\varsigma A_\varsigma}(n) := [\Gamma(N),-i\varsigma]_a^{A'_\varsigma A_\varsigma}x^a \Rightarrow x^a = \frac{1}{2^{[N/2]}}[\Gamma(N),i\varsigma]^a_{A_\varsigma A'_\varsigma}x^{A'_\varsigma A_\varsigma}(n)$

Thm. 1.10.2. $x_{\lambda_\varsigma}{}^{\mu_\varsigma}(n) := \gamma^a{}_{\lambda_\varsigma}{}^{\mu_\varsigma}(n)x_a \Rightarrow x^a = \frac{1}{2^{[n/2]}}\gamma^a{}_{\mu_\varsigma}{}^{\lambda_\varsigma}(n)x_{\lambda_\varsigma}{}^{\mu_\varsigma}(n), n \geq 2$

2 External spatiotemporal symmetry transformation [25, 26]

2.1 Poincare transformation

Cor. 2.1.1. $x' = e^{\varepsilon}x + \theta \Leftrightarrow x' = e^{-\frac{i}{2}\varepsilon^{ab}L_{ab}+i\theta^a p_a}x, L_{ab} = -i(x_a\partial_b - x_b\partial_a), p_a = -i\partial_a$

Proof: $x' = e^{\varepsilon}x + \theta$
$\Leftrightarrow x'_a = x_a + \varepsilon_{ab}x^b + \theta_a$
$\Leftrightarrow x'_a = [1 - \frac{1}{2}\varepsilon^{bc}(x_b\partial_c - x_c\partial_b) + \theta^b\partial_b]x_a$
$\Leftrightarrow x' = [1 - \frac{1}{2}\varepsilon^{ab}(x_a\partial_b - x_b\partial_a) + \theta^a\partial_a]x$
$\Leftrightarrow x' = [1 - \frac{i}{2}\varepsilon^{ab}(x_a p_b - x_b p_a) + i\theta^a p_a]x, p_a = -i\partial_a$
$\Leftrightarrow x' = (1 - \frac{i}{2}\varepsilon^{ab}L_{ab} + i\theta^a p_a)x, L_{ab} := -i(x_a\partial_b - x_b\partial_a)$
$\Leftrightarrow x' = e^{-\frac{i}{2}\varepsilon^{ab}L_{ab}+i\theta^a p_a}x$ □

Cor. 2.1.2. $e^{-\frac{i}{2}\varepsilon^{ab}L_{ab}+i\theta^a p_a}x = e^{\varepsilon}x + \theta$

The Poincare transformation contains two meanings. One is a conventional and intuitive meaning, and the other is a meaning that includes a part of Poincare generators. Actually, it's just like thinking of transformations as operators. It's not easy to think of.

2.2 Lorentz transformation

Cor. 2.2.1. $x' = e^{\varepsilon}x \Leftrightarrow x' = e^{-\frac{i}{2}\varepsilon^{ab}L_{ab}}x, L_{ab} = -i(x_a\partial_b - x_b\partial_a)$

Cor. 2.2.2. $\varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x) \Leftrightarrow \varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x), M_{ab} = L_{ab} - iS_{ab}$

Proof: $\varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x)$
$\Leftrightarrow \varphi'(x) = (1 + \frac{i}{2}\varepsilon^{ab}S_{ab})\varphi((1-\varepsilon)x)$
$\Leftrightarrow \varphi'(x) = (1 + \frac{i}{2}\varepsilon^{ab}S_{ab})[\varphi(x) + \frac{1}{2}\varepsilon^{ab}(x_a\partial_b - x_b\partial_a)\varphi(x)]$
$\Leftrightarrow \varphi'(x) = \varphi(x) + \frac{i}{2}\varepsilon^{ab}[-i(x_a\partial_b - x_b\partial_a) + S_{ab}]\varphi(x)$
$\Leftrightarrow \delta\varphi(x) = \frac{i}{2}\varepsilon^{ab}[-i(x_a\partial_b - x_b\partial_a) + S_{ab}]\varphi(x)$
$\Leftrightarrow \varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}[-i(x_a\partial_b - x_b\partial_a)+S_{ab}]}\varphi(x)$
$\Leftrightarrow \varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x), M_{ab} = L_{ab} + S_{ab}$ □

Cor. 2.2.3. $e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x)$

Cor. 2.2.4. $x = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}(e^{-\varepsilon}x) \Leftrightarrow x = e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}x$

The Lorentz group generator is implicit in the state transformation and coordinate identity transformation: M_{ab}The meaning of a generator is that the independent variable remains unchanged and the function changes. It is an operator.

2.3 Translational transformation

Cor. 2.3.1. $\varphi'(x) = \varphi(x+\theta) \Leftrightarrow \varphi'(x) = e^{\theta^a\partial_a}\varphi(x) \Leftrightarrow \varphi'(x) = e^{i\theta^a p_a}\varphi(x)$

Cor. 2.3.2. $e^{i\theta^a p_a}\varphi(x) = \varphi(x+\theta)$

The translation generator is implicit in a simple shift transformation: p_a and its operator transformation.

2.4 Commutative relations of Poincare groups generators

Commutative relations of Poincare group generator M_{ab}, p_a:

$$M_{ab} = L_{ab} + S_{ab}, L_{ab} = x_a p_b - x_b p_a, g_{ab} = \delta_{ab} \tag{4.1}$$

$$\begin{cases} i[M_{ab}, M_{cd}] = g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac} \\ i[M_{ab}, p_c] = g_{bc}p_a - g_{ac}p_b, [p_a, p_b] = 0 \end{cases} \tag{4.2}$$

Commutative relations of Poincare group generator L_{ab}, S_{ab}, p_a:

$$\begin{cases} L_{ab}, L_{cd}] = g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac} \\ i[L_{ab}, p_c] = g_{bc}p_a - g_{ac}p_b, [p_a, p_b] = 0 \end{cases} \tag{4.3}$$

$$i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \tag{4.4}$$

$$[S_{ab}, L_{cd}] = 0, [S_{ab}, p_c] = 0 \tag{4.5}$$

2.5 Meaning of generators

The generators can generate corresponding symmetric transformations. Generators are also generally conserved quantities of the system. Conversely, a conserved quantity is also a generator of the system nd generators also form certain closed algebras.

3 Infinite dimensional invariant tensors

3.1 Infinite dimensional invariant tensors [25, 26] in the sense of quantum mechanics

Def. 3.1.1. $\hat{M}_{ab} := \hat{L}_{ab} + S_{ab}, \hat{L}_{ab} := x_a\hat{p}_b - x_b\hat{p}_a, \hat{p}_a := -i\partial_a, g_{ab} = \delta_{ab}$

Cor. 3.1.1. $\begin{cases} i[\hat{M}_{ab}, \hat{M}_{cd}] = g_{ad}\hat{M}_{bc} - g_{ac}\hat{M}_{bd} + g_{bc}\hat{M}_{ad} - g_{bd}\hat{M}_{ac} \Leftrightarrow \hat{M}_{ab} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}}\hat{M}_{cd}e^{-\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}} \\ i[\hat{L}_{ab}, \hat{L}_{cd}] = g_{ad}\hat{L}_{bc} - g_{ac}\hat{L}_{bd} + g_{bc}\hat{L}_{ad} - g_{bd}\hat{L}_{ac} \Leftrightarrow \hat{L}_{ab} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}}\hat{L}_{cd}e^{-\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}} \\ i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Leftrightarrow S_{ab} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}S_{ef}}S_{cd}e^{-\frac{i}{2}\vartheta^{ef}S_{ef}} \end{cases}$

Cor. 3.1.2. $\begin{cases} i[\hat{p}_a, \hat{M}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow \hat{p}_a = [e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}}\hat{p}_b e^{-\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}} \\ i[\hat{p}_a, \hat{L}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow \hat{p}_a = [e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}}\hat{p}_b e^{-\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}} \end{cases}$

From the above, it can be seen that $\hat{M}_{ab}, \hat{p}_a$ are invariant tensors, but obviously not constant tensors. In another equivalent way of writing, Lorentz transformation in relativity can be associated with unitary transformation in quantum mechanics.

Cor. 3.1.3. $\begin{cases} i[\hat{M}_{ab}, \hat{M}_{cd}] = g_{ad}\hat{M}_{bc} - g_{ac}\hat{M}_{bd} + g_{bc}\hat{M}_{ad} - g_{bd}\hat{M}_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}}\hat{M}_{ab}e^{\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d\hat{M}_{cd} \\ i[\hat{L}_{ab}, \hat{L}_{cd}] = g_{ad}\hat{L}_{bc} - g_{ac}\hat{L}_{bd} + g_{bc}\hat{L}_{ad} - g_{bd}\hat{L}_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}}\hat{L}_{ab}e^{\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d\hat{L}_{cd} \\ i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}S_{ef}}S_{ab}e^{\frac{i}{2}\vartheta^{ef}S_{ef}} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d S_{cd} \end{cases}$

Cor. 3.1.4. $\begin{cases} i[\hat{p}_a, \hat{M}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}}\hat{p}_a e^{\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}} = [e^{\vartheta}]_a{}^b\hat{p}_b \\ i[\hat{p}_a, \hat{L}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}}\hat{p}_a e^{\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}} = [e^{\vartheta}]_a{}^b\hat{p}_b \end{cases}$

Cor. 3.1.5. $e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x), e^{i\theta^a p_a}\varphi(x) = \varphi(x+\theta)$

Cor. 3.1.6. $\langle'|\hat{P}_a|'\rangle = [e^{\vartheta}]_a{}^b\langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{ab}|'\rangle = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d\langle|\hat{J}_{cd}|\rangle, \langle'|S_{ab}|'\rangle = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d\langle|S_{cd}|\rangle$

Cor. 3.1.7. $\langle'|\hat{P}_a|'\rangle = [e^{\vartheta}]_a{}^b\langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{\alpha_\varsigma}|'\rangle = [e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}]_{\alpha_\varsigma}{}^{\beta_\varsigma}\langle|\hat{J}_{\beta_\varsigma}|\rangle$

From the above, we can see that the right side is a Lorentz transformation, and the left side can be seen as a unitary transformation. That is, Lorentz transformation is equivalent to unitary transformation.

Cor. 3.1.8. $[\hat{p}_a, \hat{p}_b] = 0 \Leftrightarrow \hat{p}_a = e^{-i\vartheta^b\hat{p}_b}\hat{p}_a e^{i\vartheta^b\hat{p}_b}$

3.2 Infinite dimensional invariant tensors [25, 26] in the sense of quantum field theory

Cor. 3.2.1. $\begin{cases} i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow \hat{J}_{ab} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}}\hat{J}_{cd}e^{-\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}} \\ i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}}\hat{J}_{ab}e^{\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}} = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d\hat{J}_{cd} \end{cases}$

Cor. 3.2.2. $\begin{cases} i[\hat{P}_a, \hat{J}_{cd}] = \delta_{a[c}\hat{P}_{d]} \Leftrightarrow \hat{P}_a = [e^{\vartheta}]_a{}^b e^{\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}}\hat{P}_b e^{-\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}} \\ i[\hat{P}_a, \hat{J}_{cd}] = \delta_{a[c}\hat{P}_{d]} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}}\hat{P}_a e^{\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}} = [e^{\vartheta}]_a{}^b\hat{P}_b \end{cases}$

Cor. 3.2.3. $[\hat{P}_a, \hat{P}_b] = 0 \Leftrightarrow \hat{P}_a = e^{-i\vartheta^b\hat{P}_b}\hat{P}_a e^{i\vartheta^b\hat{P}_b}$

Cor. 3.2.4. $\langle'|\hat{P}_a|'\rangle = [e^{\vartheta}]_a{}^b\langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{ab}|'\rangle = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d\langle|\hat{J}_{cd}|\rangle, \langle'|S_{ab}|'\rangle = [e^{\vartheta}]_a{}^c[e^{\vartheta}]_b{}^d\langle|S_{cd}|\rangle$

Cor. 3.2.5. $\langle'|\hat{P}_a|'\rangle = [e^{\vartheta}]_a{}^b\langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{\alpha_\varsigma}|'\rangle = [e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}]_{\alpha_\varsigma}{}^{\beta_\varsigma}\langle|\hat{J}_{\beta_\varsigma}|\rangle$

Conjected covariant equation:

Cor. 3.2.6. $[(s+\phi)\hat{P}_a + i\hat{J}_{ab}\hat{P}^b]\psi(s,\varsigma) = 0$

Cor. 3.2.7. $[(s+\phi)\hat{\partial}_a + i\hat{J}_{ab}\hat{\partial}^b]\Psi(x, F[\varphi(y)]) = 0$

Cor. 3.2.8. $(\hat{P}^a\partial_a + m)\Psi(x, F[\varphi(y)]) = 0$

Cor. 3.2.9. $\partial_a\Psi(x, F[\varphi(y)]) = \hat{P}_a\Psi(x, F[\varphi(y)])$

3.3 General invariant tensors [25, 26] in the sense of quantum field theory

Cor. 3.3.1. $\begin{cases} i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow \psi_\lambda = \Lambda_\lambda{}^\mu U^+\psi_\mu U \\ i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow U\psi_\lambda U^+ = \Lambda_\lambda{}^\mu\psi_\mu \end{cases}$

Cor. 3.3.2. $U\psi_\lambda U^+ = \Lambda_\lambda{}^\mu\psi_\mu, U = e^{iX}$

3.4 Operator algebra on quantum field theory

Cor. 3.4.1. $[x, \hat{p}_x] = i \Leftrightarrow \hat{p}_x \equiv -i\frac{\partial}{\partial x}, \Psi = \psi(x)$

$\downarrow$

Cor. 3.4.2. $[x_i, \hat{p}_j] = i\delta_{ij} \Leftrightarrow \hat{p}_i \equiv -i\frac{\partial}{\partial x_i}, \Psi = \psi(x_1, x_2, \cdots, x_n)$

$\downarrow$

Cor. 3.4.3. $[\psi(x_i), \pi(x_j)] = i\delta_{ij} \Leftrightarrow \pi(x_i) \equiv -i\frac{\partial}{\partial\psi(x_i)}, \Psi = F[\psi(x_1), \psi(x_2), \cdots, \psi(x_n), \cdots, \psi(x_\infty)]$

$\downarrow$

Cor. 3.4.4. $[\psi(x), \pi(x')] = i\delta(x - x') \Leftrightarrow \pi(x) \equiv -i\frac{\delta}{\delta\psi(x)}$
$\Psi = \int dx F[\psi(x)] = \sum_i \Delta x_i F[\psi(x_i)] = \varepsilon F[\psi(x_1), \psi(x_2), \cdots, \psi(x_n), \cdots, \psi(x_\infty)]$

3.5 How to find eigenstates of arbitrary operators

Chapter5 Constant Invariant Tensor and Representation Transformation

1 Definition of various spinors for general s-spin field

1.1 Penrose abstract symbol rules [1,2]

Cor. 1.1.1. $g^{\mathscr{A}_\varsigma\mathscr{B}_\varsigma}\psi_{\mathscr{B}_\varsigma}=\psi^{\mathscr{A}_\varsigma}=[\psi_{\mathscr{A}_{-\varsigma}}]^*, g_{\mathscr{A}_\varsigma\mathscr{B}_\varsigma}\psi^{\mathscr{B}_\varsigma}=\psi_{\mathscr{A}_\varsigma}=[\psi^{\mathscr{A}_{-\varsigma}}]^*$

The above shows that $\psi_{\mathscr{A}_\varsigma},\psi^{\mathscr{A}_\varsigma}$ are completely related, and only one of them is truly independent. I would choose $\psi_{\mathscr{A}_\varsigma}$ as the base quantity.

1.2 Introduction of field spinors $\psi(s,\varsigma;w),\tilde{\psi}(s,\varsigma;w),\hat{\psi}(s,\varsigma;w)$

Def. 1.2.1. $\psi(s,\varsigma;w)\prec\psi_{k_\varsigma}(s;w)\Leftrightarrow\psi^*(s,-\varsigma;w)\prec\psi^{k_\varsigma}(s;w)$

Def. 1.2.2. $\tilde{\psi}(s,\varsigma;w):=\psi_{A_\varsigma\otimes l_\varsigma}(s;w)\Leftrightarrow\tilde{\psi}^*(s,-\varsigma;w):=\psi^{A_\varsigma\otimes l_\varsigma}(s;w)$

Def. 1.2.3. $\hat{\psi}(s,\varsigma;w):=\psi_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes C_\varsigma\otimes\cdots}_{2s}}(s;w)\Leftrightarrow\hat{\psi}^*(s,-\varsigma;w):=\psi^{\overbrace{A_\varsigma\otimes B_\varsigma\otimes C_\varsigma\otimes\cdots}^{2s}}(s;w)$

1.3 Introduction of surce spinos $\tilde{J}(s,\varsigma;w),\hat{J}(s,\varsigma;w)$

Def. 1.3.1. $\tilde{J}(s,\varsigma;w):=J^{A'_\varsigma\otimes}{}_{l_\varsigma}(s;w)\Leftrightarrow\tilde{J}^*(s,-\varsigma;w):=J_{A'_\varsigma\otimes}{}^{l_\varsigma}(s;w)$

Def. 1.3.2. $\hat{J}(s,\varsigma;w):=J^{A'_\varsigma\otimes}{}_{\underbrace{B_\varsigma\otimes C_\varsigma\otimes\cdots}_{2s-1}}(s;w)\Leftrightarrow\hat{J}^*(s,-\varsigma;w):=J_{A'_\varsigma\otimes}{}^{\overbrace{B_\varsigma\otimes C_\varsigma\otimes\cdots}^{2s-1}}(s;w)$

1.4 Introduction of spinors $\psi_{k_\varsigma}(s;w),\psi^{k_\varsigma}(s;w)\psi_{A_\varsigma l_\varsigma}(s;w),\psi^{A_\varsigma l_\varsigma}(s;w)$

1.4.1 Relations between $\psi(s,\varsigma;w),\hat{\psi}(s,\varsigma;w)$

Def. 1.4.1.

$$\begin{cases}\psi_{k_\varsigma}(s;w):=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(s;w)\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)\Leftrightarrow\psi(s,\varsigma;w)=\bar{\Gamma}(s;w)\hat{\psi}(s,\varsigma;w)\\ \qquad\qquad\Updownarrow\qquad\qquad\qquad\qquad\qquad\qquad\Updownarrow\\ \psi^{k_\varsigma}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)\psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(s;w)\Leftrightarrow\psi^*(s,-\varsigma;w)=\bar{\Gamma}(s;w)\hat{\psi}^*(s,-\varsigma;w)\end{cases}$$

Cor. 1.4.1. $\psi_{k_\varsigma}(s;w)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(s;w)\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)\Leftrightarrow\frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma\cdots)}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)\psi_{k_\varsigma}(s;w)$

Cor. 1.4.2.

$$\begin{cases}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)=\frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma\cdots)}_{2s}}(s;w)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)\psi_{k_\varsigma}(s;w)\Leftrightarrow\hat{\psi}(s,\varsigma;w)=\Gamma(s;w)\psi(s,\varsigma;w)\\ \qquad\qquad\qquad\qquad\Updownarrow\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Updownarrow\\ \psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(s;w)=\frac{1}{(2s)!}\psi^{\overbrace{(A_\varsigma B_\varsigma\cdots)}^{2s}}(s;w)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(s;w)\psi^{k_\varsigma}(s;w)\Leftrightarrow\hat{\psi}^*(s,-\varsigma;w)=\Gamma(s;w)\psi^*(s,-\varsigma;w)\end{cases}$$

1.4.2 Relations between $\tilde{\psi}(s,\varsigma;w),\hat{\psi}(s,\varsigma;w)$

Def. 1.4.2.

$$\begin{cases}\psi_{A_\varsigma l_\varsigma}(s;w):=\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}(s-\tfrac{1}{2};w)\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Leftrightarrow\tilde{\psi}(s,\varsigma;w)=[I_{w+1}\otimes\bar{\Gamma}(s-\tfrac{1}{2};w)]\hat{\psi}(s,\varsigma;w)\\ \qquad\qquad\Updownarrow\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Updownarrow\\ \psi^{A_\varsigma l_\varsigma}(s;w)=\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\tfrac{1}{2};w)\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Leftrightarrow\tilde{\psi}^*(s,-\varsigma;w)=[I_{w+1}\otimes\bar{\Gamma}(s-\tfrac{1}{2};w)]\hat{\psi}^*(s,-\varsigma;w)\end{cases}$$

Cor. 1.4.3.

$$\begin{cases} \psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)=\frac{1}{(2s-1)!}\psi_{\underbrace{A_\varsigma(B_\varsigma C_\varsigma\cdots)}_{2s}}(s;w)=\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)\psi_{A_\varsigma l_\varsigma}(s;w)[\Leftrightarrow]\hat\psi(s,\varsigma;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde\psi(s,\varsigma;w)\\ \qquad\qquad [\Updownarrow] \qquad\qquad\qquad\qquad [\Updownarrow]\\ \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)=\frac{1}{(2s-1)!}\psi^{\overbrace{A_\varsigma(B_\varsigma C_\varsigma\cdots)}^{2s}}(s;w)=\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}(s-\frac{1}{2};w)\psi^{A_\varsigma l_\varsigma}(s;w)[\Leftrightarrow]\hat\psi^*(s,-\varsigma;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde\psi^*(s,-\varsigma; \end{cases}$$

1.4.3 Relations between $\psi(s,\varsigma;w),\tilde\psi(s,\varsigma;w)$

Cor. 1.4.4.

$$\begin{cases} \psi_{k_\varsigma}(s;w)=N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s;w)\psi_{A_\varsigma l_\varsigma}(s;w)[\Leftrightarrow]\psi(s,\varsigma;w)=\bar N(s;w)\tilde\psi(s,\varsigma;w)\\ \qquad [\Updownarrow] \qquad\qquad\qquad [\Updownarrow]\\ \psi^{k_\varsigma}(s;w)=N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s;w)\psi^{A_\varsigma l_\varsigma}(s;w)[\Leftrightarrow]\psi^*(s,-\varsigma;w)=\bar N(s;w)\tilde\psi^*(s,-\varsigma;w) \end{cases}$$

Cor. 1.4.5.

$$\begin{cases} \psi_{A_\varsigma l_\varsigma}(s;w)=N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;w)\psi_{k_\varsigma}(s;w),\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(s;w)=\frac{1}{(2s!}\psi_{\underbrace{(A_\varsigma B_\varsigma\cdots)}_{2s}}(s;w)[\Leftrightarrow]\tilde\psi(s,\varsigma;w)=N(s;w)\psi(s,\varsigma;w)\\ \qquad [\Updownarrow] \qquad\qquad\qquad [\Updownarrow]\\ \psi^{A_\varsigma l_\varsigma}(s;w)=N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s;w)\psi^{k_\varsigma}(s;w),\psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(s;w)=\frac{1}{(2s)!}\psi^{\overbrace{(A_\varsigma B_\varsigma\cdots)}^{2s}}(s;w)[\Leftrightarrow]\tilde\psi^*(s,-\varsigma;w)=N(s;w)\psi^*(s,-\varsigma;w) \end{cases}$$

1.5 Introduction of spinors $J^{A'_\varsigma}{}_{l_\varsigma}(s;w),J_{A'_\varsigma}{}^{l_\varsigma}(s;w)$ and relations between $\tilde J(s,\varsigma;w),\hat J(s,\varsigma;w)$

Def. 1.5.1.

$$\begin{cases} J^{A'_\varsigma}{}_{l_\varsigma}(s;w):=\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}(s-\frac{1}{2};w)J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)\Leftrightarrow\tilde J(s,\varsigma;w)=[I_{w+1}\otimes\bar\Gamma(s-\frac{1}{2};w)]\hat J(s,\varsigma;w)\\ \qquad \Updownarrow \qquad\qquad\qquad \Updownarrow\\ J_{A'_\varsigma}{}^{l_\varsigma}(s;w)=\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)\Leftrightarrow\tilde J^*(s,-\varsigma;w)=[I_{w+1}\otimes\bar\Gamma(s-\frac{1}{2};w)]\hat J^*(s,-\varsigma;w) \end{cases}$$

Cor. 1.5.1.

$$\begin{cases} J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s}}(s;w)=\frac{1}{(2s-1)!}J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma\cdots)}_{2s}}(s;w)=\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};w)J^{A'_\varsigma}{}_{l_\varsigma}(s;w)[\Leftrightarrow]\hat J(s,\varsigma;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde J(s,\varsigma;w)\\ \qquad\qquad [\Updownarrow] \qquad\qquad\qquad\qquad [\Updownarrow]\\ J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s}}(s;w)=\frac{1}{(2s-1)!}J_{A'_\varsigma}{}^{\overbrace{(B_\varsigma C_\varsigma\cdots)}^{2s}}(s;w)=\Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma\cdots}^{2s-1}}(s-\frac{1}{2};w)J_{A'_\varsigma}{}^{l_\varsigma}(s;w)[\Leftrightarrow]\hat J^*(s,-\varsigma;w)=[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde J^*(s,-\varsigma; \end{cases}$$

1.6 Introduction of spinors $J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma\cdots}$ and relations between $\hat{\mathcal J}(n),\hat J(n,\varsigma)$

Cor. 1.6.1.

$$\begin{cases} J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n-1}}:=(\frac{i\varsigma}{\sqrt2})^n\varsigma(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\overbrace{\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n-1}J_{\underbrace{a\beta_\varsigma\cdots}_{n}}[\Leftrightarrow]\hat J(n,\varsigma)=\overbrace{S^+_{em}(\varsigma)\otimes S^+_{em}(\pm\varsigma)\cdots}^{n}\hat{\mathcal J}(n)\\ \qquad [\Updownarrow] \qquad\qquad\qquad [\Updownarrow]\\ J_{\underbrace{a\beta_\varsigma\cdots}_{n}}=(\frac{i\varsigma}{\sqrt2})^n\varsigma(\varepsilon\sigma,-i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}\overbrace{\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}\cdots}^{n-1}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n-1}}[\Leftrightarrow]\hat{\mathcal J}(n)=\overbrace{S_{em}(\varsigma)\otimes S_{em}(\pm\varsigma)\cdots}^{n}\hat J(n,\varsigma) \end{cases}$$

$$[\Updownarrow]$$

Cor. 1.6.2.

$$\begin{cases} J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n-1}}:=(\frac{i\varsigma}{\sqrt2})^n\varsigma(\varepsilon\sigma,-i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}\overbrace{\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}\cdots}^{n-1}J^{\overbrace{a\beta_\varsigma\cdots}^{n}}[\Leftrightarrow]\hat J^*(n,-\varsigma)=\overbrace{S^T_{em}(\varsigma)\otimes S^T_{em}(\pm\varsigma)\cdots}^{n}\hat{\mathcal J}(n)\\ \qquad [\Updownarrow] \qquad\qquad\qquad [\Updownarrow]\\ J^{\overbrace{a\beta_\varsigma\cdots}^{n}}=(\frac{i\varsigma}{\sqrt2})^n\varsigma(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\overbrace{\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n-1}J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n-1}}[\Leftrightarrow]\hat{\mathcal J}(n)=\overbrace{S^*_{em}(\varsigma)\otimes S^*_{em}(\pm\varsigma)\cdots}^{n}\hat J^*(n,-\varsigma) \end{cases}$$

1.7 Definition of $\psi_{\alpha_\varsigma\beta_\varsigma\cdots}(n),\psi^{\alpha_\varsigma\beta_\varsigma\cdots}(n),\Psi(n,\varsigma),\hat\Psi(n,\varsigma)$

Def. 1.7.1. $\hat\Psi(n,\varsigma)\prec\psi_{\underbrace{\alpha_\varsigma\otimes\beta_\varsigma\otimes\cdots}_{n}}(n)\Leftrightarrow\hat\Psi^*(n,-\varsigma)\prec\psi^{\overbrace{\alpha_\varsigma\otimes\beta_\varsigma\otimes\cdots}^{n}}(n)$

Def. 1.7.2.

$$\begin{cases}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n):=\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\psi_{k_\varsigma}(n)\Rightarrow\psi_{k_\varsigma}(n)=\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\\ \qquad\Updownarrow\qquad\qquad\qquad\Updownarrow\\ \psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n):=\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{k_\varsigma}(n)\psi^{k_\varsigma}(n)\Rightarrow\psi^{k_\varsigma}(n)=\Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n)\end{cases}$$

Cor. 1.7.1.

$$\begin{cases}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)=\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2n}}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)[\Leftrightarrow]\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma\cdots)}_{2n}}(n)=\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\\ \qquad[\Updownarrow]\qquad\qquad\qquad[\Updownarrow]\\ \psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n)=\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)\psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2n}}(n)[\Leftrightarrow]\frac{1}{(2n)!}\psi^{\overbrace{(A_\varsigma B_\varsigma\cdots)}^{2n}}(n)=\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2n}}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n)\end{cases}$$

Cor. 1.7.2.

$$\begin{cases}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)=\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2n}}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)[\Leftrightarrow]\hat{\Psi}(n,\varsigma)=[\overbrace{S_{em}(\pm\varsigma)\otimes S_{em}(\pm\varsigma)\cdots}^{n}]\hat{\psi}(n,\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad\qquad[\Updownarrow]\\ \psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n)=\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)\psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2n}}(n)[\Leftrightarrow]\hat{\Psi}^*(n,-\varsigma)=[\overbrace{S^*_{em}(\mp\varsigma)\otimes S^*_{em}(\mp\varsigma)\cdots}^{n}]\hat{\psi}^*(n,-\varsigma)\end{cases}$$

Cor. 1.7.3.

$$\begin{cases}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma\cdots)}_{2n}}(n)=\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)[\Leftrightarrow]\hat{\psi}(n,\varsigma)=[\overbrace{S^+_{em}(\pm\varsigma)\otimes S^+_{em}(\pm\varsigma)\cdots}^{n}]\hat{\Psi}(n,\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad\qquad[\Updownarrow]\\ \psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2n}}(n)=\frac{1}{(2n)!}\psi^{\overbrace{(A_\varsigma B_\varsigma\cdots)}^{2n}}(n)=\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2n}}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(n)\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n)[\Leftrightarrow]\hat{\psi}^*(n,-\varsigma)=[\overbrace{S^T_{em}(\mp\varsigma)\otimes S^T_{em}(\mp\varsigma)\cdots}^{n}]\hat{\Psi}^*(n,-\varsigma)\end{cases}$$

Cor. 1.7.4. $\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(n)=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma\cdots)}_{2n}}(n)[\Leftrightarrow]\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n)=\frac{1}{n!}\psi^{\overbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}^{n}}(n),\delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}(n)=0$

1.8 Introduction of spinors $J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma\cdots}$ and relations between $\hat{\mathcal{J}}(n),\hat{J}(n,\varsigma)$

Cor. 1.8.1.

$$\begin{cases}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}:=(\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}[\Leftrightarrow]\hat{\psi}(n,\varsigma)=\overbrace{S^+_{em}(\pm\varsigma)\otimes S^+_{em}(\pm\varsigma)\cdots}^{n}\hat{\Psi}(n)\\ \qquad[\Updownarrow]\qquad\qquad\qquad[\Updownarrow]\\ \psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=(\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots}^{n}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}[\Leftrightarrow]\hat{\Psi}(n)=\overbrace{S_{em}(\pm\varsigma)\otimes S_{em}(\pm\varsigma)\cdots}^{n}\hat{\psi}(n,\varsigma)\end{cases}$$

$$[\Updownarrow]$$

Cor. 1.8.2.

$$\begin{cases}\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}:=(\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots}^{n}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}[\Leftrightarrow]\hat{\psi}^*(n,-\varsigma)=\overbrace{S^T_{em}(\mp\varsigma)\otimes S^T_{em}(\mp\varsigma)\cdots}^{n}\hat{\Psi}(n)\\ \qquad[\Updownarrow]\qquad\qquad\qquad[\Updownarrow]\\ \psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}}=(\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}[\Leftrightarrow]\hat{\Psi}(n)=\overbrace{S^*_{em}(\mp\varsigma)\otimes S^*_{em}(\mp\varsigma)\cdots}^{n}\hat{\psi}^*(n,-\varsigma)\end{cases}$$

2 Analysis of full symmetry conditions

2.1 Analysis of full symmetry conditions for field quantity

Def. 2.1.1.

$$\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n} Z_\varsigma} := (\frac{i\varsigma}{\sqrt{2}})^n(\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n})\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} \Leftrightarrow \psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^n(\overbrace{\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}\cdots}^{n})\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma}$$

$[\Downarrow]$ $[\Downarrow]$

Cor. 2.1.1.

$$\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^n(\overbrace{\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}\cdots}^{n})\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}[\Leftrightarrow]\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^n(\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n})\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma}$$

Cor. 2.1.2. $\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}$ *is fully symmetric in () and between () for* $(A_\varsigma B_\varsigma),(C_\varsigma D_\varsigma),\cdots$

$\Leftrightarrow \psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = \frac{1}{n!}\psi_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}Z_\varsigma}$

Cor. 2.1.3. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma}$ *is fully symmetric in () and between () for* $(A_\varsigma B_\varsigma),(C_\varsigma D_\varsigma),\cdots$

$\Leftrightarrow \psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma} = \frac{1}{n!}\psi^{\overbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}^{n}Z_\varsigma}$

Cor. 2.1.4. $\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = \psi_{\underbrace{A_\varsigma C_\varsigma B_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} \Leftrightarrow \delta^{\alpha_\varsigma\beta_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 2.1.5. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma} = \psi^{\overbrace{A_\varsigma C_\varsigma B_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma} \Leftrightarrow \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma} = 0$

Proof: $\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = \psi_{\underbrace{A_\varsigma C_\varsigma B_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}$

$\Leftrightarrow \varepsilon^{B_\varsigma C_\varsigma}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = 0$

$\Leftrightarrow \varepsilon^{B_\varsigma C_\varsigma}(\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n})\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow \varepsilon^{B_\varsigma C_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{C_\varsigma}\sigma^{\beta_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow \sigma^{\alpha_\varsigma}\sigma^{\beta_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow \delta^{\alpha_\varsigma\beta_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$ □

Cor. 2.1.6. $\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = \psi_{\underbrace{A_\varsigma Z_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}B_\varsigma} \Leftrightarrow \sigma^{\alpha_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}[Z_\varsigma]} = 0$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 2.1.7. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma} = \psi^{\overbrace{A_\varsigma Z_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}B_\varsigma} \Leftrightarrow \sigma^*_{\alpha_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}[Z_\varsigma]} = 0$

Proof: $\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = \psi_{\underbrace{A_\varsigma Z_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}B_\varsigma}$

$\Leftrightarrow \varepsilon^{B_\varsigma Z_\varsigma}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = 0$

$\Leftrightarrow \varepsilon^{B_\varsigma Z_\varsigma}(\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n})\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow \varepsilon^{B_\varsigma Z_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow \sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{Z_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=0$

$\Leftrightarrow \sigma^{\alpha_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}[Z_\varsigma]}=0$ □

Cor. 2.1.8. $\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2n}Z_\varsigma}\Leftrightarrow\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=\frac{1}{n!}\psi_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}Z_\varsigma},\delta^{\alpha_\varsigma\beta_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=0$

[⇕] [⇕]

Cor. 2.1.9. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma}=\frac{1}{(2n)!}\psi^{\overbrace{(A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots)}^{2n}Z_\varsigma}\Leftrightarrow\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma}=\frac{1}{n!}\psi^{\overbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}^{n}Z_\varsigma},\delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma}=0$

Cor. 2.1.10. $\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma}_{2n+1}}=\frac{1}{(2n+1)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma)}_{2n+1}}\Leftrightarrow\begin{cases}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=\frac{1}{n!}\psi_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}Z_\varsigma}\\ \delta^{\alpha_\varsigma\beta_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=0,\sigma^{\alpha_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}[Z_\varsigma]}=0\end{cases}$

[⇕] [⇕]

Cor. 2.1.11. $\psi^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma}^{2n+1}}=\frac{1}{(2n+1)!}\psi^{\overbrace{(A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma)}^{2n+1}}\Leftrightarrow\begin{cases}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma}=\frac{1}{n!}\psi^{\overbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}^{n}Z_\varsigma}\\ \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}Z_\varsigma}=0,\sigma_{\alpha_\varsigma}\psi^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{n}[Z_\varsigma]}=0\end{cases}$

2.2 Analysis of full symmetry conditions for source quantity

Def. 2.2.1. $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}:=(\frac{i\varsigma}{\sqrt{2}})^n\varsigma[\overbrace{(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}]J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}$

Cor. 2.2.1. $J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}=(\frac{i\varsigma}{\sqrt{2}})^n\varsigma[\overbrace{(\varepsilon\sigma,-i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots}^{n}]J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}$

Cor. 2.2.2. $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}$ *is fully symmetric in () and between () for* $(C_\varsigma D_\varsigma),(E_\varsigma F_\varsigma),\cdots$

$\Leftrightarrow J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}=\frac{1}{(n-1)!}J_{\underbrace{a(\beta_\varsigma\cdots)}_{n}Z_\varsigma}$

Cor. 2.2.3. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma}$ *is fully symmetric in () and between () for* $(C_\varsigma D_\varsigma),(E_\varsigma F_\varsigma),\cdots$

$\Leftrightarrow J^{\overbrace{a\beta_\varsigma\cdots}^{n}Z_\varsigma}=\frac{1}{(n-1)!}J^{\overbrace{a(\beta_\varsigma\cdots)}^{n}Z_\varsigma}$

Cor. 2.2.4. $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}=J^{A'_\varsigma}{}_{\underbrace{C_\varsigma B_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)^a\sigma^{\beta_\varsigma}J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}=0$

[⇕] [⇕]

Cor. 2.2.5. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma}=J_{A'_\varsigma}{}^{\overbrace{C_\varsigma B_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)_a\sigma_{\beta_\varsigma}J^{\overbrace{a\beta_\varsigma\cdots}^{n}Z_\varsigma}=0$

Proof: $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}=J^{A'_\varsigma}{}_{\underbrace{C_\varsigma B_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}$

$\Leftrightarrow\varepsilon^{B_\varsigma C_\varsigma}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma}=0$

$\Leftrightarrow\varepsilon^{B_\varsigma C_\varsigma}[\overbrace{(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}]J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}=0$

$\Leftrightarrow\varepsilon^{B_\varsigma C_\varsigma}(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}=0$

$\Leftrightarrow(\sigma,-i\varsigma)^{aA'_\varsigma C_\varsigma}\sigma^{\beta_\varsigma}{}_{C_\varsigma}{}^{D_\varsigma}J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}=0$

$\Leftrightarrow(\sigma,-i\varsigma)^a\sigma^{\beta_\varsigma}J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma}=0$ □

Cor. 2.2.6. $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = J^{A'_\varsigma}{}_{\underbrace{Z_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}B_\varsigma} \Leftrightarrow (\sigma,-i\varsigma)^a J_{\underbrace{a\beta_\varsigma\cdots}_{n}[Z_\varsigma]} = 0$

$[\Updownarrow] \qquad\qquad [\Updownarrow]$

Cor. 2.2.7. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}Z_\varsigma} = J_{A'_\varsigma}{}^{\overbrace{Z_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}B_\varsigma} \Leftrightarrow (\sigma^*,i\varsigma)_a J^{\overbrace{a\beta_\varsigma\cdots}^{n}[Z_\varsigma]} = 0$

Proof: $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = J^{A'_\varsigma}{}_{\underbrace{Z_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}B_\varsigma}$

$\Leftrightarrow \varepsilon^{B_\varsigma Z_\varsigma} J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}Z_\varsigma} = 0$

$\Leftrightarrow \varepsilon^{B_\varsigma Z_\varsigma} [\overbrace{(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}] J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow \varepsilon^{B_\varsigma Z_\varsigma} (\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma} J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow (\sigma,-i\varsigma)^{aA'_\varsigma Z_\varsigma} J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0$

$\Leftrightarrow (\sigma,-i\varsigma)^a J_{\underbrace{a\beta_\varsigma\cdots}_{n}[Z_\varsigma]} = 0$ □

Cor. 2.2.8. $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n-1}Z_\varsigma} = \frac{1}{(2n-1)!} J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2n-1}Z_\varsigma} \Leftrightarrow \begin{cases} J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma} = \frac{1}{(n-1)!} J_{\underbrace{a(\beta_\varsigma\cdots)}_{n}Z_\varsigma} \\ (\sigma,-i\varsigma)^a \sigma^{\beta_\varsigma} J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0 \end{cases}$

$[\Updownarrow] \qquad\qquad [\Updownarrow]$

Cor. 2.2.9. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n-1}Z_\varsigma} = \frac{1}{(2n-1)!} J_{A'_\varsigma}{}^{\overbrace{(B_\varsigma C_\varsigma D_\varsigma\cdots)}^{2n-1}Z_\varsigma} \Leftrightarrow \begin{cases} J^{\overbrace{a\beta_\varsigma\cdots}^{n}Z_\varsigma} = \frac{1}{(n-1)!} J^{\overbrace{a(\beta_\varsigma\cdots)}^{n}Z_\varsigma} \\ (\sigma,-i\varsigma)_a \sigma_{\beta_\varsigma} J^{\overbrace{a\beta_\varsigma\cdots}^{n}Z_\varsigma} = 0 \end{cases}$

Cor. 2.2.10. $J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma}_{2n}} = \frac{1}{(2n)!} J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma)}_{2n}} \Leftrightarrow \begin{cases} J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma} = \frac{1}{(n-1)!} J_{\underbrace{a(\beta_\varsigma\cdots)}_{n}Z_\varsigma} \\ (\sigma,-i\varsigma)^a \sigma^{\beta_\varsigma} J_{\underbrace{a\beta_\varsigma\cdots}_{n}Z_\varsigma} = 0, (\sigma,-i\varsigma)^a J_{\underbrace{a\beta_\varsigma\cdots}_{n}[Z_\varsigma]} = 0 \end{cases}$

$[\Updownarrow] \qquad\qquad [\Updownarrow]$

Cor. 2.2.11. $J_{A'_\varsigma}{}^{\overbrace{B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma}^{2n}} = \frac{1}{(2n)!} J_{A'_\varsigma}{}^{\overbrace{(B_\varsigma C_\varsigma D_\varsigma\cdots Z_\varsigma)}^{2n}} \Leftrightarrow \begin{cases} J^{\overbrace{a\beta_\varsigma\cdots}^{n}Z_\varsigma} = \frac{1}{(n-1)!} J^{\overbrace{a(\beta_\varsigma\cdots)}^{n}Z_\varsigma} \\ (\sigma,-i\varsigma)_a \sigma_{\beta_\varsigma} J^{\overbrace{a\beta_\varsigma\cdots}^{n}Z_\varsigma} = 0, (\sigma^*,i\varsigma)_a J^{\overbrace{a\beta_\varsigma\cdots}^{n}[Z_\varsigma]} = 0 \end{cases}$

3 Fully symmetric spinor and representation transform of electromagnetic field

3.1 Fully symmetric spinor $\psi_{A_\varsigma B_\varsigma}$ of electromagnetic field

Def. 3.1.1. $\psi_{A_\varsigma B_\varsigma} = \psi_{B_\varsigma A_\varsigma} \Leftrightarrow \hat{\psi}(1,\varsigma) = S_{ex}\hat{\psi}(1,\varsigma)$

Def. 3.1.2. $\hat{\Psi}(1,\varsigma) = \tilde{\Psi}(1,\varsigma) := [\psi_{x_\varsigma}, \psi_{y_\varsigma}, \psi_{z_\varsigma}, 0]^T, \hat{\psi}(1,\varsigma) = \tilde{\psi}(1,\varsigma) := [\psi_{1_\varsigma 1_\varsigma}, \psi_{1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma}, \psi_{2_\varsigma 2_\varsigma}]^T$

Def. 3.1.3. $\psi_{\alpha_\varsigma} \succ \Psi(1,\varsigma) := [\psi_{x_\varsigma}, \psi_{y_\varsigma}, \psi_{z_\varsigma}]^T, \psi(1,\varsigma) := [\psi_{1_\varsigma 1_\varsigma}, \sqrt{C_2^1}\psi_{1_\varsigma 2_\varsigma}, \psi_{2_\varsigma 2_\varsigma}]^T$

3.2 Relations between $\psi(1,\varsigma), \hat{\psi}(1,\varsigma)$

Cor. 3.2.1. $\begin{cases} \psi_{k_\varsigma}(1) = \Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi_{A_\varsigma B_\varsigma} [\Leftrightarrow] \psi(1,\varsigma) = \bar{\Gamma}(\frac{3}{2})\hat{\psi}(1,\varsigma) \\ \qquad [\Updownarrow] \qquad\qquad [\Updownarrow] \\ \psi^{k_\varsigma}(1) = \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi^{A_\varsigma B_\varsigma} [\Leftrightarrow] \psi^*(1,-\varsigma) = \bar{\Gamma}(\frac{3}{2})\hat{\psi}^*(1,-\varsigma) \end{cases}$

$[\Updownarrow]$

Cor. 3.2.2. $\begin{cases} \psi_{A_\varsigma B_\varsigma} = \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}(1) [\Leftrightarrow] \hat{\psi}(1,\varsigma) = \Gamma(\frac{3}{2})\psi(1,\varsigma) \\ \qquad [\Updownarrow] \qquad\qquad [\Updownarrow] \\ \psi^{A_\varsigma B_\varsigma} = \Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi^{k_\varsigma}(1) [\Leftrightarrow] \hat{\psi}^*(1,-\varsigma) = \Gamma(\frac{3}{2})\psi^*(1,-\varsigma) \end{cases}$

3.3 Relations between $\tilde{\psi}(1,\varsigma),\hat{\psi}(1,\varsigma)$

Cor. 3.3.1. $\tilde{\psi}(1,\varsigma)=\hat{\psi}(1,\varsigma)$

3.4 Relations between $\psi(1,\varsigma),\tilde{\psi}(1,\varsigma)$

Cor. 3.4.1. $\begin{cases}\psi_{k_\varsigma}(1)=N_{k_\varsigma}^{A_\varsigma B_\varsigma}(1)\psi_{A_\varsigma B_\varsigma}[\Leftrightarrow]\psi(1,\varsigma)=\bar{N}(1)\tilde{\psi}(1,\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad[\Updownarrow]\\ \psi^{k_\varsigma}(1)=N^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi^{A_\varsigma B_\varsigma}[\Leftrightarrow]\psi^*(1,-\varsigma)=\bar{N}(1)\tilde{\psi}^*(1,-\varsigma)\end{cases}$

$[\Updownarrow]$

Cor. 3.4.2. $\begin{cases}\psi_{A_\varsigma B_\varsigma}=N^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}(1)[\Leftrightarrow]\tilde{\psi}(1,\varsigma)=N(1)\psi(1,\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad[\Updownarrow]\\ \psi^{A_\varsigma B_\varsigma}=N_{k_\varsigma}^{A_\varsigma B_\varsigma}(1)\psi^{k_\varsigma}(1)[\Leftrightarrow]\tilde{\psi}^*(1,-\varsigma)=N(1)\psi^*(1,-\varsigma)\end{cases}$

3.5 Representation transform matrix is a constant invariant tensor

Def. 3.5.1. $J_{A'_\varsigma A_\varsigma}:=\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}J_a\Leftrightarrow J_a=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}J_{A'_\varsigma A_\varsigma}$

Cor. 3.5.1. $J_a=\frac{i}{\sqrt{2}}(\varepsilon\sigma,-i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}J^{A'_\varsigma}{}_{B_\varsigma}$

Proof: $J_a=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}J_{A'_\varsigma A_\varsigma}=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}(-\varsigma\varepsilon_{A'_\varsigma B'_\varsigma})J^{B'_\varsigma}{}_{A_\varsigma}=\frac{i}{\sqrt{2}}(\varepsilon\sigma,-i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}J^{A'_\varsigma}{}_{B_\varsigma}$ □

Cor. 3.5.2. $\begin{cases}\frac{i}{\sqrt{2}}(\varepsilon\sigma,-i\varsigma\varepsilon)_{aA'_\varsigma\otimes}{}^{B_\varsigma}\succ S_{em}(\varsigma)=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&0&-i\\-1&0&0&-1\\0&-i&-i&0\\0&-\varsigma&\varsigma&0\end{bmatrix}\\ \frac{i}{\sqrt{2}}(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma\otimes}{}_{B_\varsigma}\succ S^*_{em}(\varsigma)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&0&0&i\\-1&0&0&-1\\0&i&i&0\\0&-\varsigma&\varsigma&0\end{bmatrix}\end{cases}$

Cor. 3.5.3. $\begin{cases}\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}=\frac{i}{\sqrt{2}}[\varepsilon\sigma]^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\to\frac{i}{\sqrt{2}}(\varepsilon\sigma,-i\varsigma\varepsilon)^{A_\varsigma\otimes B_\varsigma}_{\alpha_\varsigma}\succ S_{em}(\varsigma)\\ \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}=\frac{i}{\sqrt{2}}[\sigma\varepsilon]^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\to\frac{i}{\sqrt{2}}(\sigma\varepsilon,-i\varsigma\varepsilon)^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\succ S^*_{em}(\varsigma)\end{cases}$

Cor. 3.5.4. $\begin{cases}\Gamma_{\alpha_\varsigma}{}^{k_\varsigma}(1)\succ S_m(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&-i\\-1&0&-1\\0&-i\sqrt{2}&0\end{bmatrix}\\ \Gamma_{k_\varsigma}{}^{\alpha_\varsigma}(1)\succ S^+_m(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&-1&0\\0&0&i\sqrt{2}\\i&-1&0\end{bmatrix}\end{cases}$ $\begin{cases}\Gamma^{\alpha_\varsigma}{}_{k_\varsigma}(1)\succ S^*_m(1)\\ \Gamma^{k_\varsigma}{}_{\alpha_\varsigma}(1)\succ S^T_m(1)\end{cases}$

Cor. 3.5.5. $S_{em}(\varsigma)=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&0&-i\\-1&0&0&-1\\0&-i&-i&0\\0&-\varsigma&\varsigma&0\end{bmatrix},S^+_{em}(\varsigma)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&-1&0&0\\0&0&i&-\varsigma\\0&0&i&\varsigma\\i&-1&0&0\end{bmatrix},S_{em}(\varsigma)S^+_{em}(\varsigma)=S^+_{em}(\varsigma)S_{em}(\varsigma)=I_4$

Cor. 3.5.6. $S_m(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&-i\\-1&0&-1\\0&-i\sqrt{2}&0\end{bmatrix},S^+_m(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&-1&0\\0&0&i\sqrt{2}\\i&-1&0\end{bmatrix},S_m(1)S^+_m(1)=S^+_m(1)S_m(1)=I_3$

3.6 Spinor relations of electromagnetic field and representation transformation

Cor. 3.6.1.

$\begin{cases}\psi_{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}[\Leftrightarrow]\hat{\Psi}(1,\varsigma)=S_{em}(\pm\varsigma)\hat{\psi}(1,\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad[\Updownarrow]\\ \psi_{A_\varsigma B_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma}[\Leftrightarrow]\hat{\psi}(1,\varsigma)=S^+_{em}(\pm\varsigma)\hat{\Psi}(1,\varsigma)\end{cases}[\Leftrightarrow]\begin{cases}\psi^{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi^{A_\varsigma B_\varsigma}[\Leftrightarrow]\hat{\Psi}^*(1,-\varsigma)=S^*_{em}(\mp\varsigma)\hat{\psi}^*(1,-\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad[\Updownarrow]\\ \psi^{A_\varsigma B_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi^{\alpha_\varsigma}[\Leftrightarrow]\hat{\psi}^*(1,-\varsigma)=S^T_{em}(\mp\varsigma)\hat{\Psi}^*(1,-\varsigma)\end{cases}$

$[\Updownarrow]$

Cor. 3.6.2.

$\begin{cases}\psi_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi_{k_\varsigma}[\Leftrightarrow]\Psi(1,\varsigma)=S_m(1)\psi(1,\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad[\Updownarrow]\\ \psi_{k_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi_{\alpha_\varsigma}[\Leftrightarrow]\psi(1,\varsigma)=S^+_m(1)\Psi(1,\varsigma)\end{cases}[\Leftrightarrow]\begin{cases}\psi^{\alpha_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi^{k_\varsigma}[\Leftrightarrow]\Psi^*(1,-\varsigma)=S^*_m(1)\psi^*(1,-\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad[\Updownarrow]\\ \psi^{k_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi^{\alpha_\varsigma}[\Leftrightarrow]\psi^*(1,-\varsigma)=S^T_m(1)\Psi^*(1,-\varsigma)\end{cases}$

3.7 Spinor relations of electromagnetic field source and representation transformation

Cor. 3.7.1. $\tilde{J}(1,\varsigma)=\hat{J}(1,\varsigma)=(J^{1'_\varsigma}{}_{1_\varsigma},J^{2'_\varsigma}{}_{1_\varsigma},J^{1'_\varsigma}{}_{2_\varsigma},J^{2'_\varsigma}{}_{2_\varsigma})$

Cor. 3.7.2. $\begin{cases}J_a=\frac{i}{\sqrt{2}}(\varepsilon\sigma,-i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}J^{A'_\varsigma}{}_{B_\varsigma}[\Leftrightarrow]\tilde{\mathcal{J}}(1)=S_{em}(\varsigma)\hat{J}(1,\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad[\Updownarrow]\\ J^{A'_\varsigma}{}_{B_\varsigma}=\frac{i}{\sqrt{2}}(\sigma\varepsilon,-i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}J_a[\Leftrightarrow]\hat{J}(1,\varsigma)=S^+_{em}(\varsigma)\hat{\mathcal{J}}(1)\end{cases}$

$$[\Updownarrow]$$

Cor. 3.7.3.
$$\begin{cases} J^a = \frac{i}{\sqrt{2}}(\sigma\varepsilon, -i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma} J_{A'_\varsigma}{}^{B_\varsigma} [\Leftrightarrow] \hat{\mathcal{J}}(1) = S^*_{em}(\varsigma)\hat{J}^*(1,-\varsigma) \\ \qquad [\Updownarrow] \qquad\qquad\qquad\qquad [\Updownarrow] \\ J^{A'_\varsigma}{}_{B_\varsigma} = \frac{i}{\sqrt{2}}(\varepsilon\sigma, -i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma} J_a [\Leftrightarrow] \hat{J}^*(1,-\varsigma) = S^T_{em}(\varsigma)\hat{\mathcal{J}}(1) \end{cases}$$

Cor. 3.7.4.
$$\Psi(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma} \Leftrightarrow \tilde{\Psi}(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot R} \Leftrightarrow \tilde{\psi}(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\frac{1}{2}\sigma} \Leftrightarrow \psi(1,\varsigma) \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(1)}$$

4 Fully symmetric spinor and representation transformation of gravitational field

4.1 Fully symmetric condition of gravitational field

Cor. 4.1.1. $\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \frac{1}{4!}\psi_{(A_\varsigma B_\varsigma C_\varsigma D_\varsigma)} \Leftrightarrow \psi_{\alpha_\varsigma\beta_\varsigma} = \frac{1}{2!}\psi_{(\alpha_\varsigma\beta_\varsigma)}, \delta^{\alpha_\varsigma\beta_\varsigma}\psi_{\alpha_\varsigma\beta_\varsigma} = 0$

Cor. 4.1.2. $J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma} = \frac{1}{3!}J^{A'_\varsigma}{}_{(B_\varsigma C_\varsigma D_\varsigma)} \Leftrightarrow (\sigma, -i\varsigma)^a\sigma^{\beta_\varsigma}J_{a\beta_\varsigma} = 0$

4.2 Fully symmetric spinor of gravitational field

Def. 4.2.1. $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma}, \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}$

Def. 4.2.2. $\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \psi_{B_\varsigma A_\varsigma} \Leftrightarrow \hat{\psi}(2,\varsigma) = S_{ex}\hat{\psi}(2,\varsigma)$

Def. 4.2.3. $\hat{\Psi}(2,\varsigma) \equiv [\psi_{x_\varsigma x_\varsigma}, \psi_{y_\varsigma x_\varsigma}, \psi_{z_\varsigma x_\varsigma}, 0, |\psi_{x_\varsigma y_\varsigma}, \psi_{y_\varsigma y_\varsigma}, \psi_{z_\varsigma y_\varsigma}, 0, |\psi_{x_\varsigma z_\varsigma}, \psi_{y_\varsigma z_\varsigma}, \psi_{z_\varsigma z_\varsigma}, 0, |0,0,0,0]^T$

Def. 4.2.4. $\hat{\psi}(2,\varsigma) \equiv [\psi_{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma}, \psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, |\psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, |$
$\psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, |\psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, \psi_{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}]^T$

Def. 4.2.5. $\Psi(2,\varsigma) \equiv [\psi_{x_\varsigma x_\varsigma}, \psi_{y_\varsigma x_\varsigma}, \psi_{z_\varsigma x_\varsigma}, \psi_{y_\varsigma y_\varsigma}, \psi_{z_\varsigma y_\varsigma}]^T$

Def. 4.2.6. $\psi(2,\varsigma) := [\psi_{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma}, \sqrt{C_4^1}\psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \sqrt{C_4^2}\psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \sqrt{C_4^3}\psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, \psi_{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}]^T$

Def. 4.2.7. $\bar{\bar{\psi}}(2,\varsigma) := [\psi_{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma}, \psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, \psi_{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}]^T$

Def. 4.2.8. $\tilde{\Psi}(2,\varsigma) \equiv [\psi_{x_\varsigma x_\varsigma}, \psi_{y_\varsigma x_\varsigma}, \psi_{z_\varsigma x_\varsigma}, 0, |\psi_{x_\varsigma y_\varsigma}, \psi_{y_\varsigma y_\varsigma}, \psi_{z_\varsigma y_\varsigma}, 0]^T$

Def. 4.2.9. $\tilde{C}(2,\varsigma) \equiv [C_{x_\varsigma x_\varsigma}, C_{y_\varsigma x_\varsigma}, C_{z_\varsigma x_\varsigma}, 0, |C_{x_\varsigma y_\varsigma}, C_{y_\varsigma y_\varsigma}, C_{z_\varsigma y_\varsigma}, 0]^T$

Def. 4.2.10.
$\tilde{\psi}(2,\varsigma) := [(\psi_{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma}, \psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}), \sqrt{C_3^1}(\psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}), \sqrt{C_3^2}(\psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}), (\psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, \psi_{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma})]^T$

Def. 4.2.11.
$\bar{\psi}(2,\varsigma) := [(\psi_{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma}, \psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}), (\psi_{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma}, \psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}), (\psi_{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma}, \psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}), (\psi_{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}, \psi_{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma})]^T$

4.3 Relations between $\psi(2,\varsigma), \hat{\psi}(2,\varsigma)$

Cor. 4.3.1.
$$\begin{cases} \psi_{k_\varsigma}(2) = \Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} [\Leftrightarrow] \psi(2,\varsigma) = \bar{\Gamma}(\frac{5}{2})\hat{\psi}(2,\varsigma) \\ \qquad [\Updownarrow] \qquad\qquad\qquad\qquad [\Updownarrow] \\ \psi^{k_\varsigma}(2) = \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} [\Leftrightarrow] \psi^*(2,-\varsigma) = \bar{\Gamma}(\frac{5}{2})\hat{\psi}^*(2,-\varsigma) \end{cases}$$

$$[\Updownarrow]$$

Cor. 4.3.2.
$$\begin{cases} \psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi_{k_\varsigma}(2) [\Leftrightarrow] \hat{\psi}(2,\varsigma) = \Gamma(\frac{5}{2})\psi(2,\varsigma) \\ \qquad [\Updownarrow] \qquad\qquad\qquad\qquad [\Updownarrow] \\ \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi^{k_\varsigma}(2) [\Leftrightarrow] \hat{\psi}^*(2,-\varsigma) = \Gamma(\frac{5}{2})\psi^*(2,-\varsigma) \end{cases}$$

4.4 Relations between $\tilde{\psi}(2,\varsigma), \hat{\psi}(2,\varsigma)$

Def. 4.4.1.
$$\begin{cases} \psi_{A_\varsigma l_\varsigma}(2) := \Gamma_{l_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma}(\frac{3}{2})\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \Leftrightarrow \tilde{\psi}(2,\varsigma) = [I_{w+1} \otimes \bar{\Gamma}(2)]\hat{\psi}(2,\varsigma) \\ \qquad \Updownarrow \qquad\qquad\qquad\qquad \Updownarrow \\ \psi^{A_\varsigma l_\varsigma}(2) = \Gamma^{l_\varsigma}_{B_\varsigma C_\varsigma D_\varsigma}(\frac{3}{2})\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \Leftrightarrow \tilde{\psi}^*(2,-\varsigma) = [I_{w+1} \otimes \bar{\Gamma}(2)]\hat{\psi}^*(2,-\varsigma) \end{cases}$$

$$[\Updownarrow]$$

Cor. 4.4.1.
$$\begin{cases} \psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \Gamma^{l_\varsigma}_{B_\varsigma C_\varsigma D_\varsigma}(\frac{3}{2})\psi_{A_\varsigma l_\varsigma}(2) [\Leftrightarrow] \hat{\psi}(2,\varsigma) = [I_{w+1} \otimes \Gamma(2)]\tilde{\psi}(2,\varsigma) \\ \qquad [\Updownarrow] \qquad\qquad\qquad\qquad [\Updownarrow] \\ \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \Gamma_{l_\varsigma}^{B_\varsigma C_\varsigma D_\varsigma}(\frac{3}{2})\psi^{A_\varsigma l_\varsigma}(2) [\Leftrightarrow] \hat{\psi}^*(2,-\varsigma) = [I_{w+1} \otimes \Gamma(2)]\tilde{\psi}^*(2,-\varsigma) \end{cases}$$

4.5 Relations between $\psi(2,\varsigma), \tilde{\psi}(2,\varsigma)$

Cor. 4.5.1. $\begin{cases} \psi_{k_\varsigma}(2) = N_{k_\varsigma}^{A_\varsigma l_\varsigma}(2)\psi_{A_\varsigma l_\varsigma}(2)[\Leftrightarrow]\psi(2,\varsigma) = \bar{N}(2)\tilde{\psi}(2,\varsigma) \\ [\Updownarrow] \qquad\qquad [\Updownarrow] \\ \psi^{k_\varsigma}(2) = N^{k_\varsigma}_{A_\varsigma l_\varsigma}(2)\psi^{A_\varsigma l_\varsigma}(2)[\Leftrightarrow]\psi^*(2,-\varsigma) = \bar{N}(2)\tilde{\psi}^*(2,-\varsigma) \end{cases}$

$[\Updownarrow]$

Cor. 4.5.2. $\begin{cases} \psi_{A_\varsigma l_\varsigma}(2) = N^{k_\varsigma}_{A_\varsigma l_\varsigma}(2)\psi_{k_\varsigma}(2)[\Leftrightarrow]\tilde{\psi}(2,\varsigma) = N(2)\psi(2,\varsigma) \\ [\Updownarrow] \qquad\qquad [\Updownarrow] \\ \psi^{A_\varsigma l_\varsigma}(2) = N_{k_\varsigma}^{A_\varsigma l_\varsigma}(2)\psi^{k_\varsigma}(2)[\Leftrightarrow]\tilde{\psi}^*(2,-\varsigma) = N(2)\psi^*(2,-\varsigma) \end{cases}$

4.6 Spinor relations of gravitational field source and representation transformation

Cor. 4.6.1. $\begin{cases} \psi_{\alpha_\varsigma\beta_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}[\Leftrightarrow]\hat{\Psi}(2,\varsigma) = S_{em}(\pm\varsigma)\otimes S_{em}(\pm\varsigma)\hat{\psi}(2,\varsigma) \\ [\Updownarrow] \qquad\qquad [\Updownarrow] \\ \psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\psi_{\alpha_\varsigma\beta_\varsigma}[\Leftrightarrow]\hat{\psi}(2,\varsigma) = S^+_{em}(\pm\varsigma)\otimes S^+_{em}(\pm\varsigma)\hat{\Psi}(2,\varsigma) \end{cases}$

$[\Updownarrow]$

Cor. 4.6.2. $\begin{cases} \psi^{\alpha_\varsigma\beta_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}[\Leftrightarrow]\hat{\Psi}^*(2,-\varsigma) = S^*_{em}(\mp\varsigma)\otimes S^*_{em}(\mp\varsigma)\hat{\psi}^*(2,-\varsigma) \\ [\Updownarrow] \qquad\qquad [\Updownarrow] \\ \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}[\Leftrightarrow]\hat{\psi}^*(2,-\varsigma) = S^T_{em}(\mp\varsigma)\otimes S^T_{em}(\mp\varsigma)\hat{\Psi}^*(2,-\varsigma) \end{cases}$

4.7 Spinor relations of gravitational field source and representation transformation

Def. 4.7.1. $\hat{\mathcal{J}}(2,\varsigma) \equiv [J_{xx_\varsigma}, J_{yx_\varsigma}, J_{zx_\varsigma}, J_{\pi x_\varsigma}, |J_{xy_\varsigma}, J_{yy_\varsigma}, J_{zy_\varsigma}, J_{\pi y_\varsigma}, |J_{xz_\varsigma}, J_{yz_\varsigma}, J_{zz_\varsigma}, J_{\pi z_\varsigma}, |0,0,0,0]^T$

Def. 4.7.2. $\hat{J}(2,\varsigma) \equiv [J^{1'_\varsigma}{}_{1_\varsigma 1_\varsigma 1_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 1_\varsigma 1_\varsigma}, J^{1'_\varsigma}{}_{2_\varsigma 1_\varsigma 1_\varsigma}, J^{2'_\varsigma}{}_{2_\varsigma 1_\varsigma 1_\varsigma}, |J^{1'_\varsigma}{}_{1_\varsigma 2_\varsigma 1_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 2_\varsigma 1_\varsigma}, J^{1'_\varsigma}{}_{2_\varsigma 2_\varsigma 1_\varsigma}, J^{2'_\varsigma}{}_{2_\varsigma 2_\varsigma 1_\varsigma}, | J^{1'_\varsigma}{}_{1_\varsigma 1_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 1_\varsigma 2_\varsigma}, J^{1'_\varsigma}{}_{2_\varsigma 1_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{2_\varsigma 1_\varsigma 2_\varsigma}, |J^{1'_\varsigma}{}_{1_\varsigma 2_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 2_\varsigma 2_\varsigma}, J^{1'_\varsigma}{}_{2_\varsigma 2_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{2_\varsigma 2_\varsigma 2_\varsigma},]^T$

Def. 4.7.3. $\tilde{\mathcal{J}}(2,\varsigma) \equiv [J_{xx_\varsigma}, J_{yx_\varsigma}, J_{zx_\varsigma}, J_{\pi x_\varsigma}, |J_{xy_\varsigma}, J_{yy_\varsigma}, J_{zy_\varsigma}, J_{\pi y_\varsigma}]^T$

Def. 4.7.4.
$\tilde{J}(2,\varsigma) := [(J^{1'_\varsigma}{}_{1_\varsigma 1_\varsigma 1_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 1_\varsigma 1_\varsigma}), \sqrt{C_3^1}(J^{1'_\varsigma}{}_{1_\varsigma 1_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 1_\varsigma 2_\varsigma}), \sqrt{C_3^2}(J^{1'_\varsigma}{}_{1_\varsigma 2_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 2_\varsigma 2_\varsigma}), (J^{1'_\varsigma}{}_{2_\varsigma 2_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{2_\varsigma 2_\varsigma 2_\varsigma})]^T$

Def. 4.7.5.
$\bar{J}(2,\varsigma) := [(J^{1'_\varsigma}{}_{1_\varsigma 1_\varsigma 1_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 1_\varsigma 1_\varsigma}), (J^{1'_\varsigma}{}_{1_\varsigma 1_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 1_\varsigma 2_\varsigma}), (J^{1'_\varsigma}{}_{1_\varsigma 2_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{1_\varsigma 2_\varsigma 2_\varsigma}), (J^{1'_\varsigma}{}_{2_\varsigma 2_\varsigma 2_\varsigma}, J^{2'_\varsigma}{}_{2_\varsigma 2_\varsigma 2_\varsigma})]^T$

Cor. 4.7.1. $\begin{cases} J_{a\beta_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\varsigma(\varepsilon\sigma, -i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}[\Leftrightarrow]\hat{\mathcal{J}}(2) = S_{em}(\varsigma)\otimes S_{em}(\pm\varsigma)\hat{J}(2,\varsigma) \\ [\Updownarrow] \qquad\qquad [\Updownarrow] \\ J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\varsigma(\sigma\varepsilon, -i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}J_{a\beta_\varsigma}[\Leftrightarrow]\hat{J}(2,\varsigma) = S^+_{em}(\varsigma)\otimes S^+_{em}(\pm\varsigma)\hat{\mathcal{J}}(2) \end{cases}$

$[\Updownarrow]$

Cor. 4.7.2. $\begin{cases} J^{a\beta_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\varsigma(\sigma\varepsilon, -i\varsigma\varepsilon)^{aA'_\varsigma}{}_{B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}[\Leftrightarrow]\hat{\mathcal{J}}(2) = S^*_{em}(\varsigma)\otimes S^*_{em}(\pm\varsigma)\hat{J}^*(2,-\varsigma) \\ [\Updownarrow] \qquad\qquad [\Updownarrow] \\ J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma} = (\frac{i\varsigma}{\sqrt{2}})^2\varsigma(\varepsilon\sigma, -i\varsigma\varepsilon)_{aA'_\varsigma}{}^{B_\varsigma}\sigma_{\beta_\varsigma}^{C_\varsigma D_\varsigma}J^{a\beta_\varsigma}[\Leftrightarrow]\hat{J}^*(2,-\varsigma) = S^T_{em}(\varsigma)\otimes S^T_{em}(\pm\varsigma)\hat{\mathcal{J}}(2) \end{cases}$

4.8 Relations between representation transformations

Cor. 4.8.1. $\hat{\Psi}(2,\varsigma) = S_{em}(\pm\varsigma)\otimes S_{em}(\pm\varsigma)\hat{\psi}(2,\varsigma)[\Leftrightarrow]\tilde{\Psi}(2,\varsigma) = S_{em}(\pm\varsigma)\otimes S_{em}(\frac{1}{2})\bar{\psi}(2,\varsigma)[\Leftrightarrow]\Psi(2,\varsigma) = S_m(2)\bar{\bar{\psi}}(2,\varsigma)$

Cor. 4.8.2. $\hat{\mathcal{J}}(2) = S_{em}(\varsigma)\otimes S_{em}(\pm\varsigma)\hat{J}(2,\varsigma)[\Leftrightarrow]\tilde{\mathcal{J}}(2) = S_{em}(\varsigma)\otimes S_{em}(\frac{1}{2})\bar{J}(2,\varsigma)$

Cor. 4.8.3. $S_{em}(\frac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}-i & i\\ 1 & 1\end{bmatrix}, S^+_{em}(\frac{1}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1\\ -i & 1\end{bmatrix}$

4.9 Pure virtual similarity transformation

Def. 4.9.1. $\Psi(2,\varsigma) \equiv [\psi_{x_\varsigma x_\varsigma}, \psi_{y_\varsigma x_\varsigma}, \psi_{z_\varsigma x_\varsigma}, \psi_{y_\varsigma y_\varsigma}, \psi_{z_\varsigma y_\varsigma}]^T$

Cor. 4.9.1. $\Psi(2,\varsigma) = S_m(2)\bar{\bar{\psi}}(2,\varsigma)$

$$S_m(2) = -\tfrac{1}{2}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 \\ -i & 0 & 0 & 0 & i \\ 0 & 2 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2i & 0 & 2i & 0 \end{bmatrix}, S_m^-(2) = -\tfrac{1}{2}\begin{bmatrix} -1 & 2i & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -i \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -i \\ -1 & -2i & 0 & 1 & 0 \end{bmatrix} \quad (5.1a)$$

$$G_m = S_m(2)\tau(2)S_m^-(2) \qquad S_m(2)S_m^-(2) = S_m^-(2)S_m(2) = I \quad (5.1b)$$

$$G_m = \{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2i \\ i & 0 & 0 & 2i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ -2i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{bmatrix}\} \quad (5.1c)$$

Cor. 4.9.2. $\begin{cases} [G_m, -2i\varsigma]^a\partial_a\Psi = 0 \\ \partial_x\Psi_1 + \partial_y\Psi_2 + \partial_z\Psi_3 = 0 \\ \partial_x\Psi_2 + \partial_y\Psi_4 + \partial_z\Psi_5 = 0 \\ \partial_x\Psi_3 + \partial_y\Psi_5 - \partial_z(\Psi_1 + \Psi_4) = 0 \end{cases} \Leftrightarrow \begin{cases} [G_m, -2i\varsigma]^a\partial_a\Psi = 0 \\ \nabla\cdot\vec{\psi}^{\beta_\varsigma} = 0 \end{cases}$

4.10 Pure virtual representation transformation

Cor. 4.10.1. $\Psi_{im}(2,\varsigma) \equiv -\sqrt{2}[\psi^{y_\varsigma x_\varsigma}, -\frac{1}{2}(\psi^{x_\varsigma x_\varsigma} - \psi^{y_\varsigma y_\varsigma}), \psi^{z_\varsigma y_\varsigma}, \varsigma\psi^{z_\varsigma x_\varsigma}, \frac{\sqrt{3}}{2}(\psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma})]^T$

Cor. 4.10.2. $\Psi_{im}(2,\varsigma) = S_{im}(2,\varsigma)\psi(2,\varsigma)$

$$S_{im}(2,\varsigma) = \tfrac{1}{\sqrt{2}}\begin{bmatrix} i & 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 & -1 \\ 0 & -i & 0 & -i & 0 \\ 0 & -\varsigma & 0 & \varsigma & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 \end{bmatrix}, S_{im}^+(2,\varsigma) = \tfrac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 & 0 & 0 & 0 \\ 0 & 0 & i & -\varsigma & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & i & \varsigma & 0 \\ i & -1 & 0 & 0 & 0 \end{bmatrix} \quad (5.2)$$

$$G_{im}(\varsigma) = S_{im}(2,\varsigma)\sigma(2)S_{im}^+(2,\varsigma) \qquad S_{im}(2,\varsigma)S_{im}^+(2,\varsigma) = S_{im}^+(2,\varsigma)S_{im}(2,\varsigma) = I \quad (5.3)$$

$$G_{im}(\varsigma) = \{\begin{bmatrix} 0 & 0 & 0 & -i\varsigma & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & i\sqrt{3} \\ i\varsigma & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{3} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i\varsigma & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & i\varsigma & 0 & 0 & -i\varsigma\sqrt{3} \\ 0 & 0 & 0 & i\varsigma\sqrt{3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\varsigma & 0 \\ 0 & 0 & -i\varsigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\} \quad (5.4)$$

$$G_{im}(+) = \{\begin{bmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & i\sqrt{3} \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{3} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & -i\sqrt{3} \\ 0 & 0 & 0 & i\sqrt{3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\} \quad (5.5)$$

Cor. 4.10.3.

$$\begin{cases} \varsigma\partial_\pi\Psi_1 = \frac{1}{2}\partial_y\Psi_3 - \partial_z\Psi_2 - \frac{1}{2}\partial_x\Psi_4 \\ \varsigma\partial_\pi\Psi_2 = \partial_z\Psi_1 - \frac{1}{2}\partial_x\Psi_3 - \frac{1}{2}\partial_y\Psi_4 \\ \varsigma\partial_\pi\Psi_3 = -\frac{1}{2}\partial_y\Psi_1 + \frac{1}{2}\partial_z\Psi_4 + \partial_x(\frac{1}{2}\Psi_2 + \frac{\sqrt{3}}{2}\Psi_5) \\ \varsigma\partial_\pi\Psi_4 = \frac{1}{2}\partial_x\Psi_1 - \frac{1}{2}\partial_z\Psi_3 + \partial_y(\frac{1}{2}\Psi_2 - \frac{\sqrt{3}}{2}\Psi_5) \\ \varsigma\partial_\pi\Psi_5 = -\frac{\sqrt{3}}{2}\partial_x\Psi_3 + \frac{\sqrt{3}}{2}\partial_y\Psi_4 \end{cases} \Leftrightarrow [G_{im}(+), -2i\varsigma]^a\partial_a\Psi = 0$$

$$\begin{cases} \partial_x(-\Psi_3 + \frac{1}{\sqrt{3}}\Psi_5) + \partial_y\Psi_1 + \partial_z\Psi_4 = 0 \\ \partial_x\Psi_1 + \partial_y(\Psi_2 + \frac{1}{\sqrt{3}}\Psi_5) + \partial_z\Psi_3 = 0 \\ \partial_x\Psi_4 + \partial_y\Psi_3 - \frac{2}{\sqrt{3}}\partial_z\Psi_5 = 0 \end{cases} \Leftrightarrow \nabla\cdot\vec{\psi}^{\beta_\varsigma} = 0$$

5 Relations between various field quantities of s-spin particles

5.1 Identical representation transform relations between $\hat{\psi}(s,\varsigma;w), \tilde{\psi}(s,\varsigma;w), \psi(s,\varsigma;w)$

Cor. 5.1.1.

$$\begin{cases} \hat{\psi}(s,\varsigma;w) \equiv \Gamma(s;w)\psi(s,\varsigma;w) \\ \psi(s,\varsigma;w) \equiv \bar{\Gamma}(s;w)\hat{\psi}(s,\varsigma;w) \end{cases} \quad \begin{cases} \tilde{\psi}(s,\varsigma;w) \equiv N(s;w)\psi(s,\varsigma;w) \\ \psi(s,\varsigma;w) \equiv \bar{N}(s;w)\tilde{\psi}(s,\varsigma;w) \end{cases}$$

Cor. 5.1.2.
$\begin{cases}\hat{\psi}(s,\varsigma;w)\equiv\Gamma(s;w)\bar{\Gamma}(s;w)\hat{\psi}(s,\varsigma;w)\\ \psi(s,\varsigma;w)\equiv\bar{\Gamma}(s;w)\Gamma(s;w)\psi(s,\varsigma;w)\end{cases}$ $\begin{cases}\tilde{\psi}(s,\varsigma;w)\equiv N(s;w)\bar{N}(s;w)\tilde{\psi}(s,\varsigma;w)\\ \psi(s,\varsigma;w)\equiv\bar{N}(s;w)N(s;w)\psi(s,\varsigma;w)\end{cases}$

Cor. 5.1.3.
$\begin{cases}\hat{\psi}(s,\varsigma;w)\equiv[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{\psi}(s,\varsigma;w)\\ \tilde{\psi}(s,\varsigma;w)\equiv[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\hat{\psi}(s,\varsigma;w)\end{cases}$

Cor. 5.1.4.
$\begin{cases}\hat{\psi}(s,\varsigma;w)\equiv[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\hat{\psi}(s,\varsigma;w)\\ \tilde{\psi}(s,\varsigma;w)\equiv[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{\psi}(s,\varsigma;w)\end{cases}$

5.2 Identical representation transform relations between source spinors $\hat{J}(s,\varsigma;w),\tilde{J}(s,\varsigma;w)$

Cor. 5.2.1.
$\begin{cases}\hat{J}(s,\varsigma;w)\equiv[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{J}(s,\varsigma;w)\\ \tilde{J}(s,\varsigma;w)\equiv[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\hat{J}(s,\varsigma;w)\end{cases}$ $\begin{cases}\hat{J}(s,\varsigma;w)\equiv[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)][I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\hat{J}(s,\varsigma;w)\\ \tilde{J}(s,\varsigma;w)\equiv[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)][I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{J}(s,\varsigma;w)\end{cases}$

5.3 Transformation relations between various field quantities of s-spin particles

Thm. 5.3.1. $\psi(s,\varsigma)\sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\Leftrightarrow\hat{\psi}(s,\varsigma)\sim e^{(i\omega+\varsigma\epsilon)\cdot\bar{\Omega}(s)}\Leftrightarrow\tilde{\psi}(s,\varsigma)\sim e^{(i\omega+\varsigma\epsilon)\cdot[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(s-\frac{1}{2})]}$

Proof: $\psi'(s,\varsigma;w)$
$=\bar{\Gamma}(s;w)\hat{\psi}'(s,\varsigma;w)$
$=\bar{\Gamma}(s;w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s;w)}\hat{\psi}(s,\varsigma;w)$
$=\bar{\Gamma}(s;w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s;w)}\Gamma(s;w)\psi(s,\varsigma;w)$
$=e^{\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s;w)\Omega_{ab}(s;w)\Gamma(s;w)}\psi(s,\varsigma;w)$
$=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;w)}\psi(s,\varsigma;w)$ □

Proof: $\tilde{\psi}'(s,\varsigma;w)$
$=[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\hat{\psi}'(s,\varsigma;w)$
$=[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s;w)}\hat{\psi}(s,\varsigma;w)$
$=[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s;w)}[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{\psi}(s,\varsigma;w)$
$=e^{\frac{i}{2}\vartheta^{ab}[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]\Omega_{ab}(s;w)[I_{w+1}\otimes\bar{\Gamma}(s-\frac{1}{2};w)]}\tilde{\psi}(s,\varsigma;w)$
$=e^{\frac{i}{2}\vartheta^{ab}[S_{ab}\otimes I_{4^{2s-1}}+I_{w+1}\otimes S_{ab}(\varsigma;s-\frac{1}{2})]}\tilde{\psi}(s,\varsigma;w)$ □

Proof: $\psi'(s,\varsigma;w)=\bar{N}(s;w)\tilde{\psi}'(s,\varsigma;w)$
$=\bar{N}(s;w)e^{\frac{i}{2}\vartheta^{ab}[S_{ab}\otimes I_{4^{2s-1}}+I_{w+1}\otimes S_{ab}(\varsigma;s-\frac{1}{2})]}\tilde{\psi}(s,\varsigma;w)$ □

Proof: $\bar{N}(s;w)[S_{ab}\otimes I_{4^{2s-1}}+I_{w+1}\otimes S_{ab}(\varsigma;s-\frac{1}{2})]N(s;w)=S_{ab}(s,\varsigma;w)$
$\bar{N}(s;w)[S_{ab}\otimes I_{4^{2s-1}}]N(s;w)=\frac{1}{2s}S_{ab}(s,\varsigma;w)$
$\bar{N}(s;w)[I_{w+1}\otimes S_{ab}(e,\varsigma;s-\frac{1}{2})]N(s;w)=(1-\frac{1}{2s})S_{ab}(s,\varsigma;w)$ □

5.4 Synchronous representation transformation

Cor. 5.4.1. $\sigma'=S\sigma S^{+}=c^k\sigma_k$
$\Leftrightarrow\sigma'(s;w)=[\bar{\Gamma}(s;w)(S\otimes S\otimes\cdot\cdot\otimes S)\Gamma(s;w)]\sigma(s;w)[\bar{\Gamma}(s;w)(S\otimes S\otimes\cdot\cdot\otimes S)\Gamma(s;w)]^{+}=c^k\sigma_k(s;w)$

Proof: $\sigma'=S\sigma S^{+}\Leftrightarrow\sigma'(s;w)=c^k\sigma_k(s;w)=c^k\sigma_k$
$\Leftrightarrow(S\otimes S\otimes\cdot\cdot\otimes S)\Omega(s;w)(S^{+}\otimes S^{+}\otimes\cdot\cdot\otimes S^{+})\Gamma(s;w)=\Gamma(s;w)\sigma'(s;w)$
$\Leftrightarrow(S\otimes S\otimes\cdot\cdot\otimes S)\Omega(s;w)\Gamma(s;w)\bar{\Gamma}(s;w)(S^{+}\otimes S^{+}\otimes\cdot\cdot\otimes S^{+})\Gamma(s;w)=\Gamma(s;w)\sigma'(s;w)$
$\Leftrightarrow\sigma'(s;w)=\bar{\Gamma}(s;w)(S\otimes S\otimes\cdot\cdot\otimes S)\Gamma(s;w)\sigma(s;w)\bar{\Gamma}(s;w)(S^{+}\otimes S^{+}\otimes\cdot\cdot\otimes S^{+})\Gamma(s;w)$
$\Leftrightarrow\sigma'(s;w)=[\bar{\Gamma}(s;w)(S\otimes S\otimes\cdot\cdot\otimes S)\Gamma(s;w)]\sigma(s;w)[\bar{\Gamma}(s;w)(S\otimes S\otimes\cdot\cdot\otimes S)\Gamma(s;w)]^{+}$
$\Leftrightarrow\sigma'=S\sigma S^{+}=c^k\sigma_k\Leftrightarrow\sigma'(s;w)=S'\sigma(s;w)S'^{+}=c^k\sigma_k(s;w),S'=[\bar{\Gamma}(s;w)(S\otimes S\otimes\cdot\cdot\otimes S)\Gamma(s;w)]$ □

6 Summary of common matrices

6.1 A non hermitian representation of spin matrix

Starting from the Lorentz transformation property of the fully symmetric two component Weyl spin tensor [5], a special representation of the spin matrix can be obtained.

$$\tau(s)=(\frac{1}{2}\begin{bmatrix}0&2s&0&0&0\\1&0&2s-1&0&0\\0&2&0&\cdots&0\\0&0&\cdots&0&1\\0&0&0&2s&0\end{bmatrix},\frac{i}{2}\begin{bmatrix}0&-2s&0&0&0\\1&0&-(2s-1)&0&0\\0&2&0&\cdots&0\\0&0&\cdots&0&-1\\0&0&0&2s&0\end{bmatrix},\begin{bmatrix}s&0&0&0&0\\0&s-1&0&0&0\\0&0&\cdots&0&0\\0&0&0&-(s-1)&0\\0&0&0&0&-s\end{bmatrix}) \tag{5.6a}$$

$$\sigma(s)=\mathbb{S}(s)\tau(s)\mathbb{S}^{-1}(s),[\tau_{\alpha_\varsigma}(s),\tau_{\beta_\varsigma}(s)]=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\tau_{\gamma_\varsigma}(s),s=\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},\cdots \tag{5.6b}$$

$$\tau^2(s)=s(s+1) \tag{5.6c}$$

$$\tau_{\alpha_\varsigma}(s)\prec\tau_{\alpha_\varsigma}{}^{A_\varsigma}{}_{B_\varsigma}(s),\alpha_\varsigma\sim e^{(i\omega+\varsigma\epsilon)\cdot\gamma},{}^{A_\varsigma}\sim e^{(i\omega+\varsigma\epsilon)\cdot\tau(s)},{}_{B_\varsigma}\sim e^{-(i\omega+\varsigma\epsilon)\cdot\tau^T(s)} \tag{5.6d}$$

The metric tensor corresponding to this spin matrix: $\epsilon(s)\bar{\epsilon}(s)=\bar{\epsilon}(s)\epsilon(s)=I$

$$\epsilon_{A_\varsigma B_\varsigma}(s)\succ\epsilon(s)=\begin{bmatrix}0&0&0&0&(-1)^0C_n^0\\0&0&0&(-1)^1C_n^1&0\\0&0&(-1)^2C_n^2&0&0\\0&\cdots&0&0&0\\(-1)^nC_n^n&0&0&0&0\end{bmatrix},C_n^{-k}\equiv(C_n^k)^{-1} \tag{5.7a}$$

$$\bar{\epsilon}^{A_\varsigma B_\varsigma}(s)\succ\bar{\epsilon}(s)=\begin{bmatrix}0&0&0&0&(-1)^nC_n^{-0}\\0&0&0&(-1)^{n-1}C_n^{-1}&0\\0&0&(-1)^{n-2}C_n^{-2}&0&0\\0&\cdots&0&0&0\\(-1)^0C_n^{-n}&0&0&0&0\end{bmatrix} \tag{5.7b}$$

$$\epsilon(s)=\mathbb{S}^T(s)\varepsilon(s)\mathbb{S}(s),\bar{\epsilon}(s)=\mathbb{S}^{-T}(s)\bar{\varepsilon}(s)\mathbb{S}^{-1}(s),\bar{\epsilon}(s)=(-1)^{2s}\mathbb{S}^{2T}(s)\epsilon(s)\mathbb{S}^2(s) \tag{5.7c}$$

$$\mathbb{S}(s)=\begin{bmatrix}\sqrt{C_{2s}^0}&0&0&0&0\\0&\sqrt{C_{2s}^1}&0&0&0\\0&0&\sqrt{C_{2s}^2}&0&0\\0&0&0&\cdots&0\\0&0&0&0&\sqrt{C_{2s}^{2s}}\end{bmatrix},\mathbb{S}^{-1}(s)=\begin{bmatrix}\sqrt{C_{2s}^{-0}}&0&0&0&0\\0&\sqrt{C_{2s}^{-1}}&0&0&0\\0&0&\sqrt{C_{2s}^{-2}}&0&0\\0&0&0&\cdots&0\\0&0&0&0&\sqrt{C_{2s}^{-2s}}\end{bmatrix} \tag{5.7d}$$

6.2 σ cyclic order representation transform matrix

Cor. 6.2.1.
$S_c(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}i&1\\i&-1\end{bmatrix},S_c^+(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&-i\\1&-1\end{bmatrix},S_c(\frac{1}{2})S_c^+(\frac{1}{2})=S_c^+(\frac{1}{2})S_c(\frac{1}{2})=I,S_c(\frac{1}{2})=ke^{i\frac{\pi}{4}\sigma_y(\frac{1}{2})}e^{i\frac{\pi}{4}\sigma_z(\frac{1}{2})}$

Cor. 6.2.2. $S_c(\frac{1}{2})=e^{i\varphi}e^{-i\frac{\pi}{2}\sigma_y(\frac{1}{2})}e^{-i\frac{\pi}{2}\sigma_z(\frac{1}{2})},S_c(\frac{1}{2})=e^{-i\varphi}e^{i\frac{\pi}{2}\sigma_z(\frac{1}{2})}e^{i\frac{\pi}{2}\sigma_y(\frac{1}{2})}$

Cor. 6.2.3. $S_c(\frac{1}{2})(\sigma_x,\sigma_y,\sigma_z)S_c^+(\frac{1}{2})=(\sigma_y,\sigma_z,\sigma_x),S_c^+(\frac{1}{2})(\sigma_x,\sigma_y,\sigma_z)S_c(\frac{1}{2})=(\sigma_z,\sigma_x,\sigma_y)$

Cor. 6.2.4. $S_{em}(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&i\\1&1\end{bmatrix},S_{em}^+(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}i&1\\-i&1\end{bmatrix},S_{em}(\frac{1}{2})S_{em}^+(\frac{1}{2})=S_{em}^+(\frac{1}{2})S_{em}(\frac{1}{2})=I$

Cor. 6.2.5. $S_{em}(\frac{1}{2})(\sigma_x,\sigma_y,\sigma_z)S_{em}^+(\frac{1}{2})=(-\sigma_z,-\sigma_x,\sigma_y)$

Cor. 6.2.6. $S_{xy}(\sigma_x,\sigma_y,\sigma_z)S_{xy}^+=(-\sigma_y,\sigma_x,\sigma_z),S_{xy}=\begin{bmatrix}1&0\\0&-i\end{bmatrix},S_{xy}^+=\begin{bmatrix}1&0\\0&i\end{bmatrix}$

6.3 $\sigma(s)$ cyclic order representation transform matrix

Cor. 6.3.1. $\sigma^{\alpha_\varsigma}(s)=[e^{(i\omega+\varsigma\epsilon)\cdot\gamma}]^{\alpha_\varsigma}{}_{\beta_\varsigma}e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}\sigma^{\beta_\varsigma}(s)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s)},\sigma^{\alpha_\varsigma}{}_{k_\varsigma}{}^{l_\varsigma}(s)$ *is a constant invariant tensor.*

$$[\Downarrow]$$

Cor. 6.3.2. $S_c(s)=e^{i\varphi}e^{-i\frac{\pi}{2}\sigma_y(s)}e^{-i\frac{\pi}{2}\sigma_z(s)},S_c^+(s)=e^{-i\varphi}e^{i\frac{\pi}{2}\sigma_z(s)}e^{i\frac{\pi}{2}\sigma_y(s)}$

Cor. 6.3.3. $S_c^+(s)[\sigma_x(s),\sigma_y(s),\sigma_z(s)]S_c(s)=[\sigma_z(s),\sigma_x(s),\sigma_y(s)]$

Cor. 6.3.4. $S_c^+(s)[\sigma_x(s),\sigma_y(s),\sigma_z(s)]S_c(s)=[\sigma_z(s),\sigma_x(s),\sigma_y(s)]$

Cor. 6.3.5. $[\sigma_x(s),\sigma_y(s),\sigma_z(s)]\simeq[\hat{e}_x,\hat{e}_y,\hat{e}_z]$

6.4 Electromagnetic pure virtual representation transform matrix and exchange matrix

Cor. 6.4.1. $S_{em}(\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\varsigma & \varsigma & 0 \end{bmatrix}, S_{em}^{+}(\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 & 0 \\ 0 & 0 & i & -\varsigma \\ 0 & 0 & i & \varsigma \\ i & -1 & 0 & 0 \end{bmatrix}, S_{em}(\varsigma)S_{em}^{+}(\varsigma) = S_{em}^{+}(\varsigma)S_{em}(\varsigma) = I_4$

Cor. 6.4.2. $S_{em}(\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\varsigma & \varsigma & 0 \end{bmatrix}, S_{em}^{T}(\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 & 0 & 0 \\ 0 & 0 & -i & -\varsigma \\ 0 & 0 & -i & \varsigma \\ -i & -1 & 0 & 0 \end{bmatrix}, S_{em}^{T}(\varsigma)S_{em}(\varsigma) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = -\sigma_y \otimes \sigma_y$

Cor. 6.4.3. $S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}, S_m^{+}(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}, S_m(1)S_m^{+}(1) = S_m^{+}(1)S_m(1) = I_3$

Cor. 6.4.4. $S_{ex} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_{ex}^2 = I, S_{em}(\varsigma)S_{ex} = S_{em}(-\varsigma), S_{ex}S_{em}^{+}(\varsigma) = S_{em}^{+}(-\varsigma)$

Cor. 6.4.5. $(\sigma \otimes I) = S_{ex}(I \otimes \sigma)S_{ex}, (I \otimes \sigma) = S_{ex}(\sigma \otimes I)S_{ex}$

Cor. 6.4.6. $\sigma_{-\varsigma} = S_{em}(\varsigma)(\sigma \otimes I)S_{em}^{+}(\varsigma), \sigma_{+\varsigma} = S_{em}(\varsigma)(I \otimes \sigma)S_{em}^{+}(\varsigma), \gamma = S_m(1)\sigma(1)S_m^{-}(1)$

6.5 Gravitational pure virtual representation transformation

Def. 6.5.1. $\Psi_{im}(2,\varsigma) := S_{im}(2,\varsigma)\psi(2,\varsigma)$

Cor. 6.5.1. $\Psi_{im}(2,\varsigma) = -\sqrt{2}[\psi^{y_\varsigma x_\varsigma}, -\frac{1}{2}(\psi^{x_\varsigma x_\varsigma} - \psi^{y_\varsigma y_\varsigma}), \psi^{z_\varsigma y_\varsigma}, \varsigma\psi^{z_\varsigma x_\varsigma}, \frac{\sqrt{3}}{2}(\psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma})]^T$

$$S_{im}(2,\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 & -1 \\ 0 & -i & 0 & -i & 0 \\ 0 & -\varsigma & 0 & \varsigma & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 \end{bmatrix}, S_{im}^{+}(2,\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 & 0 & 0 \\ 0 & 0 & i & -\varsigma & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & i & \varsigma & 0 \\ i & -1 & 0 & 0 & 0 \end{bmatrix} \tag{5.8}$$

$$G_{im}(\varsigma) = S_{im}(2,\varsigma)\sigma(2)S_{im}^{+}(2,\varsigma) \qquad S_{im}(2,\varsigma)S_{im}^{+}(2,\varsigma) = S_{im}^{+}(2,\varsigma)S_{im}(2,\varsigma) = I \tag{5.9}$$

$$G_{im}(\varsigma) = \{ \begin{bmatrix} 0 & 0 & 0 & -i\varsigma & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & i\sqrt{3} \\ i\varsigma & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{3} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i\varsigma & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & i\varsigma & 0 & 0 & -i\varsigma\sqrt{3} \\ 0 & 0 & 0 & i\varsigma\sqrt{3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\varsigma & 0 \\ 0 & 0 & -i\varsigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \} \tag{5.10}$$

6.6 Gravitational pure virtual similarity transform matrix

Def. 6.6.1. $\Psi(2,\varsigma) \equiv [\psi_{x_\varsigma x_\varsigma}, \psi_{y_\varsigma x_\varsigma}, \psi_{z_\varsigma x_\varsigma}, \psi_{y_\varsigma y_\varsigma}, \psi_{z_\varsigma y_\varsigma}]^T$

Cor. 6.6.1. $\Psi(2,\varsigma) = S_m(2)\overline{\psi}(2,\varsigma)$

$$S_m(2) = -\frac{1}{2} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 \\ -i & 0 & 0 & 0 & i \\ 0 & 2 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2i & 0 & 2i & 0 \end{bmatrix}, S_m^{-}(2) = -\frac{1}{2} \begin{bmatrix} -1 & 2i & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -i \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -i \\ -1 & -2i & 0 & 1 & 0 \end{bmatrix} \tag{5.11a}$$

$$G_m = S_m(2)\tau(2)S_m^{-}(2) \qquad S_m(2)S_m^{-}(2) = S_m^{-}(2)S_m(2) = I \tag{5.11b}$$

$$G_m = \{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2i \\ i & 0 & 0 & 2i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ -2i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{bmatrix} \} \tag{5.11c}$$

6.7 Conditional matrix for fully symmetric spinor

$$T(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, 2s-1\tau, \tau = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tau^n = \tau, T^n(s) = T(s) \tag{5.12a}$$

Fully symmetric spinor condition : $\tilde{\psi} = T(s)\tilde{\psi}$(**Similar to Majorana and Weyl condition** [4,5]). (5.12b)

Chapter6 New Expressions of Electromagnetic Field Equation

1 Using constant invariant tensors to define various spinors of electromagnetic field [7]

1.1 Classical description of electromagnetic field strength

$$\textbf{Electromagnetic tensor:}F_{ab}=\begin{bmatrix}0 & B_z & -B_y & -iE_x\\ -B_z & 0 & B_x & -iE_y\\ B_y & -B_x & 0 & -iE_z\\ iE_x & iE_y & iE_z & 0\end{bmatrix},\textbf{Dual tensor:}*F_{ab}=\begin{bmatrix}0 & -iE_z & iE_y & B_x\\ iE_z & 0 & -iE_x & B_y\\ -iE_y & iE_x & 0 & B_z\\ -B_x & -B_y & -B_z & 0\end{bmatrix} \tag{6.1}$$

1.2 Complex vector description of electromagnetic field strength

The first definition, it is adopted in this chapter. As follows.

Def. 1.2.1. *The first definition* $\psi_{\alpha_\varsigma}:=\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}=i\varsigma(E-i\varsigma B)_{\alpha_\varsigma}=(i\varsigma E+B)_{\alpha_\varsigma}$

The second definition, it will be used in the subsequent chapters on the separate quantization of electromagnetic fields. As follows.

Def. 1.2.2. *The second definition* $\Psi_{\alpha_\varsigma}:=\frac{\varsigma}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}=\frac{1}{\sqrt{2}}(E-i\varsigma B)_{\alpha_\varsigma}$

The third definition will be used in the subsequent chapter on B-G quantization. As follows.

Def. 1.2.3. *The third definition* $\psi_{\alpha_\varsigma}:=-\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}=-\frac{\varsigma}{\sqrt{2}}(E-i\varsigma B)_{\alpha_\varsigma}$

In the future, there will be time to unify them all into the second definition. The later Penrose and B-G quantization commutation rules are defined by using the third definition.

1.3 Basic properties of electromagnetic field strength

Cor. 1.3.1. $\frac{1}{2}(F_{ab}-\varsigma*F_{ab})=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}$

Proof: $F_{ab}=-F_{ba}$
$\Leftrightarrow F_{ab}=\frac{1}{2}S_{abcd}F^{cd},*F_{ab}:=\frac{1}{2}\varepsilon_{abcd}F^{cd}$
$\Leftrightarrow F_{ab}-\varsigma*F_{ab}=\frac{1}{2}(S_{abcd}-\varsigma\varepsilon_{abcd})F^{cd}$
$\Leftrightarrow F_{ab}-\varsigma*F_{ab}=-\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}F^{cd}$
$\Leftrightarrow F_{ab}-\varsigma*F_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}$
$\Leftrightarrow \frac{1}{2}(F_{ab}-\varsigma*F_{ab})=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}$ □

Cor. 1.3.2. $\psi_{\alpha_\varsigma}=\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\frac{1}{2}(F_{ab}-\varsigma*F_{ab})$

Cor. 1.3.3. $\psi_{\alpha_\varsigma}=-\frac{i}{2}\varsigma\sigma^{ab}_{\varsigma\alpha_\varsigma}*F_{ab}$

Cor. 1.3.4. $\sigma^{ab}_{\varsigma\alpha_\varsigma}(F_{ab}+\varsigma*F_{ab})=0$

Cor. 1.3.5. $F_{ab}-\varsigma*F_{ab}=-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{cd}_{\varsigma\alpha_\varsigma}(F_{cd}-\varsigma*F_{cd})$

Cor. 1.3.6. $F_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'}+\sigma^{\alpha}_{+ab}\psi_\alpha),*F_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'}-\sigma^{\alpha}_{+ab}\psi_\alpha)$

Proof: $F_{ab}-\varsigma*F_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}$
$\Leftrightarrow F_{ab}-*F_{ab}=i\sigma^{\alpha}_{+ab}\psi_\alpha,F_{ab}+*F_{ab}=i\sigma^{\alpha'}_{-ab}\psi_{\alpha'}$
$\Leftrightarrow F_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'}+\sigma^{\alpha}_{+ab}\psi_\alpha),*F_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'}-\sigma^{\alpha}_{+ab}\psi_\alpha)$
$\Leftrightarrow F_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'}+\sigma^{\alpha}_{+ab}\psi_\alpha)$
$\Leftrightarrow *F_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'}-\sigma^{\alpha}_{+ab}\psi_\alpha)$ □

Cor. 1.3.7. $F_{ab}=-F_{ba}\Leftrightarrow F_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'}+\sigma^{\alpha}_{+ab}\psi_\alpha)$

1.4 $\frac{1}{2}$-spinor description of electromagnetic field strength [1,2]

Def. 1.4.1. $\frac{1}{2}$*-spinor tensor of electromagnetic field* $\psi_{A_\varsigma B_\varsigma} := \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}F_{ab}$

Cor. 1.4.1. $\psi_{A_\varsigma B_\varsigma} = \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma} \Leftrightarrow \psi_{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}$

Cor. 1.4.2. $\psi_{A_\varsigma B_\varsigma} = \psi_{B_\varsigma A_\varsigma}$

Cor. 1.4.3. $\psi_{A_\varsigma B_\varsigma} = \frac{-i}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma} * F_{ab}$

Cor. 1.4.4. $\frac{1}{2}(F_{ab} - \varsigma * F_{ab}) = \frac{i\varsigma}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma}\psi_{A_\varsigma B_\varsigma} \Leftrightarrow \psi_{A_\varsigma B_\varsigma} = \frac{i\varsigma}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}\frac{1}{2}(F_{ab} - \varsigma * F_{ab})$

Cor. 1.4.5. $F_{ab} - \varsigma * F_{ab} = -\frac{1}{2}S_{ab}{}^{A_\varsigma B_\varsigma}S^{cd}{}_{A_\varsigma B_\varsigma}(F_{cd} - \varsigma * F_{cd})$

Cor. 1.4.6. $F_{ab} = \frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'} + S_{ab}{}^{AB}\psi_{AB}), *F_{ab} = \frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'} - S_{ab}{}^{AB}\psi_{AB})$

Cor. 1.4.7. $F_{ab} = -F_{ba} \Leftrightarrow F_{ab} = \frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'} + S_{ab}{}^{AB}\psi_{AB})$

combine corollaries (1.3.6),(1.274), (1.275)**, I can get the Penrose correspondence notation** [1,2]

Cor. 1.4.8. $F_{ab} \overset{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}\varepsilon_{AB} + \psi_{AB}\varepsilon_{A'B'}), *F_{ab} \overset{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}\varepsilon_{AB} - \psi_{AB}\varepsilon_{A'B'})$

1.5 1-spinor description of electromagnetic field strength

Def. 1.5.1. 1*-spinor description of electromagnetic field*
$\psi_{k_\varsigma}(1) := \Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi_{A_\varsigma B_\varsigma} = \Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi_{\alpha_\varsigma}, \psi^{k_\varsigma}(1) := \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi^{A_\varsigma B_\varsigma} = \Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi^{\alpha_\varsigma}$

Cor. 1.5.1. $\psi_{A_\varsigma B_\varsigma} = \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}(1), \psi_{\alpha_\varsigma} = \Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi_{k_\varsigma}(1)$

Cor. 1.5.2. $\psi^{A_\varsigma B_\varsigma} = \Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi^{k_\varsigma}(1), \psi^{\alpha_\varsigma} = \Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi^{k_\varsigma}(1)$

From corollary (1.246) **to get** $[\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)]^* \simeq \Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)$

Cor. 1.5.3. $\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1) \simeq \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1) \succ \{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\} = \{\frac{1}{2}(\sigma_z + I), \frac{1}{\sqrt{2}}\sigma_x, \frac{1}{2}(-\sigma_z + I)\}$

Combine the above formula and (1.246) **to get:**

Cor. 1.5.4. $\Gamma^{k_\varsigma}{}_{\alpha_\varsigma}(1) \succ \frac{1}{\sqrt{2}}\begin{bmatrix}i&-1&0\\0&0&-i\sqrt{2}\\-i&-1&0\end{bmatrix}, \Gamma^{\alpha_\varsigma}{}_{k_\varsigma}(1) \succ \frac{1}{\sqrt{2}}\begin{bmatrix}-i&0&i\\-1&0&-1\\0&i\sqrt{2}&0\end{bmatrix}$

1.6 $\frac{1}{2}$-spinor description of electromagnetic field source [1,2]

Def. 1.6.1. $\frac{1}{2}$*-spinor tensor of electromagnetic source*
$J_{A_\varsigma A'_\varsigma} := \frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}J_a, J^{A'_\varsigma A_\varsigma} := \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_a J^a$

Penrose notation: $J_a \overset{P}{=} J_{AA'}, J^a \overset{P}{=} J^{A'A}$

2 Several equivalent expressions of electromagnetic field equation

2.1 Standard description of electromagnetic field gauge theory

$$F_{uv} = \partial_u A_v - \partial_v A_u \tag{6.2}$$

Gauge transformation:

$$\begin{cases}\psi \to U(\theta)\psi, U(\theta) = e^{ig\theta}, \psi\textbf{With charge g}\\ A_u \to U(\theta)A_u U^{-1}(\theta) + \frac{i}{g}[\partial_u U(\theta)]U^{-1}(\theta) = A_u - \partial_u\theta\end{cases} \tag{6.3}$$

Cor. 2.1.1. $D_u\psi \to UD_u\psi, D_u = \partial_u + igA_u$

Proof: $D_u\psi = (\partial_u + igA_u)\psi \to [\partial_u + UigA_uU^{-1} - (\partial_u U)U^{-1}](U\psi)$
$\Leftrightarrow D_u\psi \to [\partial_u(U\psi) + UigA_u\psi - (\partial_u U)\psi]$
$\Leftrightarrow D_u\psi \to U(\partial_u + igA_u)\psi$
$\Leftrightarrow D_u\psi \to UD_u\psi, D_u = \partial_u + igA_u$ □

Cor. 2.1.2. $F_{uv} \to UF_{uv}U^{-1} = F_{uv}$

Cor. 2.1.3. $D_wF_{uv} \to UD_wF_{uv}U^{-1} = D_wF_{uv}, D_w = \partial_w + ig[A_w, \quad] = \partial_w$

2.2 Classical form of electromagnetic field equation

$$\begin{cases}\nabla\cdot\vec{E}=\rho,\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\nabla\times\vec{B}=\vec{J}+\partial_t\vec{E}\end{cases}\Leftrightarrow\begin{cases}\partial^aF_{ab}=-J_b,\partial^a*F_{ab}\equiv 0\\ F_{ab}=\partial_aA_b-\partial_bA_a\end{cases}\tag{6.4}$$

2.3 Complex vector expression of electromagnetic field equation

Complex vector tensor form:

Thm. 2.3.1. $\partial^aF_{ab}=-J_b\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}\partial_a\tilde{\Psi}^{\alpha_\varsigma}=iJ_b;F_{ab}=\partial_aA_b-\partial_bA_a,\tilde{\Psi}^{\alpha_\varsigma}=\begin{bmatrix}\psi^{\alpha_\varsigma}=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}F^{ab}\\ 0\end{bmatrix}$

Proof: $\partial^aF_{ab}=-J_b$
$\Leftrightarrow\partial^aF_{ab}=-J_b,\partial^a*F_{ab}\equiv 0$
$\Leftrightarrow\partial^a(F_{ab}-\varsigma*F_{ab})=-J_b$
$\Leftrightarrow\partial^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma})=-J^b,\alpha_\varsigma=1,2,3$
$\Leftrightarrow\partial^a[(\sigma_\varsigma,-i\varsigma)^{\alpha_\varsigma}|_{ab}\tilde{\Psi}_{\alpha_\varsigma}]=iJ_b,\alpha_\varsigma=1,2,3,4$
$\Leftrightarrow\partial^a[(\sigma_{-\varsigma},-i\varsigma)_a|_b{}^{\alpha_\varsigma}\tilde{\Psi}_{\alpha_\varsigma}]=iJ_b,\alpha_\varsigma=1,2,3,4$
$\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)_a|_b{}^{\alpha_\varsigma}\partial^a\tilde{\Psi}_{\alpha_\varsigma}=iJ_b,\alpha_\varsigma=1,2,3,4$
$\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}\partial_a\tilde{\Psi}^{\alpha_\varsigma}=iJ_b,\alpha_\varsigma=1,2,3,4$ □

Complex vector matrix form:

Cor. 2.3.1. $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}\partial_a\tilde{\Psi}^{\alpha_\varsigma}=iJ_b\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^a\partial_a\tilde{\Psi}(1,\varsigma)=iJ$

Representation transformation:

Cor. 2.3.2. $(\sigma_{-\varsigma},-i\varsigma)^a\partial_a\tilde{\Psi}(1,\varsigma)=iJ\Leftrightarrow(\sigma\otimes I,-i\varsigma)^a\partial_a\tilde{\psi}(1,\varsigma)=i\tilde{J}(1,\varsigma)$

2.4 $\frac{1}{2}$-spinor expression of electromagnetic field equation

$\frac{1}{2}$-spinor Penrose abstract index form [1, 2]

Thm. 2.4.1. $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}\partial_a\tilde{\Psi}^{\alpha_\varsigma}=iJ_b\Leftrightarrow\nabla^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma B_\varsigma}=\frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma},\nabla^{A'_\varsigma A_\varsigma}=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a$

Proof: $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}\partial_a\tilde{\Psi}^{\alpha_\varsigma}=iJ_b$
$\Leftrightarrow\partial^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma})=-J_b$
$\Leftrightarrow\partial^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\cdot\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma})=-J_b$
$\Leftrightarrow iS_{ab}{}^{A_\varsigma B_\varsigma}\partial^a\psi_{A_\varsigma B_\varsigma}=\frac{-\varsigma}{\sqrt{2}}J_b$
$\Leftrightarrow(\frac{\varsigma}{2}\delta_{ab}\varepsilon^{A_\varsigma B_\varsigma}+iS_{ab}{}^{A_\varsigma B_\varsigma})\partial^a\psi_{A_\varsigma B_\varsigma}=\frac{-\varsigma}{\sqrt{2}}J_b$
$\Leftrightarrow\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}\cdot\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}\partial^a\psi_{A_\varsigma B_\varsigma}=\frac{-1}{\sqrt{2}}J_b$
$\Leftrightarrow\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}\partial^a\psi_{A_\varsigma B_\varsigma}=\frac{-1}{\sqrt{2}}J_b\cdot\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\psi_{A_\varsigma B_\varsigma}=\frac{-\varsigma}{\sqrt{2}}\varsigma\varepsilon^{A'_\varsigma B'_\varsigma}J_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow\nabla^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma B_\varsigma}=\frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma},\nabla^{A'_\varsigma A_\varsigma}=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a$ □

$\frac{1}{2}$-spinor tensor form:

Cor. 2.4.1. $\nabla^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma B_\varsigma}=\frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\psi_{A_\varsigma B_\varsigma}=iJ^{A'_\varsigma}{}_{B_\varsigma}$

$\frac{1}{2}$-spinor matrix form:

Cor. 2.4.2. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\psi_{A_\varsigma B_\varsigma}=iJ^{A'_\varsigma}{}_{B_\varsigma}\Leftrightarrow(\sigma\otimes I,-i\varsigma)_a\partial^a\tilde{\psi}(1,\varsigma)=i\tilde{J}(1,\varsigma)$

$\frac{1}{2}$-spinor square matrix form:

Cor. 2.4.3. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\psi_{A_\varsigma B_\varsigma}=iJ^{A'_\varsigma}{}_{B_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)_a\partial^a[\psi]=i[J]$

$\frac{1}{2}$-spinor tensor expression form:(Proof for later.)

Cor. 2.4.4. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{A_\varsigma B_\varsigma}=iJ^{A'_\varsigma}{}_{B_\varsigma}\Leftrightarrow[\partial_a+iS_{ab}(1,\varsigma)\partial^b]_{k_\varsigma}{}^{l_\varsigma}\psi_{l_\varsigma}(1,\varsigma)=\mathbb{J}_{ak_\varsigma}(1,\varsigma)$

Cor. 2.4.5. $\begin{cases}\partial^aF_{ab}=-J_b\\ \partial^a*F_{ab}\equiv 0\end{cases}\Leftrightarrow[\partial_a+iS_{ab}(1,\varsigma)\partial^b]_{k_\varsigma}{}^{l_\varsigma}\psi_{l_\varsigma}(1,\varsigma)=\mathbb{J}_{ak_\varsigma}(1,\varsigma)$

2.5 Conjecture

Thm. 2.5.1. $\partial^a * F_{ab} = 0 \Leftrightarrow F_{ab} = \partial_a A_b - \partial_b A_a \Leftrightarrow \partial^a * F_{ab} \equiv 0$

Thm. 2.5.2. $\partial^a F_{ab} = -J_b, \partial^a * F_{ab} = 0 \Leftrightarrow \partial^a F_{ab} = -J_b, F_{ab} = \partial_a A_b - \partial_b A_a$

2.6 Spin tensor expression form of electromagnetic field equation [7]

Spin tensor matrix of electromagnetic field: $S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma} \succ \begin{bmatrix} 0 & \gamma_z & -\gamma_y & -\varsigma\gamma_x \\ -\gamma_z & 0 & \gamma_x & -\varsigma\gamma_y \\ \gamma_y & -\gamma_x & 0 & -\varsigma\gamma_z \\ \varsigma\gamma_x & \varsigma\gamma_y & \varsigma\gamma_z & 0 \end{bmatrix}$ (6.5)

Thm. 2.6.1. $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma}(1,\varsigma) = -i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\Psi}(1,\varsigma) = iJ$

An intuitive proof method is as follows:

Proof: $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\partial_x + i\gamma_z\partial_y - i\gamma_y\partial_z - i\varsigma\gamma_x\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma xb}^{\beta_\varsigma}J^b \\ (\partial_y + i\gamma_x\partial_z - i\gamma_z\partial_x - i\varsigma\gamma_y\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma yb}^{\beta_\varsigma}J^b \\ (\partial_z + i\gamma_y\partial_x - i\gamma_x\partial_y - i\varsigma\gamma_z\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma zb}^{\beta_\varsigma}J^b \\ (\partial_\pi + i\varsigma\gamma_x\partial_x + i\varsigma\gamma_y\partial_y + i\varsigma\gamma_z\partial_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma\pi b}^{\beta_\varsigma}J^b \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ -\partial_y & \partial_x & -\varsigma\partial_\pi \\ -\partial_z & \varsigma\partial_\pi & \partial_x \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma} \\ \psi^{y_\varsigma} \\ \psi^{z_\varsigma} \end{bmatrix} = \begin{bmatrix} \varsigma J^\pi \\ J^z \\ -J^y \end{bmatrix}, \begin{bmatrix} \partial_y & -\partial_x & \varsigma\partial_\pi \\ \partial_x & \partial_y & \partial_z \\ -\varsigma\partial_\pi & -\partial_z & \partial_y \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma} \\ \psi^{y_\varsigma} \\ \psi^{z_\varsigma} \end{bmatrix} = \begin{bmatrix} -J^z \\ \varsigma J^\pi \\ J^x \end{bmatrix} \\ \begin{bmatrix} \partial_z & -\varsigma\partial_\pi & -\partial_x \\ \varsigma\partial_\pi & \partial_z & -\partial_y \\ \partial_x & \partial_y & \partial_z \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma} \\ \psi^{y_\varsigma} \\ \psi^{z_\varsigma} \end{bmatrix} = \begin{bmatrix} J^y \\ -J^x \\ \varsigma J^\pi \end{bmatrix}, i\partial_\pi\Psi(1,\varsigma) = \varsigma\gamma\cdot\nabla\Psi(1,\varsigma) - i\varsigma\vec{J} \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi\Psi(1,\varsigma) = i\varsigma\nabla\times\Psi(1,\varsigma) - i\varsigma\vec{J} \\ \nabla\cdot\Psi(1,\varsigma) = \varsigma J^\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi\Psi(1,\varsigma) = \varsigma\gamma\cdot\nabla\Psi(1,\varsigma) - i\varsigma\vec{J} \\ \nabla\cdot\Psi(1,\varsigma) = \varsigma J^\pi \end{cases}$$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\Psi}(1,\varsigma) = iJ$ □

Another more analytical and abstract proof is as follows:

Proof: $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}$

$\Leftrightarrow \sigma_{\varsigma a}^{\beta_\varsigma}{}^c\sigma_{\varsigma\gamma_\varsigma cb}\partial^b\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b$

$\Leftrightarrow \sigma_{\varsigma ac}^{\beta_\varsigma}\sigma_{\varsigma\gamma_\varsigma}^{cb}\partial_b\psi^{\gamma_\varsigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b$

$\Leftrightarrow \sigma_{\beta_\varsigma}^{\varsigma ad}\sigma_{\varsigma ac}^{\beta_\varsigma}\sigma_{\varsigma\gamma_\varsigma}^{cb}\partial_b\psi^{\gamma_\varsigma} = -i\sigma_{\beta_\varsigma}^{\varsigma ad}\sigma_{\varsigma ab}^{\beta_\varsigma}J^b$

$\Leftrightarrow \sigma_{\varsigma\gamma_\varsigma}^{db}\partial_b\psi^{\gamma_\varsigma} = -iJ^d$

$\Leftrightarrow \sigma_{\varsigma\alpha_\varsigma}^{ab}\partial_a\psi^{\alpha_\varsigma} = iJ^b, \alpha_\varsigma = 1,2,3$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}\partial_a\tilde{\Psi}^{\alpha_\varsigma} = iJ_b, \alpha_\varsigma = 1,2,3,4$ □

This equation (3.3.2) is completely equivalent to the electromagnetic field equation. It is just the spin tensor expression of the electromagnetic field equation.

Lem. 2.6.1. $\mathbb{J}_a^{\beta_\varsigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b \Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\varsigma} = -\mathbb{J}_z^{y_\varsigma} = -\varsigma\mathbb{J}_\pi^{x_\varsigma} = J^x \\ \mathbb{J}_z^{x_\varsigma} = -\mathbb{J}_x^{z_\varsigma} = -\varsigma\mathbb{J}_\pi^{y_\varsigma} = J^y \\ \mathbb{J}_x^{y_\varsigma} = -\mathbb{J}_y^{x_\varsigma} = -\varsigma\mathbb{J}_\pi^{z_\varsigma} = J^z \\ \mathbb{J}_x^{x_\varsigma} = \mathbb{J}_y^{y_\varsigma} = \mathbb{J}_z{}^{z_\varsigma} = \varsigma J^\pi \end{cases}$

Expand and then we can prove it by expanding. The above spin equation is about special source terms, so what happens to general source terms? Please look at the following theorem.

Thm. 2.6.2. $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = \mathbb{J}_a^{\beta_\varsigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\Psi}(1,\varsigma) = iJ, \mathbb{J}_a^{\beta_\varsigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b$

Proof: $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = \mathbb{J}_a^{\beta_\varsigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\partial_x + i\gamma_z\partial_y - i\gamma_y\partial_z - i\varsigma\gamma_x\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = \mathbb{J}_x{}^{\beta_\varsigma} \\ (\partial_y + i\gamma_x\partial_z - i\gamma_z\partial_x - i\varsigma\gamma_y\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = \mathbb{J}_y^{\beta_\varsigma} \\ (\partial_z + i\gamma_y\partial_x - i\gamma_x\partial_y - i\varsigma\gamma_z\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = \mathbb{J}_z{}^{\beta_\varsigma} \\ (\partial_\pi + i\varsigma\gamma_x\partial_x + i\varsigma\gamma_y\partial_y + i\varsigma\gamma_z\partial_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma} = \mathbb{J}_\pi^{\beta_\varsigma} \end{cases}$$

$$\Leftrightarrow\begin{cases}\begin{bmatrix}\partial_x & \partial_y & \partial_z\\ -\partial_y & \partial_x & -\varsigma\partial_\pi\\ -\partial_z & \varsigma\partial_\pi & \partial_x\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma}\\ \psi^{y_\varsigma}\\ \psi^{z_\varsigma}\end{bmatrix}=\begin{bmatrix}\mathbb{J}_x^{x_\varsigma}\\ \mathbb{J}_x^{y_\varsigma}\\ \mathbb{J}_x^{z_\varsigma}\end{bmatrix}\Leftrightarrow\begin{cases}\nabla\cdot\Psi(1,\varsigma)=\mathbb{J}_x^{x_\varsigma}\\ [\nabla\times\Psi(1,\varsigma)]^{z_\varsigma}-\varsigma\partial_\pi\psi^{z_\varsigma}(1,\varsigma)=\mathbb{J}_x^{y_\varsigma}\\ -[\nabla\times\Psi(1,\varsigma)]^{y_\varsigma}+\varsigma\partial_\pi\psi^{y_\varsigma}(1,\varsigma)=\mathbb{J}_x^{z_\varsigma}\end{cases}\\ \begin{bmatrix}\partial_y & -\partial_x & \varsigma\partial_\pi\\ \partial_x & \partial_y & \partial_z\\ -\varsigma\partial_\pi & -\partial_z & \partial_y\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma}\\ \psi^{y_\varsigma}\\ \psi^{z_\varsigma}\end{bmatrix}=\begin{bmatrix}\mathbb{J}_y^{x_\varsigma}\\ \mathbb{J}_y^{y_\varsigma}\\ \mathbb{J}_y^{z_\varsigma}\end{bmatrix}\Leftrightarrow\begin{cases}-[\nabla\times\Psi(1,\varsigma)]^{z_\varsigma}+\varsigma\partial_\pi\psi^{z_\varsigma}(1,\varsigma)=\mathbb{J}_y^{x_\varsigma}\\ \nabla\cdot\Psi(1,\varsigma)=\mathbb{J}_y^{y_\varsigma}\\ [\nabla\times\Psi(1,\varsigma)]^{x_\varsigma}-\varsigma\partial_\pi\psi^{x_\varsigma}(1,\varsigma)=\mathbb{J}_y^{z_\varsigma}\end{cases}\\ \begin{bmatrix}\partial_z & -\varsigma\partial_\pi & -\partial_x\\ \varsigma\partial_\pi & \partial_z & -\partial_y\\ \partial_x & \partial_y & \partial_z\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma}\\ \psi^{y_\varsigma}\\ \psi^{z_\varsigma}\end{bmatrix}=\begin{bmatrix}\mathbb{J}_z^{x_\varsigma}\\ \mathbb{J}_z^{y_\varsigma}\\ \mathbb{J}_z^{z_\varsigma}\end{bmatrix}\Leftrightarrow\begin{cases}[\nabla\times\Psi(1,\varsigma)]^{y_\varsigma}-\varsigma\partial_\pi\psi^{y_\varsigma}(1,\varsigma)=\mathbb{J}_z^{x_\varsigma}\\ -[\nabla\times\Psi(1,\varsigma)]^{x_\varsigma}+\varsigma\partial_\pi\psi^{x_\varsigma}(1,\varsigma)=\mathbb{J}_z^{y_\varsigma}\\ \nabla\cdot\Psi(1,\varsigma)=\mathbb{J}_z^{z_\varsigma}\end{cases}\\ \partial_\pi\Psi(1,\varsigma)+i\varsigma\gamma\cdot\nabla\psi=\mathbb{J}_\pi\Leftrightarrow\partial_\pi\Psi(1,\varsigma)-\varsigma\nabla\times\Psi(1,\varsigma)=\mathbb{J}_\pi\end{cases}$$

$$\Leftrightarrow\begin{cases}\mathbb{J}_y^{z_\varsigma}=-\mathbb{J}_z^{y_\varsigma}=-\varsigma\mathbb{J}_\pi^{x_\varsigma}:=J^x\\ \mathbb{J}_z^{x_\varsigma}=-\mathbb{J}_x^{z_\varsigma}=-\varsigma\mathbb{J}_\pi^{y_\varsigma}:=J^y\\ \mathbb{J}_x^{y_\varsigma}=-\mathbb{J}_y^{x_\varsigma}=-\varsigma\mathbb{J}_\pi^{z_\varsigma}:=J^z\\ \mathbb{J}_x^{x_\varsigma}=\mathbb{J}_y^{y_\varsigma}=\mathbb{J}_z{}^{z_\varsigma}:=\varsigma J^\pi\\ \partial_\pi\Psi(1,\varsigma)-\varsigma\nabla\times\Psi(1,\varsigma)=i\vec{J}\\ \nabla\cdot\Psi(1,\varsigma)=-iJ^\pi\end{cases}$$

$$\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^a\partial_a\tilde{\Psi}(1,\varsigma)=iJ,\mathbb{J}_a^{\beta_\varsigma}=-i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b$$ □

Another more analytical and abstract proof is as follows:

Thm. 2.6.3. $(\partial_a+iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma}=\mathbb{J}_a^{\beta_\varsigma},S_{ab}=i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}\Leftrightarrow\mathbb{J}_a^{\beta_\varsigma}=\sigma_{\varsigma ab}^{\beta_\varsigma}\sigma_{\varsigma\gamma_\varsigma}^{bc}\partial_c\psi^{\gamma_\varsigma}$

Proof: $(\partial_a+iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma}=\mathbb{J}_a^{\beta_\varsigma},S_{ab}=i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow\sigma_{\varsigma a}^{\beta_\varsigma}{}^c\sigma_{\varsigma\gamma_\varsigma cb}\partial^b\psi^{\gamma_\varsigma}=\mathbb{J}_a^{\beta_\varsigma}$
$\Leftrightarrow\mathbb{J}_a^{\beta_\varsigma}=\sigma_{\varsigma ab}^{\beta_\varsigma}\sigma_{\varsigma\alpha_\varsigma}^{bc}\partial_c\psi^{\alpha_\varsigma}$

$$\Leftrightarrow\begin{cases}\mathbb{J}_y^{z_\varsigma}=-\mathbb{J}_z^{y_\varsigma}=-\varsigma\mathbb{J}_\pi^{x_\varsigma}=i\sigma_{\varsigma\alpha_\varsigma}^{xb}\partial_b\psi^{\alpha_\varsigma}\\ \mathbb{J}_z^{x_\varsigma}=-\mathbb{J}_x^{z_\varsigma}=-\varsigma\mathbb{J}_\pi^{y_\varsigma}=i\sigma_{\varsigma\alpha_\varsigma}^{yb}\partial_b\psi^{\alpha_\varsigma}\\ \mathbb{J}_x^{y_\varsigma}=-\mathbb{J}_y^{x_\varsigma}=-\varsigma\mathbb{J}_\pi^{z_\varsigma}=i\sigma_{\varsigma\alpha_\varsigma}^{zb}\partial_b\psi^{\alpha_\varsigma}\\ \mathbb{J}_x^{x_\varsigma}=\mathbb{J}_y^{y_\varsigma}=\mathbb{J}_z^{z_\varsigma}=i\varsigma\sigma_{\varsigma\alpha_\varsigma}^{\pi b}\partial_b\psi^{\alpha_\varsigma}\end{cases}$$ □

This theorem indicates that the source term of this spin equation is limited and not arbitrary. Only the source term case described in the previous theorem has a solution, while the other cases have no solution.

Cor. 2.6.1. $(\partial_a+iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma}=\mathbb{J}_a^{\beta_\varsigma},S_{ab}=i\sigma_{\varsigma ab}^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}$ *have solutions.* $\Leftrightarrow\mathbb{J}_a^{\beta_\varsigma}=-i\sigma_{\varsigma ab}^{\beta_\varsigma}J^b,\exists J^b$

2.7 Classical separated form of electromagnetic field equation

Cor. 2.7.1. $(\sigma_{-\varsigma},-i\varsigma)^a\partial_a\tilde{\Psi}(1,\varsigma)=iJ\Leftrightarrow(\gamma,-i\varsigma)^a\partial_a\Psi(1,\varsigma)=i\vec{J},i\varsigma\nabla\cdot\Psi(1,\varsigma)=iJ_\pi$

Cor. 2.7.2. $S:=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&\sqrt{1}&0\\0&0&0&\sqrt{2}\\0&-\varsigma\sqrt{1}&\varsigma\sqrt{1}&0\end{bmatrix}$

Cor. 2.7.3. $\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&\sqrt{1}&0\\0&0&0&\sqrt{2}\\0&-\varsigma\sqrt{1}&\varsigma\sqrt{1}&0\end{bmatrix}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&-\varsigma\sqrt{1}\\0&\sqrt{1}&0&\varsigma\sqrt{1}\\0&0&\sqrt{2}&0\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0&-\varsigma\\1&0&1&0\\0&1&0&\varsigma\\-\varsigma&0&\varsigma&0\end{bmatrix}$

Cor. 2.7.4. $\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&\sqrt{1}&0\\0&0&0&\sqrt{2}\\0&-\varsigma\sqrt{1}&\varsigma\sqrt{1}&0\end{bmatrix}i\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&-\varsigma\sqrt{1}\\0&\sqrt{1}&0&\varsigma\sqrt{1}\\0&0&\sqrt{2}&0\end{bmatrix}=\frac{i}{\sqrt{2}}\begin{bmatrix}0&-1&0&\varsigma\\1&0&-1&0\\0&1&0&\varsigma\\-\varsigma&0&-\varsigma&0\end{bmatrix}$

Cor. 2.7.5. $\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&\sqrt{1}&0\\0&0&0&\sqrt{2}\\0&-\varsigma\sqrt{1}&\varsigma\sqrt{1}&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&-\varsigma\sqrt{1}\\0&\sqrt{1}&0&\varsigma\sqrt{1}\\0&0&\sqrt{2}&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&0&0&\varsigma\\0&0&-1&0\\0&\varsigma&0&0\end{bmatrix}$

Cor. 2.7.6.
$(\sigma\otimes I,-i\varsigma)^a\partial_a\tilde{\psi}(1,\varsigma)=i\tilde{J}(1,\varsigma)\overset{S}{\Leftrightarrow}\begin{cases}[\sigma(1),-i\varsigma]^a\partial_a\psi(1,\varsigma)=i\bar{N}(1)\tilde{J}(1,\varsigma)\\ i\varsigma\nabla\cdot S_m(1)\psi(1,\varsigma)=iJ_\pi\end{cases}\quad\begin{cases}\begin{bmatrix}\bar{N}(1)\tilde{J}(1,\varsigma)\\J_\pi\end{bmatrix}=S\tilde{J}(1,\varsigma)\\ \begin{bmatrix}\psi(1,\varsigma)\\0\end{bmatrix}=S\tilde{\psi}(1,\varsigma)\end{cases}$

Cor. 2.7.7. $S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}, S_m^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}, S_m(1)S_m^+(1) = S_m^+(1)S_m(1) = I_3$

Cor. 2.7.8. $\begin{cases} [\sigma(1), -i\varsigma]^a \partial_a \psi(1,\varsigma) = i\bar{N}(1)\tilde{J}(1,\varsigma) \\ i\varsigma\nabla \cdot S_m(1)\psi(1,\varsigma) = iJ_\pi \end{cases} \overset{S_m(1)}{\Leftrightarrow} \begin{cases} (\gamma, -i\varsigma)^a \partial_a \Psi(1,\varsigma) = i\vec{J}, \vec{J} = S_m(1)\bar{N}(1)\tilde{J}(1,\varsigma) \\ i\varsigma\nabla \cdot \Psi(1,\varsigma) = iJ_\pi, \Psi(1,\varsigma) = S_m(1)\bar{N}(1)\tilde{\psi}(1,\varsigma) \end{cases}$

Chapter7 New Expressions of Yang-Mills Field Equation

1 Using constant invariant tensors to define various spinors of Yang-Mills field [7]

1.1 Classical description of Yang-Mills field strength

$$\textbf{Yang-Mills tensor:}F^\sigma_{ab}=\begin{bmatrix}0 & B^\sigma_z & -B^\sigma_y & -iE^\sigma_x\\ -B^\sigma_z & 0 & B^\sigma_x & -iE^\sigma_y\\ B^\sigma_y & -B^\sigma_x & 0 & -iE^\sigma_z\\ iE^\sigma_x & iE^\sigma_y & iE^\sigma_z & 0\end{bmatrix},\textbf{Dual tensor:}*F^\sigma_{ab}=\begin{bmatrix}0 & -iE^\sigma_z & iE^\sigma_y & B^\sigma_x\\ iE^\sigma_z & 0 & -iE^\sigma_x & B^\sigma_y\\ -iE^\sigma_y & iE^\sigma_x & 0 & B^\sigma_z\\ -B^\sigma_x & -B^\sigma_y & -B^\sigma_z & 0\end{bmatrix} \tag{7.1}$$

1.2 Complex vector description of Yang-Mills field strength

Def. 1.2.1. *Yang-Mills complex vector* $\psi^\sigma_{\alpha_\varsigma}:=\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F^\sigma_{ab}=i\varsigma(E-i\varsigma B)^\sigma_{\alpha_\varsigma}=(i\varsigma E+B)^\sigma_{\alpha_\varsigma}$

Cor. 1.2.1. $\frac{1}{2}(F^\sigma_{ab}-\varsigma*F^\sigma_{ab})=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^\sigma_{\alpha_\varsigma}$

Proof: $F^\sigma_{ab}=-F^\sigma_{ba}$
$\Leftrightarrow F^\sigma_{ab}=\frac{1}{2}S_{abcd}F^{cd},*F^\sigma_{ab}:=\frac{1}{2}\varepsilon_{abcd}F^{cd}$
$\Leftrightarrow F^\sigma_{ab}-\varsigma*F^\sigma_{ab}=\frac{1}{2}(S_{abcd}-\varsigma\varepsilon_{abcd})F^{cd}$
$\Leftrightarrow F^\sigma_{ab}-\varsigma*F^\sigma_{ab}=-\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}F^{cd}$
$\Leftrightarrow F^\sigma_{ab}-\varsigma*F^\sigma_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^\sigma_{\alpha_\varsigma}$
$\Leftrightarrow \frac{1}{2}(F^\sigma_{ab}-\varsigma*F^\sigma_{ab})=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^\sigma_{\alpha_\varsigma}$ □

Cor. 1.2.2. $\psi^\sigma_{\alpha_\varsigma}=\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\frac{1}{2}(F^\sigma_{ab}-\varsigma*F^\sigma_{ab})$

Cor. 1.2.3. $\psi^\sigma_{\alpha_\varsigma}=-\frac{i}{2}\varsigma\sigma^{ab}_{\varsigma\alpha_\varsigma}*F^\sigma_{ab}$

Cor. 1.2.4. $\sigma^{ab}_{\varsigma\alpha_\varsigma}(F^\sigma_{ab}+\varsigma*F^\sigma_{ab})=0$

Cor. 1.2.5. $F^\sigma_{ab}-\varsigma*F^\sigma_{ab}=-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{cd}_{\varsigma\alpha_\varsigma}(F^\sigma_{cd}-\varsigma*F^\sigma_{cd})$

Cor. 1.2.6. $F^\sigma_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha}),*F^\sigma_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}-\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha})$

Proof: $F^\sigma_{ab}-\varsigma*F^\sigma_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^\sigma_{\alpha_\varsigma}$
$\Leftrightarrow F^\sigma_{ab}-*F^\sigma_{ab}=i\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha},F^\sigma_{ab}+*F^\sigma_{ab}=i\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}$
$\Leftrightarrow F^\sigma_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha}),*F^\sigma_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}-\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha})$
$\Leftrightarrow F^\sigma_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha})$
$\Leftrightarrow *F^\sigma_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}-\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha})$ □

Cor. 1.2.7. $F^\sigma_{ab}=-F^\sigma_{ba}\Leftrightarrow F^\sigma_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^\sigma_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^\sigma_{\alpha})$

1.3 $\frac{1}{2}$-spinor description of Yang-Mills field strength [1,2]

Def. 1.3.1. $\frac{1}{2}$*-spinor tensor of Yang-Mills field* $\psi^\sigma_{A_\varsigma B_\varsigma}:=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi^\sigma_{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}F^\sigma_{ab}$

Cor. 1.3.1. $\psi^\sigma_{A_\varsigma B_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi^\sigma_{\alpha_\varsigma}\Leftrightarrow\psi^\sigma_{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi^\sigma_{A_\varsigma B_\varsigma}$

Cor. 1.3.2. $\psi^\sigma_{A_\varsigma B_\varsigma}=\psi^\sigma_{B_\varsigma A_\varsigma}$

Cor. 1.3.3. $\psi^\sigma_{A_\varsigma B_\varsigma}=\frac{-i}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}*F^\sigma_{ab}$

Cor. 1.3.4. $\frac{1}{2}(F^\sigma_{ab}-\varsigma*F^\sigma_{ab})=\frac{i\varsigma}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma}\psi^\sigma_{A_\varsigma B_\varsigma}\Leftrightarrow\psi^\sigma_{A_\varsigma B_\varsigma}=\frac{i\varsigma}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}\frac{1}{2}(F^\sigma_{ab}-\varsigma*F^\sigma_{ab})$

Cor. 1.3.5. $F^\sigma_{ab}-\varsigma*F^\sigma_{ab}=-\frac{1}{2}S_{ab}{}^{A_\varsigma B_\varsigma}S^{cd}{}_{A_\varsigma B_\varsigma}(F^\sigma_{cd}-\varsigma*F^\sigma_{cd})$

Cor. 1.3.6. $F^\sigma_{ab}=\frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi^\sigma_{A'B'}+S_{ab}{}^{AB}\psi^\sigma_{AB}),*F^\sigma_{ab}=\frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi^\sigma_{A'B'}-S_{ab}{}^{AB}\psi^\sigma_{AB})$

Cor. 1.3.7. $F^{\sigma}_{ab}=-F^{\sigma}_{ba}\Leftrightarrow F^{\sigma}_{ab}=\frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi^{\sigma}_{A'B'}+S_{ab}{}^{AB}\psi^{\sigma}_{AB})$

Combine corollaries 1.3.6 **and** (1.274), (1.275)**, I can get the Penrose correspondence notation** [1, 2]

Cor. 1.3.8. $F^{\sigma}_{ab}\overset{P}{=}\frac{1}{\sqrt{2}}(\psi^{\sigma}_{A'B'}\varepsilon_{AB}+\psi^{\sigma}_{AB}\varepsilon_{A'B'}),*F^{\sigma}_{ab}\overset{P}{=}\frac{1}{\sqrt{2}}(\psi^{\sigma}_{A'B'}\varepsilon_{AB}-\psi^{\sigma}_{AB}\varepsilon_{A'B'})$

1.4 1-spinor description of Yang-Mills field strength

Def. 1.4.1. 1*-spinor description of Yang-Mills field* $\psi^{\sigma}_{k_\varsigma}(1):=\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi^{\sigma}_{A_\varsigma B_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi^{\sigma}_{\alpha_\varsigma}$

Cor. 1.4.1. $\psi^{\sigma}_{A_\varsigma B_\varsigma}=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi^{\sigma}_{k_\varsigma}(1),\psi^{\sigma}_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi^{\sigma}_{k_\varsigma}(1)$

1.5 $\frac{1}{2}$-spinor description of Yang-Mills field source [1, 2]

Def. 1.5.1. $\frac{1}{2}$*-spinor tensor of Yang-Mills source*
$J^{A'_\varsigma A_\varsigma\sigma}:=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}J^{a\sigma},J^{\sigma}_{A_\varsigma A'_\varsigma}:=\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}J^{\sigma}_a$

Penrose notation: $J^{a\sigma}\overset{P}{=}J^{A'A\sigma},J_a\overset{P}{=}J^{\sigma}_{AA'}$

2 Several equivalent expressions of Yang-Mills field equation

2.1 Standard description of Yang-Mills theory

$$\begin{cases}F^{\sigma}_{uv}T_\sigma=\partial_uA^{\sigma}_vT_\sigma-\partial_vA^{\sigma}_uT_\sigma+ig[A^{\tau}_uT_\tau,A^{\rho}_vT_\rho]\\ [T_\tau,T_\rho]=if^{\sigma}_{\tau\rho}T_\sigma,c^\sigma T_\sigma=0\Leftrightarrow c^\sigma=0\end{cases}\tag{7.2}$$

Gauge transformation:

$$\begin{cases}\psi\to U(\theta)\psi,U(\theta)=e^{ig\theta^\sigma T_\sigma},\psi\ \textbf{with YangMills charge g}\\ A^{\sigma}_uT_\sigma\to U(\theta)A^{\sigma}_uT_\sigma U^{-1}(\theta)+\frac{i}{g}[\partial_uU(\theta)]U^{-1}(\theta)\end{cases}\tag{7.3}$$

Cor. 2.1.1. $D_u\psi\to UD_u\psi,D_u=\partial_u+igA^{\sigma}_uT_\sigma$

Proof: $D_u\psi=(\partial_u+igA^{\sigma}_uT_\sigma)\psi\to[\partial_u+UigA^{\sigma}_uT_\sigma U^{-1}-(\partial_uU)U^{-1}](U\psi)$
$\Leftrightarrow D_u\psi\to[\partial_u(U\psi)+UigA^{\sigma}_uT_\sigma\psi-(\partial_uU)\psi]$
$\Leftrightarrow D_u\psi\to U(\partial_u+igA^{\sigma}_uT_\sigma)\psi$
$\Leftrightarrow D_u\psi\to UD_u\psi,D_u=\partial_u+igA^{\sigma}_uT_\sigma$ □

Lem. 2.1.1. $\partial_u(U^{-1})=-U^{-1}\partial_u(U)U^{-1}$

Proof: $\partial_u(UU^{-1})=\partial_u(I)$
$\Leftrightarrow\partial_u(U)U^{-1}+U\partial_u(U^{-1})=0$
$\Leftrightarrow U\partial_u(U^{-1})=-\partial_u(U)U^{-1}$
$\Leftrightarrow\partial_u(U^{-1})=-U^{-1}\partial_u(U)U^{-1}$ □

Cor. 2.1.2. $F^{\sigma}_{uv}T_\sigma\to UF^{\sigma}_{uv}T_\sigma U^{-1}$

Proof: $F^{\sigma}_{uv}T_\sigma=\partial_uA^{\sigma}_vT_\sigma-\partial_vA^{\sigma}_uT_\sigma+ig[A^{\rho}_uT_\rho,A^{\tau}_vT_\tau]$
$\to\partial_u[UA^{\sigma}_vT_\sigma U^{-1}+\frac{i}{g}(\partial_vU)U^{-1}]-\partial_v[UA^{\sigma}_uT_\sigma U^{-1}+\frac{i}{g}(\partial_uU)U^{-1}]$
$\quad+ig[UA^{\rho}_uT_\rho U^{-1}+\frac{i}{g}(\partial_uU)U^{-1},UA^{\tau}_vT_\tau U^{-1}+\frac{i}{g}(\partial_vU)U^{-1}]$
$\Leftrightarrow F^{\sigma}_{uv}T_\sigma\to U(\partial_uA^{\sigma}_vT_\sigma-\partial_vA^{\sigma}_uT_\sigma+ig[A^{\rho}_uT_\rho,A^{\tau}_vT_\tau])U^{-1}$
$\Leftrightarrow F^{\sigma}_{uv}T_\sigma\to UF^{\sigma}_{uv}T_\sigma U^{-1}$ □

Cor. 2.1.3. $D_wF^{\sigma}_{uv}T_\sigma\to UD_wF^{\sigma}_{uv}T_\sigma U^{-1},D_w=\nabla_w+ig[A^{\sigma}_wT_\sigma,\quad]$

Proof: $D_wF^{\sigma}_{uv}T_\sigma=\partial_wF^{\sigma}_{uv}T_\sigma+ig[A^{\rho}_wT_\rho,F^{\tau}_{uv}T_\tau]$
$\to\partial_w(UF^{\sigma}_{uv}T_\sigma U^{-1})+ig[UA^{\rho}_wT_\rho U^{-1}+\frac{i}{g}(\partial_wU)U^{-1},UF^{\tau}_{uv}T_\tau U^{-1}]$
$\Leftrightarrow D_wF^{\sigma}_{uv}T_\sigma\to U(\partial_wF^{\sigma}_{uv}T_\sigma+ig[A^{\rho}_wT_\rho,F^{\tau}_{uv}T_\tau])U^{-1}$
$\Leftrightarrow D_wF^{\sigma}_{uv}T_\sigma\to UD_wF^{\sigma}_{uv}T_\sigma U^{-1}$ □

Cor. 2.1.4. $D_wF^{\sigma}_{uv}=\nabla_wF^{\sigma}_{uv}-gf^{\sigma}_{\rho\tau}A^{\rho}_wF^{\tau}_{uv}$

Cor. 2.1.5. $D_wF^{\sigma}_{uv}=\nabla_wF^{\sigma}_{uv}+igA^{\rho}_w(-if_\rho{}^\sigma{}_\tau)F^{\tau}_{uv}$

Cor. 2.1.6. $D_wF_{uv}=[\nabla_w+igA^{\rho}_w(-if_\rho)]F_{uv},D_w=\nabla_w+igA^{\rho}_w(-if_\rho)$

2.2 Component form of Yang-Mills equation

Cor. 2.2.1. $F^\sigma_{uv}T_\sigma = \partial_u A^\sigma_v T_\sigma - \partial_v A^\sigma_u T_\sigma + ig[A^\rho_u T_\rho, A^\tau_v T_\tau] \Leftrightarrow F^\sigma_{uv} = \partial_u A^\sigma_v - \partial_v A^\sigma_u - g f^\sigma_{\rho\tau} A^\rho_u A^\tau_v$

Cor. 2.2.2.
$F^\sigma_{uv}T_\sigma = (\partial_u + igA^\rho_u T_\rho)A^\sigma_v T_\sigma - (\partial_v + igA^\rho_v T_\rho)A^\sigma_u T_\sigma \Leftrightarrow F_{uv} = [\partial_u + \frac{1}{2}igA^\rho_u(-if_\rho)]A_v - [\partial_v + \frac{1}{2}igA^\rho_v(-if_\rho)]A_u$

Cor. 2.2.3. *Gauge transformation:* $\delta\psi = ig\theta^\sigma T_\sigma \psi, \delta A_u = ig\theta^\rho(-if_\rho)A_u - \partial_u\theta$

Cor. 2.2.4. $\delta F_{uv} = ig\theta^\rho(-if_\rho)F_{uv}$

2.3 Frame description of Yang-Mills equation

Def. 2.3.1. $F^\sigma_{ab} := e^u_a e^v_b F^\sigma_{uv}, A^\sigma_a := e^u_a A^\sigma_u$

Frame description of Yang-Mills equation

$$D^a F^\sigma_{ab} = -J^\sigma_b, D^a * F^\sigma_{ab} \equiv 0 \tag{7.4}$$

2.4 Classical separated form of Yang-Mills field equation

$$\begin{cases}\nabla_d \cdot \vec{E}^\sigma = \rho^\sigma, \nabla_d \times \vec{E}^\sigma = -D_t\vec{B}^\sigma \\ \nabla_d \cdot \vec{B}^\sigma = 0, \nabla_d \times \vec{B}^\sigma = \vec{J}^\sigma + D_t\vec{E}^\sigma\end{cases} \Leftrightarrow \quad D^a F^\sigma_{ab} = -J^\sigma_b, D^a * F^\sigma_{ab} \equiv 0 \tag{7.5}$$

2.5 Complex vector expression of Yang-Mills field equation

Complex vector tensor form:

Thm. 2.5.1. $D^a F^\sigma_{ab} = -J^\sigma_b \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ^\sigma_b; F^\sigma_{ab} = D_a A_b - D_b A_a, \tilde{\Psi}^{\alpha_\varsigma\sigma} = \begin{bmatrix} \psi^{\alpha_\varsigma\sigma} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}F^{ab\sigma} \\ 0 \end{bmatrix}$

Proof: $D^a F^\sigma_{ab} = -J^\sigma_b$
$\Leftrightarrow D^a F^\sigma_{ab} = -J^\sigma_b, D^a * F^\sigma_{ab} \equiv 0$
$\Leftrightarrow D^a(F^\sigma_{ab} - \varsigma * F^\sigma_{ab}) = -J^\sigma_b$
$\Leftrightarrow D^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^\sigma_{\alpha_\varsigma}) = -J^{b\sigma}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow D^a[(\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}|_{ab}\tilde{\Psi}^{\alpha_\varsigma\sigma}] = iJ^\sigma_b, \alpha_\varsigma = 1,2,3,4$
$\Leftrightarrow D^a[(\sigma_{-\varsigma}, -i\varsigma)_a|_b{}^{\alpha_\varsigma}\tilde{\Psi}^\sigma_{\alpha_\varsigma}] = iJ^\sigma_b, \alpha_\varsigma = 1,2,3,4$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ^\sigma_b, \alpha_\varsigma = 1,2,3,4$ □

Complex vector matrix form:

Cor. 2.5.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ^\sigma_b \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = iJ^\sigma$

Representation transformation:

Cor. 2.5.2. $(\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = iJ^\sigma \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = i\tilde{J}^\sigma(1,\varsigma)$

2.6 $\frac{1}{2}$-spinor expression of Yang-Mills field equation

$\frac{1}{2}$-spinor Penrose abstract index form [1,2]

Thm. 2.6.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ^\sigma_b \Leftrightarrow \nabla^{A'_\varsigma A_\varsigma}_d \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma, \nabla^{A'_\varsigma A_\varsigma}_d = \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_a D^a$

Proof: $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ^\sigma_b$
$\Leftrightarrow D^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^\sigma_{\alpha_\varsigma}) = -J^\sigma_b$
$\Leftrightarrow D^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab} \cdot \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi^\sigma_{A_\varsigma B_\varsigma}) = -J^\sigma_b$
$\Leftrightarrow iS_{ab}{}^{A_\varsigma B_\varsigma} D^a \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^\sigma_b$
$\Leftrightarrow (\frac{\varsigma}{2}\delta_{ab}\varepsilon^{A_\varsigma B_\varsigma} + iS_{ab}{}^{A_\varsigma B_\varsigma}) D^a \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^\sigma_b$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_a \varepsilon_{A'_\varsigma B'_\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{B'_\varsigma B_\varsigma}_b D^a \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-1}{\sqrt{2}} J^\sigma_b$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_a \varepsilon_{A'_\varsigma B'_\varsigma} D^a \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-1}{\sqrt{2}} J^\sigma_b \cdot \frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^b_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_a D^a \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} \varsigma \varepsilon^{A'_\varsigma B'_\varsigma} J_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow \nabla^{A'_\varsigma A_\varsigma}_d \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma, \nabla^{A'_\varsigma A_\varsigma}_d = \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_a D^a$ □

$\frac{1}{2}$-spinor tensor form:

Cor. 2.6.1. $\nabla^{A'_\varsigma A_\varsigma}_d \psi^\sigma_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma \Leftrightarrow (\sigma, -i\varsigma)^{A'_\varsigma A_\varsigma}_a D_a \psi^\sigma_{A_\varsigma B_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma$

$\frac{1}{2}$-spinor matrix form:

Cor. 2.6.2. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma} D_a\psi^\sigma_{A_\varsigma B_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma \Leftrightarrow (\sigma\otimes I,-i\varsigma)^a D_a\tilde{\Psi}^\sigma(1,\varsigma) = i\tilde{J}^\sigma(1,\varsigma)$

$\frac{1}{2}$-spinor square matrix form:

Cor. 2.6.3. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma} D_a\psi^\sigma_{A_\varsigma B_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma \Leftrightarrow (\sigma,-i\varsigma)^a D_a[\psi]^\sigma = i[J]^\sigma$

$\frac{1}{2}$-spinor tensor expression form:(Proof for later.)

Cor. 2.6.4. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a\psi^\sigma_{A_\varsigma B_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma \Leftrightarrow [\partial_a + iS_{ab}(1,\varsigma)\partial^b]_{k_\varsigma}{}^{l_\varsigma}\psi^\sigma_{l_\varsigma}(1,\varsigma) = \mathbb{J}^\sigma_{ak_\varsigma}(1,\varsigma)$

Cor. 2.6.5. $\begin{cases}\partial^a F^\sigma_{ab} = -J^\sigma_b\\ \partial^a * F^\sigma_{ab} \equiv 0\end{cases} \Leftrightarrow [\partial_a + iS_{ab}(1,\varsigma)\partial^b]_{k_\varsigma}{}^{l_\varsigma}\psi^\sigma_{l_\varsigma}(1,\varsigma) = \mathbb{J}^\sigma_{ak_\varsigma}(1,\varsigma)$

2.7 Conjecture

Thm. 2.7.1. $D^a * F^\sigma_{ab} = 0 \Leftrightarrow F^\sigma_{ab} = D_aA_b - D_bA_a \Leftrightarrow D^a * F^\sigma_{ab} \equiv 0$

Thm. 2.7.2. $D^aF^\sigma_{ab} = -J^\sigma_b, D^a * F^\sigma_{ab} = 0 \Leftrightarrow D^aF^\sigma_{ab} = -J^\sigma_b, F^\sigma_{ab} = D_aA_b - D_bA_a$

2.8 Spinor tensor expression form of Yang-Mills field equation [7]

Spinor tensor matrix of Yang-Mills field: $S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \succ \begin{bmatrix} 0 & \gamma_z & -\gamma_y & -\varsigma\gamma_x\\ -\gamma_z & 0 & \gamma_x & -\varsigma\gamma_y\\ \gamma_y & -\gamma_x & 0 & -\varsigma\gamma_z\\ \varsigma\gamma_x & \varsigma\gamma_y & \varsigma\gamma_z & 0\end{bmatrix}$ (7.6)

Thm. 2.8.1. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma}(1,\varsigma) = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{\Psi}^\sigma(1,\varsigma) = iJ^\sigma$

An intuitive proof method is as follows:

Proof: $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases}(D_x + i\gamma_zD_y - i\gamma_yD_z - i\varsigma\gamma_xD_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma xb}J^{b\sigma}\\ (D_y + i\gamma_xD_z - i\gamma_zD_x - i\varsigma\gamma_yD_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma yb}J^{b\sigma}\\ (D_z + i\gamma_yD_x - i\gamma_xD_y - i\varsigma\gamma_zD_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma zb}J^{b\sigma}\\ (D_\pi + i\varsigma\gamma_xD_x + i\varsigma\gamma_yD_y + i\varsigma\gamma_zD_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma\pi b}J^{b\sigma}\end{cases}$$

$$\Leftrightarrow \begin{cases}\begin{bmatrix}D_x & D_y & D_z\\ -D_y & D_x & -\varsigma D_\pi\\ -D_z & \varsigma D_\pi & D_x\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma\sigma}\\ \psi^{y_\varsigma\sigma}\\ \psi^{z_\varsigma\sigma}\end{bmatrix} = \begin{bmatrix}\varsigma J^{\pi\sigma}\\ J^{z\sigma}\\ -J^{y\sigma}\end{bmatrix}, \begin{bmatrix}D_y & -D_x & \varsigma D_\pi\\ D_x & D_y & D_z\\ -\varsigma D_\pi & -D_z & D_y\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma\sigma}\\ \psi^{y_\varsigma\sigma}\\ \psi^{z_\varsigma\sigma}\end{bmatrix} = \begin{bmatrix}-J^{z\sigma}\\ \varsigma J^{\pi\sigma}\\ J^{x\sigma}\end{bmatrix}\\ \begin{bmatrix}D_z & -\varsigma D_\pi & -D_x\\ \varsigma D_\pi & D_z & -D_y\\ D_x & D_y & D_z\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma\sigma}\\ \psi^{y_\varsigma\sigma}\\ \psi^{z_\varsigma\sigma}\end{bmatrix} = \begin{bmatrix}J^{y\sigma}\\ -J^{x\sigma}\\ \varsigma J^{\pi\sigma}\end{bmatrix}, iD_\pi\Psi^\sigma(1,\varsigma) = \varsigma\gamma\cdot\nabla_d\Psi^\sigma(1,\varsigma) - i\varsigma\vec{J}^\sigma\end{cases}$$

$$\Leftrightarrow \begin{cases}iD_\pi\Psi^\sigma(1,\varsigma) = i\varsigma\nabla_d\times\Psi^\sigma(1,\varsigma) - i\varsigma\vec{J}^\sigma\\ \nabla_d\cdot\Psi^\sigma(1,\varsigma) = \varsigma J^{\pi\sigma}\end{cases}$$

$$\Leftrightarrow \begin{cases}iD_\pi\Psi^\sigma(1,\varsigma) = \varsigma\gamma\cdot\nabla_d\Psi^\sigma(1,\varsigma) - i\varsigma\vec{J}^\sigma\\ \nabla_d\cdot\Psi^\sigma(1,\varsigma) = \varsigma J^{\pi\sigma}\end{cases}$$

$\Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{\Psi}^\sigma(1,\varsigma) = iJ$ □

Another more analytical and abstract proof is as follows:

Proof: $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma a}{}^c\sigma_{\varsigma\gamma_\varsigma cb}D^b\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}$

$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma ac}\sigma^{cb}_{\varsigma\gamma_\varsigma}D_b\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}$

$\Leftrightarrow \sigma^{\varsigma ad}_{\beta_\varsigma}\sigma^{\beta_\varsigma}_{\varsigma ac}\sigma^{cb}_{\varsigma\gamma_\varsigma}D_b\psi^{\gamma_\varsigma\sigma} = -i\sigma^{\varsigma ad}_{\beta_\varsigma}\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}$

$\Leftrightarrow \sigma^{db}_{\varsigma\gamma_\varsigma}D_b\psi^{\gamma_\varsigma\sigma} = -iJ^{d\sigma}$

$\Leftrightarrow \sigma^{ab}_{\varsigma\alpha_\varsigma}D_a\psi^{\alpha_\varsigma\sigma} = iJ^{b\sigma}, \alpha_\varsigma = 1,2,3$

$\Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ^\sigma_b, \alpha_\varsigma = 1,2,3,4$ □

This equation (3.3.2) is completely equivalent to the Yang-Mills field equation. It is just the spin tensor expression of the Yang-Mills field equation.

Lem. 2.8.1. $\mathbb{J}^{\beta_\varsigma\sigma}_a = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma} \Leftrightarrow \begin{cases}\mathbb{J}^{z_\varsigma\sigma}_y = -\mathbb{J}^{y_\varsigma\sigma}_z = -\varsigma\mathbb{J}^{x_\varsigma\sigma}_\pi = J^{x\sigma}\\ \mathbb{J}^{x_\varsigma\sigma}_z = -\mathbb{J}^{z_\varsigma\sigma}_x = -\varsigma\mathbb{J}^{y_\varsigma\sigma}_\pi = J^{y\sigma}\\ \mathbb{J}^{y_\varsigma\sigma}_x = -\mathbb{J}^{x_\varsigma\sigma}_y = -\varsigma\mathbb{J}^{z_\varsigma\sigma}_\pi = J^{z\sigma}\\ \mathbb{J}^{x_\varsigma\sigma}_x = \mathbb{J}^{y_\varsigma\sigma}_y = \mathbb{J}^{z_\varsigma\sigma}_z = \varsigma J^{\pi\sigma}\end{cases}$

Expand and then we can prove it by expanding. The above spin equation is about special source terms, so what happens to general source terms? Please look at the following theorem.

Thm. 2.8.2. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = iJ^\sigma, \mathbb{J}_a^{\beta_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}$

Proof: $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma\gamma_x D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_x^{\beta_\varsigma\sigma} \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma\gamma_y D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_y^{\beta_\varsigma\sigma} \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma\gamma_z D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_z^{\beta_\varsigma\sigma} \\ (D_\pi + i\varsigma\gamma_x D_x + i\varsigma\gamma_y D_y + i\varsigma\gamma_z D_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_\pi^{\beta_\varsigma\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_x^{x_\varsigma\sigma} \\ \mathbb{J}_x^{y_\varsigma\sigma} \\ \mathbb{J}_x^{z_\varsigma\sigma} \end{bmatrix} \Leftrightarrow \begin{cases} \nabla_d \cdot \Psi^\sigma(1,\varsigma) = \mathbb{J}_x^{x_\varsigma\sigma} \\ [\nabla_d \times \Psi^\sigma(1,\varsigma)]^{z_\varsigma\sigma} - \varsigma D_\pi\psi^{z_\varsigma\sigma}(1,\varsigma) = \mathbb{J}_x^{y_\varsigma\sigma} \\ -[\nabla_d \times \Psi^\sigma(1,\varsigma)]^{y_\varsigma\sigma} + \varsigma D_\pi\psi^{y_\varsigma\sigma}(1,\varsigma) = \mathbb{J}_x^{z_\varsigma\sigma} \end{cases} \\ \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_y^{x_\varsigma\sigma} \\ \mathbb{J}_y^{y_\varsigma\sigma} \\ \mathbb{J}_y^{z_\varsigma\sigma} \end{bmatrix} \Leftrightarrow \begin{cases} -[\nabla_d \times \Psi^\sigma(1,\varsigma)]^{z_\varsigma\sigma} + \varsigma D_\pi\psi^{z_\varsigma\sigma}(1,\varsigma) = \mathbb{J}_y^{x_\varsigma\sigma} \\ \nabla_d \cdot \Psi^\sigma(1,\varsigma) = \mathbb{J}_y^{y_\varsigma\sigma} \\ [\nabla_d \times \Psi^\sigma(1,\varsigma)]^{x_\varsigma\sigma} - \varsigma D_\pi\psi^{x_\varsigma\sigma}(1,\varsigma) = \mathbb{J}_y^{z_\varsigma\sigma} \end{cases} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_z^{x_\varsigma\sigma} \\ \mathbb{J}_z^{y_\varsigma\sigma} \\ \mathbb{J}_z^{z_\varsigma\sigma} \end{bmatrix} \Leftrightarrow \begin{cases} [\nabla_d \times \Psi^\sigma(1,\varsigma)]^{y_\varsigma\sigma} - \varsigma D_\pi\psi^{y_\varsigma\sigma}(1,\varsigma) = \mathbb{J}_z^{x_\varsigma\sigma} \\ -[\nabla_d \times \Psi^\sigma(1,\varsigma)]^{x_\varsigma\sigma} + \varsigma D_\pi\psi^{x_\varsigma\sigma}(1,\varsigma) = \mathbb{J}_z^{y_\varsigma\sigma} \\ \nabla_d \cdot \Psi^\sigma(1,\varsigma) = \mathbb{J}_z^{z_\varsigma\sigma} \end{cases} \\ D_\pi\Psi^\sigma(1,\varsigma) + i\varsigma\gamma\cdot\nabla_d\psi^\sigma = \mathbb{J}_\pi^\sigma \Leftrightarrow D_\pi\Psi^\sigma(1,\varsigma) - \varsigma\nabla_d \times \Psi^\sigma(1,\varsigma) = \mathbb{J}_\pi^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\varsigma\sigma} = -\mathbb{J}_z^{y_\varsigma\sigma} = -\varsigma\mathbb{J}_\pi^{x_\varsigma\sigma} := J^{x\sigma} \\ \mathbb{J}_z^{x_\varsigma\sigma} = -\mathbb{J}_x^{z_\varsigma\sigma} = -\varsigma\mathbb{J}_\pi^{y_\varsigma\sigma} := J^{y\sigma} \\ \mathbb{J}_x^{y_\varsigma\sigma} = -\mathbb{J}_y^{x_\varsigma\sigma} = -\varsigma\mathbb{J}_\pi^{z_\varsigma\sigma} := J^{z\sigma} \\ \mathbb{J}_x^{x_\varsigma\sigma} = \mathbb{J}_y^{y_\varsigma\sigma} = \mathbb{J}_z^{z_\varsigma\sigma} := \varsigma J^{\pi\sigma} \\ D_\pi\Psi^\sigma(1,\varsigma) - \varsigma\nabla_d \times \Psi^\sigma(1,\varsigma) = i\vec{J}^\sigma \\ \nabla_d \cdot \Psi^\sigma(1,\varsigma) = -iJ^{\pi\sigma} \end{cases}$$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = iJ^\sigma, \mathbb{J}_a^{\beta_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}$ □

Another more analytical and abstract proof is as follows:

Thm. 2.8.3. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\sigma} = \sigma^{\beta_\varsigma}_{\varsigma ab}\sigma^{bc}_{\varsigma\gamma_\varsigma}D_c\psi^{\gamma_\varsigma\sigma}$

Proof: $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma a}{}^c\sigma_{\varsigma\gamma_\varsigma cb}D^b\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}$

$\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\sigma} = \sigma^{\beta_\varsigma}_{\varsigma ab}\sigma^{bc}_{\varsigma\alpha_\varsigma}D_c\psi^{\alpha_\varsigma\sigma}$

$$\Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\varsigma\sigma} = -\mathbb{J}_z^{y_\varsigma\sigma} = -\varsigma\mathbb{J}_\pi^{x_\varsigma\sigma} = i\sigma^{xb}_{\varsigma\alpha_\varsigma}D_b\psi^{\alpha_\varsigma\sigma} \\ \mathbb{J}_z^{x_\varsigma\sigma} = -\mathbb{J}_x^{z_\varsigma\sigma} = -\varsigma\mathbb{J}_\pi^{y_\varsigma\sigma} = i\sigma^{yb}_{\varsigma\alpha_\varsigma}D_b\psi^{\alpha_\varsigma\sigma} \\ \mathbb{J}_x^{y_\varsigma\sigma} = -\mathbb{J}_y^{x_\varsigma\sigma} = -\varsigma\mathbb{J}_\pi^{z_\varsigma\sigma} = i\sigma^{zb}_{\varsigma\alpha_\varsigma}D_b\psi^{\alpha_\varsigma\sigma} \\ \mathbb{J}_x^{x_\varsigma\sigma} = \mathbb{J}_y^{y_\varsigma\sigma} = \mathbb{J}_z^{z_\varsigma\sigma} = i\varsigma\sigma^{\pi b}_{\varsigma\alpha_\varsigma}D_b\psi^{\alpha_\varsigma\sigma} \end{cases}$$ □

This theorem indicates that the source term of this spin equation is limited and not arbitrary. Only the source term case described in the previous theorem has a solution, while the other cases have no solution.

Cor. 2.8.1. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$ *have solutions.* $\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\sigma} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}, \exists J^{b\sigma}$

2.9 Classical separated form of Yang-Mills field equation

Cor. 2.9.1. $(\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1,\varsigma) = iJ^\sigma \Leftrightarrow (\gamma, -i\varsigma)^a D_a \Psi^\sigma(1,\varsigma) = i\vec{J}^\sigma, \nabla_d \cdot \Psi^\sigma(1,\varsigma) = \varsigma J^\sigma_{\pi_\varsigma}$

Chapter8 New Expressions of Gravitational Field Equation

1 Various descriptions of physical quantities of gravitational field [11–14]

1.1 Classical description of physical quantities of gravitational field [11–14]

1.1.1 Curvature tensor of gravitational field

Symmetrical properties of curvature tensor:

Antisymmetry: $R^{abcd} = -R^{bacd}, R^{abcd} = -R^{abdc}$ (8.1)

Symmetry: $R^{abcd} = R^{cdab}$ (8.2)

Cyclic symmetry: $R^{abcd} + R^{adbc} + R^{acdb} = 0$ (8.3)

1.1.2 Ricci tensor and scalar curvature of gravitational field

Def. 1.1.1. *Ricci tensor* $R^{ab} := g_{cd}R^{cadb}, R^{a*b} := g_{cd}R^{ca(*db)} \equiv 0$, *Scalar curvature* $R := g_{ab}R^{ab} = R_{ab}{}^{ab}$

Cor. 1.1.1. $R^{abcd} + R^{adbc} + R^{acdb} = 0 \Rightarrow R^{a*b} = 0$

Proof: $R^{abcd} + R^{adbc} + R^{acdb} = 0$
$\Rightarrow \varepsilon_{ebcd}(R^{abcd} + R^{adbc} + R^{acdb}) = 0$
$\Rightarrow \varepsilon_{ebcd}R^{abcd} + \varepsilon_{edbc}R^{adbc} + \varepsilon_{ecdb}R^{acdb} = 0$
$\Rightarrow 3\varepsilon_{ebcd}R^{abcd} = 0$
$\Rightarrow g_{cd}R^{ca(*db)} = 0$
$\Rightarrow R^{a*b} = 0$ □

1.1.3 Weyl tensor of gravitational field

Def. 1.1.2. $C^{abcd} := R^{abcd} + \frac{1}{2}g^{a[d}R^{c]b} + \frac{1}{2}g^{b[c}R^{d]a} + \frac{1}{6}g^{a[c}g^{d]b}R$

Symmetrical properties of Weyl tensor:

Antisymmetry: $C^{abcd} = -C^{bacd}, C^{abcd} = -C^{abdc}$ (8.4)

Symmetry: $C^{abcd} = C^{cdab}$ (8.5)

Cyclic symmetry: $C^{abcd} + C^{adbc} + C^{acdb} = 0$ (8.6)

Cor. 1.1.2. $R^{abcd} = C^{abcd} - \frac{1}{2}g^{a[d}R^{c]b} - \frac{1}{2}g^{b[c}R^{d]a} - \frac{1}{6}g^{a[c}g^{d]b}R$

Cor. 1.1.3. $C^{ab} = g_{cd}C^{cadb} = 0, C^{a*b} = g_{cd}C^{ca(*db)} = 0$

1.2 Yang-Mills description of physical quantities in gravitational field [6]

1.2.1 Yang-Mills description of gravitational field curvature tensor

Def. 1.2.1. *Gravitational field YM curvature tensor:* $F^{ab\alpha_\varsigma} := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}R^{abcd}$

Following the reasoning of the electromagnetic field situation, there are completely similar conclusions.

Cor. 1.2.1. $\frac{1}{2}[R^{abcd} - \varsigma R^{ab(*cd)}] = \frac{i}{2}\sigma^{cd}_{\varsigma\alpha_\varsigma}F^{ab\alpha_\varsigma}$

Cor. 1.2.2. $F^{ab\alpha_\varsigma} = -\frac{i\varsigma}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}R^{ab(*cd)}$

Cor. 1.2.3. $\sigma^{\alpha_\varsigma}_{\varsigma cd}[R^{abcd} + \varsigma R^{ab(*cd)}] = 0$

Cor. 1.2.4. $F^{ab\alpha_\varsigma} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}\frac{1}{2}[R^{abcd} - \varsigma R^{ab(*cd)}]$

Cor. 1.2.5. $R^{abcd} - \varsigma R^{ab(*cd)} = -\frac{1}{4}\sigma^{cd}_{\varsigma\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{\varsigma ef}(R^{abef} - \varsigma R^{ab(*ef)})$

Cor. 1.2.6. $R^{abcd} = \frac{i}{2}(\sigma^{cd}_{-\alpha'}F^{ab\alpha'} + \sigma^{cd}_{+\alpha}F^{ab\alpha}), R^{ab(*cd)} = \frac{i}{2}(\sigma^{cd}_{-\alpha'}F^{ab\alpha'} - \sigma^{cd}_{+\alpha}F^{ab\alpha})$

Unlike electromagnetic field, gravitational field has the following different conclusions.

Cor. 1.2.7. $R^{ab} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} + F^{\alpha}\sigma_{+\alpha})^{ab}, 0 = R^{a*b} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} - F^{\alpha}\sigma_{+\alpha})^{ab}$

Cor. 1.2.8. $R^{ab} = -i(F^{\alpha'}\sigma_{-\alpha'})^{ab} = -i(F^{\alpha}\sigma_{+\alpha})^{ab}, F^{\alpha'}\sigma_{-\alpha'} = F^{\alpha}\sigma_{+\alpha}$

Cor. 1.2.9. $R^{ab} = -i(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}, F^{\alpha'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma} = F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma}$

Cor. 1.2.10. $R = i\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}{}^{\alpha_\varsigma}$

1.2.2 Yang-Mills ddescription of gravitational field Weyl tensor

Def. 1.2.2. *gravitational field YM Weyl tensor* $C^{ab\alpha_\varsigma} := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}C^{abcd}$

Following the reasoning of the curvature tensor situation, there are completely similar conclusions.

Cor. 1.2.11. $\frac{1}{2}[C^{abcd} - \varsigma C^{ab(*cd)}] = \frac{i}{2}\sigma^{cd}_{\varsigma\alpha_\varsigma}C^{ab\alpha_\varsigma}$

Cor. 1.2.12. $C^{ab\alpha_\varsigma} = -\frac{i\varsigma}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}C^{ab(*cd)}$

Cor. 1.2.13. $\sigma^{\alpha_\varsigma}_{\varsigma cd}[C^{abcd} + \varsigma C^{ab(*cd)}] = 0$

Cor. 1.2.14. $C^{ab\alpha_\varsigma} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}\frac{1}{2}[C^{abcd} - \varsigma C^{ab(*cd)}]$

Cor. 1.2.15. $C^{abcd} - \varsigma C^{ab(*cd)} = -\frac{1}{4}\sigma^{cd}_{\varsigma\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{\varsigma ef}(C^{abef} - \varsigma C^{ab(*ef)})$

Cor. 1.2.16. $C^{abcd} = \frac{i}{2}(\sigma^{cd}_{-\alpha'}C^{ab\alpha'} + \sigma^{cd}_{+\alpha}C^{ab\alpha}), C^{ab(*cd)} = \frac{i}{2}(\sigma^{cd}_{-\alpha'}C^{ab\alpha'} - \sigma^{cd}_{+\alpha}C^{ab\alpha})$

Unlike the curvature tensor case, the Weyl tensor has the following different conclusions.

Cor. 1.2.17. $0 = C^{ab} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} + F^{\alpha}\sigma_{+\alpha})^{ab}, 0 = C^{a*b} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} - F^{\alpha}\sigma_{+\alpha})^{ab}$

Cor. 1.2.18. $C^{\alpha'}\sigma_{-\alpha'} = C^{\alpha}\sigma_{+\alpha} = 0, C^{\alpha'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma} = C^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma} = 0$

Cor. 1.2.19. $C = i\sigma^{ab}_{\varsigma\alpha_\varsigma}C^{\alpha_\varsigma}_{ab} = 0$

The relation between curvature tensor and Weyl tensor:

Cor. 1.2.20.
$R^{abcd} = C^{abcd} - \frac{1}{2}g^{a[d}R^{c]b} - \frac{1}{2}g^{b[c}R^{d]a} - \frac{1}{6}g^{a[c}g^{d]b}R \Rightarrow F^{ab\alpha_\varsigma} = C^{ab\alpha_\varsigma} + \frac{i}{2}\sigma^{\alpha_\varsigma a}_{\varsigma}{}_{c}R^{cb} - \frac{i}{2}\sigma^{\alpha_\varsigma b}_{\varsigma}{}_{c}R^{ca} - \frac{i}{6}\sigma^{\alpha_\varsigma ab}_{\varsigma}R$

Cor. 1.2.21. $C^{ab\alpha_\varsigma} = F^{ab\alpha_\varsigma} - \frac{1}{2}(\sigma^{\alpha_\varsigma c[a}_{\varsigma}\sigma^{b]d}_{\varsigma\beta_\varsigma} + \frac{1}{3}\sigma^{\alpha_\varsigma ab}_{\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma})F^{\beta_\varsigma}_{cd}$

1.3 Ashtekar gauge representation of gravitational field curvature tensor [34]

1.3.1 Preparation

X, Y are real four dimensional vectors or tensors in an orthogonal frame.

Lem. 1.3.1. $X^*_{a'_\varsigma} = \eta^{a_\varsigma}_{a'_\varsigma}X_{a_\varsigma}, X^*_{a'_\varsigma b'_\varsigma} = \eta^{a_\varsigma}_{a'_\varsigma}\eta^{b_\varsigma}_{b'_\varsigma}X_{a_\varsigma b_\varsigma}, X^*_{a'_\varsigma b'_\varsigma c'_\varsigma} = \eta^{a_\varsigma}_{a'_\varsigma}\eta^{b_\varsigma}_{b'_\varsigma}\eta^{c_\varsigma}_{c'_\varsigma}X_{a_\varsigma b_\varsigma c_\varsigma}\cdots$

Lem. 1.3.2. $X^*_{a'_\varsigma}Y^{*a'_\varsigma} = X_{a_\varsigma}Y^{a_\varsigma}, X^*_{a'_\varsigma b'_\varsigma}Y^{*a'_\varsigma b'_\varsigma} = X_{a_\varsigma b_\varsigma}Y^{a_\varsigma b_\varsigma}, X^*_{a'_\varsigma b'_\varsigma c'_\varsigma}Y^{*a'_\varsigma b'_\varsigma c'_\varsigma} = X_{a_\varsigma b_\varsigma c_\varsigma}Y^{a_\varsigma b_\varsigma c_\varsigma}, \cdots$

1.3.2 Ashtekar gauge representation of gravitational field curvature tensor [34]

Gravitational field curvature tensor $R_{uv}{}^{cd} = \partial_u\omega^{cd}_v - \partial_v\omega^{cd}_u + \omega^{ce}_{[u}\omega_{v]e}{}^d$ (8.7)

Def. 1.3.1. *Introduce Ashtekar variable* [34] $A^{\alpha_\varsigma}_u := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}\omega^{cd}_u$

Cor. 1.3.1. $[A^{\alpha_\varsigma}_u]^* = A^{*\alpha'_\varsigma}_{u'} = \eta^u_{u'}A^{\alpha'_\varsigma}_u$

Proof: $[A^{\alpha_\varsigma}_u]^* = \frac{i}{2}\sigma^{\alpha'_\varsigma}_{\varsigma c'd'}\omega^*_{u'}{}^{c'd'} = \eta^u_{u'}\frac{i}{2}\sigma^{\alpha'_\varsigma}_{-\varsigma cd}\omega^{cd}_u = \eta^u_{u'}A^{\alpha'_\varsigma}_u$ □

Cor. 1.3.2. $[F^{\alpha_\varsigma}_{uv}]^* = F^{*\alpha'_\varsigma}_{u'v'} = \eta^u_{u'}\eta^v_{v'}F^{\alpha'_\varsigma}_{uv}$

Proof: $[F^{\alpha_\varsigma}_{uv}]^* = F^{*\alpha'_\varsigma}_{u'v'} = \frac{i}{2}\sigma^{\alpha'_\varsigma}_{\varsigma c'd'}R^*_{u'v'}{}^{c'd'} = \frac{i}{2}\eta^u_{u'}\eta^v_{v'}\sigma^{\alpha'_\varsigma}_{-\varsigma cd}R_{uv}{}^{cd} = \eta^u_{u'}\eta^v_{v'}F^{\alpha'_\varsigma}_{uv}$ □

Cor. 1.3.3. $\omega^{ce}_{[u}\omega_{v]e}{}^d = -\frac{i}{2}(\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}\sigma^{cd}_{-\varsigma\alpha'_\varsigma}A^{\beta'_\varsigma}_uA^{\gamma'_\varsigma}_v + \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\sigma^{cd}_{\varsigma\alpha_\varsigma}A^{\beta_\varsigma}_uA^{\gamma_\varsigma}_v)$

Proof: $\omega_{[u}^{ce}\omega_{v]e}{}^{d} = \omega_{u}^{ce}\omega_{ve}{}^{d} - \omega_{v}^{ce}\omega_{ue}{}^{d}$

$\Leftrightarrow \omega_{[u}^{ce}\omega_{v]e}{}^{d} = \delta_{ef}\frac{i}{2}(\sigma^{ce}_{-\varsigma\alpha'_\varsigma}A_u^{\alpha'_\varsigma} + \sigma^{ce}_{\varsigma\alpha_\varsigma}A_u^{\alpha_\varsigma})\frac{i}{2}(\sigma^{fd}_{-\varsigma\beta'_\varsigma}A_v^{\beta'_\varsigma} + \sigma^{fd}_{\varsigma\beta_\varsigma}A_v^{\beta_\varsigma})$

$\quad - \delta_{ef}\frac{i}{2}(\sigma^{ce}_{-\varsigma\alpha'_\varsigma}A_v^{\alpha'_\varsigma} + \sigma^{ce}_{\varsigma\alpha_\varsigma}A_v^{\alpha_\varsigma})\frac{i}{2}(\sigma^{fd}_{-\varsigma\beta'_\varsigma}A_u^{\beta'_\varsigma} + \sigma^{fd}_{\varsigma\beta_\varsigma}A_u^{\beta_\varsigma})$

$\Leftrightarrow \omega_{[u}^{ce}\omega_{v]e}{}^{d} = -\frac{1}{4}\delta_{ef}(\sigma^{ce}_{-\varsigma[\alpha'_\varsigma}\sigma^{fd}_{-\varsigma\beta'_\varsigma]}A_u^{\alpha'_\varsigma}A_v^{\beta'_\varsigma} + \sigma^{ce}_{\varsigma[\alpha_\varsigma}\sigma^{fd}_{-\varsigma\beta'_\varsigma]}A_u^{\alpha_\varsigma}A_v^{\beta'_\varsigma}$

$\quad + \sigma^{ce}_{-\varsigma[\alpha'_\varsigma}\sigma^{fd}_{\varsigma\beta_\varsigma]}A_u^{\alpha'_\varsigma}A_v^{\beta_\varsigma} + \sigma^{ce}_{\varsigma[\alpha_\varsigma}\sigma^{fd}_{\varsigma\beta_\varsigma]}A_u^{\alpha_\varsigma}A_v^{\beta_\varsigma})$

$\Leftrightarrow \omega_{[u}^{ce}\omega_{v]e}{}^{d} = -\frac{1}{4}(2i\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma^{cd}_{-\varsigma\gamma'_\varsigma}A_u^{\alpha'_\varsigma}A_v^{\beta'_\varsigma} + 0 + 0 + 2i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma^{cd}_{\varsigma\gamma_\varsigma}A_u^{\alpha_\varsigma}A_v^{\beta_\varsigma})$

$\Leftrightarrow \omega_{[u}^{ce}\omega_{v]e}{}^{d} = -\frac{i}{2}(\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma^{cd}_{-\varsigma\gamma'_\varsigma}A_u^{\alpha'_\varsigma}A_v^{\beta'_\varsigma} + \varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma^{cd}_{\varsigma\gamma_\varsigma}A_u^{\alpha_\varsigma}A_v^{\beta_\varsigma})$

$\Leftrightarrow \omega_{[u}^{ce}\omega_{v]e}{}^{d} = -\frac{i}{2}(\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}\sigma^{cd}_{-\varsigma\alpha'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma} + \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\sigma^{cd}_{\varsigma\alpha_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma})$ □

Cor. 1.3.4. $R_{uv}{}^{cd} = \partial_u\omega_v^{cd} - \partial_v\omega_u^{cd} + \omega_{[u}^{ce}\omega_{v]e}{}^{d} \Leftrightarrow \begin{cases} F_{uv}^{\alpha_\varsigma} = \partial_u A_v^{\alpha_\varsigma} - \partial_v A_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma} \\ F_{uv}^{\alpha'_\varsigma} = \partial_u A_v^{\alpha'_\varsigma} - \partial_v A_u^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma} \end{cases}$

Proof: $R_{uv}{}^{cd} = \partial_u\omega_v^{cd} - \partial_v\omega_u^{cd} + \omega_{[u}^{ce}\omega_{v]e}{}^{d}$

$\Leftrightarrow R_{uv}{}^{cd} = \frac{i}{2}(\sigma^{cd}_{-\varsigma\alpha'_\varsigma}\partial_u A_v^{\alpha'_\varsigma} + \sigma^{cd}_{\varsigma\alpha_\varsigma}\partial_u A_v^{\alpha_\varsigma}) - \frac{i}{2}(\sigma^{cd}_{-\varsigma\alpha'_\varsigma}\partial_v A_u^{\alpha'_\varsigma} + \sigma^{cd}_{\varsigma\alpha_\varsigma}\partial_v A_u^{\alpha_\varsigma})$

$\quad - \frac{i}{2}(\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}\sigma^{cd}_{-\varsigma\alpha'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma} + \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\sigma^{cd}_{\varsigma\alpha_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma})$

$\Leftrightarrow \frac{i}{2}(\sigma^{cd}_{-\varsigma\alpha'_\varsigma}F_{uv}^{\alpha'_\varsigma} + \sigma^{cd}_{\varsigma\alpha_\varsigma}F_{uv}^{\alpha_\varsigma}) = \frac{i}{2}\sigma^{cd}_{-\varsigma\alpha'_\varsigma}(\partial_{[u}A_{v]}{}^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma}) + \frac{i}{2}\sigma^{cd}_{\varsigma\alpha_\varsigma}(\partial_{[u}A_{v]}{}^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma})$

$\Leftrightarrow \begin{cases} F_{uv}^{\alpha_\varsigma} = \partial_u A_v^{\alpha_\varsigma} - \partial_v A_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma} \\ F_{uv}^{\alpha'_\varsigma} = \partial_u A_v^{\alpha'_\varsigma} - \partial_v A_u^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma} \end{cases}$ □

Cor. 1.3.5. $\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}\omega_{[u}^{ce}\omega_{v]e}{}^{d} = -\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma}$

Proof: $\omega_{[u}^{ce}\omega_{v]e}{}^{d} = -\frac{i}{2}(\varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}\sigma^{cd}_{-\varsigma\alpha'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma} + \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\sigma^{cd}_{\varsigma\alpha_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma})$

$\Rightarrow \frac{i}{2}\sigma^{\rho_\varsigma}_{\varsigma cd}\omega_{[u}^{ce}\omega_{v]e}{}^{d} = \frac{1}{4}\sigma^{\rho_\varsigma}_{\varsigma cd}(\varepsilon_{\alpha'_\varsigma\beta'_\varsigma}{}^{\gamma'_\varsigma}\sigma^{cd}_{-\varsigma\gamma'_\varsigma}A_u^{\alpha'_\varsigma}A_v^{\beta'_\varsigma} + \varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma^{cd}_{\varsigma\gamma_\varsigma}A_u^{\alpha_\varsigma}A_v^{\beta_\varsigma})$

$\Leftrightarrow \frac{i}{2}\sigma^{\rho_\varsigma}_{\varsigma cd}\omega_{[u}^{ce}\omega_{v]e}{}^{d} = 0 - \varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\delta^{\rho_\varsigma}{}_{\gamma_\varsigma}A_u^{\alpha_\varsigma}A_v^{\beta_\varsigma}$

$\Leftrightarrow \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}\omega_{[u}^{ce}\omega_{v]e}{}^{d} = -\varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma}$ □

Cor. 1.3.6. $F_{uv}^{\alpha_\varsigma} = \partial_u A_v^{\alpha_\varsigma} - \partial_v A_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma} \Leftrightarrow F_{uv}^{\alpha'_\varsigma} = \partial_u A_v^{\alpha'_\varsigma} - \partial_v A_u^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma}$

Proof: $F_{uv}^{\alpha_\varsigma} = \partial_u A_v^{\alpha_\varsigma} - \partial_v A_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma}$

$\Leftrightarrow [F_{uv}^{\alpha_\varsigma}]^* = F_{u'v'}^{*\alpha'_\varsigma} = \partial_{u'}A_{v'}^{*\alpha'_\varsigma} - \partial_{v'}A_{u'}^{*\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_{u'}^{*\beta'_\varsigma}A_{v'}^{*\gamma'_\varsigma}$

$\Leftrightarrow F_{u'v'}^{*\alpha'_\varsigma} = \eta_{u'}^{u}\eta_{v'}^{v}(\partial_u A_v^{\alpha'_\varsigma} - \partial_v A_u^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma})$

$\Leftrightarrow \eta_{u'}^{u}\eta_{v'}^{v}F_{uv}^{\alpha'_\varsigma} = \eta_{u'}^{u}\eta_{v'}^{v}(\partial_u A_v^{\alpha'_\varsigma} - \partial_v A_u^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma})$

$\Leftrightarrow F_{uv}^{\alpha'_\varsigma} = \partial_u A_v^{\alpha'_\varsigma} - \partial_v A_u^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma}$ □

Cor. 1.3.7. $R_{uv}{}^{cd} = \partial_u\omega_v^{cd} - \partial_v\omega_u^{cd} + \omega_{[u}^{ce}\omega_{v]e}{}^{d} \Leftrightarrow F_{uv}^{\alpha_\varsigma} = \partial_u A_v^{\alpha_\varsigma} - \partial_v A_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma}$

Cor. 1.3.8. $R_{uv}{}^{cd} = \partial_u\omega_v^{cd} - \partial_v\omega_u^{cd} + \omega_{[u}^{ce}\omega_{v]e}{}^{d} \Leftrightarrow F_{uv}^{\alpha'_\varsigma} = \partial_u A_v^{\alpha'_\varsigma} - \partial_v A_u^{\alpha'_\varsigma} - \varepsilon^{\alpha'_\varsigma}{}_{\beta'_\varsigma\gamma'_\varsigma}A_u^{\beta'_\varsigma}A_v^{\gamma'_\varsigma}$

1.4 Complex tensor description of physical quantities in gravitational field

1.4.1 Complex tensor of gravitational field

Def. 1.4.1. $\begin{cases} \textit{Curvature complex tensor } \psi^{\alpha_\varsigma\beta_\kappa} := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}F^{ab\beta_\kappa} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\frac{i}{2}\sigma^{\beta_\kappa}_{\kappa cd}R^{abcd} \\ \textit{Weyl complex tensor } C^{\alpha_\varsigma\beta_\kappa} := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}C^{ab\beta_\kappa} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\frac{i}{2}\sigma^{\beta_\kappa}_{\kappa cd}C^{abcd} \end{cases}$

Cor. 1.4.1. $\psi^{\alpha_\varsigma\beta_\kappa} = C^{\alpha_\varsigma\beta_\kappa} - \frac{1}{2}\varsigma\kappa\sigma^{\alpha_\varsigma}_{\varsigma ac}\sigma^{\beta_\kappa c}_{\kappa}{}_{b}R^{ab} + \frac{1}{3}\varsigma\kappa\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa}R$

Proof: $\psi^{\alpha_\varsigma\beta_\kappa} := \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}F^{ab\beta_\kappa}$

$\Rightarrow \psi^{\alpha_\varsigma\beta_\kappa} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}(C^{ab\beta_\kappa} + \frac{i}{2}\sigma^{\beta_\kappa a}_{\kappa}{}_{c}R^{cb} - \frac{i}{2}\sigma^{\beta_\kappa b}_{\kappa}{}_{c}R^{ca} - \frac{i}{6}\sigma^{\beta_\kappa ab}_{\kappa}R)$

$\Rightarrow \psi^{\alpha_\varsigma\beta_\kappa} = C^{\alpha_\varsigma\beta_\kappa} + \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa b}_{\kappa}{}_{c}R^{ca} - \frac{1}{3}\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa}R$

$\Rightarrow \psi^{\alpha_\varsigma\beta_\kappa} = C^{\alpha_\varsigma\beta_\kappa} + \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ac}\sigma^{\beta_\kappa c}_{\kappa}{}_{b}R^{ab} - \frac{1}{3}\delta_{\varsigma\kappa}\delta^{\alpha_\varsigma\beta_\kappa}R$ □

Cor. 1.4.2. $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} + \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ac}\sigma^{\beta_\varsigma c}_{\varsigma}{}_{b}R^{ab} - \frac{1}{3}\delta^{\alpha_\varsigma\beta_\varsigma}R$

Cor. 1.4.3. $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} + \frac{1}{6}\delta^{\alpha_\varsigma\beta_\varsigma}R$

Proof: $\psi^{\alpha_\varsigma\beta_\varsigma}=C^{\alpha_\varsigma\beta_\varsigma}+\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ac}\sigma^{\beta_\varsigma c}_{\varsigma\ \ b}R^{ab}-\frac{1}{3}\delta^{\alpha_\varsigma\beta_\varsigma}R$
$\Leftrightarrow\psi^{\alpha_\varsigma\beta_\varsigma}=C^{\alpha_\varsigma\beta_\varsigma}+\frac{1}{2}(\delta^{\alpha_\varsigma\beta_\varsigma}\delta_{ab}+i\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\sigma^{\gamma_\varsigma}_{\varsigma ab})R^{ab}-\frac{1}{3}\delta^{\alpha_\varsigma\beta_\varsigma}R$
$\Leftrightarrow\psi^{\alpha_\varsigma\beta_\varsigma}=C^{\alpha_\varsigma\beta_\varsigma}+\frac{1}{6}\delta^{\alpha_\varsigma\beta_\varsigma}R$ □

Cor. 1.4.4. $\frac{1}{2}(F^{ab\beta_\kappa}-\varsigma*F^{ab\beta_\kappa})=\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\psi^{\alpha_\varsigma\beta_\kappa}$

Cor. 1.4.5. $\psi^{\alpha_\varsigma\beta_\kappa}=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\frac{1}{2}(F^{ab\beta_\kappa}-\varsigma*F^{ab\beta_\kappa})$

Cor. 1.4.6. $\psi^{\alpha_\varsigma\beta_\kappa}=-\frac{i}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma ab}*F^{ab\beta_\kappa}$

Cor. 1.4.7. $\sigma^{\alpha_\varsigma}_{\varsigma ab}(F^{ab\beta_\kappa}+\varsigma*F^{ab\beta_\kappa})=0$

Cor. 1.4.8. $F^{ab\beta_\kappa}-\varsigma*F^{ab\beta_\kappa}=-\frac{1}{4}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{\varsigma cd}(F^{cd\beta_\kappa}-\varsigma*F^{cd\beta_\kappa})$

Cor. 1.4.9. $F^{ab\beta_\kappa}=\frac{i}{2}(\sigma^{ab}_{-\alpha'}\psi^{\alpha'\beta_\kappa}+\sigma^{ab}_{+\alpha}\psi^{\alpha\beta_\kappa}),*F^{ab\beta_\kappa}=\frac{i}{2}(\sigma^{ab}_{-\alpha'}\psi^{\alpha'\beta_\kappa}-\sigma^{ab}_{+\alpha}\psi^{\alpha\beta_\kappa})$

Cor. 1.4.10. $\psi^{\alpha_\varsigma\beta_\kappa}=-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{abcd}=\frac{1}{4}\kappa\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{ab(*cd)}=\frac{1}{4}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{(*ab)(*cd)}=-\frac{1}{4}\varsigma\kappa\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{(*ab)cd}$

1.4.2 Properties of gravitational field complex tensor $\psi^{\alpha_\varsigma\beta_\kappa}$

Cor. 1.4.11. $\psi^{\alpha_\varsigma\beta_\kappa}=\psi^{\beta_\kappa\alpha_\varsigma}$

Proof: $R^{abcd}=R^{cdab}$
$\Rightarrow-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{abcd}=-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{cdab}$
$\Rightarrow\psi^{\alpha_\varsigma\beta_\kappa}=\psi^{\beta_\kappa\alpha_\varsigma}$ □

Cor. 1.4.12. $\psi^{x_\varsigma x_\varsigma}+\psi^{y_\varsigma y_\varsigma}+\psi^{z_\varsigma z_\varsigma}=\frac{1}{2}R$

Proof: $\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}=-(S_{abcd}-\varsigma\varepsilon_{abcd})$
$\Rightarrow-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}R^{abcd}=\frac{1}{4}(S_{abcd}-\varsigma\varepsilon_{abcd})R^{abcd}$
$\Rightarrow-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}R^{abcd}=\frac{1}{2}(R_{ab}{}^{ab}-\varsigma R_{*ab}{}^{ab})$
$\Rightarrow\psi^{x_\varsigma x_\varsigma}+\psi^{y_\varsigma y_\varsigma}+\psi^{z_\varsigma z_\varsigma}=\frac{1}{2}(R_{ab}{}^{ab}-\varsigma R_{*ab}{}^{ab})$
$\Rightarrow\psi^{x_\varsigma x_\varsigma}+\psi^{y_\varsigma y_\varsigma}+\psi^{z_\varsigma z_\varsigma}=\frac{1}{2}R$ □

Cor. 1.4.13. $C^{\alpha_\varsigma\beta_\varsigma}=\psi^{\alpha_\varsigma\beta_\varsigma}-\frac{1}{3}\delta^{\alpha_\varsigma\beta_\varsigma}\psi^{\gamma_\varsigma}{}_{\gamma_\varsigma}$

Cor. 1.4.14. $\psi^{\alpha'_\varsigma\beta'_\kappa}=(\psi^{\alpha_\varsigma\beta_\kappa})^*$

Proof: $\psi^{\alpha_\varsigma\beta_\kappa}=-\frac{1}{4}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{abcd}$
$\Leftrightarrow(\psi^{\alpha_\varsigma\beta_\kappa})^*=-\frac{1}{4}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\kappa}_{\kappa cd}R^{abcd})^*=-\frac{1}{4}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma^{\beta'_\kappa}_{\kappa c'd'}\eta^{a'}_a\eta^{b'}_b\eta^{c'}_c\eta^{d'}_dR^{abcd}$
$\Leftrightarrow(\psi^{\alpha_\varsigma\beta_\kappa})^*=-\frac{1}{4}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\kappa}_{-\kappa cd}R^{abcd}$
$\Leftrightarrow\psi^{\alpha'_\varsigma\beta'_\kappa}=(\psi^{\alpha_\varsigma\beta_\kappa})^*$ □

1.4.3 Expansion of gravitational field curvature tensor

Cor. 1.4.15. $R^{abcd}=-\frac{1}{4}(\sigma^{ab}_{-\alpha'}\sigma^{cd}_{-\beta'}\psi^{\alpha'\beta'}+\sigma^{ab}_{+\alpha}\sigma^{cd}_{-\beta'}\psi^{\alpha\beta'}+\sigma^{ab}_{-\alpha'}\sigma^{cd}_{+\beta}\psi^{\alpha'\beta}+\sigma^{ab}_{+\alpha}\sigma^{cd}_{+\beta}\psi^{\alpha\beta})$

Proof: $R^{abcd}=\frac{i}{2}(\sigma^{cd}_{-\beta'}F^{ab\beta'}+\sigma^{cd}_{+\beta}F^{ab\beta})$
$\Leftrightarrow R^{abcd}=-\frac{1}{4}[\sigma^{cd}_{-\beta'}(\sigma^{ab}_{-\alpha'}\psi^{\alpha'\beta'}+\sigma^{ab}_{+\alpha}\psi^{\alpha\beta'})+\sigma^{cd}_{+\beta}(\sigma^{ab}_{-\alpha'}\psi^{\alpha'\beta}+\sigma^{ab}_{+\alpha}\psi^{\alpha\beta})]$
$\Leftrightarrow R^{abcd}=-\frac{1}{4}(\sigma^{ab}_{-\alpha'}\sigma^{cd}_{-\beta'}\psi^{\alpha'\beta'}+\sigma^{ab}_{+\alpha}\sigma^{cd}_{-\beta'}\psi^{\alpha\beta'}+\sigma^{ab}_{-\alpha'}\sigma^{cd}_{+\beta}\psi^{\alpha'\beta}+\sigma^{ab}_{+\alpha}\sigma^{cd}_{+\beta}\psi^{\alpha\beta})$ □

Cor. 1.4.16. $R^{ab(*cd)}=-\frac{1}{4}(\sigma^{ab}_{-\alpha'}\sigma^{cd}_{-\beta'}\psi^{\alpha'\beta'}+\sigma^{ab}_{+\alpha}\sigma^{cd}_{-\beta'}\psi^{\alpha\beta'}-\sigma^{ab}_{-\alpha'}\sigma^{cd}_{+\beta}\psi^{\alpha'\beta}-\sigma^{ab}_{+\alpha}\sigma^{cd}_{+\beta}\psi^{\alpha\beta})$

Cor. 1.4.17. $R^{(*ab)cd}=-\frac{1}{4}(\sigma^{ab}_{-\alpha'}\sigma^{cd}_{-\beta'}\psi^{\alpha'\beta'}-\sigma^{ab}_{+\alpha}\sigma^{cd}_{-\beta'}\psi^{\alpha\beta'}+\sigma^{ab}_{-\alpha'}\sigma^{cd}_{+\beta}\psi^{\alpha'\beta}-\sigma^{ab}_{+\alpha}\sigma^{cd}_{+\beta}\psi^{\alpha\beta})$

Cor. 1.4.18. $R^{(*ab)(*cd)}=-\frac{1}{4}(\sigma^{ab}_{-\alpha'}\sigma^{cd}_{-\beta'}\psi^{\alpha'\beta'}-\sigma^{ab}_{+\alpha}\sigma^{cd}_{-\beta'}\psi^{\alpha\beta'}-\sigma^{ab}_{-\alpha'}\sigma^{cd}_{+\beta}\psi^{\alpha'\beta}+\sigma^{ab}_{+\alpha}\sigma^{cd}_{+\beta}\psi^{\alpha\beta})$

Cor. 1.4.19. $R^{ab}=\frac{1}{4}\delta^{ab}R+\frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}=\frac{1}{4}\delta^{ab}R-\delta_{cd}S^{ac}{}_{AB}S^{db}{}_{C'D'}\psi^{ABC'D'}$

Proof: $R^{ab}=\frac{1}{4}(\sigma_{-\alpha'}\sigma_{-\beta'}\psi^{\alpha'\beta'}+\sigma_{+\alpha}\sigma_{-\beta'}\psi^{\alpha\beta'}+\sigma_{-\alpha'}\sigma_{+\beta}\psi^{\alpha'\beta}+\sigma_{+\alpha}\sigma_{+\beta}\psi^{\alpha\beta})^{ab}$
$\Leftrightarrow R^{ab}=\frac{1}{8}(\{\sigma_{-\alpha'},\sigma_{-\beta'}\}\psi^{\alpha'\beta'}+2\{\sigma_{+\alpha},\sigma_{-\beta'}\}\psi^{\alpha\beta'}+\{\sigma_{+\alpha},\sigma_{+\beta}\}\psi^{\alpha\beta})^{ab})$
$\Leftrightarrow R^{ab}=\frac{1}{8}(R+4\sigma_{+\alpha}\sigma_{-\beta'}\psi^{\alpha\beta'}+R)^{ab}$
$\Leftrightarrow R^{ab}=\frac{1}{4}\delta^{ab}R+\frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}=\frac{1}{4}\delta^{ab}R-\delta_{cd}S^{ac}{}_{AB}S^{db}{}_{C'D'}\psi^{ABC'D'}$ □

Cor. 1.4.20. $\frac{1}{4}(\sigma^{\alpha}_{+}\sigma^{\beta'}_{-})_{ab}(\sigma_{+\rho}\sigma_{-\sigma'})^{ab}=\delta^{\alpha}{}_{\rho}\delta^{\beta'}{}_{\sigma'}$

Cor. 1.4.21. $\psi^{\alpha_\varsigma\beta'_\varsigma} = \frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ac}\sigma^{\beta'_\varsigma c}_{-\varsigma\ b}R^{ab} = \frac{1}{2}(\sigma^{\alpha_\varsigma}_{\varsigma}\sigma^{\beta'_\varsigma}_{-\varsigma})_{ab}R^{ab}$

More general proof, it does not rely on the definition of various quantities.

Cor. 1.4.22. $\psi^{\alpha\beta'} = \frac{1}{2}(\sigma^{\alpha}_{+}\sigma^{\beta'}_{-})_{ab}R^{ab} \Leftrightarrow R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}$

Proof: $\psi^{\alpha\beta'} = \frac{1}{2}(\sigma^{\alpha}_{+}\sigma^{\beta'}_{-})_{ab}R^{ab}$
$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}(\sigma^{\alpha}_{+}\sigma^{\beta'}_{-})_{cd}R^{cd}$
$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}(\sigma^{ae}_{+\alpha}\sigma_{-\beta' e}{}^{b})(\sigma^{\alpha}_{+cf}\sigma^{\beta' f}_{-\ \ d})R^{cd}$
$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha\beta'} = \frac{1}{4}(\sigma^{ae}_{+\alpha}\sigma^{\alpha cf}_{+})(\sigma_{-\beta' eb}\sigma^{\beta'}_{-\ fd})R_{c}{}^{d}$
$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha\beta'} = \frac{1}{4}(S^{aecf} - \varepsilon^{aecf})(S_{ebfd} + \varepsilon_{ebfd})R_{c}{}^{d}$
$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha\beta'} = \frac{1}{4}(2\delta^{ac}\delta_{bd} + 2\delta^{a}{}_{d}\delta_{b}{}^{c} - \delta^{a}{}_{b}\delta^{c}{}_{d})R_{c}{}^{d}$
$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{a}{}_{b}\psi^{\alpha\beta'} = \frac{1}{4}(4R^{a}{}_{b} - \delta^{a}{}_{b}R)$
$\Leftrightarrow R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}$ □

Thm. 1.4.1. $R^{ab}{}_{;b} \equiv \frac{1}{2}R^{;a} \Leftrightarrow (\sigma_{+\alpha}\sigma_{-\beta'})^{ab}D_b\psi^{\alpha\beta'} \equiv \frac{1}{2}R^{;a}$

Proof: $R^{ab}{}_{;b} \equiv \frac{1}{2}R^{;a}$
$\Leftrightarrow [\frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}]_{;b} \equiv \frac{1}{2}R^{;a}$
$\Leftrightarrow (\sigma_{+\alpha}\sigma_{-\beta'})^{ab}D_b\psi^{\alpha\beta'} \equiv \frac{1}{2}R^{;a}$ □

1.4.4 Synthesis of gravitational field curvature tensor

Cor. 1.4.23. $R^{abcd} + R^{ab(*cd)} + R^{(*ab)cd} + R^{(*ab)(*cd)} = i\sigma^{ab}_{-\alpha'}i\sigma^{cd}_{-\beta'}\psi^{\alpha'\beta'}$

Cor. 1.4.24. $R^{abcd} + R^{ab(*cd)} - R^{(*ab)cd} - R^{(*ab)(*cd)} = i\sigma^{ab}_{+\alpha}i\sigma^{cd}_{-\beta'}\psi^{\alpha\beta'}$

Cor. 1.4.25. $R^{abcd} - R^{ab(*cd)} + R^{(*ab)cd} - R^{(*ab)(*cd)} = i\sigma^{ab}_{-\alpha'}i\sigma^{cd}_{+\beta}\psi^{\alpha'\beta}$

Cor. 1.4.26. $R^{abcd} - R^{ab(*cd)} - R^{(*ab)cd} + R^{(*ab)(*cd)} = i\sigma^{ab}_{+\alpha}i\sigma^{cd}_{+\beta}\psi^{\alpha\beta}$

Unified description:

Cor. 1.4.27. $R^{abcd} - \kappa R^{ab(*cd)} - \varsigma R^{(*ab)cd} + \varsigma\kappa R^{(*ab)(*cd)} = i\sigma^{ab}_{\varsigma\alpha_\varsigma}i\sigma^{cd}_{\kappa\beta_\kappa}\psi^{\alpha_\varsigma\beta_\kappa}$

1.4.5 Expansion of gravitational field Weyl tensor

Cor. 1.4.28. $0 = C^{ab} = \frac{1}{4}\delta^{ab}C + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}C^{\alpha\beta'} \Rightarrow C^{\alpha\beta'} = 0, C^{\alpha'\beta} = 0$

Cor. 1.4.29. $C^{abcd} = -\frac{1}{4}(\sigma^{ab}_{-\alpha'}\sigma^{cd}_{-\beta'}C^{\alpha'\beta'} + \sigma^{ab}_{+\alpha}\sigma^{cd}_{+\beta}C^{\alpha\beta}), C^{\alpha'\beta'} = (C^{\alpha\beta})^*$

Cor. 1.4.30. $C^{(*ab)cd} = C^{ab(*cd)}, C^{abcd} = C^{(*ab)(*cd)}$

1.4.6 Complex tensor description of gravitational field source

Def. 1.4.2. *Gravitational field source spinor* $J^{\alpha_\varsigma}_{a} := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd}J_a{}^{cd}$

Following the reasoning of the electromagnetic field situation, there are completely similar conclusions.

Cor. 1.4.31. $[J^{\alpha_\varsigma}_{a}]^* = J^{*\alpha'_\varsigma}_{a'} = \eta^{a}_{a'}J^{\alpha'_\varsigma}_{a}$

Cor. 1.4.32. $\frac{1}{2}(J_a{}^{cd} - \varsigma * J_a{}^{cd}) = \frac{i}{2}\sigma^{cd}_{\varsigma\alpha_\varsigma}J^{\alpha_\varsigma}_{a}$

Cor. 1.4.33. $J^{\alpha_\varsigma}_{a} = -\frac{i}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd} * J_a{}^{cd}$

Cor. 1.4.34. $\sigma^{\alpha_\varsigma}_{\varsigma cd}(J_a{}^{cd} + \varsigma * J_a{}^{cd}) = 0$

Cor. 1.4.35. $J^{\alpha_\varsigma}_{a} = \frac{i}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd}\frac{1}{2}(J_a{}^{cd} - \varsigma * J_a{}^{cd})$

Cor. 1.4.36. $J_a{}^{cd} - \varsigma * J_a{}^{cd} = -\frac{1}{4}\sigma^{cd}_{\varsigma\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{\varsigma ef}(J_a{}^{ef} - \varsigma * J_a{}^{ef})$

Cor. 1.4.37. $J_a{}^{cd} = \frac{i}{2}(\sigma^{cd}_{-\alpha'}J^{\alpha'}_{a} + \sigma^{cd}_{+\alpha}J^{\alpha}_{a}), *J_a{}^{cd} = \frac{i}{2}(\sigma^{cd}_{-\alpha'}J^{\alpha'}_{a} - \sigma^{cd}_{+\alpha}J^{\alpha}_{a})$

Cor. 1.4.38. $J_a{}^{cd} = -J_a{}^{dc} \Leftrightarrow J_a{}^{cd} = \frac{i}{2}(\sigma^{cd}_{-\alpha'}J^{\alpha'}_{a} + \sigma^{cd}_{+\alpha}J^{\alpha_\varsigma}_{a})$

1.5 $\frac{1}{2}$-spinor description of physical quantities in gravitational field [1,2]

1.5.1 Curvature spinor of gravitational field [1,2]

Def. 1.5.1. *gravitational curvature spinor:*
$\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} := \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{C_\kappa D_\kappa}_{\beta_\kappa}\psi^{\alpha_\varsigma\beta_\kappa} = \frac{i\varsigma}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{C_\kappa D_\kappa}_{\beta_\kappa}F^{ab\beta_\kappa} = \frac{i\varsigma}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma}\frac{i\kappa}{\sqrt{2}}S_{cd}{}^{C_\kappa D_\kappa}R^{abcd}$

Cor. 1.5.1. $\psi^{\alpha_\varsigma\beta_\kappa} = \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{\beta_\kappa}_{C_\kappa D_\kappa}\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa}$

Cor. 1.5.2. $\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} + \frac{1}{12}(\varepsilon^{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} - \varepsilon^{A_\varsigma D_\varsigma}\varepsilon^{C_\varsigma B_\varsigma})R$

Proof: $\psi^{\alpha_\varsigma\beta_\varsigma} = C^{\alpha_\varsigma\beta_\varsigma} + \frac{1}{6}\delta^{\alpha_\varsigma\beta_\varsigma}R$
$\Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} - \frac{1}{12}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\delta^{\alpha_\varsigma\beta_\varsigma}R$
$\Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} - \frac{1}{12}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{\alpha_\varsigma C_\varsigma D_\varsigma}R$
$\Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} + \frac{1}{12}(\varepsilon^{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} - \varepsilon^{A_\varsigma D_\varsigma}\varepsilon^{C_\varsigma B_\varsigma})R$ □

Cor. 1.5.3. $\psi^{\alpha_\varsigma\beta_\kappa} = \psi^{\beta_\kappa\alpha_\varsigma} \Leftrightarrow \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \psi^{B_\varsigma A_\varsigma C_\kappa D_\kappa}, \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \psi^{A_\varsigma B_\varsigma D_\kappa C_\kappa}, \psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = \psi^{C_\kappa D_\kappa A_\varsigma B_\varsigma}$

Cor. 1.5.4. $\psi^{\alpha_\varsigma\beta_\varsigma} = \psi^{\beta_\varsigma\alpha_\varsigma} \Leftrightarrow \begin{cases}\psi^{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma}\\ \psi^{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma} = \psi^{1_\varsigma 1_\varsigma 2_\varsigma 1_\varsigma} = \psi^{1_\varsigma 2_\varsigma 1_\varsigma 1_\varsigma} = \psi^{2_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma}\\ \psi^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = \psi^{2_\varsigma 2_\varsigma 1_\varsigma 1_\varsigma}, \psi^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} = \psi^{1_\varsigma 2_\varsigma 2_\varsigma 1_\varsigma} = \psi^{2_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma} = \psi^{2_\varsigma 1_\varsigma 2_\varsigma 1_\varsigma}\\ \psi^{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma} = \psi^{2_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = \psi^{2_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} = \psi^{2_\varsigma 2_\varsigma 2_\varsigma 1_\varsigma}\\ \psi^{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma}\end{cases}$

Cor. 1.5.5. $\psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = (-\varsigma)\varepsilon_{A_\varsigma C_\varsigma}(-\varsigma)\varepsilon_{B_\varsigma D_\varsigma}\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = 2(\psi^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} - \psi^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma})$

Cor. 1.5.6. $\psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma} = \frac{1}{2}R$

Proof: $\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} := \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{C_\kappa D_\kappa}_{\beta_\kappa}\psi^{\alpha_\varsigma\beta_\kappa}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = -\frac{1}{2}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma_{\beta_\varsigma A_\varsigma B_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}$
$\Rightarrow \psi^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}$ □

Cor. 1.5.7. $C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} - \frac{1}{6}(\varepsilon^{A_\varsigma C_\varsigma}\varepsilon^{B_\varsigma D_\varsigma} - \varepsilon^{A_\varsigma D_\varsigma}\varepsilon^{C_\varsigma B_\varsigma})\psi^{E_\varsigma F_\varsigma}{}_{E_\varsigma F_\varsigma}$

Cor. 1.5.8. $\psi^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} - \psi^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = \frac{1}{4}R$

Cor. 1.5.9. $\psi^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}$ *is a fully symmetric spinor.* $\Leftrightarrow \psi^{\alpha_\varsigma\beta_\varsigma}$ *is an traceless symmetric tensor, i.e* $\psi^{\alpha_\varsigma\beta_\varsigma} = \psi^{\beta_\varsigma\alpha_\varsigma}, \psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma} + \psi^{z_\varsigma z_\varsigma} = 0$

1.5.2 Constraints for YM curvature tensor $F^{\alpha_\varsigma}_{ab}$ of gravitational field

Thm. 1.5.1. $\begin{cases}\psi^{\alpha_\varsigma\beta_\varsigma} = \psi^{\beta_\varsigma\alpha_\varsigma}\\ \psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma} + \psi^{z_\varsigma z_\varsigma} = \frac{1}{2}R\end{cases} \Leftrightarrow \begin{cases}F^{y_\varsigma}_{yz} - \varsigma F^{y_\varsigma}_{x\pi} = F^{x_\varsigma}_{zx} - \varsigma F^{x_\varsigma}_{y\pi}\\ F^{z_\varsigma}_{zx} - \varsigma F^{z_\varsigma}_{y\pi} = F^{y_\varsigma}_{xy} - \varsigma F^{y_\varsigma}_{z\pi}\\ F^{x_\varsigma}_{xy} - \varsigma F^{x_\varsigma}_{z\pi} = F^{z_\varsigma}_{yz} - \varsigma F^{z_\varsigma}_{x\pi}\\ F^{x_\varsigma}_{yz} - \varsigma F^{x_\varsigma}_{x\pi} + F^{y_\varsigma}_{zx} - \varsigma F^{y_\varsigma}_{y\pi} + F^{z_\varsigma}_{xy} - \varsigma F^{z_\varsigma}_{z\pi} = R\end{cases}$

Proof: $\psi^{x_\varsigma y_\varsigma} = \psi^{y_\varsigma x_\varsigma}$
$\Leftrightarrow \sigma^{x_\varsigma}_{\varsigma cd}F^{cdy_\varsigma} = \sigma^{y_\varsigma}_{\varsigma cd}F^{cdx_\varsigma}$
$\Leftrightarrow F^{y_\varsigma}_{yz} - \varsigma F^{y_\varsigma}_{x\pi} = F^{x_\varsigma}_{zx} - \varsigma F^{x_\varsigma}_{y\pi}$ □

Proof: $\psi^{y_\varsigma z_\varsigma} = \psi^{z_\varsigma y_\varsigma}$
$\Leftrightarrow \sigma^{y_\varsigma}_{\varsigma cd}F^{cdz_\varsigma} = \sigma^{z_\varsigma}_{\varsigma cd}F^{cdy_\varsigma}$
$\Leftrightarrow F^{z_\varsigma}_{zx} - \varsigma F^{z_\varsigma}_{y\pi} = F^{y_\varsigma}_{xy} - \varsigma F^{y_\varsigma}_{z\pi}$ □

Proof: $\psi^{z_\varsigma x_\varsigma} = \psi^{x_\varsigma z_\varsigma}$
$\Leftrightarrow \sigma^{z_\varsigma}_{\varsigma cd}F^{cdx_\varsigma} = \sigma^{x_\varsigma}_{\varsigma cd}F^{cdz_\varsigma}$
$\Leftrightarrow F^{x_\varsigma}_{xy} - \varsigma F^{x_\varsigma}_{z\pi} = F^{z_\varsigma}_{yz} - \varsigma F^{z_\varsigma}_{x\pi}$ □

Proof: $\psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma} + \psi^{z_\varsigma z_\varsigma} = \frac{1}{2}R$
$\Leftrightarrow \frac{i}{2}[\sigma^{x_\varsigma}_{\varsigma cd}F^{cdx_\varsigma} + \sigma^{y_\varsigma}_{\varsigma cd}F^{cdy_\varsigma} + \sigma^{z_\varsigma}_{\varsigma cd}F^{cdz_\varsigma}] = \frac{1}{2}R$
$\Leftrightarrow F^{x_\varsigma}_{yz} - \varsigma F^{x_\varsigma}_{x\pi} + F^{y_\varsigma}_{zx} - \varsigma F^{y_\varsigma}_{y\pi} + F^{z_\varsigma}_{xy} - \varsigma F^{z_\varsigma}_{z\pi} = R$ □

1.5.3 Constraints for Ashtekar variable $A_u^{\alpha_\varsigma}$ of gravitational field

Thm. 1.5.2.

$$\begin{cases} F_{yz}^{y_\varsigma} - \varsigma F_{x\pi}^{y_\varsigma} = F_{zx}^{x_\varsigma} - \varsigma F_{y\pi}^{x_\varsigma} \\ F_{zx}^{z_\varsigma} - \varsigma F_{y\pi}^{z_\varsigma} = F_{xy}^{y_\varsigma} - \varsigma F_{z\pi}^{y_\varsigma} \\ F_{xy}^{x_\varsigma} - \varsigma F_{z\pi}^{x_\varsigma} = F_{yz}^{z_\varsigma} - \varsigma F_{x\pi}^{z_\varsigma} \\ F_{yz}^{x_\varsigma} - \varsigma F_{x\pi}^{x_\varsigma} + F_{zx}^{y_\varsigma} - \varsigma F_{y\pi}^{y_\varsigma} \\ +F_{xy}^{z_\varsigma} - \varsigma F_{z\pi}^{z_\varsigma} = R \end{cases} \Leftrightarrow \begin{cases} F_{uv}^{\alpha_\varsigma} = \partial_u A_v^{\alpha_\varsigma} - \partial_v A_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma} A_u^{\beta_\varsigma} A_v^{\gamma_\varsigma} \\ (\partial_y A_z^{y_\varsigma} - \partial_z A_y^{y_\varsigma} - A_y^{[z_\varsigma} A_z^{x_\varsigma]}) - \varsigma(\partial_x A_\pi^{y_\varsigma} - \partial_\pi A_x^{y_\varsigma} - A_x^{[z_\varsigma} A_\pi^{x_\varsigma]}) \\ = (\partial_z A_x^{x_\varsigma} - \partial_x A_z^{x_\varsigma} - A_z^{[y_\varsigma} A_x^{z_\varsigma]}) - \varsigma(\partial_y A_\pi^{x_\varsigma} - \partial_\pi A_y^{x_\varsigma} - A_y^{[y_\varsigma} A_\pi^{z_\varsigma]}) \\ (\partial_z A_x^{z_\varsigma} - \partial_x A_z^{z_\varsigma} - A_z^{[x_\varsigma} A_x^{y_\varsigma]}) - \varsigma(\partial_y A_\pi^{z_\varsigma} - \partial_\pi A_y^{z_\varsigma} - A_y^{[x_\varsigma} A_\pi^{y_\varsigma]}) \\ = (\partial_x A_y^{y_\varsigma} - \partial_y A_x^{y_\varsigma} - A_x^{[z_\varsigma} A_y^{x_\varsigma]}) - \varsigma(\partial_z A_\pi^{y_\varsigma} - \partial_\pi A_z^{y_\varsigma} - A_z^{[z_\varsigma} A_\pi^{x_\varsigma]}) \\ (\partial_x A_y^{x_\varsigma} - \partial_y A_x^{x_\varsigma} - A_x^{[y_\varsigma} A_y^{z_\varsigma]}) - \varsigma(\partial_z A_\pi^{x_\varsigma} - \partial_\pi A_z^{x_\varsigma} - A_z^{[y_\varsigma} A_\pi^{z_\varsigma]}) \\ = (\partial_y A_z^{z_\varsigma} - \partial_z A_y^{z_\varsigma} - A_y^{[x_\varsigma} A_z^{y_\varsigma]}) - \varsigma(\partial_x A_\pi^{z_\varsigma} - \partial_\pi A_x^{z_\varsigma} - A_x^{[x_\varsigma} A_\pi^{y_\varsigma]}) \\ (\partial_y A_z^{x_\varsigma} - \partial_z A_y^{x_\varsigma} - A_y^{[y_\varsigma} A_z^{z_\varsigma]}) - \varsigma(\partial_x A_\pi^{x_\varsigma} - \partial_\pi A_x^{x_\varsigma} - A_x^{[y_\varsigma} A_\pi^{z_\varsigma]}) \\ +(\partial_z A_x^{y_\varsigma} - \partial_x A_z^{y_\varsigma} - A_z^{[z_\varsigma} A_x^{x_\varsigma]}) - \varsigma(\partial_y A_\pi^{y_\varsigma} - \partial_\pi A_y^{y_\varsigma} - A_y^{[z_\varsigma} A_\pi^{x_\varsigma]}) \\ +(\partial_x A_y^{z_\varsigma} - \partial_y A_x^{z_\varsigma} - A_x^{[x_\varsigma} A_y^{y_\varsigma]}) - \varsigma(\partial_z A_\pi^{z_\varsigma} - \partial_\pi A_z^{z_\varsigma} - A_z^{[x_\varsigma} A_\pi^{y_\varsigma]}) = R \end{cases}$$

Guage conditions for Ashtekar variable $A_u^{\alpha_\varsigma}$ of gravitational field: $\partial^u A_u^{\alpha_\varsigma} = 0, A_\pi^{\alpha_\varsigma} = 0$

1.5.4 Weyl spinor of gravitational field [1,2]

Def. 1.5.2. *Gravitational Weyl spinor:*
$C^{A_\varsigma B_\varsigma C_\kappa D_\kappa} := \frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma} \frac{i\kappa}{\sqrt{2}}\sigma_{\beta_\kappa}^{C_\kappa D_\kappa} C^{\alpha_\varsigma\beta_\kappa} = \frac{i\varsigma}{\sqrt{2}} S_{ab}{}^{A_\varsigma B_\varsigma} \frac{i\kappa}{\sqrt{2}}\sigma_{\beta_\kappa}^{C_\kappa D_\kappa} C^{ab\beta_\kappa} = \frac{i\varsigma}{\sqrt{2}} S_{ab}{}^{A_\varsigma B_\varsigma} \frac{i\kappa}{\sqrt{2}} S_{cd}{}^{C_\kappa D_\kappa} C^{abcd}$

Cor. 1.5.10. $C^{\alpha_\varsigma\beta_\kappa} = \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} \frac{i\kappa}{\sqrt{2}}\sigma^{\beta_\kappa}_{C_\kappa D_\kappa} C^{A_\varsigma B_\varsigma C_\kappa D_\kappa}$

Cor. 1.5.11. $C = 0$

Cor. 1.5.12. $C^{\alpha\beta'} = 0, C^{\alpha'\beta} = 0 \Leftrightarrow C^{ABC'D'} = 0, C^{A'B'CD} = 0$

Cor. 1.5.13. $C^{\alpha_\varsigma\beta_\kappa} = C^{\beta_\kappa\alpha_\varsigma} \Leftrightarrow C^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = C^{B_\varsigma A_\varsigma C_\kappa D_\kappa}, C^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = C^{A_\varsigma B_\varsigma D_\kappa C_\kappa}, C^{A_\varsigma B_\varsigma C_\kappa D_\kappa} = C^{C_\kappa D_\kappa A_\varsigma B_\varsigma}$

Cor. 1.5.14. $C^{\alpha_\varsigma\beta_\varsigma} = C^{\beta_\varsigma\alpha_\varsigma} \Leftrightarrow \begin{cases} C^{1_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma} \\ C^{1_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma} = C^{1_\varsigma 1_\varsigma 2_\varsigma 1_\varsigma} = C^{1_\varsigma 2_\varsigma 1_\varsigma 1_\varsigma} = C^{2_\varsigma 1_\varsigma 1_\varsigma 1_\varsigma} \\ C^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = C^{2_\varsigma 2_\varsigma 1_\varsigma 1_\varsigma}, C^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} = C^{1_\varsigma 2_\varsigma 2_\varsigma 1_\varsigma} = C^{2_\varsigma 1_\varsigma 1_\varsigma 2_\varsigma} = C^{2_\varsigma 1_\varsigma 2_\varsigma 1_\varsigma} \\ C^{1_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma} = C^{2_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = C^{2_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} = C^{2_\varsigma 2_\varsigma 2_\varsigma 1_\varsigma} \\ C^{2_\varsigma 2_\varsigma 2_\varsigma 2_\varsigma} \end{cases}$

Cor. 1.5.15. $C^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = (-\varsigma)\varepsilon_{A_\varsigma C_\varsigma}(-\varsigma)\varepsilon_{B_\varsigma D_\varsigma} C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = 2(C^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} - C^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma})$

Cor. 1.5.16. $C^{A_\varsigma B_\varsigma}{}_{A_\varsigma B_\varsigma} = \delta_{\alpha_\varsigma\beta_\varsigma} C^{\alpha_\varsigma\beta_\varsigma} = 0$

Cor. 1.5.17. $C^{1_\varsigma 2_\varsigma 1_\varsigma 2_\varsigma} - C^{1_\varsigma 1_\varsigma 2_\varsigma 2_\varsigma} = 0$

Cor. 1.5.18. $\delta_{\alpha_\varsigma\beta_\varsigma} C^{\alpha_\varsigma\beta_\varsigma} = 0 \Leftrightarrow \sigma_{\alpha_\varsigma}\sigma_{\beta_\varsigma} C^{\alpha_\varsigma\beta_\varsigma} = 0 \Leftrightarrow (\sigma, -i\varsigma)_{\alpha_\varsigma}\sigma_{\beta_\varsigma}\tilde{C}^{\alpha_\varsigma\beta_\varsigma} = 0$

Cor. 1.5.19. $C^{\alpha_\varsigma\beta_\varsigma} = C^{\beta_\varsigma\alpha_\varsigma}, C^{x_\varsigma x_\varsigma} + C^{y_\varsigma y_\varsigma} + C^{z_\varsigma z_\varsigma} = 0$

Cor. 1.5.20. $C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}$ *is a fully symmetric spinor.*

1.5.5 Weyl 2-spinor of gravitational field

Def. 1.5.3. $C^{k_\varsigma} := \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2) C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}, C_{k_\varsigma} := \Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2) C_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}$

Cor. 1.5.21. $C^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2) C^{k_\varsigma}, C_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2) C_{k_\varsigma}$

Cor. 1.5.22. $C^{\alpha_\varsigma\beta_\varsigma} = \Gamma_{k_\varsigma}^{\alpha_\varsigma\beta_\varsigma}(2) C^{k_\varsigma}, C^{k_\varsigma} = \Gamma^{k_\varsigma}_{\alpha_\varsigma\beta_\varsigma}(2) C^{\alpha_\varsigma\beta_\varsigma}$

1.5.6 Vector-spinor description of gravitational field source [1,2]

Def. 1.5.4. *Gravitational source vector-spinor* $J_a^{A_\varsigma B_\varsigma} := \frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma} J_a^{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}} S_{cd}{}^{A_\varsigma B_\varsigma} J_a{}^{cd}$

Following the reasoning of the electromagnetic field situation, there are completely similar conclusions.

Cor. 1.5.23. $J_a^{A_\varsigma B_\varsigma} = J_a^{B_\varsigma A_\varsigma}$

Cor. 1.5.24. $J_a^{A_\varsigma B_\varsigma} := \frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma} J_a^{\alpha_\varsigma} \Leftrightarrow J_a^{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} J_a^{A_\varsigma B_\varsigma}$

Cor. 1.5.25. $\frac{1}{2}(J_a{}^{cd} - \varsigma * J_a{}^{cd}) = \frac{i\varsigma}{\sqrt{2}} S^{cd}{}_{A_\varsigma B_\varsigma} J_a^{A_\varsigma B_\varsigma} \Leftrightarrow J_a^{A_\varsigma B_\varsigma} = \frac{i\varsigma}{\sqrt{2}} S_{cd}{}^{A_\varsigma B_\varsigma} \frac{1}{2}(J_a{}^{cd} - \varsigma * J_a{}^{cd})$

Cor. 1.5.26. $J_a^{A_\varsigma B_\varsigma} = \frac{-i}{\sqrt{2}} S_{cd}{}^{A_\varsigma B_\varsigma} * J_a{}^{cd}$

Cor. 1.5.27. $J_a{}^{cd} - \varsigma * J_a{}^{cd} = -\frac{1}{2} S^{cd}{}_{A_\varsigma B_\varsigma} S_{ef}{}^{A_\varsigma B_\varsigma}(J_a{}^{ef} - \varsigma * J_a{}^{ef})$

Cor. 1.5.28. $J_a{}^{cd} = \frac{i}{\sqrt{2}}(S^{cdA'B'} J_{aA'B'} + S^{cd}{}_{AB} J_a{}^{AB}), *J_a{}^{cd} = \frac{i}{\sqrt{2}}(S^{cdA'B'} J_{aA'B'} - S^{cd}{}_{AB} J_a{}^{AB})$

Cor. 1.5.29. $J_a{}^{cd} = -J_a{}^{dc} \Leftrightarrow J_a{}^{cd} = \frac{i}{\sqrt{2}}(S^{cdA'B'} J_{aA'B'} + S^{cd}{}_{AB} J_a{}^{AB})$

1.5.7 $\frac{1}{2}$-spinor description of gravitational field source [1, 2]

Def. 1.5.5. *Gravitational source spinor* $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} := \varsigma\varepsilon^{B_\varsigma A_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}J_a^{C_\kappa D_\kappa}$

Cor. 1.5.30. $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} = \varsigma\varepsilon^{B_\varsigma A_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{C_\kappa D_\kappa}_{\alpha_\kappa}J_a^{\alpha_\varsigma} = \varsigma\varepsilon^{B_\varsigma A_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\frac{i\kappa}{\sqrt{2}}S_{cd}{}^{C_\kappa D_\kappa}J_a{}^{cd}$

Cor. 1.5.31. $J_a{}^{C_\kappa D_\kappa} = \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}(-\varsigma)\varepsilon_{A_\varsigma B_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa}$

Cor. 1.5.32. $J_{A'_\varsigma}{}^{B_\varsigma C_\kappa D_\kappa} = J_{A'_\varsigma}{}^{B_\varsigma D_\kappa C_\kappa}$

Cor. 1.5.33. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma} = \frac{\varsigma}{2}\varepsilon^{B_\varsigma A_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}J_a^{\alpha_\varsigma} = \frac{\varsigma}{2}\varepsilon^{B_\varsigma A_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}S_{cd}{}^{C_\varsigma D_\varsigma}J_a{}^{cd}$

Cor. 1.5.34. $J_a^{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}(-\varsigma)\varepsilon_{A_\varsigma B_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{C_\varsigma D_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma} = \frac{\varsigma}{2}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}\varepsilon_{A_\varsigma B_\varsigma}\sigma^{\alpha_\varsigma}_{C_\varsigma D_\varsigma}J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$

Cor. 1.5.35. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma} = J_{A'_\varsigma}{}^{B_\varsigma D_\varsigma C_\varsigma} \Leftrightarrow \begin{cases} J_{1'_\varsigma}{}^{1_\varsigma 1_\varsigma 1_\varsigma} \\ J_{1'_\varsigma}{}^{1_\varsigma 1_\varsigma 2_\varsigma} = J_{1'_\varsigma}{}^{1_\varsigma 2_\varsigma 1_\varsigma}, J_{1'_\varsigma}{}^{2_\varsigma 1_\varsigma 2_\varsigma} = J_{1'_\varsigma}{}^{2_\varsigma 2_\varsigma 1_\varsigma} \\ J_{2'_\varsigma}{}^{1_\varsigma 1_\varsigma 2_\varsigma} = J_{2'_\varsigma}{}^{1_\varsigma 2_\varsigma 1_\varsigma}, J_{2'_\varsigma}{}^{2_\varsigma 1_\varsigma 2_\varsigma} = J_{2'_\varsigma}{}^{2_\varsigma 2_\varsigma 1_\varsigma} \\ J_{2'_\varsigma}{}^{2_\varsigma 2_\varsigma 2_\varsigma} \end{cases}$

Cor. 1.5.36. $J_{A'_\varsigma}{}^{1_\varsigma 2_\varsigma D_\varsigma} = J_{A'_\varsigma}{}^{2_\varsigma 1_\varsigma D_\varsigma} \Leftrightarrow \begin{cases} \varsigma J_\pi^{x_\varsigma} = J_y^{z_\varsigma} - J_z^{y_\varsigma}, \varsigma J_\pi^{y_\varsigma} = J_z^{x_\varsigma} - J_x^{z_\varsigma}, \varsigma J_\pi^{z_\varsigma} = J_x^{y_\varsigma} - J_y^{x_\varsigma} \\ J_x^{x_\varsigma} + J_y^{y_\varsigma} + J_z^{z_\varsigma} = 0 \end{cases}$

Proof: $J_{A'_\varsigma}{}^{1_\varsigma 2_\varsigma D_\varsigma} = J_{A'_\varsigma}{}^{2_\varsigma 1_\varsigma D_\varsigma}$
$\Leftrightarrow \frac{\varsigma}{2}\varepsilon^{1_\varsigma A_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\sigma^{2_\varsigma D_\varsigma}_{\alpha_\varsigma}J_a^{\alpha_\varsigma} = \frac{\varsigma}{2}\varepsilon^{2_\varsigma A_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}\sigma^{1_\varsigma D_\varsigma}_{\alpha_\varsigma}J_a^{\alpha_\varsigma}$
$\Leftrightarrow \varepsilon^{1_\varsigma 2_\varsigma}(\sigma, i\varsigma)^a_{2_\varsigma A'_\varsigma}\sigma^{2_\varsigma D_\varsigma}_{\alpha_\varsigma}J_a^{\alpha_\varsigma} = \varepsilon^{2_\varsigma 1_\varsigma}(\sigma, i\varsigma)^a_{1_\varsigma A'_\varsigma}\sigma^{1_\varsigma D_\varsigma}_{\alpha_\varsigma}J_a^{\alpha_\varsigma}$
$\Leftrightarrow (\sigma, i\varsigma)^a_{2_\varsigma A'_\varsigma}\sigma^{D_\varsigma 2_\varsigma}_{\alpha_\varsigma}J_a^{\alpha_\varsigma} = -(\sigma, i\varsigma)^a_{1_\varsigma A'_\varsigma}\sigma^{D_\varsigma 1_\varsigma}_{\alpha_\varsigma}J_a^{\alpha_\varsigma}$
$\Leftrightarrow [\sigma^{D_\varsigma 1_\varsigma}_{\alpha_\varsigma}(\sigma, i\varsigma)^a_{1_\varsigma A'_\varsigma} + \sigma^{D_\varsigma 2_\varsigma}_{\alpha_\varsigma}(\sigma, i\varsigma)^a_{2_\varsigma A'_\varsigma}]J_a^{\alpha_\varsigma} = 0$
$\Leftrightarrow \sigma^{D_\varsigma A_\varsigma}_{\alpha_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}J_a^{\alpha_\varsigma} = 0$
$\Leftrightarrow \sigma_{\alpha_\varsigma D_\varsigma}{}^{A_\varsigma}(\sigma, i\varsigma)^a_{A_\varsigma A'_\varsigma}J_a^{\alpha_\varsigma} = 0$
$\Leftrightarrow \sigma_{\alpha_\varsigma}(\sigma, i\varsigma)^a J_a^{\alpha_\varsigma} = 0$
$\Leftrightarrow (\sigma, i\varsigma)^{Ta}\sigma^T_{\alpha_\varsigma}J_a^{\alpha_\varsigma} = 0$
$\Leftrightarrow \sigma_y(\sigma, i\varsigma)^{Ta}\sigma_y\sigma_y\sigma^T_{\alpha_\varsigma}\sigma_y J_a^{\alpha_\varsigma} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a\sigma_{\alpha_\varsigma}J_a^{\alpha_\varsigma} = 0$
$\Leftrightarrow (J_x^{x_\varsigma} + J_y^{y_\varsigma} + J_z^{z_\varsigma})I + i(-\varsigma J_\pi^{x_\varsigma} + J_y^{z_\varsigma} - J_z^{y_\varsigma})\sigma_x + i(-\varsigma J_\pi^{y_\varsigma} + J_z^{x_\varsigma} - J_x^{z_\varsigma})\sigma_y + i(-\varsigma J_\pi^{z_\varsigma} + J_x^{y_\varsigma} - J_y^{x_\varsigma})\sigma_z = 0$
$\Leftrightarrow \begin{cases} \varsigma J_\pi^{x_\varsigma} = J_y^{z_\varsigma} - J_z^{y_\varsigma}, \varsigma J_\pi^{y_\varsigma} = J_z^{x_\varsigma} - J_x^{z_\varsigma}, \varsigma J_\pi^{z_\varsigma} = J_x^{y_\varsigma} - J_y^{x_\varsigma} \\ J_x^{x_\varsigma} + J_y^{y_\varsigma} + J_z^{z_\varsigma} = 0 \end{cases}$ □

Cor. 1.5.37. $[(\sigma, -i\varsigma)^a\sigma_{\alpha_\varsigma}]J_a^{\alpha_\varsigma} = 0 \Leftrightarrow \begin{cases} \varsigma J_\pi^{x_\varsigma} = J_y^{z_\varsigma} - J_z^{y_\varsigma}, \varsigma J_\pi^{y_\varsigma} = J_z^{x_\varsigma} - J_x^{z_\varsigma}, \varsigma J_\pi^{z_\varsigma} = J_x^{y_\varsigma} - J_y^{x_\varsigma} \\ J_x^{x_\varsigma} + J_y^{y_\varsigma} + J_z^{z_\varsigma} = 0 \end{cases}$

Cor. 1.5.38. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$ *is fully symmetric for indices* $B_\varsigma C_\varsigma D_\varsigma$. $\Leftrightarrow [(\sigma, -i\varsigma)^a\sigma_{\alpha_\varsigma}]J_a^{\alpha_\varsigma} = 0$

Cor. 1.5.39. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma D_\varsigma}$ *is fully symmetric for indices* $B_\varsigma C_\varsigma D_\varsigma$. $\Leftrightarrow [(\sigma, -i\varsigma)^a S_{cd}(\frac{1}{2}, \varsigma)]J_a{}^{cd} = 0$

2 Various expressions of Bianchi identities for gravitational field

2.1 Classical expressions of gravitational field equation

2.1.1 Bianchi identities for torsionless gravitational field [11–14]

Bianchi identity: $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$ (8.8)

Cor. 2.1.1. $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow R^{(*ab)cd}{}_{;a} \equiv 0$

Proof: $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$
$\Rightarrow \varepsilon_{fcde}(R^{abcd;e} + R^{abde;c} + R^{abec;d}) \equiv 0$
$\Rightarrow 3\varepsilon_{fcde}R^{abcd;e} \equiv 0$
$\Rightarrow R^{ab(*cd)}{}_{;d} \equiv 0$
$\Rightarrow R^{(*ab)cd}{}_{;a} \equiv 0$ □

Cor. 2.1.2. $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}$

Proof: $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$
$\Rightarrow R^{abcd}{}_{;a} - R^{bd;c} + R^{bc;d} \equiv 0$
$\Rightarrow R^{abcd}{}_{;a} = R^{bd;c} - R^{bc;d}$
$\Rightarrow R^{cdba}{}_{;a} \equiv R^{b[c;d]}$
$\Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}$ □

Cor. 2.1.3. $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow (R^{ab} - \frac{1}{2}g^{ab}R)_{;b} \equiv 0$

Proof: $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$
$\Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}$
$\Rightarrow R^{ac}{}_{;a} \equiv R^{;c} - R^{ac}{}_{;a}$
$\Rightarrow R^{ac}{}_{;a} \equiv \frac{1}{2}R^{;c}$
$\Rightarrow (R^{ab} - \frac{1}{2}g^{ab}R)_{;b} \equiv 0$ □

2.1.2 Classical form of gravitational field equation

$$\begin{cases} R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \\ R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \end{cases} \Leftrightarrow \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{(*ab)cd}{}_{;a} \equiv 0, (R^{ab} - \frac{1}{2}g^{ab}R)_{;b} \equiv 0 \\ R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \end{cases} \tag{8.9}$$

2.2 Yang-Mills Form of Bianchi identity for gravitational field

2.2.1 Yang-Mills gauge theory explanation of gravity [46–49]

Def. 2.2.1. $\theta^{\alpha_\varsigma}(\varsigma) := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\vartheta^{ab} = -i(i\omega + \varsigma\epsilon)^{\alpha_\varsigma}$

Cor. 2.2.1. $\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma) = i\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = (i\omega + \varsigma\epsilon)\cdot\sigma(s), \frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma) = iA_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)$

Linear independence:

Lem. 2.2.1. $c^{cd}S_{cd} = 0 \Leftrightarrow c^{cd} = 0$

Lem. 2.2.2. $[\omega_u{}^{cd}(\frac{i}{2}S_{cd}), \omega_v{}^{ef}(\frac{i}{2}S_{ef})] = \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}(\frac{i}{2}S_{cd})$

Proof: $[\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = \omega_u{}^{cd}\omega_v{}^{ef}[iS_{cd}, iS_{ef}]$
$\Leftrightarrow [\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = \omega_u{}^{cd}\omega_v{}^{ef}[\delta_{cf}iS_{de} - \delta_{ce}iS_{df} + \delta_{de}iS_{cf} - \delta_{df}iS_{ce}]$
$\Leftrightarrow [\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = 4\omega_u{}^{ce}\omega_{ve}{}^{d}iS_{cd}$
$\Leftrightarrow [\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = 2\omega_{[u}{}^{ce}\omega_{v]e}{}^{d}iS_{cd}$
$\Leftrightarrow [\omega_u{}^{cd}(\frac{i}{2}S_{cd}), \omega_v{}^{ef}(\frac{i}{2}S_{ef})] = \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}(\frac{i}{2}S_{cd})$ □

Cor. 2.2.2. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow R_{uv}{}^{cd}(\frac{i}{2}S_{cd}) = \partial_u\omega_v{}^{cd}(\frac{i}{2}S_{cd}) - \partial_v\omega_u{}^{cd}(\frac{i}{2}S_{cd}) + [\omega_u{}^{cd}(\frac{i}{2}S_{cd}), \omega_v{}^{ef}(\frac{i}{2}S_{ef})]$

Cor. 2.2.3. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow R_{uv}{}^{<cd>} = \partial_u\omega_v{}^{<cd>} - \partial_v\omega_u{}^{<cd>} + [\omega_u{}^{<cd>}, \omega_v{}^{<ef>}]$

Cor. 2.2.4. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow R_{uv}{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) = \partial_u\omega_v{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) - \partial_v\omega_u{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) + [\omega_u{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma), \omega_v{}^{ef}\frac{i}{2}S_{ef}(s,\varsigma)]$

Cor. 2.2.5. $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^{d}$
$\Leftrightarrow F^{\alpha_\varsigma}_{uv}\sigma_{\alpha_\varsigma}(s) = \partial_u A_v^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) - \partial_v A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) + i[A_u^{\beta_\varsigma}\sigma_{\beta_\varsigma}(s), A_v^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s)]$

Cor. 2.2.6. $\frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma) \to U(\theta)\frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)U^{-1}(\theta) + [\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) - i[\partial_u U(\theta)]U^{-1}(\theta)$

Cor. 2.2.7. $i[S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)] = \delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma)$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s)$

Proof: $i[S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)] = \delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma)$
$\Leftrightarrow \frac{1}{16}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}[iS_{ab}(s,\varsigma), iS_{cd}(s,\varsigma)] = \frac{1}{16}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}[\delta_{ad}iS_{bc}(s,\varsigma) - \delta_{ac}iS_{bd}(s,\varsigma) + \delta_{bc}iS_{ad}(s,\varsigma) - \delta_{bd}iS_{ac}(s,\varsigma)]$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = \frac{1}{16}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}[\delta_{ad}iS_{bc}(s,\varsigma) - \delta_{ac}iS_{bd}(s,\varsigma) + \delta_{bc}iS_{ad}(s,\varsigma) - \delta_{bd}iS_{ac}(s,\varsigma)]$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = \frac{1}{4}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}\delta_{ad}iS_{bc}(s,\varsigma)$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = \frac{1}{4}[\delta_{\alpha_\varsigma\beta_\varsigma}\delta^{bc} + i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\varsigma\gamma_\varsigma}{}^{bc}(s)]iS_{bc}(s,\varsigma)$
$\Leftrightarrow [\sigma_{\alpha_\varsigma}(s), \sigma_{\beta_\varsigma}(s)] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s)$ □

2.2.2 Yang-Mills component form of Bianchi identity for gravitational field

Cor. 2.2.8. $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases}$

Lem. 2.2.3. $D^a F_{ab}^{\alpha_\varsigma} = -J_b^{\alpha_\varsigma} \Leftrightarrow D^a F_{ab}^{\alpha'_\varsigma} = -J_b^{\alpha'_\varsigma}$

Proof: $D^a F_{ab}^{\alpha_\varsigma} = -J_b^{\alpha_\varsigma}$
$\Leftrightarrow (D^a F_{ab}^{\alpha_\varsigma})^* = -(J_b^{\alpha_\varsigma})^*$
$\Leftrightarrow \eta_c^{a'} D^c(\eta_{a'}^a \eta_{b'}^b F_{ab}^{\alpha'_\varsigma}) = -\eta_{b'}^b J_b^{\alpha'_\varsigma}$
$\Leftrightarrow \eta_{b'}^b D^a(F_{ab}^{\alpha'_\varsigma}) = -\eta_{b'}^b J_b^{\alpha'_\varsigma}$
$\Leftrightarrow D^a(F_{ab}^{\alpha'_\varsigma}) = -J_b^{\alpha'_\varsigma}$ □

Lem. 2.2.4. $D^a * F_{ab}^{\alpha_\varsigma} \equiv 0 \Leftrightarrow D^a * F_{ab}^{\alpha'_\varsigma} \equiv 0$

Proof: $D^a * F_{ab}^{\alpha_\varsigma} \equiv 0$
$\Leftrightarrow (D^a * F_{ab}^{\alpha_\varsigma})^* \equiv 0$
$\Leftrightarrow \eta_c^{a'} D^c(\eta_{a'}^a \eta_{b'}^b * F_{ab}^{\alpha'_\varsigma}) \equiv 0$
$\Leftrightarrow \eta_{b'}^b D^a * F_{ab}^{\alpha'_\varsigma} \equiv 0$
$\Leftrightarrow D^a * F_{ab}^{\alpha'_\varsigma} \equiv 0$ □

Thm. 2.2.1. $\begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha_\varsigma} \equiv -J_b^{\alpha_\varsigma}, J^{b\alpha_\varsigma} := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd} R^{b[c;d]} \\ D^a * F_{ab}^{\alpha_\varsigma} \equiv 0 \end{cases}$

Proof: $\begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} R_{ab}{}^{cd;a} \equiv -R_b{}^{[c;d]} \\ R_{*ab}{}^{cd;a} \equiv 0 \end{cases}$

$\Leftrightarrow \begin{cases} \frac{i}{2}(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd} F_{ab}^{\alpha'_\varsigma} + \sigma^{cd}_{\varsigma\alpha_\varsigma} F_{ab}^{\alpha_\varsigma})^{;a} \equiv -\frac{i}{2}(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd} J_b^{\alpha'_\varsigma} + \sigma^{cd}_{\varsigma\alpha_\varsigma} J_b^{\alpha_\varsigma}) \\ \frac{i}{2}(\sigma_{-\varsigma\alpha'_\varsigma}{}^{cd} * F_{ab}^{\alpha'_\varsigma} + \sigma^{cd}_{\varsigma\alpha_\varsigma} * F_{ab}^{\alpha_\varsigma})^{;a} \equiv 0 \end{cases}$

$\Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha_\varsigma} \equiv -J_b^{\alpha_\varsigma}, D^a F_{ab}^{\alpha'_\varsigma} \equiv -J_b^{\alpha_\varsigma} \\ D^a * F_{ab}^{\alpha_\varsigma} \equiv 0, D^a * F_{ab}^{\alpha'_\varsigma} \equiv 0 \end{cases}$

$\Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha_\varsigma} \equiv -J_b^{\alpha_\varsigma}, J^{b\alpha_\varsigma} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma cd} R^{b[c;d]} \\ D^a * F_{ab}^{\alpha_\varsigma} \equiv 0 \end{cases}$ □

Thm. 2.2.2. $\begin{cases} D^a F_{ab}^{\alpha_\varsigma} \equiv -J_b^{\alpha_\varsigma} \\ D^a * F_{ab}^{\alpha_\varsigma} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} \nabla_u F^{uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma} A_u^{\beta_\varsigma} F^{uv\gamma_\varsigma} \equiv -J^{v\alpha_\varsigma} \\ \nabla_u F^{*uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma} A_u^{\beta_\varsigma} F^{*uv\gamma_\varsigma} \equiv 0 \end{cases}$

2.2.3 Matrix description of Yang-Mills theory of Bianchi identity

1. General matrix description of Yang-Mills theory of Bianchi identity:

$$\begin{cases} R_{uv}{}^{cd}\frac{i}{2}S_{cd} = \partial_u \omega_v{}^{cd}\frac{i}{2}S_{cd} - \partial_v \omega_u{}^{cd}\frac{i}{2}S_{cd} + [\omega_u{}^{cd}\frac{i}{2}S_{cd}, \omega_v{}^{ef}\frac{i}{2}S_{ef}] \\ i[S_{ab}, S_{cd}] = \delta_{ad}S_{bc} - \delta_{ac}S_{bd} + \delta_{bc}S_{ad} - \delta_{bd}S_{ac} \\ c^{ab}S_{ab} = 0, c^{ab} = -c^{ba} \Leftrightarrow c^{ab} = 0 \end{cases} \tag{8.10}$$

Gauge transformation:

$$\begin{cases} \psi \to U(\theta)\psi, U(\theta) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}} \\ \frac{i}{2}\omega_u{}^{ab}S_{ab} \to U(\theta)\frac{i}{2}\omega_u{}^{ab}S_{ab}U^{-1}(\theta) - [\partial_u U(\theta)]U^{-1}(\theta) \end{cases} \tag{8.11}$$

Cor. 2.2.9. $D_u\psi \to U(\theta)D_u\psi, D_u = \partial_u + \frac{i}{2}\omega_u{}^{cd}S_{cd}$

Cor. 2.2.10. $R_{uv}{}^{cd}\frac{i}{2}S_{cd} \to U(\theta)R_{uv}{}^{cd}\frac{i}{2}S_{cd}U^{-1}(\theta)$

Cor. 2.2.11. $D_w R_{uv}{}^{cd}\frac{i}{2}S_{cd} \to U(\theta)D_w R_{uv}{}^{cd}\frac{i}{2}S_{cd}U^{-1}(\theta), D_w = \nabla_w + [\frac{i}{2}\omega_w{}^{cd}S_{cd}, \quad]$

The guage equation form of the Bianchi identity:

Cor. 2.2.12. $\begin{cases} \nabla_u R^{uvcd}\frac{i}{2}S_{cd} + [\frac{i}{2}\omega_u{}^{cd}S_{cd}, R^{uvcd}\frac{i}{2}S_{cd}] = 0 \\ \nabla_u R^{(*uv)cd}\frac{i}{2}S_{cd} + [\frac{i}{2}\omega_u{}^{cd}S_{cd}, R^{(*uv)cd}\frac{i}{2}S_{cd}] \equiv 0 \end{cases}$

Guage equation:

Cor. 2.2.13. $\begin{cases}\nabla_u R^{uv<cd>} + [\omega_u{}^{<cd>}, R^{uv<cd>}] \equiv -R_u{}^{<c;d>} \\ \nabla_u R^{*uv<cd>} + [\omega_u{}^{<cd>}, R^{*uv<cd>}] \equiv 0\end{cases} \Leftrightarrow \begin{cases}R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0\end{cases}$

2. Special matrix description of Yang-Mills theory of Bianchi identity:

$$\begin{cases}R_{uv}{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) = \partial_u\omega_v{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) - \partial_v\omega_u{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) + [\omega_u{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma), \omega_v{}^{ef}\frac{i}{2}S_{ef}(s,\varsigma)] \\ i[S_{ab}(s,\varsigma), S_{cd}(s,\varsigma)] = \delta_{ad}S_{bc}(s,\varsigma) - \delta_{ac}S_{bd}(s,\varsigma) + \delta_{bc}S_{ad}(s,\varsigma) - \delta_{bd}S_{ac}(s,\varsigma) \\ c^{ab}iS_{ab}(s,\varsigma) = 0, c^{ab} = -c^{ba} \Leftrightarrow c^{ab} = 0\end{cases} \tag{8.12}$$

Gauge transformation:

$$\begin{cases}\psi(s,\varsigma) \to U(\theta)\psi(s,\varsigma), U(\theta) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma)} \\ \frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma) \to U(\theta)\frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)U^{-1}(\theta) - [\partial_u U(\theta)]U^{-1}(\theta)\end{cases} \tag{8.13}$$

Cor. 2.2.14. $D_u\psi(s,\varsigma) \to U(\theta)D_u\psi(s,\varsigma), D_u = \partial_u + \frac{i}{2}\omega_u{}^{cd}S_{cd}(s,\varsigma)$

Cor. 2.2.15. $R_{uv}{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) \to U(\theta)R_{uv}{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma)U^{-1}(\theta)$

Cor. 2.2.16. $D_w R_{uv}{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma) \to UD_w R_{uv}{}^{cd}\frac{i}{2}S_{cd}(s,\varsigma)U^{-1}, D_w = \nabla_w + [\frac{i}{2}\omega_w{}^{cd}S_{cd}(s,\varsigma),\quad]$

The guage equation form of the Bianchi identity:

Cor. 2.2.17. $\begin{cases}\nabla_u R^{uvcd}\frac{i}{2}S_{cd}(s,\varsigma) + [\frac{i}{2}\omega_u{}^{cd}S_{cd}(s,\varsigma), R^{uvcd}\frac{i}{2}S_{cd}(s,\varsigma)] \equiv -R^{v[c;d]}\frac{i}{2}S_{cd}(s,\varsigma) \\ \nabla_u R^{(*uv)cd}\frac{i}{2}S_{cd}(s,\varsigma) + [\frac{i}{2}\omega_u{}^{cd}S_{cd}(s,\varsigma), R^{(*uv)cd}\frac{i}{2}S_{cd}(s,\varsigma)] \equiv 0\end{cases}$

$\Leftrightarrow \begin{cases}\nabla_u F^{uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}F^{uv\gamma_\varsigma} \equiv -J^{v\alpha_\varsigma} \\ \nabla_u F^{*uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}F^{*uv\gamma_\varsigma} \equiv 0\end{cases}$

3. Standard matrix description of Yang-Mills theory of Bianchi identity:

$$\begin{cases}F_{uv}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = \partial_u A_v^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) - \partial_v A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) + i[A_u^{\beta_\varsigma}\sigma_{\beta_\varsigma}(s), A_v^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(s)] \\ [\sigma_{\beta_\varsigma}(s), \sigma_{\gamma_\varsigma}(s)] = i\varepsilon_{\beta_\varsigma\gamma_\varsigma}{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s), c^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = 0 \Leftrightarrow c^{\alpha_\varsigma} = 0\end{cases} \tag{8.14}$$

Gauge transformation:

$$\begin{cases}\psi(s,\varsigma) \to U(\theta)\psi(s,\varsigma), U(\theta) = e^{i\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)} = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma)} = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)} \\ A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) + i[\partial_u U(\theta)]U^{-1}(\theta)\end{cases} \tag{8.15}$$

The above is just the standard Yang-Mills theory with $g = 1$ and $T = \sigma(s)$. Therefore, there are similar conclusions as follows.

Cor. 2.2.18. $D_u\psi(s,\varsigma) \to U(\theta)D_u\psi(s,\varsigma), D_u = \partial_u + iA_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) = \partial_u + \frac{i}{2}\omega_u{}^{cd}S_{cd}(s,\varsigma)$

Cor. 2.2.19. $F_{uv}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)F_{uv}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta)$

Cor. 2.2.20. $D_w F_{uv}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to UD_w F_{uv}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}, D_w = \nabla_w + [iA_w{}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s),\quad]$

The guage equation form of the Bianchi identity:

Cor. 2.2.21.
$\begin{cases}\nabla_u F^{uv\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) + [iA_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s), F^{uv\beta_\varsigma}\sigma_{\beta_\varsigma}(s)] \equiv -J^{v\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \\ \nabla_u F^{*uv\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) + [iA_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s), F^{*uv\beta_\varsigma}\sigma_{\beta_\varsigma}(s)] \equiv 0\end{cases} \Leftrightarrow \begin{cases}\nabla_u F^{uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}F^{uv\gamma_\varsigma} \equiv -J^{v\alpha_\varsigma} \\ \nabla_u F^{*uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}F^{*uv\gamma_\varsigma} \equiv 0\end{cases}$

Matrix description of Yang-Mills theory of Bianchi identity:

Cor. 2.2.22.
$\begin{cases}\nabla_u F^{uv} + i[A_u, F^{uv}] \equiv -J^v, A_u := A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \\ \nabla_u F^{*uv} + i[A_u, F^{*uv}] \equiv 0, F^{uv} := F^{uv\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)\end{cases} \Leftrightarrow \begin{cases}D^a F_{ab}^{\alpha_\varsigma} \equiv -J_b^{\alpha_\varsigma} \\ D^a * F_{ab}^{\alpha_\varsigma} \equiv 0\end{cases} \Leftrightarrow \begin{cases}R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0\end{cases}$

2.2.4 Component description of Yang-Mills theory of gravitational field

Thm. 2.2.3. $A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) + i[\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow \delta A_u^{\alpha_\varsigma} = i\theta^{\beta_\varsigma}(-i\varepsilon_{\beta_\varsigma}{}^{\alpha_\varsigma}{}_{\gamma_\varsigma})A_u^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma}$
$\Leftrightarrow \delta A_u = i\theta^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}A_u - \partial_u\theta$

Proof: $A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) + i[\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow$ □

Thm. 2.2.4. $\delta A_u^{\alpha_\varsigma} = \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}\theta^{\beta_\varsigma}A_u^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma} \Leftrightarrow \delta\omega_u{}^{ab} = \vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb} - \partial_u\vartheta^{ab}$

Proof: $\delta A_u^{\alpha_\varsigma} = i\theta^{\beta_\varsigma}(-i\varepsilon_{\beta_\varsigma}{}^{\alpha_\varsigma}{}_{\gamma_\varsigma})A_u^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma}$
$\Leftrightarrow A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s) \to U(\theta)A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)U^{-1}(\theta) + i[\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow [-\frac{1}{2}\omega_u{}^{ab}iS_{ab}(s,\varsigma)] \to U(\theta)[-\frac{1}{2}\omega_u{}^{ab}iS_{ab}(s,\varsigma)]U^{-1}(\theta) + [\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow [\frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)] \to U(\theta)[\frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)]U^{-1}(\theta) - [\partial_u U(\theta)]U^{-1}(\theta)$
$\Leftrightarrow [\frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)] \to \frac{1}{2}(\omega_u{}^{ab} - \partial_u\vartheta^{ab})iS_{ab}(s,\varsigma) + \frac{1}{4}\vartheta^{ab}\omega_u{}^{cd}[iS_{ab}(s,\varsigma), iS_{cd}(s,\varsigma)]$
$\Leftrightarrow [\frac{i}{2}\omega_u{}^{ab}S_{ab}(s,\varsigma)] \to \frac{1}{2}(\omega_u{}^{ab} - \partial_u\vartheta^{ab})iS_{ab}(s,\varsigma) + \frac{1}{2}(\vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb})iS_{ab}(s,\varsigma)$
$\Leftrightarrow \omega_u{}^{ab} \to \omega_u{}^{ab} + \vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb} - \partial_u\vartheta^{ab}$
$\Leftrightarrow \delta\omega_u{}^{ab} = \vartheta^{ac}\omega_u{}^{cb} - \omega_u{}^{ac}\vartheta^{cb} - \partial_u\vartheta^{ab}$ □

Cor. 2.2.23. *Gauge transformation:* $\begin{cases}\delta\psi(s,\varsigma) = i\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)\psi(s,\varsigma)\\ \delta A_u^{\alpha_\varsigma} = i\theta^{\beta_\varsigma}(-i\varepsilon_{\beta_\varsigma}{}^{\alpha_\varsigma}{}_{\gamma_\varsigma})A_u^{\gamma_\varsigma} - \partial_u\theta^{\alpha_\varsigma}\end{cases}$

Cor. 2.2.24. $\delta F_{uv}^{\alpha_\varsigma} = i\theta^{\beta_\varsigma}(-i\varepsilon_{\beta_\varsigma}{}^{\alpha_\varsigma}{}_{\gamma_\varsigma})F_{uv}^{\gamma_\varsigma}, \delta F_{uv}^{[\alpha_\varsigma]} = i\theta^{\beta_\varsigma}\gamma_{\beta_\varsigma}F_{uv}^{[\alpha_\varsigma]} = (i\omega + \varsigma\epsilon)\cdot\gamma F_{uv}^{[\alpha_\varsigma]}$

Cor. 2.2.25. $\delta\omega_u^{ab} = \frac{i}{2}(\sigma^{ab}_{-\varsigma\alpha'_\varsigma}\delta A_u^{\alpha'_\varsigma} + \sigma^{ab}_{\varsigma\alpha_\varsigma}\delta A_u^{\alpha_\varsigma})$

2.2.5 Similar electromagnetic field equation form of Bianchi identity

$$\begin{cases}\nabla_d\cdot\vec{E}^{\beta_\kappa} \equiv \rho^{\beta_\kappa}, \nabla_d\times\vec{E}^{\beta_\kappa} \equiv -D_t\vec{B}^{\beta_\kappa}\\ \nabla_d\cdot\vec{B}^{\beta_\kappa} \equiv 0, \nabla_d\times\vec{B}^{\beta_\kappa} \equiv \vec{J}^{\beta_\kappa} + D_t\vec{E}^{\beta_\kappa}\end{cases} \Leftrightarrow \begin{cases}D^u F_{uv}^{\beta_\kappa} \equiv -J_v^{\beta_\kappa}\\ D^u * F_{uv}^{\beta_\kappa} \equiv 0\end{cases} \tag{8.16}$$

Cor. 2.2.26. $F_{uv}^{\beta_\kappa} = \partial_u A_v^{\beta_\kappa} - \partial_v A_u^{\beta_\kappa} - \varepsilon^{\beta_\kappa}{}_{\gamma_\kappa\delta_\kappa}A_u^{\gamma_\kappa}A_v^{\delta_\kappa} \Leftrightarrow D^a * F_{ab}^{\beta_\kappa} \equiv 0; F_{ab}^{\beta_\kappa} = e_a^u e_b^v F_{uv}^{\beta_\kappa}$

2.3 Complex vector expression of Bianchi identity

Complex vector tensor form:

Thm. 2.3.1. $D^a F_{ab}^{\beta_\kappa} \equiv -J_b^{\beta_\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa} \equiv iJ_b^{\beta_\kappa}; F_{ab}^{\beta_\kappa} = e_a^u e_b^v F_{uv}^{\beta_\kappa}, \tilde{\Psi}^{\alpha_\varsigma\beta_\kappa} = \begin{bmatrix}\psi^{\alpha_\varsigma\beta_\kappa} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}F^{ab\beta_\kappa}\\ 0\end{bmatrix}$

Proof: $D^a F_{ab}^{\beta_\kappa} \equiv -J_b^{\beta_\kappa}$
$\Leftrightarrow D^a F_{ab}^{\beta_\kappa} \equiv -J_b^{\beta_\kappa}, D^a * F_{ab}^{\beta_\kappa} \equiv 0$
$\Leftrightarrow D^a(F_{ab}^{\beta_\kappa} - \varsigma * F_{ab}^{\beta_\kappa}) \equiv -J_b^{\beta_\kappa}$
$\Leftrightarrow D^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^{\beta_\kappa}_{\alpha_\varsigma}) \equiv -J^{b\beta_\kappa}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow D^a[(\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}|_{ab}\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa}] \equiv iJ_b^{\beta_\kappa}, \alpha_\varsigma = 1,2,3,4$
$\Leftrightarrow D^a[(\sigma_{-\varsigma}, -i\varsigma)_a|_b{}^{\alpha_\varsigma}\tilde{\Psi}^{\beta_\kappa}_{\alpha_\varsigma}] \equiv iJ_b^{\beta_\kappa}, \alpha_\varsigma = 1,2,3,4$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa} \equiv iJ_b^{\beta_\kappa}, \alpha_\varsigma = 1,2,3,4$ □

Complex vector matrix form:

Cor. 2.3.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa} \equiv iJ_b^{\beta_\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}^{\beta_\kappa}(1,\varsigma) \equiv iJ^{\beta_\kappa}$

Complex vector square matrix form:

Cor. 2.3.2. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa} \equiv iJ_b^{\beta_\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a[\tilde{\Psi}(1,\varsigma)] \equiv i[J]$

Representation transformation:

Cor. 2.3.3. $(\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}^{\beta_\kappa}(1,\varsigma) \equiv iJ^{\beta_\kappa} \Leftrightarrow (\sigma\otimes I, -i\varsigma)^a D_a\tilde{\psi}^{\beta_\kappa}(1,\varsigma) \equiv i\tilde{J}^{\beta_\kappa}(1,\varsigma)$

Cor. 2.3.4. $(\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}^{\beta_\kappa}(1,\varsigma) \equiv iJ^{\beta_\kappa} \Leftrightarrow (\sigma\otimes I, -i\varsigma)^a D_a[\tilde{\psi}(1,\varsigma)] \equiv i[\tilde{J}]$

2.4 $\frac{1}{2}$-spinor expression of Bianchi identity [1, 2]

$\frac{1}{2}$-spinor Penrose abstract index form:

Thm. 2.4.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa} \equiv iJ_b^{\beta_\kappa} \Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma}\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa}, \nabla_d^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a$

Proof: $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\kappa} \equiv iJ_b^{\beta_\kappa}$
$\Leftrightarrow D^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^{\beta_\kappa}_{\alpha_\varsigma}) \equiv -J_b^{\beta_\kappa}$
$\Leftrightarrow D^a(i\sigma^{\alpha_\varsigma}_{\varsigma ab}\cdot\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma}) \equiv -J_b^{\beta_\kappa}$
$\Leftrightarrow iS_{ab}{}^{A_\varsigma B_\varsigma}D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J_b^{\beta_\kappa}$
$\Leftrightarrow (\frac{\varsigma}{2}\delta_{ab}\varepsilon^{A_\varsigma B_\varsigma} + iS_{ab}{}^{A_\varsigma B_\varsigma})D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J_b^{\beta_\kappa}$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}\cdot\frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_b^{B'_\varsigma B_\varsigma}D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-1}{\sqrt{2}}J_b^{\beta_\kappa}$

$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-1}{\sqrt{2}}J^{\beta_\kappa}_b \cdot \frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}\varsigma\varepsilon^{A'_\varsigma B'_\varsigma}J_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma}\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa}, \nabla_d^{A'_\varsigma A_\varsigma} \equiv \frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a$ □

$\frac{1}{2}$-spinor tensor form:

Cor. 2.4.1. $\nabla_d^{A'_\varsigma A_\varsigma}\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa} \Leftrightarrow (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv iJ^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa}$

$\frac{1}{2}$-spinor matrix form:

Cor. 2.4.2. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv iJ^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa} \Leftrightarrow (\sigma\otimes I,-i\varsigma)_aD^a\tilde{\psi}^{\beta_\kappa}(1,\varsigma) \equiv i\tilde{J}^{\beta_\kappa}(1,\varsigma)$

$\frac{1}{2}$-spinor square matrix form:

Cor. 2.4.3. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv iJ^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa} \Leftrightarrow (\sigma,-i\varsigma)_aD^a[\psi]^{\beta_\kappa} \equiv i[J]^{\beta_\kappa}$

2.5 Full $\frac{1}{2}$-spinor expression of Bianchi identity

Cor. 2.5.1. $\nabla_d^{A'_\varsigma A_\varsigma}\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa} \Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma C_\kappa D_\kappa}$

Cor. 2.5.2. $\nabla_d^{A'_\varsigma A_\varsigma}\psi^{\beta_\varsigma}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma} \Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}$

Cor. 2.5.3. $\nabla_d^{A'_\varsigma A_\varsigma}\psi^{\beta_\kappa}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\kappa} \Leftrightarrow (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{A_\varsigma B_\varsigma C_\kappa D_\kappa} \equiv iJ^{A'_\varsigma}{}_{B_\varsigma C_\kappa D_\kappa}$

Cor. 2.5.4. $\nabla_d^{A'_\varsigma A_\varsigma}\psi^{\beta_\varsigma}_{A_\varsigma B_\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma} \Leftrightarrow (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv iJ^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}$

The proof of the following three corollaries will be left to the future.

Cor. 2.5.5. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} \equiv iJ^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}, R=0 \Leftrightarrow [2D_a+iS_{ab}(2,\varsigma)D^b]_{k_\varsigma}{}^{l_\varsigma}\psi_{l_\varsigma}(2,\varsigma) = \mathbb{J}_{ak_\varsigma}(2,\varsigma)$

Cor. 2.5.6. $\begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0, R=0 \end{cases} \Leftrightarrow [2D_a+iS_{ab}(2,\varsigma)D^b]_{k_\varsigma}{}^{l_\varsigma}\psi_{l_\varsigma}(2,\varsigma) = \mathbb{J}_{ak_\varsigma}(2,\varsigma)$

Cor. 2.5.7. $\begin{cases} (\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\beta_\varsigma} \equiv iJ^{\beta_\varsigma}_b \\ \psi_{\alpha_\varsigma\beta_\varsigma} = \psi_{\beta_\varsigma\alpha_\varsigma}, \psi_{\alpha_\varsigma}{}^{\alpha_\varsigma} = 0, (\sigma,-i\varsigma)^a\sigma_{\beta_\varsigma}J^{\beta_\varsigma}_a = 0 \end{cases} \Leftrightarrow (\sigma\otimes I_4,-i\varsigma)^aD_a\tilde{\psi}(2,\varsigma) = i\tilde{J}(2,\varsigma)$

2.6 Conjecture

Thm. 2.6.1. $D^a * F^{\beta_\kappa}_{ab} = 0 \Leftrightarrow F^{\beta_\kappa}_{ab} \Leftrightarrow D^a * F^{\beta_\kappa}_{ab} \equiv 0$

Thm. 2.6.2. $D^aF^{\beta_\kappa}_{ab} = -J^{\beta_\kappa}_b, D^a * F^{\beta_\kappa}_{ab} = 0 \Leftrightarrow D^aF^{\beta_\kappa}_{ab} = -J^{\beta_\kappa}_b, F^{\beta_\kappa}_{ab}$

2.7 Spin tensor expression of bianchi identity [7]

Gravitational field Spin tensor matrix: $S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \succ \begin{bmatrix} 0 & \gamma_z & -\gamma_y & -\varsigma\gamma_x \\ -\gamma_z & 0 & \gamma_x & -\varsigma\gamma_y \\ \gamma_y & -\gamma_x & 0 & -\varsigma\gamma_z \\ \varsigma\gamma_x & \varsigma\gamma_y & \varsigma\gamma_z & 0 \end{bmatrix}$ (8.17)

Thm. 2.7.1. $(D_a+iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa}(1,\varsigma) \equiv -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{\Psi}^{\delta_\kappa}(1,\varsigma) \equiv iJ^{\delta_\kappa}$

An intuitive proof method is as follows:

Proof: $(D_a+iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (D_x+i\gamma_zD_y-i\gamma_yD_z-i\varsigma\gamma_xD_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma xb}J^{b\delta_\kappa} \\ (D_y+i\gamma_xD_z-i\gamma_zD_x-i\varsigma\gamma_yD_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma yb}J^{b\delta_\kappa} \\ (D_z+i\gamma_yD_x-i\gamma_xD_y-i\varsigma\gamma_zD_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma zb}J^{b\delta_\kappa} \\ (D_\pi+i\varsigma\gamma_xD_x+i\varsigma\gamma_yD_y+i\varsigma\gamma_zD_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma\pi b}J^{b\delta_\kappa} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix}\begin{bmatrix} \psi^{x_\varsigma\delta_\kappa} \\ \psi^{y_\varsigma\delta_\kappa} \\ \psi^{z_\varsigma\delta_\kappa} \end{bmatrix} \equiv \begin{bmatrix} \varsigma J^{\pi\delta_\kappa} \\ J^{z\delta_\kappa} \\ -J^{y\delta_\kappa} \end{bmatrix}, \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix}\begin{bmatrix} \psi^{x_\varsigma\delta_\kappa} \\ \psi^{y_\varsigma\delta_\kappa} \\ \psi^{z_\varsigma\delta_\kappa} \end{bmatrix} \equiv \begin{bmatrix} -J^{z\delta_\kappa} \\ \varsigma J^{\pi\delta_\kappa} \\ J^{x\delta_\kappa} \end{bmatrix} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix}\begin{bmatrix} \psi^{x_\varsigma\delta_\kappa} \\ \psi^{y_\varsigma\delta_\kappa} \\ \psi^{z_\varsigma\delta_\kappa} \end{bmatrix} \equiv \begin{bmatrix} J^{y\delta_\kappa} \\ -J^{x\delta_\kappa} \\ \varsigma J^{\pi\delta_\kappa} \end{bmatrix}, iD_\pi\Psi^{\delta_\kappa}(1,\varsigma) \equiv \varsigma\gamma\cdot\nabla_d\Psi^{\delta_\kappa}(1,\varsigma) - i\varsigma\vec{J}^{\delta_\kappa} \end{cases}$$

$$\Leftrightarrow \begin{cases} iD_\pi\Psi^{\delta_\kappa}(1,\varsigma) \equiv i\varsigma\nabla_d \times \Psi^{\delta_\kappa}(1,\varsigma) - i\varsigma\vec{J}^{\delta_\kappa} \\ \nabla_d \cdot \Psi^{\delta_\kappa}(1,\varsigma) \equiv \varsigma J^{\pi\delta_\kappa} \end{cases}$$
$$\Leftrightarrow \begin{cases} iD_\pi\Psi^{\delta_\kappa}(1,\varsigma) \equiv \varsigma\gamma\cdot\nabla_d\Psi^{\delta_\kappa}(1,\varsigma) - i\varsigma\vec{J}^{\delta_\kappa} \\ \nabla_d \cdot \Psi^{\delta_\kappa}(1,\varsigma) \equiv \varsigma J^{\pi\delta_\kappa} \end{cases}$$
$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}^{\delta_\kappa}(1,\varsigma) \equiv iJ \qquad \square$$

Another more analytical and abstract proof is as follows:

Proof: $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma a}{}^{c}\sigma_{\varsigma\gamma_\varsigma cb}D^b\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}$
$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma ac}\sigma^{cb}_{\varsigma\gamma_\varsigma}D_b\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}$
$\Leftrightarrow \sigma^{\varsigma ad}_{\beta_\varsigma}\sigma^{\beta_\varsigma}_{\varsigma ac}\sigma^{cb}_{\varsigma\gamma_\varsigma}D_b\psi^{\gamma_\varsigma\delta_\kappa} \equiv -i\sigma^{\varsigma ad}_{\beta_\varsigma}\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}$
$\Leftrightarrow \sigma^{db}_{\varsigma\gamma_\varsigma}D_b\psi^{\gamma_\varsigma\delta_\kappa} \equiv -iJ^{d\delta_\kappa}$
$\Leftrightarrow \sigma^{ab}_{\varsigma\alpha_\varsigma}D_a\psi^{\alpha_\varsigma\delta_\kappa} \equiv iJ^{b\delta_\kappa}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{\Psi}^{\alpha_\varsigma\delta_\kappa} \equiv iJ^{\delta_\kappa}_b, \alpha_\varsigma = 1,2,3,4$ $\square$

The equation (3.3.2) is just the spin tensor expression of Bianchi identity.

Lem. 2.7.1. $\mathbb{J}^{\beta_\varsigma\delta_\kappa}_a = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa} \Leftrightarrow \begin{cases} \mathbb{J}^{z_\varsigma\delta_\kappa}_y = -\mathbb{J}^{y_\varsigma\delta_\kappa}_z = -\varsigma\mathbb{J}^{x_\varsigma\delta_\kappa}_\pi = J^{x\delta_\kappa} \\ \mathbb{J}^{x_\varsigma\delta_\kappa}_z = -\mathbb{J}^{z_\varsigma\delta_\kappa}_x = -\varsigma\mathbb{J}^{y_\varsigma\delta_\kappa}_\pi = J^{y\delta_\kappa} \\ \mathbb{J}^{y_\varsigma\delta_\kappa}_x = -\mathbb{J}^{x_\varsigma\delta_\kappa}_y = -\varsigma\mathbb{J}^{z_\varsigma\delta_\kappa}_\pi = J^{z\delta_\kappa} \\ \mathbb{J}^{x_\varsigma\delta_\kappa}_x = \mathbb{J}^{y_\varsigma\delta_\kappa}_y = \mathbb{J}^{z_\varsigma\delta_\kappa}_z = \varsigma J^{\pi\delta_\kappa} \end{cases}$

Expand and then we can prove it by expanding. The above spin equation is about special source terms, so what happens to general source terms? Please look at the following theorem.

Thm. 2.7.2.
$(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}^{\beta_\varsigma\delta_\kappa}_a, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}^{\delta_\kappa}(1,\varsigma) = iJ^{\delta_\kappa}, \mathbb{J}^{\beta_\varsigma\delta_\kappa}_a = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}$

Proof: $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}^{\beta_\varsigma\delta_\kappa}_a, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma\gamma_x D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}^{\beta_\varsigma\delta_\kappa}_x \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma\gamma_y D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}^{\beta_\varsigma\delta_\kappa}_y \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma\gamma_z D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}^{\beta_\varsigma\delta_\kappa}_z \\ (D_\pi + i\varsigma\gamma_x D_x + i\varsigma\gamma_y D_y + i\varsigma\gamma_z D_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}^{\beta_\varsigma\delta_\kappa}_\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix}\begin{bmatrix} \psi^{x_\varsigma\delta_\kappa} \\ \psi^{y_\varsigma\delta_\kappa} \\ \psi^{z_\varsigma\delta_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}^{x_\varsigma\delta_\kappa}_x \\ \mathbb{J}^{y_\varsigma\delta_\kappa}_x \\ \mathbb{J}^{z_\varsigma\delta_\kappa}_x \end{bmatrix} \Leftrightarrow \begin{cases} \nabla_d\cdot\Psi^{\delta_\kappa}(1,\varsigma) = \mathbb{J}^{x_\varsigma\delta_\kappa}_x \\ [\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma)]^{z_\varsigma\delta_\kappa} - \varsigma D_\pi\psi^{z_\varsigma\delta_\kappa}(1,\varsigma) = \mathbb{J}^{y_\varsigma\delta_\kappa}_x \\ -[\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma)]^{y_\varsigma\delta_\kappa} + \varsigma D_\pi\psi^{y_\varsigma\delta_\kappa}(1,\varsigma) = \mathbb{J}^{z_\varsigma\delta_\kappa}_x \end{cases} \\ \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix}\begin{bmatrix} \psi^{x_\varsigma\delta_\kappa} \\ \psi^{y_\varsigma\delta_\kappa} \\ \psi^{z_\varsigma\delta_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}^{x_\varsigma\delta_\kappa}_y \\ \mathbb{J}^{y_\varsigma\delta_\kappa}_y \\ \mathbb{J}^{z_\varsigma\delta_\kappa}_y \end{bmatrix} \Leftrightarrow \begin{cases} -[\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma)]^{z_\varsigma\delta_\kappa} + \varsigma D_\pi\psi^{z_\varsigma\delta_\kappa}(1,\varsigma) = \mathbb{J}^{x_\varsigma\delta_\kappa}_y \\ \nabla_d\cdot\Psi^{\delta_\kappa}(1,\varsigma) = \mathbb{J}^{y_\varsigma\delta_\kappa}_y \\ [\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma)]^{x_\varsigma\delta_\kappa} - \varsigma D_\pi\psi^{x_\varsigma\delta_\kappa}(1,\varsigma) = \mathbb{J}^{z_\varsigma\delta_\kappa}_y \end{cases} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix}\begin{bmatrix} \psi^{x_\varsigma\delta_\kappa} \\ \psi^{y_\varsigma\delta_\kappa} \\ \psi^{z_\varsigma\delta_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}^{x_\varsigma\delta_\kappa}_z \\ \mathbb{J}^{y_\varsigma\delta_\kappa}_z \\ \mathbb{J}^{z_\varsigma\delta_\kappa}_z \end{bmatrix} \Leftrightarrow \begin{cases} [\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma)]^{y_\varsigma\delta_\kappa} - \varsigma D_\pi\psi^{y_\varsigma\delta_\kappa}(1,\varsigma) = \mathbb{J}^{x_\varsigma\delta_\kappa}_z \\ -[\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma)]^{x_\varsigma\delta_\kappa} + \varsigma D_\pi\psi^{x_\varsigma\delta_\kappa}(1,\varsigma) = \mathbb{J}^{y_\varsigma\delta_\kappa}_z \\ \nabla_d\cdot\Psi^{\delta_\kappa}(1,\varsigma) = \mathbb{J}^{z_\varsigma\delta_\kappa}_z \end{cases} \\ D_\pi\Psi^{\delta_\kappa}(1,\varsigma) + i\varsigma\gamma\cdot\nabla_d\psi^{\delta_\kappa} = \mathbb{J}^{\delta_\kappa}_\pi \Leftrightarrow D_\pi\Psi^{\delta_\kappa}(1,\varsigma) - \varsigma\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma) = \mathbb{J}^{\delta_\kappa}_\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathbb{J}^{z_\varsigma\delta_\kappa}_y = -\mathbb{J}^{y_\varsigma\delta_\kappa}_z = -\varsigma\mathbb{J}^{x_\varsigma\delta_\kappa}_\pi := J^{x\delta_\kappa} \\ \mathbb{J}^{x_\varsigma\delta_\kappa}_z = -\mathbb{J}^{z_\varsigma\delta_\kappa}_x = -\varsigma\mathbb{J}^{y_\varsigma\delta_\kappa}_\pi := J^{y\delta_\kappa} \\ \mathbb{J}^{y_\varsigma\delta_\kappa}_x = -\mathbb{J}^{x_\varsigma\delta_\kappa}_y = -\varsigma\mathbb{J}^{z_\varsigma\delta_\kappa}_\pi := J^{z\delta_\kappa} \\ \mathbb{J}^{x_\varsigma\delta_\kappa}_x = \mathbb{J}^{y_\varsigma\delta_\kappa}_y = \mathbb{J}^{z_\varsigma\delta_\kappa}_z := \varsigma J^{\pi\delta_\kappa} \\ D_\pi\Psi^{\delta_\kappa}(1,\varsigma) - \varsigma\nabla_d\times\Psi^{\delta_\kappa}(1,\varsigma) = i\vec{J}^{\delta_\kappa} \\ \nabla_d\cdot\Psi^{\delta_\kappa}(1,\varsigma) = -iJ^{\pi\delta_\kappa} \end{cases}$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a\tilde{\Psi}^{\delta_\kappa}(1,\varsigma) = iJ^{\delta_\kappa}, \mathbb{J}^{\beta_\varsigma\delta_\kappa}_a = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa} \qquad \square$$

Another more analytical and abstract proof is as follows:

Thm. 2.7.3. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}^{\beta_\varsigma\delta_\kappa}_a, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow \mathbb{J}^{\beta_\varsigma\delta_\kappa}_a = \sigma^{\beta_\varsigma}_{\varsigma ab}(\sigma^{bc}_{\varsigma\gamma_\varsigma}D_c\psi^{\gamma_\varsigma\delta_\kappa})$

Proof: $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma\delta_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma a}{}^c\sigma_{\varsigma\gamma_\varsigma cb}D^b\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma\delta_\kappa}$
$\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\delta_\kappa} = \sigma^{\beta_\varsigma}_{\varsigma ab}(\sigma^{bc}_{\varsigma\alpha_\varsigma}D_c\psi^{\alpha_\varsigma\delta_\kappa})$ □

This theorem indicates that the source term of this spin equation is limited and not arbitrary. Only the source term case described in the previous theorem has a solution, while the other cases have no solution.

Cor. 2.7.1. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma\delta_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$ *have solutions.* $\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\delta_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\delta_\kappa}, \exists J^{b\delta_\kappa}$

2.8 Weyl expression of Bianchi identity

2.8.1 Classical Bianchi identities Satisfied by Weyl tensor of gravitational field [14]

Def. 2.8.1. $C^{abcd} \equiv R^{abcd} + \frac{1}{2}g^{a[d}R^{c]b} + \frac{1}{2}g^{b[c}R^{d]a} + \frac{1}{6}g^{a[c}g^{d]b}R$

Cor. 2.8.1. $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow C^{abcd}{}_{;a} \equiv -\frac{1}{2}R^{b[c;d]} + \frac{1}{12}g^{b[c}R^{;d]}$

Proof: $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$
$\Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{ba}{}_{;a} \equiv \frac{1}{2}R^{;b}$
$\Rightarrow C^{abcd}{}_{;a} \equiv R^{abcd}{}_{;a} + \frac{1}{2}g^{a[d}R^{c]b}{}_{;a} + \frac{1}{2}g^{b[c}R^{d]a}{}_{;a} + \frac{1}{6}g^{a[c}g^{d]b}R_{;a}$
$\Rightarrow C^{abcd}{}_{;a} \equiv -R^{b[c;d]} + \frac{1}{2}R^{b[c;d]} + \frac{1}{4}g^{b[c}R^{;d]} - \frac{1}{6}g^{b[c}R^{;d]}$
$\Leftrightarrow C^{abcd}{}_{;a} \equiv -\frac{1}{2}R^{b[c;d]} + \frac{1}{12}g^{b[c}R^{;d]}$ □

Cor. 2.8.2. $C^{(*ab)cd} \equiv R^{(*ab)cd} + \frac{1}{2}\varepsilon^{abe[c}R^{d]}{}_e + \frac{1}{6}\varepsilon^{abcd}R$

Weyl tensor form of Bianchi identity:

Cor. 2.8.3. $\begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} C^{abcd}{}_{;a} \equiv -R^{b[c;d]} + \frac{1}{2}g^{a[d}R^{c]b}_{;a} + \frac{1}{2}g^{b[c}R^{d]a}_{;a} + \frac{1}{6}g^{a[c}g^{d]b}R_{;a} \\ C^{(*ab)cd}{}_{;a} \equiv \frac{1}{2}\varepsilon^{abe[c}R^{d]}{}_{e;a} + \frac{1}{6}\varepsilon^{abcd}R_{;a} \end{cases}$

Cor. 2.8.4. $\begin{cases} C^{abcd}{}_{;a} \equiv -\frac{1}{2}R^{b[c;d]} + \frac{1}{12}g^{b[c}R^{;d]} \\ C^{(*ab)cd}{}_{;a} \equiv \frac{1}{2}\varepsilon^{abe[c}R^{d]}{}_{e;a} + \frac{1}{6}\varepsilon^{abcd}R_{;a} \end{cases}$

2.8.2 Weyl complex vector expression of Bianchi identity

Def. 2.8.2. $\tilde{C}^{\alpha_\varsigma\beta_\varsigma}(1,\varsigma) \equiv [C^{\alpha_\varsigma\beta_\varsigma}, 0^{\beta_\varsigma}]$

Thm. 2.8.1. $D^aF^{\beta_\varsigma}_{ab} \equiv -J^{\beta_\varsigma}_b \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma} \equiv i\frac{i}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}(R^{[c;d]}_b - \frac{1}{6}\delta^{[c}_bR^{;d]})$

Proof: $D^aF^{\beta_\varsigma}_{ab} \equiv -J^{\beta_\varsigma}_b$
$\Leftrightarrow D_a(i\sigma^{ab}_{\varsigma\alpha_\varsigma}\psi^{\alpha_\varsigma\beta_\varsigma}) \equiv -J^{b\beta_\varsigma}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow D_a[\sigma^{ab}_{\varsigma\alpha_\varsigma}(C^{\alpha_\varsigma\beta_\varsigma} + \frac{1}{6}\delta^{\alpha_\varsigma\beta_\varsigma}R)] \equiv iJ^{b\beta_\varsigma}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow D_a(\sigma^{ab}_{\varsigma\alpha_\varsigma}C^{\alpha_\varsigma\beta_\varsigma}) \equiv -\frac{1}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}R^{b[c;d]} - \frac{1}{6}\sigma^{ab}_{\varsigma\alpha_\varsigma}\delta^{\alpha_\varsigma\beta_\varsigma}R_{;a}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow D_a(\sigma^{ab}_{\varsigma\alpha_\varsigma}C^{\alpha_\varsigma\beta_\varsigma}) \equiv -\frac{1}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}R^{b[c;d]} + \frac{1}{6}\sigma^{\beta_\varsigma}_{\varsigma cd}\delta^{b[c}R^{;d]}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma} \equiv i\frac{i}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}(R^{[c;d]}_b - \frac{1}{6}\delta^{[c}_bR^{;d]}), \alpha_\varsigma = 1,2,3,4$ □

Def. 2.8.3. $\bar{J}^{bcd} \equiv R^{b[c;d]} - \frac{1}{6}g^{b[c}R^{;d]}, \bar{J}^{b\beta_\varsigma} \equiv \frac{i}{2}\sigma^{\beta_\varsigma}_{\varsigma cd}\bar{J}^{bcd}$

2.8.3 Weyl complex vector matrix expression of Bianchi identity

Complex vector matrix form:

Cor. 2.8.5. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma} \equiv i\bar{J}^{\beta_\varsigma}_b \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^aD_a\tilde{\Psi}^{\beta_\varsigma}(1,\varsigma) \equiv iJ^{\beta_\varsigma}$

Complex vector square matrix form:

Cor. 2.8.6. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma} \equiv i\bar{J}^{\beta_\varsigma}_b \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^aD_a[\tilde{C}(1,\varsigma)] \equiv i[\bar{J}]$

Representation transformation:

Cor. 2.8.7. $(\sigma_{-\varsigma}, -i\varsigma)^aD_a\tilde{C}^{\beta_\varsigma}(1,\varsigma) \equiv i\bar{J}^{\beta_\varsigma} \Leftrightarrow (\sigma\otimes I, -i\varsigma)^aD_a\tilde{c}^{\beta_\varsigma}(1,\varsigma) \equiv i\tilde{\bar{J}}^{\beta_\varsigma}(1,\varsigma)$

Cor. 2.8.8. $(\sigma_{-\varsigma}, -i\varsigma)^aD_a\tilde{C}^{\beta_\varsigma}(1,\varsigma) \equiv i\bar{J}^{\beta_\varsigma} \Leftrightarrow (\sigma\otimes I, -i\varsigma)^aD_a[\tilde{c}(1,\varsigma)] \equiv i[\tilde{\bar{J}}]$

2.8.4 Weyl spinor expression of Bianchi identity [1,2]

$\frac{1}{2}$-spinor Penrose abstract indix form:

Thm. 2.8.2. $(\sigma_{-\varsigma},-i\varsigma)^{a}{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma}\equiv i\bar{J}_b^{\beta_\varsigma}\Leftrightarrow\nabla_d^{A'_\varsigma A_\varsigma}C^{\beta_\varsigma}_{A_\varsigma B_\varsigma}\equiv\frac{-\varsigma}{\sqrt{2}}\bar{J}^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma}$

$\frac{1}{2}$-spinor tensor form:

Cor. 2.8.9. $\nabla_d^{A'_\varsigma A_\varsigma}C^{\beta_\varsigma}_{A_\varsigma B_\varsigma}\equiv\frac{-\varsigma}{\sqrt{2}}\bar{J}^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^aC^{\beta_\varsigma}_{A_\varsigma B_\varsigma}\equiv i\bar{J}^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma}$

$\frac{1}{2}$-spinor matrix form:

Cor. 2.8.10. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^aC^{\beta_\varsigma}_{A_\varsigma B_\varsigma}\equiv i\bar{J}^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma}\Leftrightarrow(\sigma\otimes I,-i\varsigma)_aD^a\tilde{C}^{\beta_\varsigma}(1,\varsigma)\equiv i\tilde{\bar{J}}^{\beta_\varsigma}(1,\varsigma)$

$\frac{1}{2}$-spinor square matrix form:

Cor. 2.8.11. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^aC^{\beta_\varsigma}_{A_\varsigma B_\varsigma}\equiv i\bar{J}^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)_aD^a[C]^{\beta_\varsigma}\equiv i[\bar{J}]^{\beta_\varsigma}$

2.8.5 Complete $\frac{1}{2}$-Weyl spinor expression of Bianchi identities

Cor. 2.8.12. $\nabla_d^{A'_\varsigma A_\varsigma}C^{\beta_\varsigma}_{A_\varsigma B_\varsigma}\equiv\frac{-\varsigma}{\sqrt{2}}\bar{J}^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma}\Leftrightarrow\nabla_d^{A'_\varsigma A_\varsigma}C_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}\equiv\frac{-\varsigma}{\sqrt{2}}\bar{J}^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}$

Cor. 2.8.13. $\nabla_d^{A'_\varsigma A_\varsigma}C^{\beta_\varsigma}_{A_\varsigma B_\varsigma}\equiv\frac{-\varsigma}{\sqrt{2}}\bar{J}^{A'_\varsigma}{}_{B_\varsigma}{}^{\beta_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^aC_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}\equiv i\bar{J}^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}$

2.8.6 Spin tensor Weyl expression of Bianchi identity

Thm. 2.8.3. $(D_a+iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}C^{\gamma_\varsigma\delta_\varsigma}(1,\varsigma)\equiv-i\sigma^{\beta_\varsigma}_{\varsigma ab}\bar{J}^{b\delta_\varsigma},S_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{C}^{\delta_\varsigma}(1,\varsigma)\equiv i\bar{J}^{\delta_\varsigma}$

The equation (2.8.3) is just the spin tensor Weyl expression of Bianchi identity.

Thm. 2.8.4.
$(D_a+iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}C^{\gamma_\varsigma\delta_\varsigma}=\mathbb{J}_a^{\beta_\varsigma\delta_\varsigma},S_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}\Leftrightarrow(\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{C}^{\delta_\varsigma}(1,\varsigma)=i\bar{J}^{\delta_\varsigma},\mathbb{J}_a^{\beta_\varsigma\delta_\varsigma}=-i\sigma^{\beta_\varsigma}_{\varsigma ab}\bar{J}^{b\delta_\varsigma}$

This theorem indicates that the source term of this spin equation is limited and not arbitrary. Only the source term case described in the previous theorem has a solution, while the other cases have no solution.

Cor. 2.8.14. $(D_a+iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}C^{\gamma_\varsigma\delta_\varsigma}=\bar{\mathbb{J}}_a^{\beta_\varsigma\delta_\varsigma},S_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}\Leftrightarrow\bar{\mathbb{J}}_a^{\beta_\varsigma\delta_\varsigma}=\sigma^{\beta_\varsigma}_{\varsigma ab}(\sigma^{bc}_{\varsigma\gamma_\varsigma}D_cC^{\gamma_\varsigma\delta_\varsigma})$

Cor. 2.8.15. $(D_a+iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}C^{\gamma_\varsigma\delta_\varsigma}=\bar{\mathbb{J}}_a^{\beta_\varsigma\delta_\varsigma},S_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$ *have solutions.* $\Leftrightarrow\bar{\mathbb{J}}_a^{\beta_\varsigma\delta_\varsigma}=-i\sigma^{\beta_\varsigma}_{\varsigma ab}\bar{J}^{b\delta_\varsigma},\exists\bar{J}^{b\delta_\varsigma}$

2.8.7 Full spin tensor Weyl expression of Bianchi identity

Cor. 2.8.16. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^aC_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}\equiv i\bar{J}^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}\Leftrightarrow[2D_a+iS_{ab}(2,\varsigma)D^b]_{k_\varsigma}{}^{l_\varsigma}c_{l_\varsigma}(2,\varsigma)\equiv\mathbb{J}_{ak_\varsigma}(2,\varsigma)$

Cor. 2.8.17. $(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^aC_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}\equiv i\bar{J}^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}\Leftrightarrow(\sigma\otimes I_4,-i\varsigma)^aD_a\tilde{c}(2,\varsigma)\equiv i\tilde{J}(2,\varsigma)$

The proof of the above two propositions will be supplemented in subsequent chapters and will be omitted here.

Cor. 2.8.18. $(\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{C}^{\beta_\varsigma}(1,\varsigma)\equiv i\bar{J}^{\beta_\varsigma}\Leftrightarrow(\sigma\otimes I_4,-i\varsigma)^aD_a\tilde{c}(2,\varsigma)\equiv i\tilde{J}(2,\varsigma)$

2.9 Classical separated form of Bianchi identity

Cor. 2.9.1. $(\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{\Psi}^{\beta_\varsigma}(1,\varsigma)=iJ^{\beta_\varsigma}\Leftrightarrow(\gamma,-i\varsigma)^aD_a\Psi^{\beta_\varsigma}(1,\varsigma)=i\vec{J}^{\beta_\varsigma},i\varsigma\nabla_d\cdot\Psi^{\beta_\varsigma}(1,\varsigma)=iJ^{\beta_\varsigma}_{\pi_\varsigma}$

Cor. 2.9.2.
$$\begin{cases}(\sigma_{-\varsigma},-i\varsigma)^aD_a\tilde{\Psi}^{\beta_\varsigma}(1,\varsigma)=iJ^{\beta_\varsigma}\\ \psi_{\alpha_\varsigma\beta_\varsigma}=\psi_{\beta_\varsigma\alpha_\varsigma},\psi_{\alpha_\varsigma}{}^{\alpha_\varsigma}=0,(\sigma,-i\varsigma)^a\sigma_{\beta_\varsigma}J_a^{\beta_\varsigma}=0\end{cases}\Leftrightarrow\begin{cases}(\frac{1}{2}G_m,-i\varsigma)^aD_a\Psi(2,\varsigma)=i\vec{\mathcal{J}}(2,\varsigma)\\ i\varsigma\nabla_d\cdot\Psi^{\beta_\varsigma}(1,\varsigma)=iJ^{\beta_\varsigma}_{\pi_\varsigma}\end{cases}$$

Cor. 2.9.3.
$$\begin{cases}(\frac{1}{2}G_m,-i\varsigma)^aD_a\Psi(2,\varsigma)=i\vec{\mathcal{J}}(2,\varsigma)\\ i\varsigma\nabla_d\cdot\Psi^{\beta_\varsigma}(1,\varsigma)=iJ^{\beta_\varsigma}_{\pi_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\frac{1}{2}\sigma(2),-i\varsigma]^aD_a\psi(2,\varsigma)=i\vec{J}(2,\varsigma)\\ i\varsigma\nabla_d\cdot\Psi^{\beta_\varsigma}(1,\varsigma)=iJ^{\beta_\varsigma}_{\pi_\varsigma}\end{cases}$$

Cor. 2.9.4.
$$\begin{cases}[\sigma(s),-is\varsigma]^aD_a\psi(s,\varsigma)=is\vec{J}(s,\varsigma)\\ i\varsigma\nabla_d\cdot\Psi^{l_\varsigma}(1,\varsigma)=iJ^{l_\varsigma}_{\pi_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\frac{1}{s}\sigma(s),-i\varsigma]^aD_a\psi(s,\varsigma)=i\vec{J}(s,\varsigma)\\ i\varsigma\nabla_d\cdot\Psi^{l_\varsigma}(1,\varsigma)=iJ^{l_\varsigma}_{\pi_\varsigma}\end{cases}$$

Cor. 2.9.5.

$$S=\frac{1}{\sqrt{4}}\begin{bmatrix}\sqrt{4}&0&0&0&0&0&0&0\\0&\sqrt{1}&\sqrt{3}&0&0&0&0&0\\0&0&0&\sqrt{2}&\sqrt{2}&0&0&0\\0&0&0&0&0&\sqrt{3}&\sqrt{1}&0\\0&0&0&0&0&0&0&\sqrt{4}\\0&\sqrt{3/2}i&-\sqrt{1/2}i&0&0&-\sqrt{1/2}i&\sqrt{3/2}i&0\\0&-\sqrt{3/2}&\sqrt{1/2}&0&0&-\sqrt{1/2}&\sqrt{3/2}&0\\0&0&0&-\sqrt{2}i&\sqrt{2}i&0&0&0\end{bmatrix},S^{+}=\frac{1}{\sqrt{4}}\begin{bmatrix}\sqrt{4}&0&0&0&0&0&0&0\\0&\sqrt{1}&0&0&0&-\sqrt{3/2}i&-\sqrt{3/2}&0\\0&\sqrt{3}&0&0&0&\sqrt{1/2}i&\sqrt{1/2}&0\\0&0&\sqrt{2}&0&0&0&0&\sqrt{2}i\\0&0&\sqrt{2}&0&0&0&0&-\sqrt{2}i\\0&0&0&\sqrt{3}&0&\sqrt{1/2}i&-\sqrt{1/2}&0\\0&0&0&\sqrt{1}&0&-\sqrt{3/2}i&\sqrt{3/2}&0\\0&0&0&0&\sqrt{4}&0&0&0\end{bmatrix}$$

Cor. 2.9.6.

$$(\sigma\otimes I_4,-i\varsigma)^a\partial_a\tilde{\psi}(2,\varsigma)=i\tilde{J}(2,\varsigma)\overset{S}{\Leftrightarrow}\begin{cases}[\sigma(2),-i\varsigma]^a\partial_a\psi(2,\varsigma)=i\bar{N}(2)\tilde{J}(2,\varsigma)\\ i\varsigma\nabla\cdot S^{l_\varsigma}_m(2)S_{im}(2,+)\psi(2,\varsigma)=iJ^{l_\varsigma}_\pi,J^{l_\varsigma}_\pi\succ J_\pi\end{cases}\quad\begin{cases}\begin{bmatrix}\bar{N}(2)\tilde{J}(2,\varsigma)\\J_\pi\end{bmatrix}=S\tilde{J}(2,\varsigma)\\\begin{bmatrix}\psi(2,\varsigma)\\0_3\end{bmatrix}=S\tilde{\psi}(2,\varsigma)\end{cases}$$

Cor. 2.9.7. $S^{l_\varsigma}_{im}(2)=(\begin{bmatrix}0&0&-1&0&\frac{1}{\sqrt{3}}\\1&0&0&0&0\\0&0&0&1&0\end{bmatrix},\begin{bmatrix}1&0&0&0&\frac{1}{\sqrt{3}}\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\begin{bmatrix}0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&-\frac{2}{\sqrt{3}}\end{bmatrix})$

Cor. 2.9.8.

$$\begin{cases}[\sigma(2),-i\varsigma]^a\partial_a\psi(2,\varsigma)=i\bar{N}(2)\tilde{J}(2,\varsigma)\\ i\varsigma\nabla\cdot S^{l_\varsigma}_m(2)S_{im}(2,+)\psi(2,\varsigma)=iJ^{l_\varsigma}_\pi\end{cases}\overset{S_{im}(2,+)}{\Leftrightarrow}\begin{cases}(\frac{1}{2}G_{im}(+),-i\varsigma)^a\partial_a\Psi(2,\varsigma)=i\vec{\mathcal{J}}(2),\vec{\mathcal{J}}(2)=S_{im}(2,+)\bar{N}(2)\tilde{J}(2,\varsigma)\\ i\varsigma\nabla\cdot S^{l_\varsigma}_m(2)\Psi(2,\varsigma)=iJ^{l_\varsigma}_\pi,\Psi(2,\varsigma)=S_{im}(2,+)\bar{N}(2)\tilde{\psi}(2,\varsigma)\end{cases}$$

Cor. 2.9.9.

$$(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\tilde{\psi}(s,\varsigma)=i\tilde{J}(s,\varsigma)\overset{S}{\Leftrightarrow}\begin{cases}[\sigma(s),-i\varsigma]^a\partial_a\psi(s,\varsigma)=i\bar{N}(s)\tilde{J}(s,\varsigma)\\ i\varsigma\nabla\cdot S^{l_\varsigma}(s)\psi(s,\varsigma)=iJ^{l_\varsigma}_\pi,J^{l_\varsigma}_\pi\succ J_\pi\end{cases}\quad\begin{cases}\begin{bmatrix}\bar{N}(s)\tilde{J}(s,\varsigma)\\J_\pi\end{bmatrix}=S\tilde{J}(s,\varsigma)\\\begin{bmatrix}\psi(s,\varsigma)\\0\end{bmatrix}=S\tilde{\psi}(s,\varsigma)\end{cases}$$

2.10 Special similar electromagnetic field expression of Bianchi identity

2.10.1 Dual electromagnetic field expression of Bianchi identity???

Cor. 2.10.1. $(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a D_a C_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}\equiv i\bar{J}^{A'_\varsigma}_{B_\varsigma C_\varsigma D_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a D_a C_{A_\varsigma l_\varsigma}(\frac{3}{2})\equiv i\bar{J}^{A'_\varsigma}_{l_\varsigma}(\frac{3}{2})$

Cor. 2.10.2. $(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a D_a C_{A_\varsigma l_\varsigma}(\frac{3}{2})\equiv i\bar{J}^{A'_\varsigma}_{l_\varsigma}(\frac{3}{2})\Leftrightarrow(\sigma_{-\varsigma}\otimes I,-i\varsigma)_a D^a\tilde{C}(2,\varsigma)\equiv i\tilde{\mathcal{J}}(2,\varsigma)$

The latter equation is formally equivalent to two electromagnetic field equations with both electric and magnetic charges.It satisfies Lorentz covariant and characterizes a torsion free gravitational field. It has nothing to do with whether the Einstein equation is established or not. Therefore, some analytical techniques for electromagnetic fields can be used here. So that we can obtain some properties of gravity.

Def. 2.10.1. $\Omega(\varsigma)=(\begin{bmatrix}0&0\\-\sigma_{\varsigma y}&\sigma_{\varsigma x}\end{bmatrix},\begin{bmatrix}\sigma_{\varsigma y}&-\sigma_{\varsigma x}\\0&0\end{bmatrix},\begin{bmatrix}0&-i\\i&0\end{bmatrix})$

Cor. 2.10.3. $\tilde{C}(2,\varsigma)\sim e^{(i\omega+\varsigma\epsilon)\cdot R\otimes I_4+(i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$

Proof: $\Lambda[\tilde{C}(2,\varsigma)]=S_{em}(\varsigma)\otimes S_{em}(\frac{1}{2})e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{3}{2})}S^{+}_{em}(\varsigma)\otimes S^{+}_{em}(\frac{1}{2})$
$=e^{(i\omega+\varsigma\epsilon)\cdot[R\otimes I_4+\Omega(\varsigma)]}=e^{(i\omega+\varsigma\epsilon)\cdot R\otimes I_4+(i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$ □

Cor. 2.10.4. $\tilde{\mathcal{J}}(2,\varsigma)\sim e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)\otimes I_4+(i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$

Proof: $\Lambda[\tilde{\mathcal{J}}(2,\varsigma)]=S_{em}(\varsigma)\otimes S_{em}(\frac{1}{2})e^{(i\omega-\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{3}{2})}S^{+}_{em}(\varsigma)\otimes S^{+}_{em}(\frac{1}{2})$
$=e^{(i\omega\cdot R-\varsigma\epsilon\cdot L)\otimes I_4+(i\omega+\varsigma\epsilon)\cdot\Omega(\varsigma)}$ □

2.10.2 Special similar electromagnetic field expression of Bianchi identity

Cor. 2.10.5. $J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma D_\varsigma}$ *is fully symmetric for* $B_\varsigma C_\varsigma D_\varsigma\Leftrightarrow[(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}]J^{\alpha_\varsigma}_a=0$

Cor. 2.10.6. $X^{\alpha_\varsigma}_l=0\Leftrightarrow X^{\alpha_\varsigma}_a=0;[(\sigma,-i\varsigma)^a\sigma_{\alpha_\varsigma}]X^{\alpha_\varsigma}_a=0,l=x,y,z$

Cor. 2.10.7. $(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\tilde{C}^{\alpha_\varsigma\beta_\varsigma}\equiv i\bar{J}^{\beta_\varsigma}_b\Leftrightarrow(\gamma,-i\varsigma)^a{}_{l\alpha_\varsigma}D_aC^{\alpha_\varsigma\beta_\varsigma}\equiv i\bar{J}^{\beta_\varsigma}_l$

The above covariant equation shows that $(\gamma,-i\varsigma)_a$ exhibits some covariance under certain special circumstances. The general covariant equation is constructed based on the Pauli matrix. But this equation uses the photon spin matrix to have constructed a complete covariant equation. This is the first time I have seen such a situation. The reason why this happens is due to the complete symmetry of the field and source.

3 Physical Yang-Mills gauge equation for gravitational field

3.1 Einstein equation [11] and Yang-Mills gauge equation for gravitational field

Einstein equation: $R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab}$ (8.18)

Cor. 3.1.1. $R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow T^{ab}{}_{;b} = 0$

Proof: $R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab}$
$\Rightarrow (R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab}); b = -8\pi G T^{ab}; b$
$\Rightarrow 0 = -8\pi G T^{ab}; b$
$\Rightarrow T^{ab}{}_{;b} = 0$ □

Cor. 3.1.2. $R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$

Proof: $R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \to R = 8\pi G T + 4\Lambda$
$\Leftrightarrow R^{ab} - \frac{1}{2}g^{ab}(8\pi G T + 4\Lambda) + \Lambda g^{ab} = -8\pi G T^{ab}$
$\Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \to R = 8\pi G T + 4\Lambda$ □

Cor. 3.1.3. $\begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \\ R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \\ R^{abcd}{}_{;a} = 8\pi G(T^{b[c;d]} - \frac{1}{2}g^{b[c}T^{;d]}), R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases}$

Cor. 3.1.4. $\begin{cases} R^{abcd}{}_{;a} = -J^{bcd} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \\ J^{bcd} \equiv -8\pi G(T^{b[c;d]} - \frac{1}{2}g^{b[c}T^{;d]}) \end{cases} \Leftrightarrow \begin{cases} D^a F^{\alpha_\varsigma}_{ab} = -J^{b\alpha_\varsigma} \\ D^a * F^{\alpha_\varsigma}_{ab} \equiv 0 \\ J^{b\alpha_\varsigma} \equiv \frac{1}{2}\varsigma\sigma^{\alpha_\varsigma}_{\varsigma cd}J^{bcd}, J^{bcd} \equiv -8\pi G(T^{b[c;d]} - \frac{1}{2}g^{b[c}T^{;d]}) \end{cases}$

Cor. 3.1.5. $\begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \\ R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \end{cases} \Rightarrow \begin{cases} R^{abcd}{}_{;a} = -J^{bcd} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F^{\alpha_\varsigma}_{ab} = -J^{b\alpha_\varsigma} \\ D^a * F^{\alpha_\varsigma}_{ab} \equiv 0 \end{cases}$

Cor. 3.1.6. $\begin{cases} D^a F^{\alpha_\varsigma}_{ab} = -J^{b\alpha_\varsigma} \\ D^a * F^{\alpha_\varsigma}_{ab} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F^{[\alpha_\varsigma]}_{ab} = -J^{b[\alpha_\varsigma]} \\ D^a * F^{[\alpha_\varsigma]}_{ab} \equiv 0 \end{cases}$

3.2 Spinor expression of Yang-Mills gauge equation for gravitational field

As long as the Einstein equation $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$ is substituted into the source terms of various representations of the Bianchi identity and the corresponding identity sign is replaced with an equal sign, various representations of the physical Yang-Mills gauge equation for gravitational field can be obtained. Formally, it is completely consistent with various expressions of the Bianchi identity and will not be written repeatedly. In essence, the Yang-Mills gauge equation for gravitational field is only an identity for gravitational field. It is irrelevant whether the Einstein equation is established or not. But what really describes physics is the Einstein equation. Only after applying the Einstein equation to the source term of the gravitational field gauge identity. And the Yang-Mills gauge equation of the gravitational field truly carries the physical gravitational source term. At this time, the Yang-Mills gauge equation of the gravitational field became a real physical equation. So this is completely different from the case of electromagnetic field and Yang-Mills field. The gauge equations for electromagnetic field and Yang-Mills field are not just identities, but directly describe real physics.

3.3 Self review

In fact, both electromagnetic and gravitational fields can be attributed to the Yang-Mills field case. When σ is empty, it is an electromagnetic field; When $\sigma = \beta_\kappa$, it is a gravitational field; When σ is multiple letters, a more general situation can be described. Therefore, the Yang-Mills field is already a very general case in mathematical form.

4 Equivalent matrix form of Einstein equation of general relativity [11–14]

4.1 Preparation

Cor. 4.1.1. $R^{ab} = \varsigma(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}, (F^{\alpha'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma})^{ab} = -(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}, F^{\alpha_\varsigma}{}_{ab} = F_{ab}{}^{\alpha_\varsigma}, R = -\varsigma\sigma_{\varsigma\alpha_\varsigma}{}^{ab}F_{ab}{}^{\alpha_\varsigma}$

Cor. 4.1.2. $R^{ab} = -i(F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab}, F^{\alpha'_\varsigma}\sigma_{-\varsigma\alpha'_\varsigma} = F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma}$

Cor. 4.1.3. $R = i\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}{}^{\alpha_\varsigma}$

Cor. 4.1.4. $R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$

Def. 4.1.1. $\bar{T}^{ab} := 8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) - \Lambda g^{ab}, \bar{\mathcal{T}}^b \equiv [\bar{T}_x{}^b, \bar{T}_y{}^b, \bar{T}_z{}^b, \bar{T}_\pi{}^b]^T$

Def. 4.1.2. $\mathcal{F}_{ab}(2,\varsigma) \equiv [F^{x_\varsigma}_{ab}, F^{y_\varsigma}_{ab}, F^{z_\varsigma}_{ab}, 0_{ab}]^T, F_{ab}(2,\varsigma) \equiv F^{[\alpha_\varsigma]}_{ab}, \mathcal{R} = [R, 0]$

Def. 4.1.3. $\mathcal{A}_u(\varsigma) \equiv [A_u^{x_\varsigma}, A_u^{y_\varsigma}, A_u^{z_\varsigma}, 0_u]^T = A_u^{[\alpha_\varsigma]}(\varsigma), \mathcal{J}_a(\varsigma) \equiv [J_a^{x_\varsigma}, J_a^{y_\varsigma}, J_a^{z_\varsigma}, 0_a]^T = J_a^{[\alpha_\varsigma]}$

Cor. 4.1.5. $F_{uv}^{\alpha_\varsigma} = \partial_u A_v^{\alpha_\varsigma} - \partial_v A_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma} A_u^{\beta_\varsigma} A_v^{\gamma_\varsigma}$
$\Leftrightarrow \mathcal{F}_{uv}(\varsigma) = \partial_u \mathcal{A}_v(\varsigma) - \partial_v \mathcal{A}_u(\varsigma) + i\mathcal{A}_u^T(\varsigma)\mathcal{R}\mathcal{A}_v(\varsigma) = [\partial_u + \frac{i}{2}\mathcal{A}_u^T(\varsigma)\mathcal{R}]\mathcal{A}_v(\varsigma) - [\partial_v + \frac{i}{2}\mathcal{A}_v^T(\varsigma)\mathcal{R}]\mathcal{A}_u(\varsigma)$

4.2 Equivalent matrix form of Einstein equation

Cor. 4.2.1. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a \mathcal{F}^{ab}(2,\varsigma) = i\bar{\mathcal{T}}^b$

Proof: $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$
$\Leftrightarrow (F^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma})^{ab} = -i\bar{T}^{ab}$
$\Leftrightarrow (\sigma_{\varsigma\alpha_\varsigma}F^{\alpha_\varsigma})^{ab} = -i\bar{T}^{ab}$
$\Leftrightarrow [(\sigma_\varsigma, -i\varsigma)_{\alpha_\varsigma}F^{\alpha_\varsigma}]^{ab} = -i\bar{T}^{ab}$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ac}F_{\alpha_\varsigma}^{cb} = -i\bar{T}_a{}^b$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ca}F_{\alpha_\varsigma}^{cb} = i\bar{T}_a{}^b$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_{ca}{}^{\alpha_\varsigma}F_{\alpha_\varsigma}^{cb} = i\bar{T}_a{}^b$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_{ac}{}^{\alpha_\varsigma}F_{\alpha_\varsigma}^{ab} = i\bar{T}_c{}^b$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a \mathcal{F}^{ab}(2,\varsigma) = i\bar{\mathcal{T}}^b$ □

Cor. 4.2.2. $R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{\varsigma\alpha_\varsigma}\sigma_{-\varsigma\beta'_\varsigma})^{ab}\psi^{\alpha_\varsigma\beta'_\varsigma}$

Cor. 4.2.3. $R_{ab} = -\bar{T}_{ab} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a * \mathcal{F}^{ab}(2,\varsigma) = i\varsigma\bar{\mathcal{T}}^b - \frac{i\varsigma}{2}\delta^b\bar{T}$

Proof: $R_{ab} = -\bar{T}_{ab}$
$\Leftrightarrow i\varsigma(R_a{}^b - \frac{1}{2}\delta_a^b R) = -i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow \frac{i\varsigma}{2}(2R_a{}^b - \frac{1}{2}\delta_a^b R - \frac{1}{2}\delta_a^b R) = -i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow \frac{i\varsigma}{2}(\sigma_{\varsigma ac}^{\alpha_\varsigma}\sigma_{-\varsigma}^{\beta'_\varsigma cb}\psi_{\beta'_\varsigma\alpha_\varsigma} - \delta^{\alpha_\varsigma\beta_\varsigma}\delta_a^b\psi_{\alpha_\varsigma\beta_\varsigma}) = -i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow \sigma_{\varsigma ac}^{\alpha_\varsigma}\frac{i\varsigma}{2}(\sigma_{-\varsigma}^{\beta'_\varsigma cb}\psi_{\beta'_\varsigma\alpha_\varsigma} - \sigma_\varsigma^{\beta_\varsigma cb}\psi_{\beta_\varsigma\alpha_\varsigma}) = -i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow \sigma_{\varsigma ac}^{\alpha_\varsigma} * F_{\alpha_\varsigma}^{cb} = -i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ac} * F_{\alpha_\varsigma}^{cb} = -i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow (\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}{}_{ca} * F_{\alpha_\varsigma}^{cb} = i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_{ca}{}^{\alpha_\varsigma} * F_{\alpha_\varsigma}^{cb} = i\varsigma(\bar{T}_a{}^b - \frac{1}{2}\delta_a^b\bar{T})$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_{ac}{}^{\alpha_\varsigma} * F_{\alpha_\varsigma}^{ab} = i\varsigma(\bar{T}_c{}^b - \frac{1}{2}\delta_c^b\bar{T})$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a * \mathcal{F}^{ab}(2,\varsigma) = i\varsigma\bar{\mathcal{T}}^b - \frac{i\varsigma}{2}\delta^b\bar{T}$ □

Cor. 4.2.4. $(\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(2,\varsigma) = i\bar{\mathcal{T}}^b \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a * \mathcal{F}^{ab}(2,\varsigma) = i\varsigma\bar{\mathcal{T}}^b - \frac{i\varsigma}{2}\delta^b\bar{T}$

Self comment: Here is a wonderful and concise pair of spinor equations that are all equivalent to Einstein's equation, which is very interesting.

Cor. 4.2.5. $\begin{cases} D^a F_{ab}^{\alpha_\varsigma} \equiv -J^{b\alpha_\varsigma} \\ D^a * F_{ab}^{\alpha_\varsigma} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D_a\mathcal{F}^{ab}(2,\varsigma) \equiv -\bar{\mathcal{J}}^b(\varsigma) \\ D_a * \mathcal{F}^{ab}(2,\varsigma) \equiv 0 \end{cases}$

Cor. 4.2.6. $\begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \\ R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} \textit{Einstein equation: } (\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(2,\varsigma) = -i\bar{\mathcal{T}}^b \\ \textit{Bianchi identity: } D_a\mathcal{F}^{ab}(2,\varsigma) \equiv -\bar{\mathcal{J}}^b(\varsigma), D_a * \mathcal{F}^{ab}(2,\varsigma) \equiv 0 \end{cases}$

Cor. 4.2.7. $(\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(2,\varsigma) = i\bar{\mathcal{T}}^b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0 (Gauge\ condition)$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a e_a^u e_b^v[\partial_u + i\mathcal{A}_u^T(\varsigma)\mathcal{R}]\mathcal{A}_v(\varsigma) = i\bar{\mathcal{T}}_b, (\sigma_{-\varsigma}, -i\varsigma)^a\mathcal{A}_a(\varsigma) = 0$

4.3 New Form of spin tensor equivalent to Einstein equation with lower first derivative

Cor. 4.3.1. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(\gamma,\varsigma)]F^{bc}(2,\varsigma) = -i\sigma_{\varsigma ab}^{[\beta_\varsigma]}\bar{T}^{bc}$

Cor. 4.3.2. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(\gamma,\varsigma)] * F^{bc}(2,\varsigma) = -i\varsigma\sigma_{\varsigma ab}^{[\beta_\varsigma]}(\bar{T}^{bc} - \frac{1}{2}\delta^{bc}\bar{T})$

Chapter9 New Expression of Gravitino Field Equation

1 Using constant invariant tensors to define various spinors of gravitino field [7]

1.1 Field strength description of gravitino theory

Def. 1.1.1. $F_{uv}(\frac{3}{2},\varsigma) := D_u\psi_v(\varsigma) - D_v\psi_u(\varsigma)$

Cor. 1.1.1. $F_{uv}(\frac{3}{2},\varsigma) = (\partial_u + \frac{i}{2}\sigma_{\alpha_\varsigma}A_u^{\alpha_\varsigma})\psi_v(\varsigma) - (\partial_v + \frac{i}{2}\sigma_{\alpha_\varsigma}A_v^{\alpha_\varsigma})\psi_u(\varsigma)$

Proof: $F_{uv}(\frac{3}{2},\varsigma) := D_u\psi_v(\varsigma) - D_v\psi_u(\varsigma)$
$= [\partial_u\psi_v(\varsigma) + \Gamma_{uv}^\lambda\psi_\lambda(\varsigma) + \frac{i}{2}A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}\psi_v(\varsigma)] - [\partial_v\psi_u(\varsigma) + \Gamma_{vu}^\lambda\psi_\lambda(\varsigma) + \frac{i}{2}A_v^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}\psi_u(\varsigma)]$
$= \partial_u\psi_v(\varsigma) - \partial_v\psi_u(\varsigma) + \frac{i}{2}\sigma_{\alpha_\varsigma}[A_u^{\alpha_\varsigma}\psi_v(\varsigma) - A_v^{\alpha_\varsigma}\psi_u(\varsigma)]$
$= (\partial_u + \frac{i}{2}\sigma_{\alpha_\varsigma}A_u^{\alpha_\varsigma})\psi_v(\varsigma) - (\partial_v + \frac{i}{2}\sigma_{\alpha_\varsigma}A_v^{\alpha_\varsigma})\psi_u(\varsigma)$ □

Cor. 1.1.2. $\delta\psi_u(\varsigma) = i\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(\frac{1}{2})\psi_u(\varsigma), \delta F_{uv}(\frac{3}{2},\varsigma) = i\theta^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(\frac{1}{2})F_{uv}(\frac{3}{2},\varsigma)$

Comparison with gravitational field:

Cor. 1.1.3. $F_{uv}(2,\varsigma) = (\partial_u + \frac{i}{2}\gamma_{\alpha_\varsigma}A_u^{\alpha_\varsigma})\psi_v(\varsigma) - (\partial_v + \frac{i}{2}\gamma_{\alpha_\varsigma}A_v^{\alpha_\varsigma})\psi_u(\varsigma)$

Proof: $F_{uv}(2,\varsigma) := \tilde{D}_uA_v(\varsigma) - \tilde{D}_vA_u(\varsigma)$
$= [\partial_uA_v(\varsigma) + \Gamma_{uv}^\lambda A_\lambda(\varsigma) + \frac{i}{2}A_u^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}A_v(\varsigma)] - [\partial_vA_u(\varsigma) + \Gamma_{vu}^\lambda A_\lambda(\varsigma) + \frac{i}{2}A_v^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}A_u(\varsigma)]$
$= \partial_uA_v(\varsigma) - \partial_vA_u(\varsigma) + \frac{i}{2}\gamma_{\alpha_\varsigma}[A_u^{\alpha_\varsigma}A_v(\varsigma) - A_v^{\alpha_\varsigma}A_u(\varsigma)]$
$= (\partial_u + \frac{i}{2}\gamma_{\alpha_\varsigma}A_u^{\alpha_\varsigma})A_v(\varsigma) - (\partial_v + \frac{i}{2}\gamma_{\alpha_\varsigma}A_v^{\alpha_\varsigma})A_u(\varsigma)$ □

Cor. 1.1.4. $F_{uv}(2,\varsigma) = (\partial_u + \frac{i}{2}\gamma_{\alpha_\varsigma}A_u^{\alpha_\varsigma})A_v(\varsigma) - (\partial_v + \frac{i}{2}\gamma_{\alpha_\varsigma}A_v^{\alpha_\varsigma})A_u(\varsigma) \Leftrightarrow F_{uv}^{\alpha_\varsigma} = \partial_uA_v^{\alpha_\varsigma} - \partial_vA_u^{\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma\gamma_\varsigma}A_u^{\beta_\varsigma}A_v^{\gamma_\varsigma}$

Cor. 1.1.5. $\delta A_u(\varsigma) = i\theta^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}A_u - \partial_u\theta, \delta F_{uv}(2,\varsigma) = i\theta^{\alpha_\varsigma}\gamma_{\alpha_\varsigma}F_{uv}(2,\varsigma)$

1.2 Classical description of gravitino field strength

$$F_{ab}^{Z_\kappa} = \begin{bmatrix} 0 & B_z^{Z_\kappa} & -B_y^{Z_\kappa} & -iE_x^{Z_\kappa} \\ -B_z^{Z_\kappa} & 0 & B_x^{Z_\kappa} & -iE_y^{Z_\kappa} \\ B_y^{Z_\kappa} & -B_x^{Z_\kappa} & 0 & -iE_z^{Z_\kappa} \\ iE_x^{Z_\kappa} & iE_y^{Z_\kappa} & iE_z^{Z_\kappa} & 0 \end{bmatrix}, *F_{ab}^{Z_\kappa} = \begin{bmatrix} 0 & -iE_z^{Z_\kappa} & iE_y^{Z_\kappa} & B_x^{Z_\kappa} \\ iE_z^{Z_\kappa} & 0 & -iE_x^{Z_\kappa} & B_y^{Z_\kappa} \\ -iE_y^{Z_\kappa} & iE_x^{Z_\kappa} & 0 & B_z^{Z_\kappa} \\ -B_x^{Z_\kappa} & -B_y^{Z_\kappa} & -B_z^{Z_\kappa} & 0 \end{bmatrix} \tag{9.1}$$

$$(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_bF^{ab}(\tfrac{3}{2},\varsigma) = 0, (\sigma,i\varsigma)_a(\sigma,-i\varsigma)_b * F^{ab}(\tfrac{3}{2},\varsigma) = 0 \tag{9.2}$$

1.3 Complex vector description of gravitino field strength

Def. 1.3.1. *Gravitino field complex vector* $\psi_{\alpha_\varsigma}^{Z_\kappa} := \frac{i}{2}\sigma_{\varsigma\alpha_\varsigma}^{ab}F_{ab}^{Z_\kappa} = i\varsigma(E - i\varsigma B)_{\alpha_\varsigma}^{Z_\kappa} = (i\varsigma E + B)_{\alpha_\varsigma}^{Z_\kappa}$

Cor. 1.3.1. $\frac{1}{2}(F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa}) = \frac{i}{2}\sigma_{\varsigma ab}^{\alpha_\varsigma}\psi_{\alpha_\varsigma}^{Z_\kappa}$

Proof: $F_{ab}^{Z_\kappa} = -F_{ba}^{Z_\kappa}$
$\Leftrightarrow F_{ab}^{Z_\kappa} = \frac{1}{2}S_{abcd}F^{cd}, *F_{ab}^{Z_\kappa} := \frac{1}{2}\varepsilon_{abcd}F^{cd}$
$\Leftrightarrow F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa} = \frac{1}{2}(S_{abcd} - \varsigma\varepsilon_{abcd})F^{cd}$
$\Leftrightarrow F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa} = -\frac{1}{2}\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma cd}F^{cd}$
$\Leftrightarrow F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\psi_{\alpha_\varsigma}^{Z_\kappa}$
$\Leftrightarrow \frac{1}{2}(F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa}) = \frac{i}{2}\sigma_{\varsigma ab}^{\alpha_\varsigma}\psi_{\alpha_\varsigma}^{Z_\kappa}$ □

Cor. 1.3.2. $\psi_{\alpha_\varsigma}^{Z_\kappa} = \frac{i}{2}\sigma_{\varsigma\alpha_\varsigma}^{ab}\frac{1}{2}(F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa})$

Cor. 1.3.3. $\psi_{\alpha_\varsigma}^{Z_\kappa} = -\frac{i}{2}\varsigma\sigma_{\varsigma\alpha_\varsigma}^{ab} * F_{ab}^{Z_\kappa}$

Cor. 1.3.4. $\sigma_{\varsigma\alpha_\varsigma}^{ab}(F_{ab}^{Z_\kappa} + \varsigma * F_{ab}^{Z_\kappa}) = 0$

Cor. 1.3.5. $F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa} = -\frac{1}{4}\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\varsigma\alpha_\varsigma}^{cd}(F_{cd}^{Z_\kappa} - \varsigma * F_{cd}^{Z_\kappa})$

Cor. 1.3.6. $F^{Z_\kappa}_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha}),*F^{Z_\kappa}_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}-\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha})$

Proof: $F^{Z_\kappa}_{ab}-\varsigma*F^{Z_\kappa}_{ab}=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi^{Z_\kappa}_{\alpha_\varsigma}$
$\Leftrightarrow F^{Z_\kappa}_{ab}-*F^{Z_\kappa}_{ab}=i\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha},F^{Z_\kappa}_{ab}+*F^{Z_\kappa}_{ab}=i\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}$
$\Leftrightarrow F^{Z_\kappa}_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha}),*F^{Z_\kappa}_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}-\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha})$
$\Leftrightarrow F^{Z_\kappa}_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha})$
$\Leftrightarrow *F^{Z_\kappa}_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}-\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha})$ □

Cor. 1.3.7. $F^{Z_\kappa}_{ab}=-F^{Z_\kappa}_{ba}\Leftrightarrow F^{Z_\kappa}_{ab}=\frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi^{Z_\kappa}_{\alpha'}+\sigma^{\alpha}_{+ab}\psi^{Z_\kappa}_{\alpha})$

Cor. 1.3.8. $(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_bF^{ab}(\varsigma)=0\Leftrightarrow\sigma_{\alpha_\varsigma}\psi^{\alpha_\varsigma[Z_\varsigma]}=0$

1.4 $\frac{1}{2}$-spinor description of gravitino field strength [1,2]

Def. 1.4.1. $\frac{1}{2}$*-spinor tensor of gravitino field* $\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}:=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi^{Z_\kappa}_{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}F^{Z_\kappa}_{ab}$

Cor. 1.4.1. $\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi^{Z_\kappa}_{\alpha_\varsigma}\Leftrightarrow\psi^{Z_\kappa}_{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}$

Cor. 1.4.2. $\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}=\psi^{Z_\kappa}_{B_\varsigma A_\varsigma}$

Cor. 1.4.3. $\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}=\frac{-i}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}*F^{Z_\kappa}_{ab}$

Cor. 1.4.4. $\frac{1}{2}(F^{Z_\kappa}_{ab}-\varsigma*F^{Z_\kappa}_{ab})=\frac{i\varsigma}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma}\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}\Leftrightarrow\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}=\frac{i\varsigma}{\sqrt{2}}S^{ab}{}_{A_\varsigma B_\varsigma}\frac{1}{2}(F^{Z_\kappa}_{ab}-\varsigma*F^{Z_\kappa}_{ab})$

Cor. 1.4.5. $F^{Z_\kappa}_{ab}-\varsigma*F^{Z_\kappa}_{ab}=-\frac{1}{2}S_{ab}{}^{A_\varsigma B_\varsigma}S^{cd}{}_{A_\varsigma B_\varsigma}(F^{Z_\kappa}_{cd}-\varsigma*F^{Z_\kappa}_{cd})$

Cor. 1.4.6. $F^{Z_\kappa}_{ab}=\frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi^{Z_\kappa}_{A'B'}+S_{ab}{}^{AB}\psi^{Z_\kappa}_{AB}),*F^{Z_\kappa}_{ab}=\frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi^{Z_\kappa}_{A'B'}-S_{ab}{}^{AB}\psi^{Z_\kappa}_{AB})$

Cor. 1.4.7. $F^{Z_\kappa}_{ab}=-F^{Z_\kappa}_{ba}\Leftrightarrow F^{Z_\kappa}_{ab}=\frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi^{Z_\kappa}_{A'B'}+S_{ab}{}^{AB}\psi^{Z_\kappa}_{AB})$

combine corollaries (1.3.6) **and** (1.274), (1.275)**, I can get the Penrose correspondence notation** [1,2]

Cor. 1.4.8. $F^{Z_\kappa}_{ab}\overset{P}{=}\frac{1}{\sqrt{2}}(\psi^{Z_\kappa}_{A'B'}\varepsilon_{AB}+\psi^{Z_\kappa}_{AB}\varepsilon_{A'B'}),*F^{Z_\kappa}_{ab}\overset{P}{=}\frac{1}{\sqrt{2}}(\psi^{Z_\kappa}_{A'B'}\varepsilon_{AB}-\psi^{Z_\kappa}_{AB}\varepsilon_{A'B'})$

Cor. 1.4.9. $(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_bF^{ab}(\varsigma)=0\Leftrightarrow\sigma_{\alpha_\varsigma}\psi^{\alpha_\varsigma[Z_\varsigma]}=0\Leftrightarrow\psi_{A_\varsigma B_\varsigma C_\varsigma}=\frac{1}{3!}\psi_{(A_\varsigma B_\varsigma C_\varsigma)}$

1.5 1-spinor description of gravitino field strength

Def. 1.5.1. 1*-spinor description of gravitino field* $\psi^{Z_\kappa}_{k_\varsigma}(1):=\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi^{Z_\kappa}_{\alpha_\varsigma}$

Cor. 1.5.1. $\psi^{Z_\kappa}_{A_\varsigma B_\varsigma}=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi^{Z_\kappa}_{k_\varsigma}(1),\psi^{Z_\kappa}_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi^{Z_\kappa}_{k_\varsigma}(1)$

1.6 $\frac{1}{2}$-spinor description of gravitino field source [1,2]

Def. 1.6.1. $\frac{1}{2}$*-spinor tensor of gravitino source*
$J^{A'_\varsigma A_\varsigma Z_\kappa}:=\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_aJ^{aZ_\kappa},J^{Z_\kappa}_{A_\varsigma A'_\varsigma}:=\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}J^{Z_\kappa}_a$

Penrose notation: $J^{aZ_\kappa}\overset{P}{=}J^{A'AZ_\kappa},J^{Z_\kappa}_a\overset{P}{=}J^{Z_\kappa}_{AA'}$

1.7 Proof of symmetry conditions for gravitino field

Cor. 1.7.1. $\psi^{A_\varsigma B_\varsigma C_\varsigma}=\psi^{A_\varsigma C_\varsigma B_\varsigma}\Leftrightarrow(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_bF^{ab}(\varsigma)=0$

Proof: $\psi^{A_\varsigma B_\varsigma C_\varsigma}=\psi^{A_\varsigma C_\varsigma B_\varsigma}$
$\Leftrightarrow\varepsilon_{B_\varsigma C_\varsigma}\psi^{A_\varsigma B_\varsigma C_\varsigma}=0$
$\Leftrightarrow-\frac{1}{\sqrt{2}}\varsigma\varepsilon_{B_\varsigma C_\varsigma}iS_{ab}{}^{A_\varsigma B_\varsigma}F^{abC_\varsigma}=0$
$\Leftrightarrow\varepsilon_{B_\varsigma C_\varsigma}iS_{ab}{}^{A_\varsigma}{}_{D_\varsigma}\bar{\varepsilon}^{D_\varsigma B_\varsigma}F^{abC_\varsigma}=0$
$\Leftrightarrow iS_{ab}{}^{A_\varsigma}{}_{D_\varsigma}\delta^{D_\varsigma}{}_{C_\varsigma}F^{abC_\varsigma}=0$
$\Leftrightarrow iS_{ab}{}^{A_\varsigma}{}_{C_\varsigma}F^{abC_\varsigma}=0$
$\Leftrightarrow\frac{1}{4}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}F^{ab[C_\varsigma]}=0$
$\Leftrightarrow(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_bF^{ab[C_\varsigma]}=0$
$\Leftrightarrow(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_bF^{ab}(\frac{3}{2},\varsigma)=0$ □

Cor. 1.7.2. $\psi_{A_\varsigma B_\varsigma C_\varsigma}=\frac{1}{3!}\psi_{(A_\varsigma B_\varsigma C_\varsigma)}\Leftrightarrow(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_bF^{ab}(\frac{3}{2},\varsigma)=0\Leftrightarrow(\sigma,i\varsigma)_a(\sigma,-i\varsigma)_b*F^{ab}(\frac{3}{2},\varsigma)=0$

Cor. 1.7.3. $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma}=J_{A'_\varsigma}{}^{C_\varsigma B_\varsigma}\Leftrightarrow(\sigma,-i\varsigma)^aJ_a(\varsigma)=0$

Proof: $J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} = J_{A'_\varsigma}{}^{C_\varsigma B_\varsigma}$
$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma} J_{A'_\varsigma}{}^{B_\varsigma C_\varsigma} = 0$
$\Leftrightarrow \varepsilon_{B_\varsigma C_\varsigma} \frac{1}{\sqrt{2}}(\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} \bar{\varepsilon}^{A_\varsigma B_\varsigma} J_a{}^{C_\varsigma} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma A_\varsigma} \delta^{A_\varsigma}{}_{C_\varsigma} J_a{}^{C_\varsigma} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a{}_{A'_\varsigma C_\varsigma} J_a{}^{C_\varsigma} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a J_a{}^{[C_\varsigma]} = 0$
$\Leftrightarrow (\sigma, -i\varsigma)^a J_a(\varsigma) = 0$ □

2 Equivalent expressions of Penrose type gravitino field equation in flat space-time

2.1 Frame description of gravitino equation

Def. 2.1.1. $F_{ab}^{Z_\kappa} := e_a^u e_b^v F_{uv}^{Z_\kappa}, \psi_a^{Z_\kappa} := e_a^u \psi_u^{Z_\kappa}$

Frame description of gravitino equation:

$$\partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, \partial^a * F_{ab}^{Z_\kappa} \equiv 0 \quad (9.3)$$

2.2 Classical description of gravitino field equation

$$\begin{cases} \nabla \cdot \vec{E}^{Z_\kappa} = \rho^{Z_\kappa}, \nabla \times \vec{E}^{Z_\kappa} = -\partial_t \vec{B}^{Z_\kappa} \\ \nabla \cdot \vec{B}^{Z_\kappa} = 0, \nabla \times \vec{B}^{Z_\kappa} = \vec{J}^{Z_\kappa} + \partial_t \vec{E}^{Z_\kappa} \end{cases} \Leftrightarrow \partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, \partial^a * F_{ab}^{Z_\kappa} \equiv 0 \quad (9.4)$$

2.3 Complex vector representation of gravitino field equation

Complex vector tensor form:

Thm. 2.3.1. $\partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} \partial_a \tilde{\Psi}^{\alpha_\varsigma \sigma} = iJ_b^{Z_\kappa}; F_{ab}^{Z_\kappa} = \partial_a A_b - \partial_b A_a, \tilde{\Psi}^{\alpha_\varsigma \sigma} = \begin{bmatrix} \psi^{\alpha_\varsigma \sigma} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab} F^{ab\sigma} \\ 0 \end{bmatrix}$

Proof: $\partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}$
$\Leftrightarrow \partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, \partial^a * F_{ab}^{Z_\kappa} \equiv 0$
$\Leftrightarrow \partial^a (F_{ab}^{Z_\kappa} - \varsigma * F_{ab}^{Z_\kappa}) = -J_b^{Z_\kappa}$
$\Leftrightarrow \partial^a (i\sigma^{\alpha_\varsigma}_{\varsigma ab} \psi^{Z_\kappa}_{\alpha_\varsigma}) = -J^{b\sigma}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow \partial^a [(\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}|_{ab} \tilde{\Psi}^{\alpha_\varsigma \sigma}] = iJ_b^{Z_\kappa}, \alpha_\varsigma = 1,2,3,4$
$\Leftrightarrow \partial^a [(\sigma_{-\varsigma}, -i\varsigma)_a|_b{}^{\alpha_\varsigma} \tilde{\Psi}^{Z_\kappa}_{\alpha_\varsigma}] = iJ_b^{Z_\kappa}, \alpha_\varsigma = 1,2,3,4$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} \partial_a \tilde{\Psi}^{\alpha_\varsigma \sigma} = iJ_b^{Z_\kappa}, \alpha_\varsigma = 1,2,3,4$ □

Complex vector matrix form:

Cor. 2.3.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} \partial_a \tilde{\Psi}^{\alpha_\varsigma \sigma} = iJ_b^{Z_\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z_\kappa}(1,\varsigma) = iJ^{Z_\kappa}$

Representation transformation:

Cor. 2.3.2. $(\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z_\kappa}(1,\varsigma) = iJ^{Z_\kappa} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z_\kappa}(1,\varsigma) = i\tilde{J}^{Z_\kappa}(1,\varsigma)$

2.4 $\frac{1}{2}$-spinor description of gravitino field strength [1,2]

$\frac{1}{2}$-spinor Penrose abstract index form:

Thm. 2.4.1. $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} \partial_a \tilde{\Psi}^{\alpha_\varsigma \sigma} = iJ_b^{Z_\kappa} \Leftrightarrow \nabla^{A'_\varsigma A_\varsigma} \psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma, \nabla^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a$

Proof: $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} \partial_a \tilde{\Psi}^{\alpha_\varsigma \sigma} = iJ_b^{Z_\kappa}$
$\Leftrightarrow \partial^a (i\sigma^{\alpha_\varsigma}_{\varsigma ab} \psi^{Z_\kappa}_{\alpha_\varsigma}) = -J_b^{Z_\kappa}$
$\Leftrightarrow \partial^a (i\sigma^{\alpha_\varsigma}_{\varsigma ab} \cdot \frac{i\varsigma}{\sqrt{2}} \sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} \psi^{Z_\kappa}_{A_\varsigma B_\varsigma}) = -J_b^{Z_\kappa}$
$\Leftrightarrow iS_{ab}{}^{A_\varsigma B_\varsigma} \partial^a \psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J_b^{Z_\kappa}$
$\Leftrightarrow (\frac{\varsigma}{2}\delta_{ab}\varepsilon^{A_\varsigma B_\varsigma} + iS_{ab}{}^{A_\varsigma B_\varsigma}) \partial^a \psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J_b^{Z_\kappa}$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_b^{B'_\varsigma B_\varsigma} \partial^a \psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-1}{\sqrt{2}} J_b^{Z_\kappa}$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} \partial^a \psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-1}{\sqrt{2}} J_b^{Z_\kappa} \cdot \frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^b_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a \psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} \varsigma \varepsilon^{A'_\varsigma B'_\varsigma} J_{B'_\varsigma B_\varsigma}$
$\Leftrightarrow \nabla^{A'_\varsigma A_\varsigma} \psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^\sigma, \nabla^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a$ □

$\frac{1}{2}$-spinor tensor form:

Cor. 2.4.1. $\nabla^{A'_\varsigma A_\varsigma}\psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^{Z_\kappa} \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial_a\psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma}{}^{Z_\kappa}$

$\frac{1}{2}$-spinor matrix form:

Cor. 2.4.2. $(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial_a\psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma}{}^{Z_\kappa} \Leftrightarrow (\sigma\otimes I, -i\varsigma)^a\partial_a\tilde{\Psi}^{Z_\kappa}(1,\varsigma) = i\tilde{J}^{Z_\kappa}(1,\varsigma)$

$\frac{1}{2}$-spinor square matrix form:

Cor. 2.4.3. $(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial_a\psi^{Z_\kappa}_{A_\varsigma B_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma}{}^{Z_\kappa} \Leftrightarrow (\sigma, -i\varsigma)^a\partial_a[\psi]^{Z_\kappa} = i[J]^{Z_\kappa}$

2.5 Full $\frac{1}{2}$-spinor expression of gravitino field

Cor. 2.5.1. $\nabla^{A'_\varsigma A_\varsigma}\psi^{Z_\varsigma}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^{Z_\varsigma} \Leftrightarrow \nabla^{A'_\varsigma A_\varsigma}\partial_a\psi_{A_\varsigma B_\varsigma C_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma}$

Cor. 2.5.2. $\nabla^{A'_\varsigma A_\varsigma}\psi^{Z_\varsigma}_{A_\varsigma B_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}{}_{B_\varsigma}{}^{Z_\varsigma} \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial_a\psi_{A_\varsigma B_\varsigma C_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma C_\varsigma}$

2.6 Fully symmetric equation (generalized covariant extension)

Cor. 2.6.1.

$$\begin{cases}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}D_a\psi_{A_\varsigma B_\varsigma C_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma C_\varsigma}\\ \psi_{A_\varsigma B_\varsigma C_\varsigma} = \frac{1}{3!}\psi_{(A_\varsigma B_\varsigma C_\varsigma)}, J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma} = \frac{1}{2!}J^{A'_\varsigma}{}_{(B_\varsigma C_\varsigma)}\end{cases} \Leftrightarrow \begin{cases}D^aF_{ab}{}^{[C_\varsigma]} = -J_b{}^{[C_\varsigma]}, D^a * F_{ab}{}^{[C_\varsigma]} \equiv 0\\ (\sigma, i\varsigma)_a(\sigma, -i\varsigma)_bF^{ab[C_\varsigma]} = 0, (\sigma, -i\varsigma)_aJ^{a[C_\varsigma]} = 0\end{cases}$$

The proof of the following two corollaries will be left to the future.

Cor. 2.6.2.

$$\begin{cases}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}D_a\psi_{A_\varsigma B_\varsigma C_\varsigma} = iJ^{A'_\varsigma}{}_{B_\varsigma C_\varsigma}\\ \psi_{A_\varsigma B_\varsigma C_\varsigma} = \frac{1}{3!}\psi_{(A_\varsigma B_\varsigma C_\varsigma)}, J^{A'_\varsigma}{}_{B_\varsigma C_\varsigma} = \frac{1}{2!}J^{A'_\varsigma}{}_{(B_\varsigma C_\varsigma)}\end{cases} \Leftrightarrow [\tfrac{3}{2}D_a + iS_{ab}(\tfrac{3}{2},\varsigma)D^b]_{k_\varsigma}{}^{l_\varsigma}(\tfrac{3}{2},\varsigma)\psi_{l_\varsigma} = \mathbb{J}_{ak_\varsigma}(\tfrac{3}{2},\varsigma)$$

Cor. 2.6.3.

$$\begin{cases}D^aF_{ab}{}^{[C_\varsigma]} = -J_b{}^{[C_\varsigma]}, D^a * F_{ab}{}^{[C_\varsigma]} \equiv 0\\ (\sigma, i\varsigma)_a(\sigma, -i\varsigma)_bF^{ab[C_\varsigma]} = 0, (\sigma, -i\varsigma)_aJ^{a[C_\varsigma]} = 0\end{cases} \Leftrightarrow [\tfrac{3}{2}D_a + iS_{ab}(\tfrac{3}{2},\varsigma)D^b]_{k_\varsigma}{}^{l_\varsigma}(\tfrac{3}{2},\varsigma)\psi_{l_\varsigma} = \mathbb{J}_{ak_\varsigma}(\tfrac{3}{2},\varsigma)$$

2.7 Conjecture

Thm. 2.7.1. $\partial^a * F^{Z_\kappa}_{ab} = 0 \Leftrightarrow F^{Z_\kappa}_{ab} = \partial_aA^{Z_\kappa}_b - \partial_bA^{Z_\kappa}_a \Leftrightarrow \partial^a * F^{Z_\kappa}_{ab} \equiv 0$

Thm. 2.7.2. $\partial^aF^{Z_\kappa}_{ab} = -J^{Z_\kappa}_b, \partial^a * F^{Z_\kappa}_{ab} = 0 \Leftrightarrow \partial^aF^{Z_\kappa}_{ab} = -J^{Z_\kappa}_b, F^{Z_\kappa}_{ab} = \partial_aA^{Z_\kappa}_b - \partial_bA^{Z_\kappa}_a$

2.8 Spin tensor expression of gravitino field [7]

Spin tensor matrix of gravitino field:
$$S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \succ \begin{bmatrix}0 & \gamma_z & -\gamma_y & -\varsigma\gamma_x\\ -\gamma_z & 0 & \gamma_x & -\varsigma\gamma_y\\ \gamma_y & -\gamma_x & 0 & -\varsigma\gamma_z\\ \varsigma\gamma_x & \varsigma\gamma_y & \varsigma\gamma_z & 0\end{bmatrix} \tag{9.5}$$

Thm. 2.8.1. $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa}(1,\varsigma) = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\Psi}^{Z_\kappa}(1,\varsigma) = iJ^{Z_\kappa}$

An intuitive proof method is as follows:

Proof: $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases}(\partial_x + i\gamma_z\partial_y - i\gamma_y\partial_z - i\varsigma\gamma_x\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma xb}J^{bZ_\kappa}\\ (\partial_y + i\gamma_x\partial_z - i\gamma_z\partial_x - i\varsigma\gamma_y\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma yb}J^{bZ_\kappa}\\ (\partial_z + i\gamma_y\partial_x - i\gamma_x\partial_y - i\varsigma\gamma_z\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma zb}J^{bZ_\kappa}\\ (\partial_\pi + i\varsigma\gamma_x\partial_x + i\varsigma\gamma_y\partial_y + i\varsigma\gamma_z\partial_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma\pi b}J^{bZ_\kappa}\end{cases}$$

$$\Leftrightarrow \begin{cases}\begin{bmatrix}\partial_x & \partial_y & \partial_z\\ -\partial_y & \partial_x & -\varsigma\partial_\pi\\ -\partial_z & \varsigma\partial_\pi & \partial_x\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma Z_\kappa}\\ \psi^{y_\varsigma Z_\kappa}\\ \psi^{z_\varsigma Z_\kappa}\end{bmatrix} = \begin{bmatrix}\varsigma J^{\pi Z_\kappa}\\ J^{zZ_\kappa}\\ -J^{yZ_\kappa}\end{bmatrix}, \begin{bmatrix}\partial_y & -\partial_x & \varsigma\partial_\pi\\ \partial_x & \partial_y & \partial_z\\ -\varsigma\partial_\pi & -\partial_z & \partial_y\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma Z_\kappa}\\ \psi^{y_\varsigma Z_\kappa}\\ \psi^{z_\varsigma Z_\kappa}\end{bmatrix} = \begin{bmatrix}-J^{zZ_\kappa}\\ \varsigma J^{\pi Z_\kappa}\\ J^{xZ_\kappa}\end{bmatrix}\\ \begin{bmatrix}\partial_z & -\varsigma\partial_\pi & -\partial_x\\ \varsigma\partial_\pi & \partial_z & -\partial_y\\ \partial_x & \partial_y & \partial_z\end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma Z_\kappa}\\ \psi^{y_\varsigma Z_\kappa}\\ \psi^{z_\varsigma Z_\kappa}\end{bmatrix} = \begin{bmatrix}J^{yZ_\kappa}\\ -J^{xZ_\kappa}\\ \varsigma J^{\pi Z_\kappa}\end{bmatrix}, i\partial_\pi\Psi^{Z_\kappa}(1,\varsigma) = \varsigma\gamma\cdot\nabla\Psi^{Z_\kappa}(1,\varsigma) - i\varsigma\vec{J}^{Z_\kappa}\end{cases}$$

$$\Leftrightarrow \begin{cases}i\partial_\pi\Psi^{Z_\kappa}(1,\varsigma) = i\varsigma\nabla\times\Psi^{Z_\kappa}(1,\varsigma) - i\varsigma\vec{J}^{Z_\kappa}\\ \nabla\cdot\Psi^{Z_\kappa}(1,\varsigma) = \varsigma J^{\pi Z_\kappa}\end{cases}$$

$$\Leftrightarrow \begin{cases}i\partial_\pi\Psi^{Z_\kappa}(1,\varsigma) = \varsigma\gamma\cdot\nabla\Psi^{Z_\kappa}(1,\varsigma) - i\varsigma\vec{J}^{Z_\kappa}\\ \nabla\cdot\Psi^{Z_\kappa}(1,\varsigma) = \varsigma J^{\pi Z_\kappa}\end{cases}$$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\Psi}^{Z_\kappa}(1,\varsigma) = iJ^{Z_\kappa}$ □

Another more analytical and abstract proof is as follows:

Proof: $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma a}{}^{c}\sigma_{\varsigma\gamma_\varsigma cb}\partial^b\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}$
$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma ac}\sigma^{cb}_{\varsigma\gamma_\varsigma}\partial_b\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}$
$\Leftrightarrow \sigma^{\varsigma ad}_{\beta_\varsigma}\sigma^{\beta_\varsigma}_{\varsigma ac}\sigma^{cb}_{\varsigma\gamma_\varsigma}\partial_b\psi^{\gamma_\varsigma Z_\kappa} = -i\sigma^{\varsigma ad}_{\beta_\varsigma}\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}$
$\Leftrightarrow \sigma^{db}_{\varsigma\gamma_\varsigma}\partial_b\psi^{\gamma_\varsigma Z_\kappa} = -iJ^{dZ_\kappa}$
$\Leftrightarrow \sigma^{ab}_{\varsigma\alpha_\varsigma}\partial_a\psi^{\alpha_\varsigma Z_\kappa} = iJ^{bZ_\kappa}, \alpha_\varsigma = 1,2,3$
$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma}\partial_a\tilde{\Psi}^{\alpha_\varsigma Z_\kappa} = iJ^{Z_\kappa}_b, \alpha_\varsigma = 1,2,3,4$ □

The equation (3.3.2) is completely equivalent to gravitino field equation. It is just the spin tensor expression of gravitino field equation.

Lem. 2.8.1. $\mathbb{J}^{\beta_\varsigma Z_\kappa}_a = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa} \Leftrightarrow \begin{cases} \mathbb{J}^{z_\varsigma Z_\kappa}_y = -\mathbb{J}^{y_\varsigma Z_\kappa}_z = -\varsigma\mathbb{J}^{x_\varsigma Z_\kappa}_\pi = J^{xZ_\kappa} \\ \mathbb{J}^{x_\varsigma Z_\kappa}_z = -\mathbb{J}^{z_\varsigma Z_\kappa}_x = -\varsigma\mathbb{J}^{y_\varsigma Z_\kappa}_\pi = J^{yZ_\kappa} \\ \mathbb{J}^{y_\varsigma Z_\kappa}_x = -\mathbb{J}^{x_\varsigma Z_\kappa}_y = -\varsigma\mathbb{J}^{z_\varsigma Z_\kappa}_\pi = J^{zZ_\kappa} \\ \mathbb{J}^{x_\varsigma Z_\kappa}_x = \mathbb{J}^{y_\varsigma Z_\kappa}_y = \mathbb{J}^{z_\varsigma Z_\kappa}_z = \varsigma J^{\pi Z_\kappa} \end{cases}$

Expand and then we can prove it by expanding. The above spin equation is about special source terms, so what happens to general source terms? Please look at the following theorem.

Thm. 2.8.2. $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_a, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\Psi}^{Z_\kappa}(1,\varsigma) = iJ^{Z_\kappa}, \mathbb{J}^{\beta_\varsigma Z_\kappa}_a = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}$

Proof: $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_a, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\partial_x + i\gamma_z\partial_y - i\gamma_y\partial_z - i\varsigma\gamma_x\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_x \\ (\partial_y + i\gamma_x\partial_z - i\gamma_z\partial_x - i\varsigma\gamma_y\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_y \\ (\partial_z + i\gamma_y\partial_x - i\gamma_x\partial_y - i\varsigma\gamma_z\partial_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_z \\ (\partial_\pi + i\varsigma\gamma_x\partial_x + i\varsigma\gamma_y\partial_y + i\varsigma\gamma_z\partial_z)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ -\partial_y & \partial_x & -\varsigma\partial_\pi \\ -\partial_z & \varsigma\partial_\pi & \partial_x \end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma Z_\kappa} \\ \psi^{y_\varsigma Z_\kappa} \\ \psi^{z_\varsigma Z_\kappa}\end{bmatrix} = \begin{bmatrix}\mathbb{J}^{x_\varsigma Z_\kappa}_x \\ \mathbb{J}^{y_\varsigma Z_\kappa}_x \\ \mathbb{J}^{z_\varsigma Z_\kappa}_x\end{bmatrix} \Leftrightarrow \begin{cases} \nabla\cdot\Psi^{Z_\kappa}(1,\varsigma) = \mathbb{J}^{x_\varsigma Z_\kappa}_x \\ [\nabla\times\Psi^{Z_\kappa}(1,\varsigma)]^{z_\varsigma Z_\kappa} - \varsigma\partial_\pi\psi^{z_\varsigma Z_\kappa}(1,\varsigma) = \mathbb{J}^{y_\varsigma Z_\kappa}_x \\ -[\nabla\times\Psi^{Z_\kappa}(1,\varsigma)]^{y_\varsigma Z_\kappa} + \varsigma\partial_\pi\psi^{y_\varsigma Z_\kappa}(1,\varsigma) = \mathbb{J}^{z_\varsigma Z_\kappa}_x \end{cases} \\ \begin{bmatrix} \partial_y & -\partial_x & \varsigma\partial_\pi \\ \partial_x & \partial_y & \partial_z \\ -\varsigma\partial_\pi & -\partial_z & \partial_y \end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma Z_\kappa} \\ \psi^{y_\varsigma Z_\kappa} \\ \psi^{z_\varsigma Z_\kappa}\end{bmatrix} = \begin{bmatrix}\mathbb{J}^{x_\varsigma Z_\kappa}_y \\ \mathbb{J}^{y_\varsigma Z_\kappa}_y \\ \mathbb{J}^{z_\varsigma Z_\kappa}_y\end{bmatrix} \Leftrightarrow \begin{cases} -[\nabla\times\Psi^{Z_\kappa}(1,\varsigma)]^{z_\varsigma Z_\kappa} + \varsigma\partial_\pi\psi^{z_\varsigma Z_\kappa}(1,\varsigma) = \mathbb{J}^{x_\varsigma Z_\kappa}_y \\ \nabla\cdot\Psi^{Z_\kappa}(1,\varsigma) = \mathbb{J}^{y_\varsigma Z_\kappa}_y \\ [\nabla\times\Psi^{Z_\kappa}(1,\varsigma)]^{x_\varsigma Z_\kappa} - \varsigma\partial_\pi\psi^{x_\varsigma Z_\kappa}(1,\varsigma) = \mathbb{J}^{z_\varsigma Z_\kappa}_y \end{cases} \\ \begin{bmatrix} \partial_z & -\varsigma\partial_\pi & -\partial_x \\ \varsigma\partial_\pi & \partial_z & -\partial_y \\ \partial_x & \partial_y & \partial_z \end{bmatrix}\begin{bmatrix}\psi^{x_\varsigma Z_\kappa} \\ \psi^{y_\varsigma Z_\kappa} \\ \psi^{z_\varsigma Z_\kappa}\end{bmatrix} = \begin{bmatrix}\mathbb{J}^{x_\varsigma Z_\kappa}_z \\ \mathbb{J}^{y_\varsigma Z_\kappa}_z \\ \mathbb{J}^{z_\varsigma Z_\kappa}_z\end{bmatrix} \Leftrightarrow \begin{cases} [\nabla\times\Psi^{Z_\kappa}(1,\varsigma)]^{y_\varsigma Z_\kappa} - \varsigma\partial_\pi\psi^{y_\varsigma Z_\kappa}(1,\varsigma) = \mathbb{J}^{x_\varsigma Z_\kappa}_z \\ -[\nabla\times\Psi^{Z_\kappa}(1,\varsigma)]^{x_\varsigma Z_\kappa} + \varsigma\partial_\pi\psi^{x_\varsigma Z_\kappa}(1,\varsigma) = \mathbb{J}^{y_\varsigma Z_\kappa}_z \\ \nabla\cdot\Psi^{Z_\kappa}(1,\varsigma) = \mathbb{J}^{z_\varsigma Z_\kappa}_z \end{cases} \\ \partial_\pi\Psi^{Z_\kappa}(1,\varsigma) + i\varsigma\gamma\cdot\nabla\psi^{Z_\kappa} = \mathbb{J}^{Z_\kappa}_\pi \Leftrightarrow \partial_\pi\Psi^{Z_\kappa}(1,\varsigma) - \varsigma\nabla\times\Psi^{Z_\kappa}(1,\varsigma) = \mathbb{J}^{Z_\kappa}_\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathbb{J}^{z_\varsigma Z_\kappa}_y = -\mathbb{J}^{y_\varsigma Z_\kappa}_z = -\varsigma\mathbb{J}^{x_\varsigma Z_\kappa}_\pi := J^{xZ_\kappa} \\ \mathbb{J}^{x_\varsigma Z_\kappa}_z = -\mathbb{J}^{z_\varsigma Z_\kappa}_x = -\varsigma\mathbb{J}^{y_\varsigma Z_\kappa}_\pi := J^{yZ_\kappa} \\ \mathbb{J}^{y_\varsigma Z_\kappa}_x = -\mathbb{J}^{x_\varsigma Z_\kappa}_y = -\varsigma\mathbb{J}^{z_\varsigma Z_\kappa}_\pi := J^{zZ_\kappa} \\ \mathbb{J}^{x_\varsigma Z_\kappa}_x = \mathbb{J}^{y_\varsigma Z_\kappa}_y = \mathbb{J}^{z_\varsigma Z_\kappa}_z := \varsigma J^{\pi Z_\kappa} \\ \partial_\pi\Psi^{Z_\kappa}(1,\varsigma) - \varsigma\nabla\times\Psi^{Z_\kappa}(1,\varsigma) = i\vec{J}^{Z_\kappa} \\ \nabla\cdot\Psi^{Z_\kappa}(1,\varsigma) = -iJ^{\pi Z_\kappa} \end{cases}$$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\Psi}^{Z_\kappa}(1,\varsigma) = iJ^{Z_\kappa}, \mathbb{J}^{\beta_\varsigma Z_\kappa}_a = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}$ □

Another more analytical and abstract proof is as follows:

Thm. 2.8.3. $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_a, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow \mathbb{J}^{\beta_\varsigma Z_\kappa}_a = \sigma^{\beta_\varsigma}_{\varsigma ab}\sigma^{bc}_{\varsigma\gamma_\varsigma}\partial_c\psi^{\gamma_\varsigma Z_\kappa}$

Proof: $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_a, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow \sigma^{\beta_\varsigma}_{\varsigma a}{}^{c}\sigma_{\varsigma\gamma_\varsigma cb}\partial^b\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}^{\beta_\varsigma Z_\kappa}_a$
$\Leftrightarrow \mathbb{J}^{\beta_\varsigma Z_\kappa}_a = \sigma^{\beta_\varsigma}_{\varsigma ab}\sigma^{bc}_{\varsigma\alpha_\varsigma}\partial_c\psi^{\alpha_\varsigma Z_\kappa}$

$$\Leftrightarrow \begin{cases} \mathbb{J}^{z_\varsigma Z_\kappa}_y = -\mathbb{J}^{y_\varsigma Z_\kappa}_z = -\varsigma\mathbb{J}^{x_\varsigma Z_\kappa}_\pi = i\sigma^{xb}_{\varsigma\alpha_\varsigma}\partial_b\psi^{\alpha_\varsigma Z_\kappa} \\ \mathbb{J}^{x_\varsigma Z_\kappa}_z = -\mathbb{J}^{z_\varsigma Z_\kappa}_x = -\varsigma\mathbb{J}^{y_\varsigma Z_\kappa}_\pi = i\sigma^{yb}_{\varsigma\alpha_\varsigma}\partial_b\psi^{\alpha_\varsigma Z_\kappa} \\ \mathbb{J}^{y_\varsigma Z_\kappa}_x = -\mathbb{J}^{x_\varsigma Z_\kappa}_y = -\varsigma\mathbb{J}^{z_\varsigma Z_\kappa}_\pi = i\sigma^{zb}_{\varsigma\alpha_\varsigma}\partial_b\psi^{\alpha_\varsigma Z_\kappa} \\ \mathbb{J}^{x_\varsigma Z_\kappa}_x = \mathbb{J}^{y_\varsigma Z_\kappa}_y = \mathbb{J}^{z_\varsigma Z_\kappa}_z = i\varsigma\sigma^{\pi b}_{\varsigma\alpha_\varsigma}\partial_b\psi^{\alpha_\varsigma Z_\kappa} \end{cases}$$ □

This theorem indicates that the source term of this spin equation is limited and not arbitrary. Only the source term case described in the previous theorem has a solution, while the other cases have no solution.

Cor. 2.8.1. $(\partial_a + iS_{ab}\partial^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma Z_\kappa} = \mathbb{J}_a^{\beta_\varsigma Z_\kappa}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$ *have solutions.* $\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma Z_\kappa} = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{bZ_\kappa}, \exists J^{bZ_\kappa}$

3 Analysis of Rarita-Schwinger equation [17]

3.1 Preparation

Rarita-Schwinger lagrangian $\mathcal{L}_{RS} = -\bar{\psi}^a\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma)$

Lem. 3.1.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(e,\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) + \frac{1}{2}m\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma)\psi^b(e,\varsigma) = 0$
Using the formula: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma) = 2iS_{ab}(e,\varsigma)\gamma_c(\varsigma) - \gamma_{[a}(\varsigma)\delta_{b]c}, \varepsilon_{abcd}S^{cd}(e,\varsigma) = -2\gamma_5(\varsigma)iS_{ab}(e,\varsigma)$
$\Leftrightarrow [2iS_{ab}(e,\varsigma)\gamma_c(\varsigma) - \gamma_{[a}(\varsigma)\delta_{b]c}]D^b\psi^c(e,\varsigma) - m\gamma_5(\varsigma)iS_{ab}(e,\varsigma)\psi^b(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(e,\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$ □

Lem. 3.1.2. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$

$$\Rightarrow \begin{cases} m[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - m[D^a\psi_a(e,\varsigma)] = 0 \\ 2[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - 2[D^a\psi_a(e,\varsigma)] - 3m[\gamma_a(\varsigma)\psi^a(e,\varsigma)] = 0 \end{cases}$$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(e,\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$

$$\Rightarrow \begin{cases} [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m][D^a\psi_a(e,\varsigma)] - [\gamma_a(\varsigma)D^a]D_c\psi^c(e,\varsigma) \\ -D^aD_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0 \\ 4[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m][\gamma^a(\varsigma)\psi_a(e,\varsigma)] - 4D_c\psi^c(e,\varsigma) - [\gamma^a(\varsigma)D_a][\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} m[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - m[D^a\psi_a(e,\varsigma)] = 0 \\ 2[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - 2[D^a\psi_a(e,\varsigma)] - 3m[\gamma_a(\varsigma)\psi^a(e,\varsigma)] = 0 \end{cases}$$

$\Leftrightarrow \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0$ □

Lem. 3.1.3. $\begin{cases} m[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - m[D^a\psi_a(e,\varsigma)] = 0 \\ 2[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] - 2[D^a\psi_a(e,\varsigma)] - 3m[\gamma_a(\varsigma)\psi^a(e,\varsigma)] = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0, m \neq 0 \\ D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)] = 0, m = 0 \end{cases}$$

3.2 Equivalent form of Rarita-Schwinger equation with mass

Cor. 3.2.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0, m \neq 0$
$\Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0, m \neq 0$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b - m][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b + m]\psi_a(\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$
$\Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0$ □

Cor. 3.2.2. $[\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, D_a\psi^a(e,\varsigma) = 0, m \neq 0$
$\Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0, m \neq 0$

Important conclusions:

Thm. 3.2.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)[D^b + \frac{1}{2}m\gamma^b(\varsigma)]\psi^c(e,\varsigma) = 0 \Leftrightarrow [\gamma_b(\varsigma)D^b + m]\psi^a(e,\varsigma) = 0, \gamma_a(\varsigma)\psi^a(e,\varsigma) = 0; m \neq 0$

3.3 Equivalent form of Rarita-Schwinger equation without mass

Cor. 3.3.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) = 0 \Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0, D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$

Proof: $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma) = 0$
$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b][\gamma_c(\varsigma)\psi^c(e,\varsigma)] + [\gamma_b(\varsigma)D^b]\psi_a(\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e,\varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e,\varsigma)] = 0$
$\Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0, D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$ □

Cor. 3.3.2. $\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma) - D^a\psi^b(e,\varsigma)] = 0 \Rightarrow D_a\psi^a(e,\varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)],$

Cor. 3.3.3. $\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma)-D^a\psi^b(e,\varsigma)]=0$
$\Rightarrow[\gamma_b(\varsigma)D^b]\psi^a(e,\varsigma)=D^a[\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Rightarrow\gamma_a(\varsigma)\gamma_b(\varsigma)D^b\psi^a(e,\varsigma)=[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Rightarrow[2\delta_{ab}-\gamma_b(\varsigma)\gamma_a(\varsigma)]D^b\psi^a(e,\varsigma)=[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Rightarrow D_a\psi^a(e,\varsigma)=[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$,

Cor. 3.3.4. $\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma)-D^a\psi^b(e,\varsigma)]=0, D_a\psi^a(e,\varsigma)=[\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e,\varsigma)]$
$\Leftrightarrow\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma)-D^a\psi^b(e,\varsigma)]=0$

Cor. 3.3.5. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma)=0\Leftrightarrow\gamma_b(\varsigma)[D^b\psi^a(e,\varsigma)-D^a\psi^b(e,\varsigma)]=0$

Important conclusions:

Thm. 3.3.1. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma)=0\Leftrightarrow\gamma_a(\varsigma)F^{ab}(e,\varsigma)=0, F^{ab}(e,\varsigma)\equiv D^a\psi^b(e,\varsigma)-D^b\psi^a(e,\varsigma)$

Cor. 3.3.6. $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e,\varsigma)=0,\gamma_a(\varsigma)\psi^a(e,\varsigma)=0()$
$\Leftrightarrow\gamma_b(\varsigma)D^b\psi^a(e,\varsigma)=0,\gamma_a(\varsigma)\psi^a(e,\varsigma)=0$

3.4 Equivalent form of Weyl Type R-S equation

Cor. 3.4.1. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma)=0\Leftrightarrow(\sigma,-i\varsigma)_b[D^b\psi^a(\varsigma)-D^a\psi^b(\varsigma)]=0, D_a\psi^a(\varsigma)=[(\sigma,i\varsigma)_aD^a][(\sigma,-i\varsigma)_b\psi^b(\varsigma)]$

Cor. 3.4.2. $(\sigma,-i\varsigma)_b[D^b\psi^a(\varsigma)-D^a\psi^b(\varsigma)]=0\Rightarrow D_a\psi^a(\varsigma)=[(\sigma,i\varsigma)_aD^a][(\sigma,-i\varsigma)_b\psi^b(\varsigma)]$

Cor. 3.4.3. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma)=0\Leftrightarrow(\sigma,-i\varsigma)_b[D^b\psi^a(\varsigma)-D^a\psi^b(\varsigma)]=0$

Important conclusions:

Thm. 3.4.1. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma)=0\Leftrightarrow(\sigma,-i\varsigma)_aF^{ab}(\frac{3}{2},\varsigma)=0, F^{ab}(\frac{3}{2},\varsigma):=D^a\psi^b(\varsigma)-D^b\psi^a(\varsigma)$

Cor. 3.4.4. $F_{uv}(\frac{3}{2},\varsigma)\equiv D_u\psi_v(\varsigma)-D_v\psi_u(\varsigma)\Leftrightarrow F_{uv}(\frac{3}{2},\varsigma)=(\partial_u+\frac{i}{2}\sigma_{\alpha_\varsigma}A_u^{\alpha_\varsigma})\psi_v(\varsigma)-(\partial_v+\frac{i}{2}\sigma_{\alpha_\varsigma}A_v^{\alpha_\varsigma})\psi_u(\varsigma)$

Cor. 3.4.5. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma)=0,(\sigma,-i\varsigma)_a\psi^a(\varsigma)=0\Leftrightarrow(\sigma,-i\varsigma)_bD^b\psi^a(\varsigma)=0,(\sigma,-i\varsigma)_a\psi^a(\varsigma)=0$

3.5 Equivalent spin tensor form with lower first derivative for Weyl Type R-S equation

Cor. 3.5.1. $\varepsilon_{abcd}(\sigma,-i\varsigma)^dD^b\psi^c(\varsigma)=0\Leftrightarrow[\frac{1}{2}\delta_{ab}+iS_{ab}(\varsigma)]F^{bc}(\frac{3}{2},\varsigma)=0, F^{bc}(\frac{3}{2},\varsigma)\equiv D^b\psi^c(\varsigma)-D^c\psi^b(\varsigma)$

4 Comparison between equations

4.1 Comparison between Weyl type and Penrose type gravitino equation

Weyl type R-S equation: $(\sigma,-i\varsigma)_aF^{ab}(\frac{3}{2},\varsigma)=0\leftrightarrow$ **Penrose type R-S equation:** $\partial_aF^{ab}(\frac{3}{2},\varsigma)=-J^b(\varsigma)$ (9.6)

$$F_{uv}(\tfrac{3}{2},\varsigma)\equiv(\partial_u+\tfrac{i}{2}A_u^{\alpha_\varsigma}\sigma_{\alpha_\varsigma})\psi_v(\varsigma)-(\partial_v+\tfrac{i}{2}A_v^{\alpha_\varsigma}\sigma_{\alpha_\varsigma})\psi_u(\varsigma) \quad (9.7)$$

Formally it is equivalent to $(\sigma,-i\varsigma)_a\leftrightarrow\partial_a$. The gravitational field case and the gravitino case are also very similar in form.

4.2 Comparison between Einstein equation and gauge equation of gravitational field

Einstein equation of gravitational field: $(\sigma_{-\varsigma},-i\varsigma)_a\mathcal{F}^{ab}(2,\varsigma)=\varsigma\bar{\mathcal{T}}^b\leftrightarrow$ **Gauge equation of gravitational field:** $D_a\mathcal{F}^{al}$ (9.8)

$$\mathcal{F}_{uv}(2,\varsigma)=(\partial_u+\tfrac{i}{2}A_u^{\alpha_\varsigma}\mathcal{R}_{\alpha_\varsigma})\mathcal{A}_v(\varsigma)-(\partial_v+\tfrac{i}{2}A_v^{\alpha_\varsigma}\mathcal{R}_{\alpha_\varsigma})\mathcal{A}_u(\varsigma) \quad (9.9)$$

Formally it is equivalent to $(\sigma_{-\varsigma},-i\varsigma)_a\leftrightarrow D_a$

Chapter10 Spin Equations for Various Particles

1 Description of spin vector W_a

1.1 Definition of spin vectors $W_a, W_a(s,\varsigma)$

s-spin tensor matrix: $S_{(ab)}(s,\varsigma) = \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0 \end{bmatrix}$ (10.1)

Def. 1.1.1. $W_a := -i * M_{ab}p^b = \frac{-i}{2}\varepsilon_{abcd}M^{bc}p^d$

Def. 1.1.2. $W_a(s,\varsigma) := -i * M_{ab}(s,\varsigma)p^b, M_{ab}(s,\varsigma) = L_{ab} + S_{ab}(s,\varsigma)$

Pro. 1.1.1. $W_a p^a = 0, W_a(s,\varsigma)p^a = 0$

The above shows that the spin vector is orthogonal to momentum and has only three independent components.

Pro. 1.1.2. $*L_{ab}p^b = 0$

Proof: $*L_{ab}p^b = \frac{1}{2}\varepsilon_{abcd}(x^c p^d - x^d p^c)p^b = \varepsilon_{abcd}x^c p^d p^b = \varepsilon_{abcd}x^c p^b p^d = 0$ □

The above shows Orbital angular momentum has no contribution to the spin vector, so the following conclusions are obtained.

Cor. 1.1.1. $\begin{cases} W_a = -i * S_{ab}p^b \\ W_a(s,\varsigma) = -i * S_{ab}(s,\varsigma)p^b = i\varsigma S_{ab}(s,\varsigma)p^b \end{cases}$

1.2 Properties of spin vector $W_a(s,\varsigma)$

Pro. 1.2.1. $W_a(s,\varsigma)W^a(s,\varsigma) = m^2 s(s+1), m^2 = -p_a p^a$

Proof: $W_a(s,\varsigma)W^a(s,\varsigma) = [i\varsigma S_{ab}(s,\varsigma)p^b][i\varsigma S^{ac}(s,\varsigma)p_c]$
$\Leftrightarrow W_a(s,\varsigma)W^a(s,\varsigma) = -p^a S_{ca}(s,\varsigma)S^{cb}(s,\varsigma)p_b$
$\Leftrightarrow W_a(s,\varsigma)W^a(s,\varsigma) = p^a S_{ac}(s,\varsigma)S^c{}_b(s,\varsigma)p^b$
$\Leftrightarrow W_a(s,\varsigma)W^a(s,\varsigma) = -p^a s(s+1)\delta_{ab}p^b$
$\Leftrightarrow W_a(s,\varsigma)W^a(s,\varsigma) = -s(s+1)p_a p^a$
$\Leftrightarrow W_a(s,\varsigma)W^a(s,\varsigma) = m^2 s(s+1), m^2 = -p_a p^a$ □

Using property of the constant tensor S_{ab}, the following general conclusions can be proved.

Pro. 1.2.2. $W_a W^a = m^2 s(s+1), m^2 = -p_a p^a \neq 0$

Proof: $W_a W^a = [-i * S_{ab}p^b][-i * S^{ab}p_b]$
$= -[*S_{ab}(0,0,0,im)^b][*S^{ab}(0,0,0,im)_b]$
$= -[*S_{a\pi}ip][*S^{a\pi}ip]$
$= m^2 * S_{a\pi} * S^{a\pi}$
$= -m^2(S_{xy}^2 + S_{yz}^2 + S_{zx}^2)$
$= m^2 s(s+1)$ □

Pro. 1.2.3. $W_a W^a = p^2 s(s+1) - p^2(S_{x\pi}^2 + S_{y\pi}^2 + S_{xy}^2), m^2 = -p_a p^a = 0$

Proof: $W_a W^a = [-i * S_{ab}p^b][-i * S^{ab}p_b]$
$= -[*S_{ab}(0,0,p,ip)^b][*S^{ab}(0,0,p,ip)_b]$
$= -[*S_{az}p][*S^{az}p] - [*S_{a\pi}ip][*S^{a\pi}ip]$
$= -p^2 * S_{az} * S^{az} - p^2 * S_{a\pi} * S^{a\pi}$
$= p^2(S_{x\pi}^2 + S_{y\pi}^2 + S_{xy}^2) - p^2(S_{xy}^2 + S_{yz}^2 + S_{zx}^2)$
$= p^2 s(s+1) - p^2(S_{x\pi}^2 + S_{y\pi}^2 + S_{xy}^2)$ □

Pro. 1.2.4. $W_a(s,\varsigma)W^a(s,\varsigma)=0, m^2=-p_ap^a=0$

Pro. 1.2.5. $[W_a(s,\varsigma),W_b(s,\varsigma)]=\varsigma[W_a(s,\varsigma)p_b-W_b(s,\varsigma)p_a]-m^2S_{ab}(s,\varsigma)$

Pro. 1.2.6. $\vec{W}(s,\varsigma)=-i\varsigma\sigma(s)\times\vec{p}-i\sigma(s)p_\pi, W_\pi(s,\varsigma)=i\sigma(s)\cdot\vec{p}$

Pro. 1.2.7. $\vec{W}(s,\varsigma)\times\vec{W}(s,\varsigma)=\varsigma\vec{W}(s,\varsigma)\times\vec{p}+im^2\sigma(s)$

Pro. 1.2.8. $[\sigma(s),i\varsigma]_aW^a(s,\varsigma)=-is(s+1)p_\pi$

1.3 Properties of spin vector $W_a(s,\varsigma)$ in a special coordinate system
Properties of $W_a(s,\varsigma)$ for massive particles in the follow-up coordinate system:

Pro. 1.3.1. $\vec{W}(s,\varsigma)=m\sigma(s), W_\pi(s,\varsigma)=0\ for\ \vec{p}=0$

Properties of $W_a(s,\varsigma)$ for massless particles in the motional direction coordinate system:

Pro. 1.3.2. $\begin{cases}W_x(s,\varsigma)=[\sigma_x(s)-i\varsigma\sigma_y(s)]p, W_y(s,\varsigma)=[\sigma_y(s)+i\varsigma\sigma_x(s)]p\\ W_z(s,\varsigma)=\sigma_z(s)p, W_\pi(s,\varsigma)=i\sigma_z(s)p\end{cases} \quad for\begin{cases}m=0,p_x=p_y=0\\ p_z=-ip_\pi=p>0\end{cases}$

Cor. 1.3.1. $[M_{ab},p_cp^c]=0,[L_{ab},p_cp^c]=0,[S_{ab},p_cp^c]=0,[p_a,p_cp^c]=0,[p_a,W_b]=0$

1.4 Commutative relation of spin vector $W_a(s,\varsigma)$ and $p_a, S_{ab}(s,\varsigma)$
Commutative relation:

$$\begin{cases}i[S_{ab}(s,\varsigma),S_{cd}(s,\varsigma)]=g_{ad}S_{bc}(s,\varsigma)-g_{ac}S_{bd}(s,\varsigma)+g_{bc}S_{ad}(s,\varsigma)-g_{bd}S_{ac}(s,\varsigma)\\ [W_a(s,\varsigma),W_b(s,\varsigma)]=\varsigma[W_a(s,\varsigma)p_b+W_b(s,\varsigma)p_a+iS_{ab}(s,\varsigma)p_cp^c]\\ [W_a(s,\varsigma),S_{bc}(s,\varsigma)]=g_{ac}W_b(s,\varsigma)-g_{ab}W_c(s,\varsigma)-iS_{ac}(s,\varsigma)p_b+iS_{ab}(s,\varsigma)p_c\\ [p_a,W_b(s,\varsigma)]=0,[p_a,S_{bc}(s,\varsigma)]=0,[p_a,p_b]=0\end{cases} \tag{10.2}$$

1.5 Casimir operators of Poincare group with massive particles [8]
Pro. 1.5.1. $W_a(s,\varsigma)W^a(s,\varsigma)=m^2s(s+1), p_ap^a=-m^2, p_aW^a(s,\varsigma)=0$

1.6 Casimir operators of Poincare group with massless particles [8]
Pro. 1.6.1. $W_a(s,\varsigma)W^a(s,\varsigma)=0, p_ap^a=0, p_aW^a(s,\varsigma)=0$

1.7 Unified description of spin tensor $S_{ab}(s,\varsigma)$
$S_{ab}(s,\varsigma)$ is suitable for any component form.

$$tr[S_{ab}(s,\varsigma)S_{cd}(s,\varsigma)]=-\tfrac{2}{3}s(s+\tfrac{1}{2})(s+1)\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd} \tag{10.3}$$

$$tr[S_{ab}(s,-\varsigma)S_{cd}(s,-\varsigma)]=-\tfrac{2}{3}s(s+\tfrac{1}{2})(s+1)\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha'_\varsigma cd} \tag{10.4}$$

$$S_{ac}(s,\varsigma)S^c{}_b(s,\varsigma)=-s(s+1)\delta_{ab}, S_{ac}(s,-\varsigma)S^c{}_b(s,-\varsigma)=-s(s+1)\delta_{ab} \tag{10.5}$$

$$\sigma^2(s)=\tfrac{1}{4}S_{ab}(s,\varsigma)S^{ab}(s,\varsigma)=\tfrac{1}{4}S_{ab}(s,-\varsigma)S^{ab}(s,-\varsigma)=s(s+1) \tag{10.6}$$

2 Construction of spin equation

2.1 A new particle equation directly constructed by spin quantities
The following particle equation is directly constructed from the spin quantity:

$$[(s+\phi)D_a+iS_{ab}D^b]\psi=\mathbb{J}_a \tag{10.7}$$

ψ is the particle state spinor, s is the particle spin, S_{ab} is the particle spin tensor, ϕ is a scalar field, $\mathbb{J}_a$ is the spinor source and D_a is the covariant derivative.
2.2 Properties of the new particle equation

s-spin matrix: $S_{ab}(s,\varsigma)=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma}(s)\succ\begin{bmatrix}0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s)\\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s)\\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s)\\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0\end{bmatrix}$

Thm. 2.2.1. $[(s+\phi)\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi=0\Rightarrow\phi=0\ or\ \phi=-(2s+1)\ or\ \sigma(s)\cdot\nabla\psi=0,\partial_\pi\psi=0$

Proof: $[(s+\phi)\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi=0$

$$\Leftrightarrow\begin{cases}[(s+\phi)\partial_x+i\sigma_z(s)\partial_y-i\sigma_y(s)\partial_z-i\varsigma\sigma_x(s)\partial_\pi]\psi=0\\ [(s+\phi)\partial_y+i\sigma_x(s)\partial_z-i\sigma_z(s)\partial_x-i\varsigma\sigma_y(s)\partial_\pi]\psi=0\\ [(s+\phi)\partial_z+i\sigma_y(s)\partial_x-i\sigma_x(s)\partial_y-i\varsigma\sigma_z(s)\partial_\pi]\psi=0\\ [(s+\phi)\partial_\pi+i\varsigma\sigma_x(s)\partial_x+i\varsigma\sigma_y(s)\partial_y+i\varsigma\sigma_z(s)\partial_z]\psi=0\end{cases}$$

$$\Rightarrow \begin{cases} [(s+1+\phi)\sigma(s)\cdot\nabla - i\varsigma\sigma^2(s)\partial_\pi]\psi = 0 \\ [(s+\phi)\partial_\pi + i\varsigma\sigma\cdot\nabla]\psi = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sigma(s)\cdot\nabla\psi = i\varsigma(s+\phi)\partial_\pi\psi \\ [(s+\phi)(s+1+\phi) - s(s+1)]\partial_\pi\psi = 0 \end{cases}$$

$\Leftrightarrow \phi = 0$ or $\phi = -(2s+1)$ or $\sigma(s)\cdot\nabla\psi = 0, \partial_\pi\psi = 0$ □

Thm. 2.2.2. $[s\partial_a + iS_{ab}\partial^b]\psi = 0 \Rightarrow \partial_a\partial^a\psi = 0$

Proof: $[s\partial_a + iS_{ab}\partial^b]\psi = 0$
$\Rightarrow \partial^a[s\partial_a + iS_{ab}\partial^b]\psi = 0$
$\Leftrightarrow [s\partial_a\partial^a + iS_{ab}\partial^a\partial^b]\psi = 0$
$\Leftrightarrow [s\partial_a\partial^a + 0]\psi = 0$
$\Leftrightarrow \partial_a\partial^a\psi = 0$ □

This equation describes massless particles.

Thm. 2.2.3. $\begin{cases} \textit{When } \phi \neq 0, [(s+\phi)\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \textit{ has no plane wave solutions.} \\ \textit{When } \phi = 0, [(s+\phi)\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \textit{ has plane wave solutions.} \end{cases}$

Proof: Because this equation describes massless particles, the particle motion direction can always be selected as z, at this time $p_a = (0,0,p,ip)$, then

$$[(s+\phi)p_a + iS_{ab}(s,\varsigma)p^b]\psi(s,\varsigma) = 0$$

$$\Leftrightarrow \begin{cases} [(s+\phi)p_x + i\sigma_z(s)p_y - i\sigma_y(s)p_z - i\varsigma\sigma_x(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_y + i\sigma_x(s)p_z - i\sigma_z(s)p_x - i\varsigma\sigma_y(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_z + i\sigma_y(s)p_x - i\sigma_x(s)p_y - i\varsigma\sigma_z(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_\pi + i\varsigma\sigma_x(s)p_x + i\varsigma\sigma_y(s)p_y + i\varsigma\sigma_z(s)p_z]\psi(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [-i\sigma_y(s)p_z - i\varsigma\sigma_x(s)p_\pi]\psi(s,\varsigma) = 0 \\ [i\sigma_x(s)p_z - i\varsigma\sigma_y(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_z - i\varsigma\sigma_z(s)p_\pi]\psi(s,\varsigma) = 0 \\ [(s+\phi)p_\pi + i\varsigma\sigma_z(s)p_z]\psi(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [\sigma_x(s) - i\varsigma\sigma_y(s)]p\psi(s,\varsigma) = 0 \Leftrightarrow \psi_m(s,\varsigma) = 0, m = s-1,\cdots,-(s-1),\varsigma s \\ [(s+\phi) + \varsigma\sigma_z(s)]p\psi(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi_m(s,\varsigma) = 0, m = s-1,\cdots,-(s-1),\varsigma s \\ \phi\psi_{-\varsigma s}(s,\varsigma) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{When } \phi \neq 0, \psi(s,\varsigma) = 0, \text{that is, all spin components are zero.} \\ \text{When } \phi = 0, \psi(s,\varsigma) = [\frac{1}{2}(\varsigma-1)\psi_s, 0,\cdots,0,\frac{1}{2}(\varsigma+1)\psi_{-s}]^T e^{ip\cdot x} \\ \text{That is, } -\varsigma s - \text{spin component may not be zero and the remaining components are all zero.} \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{When } \phi \neq 0, [(s+\phi)\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \text{ has no plane wave solutions.} \\ \text{When } \phi = 0, [(s+\phi)\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \text{ has plane wave solutions.} \end{cases}$$ □

That is, ϕ in this equation has a similar switching effect.

Cor. 2.2.1. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0$ *has plane wave solutions:*
$\psi(s,\varsigma) = [\frac{1}{2}(\varsigma-1)\psi_s, 0,\cdots,0,\frac{1}{2}(\varsigma+1)\psi_{-s}]^T e^{ip\cdot x}$

2.3 Definition of spin equation

Def. 2.3.1. $[sD_a + iS_{ab}D^b]\psi = \mathbb{J}_a$ *is called Spin Equation.*

Cor. 2.3.1. $(s\delta_{ab} + iS_{ab})D^b\psi = \mathbb{J}_a$

2.4 An equivalent expression of spin equation

Cor. 2.4.1. $[s\hat{P}_a + \varsigma\hat{W}_a(s,\varsigma)]\psi(s,\varsigma) = -i\mathbb{J}_a(s,\varsigma), \hat{P}_a := -i\partial_a, \hat{W}_a(s,\varsigma) := \varsigma S_{ab}(s,\varsigma)\partial^b$

That is, the switch spin equation can be regarded as an equation determined by the relation between momentum and spin vector.

Thm. 2.4.1. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi = 0 \Leftrightarrow [s\varsigma\hat{p}_a + \hat{W}_a(s,\varsigma)]\psi(s,\varsigma) = 0 \Leftrightarrow \hat{W}_a(s,\varsigma)\psi(s,\varsigma) = -s\varsigma\hat{p}_a\psi(s,\varsigma)$

2.5 Definition of switch spin equation

Def. 2.5.1. $[(s+\phi)D_a + iS_{ab}D^b]\psi = \mathbb{J}_a$ *is called Switch Spin Equation,* ϕ *is called switch type scalar field.*

Cor. 2.5.1. $[(s+\phi)\delta_{ab} + iS_{ab}]D^b\psi = \mathbb{J}_a$

2.6 An equivalent expression of switch spin equation

Cor. 2.6.1. $[(s+\phi)\hat{p}_a + \varsigma\hat{W}_a(s,\varsigma)]\psi(s,\varsigma) = -i\mathbb{J}_a(s,\varsigma)$

That is, the spin equation can be regarded as an equation determined by the relation between momentum and spin vector.

3 Spin equations of various particles

3.1 Neutrino [5] spin equation

Neutrino spin matrix: $S_{ab}(\varsigma) = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma} \succ \frac{1}{2}\begin{bmatrix} 0 & \sigma_z & -\sigma_y & -\varsigma\sigma_x \\ -\sigma_z & 0 & \sigma_x & -\varsigma\sigma_y \\ \sigma_y & -\sigma_x & 0 & -\varsigma\sigma_z \\ \varsigma\sigma_x & \varsigma\sigma_y & \varsigma\sigma_z & 0 \end{bmatrix}$ (10.8)

Thm. 3.1.1. $[\frac{1}{2}D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = 0 \Leftrightarrow (\sigma,-i\varsigma)^a D_a\psi(\frac{1}{2},\varsigma) = 0$

3.2 Electron [4] spin equation in any N+1 dimensional space-time

Electron spin equation in n=N+1 dimensional space-time:

Thm. 3.2.1. $[\frac{1}{2}(D_a + m\gamma_a) + iS_{ab}D^b]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a,\gamma_b] \Leftrightarrow (\gamma^a D_a + m)\psi = 0$

Proof: $[\frac{1}{2}(D_a + m\gamma_a) + iS_{ab}D^b]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a,\gamma_b]$
$\Leftrightarrow [(2iS_{ab} + \delta_{ab})D^b + \gamma_a m]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a,\gamma_b]$
$\Leftrightarrow [\frac{1}{2}([\gamma_a,\gamma_b] + \{\gamma_a,\gamma_b\})D_b + \gamma_a m]\psi = 0$
$\Leftrightarrow \gamma_a(\gamma_b D^b + m)\psi = 0$
$\Leftrightarrow (\gamma_a D^a + m)\psi = 0$
$\Leftrightarrow (\gamma^a D_a + m)\psi = 0$ □

Electron spin equation in four dimensional space-time:

Cor. 3.2.1. $\{\frac{1}{2}[D_a + m\gamma_a(\varsigma)] + iS_{ab}(e,\varsigma)D^b\}\psi(e,\varsigma) = 0 \Leftrightarrow [\gamma^a(\varsigma)D_a + m]\psi(e,\varsigma) = 0$

3.3 Spin equation of Yang-Mills field [6]

Thm. 3.3.1. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma\sigma}(1,\varsigma) = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$
$\Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\psi}^\sigma(1,\varsigma) = i\tilde{\mathcal{J}}^\sigma(1,\varsigma)$

Thm. 3.3.2. $(D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\psi^{\gamma_\varsigma\sigma}(1,\varsigma) = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^{b\sigma}, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\psi}^\sigma(1,\varsigma) = iJ^\sigma$

3.4 Spin equation of s-spin particle: fully symmetric Penrose equation [1,2]

3.4.1 s-spin equation

s-spin matrix: $S_{ab}(s,\varsigma) = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma}(s) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0 \end{bmatrix}$ (10.9)

Thm. 3.4.1. $\begin{cases} \nabla^{A'_\varsigma A_\varsigma}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}} \\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2s}} \\ J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}} = \frac{1}{(2s-1)!}J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma\cdots)}_{2s-1}} \end{cases} \Leftrightarrow [sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s)$

Proof: $\nabla^{A'_\varsigma A_\varsigma}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}, \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2s}}, J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}} = \frac{1}{(2s-1)!}J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma\cdots)}_{2s-1}}$
$\Leftrightarrow (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$
$\Leftrightarrow (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s)D^a\psi_{k_\varsigma}(s) = iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$
$\Leftrightarrow (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2})D^a\psi_{k_\varsigma}(s) = iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$
$\Leftrightarrow (\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)D^a\psi_{k_\varsigma}(s) = iJ^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2})$
$\Leftrightarrow N^{Z_\varsigma l_\varsigma}_{j_\varsigma}(s)(\sigma,i\varsigma)_{a\,Z_\varsigma A'_\varsigma}(\sigma,-i\varsigma)_b^{A'_\varsigma A_\varsigma}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)D^b\psi_{k_\varsigma}(s) = iN^{Z_\varsigma l_\varsigma}_{j_\varsigma}(s)(\sigma,i\varsigma)_{a\,Z_\varsigma A'_\varsigma}J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2})$

$\Leftrightarrow N_{j_\varsigma}^{Z_\varsigma l_\varsigma}(s)[\delta_{ab}\delta_{Z_\varsigma}{}^{A_\varsigma} + 2iS_{abZ_\varsigma}{}^{A_\varsigma}(\frac{1}{2},\varsigma)]N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s)D^b\psi_{k_\varsigma}(s) = iN_{j_\varsigma}^{Z_\varsigma l_\varsigma}(s)(\sigma, i\varsigma)_{aZ_\varsigma A'_\varsigma}J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2})$
$\Leftrightarrow [s\delta_{ab}\delta_{j_\varsigma}{}^{k_\varsigma} + iS_{abj_\varsigma}{}^{k_\varsigma}(s,\varsigma)]D^b\psi_{k_\varsigma}(s) = is(\sigma, i\varsigma)_{aA_\varsigma A'_\varsigma}N_{j_\varsigma}^{A_\varsigma l_\varsigma}(s)J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2})$
$\Leftrightarrow [s\delta_{ab}\delta_{j_\varsigma}{}^{k_\varsigma} + iS_{abj_\varsigma}{}^{k_\varsigma}(s,\varsigma)]D^b\psi_{k_\varsigma}(s) = is\delta_{ab}(\sigma, i\varsigma)^b_{A_\varsigma A'_\varsigma}\Gamma_{j_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s)J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$
$\Leftrightarrow [s\delta_{ab}\delta_{j_\varsigma}{}^{k_\varsigma} + iS_{abj_\varsigma}{}^{k_\varsigma}(s,\varsigma)]D^b\psi_{k_\varsigma}(s) = -\sqrt{2}\varsigma sZ^{al_\varsigma}_{A'_\varsigma j_\varsigma}(s,\varsigma)J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2})$
$\Leftrightarrow [sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s), \psi(s,\varsigma) \prec \psi_{k_\varsigma}(s), \tilde{J}(s) \prec J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2})$ □

From the above, it can be seen that the s-spin equation is the spin tensor expression of the fully symmetric Penrose equation.

Cor. 3.4.1. $(\sigma\otimes I_{2^{2s-1}}, -i\varsigma)^a D_a\hat{\psi}(s,\varsigma) = i\hat{J}(s,\varsigma) \Leftrightarrow (\sigma\otimes I_{2s}, -i\varsigma)^a D_a\tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma)$

Proof: $(\sigma\otimes I_{2^{2s-1}}, -i\varsigma)^a D_a\hat{\psi}(s,\varsigma) = i\hat{J}(s,\varsigma)$
$\Leftrightarrow (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)^a\Gamma(s)D_a\psi(s,\varsigma) = i[I\otimes\Gamma(s-\frac{1}{2})]\tilde{J}(s,\varsigma)$
$\Leftrightarrow (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)^a[I_{w+1}\otimes\Gamma(s-\frac{1}{2})]N(s)D_a\psi(s,\varsigma) = i[I\otimes\Gamma(s-\frac{1}{2})]\tilde{J}(s,\varsigma)$
$\Leftrightarrow [I_{w+1}\otimes\Gamma(s-\frac{1}{2})](\sigma\otimes I_{2s}, -i\varsigma)^a N(s)D_a\psi(s,\varsigma) = i[I\otimes\Gamma(s-\frac{1}{2})]\tilde{J}(s,\varsigma)$
$\Leftrightarrow (\sigma\otimes I_{2s}, -i\varsigma)^a D_a\tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma)$ □

Cor. 3.4.2. $[sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s) \Leftrightarrow (\sigma\otimes I_{2s}, -i\varsigma)_a D^a\tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma)$

Proof: $[sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)$
$\Leftrightarrow [s\delta_{ab}I_{2s+1} + iS_{ab}(s,\varsigma)]D^b\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)$
$\Leftrightarrow 2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)D^b\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)$
$\Leftrightarrow Z_b(s,\varsigma)D^b\psi(s,\varsigma) = \frac{-\varsigma}{\sqrt{2}}\tilde{J}(s,\varsigma)$
$\Leftrightarrow (\sigma\otimes I_{2s}, -i\varsigma)^a D_a\tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma)$ □

Cor. 3.4.3.

$$\begin{cases}(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}} \\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2s}}, J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}} = \frac{1}{(2s-1)!}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}\end{cases} \Leftrightarrow (\sigma\otimes I_{2s}, -i\varsigma)^a D_a\tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma)$$

Cor. 3.4.4. $[sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = \mathbb{J}_a(s,\varsigma) \Rightarrow \mathbb{J}^a(s,\varsigma) = \frac{2s}{2s+1}\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\mathbb{J}^b(s,\varsigma)$

Proof: $[sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = \mathbb{J}_a(s,\varsigma)$
$\Leftrightarrow 2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)D^b\psi(s,\varsigma) = \mathbb{J}_a(s,\varsigma)$
$\Rightarrow Z_b(s,\varsigma)D^b\psi(s,\varsigma) = \frac{1}{2s+1}Z^a(s,\varsigma)\mathbb{J}_a(s,\varsigma)$
$\Rightarrow 2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)D^b\psi(s,\varsigma) = \frac{1}{2s+1}2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\mathbb{J}^b(s,\varsigma)$
$\Rightarrow \mathbb{J}^a(s,\varsigma) = \frac{2s}{2s+1}\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\mathbb{J}^b(s,\varsigma)$ □

Cor. 3.4.5. $[sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = \mathbb{J}_a(s,\varsigma) \Leftrightarrow \mathbb{J}^a(s,\varsigma) \neq \frac{2s}{2s+1}\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\mathbb{J}^b(s,\varsigma)$

3.4.2 Equivalence between different order spin equations (It needs to be improved).

Thm. 3.4.2. $[sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)$
$\Leftrightarrow [(s-l)D_a + iS_{ab}(s-l,\varsigma)D^b]\psi^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2l}}(s-l,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s-l,\varsigma)\tilde{J}^{\overbrace{A_\varsigma B_\varsigma\cdots}^{2l}}(s-l,\varsigma)$
$l = 0, \frac{1}{2}, 1, \cdots, s$ + *Symmetry condition.*

3.4.3 Properties of source $\mathbb{J}_a(s,\varsigma)$

Cor. 3.4.6. $\mathbb{J}^a(s,\varsigma) = \frac{2s}{2s+1}\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\mathbb{J}^b(s,\varsigma) \Leftrightarrow \exists\tilde{J}(s,\varsigma), \mathbb{J}_a(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)$

Cor. 3.4.7. $\mathbb{J}^a(s,\varsigma) = \frac{2s}{2s+1}\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\mathbb{J}^b(s,\varsigma) \Leftrightarrow \mathbb{J}_a(s,\varsigma) = \frac{1}{s+1}iS_{ab}(s,\varsigma)\mathbb{J}^b(s,\varsigma)$

Cor. 3.4.8. $\mathbb{J}_a(s,\varsigma) = \frac{1}{s+1}iS_{ab}(s,\varsigma)\mathbb{J}^b(s,\varsigma) \Leftrightarrow \begin{cases}(s+1)\mathbb{J} = -i\sigma(s)\times\mathbb{J} - i\varsigma\sigma(s)\mathbb{J}_\pi \\ \sigma(s)\cdot\mathbb{J} + i\varsigma(s+1)\mathbb{J}_\pi = 0\end{cases}$

Pro. 3.4.1. $\sigma(s)\cdot[\sigma(s)\times\mathbb{J}] = i\sigma(s)\cdot\mathbb{J}$

Cor. 3.4.9. $(s+1)\mathbb{J} = -i\sigma(s)\times\mathbb{J} - i\varsigma\sigma(s)\mathbb{J}_\pi \Rightarrow \sigma(s)\cdot\mathbb{J} + i\varsigma(s+1)\mathbb{J}_\pi = 0$

Cor. 3.4.10. $\mathbb{J}_a(s,\varsigma) = \frac{1}{s+1}iS_{ab}(s,\varsigma)\mathbb{J}^b(s,\varsigma) \Leftrightarrow (s+1)\mathbb{J} = -i\sigma(s)\times\mathbb{J} - i\varsigma\sigma(s)\mathbb{J}_\pi$

3.4.4 Helicity of massless s-spin particles

Def. 3.4.1. *Helicity of massless s-spin particles:* $\mathscr{P}(s) := \frac{\sigma(s)\cdot\vec{p}}{|\vec{p}|}$

Cor. 3.4.11. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma) = 0 \Rightarrow [\sigma(s), -is\varsigma]^a\partial_a\varphi(s,\varsigma) = 0, \partial^a\partial_a\varphi(s,\varsigma) = 0$

Cor. 3.4.12. $\begin{cases}(\vec{p}^2 - E^2)\psi(s,\varsigma) = 0\\ \sigma(s)\cdot\vec{p}\psi(s,\varsigma) = -s\varsigma E\psi(s,\varsigma)\end{cases} \Rightarrow \mathscr{P}(s)\psi(s,\varsigma) = \frac{\sigma(s)\cdot\vec{p}}{|\vec{p}|}\psi(s,\varsigma) = \begin{cases}-s\varsigma\psi(s,\varsigma), E = |\vec{p}|\\ s\varsigma\psi(s,\varsigma), E = -|\vec{p}|\end{cases}$

From the above, the eigenvalue of the helicity of a massless s-spin particle can only be $\pm s$ and no other values.

3.5 Spin equation of s-spin particles in even dimensional space-time

Penrose equation with full symmetry in even dimensional space-time [1,2]??

3.5.1 s-spin equation in even dimensional space-time

Lem. 3.5.1. $?(\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}(\Gamma, i\varsigma)^a_{B_\varsigma B'_\varsigma} = 2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}$

Lem. 3.5.2. $(\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;n)N^{B_\varsigma m_\varsigma}_{k_\varsigma}(s;n)(\Gamma, i\varsigma)^a_{B_\varsigma B'_\varsigma} = 2(1+\frac{n}{2s})\delta^{A'_\varsigma}_{B'_\varsigma}\delta^{m_\varsigma}_{l_\varsigma}$

Thm. 3.5.1. $\begin{cases}\nabla^{A'_\varsigma A_\varsigma}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2s}}\\ J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}} = \frac{1}{(2s-1)!}J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma\cdots)}_{2s-1}}\end{cases} \Leftrightarrow [sD_a + iS_{ab}(s,\varsigma;n)D^b]\psi(s,\varsigma;n) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s;n)$

Proof: $\nabla^{A'_\varsigma A_\varsigma}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{-\varsigma}{\sqrt{2}}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}, \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = \frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2s}}, J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}} = \frac{1}{(2s-1)!}J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma\cdots)}_{2s-1}}$

$\Leftrightarrow (\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}} = iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$

$\Leftrightarrow (\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s;n)D^a\psi_{k_\varsigma}(s;n) = iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$

$\Leftrightarrow (\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;n)\Gamma^{l_\varsigma}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2};n)D^a\psi_{k_\varsigma}(s;n) = iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$

$\Leftrightarrow (\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;n)D^a\psi_{k_\varsigma}(s;n) = iJ^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2};n)$

$\Leftrightarrow N^{Z_\varsigma l_\varsigma}_{j_\varsigma}(s;n)(\Gamma, i\varsigma)_{a\,Z_\varsigma A'_\varsigma}(\Gamma, -i\varsigma)_b^{A'_\varsigma A_\varsigma}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;n)D^b\psi_{k_\varsigma}(s;n) = iN^{Z_\varsigma l_\varsigma}_{j_\varsigma}(s;n)(\Gamma, i\varsigma)_{a\,Z_\varsigma A'_\varsigma}J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2};n)$

$\Leftrightarrow N^{Z_\varsigma l_\varsigma}_{j_\varsigma}(s;n)[\delta_{ab}\delta_{Z_\varsigma}{}^{A_\varsigma} + 2iS_{abZ_\varsigma}{}^{A_\varsigma}(\frac{1}{2},\varsigma)]N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s;n)D^b\psi_{k_\varsigma}(s;n) = iN^{Z_\varsigma l_\varsigma}_{j_\varsigma}(s;n)(\Gamma, i\varsigma)_{a\,Z_\varsigma A'_\varsigma}J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2};n)$

$\Leftrightarrow [s\delta_{ab}\delta_{j_\varsigma}{}^{k_\varsigma} + iS_{abj_\varsigma}{}^{k_\varsigma}(s,\varsigma)]D^b\psi_{k_\varsigma}(s;n) = is(\Gamma, i\varsigma)_{a\,A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{j_\varsigma}(s;n)J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2};n)$

$\Leftrightarrow [s\delta_{ab}\delta_{j_\varsigma}{}^{k_\varsigma} + iS_{abj_\varsigma}{}^{k_\varsigma}(s,\varsigma)]D^b\psi_{k_\varsigma}(s;n) = is\delta_{ab}(\Gamma, i\varsigma)^b_{A_\varsigma A'_\varsigma}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{j_\varsigma}(s;n)J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}$

$\Leftrightarrow [s\delta_{ab}\delta_{j_\varsigma}{}^{k_\varsigma} + iS_{abj_\varsigma}{}^{k_\varsigma}(s,\varsigma)]D^b\psi_{k_\varsigma}(s;n) = -\sqrt{2}\varsigma sZ^{al_\varsigma}_{A'_\varsigma j_\varsigma}(s,\varsigma;n)J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2};n)$

$\Leftrightarrow \begin{cases}[sD_a + iS_{ab}(s,\varsigma;n)D^b]\psi(s,\varsigma;n) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma;n)\tilde{J}(s;n)\\ \psi(s,\varsigma) \prec \psi_{k_\varsigma}(s;n), \tilde{J}(s;n) \prec J^{A'_\varsigma}{}_{l_\varsigma}(s-\frac{1}{2};n)\end{cases}$ □

From the above, it can be seen that the s-spin equation is the spin tensor expression of the fully symmetric Penrose equation.

3.6 Generalized spin equation

Thm. 3.6.1. $(\sigma\langle w\rangle \otimes I_{(w+1)^{2s-1}}, -i\varsigma)^a D_a\hat{\psi}(s,\varsigma;w) = i\hat{J}(s,\varsigma;w)$
$\Leftrightarrow (\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, -i\varsigma)^a D_a\tilde{\psi}(s,\varsigma;w) = i\tilde{J}(s,\varsigma;w)$

Proof: $(\sigma\langle w\rangle \otimes I_{(w+1)^{2s-1}}, -i\varsigma)^a D_a\hat{\psi}(s,\varsigma;w) = i\hat{J}(s,\varsigma;w)$

$\Leftrightarrow (\sigma\langle w\rangle \otimes I_{(w+1)^{2s-1}}, -i\varsigma)^a\Gamma(s;w)D_a\psi(s,\varsigma;w) = i[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{J}(s,\varsigma;w)$

$\Leftrightarrow (\sigma\langle w\rangle \otimes I_{(w+1)^{2s-1}}, -i\varsigma)^a[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]N(s;w)D_a\psi(s,\varsigma;w) = i[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{J}(s,\varsigma;w)$

$\Leftrightarrow [I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)](\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, -i\varsigma)^a N(s;w)D_a\psi(s,\varsigma;w) = i[I_{w+1}\otimes\Gamma(s-\frac{1}{2};w)]\tilde{J}(s,\varsigma;w)$

$\Leftrightarrow (\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, -i\varsigma)^a D_a\tilde{\psi}(s,\varsigma;w) = i\tilde{J}(s,\varsigma;w)$ □

Thm. 3.6.2. $(\sigma\langle w\rangle \otimes I_{(w+1)^{2s-1}}, -i\varsigma)^a D_a \hat{\psi}(s,\varsigma;w) = i\hat{J}(s,\varsigma;w)$

$$\Rightarrow \begin{cases} [sD_a + iS_{ab}(s,\varsigma;w)D^b]\psi(s,\varsigma;w) = is\bar{N}(s;w)(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, i\varsigma)_a \tilde{J}(s,\varsigma;w) \\ [\sigma(s;w), -is\varsigma]_a D^a \psi(s,\varsigma;w) = is\bar{N}(s;w)\tilde{J}(s,\varsigma;w) \end{cases}$$

Proof: $(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, -i\varsigma)_b D^b \tilde{\psi}(s,\varsigma;w) = i\tilde{J}(s,\varsigma;w)$

$\Rightarrow \bar{N}(s;w)(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, i\varsigma)_a(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, -i\varsigma)_b N(s;w)D^b\psi(s,\varsigma;w) = i\bar{N}(s;w)(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, i\varsigma)_a \tilde{J}(s,\varsigma;w)$

$\Leftrightarrow \bar{N}(s;w)[\delta_{ab} + 2iS_{ab}(\frac{1}{2},\varsigma;w) \otimes I_{C^{2s-1}_{2s-1+w}}]N(s;w)D^b\psi(s,\varsigma;w) = i\bar{N}(s;w)(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, i\varsigma)_a \tilde{J}(s,\varsigma;w)$

$\Leftrightarrow [\delta_{ab} + \frac{i}{s}S_{ab}(s,\varsigma;w)]D^b\psi(s,\varsigma;w) = i\bar{N}(s;w)(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, i\varsigma)_a \tilde{J}(s,\varsigma;w)$

$\Leftrightarrow [sD_a + iS_{ab}(s,\varsigma;w)D^b]\psi(s,\varsigma;w) = is\bar{N}(s;w)(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, i\varsigma)_a \tilde{J}(s,\varsigma;w)$ □

Proof: $(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, -i\varsigma)_a D^a \tilde{\psi}(s,\varsigma;w) = i\tilde{J}(s,\varsigma;w)$

$\Rightarrow \bar{N}(s;w)(\sigma\langle w\rangle \otimes I_{C^{2s-1}_{2s-1+w}}, -i\varsigma)_a N(s;w)D^a\psi(s,\varsigma;w) = i\bar{N}(s;w)\tilde{J}(s,\varsigma;w)$

$\Leftrightarrow [\frac{1}{s}\sigma(s;w), -i\varsigma]_a D^a\psi(s,\varsigma;w) = i\bar{N}(s;w)\tilde{J}(s,\varsigma;w)$

$\Leftrightarrow [\sigma(s;w), -is\varsigma]_a D^a\psi(s,\varsigma;w) = is\bar{N}(s;w)\tilde{J}(s,\varsigma;w)$ □

4 Switch spin equation

4.1 Neutrino switch spin equation

Thm. 4.1.1. $[(\frac{1}{2}+\phi)D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = 0$

$$\Leftrightarrow \begin{cases} (\sigma,-i\varsigma)^a D_a\psi(\frac{1}{2},\varsigma) = 0, \phi = 0 \\ \sigma_x D_x\psi(\frac{1}{2},\varsigma) = \sigma_y D_y\psi(\frac{1}{2},\varsigma) = \sigma_z D_z\psi(\frac{1}{2},\varsigma) = -i\varsigma D_\pi\psi(\frac{1}{2},\varsigma), \phi = -2 \\ \psi(\frac{1}{2},\varsigma) = \textit{constant solutions}, \phi \neq 0,-2 \end{cases}$$

Proof: $[(\frac{1}{2}+\phi)D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = 0$

$\Leftrightarrow [\frac{1}{2}D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = -\phi D_a\psi(\frac{1}{2},\varsigma)$

$\Leftrightarrow \sigma_a[\frac{1}{2}D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2},\varsigma) = -(\sigma,-i\varsigma)_a\phi D_a\psi(\frac{1}{2},\varsigma)$

$\Leftrightarrow (\sigma,-i\varsigma)^b D_b\psi(\frac{1}{2},\varsigma) = -2\phi(\sigma,-i\varsigma)_a D_a\psi(\frac{1}{2},\varsigma)$

$\Leftrightarrow (\sigma,-i\varsigma)^a D_a\psi(\frac{1}{2},\varsigma) = -2\phi\sigma_x D_x\psi(\frac{1}{2},\varsigma) = -2\phi\sigma_y D_y\psi(\frac{1}{2},\varsigma) = -2\phi\sigma_z D_z\psi(\frac{1}{2},\varsigma) = -2\phi(-i\varsigma)D_\pi\psi(\frac{1}{2},\varsigma)$

$$\Leftrightarrow \begin{cases} (\sigma,-i\varsigma)^a D_a\psi(\frac{1}{2},\varsigma) = 0, \phi = 0 \\ \sigma_x D_x\psi(\frac{1}{2},\varsigma) = \sigma_y D_y\psi(\frac{1}{2},\varsigma) = \sigma_z D_z\psi(\frac{1}{2},\varsigma) = -i\varsigma D_\pi\psi(\frac{1}{2},\varsigma), \phi = -2 \\ D_a\psi(\frac{1}{2},\varsigma) = 0, \phi \neq 0,-2 \end{cases}$$ □

Cor. 4.1.1. $[(\frac{1}{2}+\phi)\partial_a + iS_{ab}(\varsigma)\partial^b]\psi(\frac{1}{2},\varsigma) = 0$

$$\Leftrightarrow \begin{cases} (\sigma,-i\varsigma)^a \partial_a\psi(\frac{1}{2},\varsigma) = 0, \phi = 0 \\ \sigma_x \partial_x\psi(\frac{1}{2},\varsigma) = \sigma_y \partial_y\psi(\frac{1}{2},\varsigma) = \sigma_z \partial_z\psi(\frac{1}{2},\varsigma) = -i\varsigma \partial_\pi\psi(\frac{1}{2},\varsigma), \phi = -2 \\ \psi(\frac{1}{2},\varsigma) = \textit{constant solutions}, \phi \neq 0,-2 \end{cases}$$

Cor. 4.1.2. $\sigma_x\partial_x\psi(\frac{1}{2},\varsigma) = \sigma_y\partial_y\psi(\frac{1}{2},\varsigma) = \sigma_z\partial_z\psi(\frac{1}{2},\varsigma) = -i\varsigma\partial_\pi\psi(\frac{1}{2},\varsigma)$

$\Rightarrow \psi(\frac{1}{2},\varsigma) = \omega_0 + (x\sigma_x + y\sigma_y + z\sigma_z + i\varsigma\pi)\pi_0 \Leftrightarrow \psi_{A_\varsigma}(\frac{1}{2},\varsigma) = \omega_{A_\varsigma} + x_a(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\pi^{A'_\varsigma}$

$\Leftrightarrow (\sigma^*,i\varsigma)^a_{A'_\varsigma(A_\varsigma}\partial_a\omega_{B_\varsigma)} = 0$

The above conclusion is the projection relation of Penrose torsion [2,3].

4.2 Switch spin equation of electromagnetic field without sources

Thm. 4.2.1. $[(1+\phi)D_a + iS_{ab}D^b]^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = 0, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\psi}(1,\varsigma) = 0, \phi = 0 \\ \begin{cases} -D_y\Psi_{z_\varsigma} = D_z\Psi_{y_\varsigma} = \varsigma D_\pi\Psi_{x_\varsigma}, -D_z\Psi_{x_\varsigma} = D_x\Psi_{z_\varsigma} = \varsigma D_\pi\Psi_{y_\varsigma} \\ -D_x\Psi_{y_\varsigma} = D_y\Psi_{x_\varsigma} = \varsigma D_\pi\Psi_{z_\varsigma}, D_x\Psi_{x_\varsigma} = D_y\Psi_{y_\varsigma} = D_z\Psi_{z_\varsigma} \end{cases}, \phi = -3 \\ D_a\Psi_{b_\varsigma} = 0, \phi \neq 0,-3 \end{cases}$$

Proof: $[(1+\phi)D_a + iS_{ab}D^b]^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = 0, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$\Leftrightarrow (D_a + iS_{ab}D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = -\phi D_a\Psi^{\beta_\varsigma}(1,\varsigma)$

$\Leftrightarrow (\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\psi}(1,\varsigma) = i\tilde{\mathcal{J}}(1,\varsigma), -\phi D_a\Psi^{\beta_\varsigma}(1,\varsigma) = -i\sigma^{\beta_\varsigma}_{\varsigma ab}J^b$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma},-i\varsigma)^a D_a\tilde{\psi}(1,\varsigma) = 0, \phi = 0 \\ \begin{cases} -D_y\Psi_{z_\varsigma} = D_z\Psi_{y_\varsigma} = \varsigma D_\pi\Psi_{x_\varsigma}, -D_z\Psi_{x_\varsigma} = D_x\Psi_{z_\varsigma} = \varsigma D_\pi\Psi_{y_\varsigma} \\ -D_x\Psi_{y_\varsigma} = D_y\Psi_{x_\varsigma} = \varsigma D_\pi\Psi_{z_\varsigma}, D_x\Psi_{x_\varsigma} = D_y\Psi_{y_\varsigma} = D_z\Psi_{z_\varsigma} \end{cases}, \phi = -3 \\ D_a\Psi_{b_\varsigma} = 0, \phi \neq 0,-3 \end{cases}$$ □

Cor. 4.2.1. $[(1+\phi)\partial_a + iS_{ab}\partial^b]^{\beta_\varsigma}{}_{\gamma_\varsigma}\Psi^{\gamma_\varsigma}(1,\varsigma) = 0, S_{ab} = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a\partial_a\tilde{\psi}(1,\varsigma) = 0, \phi = 0 \\ \begin{cases} -\partial_y\Psi_{z_\varsigma} = \partial_z\Psi_{y_\varsigma} = \varsigma\partial_\pi\Psi_{x_\varsigma}, -\partial_z\Psi_{x_\varsigma} = \partial_x\Psi_{z_\varsigma} = \varsigma\partial_\pi\Psi_{y_\varsigma} \\ -\partial_x\Psi_{y_\varsigma} = \partial_y\Psi_{x_\varsigma} = \varsigma\partial_\pi\Psi_{z_\varsigma}, \partial_x\Psi_{x_\varsigma} = \partial_y\Psi_{y_\varsigma} = \partial_z\Psi_{z_\varsigma} \end{cases}, \phi = -3 \\ \Psi_{\alpha_\varsigma} = constant\ solutions, \phi \neq 0, -3 \end{cases}$$

Cor. 4.2.2. $\begin{cases} -\partial_y\Psi_{z_\varsigma} = \partial_z\Psi_{y_\varsigma} = \varsigma\partial_\pi\Psi_{x_\varsigma}, -\partial_z\Psi_{x_\varsigma} = \partial_x\Psi_{z_\varsigma} = \varsigma\partial_\pi\Psi_{y_\varsigma} \\ -\partial_x\Psi_{y_\varsigma} = \partial_y\Psi_{x_\varsigma} = \varsigma\partial_\pi\Psi_{z_\varsigma}, \partial_x\Psi_{x_\varsigma} = \partial_y\Psi_{y_\varsigma} = \partial_z\Psi_{z_\varsigma} \end{cases} \Rightarrow \Psi^{\alpha_\varsigma}(1,\varsigma) = x^a\sigma^{\alpha_\varsigma}_{\varsigma ab}C^b$

4.3 Vector field spin equation and switch spin equation in any N+1 dimensional space-time

Vector field spin equation in any N+1 dimensional space-time.

Thm. 4.3.1. $(D_a\delta_{cd} + S_{abcd}D^b)A^d = X_{ac} \Leftrightarrow X_{ab} = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$

Proof: $(D_a\delta_{cd} + S_{abcd}D^b)A^d = X_{ac}$
$\Leftrightarrow [D_a\delta_{cd} + (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})D^b]A^d = X_{ac}$
$\Leftrightarrow D_aA_c + \delta_{ac}D_bA^b - D_cA_a = X_{ac}$
$\Leftrightarrow D_aA_b - D_bA_a + \delta_{ab}D_cA^c = X_{ab}$
$\Leftrightarrow X_{ab} = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$ □

Cor. 4.3.1. $(D_a\delta_{cd} + S_{abcd}D^b)A^d = 0 \Leftrightarrow D_aA_b - D_bA_a = 0, D_aA^a = 0$

Cor. 4.3.2. $(\partial_a\delta_{cd} + S_{abcd}\partial^b)A^d = 0 \Leftrightarrow \partial_aA_b - \partial_bA_a = 0, \partial_aA^a = 0 \Leftrightarrow \partial^a\partial_a\phi = 0, A_a = \partial_a\phi$

Vector field switch spin equation without sources in any N+1 dimensional space-time.

Cor. 4.3.3. $[(1+\phi)D_a\delta_{cd} + S_{abcd}\partial^b]A^d = 0 \Leftrightarrow \begin{cases} D_aA_b - D_bA_a = 0, D_aA^a = 0, \phi = 0 \\ D_aA_b + D_bA_a = 0, \phi = -2 \\ D_aA_{b\neq a} = 0, D_xA_x = D_yA_y = D_zA_z = D_\pi A_\pi, \phi = -4 \\ D_aA_b = 0, \phi \neq 0, -2, -4 \end{cases}$

Proof: $[(1+\phi)D_a\delta_{cd} + S_{abcd}D^b]A^d = 0$
$\Leftrightarrow (D_a\delta_{cd} + S_{abcd}D^b)A^d = -\phi D_aA_c$
$\Leftrightarrow -\phi D_aA_b = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$
$\Leftrightarrow -\phi D_aA_a = D_cA^c, -\phi(D_aA_{b\neq a} + D_bA_{a\neq b}) = 0, (2+\phi)(D_aA_b - D_bA_a) = 0$

$$\Leftrightarrow \begin{cases} -\phi D_aA_a = D_cA^c, (4+\phi)D_aA^a = 0 \\ -\phi(D_aA_{b\neq a} + D_bA_{a\neq b}) = 0, (2+\phi)(D_aA_b - D_bA_a) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} D_aA_b - D_bA_a = 0, D_aA^a = 0, \phi = 0 \\ D_aA_b + D_bA_a = 0, \phi = -2 \\ D_aA_{b\neq a} = 0, D_xA_x = D_yA_y = D_zA_z = D_\pi A_\pi, \phi = -4 \\ D_aA_b = 0, \phi \neq 0, -2, -4 \end{cases}$$ □

Cor. 4.3.4. $[(1+\phi)\partial_a\delta_{cd} + S_{abcd}\partial^b]A^d = 0 \Leftrightarrow \begin{cases} \partial_aA_b - \partial_bA_a = 0, \partial_aA^a = 0, \phi = 0 \\ \partial_aA_b + \partial_bA_a = 0, \phi = -2 \\ \partial_aA_{b\neq a} = 0, \partial_xA_x = \partial_yA_y = \partial_zA_z = \partial_\pi A_\pi, \phi = -4 \\ A_a = constant\ solutions, \phi \neq 0, -2, -4 \end{cases}$

Cor. 4.3.5. $\partial_aA_{b\neq a} = 0, \partial_xA_x = \partial_yA_y = \partial_zA_z = \partial_\pi A_\pi \Rightarrow A_a = kx_a$

4.4 The source of scalar field in any N+1 dimensional space-time

The source of scalar field:

Cor. 4.4.1. $(\partial_a\delta_{cd} + S_{abcd}\partial^b)\partial^d\phi = m^2\phi\delta_{ac} \Leftrightarrow (\partial^a\partial_a - m^2)\phi = 0$

Proof: $(\partial_a\delta_{cd} + S_{abcd}\partial^b)\partial^d\phi = m^2\phi\delta_{ac}$
$\Leftrightarrow [\partial_a\delta_{cd} + (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\partial^b]\partial^d\phi = m^2\phi\delta_{ac}$
$\Leftrightarrow (\partial^b\partial_b - m^2)\delta_{ac}\phi = 0$
$\Leftrightarrow (\partial^a\partial_a - m^2)\phi = 0$ □

4.5 Switch electron spin equation in any N+1 dimensional space-time

Switch electron spin equation in any N+1 dimensional space-time.

Thm. 4.5.1. $[(\frac{1}{2}+\phi)(D_a+m\gamma_a)+iS_{ab}D^b]\psi=0, S_{ab}=-\frac{i}{4}[\gamma_a,\gamma_b] \Leftrightarrow (\gamma^a D_a+m)\psi=-2\phi\gamma_b D_b\psi$

Proof: $[(\frac{1}{2}+\phi)(D_a+m\gamma_a)+iS_{ab}D^b]\psi=0, S_{ab}=-\frac{i}{4}[\gamma_a,\gamma_b]$
$\Leftrightarrow [\frac{1}{2}(D_a+m\gamma_a)+iS_{ab}D^b]\psi=-\phi D_a\psi, S_{ab}=-\frac{i}{4}[\gamma_a,\gamma_b]$
$\Leftrightarrow [(2iS_{ab}+\delta_{ab})D_b+\gamma_a m]\psi=-2\phi D_a\psi, S_{ab}=-\frac{i}{4}[\gamma_a,\gamma_b]$
$\Leftrightarrow [\frac{1}{2}([\gamma_a,\gamma_b]+\{\gamma_a,\gamma_b\})D_b+\gamma_a m]\psi=-2\phi D_a\psi$
$\Leftrightarrow \gamma_a(\gamma_b D^b+m)\psi=-2\phi D_a\psi$
$\Leftrightarrow (\gamma_b D^b+m)\psi=-2\phi\gamma_a D_a\psi$
$\Leftrightarrow (\gamma^a D_a+m)\psi=-2\phi\gamma_b D_b\psi$ □

Cor. 4.5.1. $(\gamma^a D_a+m)\psi=-2\phi\gamma_b D_b\psi, \phi\neq 0$

$$\Leftrightarrow\begin{cases}\psi=0,\phi=-\frac{n}{2},m\neq 0\\ \gamma_1 D_{x_1}\psi=\gamma_2 D_{x_2}\psi=\cdots=\gamma_n D_{x_n}\psi=-(n+2\phi)^{-1}m\psi,\phi\neq-\frac{n}{2},m\neq 0\\ \gamma_1 D_{x_1}\psi=\gamma_2 D_{x_2}\psi=\cdots=\gamma_n D_{x_n}\psi,\phi=-\frac{n}{2},m=0\\ \gamma_1 D_{x_1}\psi=\gamma_2 D_{x_2}\psi=\cdots=\gamma_n D_{x_n}\psi=0,\phi\neq-\frac{n}{2},m=0\end{cases}$$

Cor. 4.5.2. $(\gamma^a D_a+m)\psi=-2\phi\gamma_b D_b\psi, \phi\neq 0 \Rightarrow \begin{cases}\psi=0,\phi=-\frac{n}{2},m\neq 0\\ \psi=0,\phi\neq-\frac{n}{2},m\neq 0\\ \psi=x^a\gamma_a\lambda,\phi=-\frac{n}{2},m=0\\ \psi=\textit{constant solutions},\phi\neq-\frac{n}{2},m=0\end{cases}$

4.6 Switch spin equation for s-spin particles without sources

Cor. 4.6.1. $[(s+\phi)D_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{J}_a(s,\varsigma)\Rightarrow(2s+1+\phi)Z_a(s,\varsigma)D^a\psi(s,\varsigma)=Z_a(s,\varsigma)\mathbb{J}^a(s,\varsigma)$

Proof: $[(s+\phi)D_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{J}_a(s,\varsigma)$
$\Leftrightarrow [sD_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=\mathbb{J}_a(s,\varsigma)-\phi D_a\psi(s,\varsigma)$
$\Leftrightarrow 2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)D^b\psi(s,\varsigma)=\mathbb{J}_a(s,\varsigma)-\phi D_a\psi(s,\varsigma)$
$\Rightarrow Z_b(s,\varsigma)D^b\psi(s,\varsigma)=Z^a(s,\varsigma)\mathbb{J}_a(s,\varsigma)-\frac{\phi}{2s+1}Z^a(s,\varsigma)D_a\psi(s,\varsigma)$
$\Rightarrow (2s+1+\phi)Z_a(s,\varsigma)D^a\psi(s,\varsigma)=Z_a(s,\varsigma)\mathbb{J}^a(s,\varsigma)$ □

Cor. 4.6.2. $[(s+\phi)D_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=0\Rightarrow(2s+1+\phi)Z_a(s,\varsigma)D^a\psi(s,\varsigma)=0$

Proof: $[(s+\phi)D_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=0$
$\Leftrightarrow [sD_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=-\phi D_a\psi(s,\varsigma)$
$\Leftrightarrow 2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)D^b\psi(s,\varsigma)=-\phi D_a\psi(s,\varsigma)$
$\Rightarrow Z_b(s,\varsigma)D^b\psi(s,\varsigma)=\frac{-\phi}{2s+1}Z^a(s,\varsigma)D_a\psi(s,\varsigma)$
$\Rightarrow (2s+1+\phi)Z_a(s,\varsigma)D^a\psi(s,\varsigma)=0$ □

Cor. 4.6.3. $[(s+\phi)D_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=0\Leftrightarrow\begin{cases}(\sigma\otimes I_{2s},-i\varsigma)^a D_a\tilde{\psi}(s,\varsigma)=0,\phi=0\\ D_a\psi(s,\varsigma)=\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma),\phi=-(2s+1)\\ D_a\psi(s,\varsigma)=0,\phi\neq 0,-(2s+1)\end{cases}$

Cor. 4.6.4. $[-(s+1)D_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=0\Leftrightarrow D_a\psi(s,\varsigma)=\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma),\forall\tilde{J}(s,\varsigma)$

Cor. 4.6.5. $[-(s+1)\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma)=0\Leftarrow\psi(s,\varsigma)=x^a\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)$

Cor. 4.6.6. $[(s+\phi)\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma)=0$

$$\Rightarrow\begin{cases}\textit{When } \phi=0,(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\tilde{\psi}(s,\varsigma)=0 \textit{ has plane wave solutions that characterize the solution of particles.}\\ \textit{When } \phi=-(2s+1),\psi(s,\varsigma)=x^a\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)\\ \textit{ has no plane wave solutions that degenerate into a solution representing space-time.}\\ \textit{When } \phi\neq 0,-(2s+1),\psi(s,\varsigma)=\textit{constant solutions}\\ \textit{Only a constant solution that degenerates into a solution representing the void.}\end{cases}$$

5 New form of spin equation with lower first derivative

5.1 New form of s-spin equation with lower first derivative

Def. 5.1.1. *Spin equation with lower first derivative:* $[s\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}^c(s,\varsigma)$

Thm. 5.1.1. $[s\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}^c(s,\varsigma) \Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a\tilde{\psi}^{ab}(s,\varsigma) = i\tilde{J}^b(s,\varsigma)$

Proof: $[s\delta_{ab}I_{2s+1} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}^c(s,\varsigma)$
$\Leftrightarrow 2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\psi^{bc}(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}^c(s,\varsigma)$
$\Leftrightarrow \mathbb{Z}_b(s,\varsigma)\psi^{bc}(s,\varsigma) = \frac{-\varsigma}{\sqrt{2}}\tilde{J}^c(s,\varsigma)$
$\Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a\tilde{\psi}^{ab}(s,\varsigma) = i\tilde{J}^b(s,\varsigma)$ □

5.2 $\frac{1}{2}$-spin equation with lower first derivative: gravitino equation

Spin $s = \frac{1}{2}$ cases: That is the matrix form of Weyl gravitino equation.

Cor. 5.2.1. $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b\psi^c(\varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)_a\psi^{ab}(\varsigma) = 0, \psi^{ab}(\varsigma) \equiv D^a\psi^b(\varsigma) - D^b\psi^a(\varsigma)$

Cor. 5.2.2. $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b\psi^c(\varsigma) = 0 \Leftrightarrow [\frac{1}{2}\delta_{ab} + iS_{ab}(\varsigma)]\psi^{bc}(\varsigma) = 0, \psi^{bc}(\varsigma) \equiv D^b\psi^c(\varsigma) - D^c\psi^b(\varsigma)$

5.3 1-spin equation with lower first derivative: Einstein equation

Spin $s = 1$ cases: That is the matrix form of Einstein equation.

Cor. 5.3.1. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a\mathcal{F}^{ab}(\varsigma) = i\bar{\mathcal{T}}^b$

Cor. 5.3.2. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow (\sigma \otimes I, -i\varsigma)_a\tilde{\psi}^{ab}(1,\varsigma) = i\tilde{J}^b$
$\tilde{\psi}^{bc}(1,\varsigma) = S^+_{em}(\varsigma)\mathcal{F}^{bc}(\varsigma), \tilde{J}^c = S^+_{em}(\varsigma)\bar{\mathcal{T}}^c$

Cor. 5.3.3. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(1,\varsigma)]\psi^{bc}(1,\varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(1,\varsigma)\tilde{J}^c(1,\varsigma)$
$\psi^{bc}(1,\varsigma) = \bar{N}(1)S^+_{em}(\varsigma)\mathcal{F}^{bc}(\varsigma), \tilde{J}^c = S^+_{em}(\varsigma)\bar{\mathcal{T}}^c$

From the above corollary, the following corollary can be directly obtained through representation transformation, but it can also be proved in the following manner.

Cor. 5.3.4. $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(\gamma,\varsigma)]F^{bc}(2,\varsigma) = -i\sigma^{[\beta_\varsigma]}_{\varsigma ab}\bar{T}^{bc}$

5.4 New form of switch spin equation with lower first derivative

Def. 5.4.1. *Switch spin equation with lower first derivative:* $[(s+\phi)\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = \mathbb{J}_a{}^c(s,\varsigma)$

Cor. 5.4.1. $[(s+\phi)\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = 0 \Rightarrow (2s+1+\phi)Z_a(s,\varsigma)\psi^{ab}(s,\varsigma) = 0$

Proof: $[(s+\phi)\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = 0$
$\Leftrightarrow [s\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = -\phi\psi_a{}^c(s,\varsigma)$
$\Leftrightarrow 2s\bar{Z}_a(s,\varsigma)Z_b(s,\varsigma)\psi^{bc}(s,\varsigma) = -\phi\psi_a{}^c(s,\varsigma)$
$\Rightarrow Z_b(s,\varsigma)\psi^{bc}(s,\varsigma) = \frac{-\phi}{2s+1}Z^a(s,\varsigma)\psi_a{}^c(s,\varsigma)$
$\Rightarrow (2s+1+\phi)Z_a(s,\varsigma)\psi^{ab}(s,\varsigma) = 0$ □

Cor. 5.4.2. $[(s+\phi)\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = 0 \Leftrightarrow \begin{cases} \mathbb{Z}_a(s,\varsigma)\psi^{ab}(s,\varsigma) = 0, \phi = 0 \\ \psi^{ab}(s,\varsigma) = \bar{Z}^a(s,\varsigma)\tilde{J}^b(s,\varsigma), \phi = -(2s+1) \\ \psi^{ab}(s,\varsigma) = 0, \phi \neq 0, -(2s+1) \end{cases}$

Cor. 5.4.3. $\begin{cases} [-(s+1)\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = 0 \\ \psi^{ab}(s,\varsigma) + \psi^{ba}(s,\varsigma) = 0 \end{cases} \Leftrightarrow \begin{cases} \psi^{ab}(s,\varsigma) = \bar{Z}^a(s,\varsigma)\tilde{J}^b(s,\varsigma) \\ \bar{Z}^a(s,\varsigma)\tilde{J}^b(s,\varsigma) + \bar{Z}^b(s,\varsigma)\tilde{J}^a(s,\varsigma) = 0 \end{cases}$

5.5 Comparison of two spin equations

Spin equation with lower first derivative:
$[s\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s,\varsigma)\tilde{J}^c(s,\varsigma) \Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a\tilde{\psi}^{ab}(s,\varsigma) = i\tilde{J}^b(s,\varsigma)$
Spin equation:
$[s\delta_{ab} + iS_{ab}(s,\varsigma)]D^b\psi(s,\varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma) \Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_aD^a\tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma)$

5.6 Comparison of two switch spin equations

Switch spin equation with lower first derivative:
$[(s+\phi)\delta_{ab} + iS_{ab}(s,\varsigma)]\psi^{bc}(s,\varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s,\varsigma)\tilde{J}^c(s,\varsigma)$
Switch spin equation:
$[(s+\phi)\delta_{ab} + iS_{ab}(s,\varsigma)]D^b\psi(s,\varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s,\varsigma)\tilde{J}(s,\varsigma)$

5.7 Guess: a new physical equation

Cor. 5.7.1. $Z_a(s,\varsigma)D^a\psi(s,\varsigma) - m^2\tilde{A}(s,\varsigma) = \tilde{J}(s,\varsigma), \psi(s,\varsigma) = \bar{Z}_a(s,\varsigma)D^a\tilde{A}(s,\varsigma)$

Equation after introducing guage condition:

Cor. 5.7.2. $Z_a(s,\varsigma)D^a\psi(s,\varsigma) - m^2\tilde{A}(s,\varsigma) = \tilde{J}(s,\varsigma), N(s)\psi(s,\varsigma) = (\sigma \otimes I_{2s}, i\varsigma)_aD^a\tilde{A}(s,\varsigma)$

Chapter11 Penrose equation and torsion equation

1 Restatement of fully symmetric Penrose equation with arbitrary spin [1,2]

1.1 Integral spinor equivalent form of fully symmetric Penrose equation with arbitrary spin

Thm. 1.1.1.

$$\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}Z_\varsigma}=iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}Z_\varsigma}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}Z_\varsigma}=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2n}Z_\varsigma}\\ J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}Z_\varsigma}=\frac{1}{(2n-1)!}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}Z_\varsigma}\end{cases}\Leftrightarrow\begin{cases}\sigma^{\alpha_\varsigma}_{\varsigma ab}D^a\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=iJ_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}Z_\varsigma}\\ \psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=\frac{1}{n!}\psi_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}Z_\varsigma}\\ J_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}Z_\varsigma}=\frac{1}{(n-1)!}J_{\underbrace{b(\beta_\varsigma\gamma_\varsigma\cdots)}_{n}Z_\varsigma}\\ \delta^{\alpha_\varsigma\beta_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=0,(\sigma,-i\varsigma)^a\sigma^{\alpha_\varsigma}J_{\underbrace{a\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=0\\ \sigma^{\alpha_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}[Z_\varsigma]}=0,(\sigma,-i\varsigma)^aJ_{\underbrace{a\alpha_\varsigma\beta_\varsigma\cdots}_{n}[Z_\varsigma]}=0\end{cases}$$

Thm. 1.1.2.

$$\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}Z_\varsigma}=iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}Z_\varsigma}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}Z_\varsigma}=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2n}Z_\varsigma}\\ J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}Z_\varsigma}=\frac{1}{(2n-1)!}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}Z_\varsigma}\end{cases}\Leftrightarrow\begin{cases}(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\Psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=iJ_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}Z_\varsigma}\\ \Psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=\frac{1}{n!}\Psi_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}Z_\varsigma}\\ J_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}Z_\varsigma}=\frac{1}{(n-1)!}J_{\underbrace{b(\beta_\varsigma\gamma_\varsigma\cdots)}_{n}Z_\varsigma}\\ \delta^{\alpha_\varsigma\beta_\varsigma}\Psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=0,(\sigma,-i\varsigma)^a(\sigma,i\varsigma)^{\alpha_\varsigma}J_{\underbrace{a\alpha_\varsigma\beta_\varsigma\cdots}_{n}Z_\varsigma}=0\\ (\sigma,-i\varsigma)^{\alpha_\varsigma}\Psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}[Z_\varsigma]}=0,(\sigma,-i\varsigma)^aJ_{\underbrace{a\alpha_\varsigma\beta_\varsigma\cdots}_{n}[Z_\varsigma]}=0\end{cases}$$

Thm. 1.1.3.

$$\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2n}}\\ J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}=\frac{1}{(2n-1)!}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\end{cases}\Leftrightarrow\begin{cases}\sigma^{\alpha_\varsigma}_{\varsigma ab}D^a\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=iJ_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}}\\ \psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=\frac{1}{n!}\psi_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}},J_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}}=\frac{1}{(n-1)!}J_{\underbrace{b(\beta_\varsigma\gamma_\varsigma\cdots)}_{n}}\\ \delta^{\alpha_\varsigma\beta_\varsigma}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=0,(\sigma,-i\varsigma)^a\sigma^{\alpha_\varsigma}J_{\underbrace{a\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=0\end{cases}$$

Thm. 1.1.4.

$$\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2n}}\\ J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}=\frac{1}{(2n-1)!}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\end{cases}\Leftrightarrow\begin{cases}(\sigma_{-\varsigma},-i\varsigma)^a{}_{b\alpha_\varsigma}D_a\Psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=iJ_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}}\\ \Psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=\frac{1}{n!}\Psi_{\underbrace{(\alpha_\varsigma\beta_\varsigma\cdots)}_{n}},J_{\underbrace{b\beta_\varsigma\gamma_\varsigma\cdots}_{n}}=\frac{1}{(n-1)!}J_{\underbrace{b(\beta_\varsigma\gamma_\varsigma\cdots)}_{n}}\\ \delta^{\alpha_\varsigma\beta_\varsigma}\Psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=0,(\sigma,-i\varsigma)^a(\sigma,i\varsigma)^{\alpha_\varsigma}J_{\underbrace{a\alpha_\varsigma\beta_\varsigma\cdots}_{n}}=0\end{cases}$$

1.2 Matrix equivalent form of fully symmetric Penrose equation with arbitrary spin

Thm. 1.2.1.

$$\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2n}},J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}=\frac{1}{(2n-1)!}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\end{cases}\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)^aD_a\hat{\psi}(s,\varsigma)=i\hat{J}(s,\varsigma)$$

The above theorem can be obtained by rewriting components into a matrix.

Thm. 1.2.2. $(\sigma\otimes I_{2^{2s-1}},-i\varsigma)^aD_a\hat{\psi}(s,\varsigma)=i\hat{J}(s,\varsigma)\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)^aD_a\tilde{\psi}(s,\varsigma)=i\tilde{J}(s,\varsigma)$

The above theorem can be obtained by expanding, removing redundant equations and sorting them out.

Cor. 1.2.1.
$$\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2n}}=\frac{1}{(2n)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2n}},J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}=\frac{1}{(2n-1)!}J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2n-1}}\end{cases}\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)^aD_a\tilde{\psi}(s,\varsigma)=i\tilde{J}(s,\varsigma)$$

Thm. 1.2.3. $(\sigma\otimes I_{2^{2n-1}},-i\varsigma)^aD_a\hat{\psi}(n,\varsigma)=i\hat{J}(n,\varsigma)\Leftrightarrow(\sigma_{-\varsigma}\otimes I_{4^{n-1}},-i\varsigma)^aD_a\hat{\Psi}(n,\varsigma)=i\hat{\mathcal{J}}(n,\varsigma)$

The above theorem can be obtained by making a representation transformation.

Thm. 1.2.4. $(\sigma\otimes I_{2n},-i\varsigma)^aD_a\tilde{\psi}(n,\varsigma)=i\tilde{J}(n,\varsigma)\Leftrightarrow(\sigma_{-\varsigma}\otimes I_n,-i\varsigma)^aD_a\tilde{\Psi}(n,\varsigma)=i\tilde{\mathcal{J}}(n,\varsigma)$

For $n=1,2$, the above theorem can be obtained by making a representation transformation. For $n>2$, it needs to be proved later.

1.3 Spin equation equivalent form of fully symmetric Penrose equation with arbitrary spin

Thm. 1.3.1.
$$\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}D^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}=iJ^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}\\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}=\frac{1}{(2s)!}\psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma\cdots)}_{2s}}\\ J^{A'_\varsigma}{}_{\underbrace{B_\varsigma C_\varsigma\cdots}_{2s-1}}=\frac{1}{(2s-1)!}J^{A'_\varsigma}{}_{\underbrace{(B_\varsigma C_\varsigma\cdots)}_{2s-1}}\end{cases}\Leftrightarrow[sD_a+iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma)=-\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s)$$

2 Restatement of torsion equation

2.1 Penrose torsion equation [2,3]

$$\nabla_{A'_\varsigma(A_\varsigma}\omega_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2s}}(s)=0,\nabla_{A'_\varsigma(A_\varsigma}\omega_{B_\varsigma)}(\tfrac{1}{2})=0\tag{11.1}$$

Cor. 2.1.1. $\nabla_{A'_\varsigma(A_\varsigma}\omega_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2s}}(s)=0\Leftrightarrow(\sigma^*,i\varsigma)^a_{A'_\varsigma(A_\varsigma}\partial_a\omega_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2s}}(s)=0$

Cor. 2.1.2. $\nabla_{A'_\varsigma(A_\varsigma}\omega_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2s}}(s)=0\Leftrightarrow(\sigma^*,i\varsigma)^a_{A'_\varsigma(A_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2s}}(s)\partial_a\omega_{k_\varsigma}(s)=0$

2.2 Equivalent form of similar torsion equation

$$\nabla_{A'_\varsigma(A_\varsigma}\omega_{\underbrace{B_\varsigma)C_\varsigma D_\varsigma\cdots}_{2s}}(s)=0,\omega_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2s}}(s)=\frac{1}{(2s)!}\omega_{\underbrace{(B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2s}}(s)\tag{11.2}$$

Cor. 2.2.1. $\nabla_{A'_\varsigma(A_\varsigma}\omega_{\underbrace{B_\varsigma)C_\varsigma D_\varsigma\cdots}_{2s}}(s)=0,\omega_{\underbrace{B_\varsigma C_\varsigma D_\varsigma\cdots}_{2s}}(s)=\frac{1}{(2s)!}\omega_{\underbrace{(B_\varsigma C_\varsigma D_\varsigma\cdots)}_{2s}}(s)\Leftrightarrow\nabla_{A'_\varsigma(A_\varsigma}N^{k_\varsigma}_{B_\varsigma)l_\varsigma}(s)\omega_{k_\varsigma}(s)=0$

Cor. 2.2.2. $\nabla_{A'_\varsigma(A_\varsigma}N^{k_\varsigma}_{B_\varsigma)l_\varsigma}(s)\omega_{k_\varsigma}(s)=0\Leftrightarrow(\sigma^*,i\varsigma)^a_{A'_\varsigma(A_\varsigma}N^{k_\varsigma}_{B_\varsigma)l_\varsigma}(s)\partial^a\psi_{k_\varsigma}(s)=0$

Cor. 2.2.3. $\nabla_{A'_\varsigma(A_\varsigma}N^{k_\varsigma}_{B_\varsigma)l_\varsigma}(s)\omega_{k_\varsigma}(s)=0\Leftrightarrow[-(s+1)\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0$

Proof: $\nabla_{A'_\varsigma(A_\varsigma}N^{k_\varsigma}_{B_\varsigma)l_\varsigma}(s)\omega_{k_\varsigma}(s)=0$
$\Leftrightarrow\partial^a\omega_{k_\varsigma}(s)=(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)\mathring{\pi}^{A'_\varsigma}_{l_\varsigma}(s),\forall\mathring{\pi}^{A'_\varsigma}_{l_\varsigma}(s)$
$\Leftrightarrow\partial^a\omega_{k_\varsigma}(s)=(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)\mathring{\pi}^{A'_\varsigma}_{l_\varsigma}(s),\mathring{\pi}^{A'_\varsigma}_{l_\varsigma}(s)=\frac{s}{2s+1}N^{k_\varsigma}_{A_\varsigma l_\varsigma}(s)(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\omega_{k_\varsigma}(s)$
$\Leftrightarrow\partial^a\omega_{k_\varsigma}(s)=(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)\frac{s}{2s+1}N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s)(\sigma,-i\varsigma)_b^{A'_\varsigma B_\varsigma}\partial^b\omega_{m_\varsigma}(s)$
$\Leftrightarrow(2s+1)\partial^a\omega^{k_\varsigma}(s)=N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)s(\delta_{ab}\delta_{A_\varsigma}{}^{B_\varsigma}+2iS_{abA_\varsigma}{}^{B_\varsigma})N^{m_\varsigma}_{B_\varsigma l_\varsigma}(s)\partial^b\omega_{m_\varsigma}(s)$
$\Leftrightarrow(s+1)\partial^a\omega^{k_\varsigma}(s)=iS_{abk_\varsigma}{}^{m_\varsigma}(s)\partial^b\omega_{m_\varsigma}(s)$
$\Leftrightarrow[-(s+1)\partial_a+iS_{ab}(s,\varsigma)\partial^b]\omega(s,\varsigma)=0$ □

Cor. 2.2.4. $\nabla_{A'_\varsigma(A_\varsigma}\omega_{B_\varsigma)}(\frac{1}{2})=0\Leftrightarrow[-\frac{3}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\omega(\frac{1}{2})=0$

2.3 Solution of similar torsion equation

Cor. 2.3.1. $\nabla_{A'_\varsigma(A_\varsigma}N^{k_\varsigma}_{B_\varsigma)l_\varsigma}(s)\omega_{k_\varsigma}(s)=0\Leftrightarrow\omega_{k_\varsigma}(s)=\mathring{\omega}_{k_\varsigma}(s)+x_a(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}N^{A_\varsigma l_\varsigma}_{k_\varsigma}(s)\mathring{\pi}^{A'_\varsigma}_{l_\varsigma}(s)$

Cor. 2.3.2. $\nabla_{A'_\varsigma(A_\varsigma}\omega_{B_\varsigma)}(\frac{1}{2})=0\Leftrightarrow\omega_{A_\varsigma}(\frac{1}{2})=\mathring{\omega}_{A_\varsigma}(\frac{1}{2})+x_a(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\mathring{\pi}^{A'_\varsigma}(\frac{1}{2})$

2.4 Relation between switch spin equation and similar torsion equation

Cor. 2.4.1.

$$[(s+\phi)\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma)=0\Rightarrow\begin{cases}\textit{Particles solution: }(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\tilde{\psi}(s,\varsigma)=0,\phi=0\\ \textit{similar torsion solution: }\psi(s,\varsigma)=\mathring{\psi}_0(s,\varsigma)+x^a\bar{Z}_a(s,\varsigma)\tilde{J}_0(s,\varsigma),\phi=-(2s+1)\\ \textit{Vacuum solution: }\psi(s,\varsigma)=constant,\phi\neq 0,-(2s+1)\end{cases}$$

Chapter12 Analysis of Bargmann-Wigner equation

1 Bargmann-Wigner equation

1.1 Bargmann-Wigner equation [16]

$$[\gamma^a(\varsigma)D_a + m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\zeta_\varsigma}_{2s}} = J_{\underbrace{\kappa_\varsigma\mu_\varsigma\cdot\cdot\zeta_\varsigma}_{2s}};\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\zeta_\varsigma}_{2s}}, J_{\underbrace{\kappa_\varsigma\mu_\varsigma\cdot\cdot\zeta_\varsigma}_{2s}} \text{ are fully symmetric except } \kappa_\varsigma. \quad (12.1)$$

2 Complete expansion of second order matrices

2.1 Complete Pauli basis of second order matrices

Complete Pauli basis of second order matrices: $\Gamma_a(\varsigma) = \{\sigma, i\varsigma\}$

Pro. 2.1.1. $x^a\Gamma_a(\varsigma) = 0 \Rightarrow x^a = 0$

Proof: $x^a\Gamma_a(\varsigma) = 0$
$\Rightarrow x^a(\sigma, i\varsigma)_a = 0$
$\Rightarrow \{x^a(\sigma, i\varsigma)_a, (\sigma, -i\varsigma)_b\} = 0$
$\Rightarrow x^a(2\delta_{ab}) = 0$
$\Rightarrow x^a = 0$ □

Cor. 2.1.1. $x^a\Gamma_a(\varsigma) = 0 \Leftrightarrow x^a = 0$

Pro. 2.1.2. $X = \frac{1}{2}tr[\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma), \forall X \in$ *second order matrices*

Proof: $X = X^{11}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + X^{12}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + X^{21}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + X^{22}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \forall X \in$ second order matrices
$\Leftrightarrow X = \frac{1}{2}[X^{11}(I + \sigma_z) + X^{12}(\sigma_x + i\sigma_y) + X^{21}(\sigma_x - i\sigma_y) + X^{22}(I - \sigma_z)], \forall X \in$ second order matrices
$\Leftrightarrow X = \frac{1}{2}(X^{12} + X^{21})\sigma_x + \frac{i}{2}(X^{12} - X^{21})\sigma_y + \frac{1}{2}(X^{11} - X^{22})\sigma_z - i\varsigma\frac{1}{2}(X^{11} + X^{22})i\varsigma I, \forall X \in$ second order matrices
$\Leftrightarrow X = \frac{1}{2}tr[\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma), \forall X \in$ second order matrices □

Cor. 2.1.2. $X = x^a\Gamma_a(\varsigma), x^a = tr[\Gamma^a(-\varsigma)X], \forall X \in$ *second order matrices*

Complete basis properties of second order matrices:

Orthogonality: $\Gamma_a(-\varsigma)\Gamma_a(\varsigma) = I, tr[\Gamma_a(-\varsigma)\Gamma_b(\varsigma)] = 2\delta_{ab}$ (12.2)

Linear independence: $x^a\Gamma_a(\varsigma) = 0 \Leftrightarrow x^a = 0$ (12.3)

Completeness: $X = x^a\Gamma_a, \forall X \in$ **second order matrices** (12.4)

Expand Uniqueness: $X = x^a\Gamma_a \Leftrightarrow x^a = \frac{1}{2}tr[\Gamma_a(-\varsigma)X], \forall X \in$ **second order matrices** (12.5)

2.2 Symmetric and antisymmetric basis expansion of second order matrices

Symmetric and antisymmetric basis of second order matrices: $\Gamma_a(\varsigma)\varepsilon = \{\sigma, i\varsigma\}\varepsilon$**,** $[\Gamma_a(\varsigma)\varepsilon]^T = \{\sigma, -i\varsigma\}\varepsilon$**,** $\sigma\varepsilon$ **is a symmetric basis.** $i\varsigma\varepsilon$ **is an antisymmetric basis.**

Pro. 2.2.1. $x^a\Gamma_a(\varsigma)\varepsilon = 0 \Leftrightarrow x^a = 0$

Pro. 2.2.2. $X = \frac{1}{2}tr[\bar{\varepsilon}\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma)\varepsilon, \forall X \in$ *second order matrices*

Proof: $X\bar{\varepsilon} = \frac{1}{2}tr[\Gamma^a(-\varsigma)X\bar{\varepsilon}]\Gamma_a(\varsigma), \forall X \in$ second order matrices
$\Leftrightarrow X = \frac{1}{2}tr[\Gamma^a(-\varsigma)X\bar{\varepsilon}]\Gamma_a(\varsigma)\varepsilon, \forall X \in$ second order matrices
$\Leftrightarrow X = \frac{1}{2}tr[\varepsilon\bar{\varepsilon}\Gamma^a(-\varsigma)X\bar{\varepsilon}]\Gamma_a(\varsigma)\varepsilon, \forall X \in$ second order matrices
$\Leftrightarrow X = \frac{1}{2}tr[\bar{\varepsilon}\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma)\varepsilon, \forall X \in$ second order matrices □

3 Complete expansion of fourth order matrices

3.1 Double Pauli basis expansions of fourth order matrices

Pro. 3.1.1. $X = \frac{1}{4}tr[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X]\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$

Proof: $X = \frac{1}{2}\begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}]\Gamma_a(\varsigma) & tr[\Gamma^a(-\varsigma)X_{12}]\Gamma_a(\varsigma) \\ tr[\Gamma^a(-\varsigma)X_{21}]\Gamma_a(\varsigma) & tr[\Gamma^a(-\varsigma)X_{22}]\Gamma_a(\varsigma) \end{bmatrix}, \forall X$
$\Leftrightarrow X = \frac{1}{2}\Gamma_a(\varsigma) \otimes \begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}] & tr[\Gamma^a(-\varsigma)X_{12}] \\ tr[\Gamma^a(-\varsigma)X_{21}] & tr[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}, \forall X$
$\Leftrightarrow X = \frac{1}{4}tr\{\Gamma^b(-\varsigma)\begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}] & tr[\Gamma^a(-\varsigma)X_{12}] \\ tr[\Gamma^a(-\varsigma)X_{21}] & tr[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}\}\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$
$\Leftrightarrow X = \frac{1}{4}tr[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X]\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$ □

Cor. 3.1.1. $tr\{\Gamma^b(-\varsigma)\begin{bmatrix} tr[\Gamma^a(-\varsigma)X_{11}] & tr[\Gamma^a(-\varsigma)X_{12}] \\ tr[\Gamma^a(-\varsigma)X_{21}] & tr[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}\} = tr[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X]$

3.2 Charge conjugation matrix C [4, 9]

Def. 3.2.1. $\bar{C}\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = -C, C^+ = \bar{C}$

Cor. 3.2.1. $\gamma_a(\varsigma)C = [\gamma_a(\varsigma)C]^T$

Proof: $\gamma_a(\varsigma)C = C\bar{C}\gamma_a(\varsigma)C = -C\gamma_a^T(\varsigma) = C^T\gamma_a^T(\varsigma) = [\gamma_a(\varsigma)C]^T$ □

Cor. 3.2.2. $\bar{C}\gamma_a(\varsigma) = [\bar{C}\gamma_a(\varsigma)]^T$

Proof: $\bar{C}\gamma_a(\varsigma) = \bar{C}\gamma_a(\varsigma)C\bar{C} = -\gamma_a^T(\varsigma)\bar{C} = -[C^*\gamma_a(\varsigma)]^T = [\bar{C}\gamma_a(\varsigma)]^T$ □

Cor. 3.2.3. $S_{ab}(e,\varsigma)C = [S_{ab}(e,\varsigma)C]^T$

Proof: $S_{ab}(e,\varsigma)C = -\frac{i}{4}[\gamma_a(\varsigma)\gamma_b(\varsigma) - \gamma_b(\varsigma)\gamma_a(\varsigma)]C$
$= -\frac{i}{4}[C\bar{C}\gamma_a(\varsigma)C\bar{C}\gamma_b(\varsigma)C - C\bar{C}\gamma_b(\varsigma)C\bar{C}\gamma_a(\varsigma)C]$
$= -\frac{i}{4}C[\gamma_a^T(\varsigma)\gamma_b^T(\varsigma) - \gamma_b^T(\varsigma)\gamma_a^T(\varsigma)] = - -\frac{i}{4}C^T[\gamma_b(\varsigma)\gamma_a(\varsigma) - \gamma_a(\varsigma)\gamma_b(\varsigma)]^T$
$= C^T S_{ab}^T(e,\varsigma) = [S_{ab}(e,\varsigma)C]^T$ □

Cor. 3.2.4. $\bar{C}S_{ab}(e,\varsigma) = [\bar{C}S_{ab}(e,\varsigma)]^T$

Proof: $\bar{C}S_{ab}(e,\varsigma) = -\frac{i}{4}\bar{C}[\gamma_a(\varsigma)\gamma_b(\varsigma) - \gamma_b(\varsigma)\gamma_a(\varsigma)]$
$= -\frac{i}{4}[\bar{C}\gamma_a(\varsigma)C\bar{C}\gamma_b(\varsigma)C\bar{C} - \bar{C}\gamma_b(\varsigma)C\bar{C}\gamma_a(\varsigma)C\bar{C}]$
$= -\frac{i}{4}[\gamma_a^T(\varsigma)\gamma_b^T(\varsigma) - \gamma_b^T(\varsigma)\gamma_a^T(\varsigma)]\bar{C} = \frac{i}{4}[\gamma_b(\varsigma)\gamma_a(\varsigma) - \gamma_a(\varsigma)\gamma_b(\varsigma)]^T\bar{C}^T$
$= S_{ab}^T(e,\varsigma)\bar{C}^T = [\bar{C}S_{ab}(e,\varsigma)]^T$ □

Cor. 3.2.5. $\bar{C}\gamma_5(\varsigma)C = \gamma_5^T(\varsigma)$

Proof: $\bar{C}\gamma_5(\varsigma)C = \bar{C}\gamma_x(\varsigma)\gamma_y(\varsigma)\gamma_z(\varsigma)\gamma_\pi(\varsigma)C$
$= \bar{C}\gamma_x(\varsigma)C\bar{C}\gamma_y(\varsigma)C\bar{C}\gamma_z(\varsigma)C\bar{C}\gamma_\pi(\varsigma)C$
$= \gamma_x^T(\varsigma)\gamma_y^T(\varsigma)\gamma_z^T(\varsigma)\gamma_\pi^T(\varsigma) = [\gamma_\pi(\varsigma)\gamma_z(\varsigma)\gamma_y(\varsigma)\gamma_x(\varsigma)]^T = \gamma_5^T(\varsigma)$ □

Cor. 3.2.6. $C = -C^T, \bar{C} = -\bar{C}^T,$

Cor. 3.2.7. $\gamma_5(\varsigma)C = -[\gamma_5(\varsigma)C]^T, \bar{C}\gamma_5(\varsigma) = -[\bar{C}\gamma_5(\varsigma)]^T$

Cor. 3.2.8. $\gamma_5(\varsigma)\gamma_a(\varsigma)C = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T$

Proof: $\gamma_5(\varsigma)\gamma_a(\varsigma)C = C\bar{C}\gamma_5(\varsigma)C\bar{C}\gamma_a(\varsigma)C = -C\gamma_5^T(\varsigma)\gamma_a^T(\varsigma)$
$= C^T\gamma_5^T(\varsigma)\gamma_a^T(\varsigma) = [\gamma_a(\varsigma)\gamma_5(\varsigma)C]^T = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T$ □

Cor. 3.2.9. $\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$

Proof: $\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = \bar{C}\gamma_5(\varsigma)C\bar{C}\gamma_a(\varsigma)C\bar{C} = -\gamma_5^T(\varsigma)\gamma_a^T(\varsigma)\bar{C}$
$= \gamma_5^T(\varsigma)\gamma_a^T(\varsigma)\bar{C}^T = [\bar{C}\gamma_a(\varsigma)\gamma_5(\varsigma)]^T = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$ □

Summary:
Symmetric basis: $\gamma_a(\varsigma)C = [\gamma_a(\varsigma)C]^T, \bar{C}\gamma_a(\varsigma) = [\bar{C}\gamma_a(\varsigma)]^T, S_{ab}(e,\varsigma)C = [S_{ab}(e,\varsigma)C]^T, \bar{C}S_{ab}(e,\varsigma) = [\bar{C}S_{ab}(e,\varsigma)]^T$
Antisymmetric basis: $C = -C^T, \bar{C} = -\bar{C}^T, \gamma_5(\varsigma)C = -[\gamma_5(\varsigma)C]^T, , \bar{C}\gamma_5(\varsigma) = -[\bar{C}\gamma_5(\varsigma)]^T,$
$\gamma_5(\varsigma)\gamma_a(\varsigma)C = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T, \bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$

3.3 Dirac matrix under special representation [4,9]

Take the Dirac matrix under special representation: $[\gamma_a(\varsigma),\gamma_5(\varsigma)]=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z]$

Detailed expansion:

$[\gamma_a(\varsigma),\gamma_5(\varsigma)]=[(\sigma_x\otimes\sigma_y,\sigma_y\otimes\sigma_y,\sigma_z\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z]$

$[\gamma_a(\varsigma),\gamma_5(\varsigma)]\gamma_5(\varsigma)=-i\varsigma[(\sigma_x\otimes\sigma_x,\sigma_y\otimes\sigma_x,\sigma_z\otimes\sigma_x,\varsigma I\otimes\sigma_y),i\varsigma I\otimes I]$

$$S_{ab}(e,\varsigma)=-\frac{i}{4}[\gamma_a(\varsigma),\gamma_b(\varsigma)]=\frac{1}{2}\begin{bmatrix}0&\sigma_z\otimes I&-\sigma_y\otimes I&-\varsigma\sigma_x\otimes\sigma_z\\-\sigma_z\otimes I&0&\sigma_x\otimes I&-\varsigma\sigma_y\otimes\sigma_z\\\sigma_y\otimes I&-\sigma_x\otimes I&0&-\varsigma\sigma_z\otimes\sigma_z\\\varsigma\sigma_x\otimes\sigma_z&\varsigma\sigma_y\otimes\sigma_z&\varsigma\sigma_z\otimes\sigma_z&0\end{bmatrix}$$

Charge conjugate matrix under special representation: $C=\gamma_y(\varsigma)\gamma_\pi(\varsigma)$

3.4 Dirac basis expansion of fourth order matrices [4,9]

Dirac complete basis expansion of fourth order matrices:

$\Gamma_A(\varsigma)=[\gamma_a(\varsigma),2S_{ab}(e,\varsigma),-I_4,-i\gamma_a(\varsigma)\gamma_5(\varsigma),-\gamma_5(\varsigma)]$

Pro. 3.4.1. $X=[im\gamma_a(\varsigma)A^a+S_{ab}(e,\varsigma)F^{ab}]-[I_4\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi],\forall X$

$$\begin{cases}\phi=-\frac{1}{4}trX\\\Phi=-\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases},\begin{cases}imA^a=\frac{1}{4}tr[\gamma^a(\varsigma)X]\\im\mathbf{A}^a=\frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases},F^{ab}=\frac{1}{2}tr[S^{ab}(e,\varsigma)X]$$

Proof: $X=\frac{1}{4}tr[\Gamma^a(-\varsigma)\otimes\Gamma^b(-\varsigma)X]\Gamma_a(\varsigma)\otimes\Gamma_b(\varsigma),\forall X$

$\Leftrightarrow X=[im\gamma_a(\varsigma)A^a-iS_{ab}(e,\varsigma)F^{ab}]-[I_4\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi],\forall X$

$$\begin{cases}im\mathbf{A}^i=\frac{i\varsigma}{4}tr[\Gamma^i(-\varsigma)\otimes\Gamma^x(-\varsigma)X]=\frac{1}{4}tr[\gamma^i(\varsigma)\gamma^5(\varsigma)X]\\imA^\pi=\frac{i}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^x(-\varsigma)X]=\frac{1}{4}tr[\gamma^\pi(\varsigma)X]\end{cases}$$

$$\begin{cases}imA^i=\frac{1}{4}tr[\Gamma^i(-\varsigma)\otimes\Gamma^y(-\varsigma)X]=\frac{1}{4}tr[\gamma^i(\varsigma)X]\\im\mathbf{A}^\pi=\frac{\varsigma}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^y(-\varsigma)X]=\frac{1}{4}tr[\gamma^\pi(\varsigma)\gamma^5(\varsigma)X]\end{cases}$$

$$\begin{cases}F^{i\pi}=-F^{\pi i}=-\frac{\varsigma}{4}tr[\Gamma^i(-\varsigma)\otimes\Gamma^z(-\varsigma)X]=-\frac{i}{2}tr[S^{i\pi}(e,\varsigma)X]\\\Phi=-\frac{i}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^z(-\varsigma)X]=-\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases}$$

$$\begin{cases}F^{yz}=-F^{zy}=\frac{i\varsigma}{4}tr[\Gamma^x(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X]=-\frac{i}{2}tr[S^{yz}(e,\varsigma)X]\\F^{zx}=-F^{xz}=\frac{i\varsigma}{4}tr[\Gamma^y(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X]=-\frac{i}{2}tr[S^{zx}(e,\varsigma)X]\\F^{xy}=-F^{yx}=\frac{i\varsigma}{4}tr[\Gamma^z(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X]=-\frac{i}{2}tr[S^{xy}(e,\varsigma)X]\\\phi=\frac{1}{4}tr[\Gamma^\pi(-\varsigma)\otimes\Gamma^\pi(-\varsigma)X]=-\frac{1}{4}trX\end{cases}$$

$\Leftrightarrow X=[im\gamma_a(\varsigma)A^a+S_{ab}(e,\varsigma)F^{ab}]-[I_4\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi],\forall X$

$$\begin{cases}\phi=-\frac{1}{4}trX\\\Phi=-\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases},\begin{cases}imA^a=\frac{1}{4}tr[\gamma^a(\varsigma)X]\\im\mathbf{A}^a=\frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases},F^{ab}=\frac{1}{2}tr[S^{ab}(e,\varsigma)X]\qquad\square$$

Cor. 3.4.1. $X=[im\gamma_a(\varsigma)A^a+S_{ab}(e,\varsigma)F^{ab}]-[I_4\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi],\forall X$

$\Leftrightarrow X=[im\gamma_a(\varsigma)A^a+S_{ab}(e,\varsigma)F^{ab}]-[I_4\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi],\forall X$

$$\begin{cases}\phi=-\frac{1}{4}trX\\\Phi=-\frac{1}{4}tr[\gamma^5(\varsigma)X]\end{cases},\begin{cases}imA^a=\frac{1}{4}tr[\gamma^a(\varsigma)X]\\im\mathbf{A}^a=\frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases},F^{ab}=\frac{1}{2}tr[S^{ab}(e,\varsigma)X]$$

3.5 Symmetric and antisymmetric basis expansion of fourth order matrices

Symmetric and antisymmetric basis of fourth order matrices:

$\Gamma_A(\varsigma)=[\gamma_a(\varsigma),2S_{ab}(e,\varsigma),|-I_4,-i\gamma_a(\varsigma)\gamma_5(\varsigma),-\gamma_5(\varsigma)]C$

Pro. 3.5.1. $X=[im\gamma_a(\varsigma)CA^a+S_{ab}(e,\varsigma)CF^{ab}]-[C\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a+\gamma_5(\varsigma)C\Phi],\forall X$

$$F^{ab}=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)X],\begin{cases}imA^a=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X]\\im\mathbf{A}^a=-\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases},\begin{cases}\phi=-\frac{1}{4}tr[\bar{C}X]\\\Phi=-\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X]\end{cases}$$

Proof: $X\bar{C}=\frac{1}{4}tr[\Gamma^a(-\varsigma)\otimes\Gamma^b(-\varsigma)X\bar{C}]\Gamma_a(\varsigma)\otimes\Gamma_b(\varsigma),\forall X$

$\Leftrightarrow X\bar{C}=[im\gamma_a(\varsigma)A^a+S_{ab}(e,\varsigma)F^{ab}]-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]$

$$\begin{cases}\phi=-\frac{1}{4}tr[X\bar{C}]\\\Phi=-\frac{1}{4}tr[\gamma^5(\varsigma)X\bar{C}]\end{cases},\begin{cases}imA^a=\frac{1}{4}tr[\gamma^a(\varsigma)X\bar{C}]\\im\mathbf{A}^a=\frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X\bar{C}]\end{cases},F^{ab}=\frac{1}{2}tr[S^{ab}(e,\varsigma)X\bar{C}]$$

$\Leftrightarrow X=[im\gamma_a(\varsigma)CA^a+S_{ab}(e,\varsigma)CF^{ab}]-[C\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a+\gamma_5(\varsigma)C\Phi]$

$$F^{ab}=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)X],\begin{cases}imA^a=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X]\\im\mathbf{A}^a=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X]\end{cases},\begin{cases}\phi=-\frac{1}{4}tr[\bar{C}X]\\\Phi=-\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X]\end{cases}\qquad\square$$

Cor. 3.5.1. $X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$
$\Leftrightarrow X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$

$$iF^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)X], \begin{cases} imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X] \end{cases}, \begin{cases} \phi = -\frac{1}{4}tr[\bar{C}X] \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X] \end{cases}$$

3.6 Expansion of symmetric fourth order matrix

Symmetric basis of fourth order matrix:
$\Gamma_A(\varsigma) = [\gamma_a(\varsigma), 2S_{ab}(e,\varsigma)]C$**,** $\bar{C}\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = \bar{C} = -C, C^+(\varsigma) = \bar{C}$

Pro. 3.6.1. $G = im\gamma_a(\varsigma)CA^a + S_{ab}(e,\varsigma)CF^{ab}, G = G^T, F^{ab} = tr[\bar{C}S^{ab}(e,\varsigma)G], imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G]$

Proof: $G = [im\gamma_a(\varsigma)CA^a + S_{ab}(e,\varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi]$

$$F^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)G], G = G^T, \begin{cases} imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G] \\ im\mathbf{A}^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)G] = 0 \end{cases}, \begin{cases} \phi = -\frac{1}{4}tr[\bar{C}G] = 0 \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)G] = 0 \end{cases}$$

$\Leftrightarrow G = im\gamma_a(\varsigma)CA^a + S_{ab}(e,\varsigma)CF^{ab}, G = G^T, F^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)G], imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G]$ □

4 1-spin Bargmann-Wigner equation [16]

4.1 Analysis of 1-spin Bargmann-Wigner equation with mass

Lem. 4.1.1. $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a{}^\sigma + S^{ab}(e,\varsigma)CF_{ab}{}^\sigma] = J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma,$

$$\Leftrightarrow \begin{cases} i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], m[F_{ab}{}^\sigma - (D_aA_b{}^\sigma - D_bA_a{}^\sigma)] = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \\ imD^aA_a{}^\sigma = \frac{1}{4}tr[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], 0 = \frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], iD^b{*}F_{ab}{}^\sigma = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \end{cases}$$

Proof: $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a{}^\sigma + S^{ab}(e,\varsigma)CF_{ab}{}^\sigma] = J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma,$
$\Leftrightarrow im\gamma^c(\varsigma)\gamma^a(\varsigma)D_cA_a{}^\sigma + \gamma^c(\varsigma)S^{ab}(e,\varsigma)D_cF_{ab}{}^\sigma + im^2\gamma^a(\varsigma)A_a{}^\sigma + mS^{ab}(e,\varsigma)F_{ab}{}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma\bar{C}$
$\Leftrightarrow im[\delta^{ca} + 2iS^{ca}(e,\varsigma)]D_cA_a{}^\sigma - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma$
$+ im^2\gamma^a(\varsigma)A_a{}^\sigma + mS^{ab}(e,\varsigma)F_{ab}{}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma\bar{C},$
$\Leftrightarrow im[D^aA_a{}^\sigma + 2iS^{ab}(e,\varsigma)D_aA_b{}^\sigma] - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma$
$+ im^2\gamma^a(\varsigma)A_a{}^\sigma + mS^{ab}(e,\varsigma)F_{ab}{}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma\bar{C}$
$\Leftrightarrow i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma)\gamma^a(\varsigma)C + m[F_{ab}{}^\sigma - (D_aA_b{}^\sigma - D_bA_a{}^\sigma)]S^{ab}(e,\varsigma)C$
$+ imD^aA_a{}^\sigma C + iD^b{*}F_{ab}{}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C = J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma$

$$\Leftrightarrow \begin{cases} i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], m[F_{ab}{}^\sigma - (D_aA_b{}^\sigma - D_bA_a{}^\sigma)] = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \\ imD^aA_a{}^\sigma = \frac{1}{4}tr[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], 0 = \frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], iD^b{*}F_{ab}{}^\sigma = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \end{cases}$$ □

Lem. 4.1.2.
$$\begin{cases} i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \\ im[F_{ab}{}^\sigma - (D_aA_b{}^\sigma - D_bA_a{}^\sigma)] = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] = 0 \\ imD^aA_a{}^\sigma = \frac{1}{4}tr[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] = 0, 0 = \frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \\ iD^b{*}F_{ab}{}^\sigma = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma = J_a{}^\sigma, D^b{*}F_{ab}{}^\sigma = 0, F_{ab}{}^\sigma = D_aA_b{}^\sigma - D_bA_a{}^\sigma, D^aA_a{}^\sigma = 0 \\ J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma = -iJ_a{}^\sigma\gamma^a(\varsigma)C \end{cases}$$

Proof:
$$\begin{cases} i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \\ im[F_{ab}{}^\sigma - (D_aA_b{}^\sigma - D_bA_a{}^\sigma)] = \frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] = 0 \\ imD^aA_a{}^\sigma = \frac{1}{4}tr[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] = 0, 0 = \frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] \\ iD^b{*}F_{ab}{}^\sigma = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], D^b{*}F_{ab}{}^\sigma = 0, F_{ab}{}^\sigma = D_aA_b{}^\sigma - D_bA_a{}^\sigma, D^aA_a{}^\sigma = 0 \\ J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma = \begin{bmatrix} 0 & J^{[1_\varsigma 2_\varsigma]\sigma} \\ J^{[2_\varsigma 1_\varsigma]\sigma} & 0 \end{bmatrix}, \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma] = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} i(D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma], J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma = \begin{bmatrix} 0 & J_a{}^\sigma\Gamma^a(\varsigma)\bar{\varepsilon} \\ J_a{}^\sigma\Gamma^a(-\varsigma)\bar{\varepsilon} & 0 \end{bmatrix} \\ D^b{*}F_{ab}{}^\sigma = 0, F_{ab}{}^\sigma = D_aA_b{}^\sigma - D_bA_a{}^\sigma, D^aA_a{}^\sigma = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma = J_a{}^\sigma, D^b{*}F_{ab}{}^\sigma = 0, F_{ab}{}^\sigma = D_aA_b{}^\sigma - D_bA_a{}^\sigma, D^aA_a{}^\sigma = 0 \\ J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma = -iJ_a{}^\sigma\gamma^a(\varsigma)C \end{cases}$$ □

Cor. 4.1.1. $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a{}^\sigma + S^{ab}(e,\varsigma)CF_{ab}{}^\sigma] = -iJ_a{}^\sigma\gamma^a(\varsigma)C$
$\Leftrightarrow D^bF_{ab}{}^\sigma + m^2A_a{}^\sigma = -J_a{}^\sigma, D^b{*}F_{ab}{}^\sigma = 0, F_{ab}{}^\sigma = D_aA_b{}^\sigma - D_bA_a{}^\sigma, D^aA_a{}^\sigma = 0$

Cor. 4.1.2. $[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)CA_a{}^\sigma+S^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=-iJ_a{}^\sigma\gamma^a(\varsigma)C$
$$\Leftrightarrow\begin{cases}[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]A_a{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma\end{cases}$$

Cor. 4.1.3. $\begin{cases}[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]A_a{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma\end{cases}$
$$\Leftrightarrow D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0$$

Cor. 4.1.4. $\begin{cases}[\gamma^c(\varsigma)\partial_c+m][im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]A_a{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ F_{ab}{}^\sigma=\partial_aA_b{}^\sigma-\partial_bA_a{}^\sigma\end{cases}$
$$\Leftrightarrow \partial^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma,F_{ab}{}^\sigma=\partial_aA_b{}^\sigma-\partial_bA_a{}^\sigma,\partial^aJ_a{}^\sigma=0$$

Cor. 4.1.5. $\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=im\gamma^a(\varsigma)CA_a{}^\sigma+S^{ab}(e,\varsigma)CF_{ab}{}^\sigma\end{cases}\Leftrightarrow\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]A_a{}^\sigma\\ F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma\end{cases}$

Cor. 4.1.6. $\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi_{\lambda_\varsigma\mu_\varsigma}{}^\sigma=\psi_{\mu_\varsigma\lambda_\varsigma}{}^\sigma\end{cases}\Leftrightarrow\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]A_a{}^\sigma\end{cases}$

Cor. 4.1.7. $\begin{cases}[\gamma^c(\varsigma)D_c+m]\psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi_{\lambda_\varsigma\mu_\varsigma}{}^\sigma=\psi_{\mu_\varsigma\lambda_\varsigma}{}^\sigma\end{cases}$
$$\Leftrightarrow\begin{cases}D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,F_{ab}{}^\sigma=D_aA_b{}^\sigma-D_bA_a{}^\sigma,D^aA_a{}^\sigma=0\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]A_a{}^\sigma\end{cases}$$

Cor. 4.1.8. $\begin{cases}[\gamma^c(\varsigma)\partial_c+m]\psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi_{\lambda_\varsigma\mu_\varsigma}{}^\sigma=\psi_{\mu_\varsigma\lambda_\varsigma}{}^\sigma\end{cases}$
$$\Leftrightarrow\begin{cases}\partial^bF_{ab}{}^\sigma+m^2A_a{}^\sigma=-J_a{}^\sigma,\partial^aJ_a{}^\sigma=0,F_{ab}{}^\sigma=\partial_aA_b{}^\sigma-\partial_bA_a{}^\sigma\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]A_a{}^\sigma\end{cases}$$

4.2 1-spin Bargmann-Wigner equation with mass

Thm. 4.2.1. $\begin{cases}[\gamma^c(\varsigma)\partial_c+m]\psi_{[\lambda_\varsigma]\mu_\varsigma}{}^\sigma=-iJ_a{}^\sigma\gamma^a(\varsigma)C\\ \psi_{\lambda_\varsigma\mu_\varsigma}{}^\sigma=\psi_{\mu_\varsigma\lambda_\varsigma}{}^\sigma\end{cases}\Leftrightarrow\begin{cases}(-\partial^b\partial_b+m^2)A_a{}^\sigma=-J_a{}^\sigma\\ \partial^aA_a{}^\sigma=0,\partial^aJ_a{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_a{}^\sigma\end{cases}$

4.3 Analysis of 1-spin Bargmann-Wigner equation without mass

Lem. 4.3.1. $[\gamma^c(\varsigma)D_c+m][im\gamma^a(\varsigma)CA_a{}^\sigma+S^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma,$
$$\Leftrightarrow\begin{cases}i(D^bF_{ab}{}^\sigma+m^2A_a{}^\sigma)=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma],m[F_{ab}{}^\sigma-(D_aA_b{}^\sigma-D_bA_a{}^\sigma)]=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma]\\ imD^aA_a{}^\sigma=\frac{1}{4}tr[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma],0=\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma],iD^b{*}F_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma]\end{cases}$$

Proof: $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a{}^\sigma+S^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma$
$\Leftrightarrow im\gamma^c(\varsigma)\gamma^a(\varsigma)D_cA_a{}^\sigma+\gamma^c(\varsigma)S^{ab}(e,\varsigma)D_cF_{ab}{}^\sigma=J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma\bar{C}$
$\Leftrightarrow im[\delta^{ca}+2iS^{ca}(e,\varsigma)]D_cA_a{}^\sigma-\frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma)-\gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma=J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma\bar{C}$
$\Leftrightarrow im[D^aA_a{}^\sigma+2iS^{ab}(e,\varsigma)D_aA_b{}^\sigma]-\frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma)-\gamma^{[a}\delta^{b]c}]D_cF_{ab}{}^\sigma=J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma\bar{C}$
$\Leftrightarrow iD^bF_{ab}{}^\sigma\gamma^a(\varsigma)C+im[(D_aA_b{}^\sigma-D_bA_a{}^\sigma)]S^{ab}(e,\varsigma)C+imD^aA_a{}^\sigma C+iD^b{*}F_{ab}{}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C=J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma$
$$\Leftrightarrow\begin{cases}iD^bF_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma],-m(D_aA_b{}^\sigma-D_bA_a{}^\sigma)=\frac{1}{2}tr[\bar{C}S^{ab}(e,\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma]\\ imD^aA_a{}^\sigma=\frac{1}{4}tr[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma],0=\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma],iD^b{*}F_{ab}{}^\sigma=\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}{}^\sigma]\end{cases}\quad\square$$

4.4 1-spin Bargmann-Wigner equation without mass

Cor. 4.4.1. $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a{}^\sigma+S^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=-iJ_a{}^\sigma\gamma^a(\varsigma)C$
$$\Leftrightarrow D^bF_{ab}{}^\sigma=-J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,D_aA_b{}^\sigma-D_bA_a{}^\sigma=0,D^aA_a{}^\sigma=0$$

Proof: $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a{}^\sigma+S^{ab}(e,\varsigma)CF_{ab}{}^\sigma]=-iJ_a{}^\sigma\gamma^a(\varsigma)C$
$\Leftrightarrow iD^bF_{ab}{}^\sigma\gamma^a(\varsigma)C-m(D_aA_b{}^\sigma-D_bA_a{}^\sigma)S^{ab}(e,\varsigma)C+imD^aA_a{}^\sigma C+iD^b{*}F_{ab}{}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C=-iJ_a{}^\sigma\gamma^a(\varsigma)C$
$\Leftrightarrow D^bF_{ab}{}^\sigma=-J_a{}^\sigma,D^b{*}F_{ab}{}^\sigma=0,D_aA_b{}^\sigma-D_bA_a{}^\sigma=0,D^aA_a{}^\sigma=0$ $\square$

Cor. 4.4.2. $\gamma^c(\varsigma)\partial_c[im\gamma^a(\varsigma)CA_a+S^{ab}(e,\varsigma)CF_{ab}]=-iJ_a\gamma^a(\varsigma)C$
$$\Leftrightarrow\partial^bF_{ab}=-J_a,\partial^b*F_{ab}=0,\partial^a\partial_a\phi=0,A_a=\partial_a\phi$$

In massless case due to complete independence of $F_{ab}{}^{\sigma}, A_a{}^{\sigma}$, it is unable to obtain more concise and meaningful conclusions. And there are redundant equations that appear to be sloppy and not concise enough. It can't be naturally generalized to the high spin case.Therefore, Bargmann Wigner equation seems not suitable for describing massless particles, but Penrose spinor equation [1,2](Spin Equation) is more suitable for describing massless particles.

5 $\frac{3}{2}, 2$-spin Bargmann-Wigner equation [16]

5.1 Analysis of $\frac{3}{2}$-spin Bargmann-Wigner equation with mass

Cor. 5.1.1. $\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma} \qquad tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0$
$\Rightarrow [im\gamma^a(\varsigma) - 2S^{ab}(e,\varsigma)D_b]A_{a[\eta_\varsigma]}{}^{\sigma} = 0$

Cor. 5.1.2. $\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma} \qquad tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0$
$\Rightarrow [im\gamma^a(\varsigma) + 2S^{ab}(e,\varsigma)D_b]A_{a[\eta_\varsigma]}{}^{\sigma} = 0$

Cor. 5.1.3. $\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma} \qquad tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0,$
$\Rightarrow [im\gamma^a(\varsigma)\gamma^c(\varsigma) + 2S^{ab}(e,\varsigma)\gamma^c(\varsigma)D_b]A_{a[\eta_\varsigma]}{}^{\sigma} = 0$

Cor. 5.1.4. $\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma} \\ tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0\end{cases}$
$\Leftrightarrow \begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma} \\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0, D^aA_{a[\eta_\varsigma]}{}^{\sigma} = 0\end{cases}$

Cor. 5.1.5. $\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = \psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}{}^{\sigma}$
$\Leftrightarrow tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0, tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}] = 0$

Cor. 5.1.6. $\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma} \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = \psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}{}^{\sigma}\end{cases}$
$\Leftrightarrow \begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma} \\ [\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}{}^{\sigma} = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0\end{cases}$

Cor. 5.1.7. $\begin{cases}A_{a\eta_\varsigma\xi_\varsigma}{}^{\sigma} = [im\gamma^b(\varsigma)C - 2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^{\sigma} \\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}{}^{\sigma} = 0\end{cases}$
$\Leftrightarrow \begin{cases}A_{a\eta_\varsigma\xi_\varsigma}{}^{\sigma} = [im\gamma^b(\varsigma)C - 2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^{\sigma} \\ \delta^{ab}A_{ab}{}^{\sigma} = 0, A_{ab}{}^{\sigma} = A_{ba}{}^{\sigma}, \partial^aA_{ab}{}^{\sigma} = 0\end{cases}$

Proof: $A_{a\eta_\varsigma\xi_\varsigma}{}^{\sigma} = [im\gamma^b(\varsigma)C - 2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^{\sigma}, \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}{}^{\sigma} = 0$
$\Leftrightarrow \begin{cases}A_{a\eta_\varsigma\xi_\varsigma}{}^{\sigma} = [im\gamma^b(\varsigma)C - 2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^{\sigma} \\ im[\delta^{ab} + 2iS^{ab}(e,\varsigma)]A_{ab}{}^{\sigma} - i\gamma^d(\varsigma)\gamma^5(\varsigma)\varepsilon^{abz}{}_d\partial_zA_{ab}{}^{\sigma} + i\gamma^z(\varsigma)(\delta^{ab}\partial_zA_{ab}{}^{\sigma} - \partial^aA_{az}{}^{\sigma}) = 0\end{cases}$
$\Leftrightarrow \begin{cases}A_{a\eta_\varsigma\xi_\varsigma}{}^{\sigma} = [im\gamma^b(\varsigma)C - 2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^{\sigma} \\ \delta^{ab}A_{ab}{}^{\sigma} = 0, A_{ab}{}^{\sigma} - A_{ba}{}^{\sigma} = 0, (\varsigma)\varepsilon^{abz}{}_d\partial_zA_{ab}{}^{\sigma} = 0, (\delta^{ab}\partial_zA_{ab}{}^{\sigma} - \partial^aA_{az}{}^{\sigma}) = 0\end{cases}$
$\Leftrightarrow \begin{cases}A_{a\eta_\varsigma\xi_\varsigma}{}^{\sigma} = [im\gamma^b(\varsigma)C - 2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^{\sigma} \\ \delta^{ab}A_{ab}{}^{\sigma} = 0, A_{ab}{}^{\sigma} = A_{ba}{}^{\sigma}, \partial^aA_{ab}{}^{\sigma} = 0\end{cases}$ □

5.2 $\frac{3}{2}$-spin Bargmann-Wigner equation with mass in curved space-time

Lem. 5.2.1. $[\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}{}^{\sigma} = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0 \Rightarrow D^aA_{a[\eta_\varsigma]}{}^{\sigma} = 0$

Proof: $[\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}{}^{\sigma} = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow \gamma^a(\varsigma)[\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}{}^{\sigma} = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow \gamma^a(\varsigma)\gamma^b(\varsigma)D_bA_{a[\eta_\varsigma]}{}^{\sigma} = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow [\gamma^a(\varsigma)\gamma^b(\varsigma) + \gamma^b(\varsigma)\gamma^a(\varsigma) - \gamma^b(\varsigma)\gamma^a(\varsigma)]D_bA_{a[\eta_\varsigma]}{}^{\sigma} = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow 2\delta^{ab}D_bA_{a[\eta_\varsigma]}{}^{\sigma} - \gamma^b(\varsigma)D_b[\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}] = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow D^aA_{a[\eta_\varsigma]}{}^{\sigma} = 0$ □

Lem. 5.2.2. $[\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}{}^{\sigma} = 0 \Rightarrow (\partial_b\partial^b - m^2)A_{a[\eta_\varsigma]}{}^{\sigma} = 0$

Proof: $[\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow [\gamma^b(\varsigma)\partial_b - m][\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow [\gamma^b(\varsigma)\gamma^c(\varsigma)\partial_b\partial_c - m^2]A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow \{[\delta^{bc} + 2iS^{ab}(e,\varsigma)]\partial_b\partial_c - m^2\}A_{a[\eta_\varsigma]}{}^{\sigma} = 0$
$\Rightarrow (\partial_b\partial^b - m^2)A_{a[\eta_\varsigma]}{}^{\sigma} = 0$ □

Thm. 5.2.1. $\begin{cases}[\gamma^a(\varsigma)D_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma}{}^{\sigma}=-iJ_{a\eta_\varsigma}{}^{\sigma}\gamma^a(\varsigma)C\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}\ \textit{full symmetric except}\ \sigma\end{cases}$

$$\Leftrightarrow\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ D^bF_{ab\eta_\varsigma}{}^{\sigma}+m^2A_{a\eta_\varsigma}{}^{\sigma}=-J_{a\eta_\varsigma}{}^{\sigma},D^b{*F}_{ab\eta_\varsigma}{}^{\sigma}=0,F_{ab\eta_\varsigma}{}^{\sigma}=D_aA_{b\eta_\varsigma}{}^{\sigma}-D_bA_{a\eta_\varsigma}{}^{\sigma}\\ [\gamma^b(\varsigma)D_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}$$

Proof: $[\gamma^a(\varsigma)D_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma}{}^{\sigma}=-iJ_{a\eta_\varsigma}{}^{\sigma}\gamma^a(\varsigma)C,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}$ full symmetric except σ

$$\Leftrightarrow\begin{cases}[\gamma^a(\varsigma)D_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=-iJ_{a\eta_\varsigma}{}^{\sigma}\gamma^a(\varsigma)C,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=\psi_{\mu_\varsigma\lambda_\varsigma\eta_\varsigma}{}^{\sigma}\\ tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}]=0,tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}]=0,tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}]=0\end{cases}$$

$$\Leftrightarrow\begin{cases}[\gamma^a(\varsigma)D_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=-iJ_{a\eta_\varsigma}{}^{\sigma}\gamma^a(\varsigma)C,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}]=0,tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}]=0,tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}{}^{\sigma}]=0\end{cases}$$

$$\Leftrightarrow\begin{cases}[\gamma^a(\varsigma)D_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=-iJ_{a\eta_\varsigma}{}^{\sigma}\gamma^a(\varsigma)C,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ [\gamma^b(\varsigma)D_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0,D^aA_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ D^bF_{ab\eta_\varsigma}{}^{\sigma}+m^2A_{a\eta_\varsigma}{}^{\sigma}=-J_{a\eta_\varsigma}{}^{\sigma},D^b{*F}_{ab\eta_\varsigma}{}^{\sigma}=0,F_{ab\eta_\varsigma}{}^{\sigma}=D_aA_{b\eta_\varsigma}{}^{\sigma}-D_bA_{a\eta_\varsigma}{}^{\sigma},D^aA_{a\eta_\varsigma}{}^{\sigma}=0\\ [\gamma^b(\varsigma)D_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0,D^aA_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ D^bF_{ab\eta_\varsigma}{}^{\sigma}+m^2A_{a\eta_\varsigma}{}^{\sigma}=-J_{a\eta_\varsigma}{}^{\sigma},D^b{*F}_{ab\eta_\varsigma}{}^{\sigma}=0,F_{ab\eta_\varsigma}{}^{\sigma}=D_aA_{b\eta_\varsigma}{}^{\sigma}-D_bA_{a\eta_\varsigma}{}^{\sigma}\\ [\gamma^b(\varsigma)D_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)J_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}\qquad\square$$

In curved space-time, the equation can't be further simplified, so more concise and meaningful conclusions can't be obtained.

5.3 Source item requirements for $\frac{3}{2}$-spin B-W equation with mass in flat space-time???

Thm. 5.3.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma}{}^{\sigma}=-iJ_{a\eta_\varsigma}{}^{\sigma}\gamma^a(\varsigma)C\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}\ \textit{full symmetric except}\ \sigma\end{cases}$

$$\Leftrightarrow\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ \partial^bF_{ab\eta_\varsigma}{}^{\sigma}+m^2A_{a\eta_\varsigma}{}^{\sigma}=0,J_{a\eta_\varsigma}{}^{\sigma}=0,F_{ab\eta_\varsigma}{}^{\sigma}=\partial_aA_{b\eta_\varsigma}{}^{\sigma}-\partial_bA_{a\eta_\varsigma}{}^{\sigma}\\ [\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}$$

Proof: $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma}{}^{\sigma}=-iJ_{a\eta_\varsigma}{}^{\sigma}\gamma^a(\varsigma)C\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}\ \text{full symmetric except}\ \sigma\end{cases}$

$$\Leftrightarrow\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ \partial^bF_{ab\eta_\varsigma}{}^{\sigma}+m^2A_{a\eta_\varsigma}{}^{\sigma}=-J_{a\eta_\varsigma}{}^{\sigma},\partial^b{*F}_{ab\eta_\varsigma}{}^{\sigma}=0,F_{ab\eta_\varsigma}{}^{\sigma}=\partial_aA_{b\eta_\varsigma}{}^{\sigma}-\partial_bA_{a\eta_\varsigma}{}^{\sigma}\\ [\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ \partial^bF_{ab\eta_\varsigma}{}^{\sigma}+m^2A_{a\eta_\varsigma}{}^{\sigma}=-J_{a\eta_\varsigma}{}^{\sigma},F_{ab\eta_\varsigma}{}^{\sigma}=\partial_aA_{b\eta_\varsigma}{}^{\sigma}-\partial_bA_{a\eta_\varsigma}{}^{\sigma}\\ [\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0,\partial^aA_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\\ \partial^bF_{ab\eta_\varsigma}{}^{\sigma}+m^2A_{a\eta_\varsigma}{}^{\sigma}=0,J_{a\eta_\varsigma}{}^{\sigma}=0,F_{ab\eta_\varsigma}{}^{\sigma}=\partial_aA_{b\eta_\varsigma}{}^{\sigma}-\partial_bA_{a\eta_\varsigma}{}^{\sigma}\\ [\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0\end{cases}\qquad\square$$

Compared to the curved space-time case, the equation has been further simplified in flat space-time. More concise and meaningful conclusions have been obtained. The self consistency of the equation itself also automatically requires that the source term must be zero.(???)

5.4 $\frac{3}{2}$-spin Bargmann-Wigner equation [19] with mass in flat space-time

Thm. 5.4.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma}{}^{\sigma}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}\ \textit{full symmetric except}\ \sigma\end{cases}\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}{}^{\sigma}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^{\sigma}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^{\sigma}\end{cases}$

Proof: $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma}{}^{\sigma}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}\ \text{full symmetric except}\ \sigma\end{cases}$

$$\Leftrightarrow\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma}{}^{\sigma}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=\psi_{\mu_\varsigma\lambda_\varsigma\eta_\varsigma}{}^{\sigma}\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^{\sigma}=\psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}{}^{\sigma}\end{cases}$$

$$\Leftrightarrow \begin{cases}(-\partial^b\partial_b+m^2)A_{a\eta_\varsigma}{}^\sigma=0,\partial^a A_{a\eta_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^\sigma\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^\sigma=\psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow \begin{cases}(-\partial^b\partial_b+m^2)A_{a\eta_\varsigma}{}^\sigma=0,\partial^a A_{a\eta_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^\sigma\\ [\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}{}^\sigma=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^\sigma=0\end{cases}$$

$$\Leftrightarrow \begin{cases}[\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}{}^\sigma=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}{}^\sigma\end{cases}$$ □

5.5 2-spin Bargmann-Wigner equation with mass in flat space-time

Thm. 5.5.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma \textit{ full symmetric except }{}^\sigma\end{cases}$

$$\Leftrightarrow \begin{cases}(-\partial^d\partial_d+m^2)A_{ab}{}^\sigma=0,\delta^{ab}A_{ab}{}^\sigma=0,A_{ab}{}^\sigma=A_{ba}{}^\sigma,\partial^a A_{ab}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ac}(e,\varsigma)C\partial_c]_{\lambda_\varsigma\mu_\varsigma}[im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma\end{cases}$$

Proof: $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma \text{ full symmetric except }{}^\sigma\end{cases}$

$$\Leftrightarrow \begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma \text{ full symmetric except }{}_{\xi_\varsigma}{}^\sigma.\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=\psi_{\lambda_\varsigma\mu_\varsigma\xi_\varsigma\eta_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow \begin{cases}[\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma\xi_\varsigma}{}^\sigma\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=\psi_{\lambda_\varsigma\mu_\varsigma\xi_\varsigma\eta_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow \begin{cases}[\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,A_{a\eta_\varsigma\xi_\varsigma}{}^\sigma=A_{a\xi_\varsigma\eta_\varsigma}{}^\sigma\\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma\xi_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow \begin{cases}(-\partial^d\partial_d+m^2)A_{ab}{}^\sigma=0,\partial^b A_{ab}{}^\sigma=0\\ A_{a\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^b(\varsigma)C-2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma\\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma\xi_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow \begin{cases}(-\partial^d\partial_d+m^2)A_{ab}{}^\sigma=0,\partial^b A_{ab}{}^\sigma=0\\ A_{a\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^b(\varsigma)C-2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma\\ \delta^{ab}A_{ab}{}^\sigma=0,A_{ab}{}^\sigma=A_{ba}{}^\sigma,\partial^a A_{ab}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma\xi_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow \begin{cases}(-\partial^d\partial_d+m^2)A_{ab}{}^\sigma=0,\delta^{ab}A_{ab}{}^\sigma=0,A_{ab}{}^\sigma=A_{ba}{}^\sigma,\partial^a A_{ab}{}^\sigma=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}{}^\sigma=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}[im\gamma^b(\varsigma)C-2S^{bz}(e,\varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma\end{cases}$$ □

Cor. 5.5.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma=0\\ \mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab}{}^\sigma=\mathbb{X}^a_{\lambda_\varsigma\eta_\varsigma}\mathbb{X}^b_{\mu_\varsigma\xi_\varsigma}A_{ab}{}^\sigma\end{cases} \Leftrightarrow \begin{cases}(-\partial^d\partial_d+m^2)A_{ab}{}^\sigma=0\\ \delta^{ab}A_{ab}{}^\sigma=0,A_{ab}{}^\sigma=A_{ba}{}^\sigma,\partial^a A_{ab}{}^\sigma=0\end{cases}$

Cor. 5.5.2. $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab}=\mathbb{X}^a_{\mu_\varsigma\eta_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\xi_\varsigma}A_{ab}\Leftrightarrow$???

6 Arbitrary spin particles Bargmann-Wigner equation in flat space-time

6.1 n-spin Bargmann-Wigner equation [16, 20, 21] with mass in flat space-time

Def. 6.1.1. $\mathbb{X}_a:=[im\gamma_a(\varsigma)-2S_{ab}(e,\varsigma)\partial^b]C,\mathbb{X}^a:=[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]C$

Thm. 6.1.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n}}{}^\sigma=0\\ \psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n}}{}^\sigma \textit{ full symmetric except }{}^\sigma\end{cases} \Leftrightarrow \begin{cases}(-\partial^d\partial_d+m^2)A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=0\\ A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma \textit{ full symmetric except }{}^\sigma\\ \delta^{ab}A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=0,\partial^a A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma\end{cases}$

Cor. 6.1.1.
$$\begin{cases} [\gamma^a(\varsigma)\partial_a+m]\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=0\Leftrightarrow(-\partial^d\partial_d+m^2)A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=0 \\ A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=\frac{1}{n!}A_{\underbrace{\{abc\cdot\cdot\}}_{n}}{}^\sigma,\delta^{ab}A_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=0,\partial^aA_{\underbrace{abc\cdot\cdot}_{n}}{}^\sigma=0 \end{cases}$$

6.2 $n+\frac{1}{2}$-spin Bargmann-Wigner equation with mass [16,17,20] in flat space-time

Thm. 6.2.1.
$$\begin{cases} [\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n+1}}{}^\sigma=0 \\ \psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n+1}}{}^\sigma\ \textit{full symmetric except}\ {}^\sigma \end{cases}\Leftrightarrow\begin{cases} [\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=0 \\ A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma\ \textit{fully symmetric except}\ {}_{\zeta_\varsigma}{}^\sigma \\ \delta^{ab}A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n+1}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma \end{cases}$$

Cor. 6.2.1.
$$\begin{cases} [\gamma^a(\varsigma)\partial_a+m]\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=0\Leftrightarrow[\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=0 \\ A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=\frac{1}{n!}A_{\underbrace{\{abc\cdot\cdot\}}_{n}[\zeta_\varsigma]}{}^\sigma,\delta^{ab}A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{abc\cdot\cdot}_{n}[\zeta_\varsigma]}{}^\sigma=0 \end{cases}$$

Using mathematical induction and the reasoning techniques of $s=\frac{3}{2}$ and $s=2$ can easily and strictly prove the above two theorems. Let's begin to prove them.

6.3 Strictly prove the above two theorems by using mathematical induction

Proof: Use mathematical induction to prove the above two theorems together.

Step 1: When $s=1/2$, the following is established:
$$\begin{cases} [\gamma^a(\varsigma)\partial_a+m]\psi_{[\lambda_\varsigma]}{}^\sigma=0 \\ \psi_{\lambda_\varsigma}{}^\sigma\ \text{full symmetric except}\ {}^\sigma \end{cases}\Leftrightarrow\begin{cases} [\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{ab\cdot\cdot}_{0}[\lambda_\varsigma]}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot}_{0}\eta_\varsigma}{}^\sigma\ \text{fully symmetric except}\ {}_{\lambda_\varsigma}{}^\sigma \\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{0}[\lambda_\varsigma]}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{ab\cdot\cdot}_{0}[\lambda_\varsigma]}{}^\sigma=0 \\ \psi_{\lambda_\varsigma}{}^\sigma=A_{\underbrace{ab\cdot\cdot}_{0}\lambda_\varsigma}{}^\sigma \end{cases}$$

Step 2: When $s=n-1/2$, the following is established.
$$\begin{cases} [\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot\eta_\varsigma}_{2n-1}}{}^\sigma=0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma}_{2n-1}}{}^\sigma\ \text{full symmetric except}\ {}^\sigma \end{cases}\Leftrightarrow\begin{cases} [\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma}{}^\sigma\ \text{fully symmetric except}\ {}_{\eta_\varsigma}{}^\sigma \\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]}{}^\sigma=0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma}_{2n-1}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n-1}A_{\underbrace{a\cdot\cdot}_{n-1}\eta_\varsigma}{}^\sigma \end{cases}$$

Step 3: When $s=n$,
$$1\begin{cases} [\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma\ \text{full symmetric except}\ \sigma \end{cases}$$
$$\Leftrightarrow\begin{cases} [\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma\ \text{full symmetric except}\ {}_{\xi_\varsigma}{}^\sigma. \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\xi_\varsigma\eta_\varsigma}_{2n}}{}^\sigma \end{cases}$$
$$\Leftrightarrow\begin{cases} [\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma\ \text{fully symmetric except}\ {}_{\eta_\varsigma\xi_\varsigma}{}^\sigma \\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n-1}A_{\underbrace{a\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\xi_\varsigma\eta_\varsigma}_{2n}}{}^\sigma \end{cases}$$

$$\Leftrightarrow\begin{cases}[\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma\text{ fully symmetric except }{}_{\eta_\varsigma\xi_\varsigma}{}^\sigma\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0\\ A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma=A_{\underbrace{ab\cdot\cdot}_{n-1}\xi_\varsigma\eta_\varsigma}{}^\sigma,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n-1}A_{\underbrace{a\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow\begin{cases}(-\partial^d\partial_d+m^2)A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma\text{ fully symmetric except }{}_{\eta_\varsigma\xi_\varsigma}{}^\sigma\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{ab\cdot\cdot}_{n-1}[\eta_\varsigma]\xi_\varsigma}{}^\sigma=0,\partial^cA_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0\\ A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma=\mathbb{X}^c_{\eta_\varsigma\xi_\varsigma}A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n-1}A_{\underbrace{a\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow\begin{cases}(-\partial^d\partial_d+m^2)A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0,\partial^cA_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma\text{ fully symmetric except }{}_{\eta_\varsigma\xi_\varsigma}{}^\sigma\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=A_{\underbrace{cb\cdot\cdot a}_{n}}{}^\sigma,\partial^aA_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0\\ A_{\underbrace{ab\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma=\mathbb{X}^c_{\eta_\varsigma\xi_\varsigma}A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n-1}A_{\underbrace{a\cdot\cdot}_{n-1}\eta_\varsigma\xi_\varsigma}{}^\sigma\end{cases}$$

$$\Leftrightarrow\begin{cases}(-\partial^d\partial_d+m^2)A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma\text{ fully symmetric except }{}^\sigma\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0,\partial^aA_{\underbrace{ab\cdot\cdot c}_{n}}{}^\sigma=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot\mathbb{X}^c_{\eta_\varsigma\xi_\varsigma}}^{n}A_{\underbrace{a\cdot\cdot c}_{n}}{}^\sigma\end{cases}$$

Step 4: When $s=n+1/2$,

$$2\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma\text{ full symmetric except }\sigma\end{cases}$$

$$\Leftrightarrow\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma\text{ fully symmetric except }{}_{\zeta_\varsigma}{}^\sigma\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\zeta_\varsigma\xi_\varsigma}_{2n+1}}{}^\sigma\end{cases}$$

$$\Leftrightarrow\begin{cases}(-\partial^d\partial_d+m^2)A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma\text{ fully symmetric except }{}_{\zeta_\varsigma}{}^\sigma\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma=0,\partial^aA_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot\mathbb{X}^c_{\eta_\varsigma\xi_\varsigma}}^{n}A_{\underbrace{a\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\zeta_\varsigma\xi_\varsigma}_{2n+1}}{}^\sigma\end{cases}$$

$$\Leftrightarrow\begin{cases}(-\partial^d\partial_d+m^2)A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma\text{ fully symmetric except }{}_{\zeta_\varsigma}{}^\sigma\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma=0,\partial^aA_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot\mathbb{X}^c_{\eta_\varsigma\xi_\varsigma}}^{n}A_{\underbrace{a\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma\\ [\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{ab\cdot\cdot c}_{n}[\zeta_\varsigma]}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{ab\cdot\cdot c}_{n}[\zeta_\varsigma]}{}^\sigma=0\end{cases}$$

$$\Leftrightarrow\begin{cases}[\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{ab\cdot\cdot c}_{n}[\zeta_\varsigma]}{}^\sigma=0,A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma\text{ fully symmetric except }{}_{\zeta_\varsigma}{}^\sigma\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma=0,\gamma^a(\varsigma)A_{\underbrace{ab\cdot\cdot c}_{n}[\zeta_\varsigma]}{}^\sigma=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\zeta_\varsigma}_{2n+1}}{}^\sigma=\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\cdot\cdot\mathbb{X}^c_{\eta_\varsigma\xi_\varsigma}}^{n}A_{\underbrace{a\cdot\cdot c}_{n}\zeta_\varsigma}{}^\sigma\end{cases}$$

This step proves that when $s=n$ and $s=n+1/2$, the proposition is established.

Step 5: Based on the above inductive reasoning, the proposition is established and proved simultaneously. □

6.4 Review of s-spin Bargmann-Wigner equation with mass

From the above, it can be seen that Bargmann Wigner equation is equivalent to Rarita Schwinger equation in semi integer spin case [17] and is equivalent to Klein Gordon equation in integer spin case [21] in a flat space-time. It reveals the profound and rich physical connotation of Bargmann Wigner equation. However, if we consider the general source term, we can't obtain this equivalent result. Only a source term that meets certain conditions can be established. And only the spins with $s=\frac{1}{2}$ and $s=1$ can have a source term. For a spin with $s=\frac{3}{2}$ or more, the intrinsic self consistency of the equation requires that the source term must be zero. In addition in curved space-time due to the existence of the generalized covariant derivative term, this equivalent conclusion no longer holds. This situation is not as good as the properties of Penrose spinor equation or the spin equation. In general, Penrose spinor equation or spin equation is more suitable for describing massless particles, while Bargmann Wigner equation is more suitable for describing massive particles.

6.5 Bargmann-Wigner spin equation form

Reduce a pair of vector indices:(On the right is Penrose notation, denoted by $\stackrel{P}{=}$.)

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\delta_b^a\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}=\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma} \qquad \delta^a_b\stackrel{P}{=}\delta^A_B\delta^{A'}_{B'} \tag{12.6}$$

$$\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\delta^{ab}\frac{i\varsigma}{\sqrt{2}}(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}=\varepsilon^{AB}\varepsilon^{A'B'} \qquad \delta^{ab}\stackrel{P}{=}\varepsilon^{AB}\varepsilon^{A'B'} \tag{12.7}$$

$$\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\delta_{ab}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}=\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma} \qquad \delta_{ab}\stackrel{P}{=}\varepsilon_{AB}\varepsilon_{A'B'} \tag{12.8}$$

Lem. 6.5.1. ${\gamma^a}_{\lambda_\varsigma}{}^{\mu_\varsigma}$
$=\begin{bmatrix} 0 & -i(\sigma,i\varsigma)^a \\ i(\sigma,-i\varsigma)_a & 0\end{bmatrix}$
$=\begin{bmatrix} 0^a_{A_\varsigma B_\varsigma} & -i(\sigma,i\varsigma)^a_{A_\varsigma B'_\varsigma} \\ i(\sigma,-i\varsigma)_a^{A'_\varsigma B_\varsigma} & 0_a^{A'_\varsigma B'_\varsigma}\end{bmatrix}$

6.6 Bargmann-Wigner spin equation form

Def. 6.6.1. $\begin{cases} {S_{abj_\varsigma}}^{k_\varsigma}(e,s):=2sN^{\lambda_\varsigma l_\varsigma}_{j_\varsigma}(s,3){S_{ab\lambda_\varsigma}}^{\mu_\varsigma}(e,\varsigma)N^{k_\varsigma}_{\mu_\varsigma l_\varsigma}(s,3) \\ Z^{ak_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3):={\gamma^a}_{\rho_\varsigma}{}^{\lambda_\varsigma}N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(s,3),\bar{Z}^{\rho_\varsigma l_\varsigma}_{aj_\varsigma}(s,3):=N^{\lambda_\varsigma l_\varsigma}_{j_\varsigma}(s,3){\gamma^a}_{\lambda_\varsigma}{}^{\rho_\varsigma}\end{cases}$

Lem. 6.6.1. $\bar{Z}^{\rho_\varsigma l_\varsigma}_{aj_\varsigma}(s,3)Z^{bk_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)=\frac{1}{s}[s{\delta_a}^b{\delta_{j_\varsigma}}^{k_\varsigma}+i{S_a}^b{}_{j_\varsigma}{}^{k_\varsigma}(e,s)]$

Proof: $\bar{Z}^{\rho_\varsigma l_\varsigma}_{aj_\varsigma}(s,3)Z^{bk_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)$
$=N^{\lambda_\varsigma l_\varsigma}_{j_\varsigma}(s,3){\gamma^a}_{\lambda_\varsigma}{}^{\rho_\varsigma}{\gamma^b}_{\rho_\varsigma}{}^{\mu_\varsigma}N^{k_\varsigma}_{\mu_\varsigma l_\varsigma}(s,3)$
$=N^{\lambda_\varsigma l_\varsigma}_{j_\varsigma}(s,3)(\gamma_a\gamma^b)_{\lambda_\varsigma}{}^{\mu_\varsigma}N^{k_\varsigma}_{\mu_\varsigma l_\varsigma}(s,3)$
$=N^{\lambda_\varsigma l_\varsigma}_{j_\varsigma}(s,3)[{\delta_a}^b+2i{S_a}^b(e,\varsigma)]_{\lambda_\varsigma}{}^{\mu_\varsigma}N^{k_\varsigma}_{\mu_\varsigma l_\varsigma}(s,3)$
$={\delta_a}^b{\delta_{j_\varsigma}}^{k_\varsigma}+N^{\lambda_\varsigma l_\varsigma}_{j_\varsigma}(s,3)2i{S_a}^b{}_{\lambda_\varsigma}{}^{\mu_\varsigma}(e,\varsigma)N^{k_\varsigma}_{\mu_\varsigma l_\varsigma}(s,3)$
$=\frac{1}{s}[s{\delta_a}^b{\delta_{j_\varsigma}}^{k_\varsigma}+i{S_a}^b{}_{j_\varsigma}{}^{k_\varsigma}(e,s)]$ □

Proof: $Z^{ak_\varsigma}_{\rho'_\varsigma l'_\varsigma}(s,3)\bar{Z}^{\rho_\varsigma l_\varsigma}_{ak_\varsigma}(s,3)$
$={\gamma^a}_{\rho'_\varsigma}{}^{\lambda'_\varsigma}N^{k_\varsigma}_{\lambda'_\varsigma l'_\varsigma}(s,3)N^{\lambda_\varsigma l_\varsigma}_{k_\varsigma}(s,3){\gamma_{a\lambda_\varsigma}}^{\rho_\varsigma}$
$={\gamma^a}_{\rho'_\varsigma}{}^{\lambda'_\varsigma}{\gamma_{a\lambda_\varsigma}}^{\rho_\varsigma}N^{k_\varsigma}_{\lambda'_\varsigma l'_\varsigma}(s,3)N^{\lambda_\varsigma l_\varsigma}_{k_\varsigma}(s,3)$ □

Thm. 6.6.1. $(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdots}_{2s}}=0\Rightarrow\begin{cases}[\gamma^a(s)\partial_a+sm]\psi(e,s)=0\\ [s\partial_a+iS_{ab}(e,s)\partial^b]\psi(e,s)=-m\gamma_a(s)\psi(e,s)\end{cases}$

Proof: $(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}=0$
$\Leftrightarrow(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdots}_{2s}}(s,3)\psi_{k_\varsigma}(e,s)=0$
$\Leftrightarrow(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(s,3)\Gamma^{l_\varsigma}_{\underbrace{\mu_\varsigma\eta_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2},3)\psi_{k_\varsigma}(e,s)=0$
$\Leftrightarrow(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)=0$
$\Rightarrow N^{\rho_\varsigma l_\varsigma}_{j_\varsigma}(s,3)(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)=0$
$\Leftrightarrow[\gamma^a(s)\partial_a+sm]_{j_\varsigma}{}^{k_\varsigma}\psi_{k_\varsigma}(e,s)=0$
$\Leftrightarrow[\gamma^a(s)\partial_a+sm]\psi(e,s)=0$ □

Proof: $(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}=0$

$\Leftrightarrow(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\cdots}_{2s}}(s,3)\psi_{k_\varsigma}(e,s)=0$

$\Leftrightarrow(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(s,3)\Gamma^{l_\varsigma}_{\underbrace{\mu_\varsigma\eta_\varsigma\cdots}_{2s-1}}(s-\frac{1}{2},3)\psi_{k_\varsigma}(e,s)=0$

$\Leftrightarrow(\gamma^a\partial_a+m)_{\rho_\varsigma}{}^{\lambda_\varsigma}N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)=0$

$\Leftrightarrow\gamma^a{}_{\rho_\varsigma}{}^{\lambda_\varsigma}N^{k_\varsigma}_{\lambda_\varsigma l_\varsigma}(s,3)\partial_a\psi_{k_\varsigma}(e,s)=-mN^{k_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)$

$\Leftrightarrow Z^{ak_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)\partial_a\psi_{k_\varsigma}(e,s)=-mN^{k_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)$

$\Rightarrow\bar{Z}^{\rho_\varsigma l_\varsigma}_{aj_\varsigma}(s,3)Z^{bk_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)\partial_b\psi_{k_\varsigma}(e,s)=-m\bar{Z}^{\rho_\varsigma l_\varsigma}_{aj_\varsigma}(s,3)N^{k_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)$

$\Leftrightarrow[s\delta_{ab}\delta_{j_\varsigma}{}^{k_\varsigma}+iS_{abj_\varsigma}{}^{k_\varsigma}(e,s)]\partial^b\psi_{k_\varsigma}(e,s)=-sm\bar{Z}^{\rho_\varsigma l_\varsigma}_{aj_\varsigma}(s,3)N^{k_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)$

$\Leftrightarrow[s\partial_a+iS_{ab}(e,s)\partial^b]_{j_\varsigma}{}^{k_\varsigma}\psi_{k_\varsigma}(e,s)=-sm\bar{Z}^{\rho_\varsigma l_\varsigma}_{aj_\varsigma}(s,3)N^{k_\varsigma}_{\rho_\varsigma l_\varsigma}(s,3)\psi_{k_\varsigma}(e,s)$

$\Leftrightarrow[s\partial_a+iS_{ab}(e,s)\partial^b]\psi(e,s)=-m\gamma_a(s)\psi(e,s)$ □

Cor. 6.6.1. $\begin{cases}[\gamma^a(s)\partial_a+sm]\psi(e,s)=0\\ [s\partial_a+iS_{ab}(e,s)\partial^b]\psi(e,s)=-m\gamma_a(s)\psi(e,s)\end{cases}\Leftrightarrow\begin{cases}[\gamma^a(s)\partial_a+sm]\psi(e,s)=0\\ \frac{1}{s}\gamma_a(s)\gamma_b(s)\partial^b\psi(e,s)=[s\delta_{ab}+iS_{ab}(e,s)]\partial^b\psi(e,s)\end{cases}$

7 Antisymmetric Dirac equation [4]

7.1 Analysis of antisymmetric Dirac equation with mass

Thm. 7.1.1. $[\gamma^c(\varsigma)\partial_c+m]F_{[\lambda_\varsigma\mu_\varsigma]}=J,F_{\lambda_\varsigma\mu_\varsigma}=-F_{\mu_\varsigma\lambda_\varsigma}$

$\Leftrightarrow\begin{cases}[-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b-\gamma_a(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)]C+[m(\Phi+i\partial_a\mathbf{A}^a)+m\gamma_5(\varsigma)\phi-\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C=-\gamma_5(\varsigma)J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

Proof: $[\gamma^a(\varsigma)\partial_a+m]F_{[\lambda_\varsigma\mu_\varsigma]}=J,F_{\lambda_\varsigma\mu_\varsigma}=-F_{\mu_\varsigma\lambda_\varsigma}$

$\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m]F_{[\lambda_\varsigma\mu_\varsigma]}=J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m][C\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a+\gamma_5(\varsigma)C\Phi]=-J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m][\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]=-J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}m\phi+\gamma_a(\varsigma)\partial^a\phi+im\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b+\gamma_a(\varsigma)\gamma_5(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)+m\gamma_5(\varsigma)\Phi=-J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}m\phi+\gamma_a(\varsigma)\partial^a\phi-2mS_{ab}(e,\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b+\gamma_a(\varsigma)\gamma_5(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)+m\gamma_5(\varsigma)(\Phi+i\partial_a\mathbf{A}^a)=-J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}m\gamma_5(\varsigma)\phi-\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b-\gamma_a(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)+m(\Phi+i\partial_a\mathbf{A}^a)=-\gamma_5(\varsigma)J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}[-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b-\gamma_a(\varsigma)(im^2\mathbf{A}^a+\partial^a\Phi)]C+[m(\Phi+i\partial_a\mathbf{A}^a)+m\gamma_5(\varsigma)\phi-\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C=-\gamma_5(\varsigma)J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$ □

7.2 Analysis of antisymmetric Dirac equation without mass

Thm. 7.2.1. $\gamma^c(\varsigma)\partial_cF_{[\lambda_\varsigma\mu_\varsigma]}=J,F_{\lambda_\varsigma\mu_\varsigma}=-F_{\mu_\varsigma\lambda_\varsigma}$

$\Leftrightarrow\begin{cases}[-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b-\gamma_a(\varsigma)\partial^a\Phi]C+[im\partial_a\mathbf{A}^a-\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C=-\gamma_5(\varsigma)J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

Proof: $\gamma^a(\varsigma)\partial_aF_{[\lambda_\varsigma\mu_\varsigma]}=J,F_{\lambda_\varsigma\mu_\varsigma}=-F_{\mu_\varsigma\lambda_\varsigma}$

$\Leftrightarrow\begin{cases}\gamma^b(\varsigma)\partial_bF_{[\lambda_\varsigma\mu_\varsigma]}=J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}\gamma^b(\varsigma)\partial_b[C\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a+\gamma_5(\varsigma)C\Phi]=-J\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$\Leftrightarrow\begin{cases}\gamma^b(\varsigma)\partial_b[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]=-J\bar{C}\\ F=-[\phi+im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a+\gamma_5(\varsigma)\Phi]C\end{cases}$

$$\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\partial^a\phi + im\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\Phi = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\partial^a\phi - 2mS_{ab}(e,\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\Phi + im\gamma_5(\varsigma)\partial_a\mathbf{A}^a = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} -\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi - 2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi + im\partial_a\mathbf{A}^a = -\gamma_5(\varsigma)J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} [-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi]C + [im\partial_a\mathbf{A}^a - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\gamma_5(\varsigma)J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$ □

7.3 Massive pseudoscalar field equation

Thm. 7.3.1. $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = \frac{j}{m}\gamma_5(\varsigma)C \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases} \Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$

Proof: $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = \frac{j}{m}\gamma_5(\varsigma)C \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases}$

$$\Leftrightarrow \begin{cases} [-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\frac{j}{m}C \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, (im^2\mathbf{A}^a + \partial^a\Phi) = 0, \phi = 0, \partial^a\phi = 0 \\ m^2(\Phi + i\partial_a\mathbf{A}^a) = -j \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ \mathbf{A}^a = im^{-2}\partial^a\Phi, \phi = 0 \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$$ □

Cor. 7.3.1. $[\gamma^a(\varsigma)\partial_a + m][\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi = j\gamma_5(\varsigma)C \Leftrightarrow (-\partial^a\partial_a + m^2)\Phi = -j$

7.4 Massless pseudoscalar field equation

Cor. 7.4.1. $\gamma^a(\varsigma)\partial_a[\gamma^b(\varsigma)\partial_b\gamma_5(\varsigma)C\Phi] = j\gamma_5(\varsigma)C \Leftrightarrow \gamma^a(\varsigma)\partial_a[\gamma^b(\varsigma)\partial_bC\Phi] = jC \Leftrightarrow \partial^a\partial_a\Phi = j$

Cor. 7.4.2. $\gamma_a(\varsigma)\partial^a[\gamma_5(\varsigma)C\Phi] = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \gamma_a(\varsigma)\partial^a[C\Phi] = \gamma_a(\varsigma)CJ^a \Leftrightarrow \partial^a\Phi = J^a$

7.5 Massive pseudovector field equation

Thm. 7.5.1. $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = i\gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases} \Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a \end{cases}$

Proof: $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = i\gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases}$

$$\Leftrightarrow \begin{cases} [-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = i\gamma_a(\varsigma)CJ^a \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \Phi = -i\partial_a\mathbf{A}^a, \phi = 0, \partial^a\phi = 0 \\ (im^2\mathbf{A}^a + \partial^a\Phi) = -iJ^a \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a \\ \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \Phi = -i\partial_a\mathbf{A}^a, \phi = 0 \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a \end{cases}$$ □

Cor. 7.5.1. $[\gamma_b(\varsigma)\partial^b + m][\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$

Proof: $[\gamma_b(\varsigma)\partial^b + m][\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a$
$\Leftrightarrow -2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b - i\gamma_a(\varsigma)(m^2\mathbf{A}^a - \partial^a\partial_b\mathbf{A}^b) = i\gamma_a(\varsigma)J^a$
$\Leftrightarrow -\partial^a\partial_b\mathbf{A}^b + m^2\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$
$\Leftrightarrow (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$ □

7.6 Massless pseudovector field equation

Cor. 7.6.1. $\gamma_b(\varsigma)\partial^b[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \partial^b\partial_b\mathbf{A}^a = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$

Proof: $\gamma_b(\varsigma)\partial^b[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a$
$\Leftrightarrow [-2mS_{ab}(e,\varsigma)\partial^a\mathbf{A}^b + i\gamma_a(\varsigma)\partial^a\partial_b\mathbf{A}^b] + im\partial_a\mathbf{A}^a = i\gamma_a(\varsigma)J^a$
$\Leftrightarrow \partial^a\partial_b\mathbf{A}^b = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$
$\Leftrightarrow \partial^b\partial_b\mathbf{A}^a = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$ □

Cor. 7.6.2. $\gamma_a(\varsigma)\partial^a[\gamma_5(\varsigma)C\partial_b\mathbf{A}^b] = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \gamma_a(\varsigma)\partial^a[C\partial_b\mathbf{A}^b] = \gamma_a(\varsigma)CJ^a \Leftrightarrow \partial^a\partial_b\mathbf{A}^b = J^a$

Cor. 7.6.3. $\gamma_b(\varsigma)\partial^b[\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a] = \gamma_5(\varsigma)[jC + J^{ab}S_{ab}(e,\varsigma)]$
$\Leftrightarrow \gamma_b(\varsigma)\partial^b[\gamma_a(\varsigma)C\mathbf{A}^a] = jC + J^{ab}S_{ab}(e,\varsigma)$
$\Leftrightarrow \partial^a\mathbf{A}^b - \partial^b\mathbf{A}^a = J^{ab}, \partial_a\mathbf{A}^a = j$

Chapter13 Advanced Representation Transformation Technology

Self comment: In this chapter, I have made a further in-depth study of representation transformation. Through various complex representation transformation techniques, some useful conclusions have been obtained. It is very useful for studying the Lorentz transformation of various spin particles.

1 Advanced representation transformation technology

1.1 Representation transformation and constant invariant tensors

Thm. 1.1.1. $\psi' = S\psi \Rightarrow \Lambda(\psi') = S\Lambda(\psi)S^{-1} \Leftrightarrow S = \Lambda(\psi')S\Lambda^{-1}(\psi) \Leftrightarrow S^{-1} = \Lambda(\psi)S^{-1}\Lambda^{-1}(\psi')$

Therefore, the representation transformation is a second order constant invariant tensor. The component form is as follows:

Cor. 1.1.1. $\psi'^{\alpha'} = S^{\alpha'}{}_{\alpha}\psi^{\alpha}, \psi^{\alpha} = S^{-1\alpha}{}_{\alpha'}\psi^{\alpha'}$

1.2 Introduction of representation transformation matrix $\tilde{S}(s)$ and constant matrix $\Sigma(s)$

1.2.1 Introduction of representation transformation matrix $\tilde{S}(s)$

Def. 1.2.1. $\tilde{S}(s) := \begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix}, \tilde{S}^{+}(s) = [N(s), X(s)]$

Cor. 1.2.1. $\begin{cases} \tilde{S}^{+}(s)\tilde{S}(s) = I_{4s} \Leftrightarrow N(s)\bar{N}(s) + X(s)\bar{X}(s) = I_{4s} \\ \tilde{S}(s)\tilde{S}^{+}(s) = I_{4s} \Leftrightarrow \bar{N}(s)N(s) = I_{2s+1}, \bar{X}(s)X(s) = I_{2s-1}, \bar{N}(s)X(s) = 0, \bar{X}(s)N(s) = 0 \end{cases}$

Cor. 1.2.2. $\begin{cases} \tilde{S}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})] = \begin{bmatrix} \sigma(s) & 0 \\ 0 & \sigma(s-1) \end{bmatrix} \tilde{S}(s) \\ [\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})]\tilde{S}^{+}(s) = \tilde{S}^{+}(s) \begin{bmatrix} \sigma(s) & 0 \\ 0 & \sigma(s-1) \end{bmatrix} \end{cases}$

Cor. 1.2.3. $\tilde{S}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})]\tilde{S}^{+}(s) = \begin{bmatrix} \sigma(s) & 0 \\ 0 & \sigma(s-1) \end{bmatrix}$

Cor. 1.2.4. $$\tilde{S}(s) = \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2s-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2s-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s-1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s} \\ 0 & -\sqrt{2s-1} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2s-2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{2s-1} & 0 \end{bmatrix} = \begin{bmatrix} N_{1_\varsigma}(s) \sqcup N_{2_\varsigma}(s) \\ \sqrt{1-\frac{1}{2s}} \cdot \bar{N}_{2_\varsigma}(s-\frac{1}{2}) \sqcup [-\bar{N}_{1_\varsigma}(s-\frac{1}{2})] \end{bmatrix}$$

Cor. 1.2.5. $$\tilde{S}^{+}(s) = \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 & 0 & -\sqrt{2s-1} & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2s-2} & 0 & 0 \\ 0 & 0 & \sqrt{2s-2} & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s-1} & 0 & 0 & 0 & 0 & -\sqrt{1} \\ 0 & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & 0 & 0 & \sqrt{2s-1} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2s} & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.2.2 Several concrete representations of representation transformation matrix $\tilde{S}(s)$

Cor. 1.2.6.

$$\tilde{S}(\tfrac{1}{2}, 1, \cdot\cdot) = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{1} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & -\sqrt{2} & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & \sqrt{2} & 0 \end{bmatrix}, \frac{1}{\sqrt{4}} \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & -\sqrt{3} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{3} & 0 \end{bmatrix}, \cdots$$

Cor. 1.2.7.

$$\tilde{S}^+(\tfrac{1}{2},1,\cdot\cdot)=\begin{bmatrix}\sqrt{1}&0\\0&\sqrt{1}\end{bmatrix},\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&-\sqrt{1}\\0&\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\end{bmatrix},\frac{1}{\sqrt{3}}\begin{bmatrix}\sqrt{3}&0&0&0&0&0\\0&\sqrt{1}&0&0&-\sqrt{2}&0\\0&\sqrt{2}&0&0&\sqrt{1}&0\\0&0&\sqrt{2}&0&0&-\sqrt{1}\\0&0&\sqrt{1}&0&0&\sqrt{2}\\0&0&0&\sqrt{3}&0&0\end{bmatrix},\frac{1}{\sqrt{4}}\begin{bmatrix}\sqrt{4}&0&0&0&0&0&0&0\\0&\sqrt{1}&0&0&0&-\sqrt{3}&0&0\\0&\sqrt{3}&0&0&0&\sqrt{1}&0&0\\0&0&\sqrt{2}&0&0&0&-\sqrt{2}&0\\0&0&\sqrt{2}&0&0&0&\sqrt{2}&0\\0&0&0&\sqrt{3}&0&0&0&-\sqrt{1}\\0&0&0&\sqrt{1}&0&0&0&\sqrt{3}\\0&0&0&0&\sqrt{4}&0&0&0\end{bmatrix},\cdots$$

1.2.3 Introduction and concrete representations of constant matrix $O(s)$

Def. 1.2.2. $\begin{cases}X^{A_\varsigma l_\varsigma}_{m_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})N^{n_\varsigma}_{B_\varsigma l_\varsigma}(s):=\frac{1}{2s}O^{\alpha_\varsigma}{}_{l_\varsigma}{}^{n_\varsigma}(s)\Leftrightarrow X^{A_\varsigma}(s)\sigma^{\alpha_\varsigma}{}_{A_\varsigma}{}^{B_\varsigma}(\frac{1}{2})\bar{N}_{B_\varsigma}(s)=\frac{1}{2s}O(s)\\ \bar{X}(s)\sigma(\frac{1}{2})\otimes I_{2s}N(s)=\frac{1}{2s}O(s)\Leftrightarrow\bar{N}(s)\sigma(\frac{1}{2})\otimes I_{2s}X(s)=\frac{1}{2s}O^+(s)\end{cases}$

Thm. 1.2.1. $\begin{cases}O^+(s)\cdot O(s)=s(2s-1)I_{2s+1},O(s)\cdot O^+(s)=s(2s+1)I_{2s-1}\\ O(s)\cdot\sigma(s)=\sigma(s-1)\cdot O(s),\sigma(s)\cdot O^+(s)=O^+(s)\cdot\sigma(s-1)\end{cases}$

Proof: $\tilde{S}(s)\sigma(\frac{1}{2})\otimes I_{2s}\tilde{S}^+(s)\cdot\tilde{S}(s)\sigma(\frac{1}{2})\otimes I_{2s}\tilde{S}^+(s)=\frac{1}{2s}\begin{bmatrix}\sigma(s)&O^+(s)\\O(s)&-\sigma(s-1)\end{bmatrix}\cdot\frac{1}{2s}\begin{bmatrix}\sigma(s)&O^+(s)\\O(s)&-\sigma(s-1)\end{bmatrix}$

$\Leftrightarrow\frac{3}{4}=\frac{1}{4s^2}\begin{bmatrix}\sigma^2(s)+O^+(s)\cdot O(s)&\sigma(s)\cdot O^+(s)-O^+(s)\cdot\sigma(s-1)\\O(s)\cdot\sigma(s)-\sigma(s-1)\cdot O(s)&O(s)\cdot O^+(s)+\sigma^2(s-1)\end{bmatrix}$

$\Leftrightarrow\begin{cases}O^+(s)\cdot O(s)=s(2s-1)I_{2s+1},O(s)\cdot O^+(s)=s(2s+1)I_{2s-1}\\ O(s)\cdot\sigma(s)=\sigma(s-1)\cdot O(s),\sigma(s)\cdot O^+(s)=O^+(s)\cdot\sigma(s-1)\end{cases}$ □

Cor. 1.2.8. $O_x(s)=-\sqrt{s(s-\frac{1}{2})}[\bar{N}_{1_\varsigma}(s-\frac{1}{2})\bar{N}_{1_\varsigma}(s)-\bar{N}_{2_\varsigma}(s-\frac{1}{2})\bar{N}_{2_\varsigma}(s)]$

$$=\frac{1}{2}\begin{bmatrix}-\sqrt{2s\cdot(2s-1)}&0&\sqrt{2\cdot1}&0&0&0\\0&-\sqrt{(2s-1)\cdot(2s-2)}&0&\sqrt{3\cdot2}&0&0\\0&0&\cdots&0&\cdots&0\\0&0&0&-\sqrt{2\cdot1}&0&\sqrt{2s\cdot(2s-1)}\end{bmatrix}$$

Cor. 1.2.9. $O_y(s)=-i\sqrt{s(s-\frac{1}{2})}[\bar{N}_{1_\varsigma}(s-\frac{1}{2})\bar{N}_{1_\varsigma}(s)+\bar{N}_{2_\varsigma}(s-\frac{1}{2})\bar{N}_{2_\varsigma}(s)]$

$$=\frac{i}{2}\begin{bmatrix}-\sqrt{2s\cdot(2s-1)}&0&-\sqrt{2\cdot1}&0&0&0\\0&-\sqrt{(2s-1)\cdot(2s-2)}&0&-\sqrt{3\cdot2}&0&0\\0&0&\cdots&0&\cdots&0\\0&0&0&-\sqrt{2\cdot1}&0&-\sqrt{2s\cdot(2s-1)}\end{bmatrix}$$

Cor. 1.2.10. $O_z(s)=\sqrt{s(s-\frac{1}{2})}[\bar{N}_{1_\varsigma}(s-\frac{1}{2})\bar{N}_{2_\varsigma}(s)+\bar{N}_{2_\varsigma}(s-\frac{1}{2})\bar{N}_{1_\varsigma}(s)]$

$$=\begin{bmatrix}0&\sqrt{1\cdot(2s-1)}&0&0&0&0\\0&0&\sqrt{2\cdot(2s-2)}&0&0&0\\0&0&0&\cdots&0&0\\0&0&0&0&\sqrt{(2s-1)\cdot1}&0\end{bmatrix},\bar{N}_{1_\varsigma}(s-\tfrac{1}{2})\bar{N}_{2_\varsigma}(s)=\bar{N}_{2_\varsigma}(s-\tfrac{1}{2})\bar{N}_{1_\varsigma}(s)$$

Cor. 1.2.11. $O(2)=\frac{1}{2}\begin{bmatrix}-\sqrt{4\cdot3}&0&\sqrt{2\cdot1}&0&0\\0&-\sqrt{3\cdot2}&0&\sqrt{3\cdot2}&0\\0&0&-\sqrt{2\cdot1}&0&\sqrt{4\cdot3}\end{bmatrix},\frac{i}{2}\begin{bmatrix}-\sqrt{4\cdot3}&0&-\sqrt{2\cdot1}&0&0\\0&-\sqrt{3\cdot2}&0&-\sqrt{3\cdot2}&0\\0&0&-\sqrt{2\cdot1}&0&-\sqrt{4\cdot3}\end{bmatrix},\begin{bmatrix}0&\sqrt{1\cdot3}&0&0&0\\0&0&\sqrt{2\cdot2}&0&0\\0&0&0&\sqrt{3\cdot1}&0\end{bmatrix}$

1.2.4 Introduction and concrete representations of constant matrix $\Sigma(s)$

Def. 1.2.3. $\Sigma(s):=\tilde{S}(s)\sigma(\frac{1}{2})\otimes I_{2s}\tilde{S}^+(s)=\frac{1}{2s}\begin{bmatrix}\sigma(s)&O^+(s)\\O(s)&-\sigma(s-1)\end{bmatrix},\tilde{S}(s)=\begin{bmatrix}\bar{N}(s)\\\bar{X}(s)\end{bmatrix}$

Cor. 1.2.12. $\Sigma(1)=\frac{1}{4}\{\begin{bmatrix}0&\sqrt{1\cdot2}&0&-\sqrt{2\cdot1}\\\sqrt{1\cdot2}&0&\sqrt{2\cdot1}&0\\0&\sqrt{2\cdot1}&0&\sqrt{2\cdot1}\\-\sqrt{2\cdot1}&0&\sqrt{2\cdot1}&0\end{bmatrix},i\begin{bmatrix}0&-\sqrt{1\cdot2}&0&\sqrt{2\cdot1}\\\sqrt{1\cdot2}&0&-\sqrt{2\cdot1}&0\\0&\sqrt{2\cdot1}&0&\sqrt{2\cdot1}\\-\sqrt{2\cdot1}&0&-\sqrt{2\cdot1}&0\end{bmatrix},\begin{bmatrix}2&0&0&0\\0&0&0&2\sqrt{1\cdot1}\\0&0&-2&0\\0&2\sqrt{1\cdot1}&0&0\end{bmatrix}\}$

Cor. 1.2.13. $O(1)=\frac{1}{2}\{\begin{bmatrix}-\sqrt{2\cdot1}&0&\sqrt{2\cdot1}\end{bmatrix},i\begin{bmatrix}-\sqrt{2\cdot1}&0&-\sqrt{2\cdot1}\end{bmatrix},\begin{bmatrix}0&2\sqrt{1\cdot1}&0\end{bmatrix}\}$

Cor. 1.2.14. $\Sigma(\frac{3}{2})$

$$=\frac{1}{6}\{\begin{bmatrix}0&\sqrt{1\cdot3}&0&0&-\sqrt{3\cdot2}&0\\\sqrt{1\cdot3}&0&\sqrt{2\cdot2}&0&0&-\sqrt{2\cdot1}\\0&\sqrt{2\cdot2}&0&\sqrt{3\cdot1}&\sqrt{2\cdot1}&0\\0&0&\sqrt{3\cdot1}&0&0&\sqrt{3\cdot2}\\-\sqrt{3\cdot2}&0&\sqrt{2\cdot1}&0&0&-\sqrt{1\cdot1}\\0&-\sqrt{2\cdot1}&0&\sqrt{3\cdot2}&-\sqrt{1\cdot1}&0\end{bmatrix},i\begin{bmatrix}0&-\sqrt{1\cdot3}&0&0&\sqrt{3\cdot2}&0\\\sqrt{1\cdot3}&0&-\sqrt{2\cdot2}&0&0&\sqrt{2\cdot1}\\0&\sqrt{2\cdot2}&0&-\sqrt{3\cdot1}&\sqrt{2\cdot1}&0\\0&0&\sqrt{3\cdot1}&0&0&\sqrt{3\cdot2}\\-\sqrt{3\cdot2}&0&-\sqrt{2\cdot1}&0&0&\sqrt{1\cdot1}\\0&-\sqrt{2\cdot1}&0&-\sqrt{3\cdot2}&-\sqrt{1\cdot1}&0\end{bmatrix},\begin{bmatrix}3&0&0&0&0&0\\0&1&0&0&2\sqrt{1\cdot2}&0\\0&0&-1&0&0&2\sqrt{2\cdot1}\\0&0&0&3&0&0\\0&2\sqrt{1\cdot2}&0&0&-1&0\\0&0&2\sqrt{2\cdot1}&0&0&1\end{bmatrix}\}$$

Cor. 1.2.15.

$$\Sigma(2)=\frac{1}{8}\{\begin{bmatrix}0&\sqrt{1\cdot4}&0&0&0&-\sqrt{4\cdot3}&0&0\\\sqrt{1\cdot4}&0&\sqrt{2\cdot3}&0&0&0&-\sqrt{3\cdot2}&0\\0&\sqrt{2\cdot3}&0&\sqrt{3\cdot2}&0&\sqrt{2\cdot1}&0&-\sqrt{2\cdot1}\\0&0&\sqrt{3\cdot2}&0&\sqrt{4\cdot1}&0&\sqrt{3\cdot2}&0\\0&0&0&\sqrt{4\cdot1}&0&0&0&\sqrt{4\cdot3}\\-\sqrt{4\cdot3}&0&\sqrt{2\cdot1}&0&0&0&-\sqrt{1\cdot2}&0\\0&-\sqrt{3\cdot2}&0&\sqrt{3\cdot2}&0&-\sqrt{1\cdot2}&0&-\sqrt{2\cdot1}\\0&0&-\sqrt{2\cdot1}&0&\sqrt{4\cdot3}&0&-\sqrt{2\cdot1}&0\end{bmatrix},i\begin{bmatrix}0&-\sqrt{1\cdot4}&0&0&0&\sqrt{4\cdot3}&0&0\\\sqrt{1\cdot4}&0&-\sqrt{2\cdot3}&0&0&0&\sqrt{3\cdot2}&0\\0&\sqrt{2\cdot3}&0&-\sqrt{3\cdot2}&0&\sqrt{2\cdot1}&0&\sqrt{2\cdot1}\\0&0&\sqrt{3\cdot2}&0&-\sqrt{4\cdot1}&0&\sqrt{3\cdot2}&0\\0&0&0&\sqrt{4\cdot1}&0&0&0&\sqrt{4\cdot3}\\-\sqrt{4\cdot3}&0&-\sqrt{2\cdot1}&0&0&0&\sqrt{1\cdot2}&0\\0&-\sqrt{3\cdot2}&0&-\sqrt{3\cdot2}&0&-\sqrt{1\cdot2}&0&\sqrt{2\cdot1}\\0&0&-\sqrt{2\cdot1}&0&-\sqrt{4\cdot3}&0&-\sqrt{2\cdot1}&0\end{bmatrix}$$

$$,\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2\sqrt{1\cdot 3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2\cdot 2} & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 2\sqrt{3\cdot 1} \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 2\sqrt{1\cdot 3} & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2\sqrt{2\cdot 2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{3\cdot 1} & 0 & 0 & 0 & 2 \end{bmatrix}\}$$

1.2.5 Equivalent separated equation for massless particles

Thm. 1.2.2. $(\sigma \otimes I_{2s}, -i\varsigma)^a \partial_a \tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma) \Leftrightarrow \begin{cases} [\frac{1}{s}\sigma(s), -i\varsigma]^a \partial_a \psi(s,\varsigma) = i\bar{N}(s)\tilde{J}(s,\varsigma) \\ \frac{1}{s}O(s)\cdot\nabla\psi(s,\varsigma) = i\bar{X}(s)\tilde{J}(s,\varsigma) \end{cases}$

Cor. 1.2.16. $\psi(s,\varsigma) = \bar{N}(s)\tilde{\psi}(s,\varsigma), 0_{2s-1} = \bar{X}(s)\tilde{\psi}(s,\varsigma)$

1.3 Introduction of representation transformation matrix $\hat{S}(s)$

Def. 1.3.1. $\hat{S}(s) = \begin{bmatrix} \tilde{S}(s) & 0 \\ 0 & I_{4^s-4s} \end{bmatrix} I \otimes \hat{S}(s-\frac{1}{2}), \tilde{S}(s) = \begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix}$

Cor. 1.3.1. $\hat{S}(s=\frac{1}{2},1,\frac{3}{2},2,\cdots) = I, \begin{bmatrix} \bar{N}(1) \\ \bar{X}(1) \end{bmatrix}, \begin{bmatrix} \bar{N}(\frac{3}{2})[I\otimes\bar{N}(1)] \\ \bar{X}(\frac{3}{2})[I\otimes\bar{N}(1)] \\ I\otimes\bar{X}(1) \end{bmatrix}, \begin{bmatrix} \bar{N}(2)[I\otimes[\bar{N}(\frac{3}{2})[I\otimes\bar{N}(1)]]] \\ \bar{X}(2)[I\otimes[\bar{N}(\frac{3}{2})[I\otimes\bar{N}(1)]]] \\ I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{N}(1)]] \\ I\otimes I\otimes\bar{X}(1) \end{bmatrix}, \cdots$

Cor. 1.3.2. $\hat{S}(s=\frac{1}{2},1,\frac{3}{2},2,\cdots) = I, \begin{bmatrix} \bar{N}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})] \\ \bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})] \end{bmatrix}, \begin{bmatrix} \bar{N}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)] \\ \bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)] \\ I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \begin{bmatrix} \bar{N}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ \bar{X}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)] \\ I\otimes I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \cdots$

Cor. 1.3.3. $\hat{S}(s=\frac{1}{2},1,\frac{3}{2},2,\cdots) = I, \begin{bmatrix} \bar{\Gamma}(1) \\ \bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})] \end{bmatrix}, \begin{bmatrix} \bar{\Gamma}(\frac{3}{2}) \\ \bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)] \\ I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \begin{bmatrix} \bar{\Gamma}(2) \\ \bar{X}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)] \\ I\otimes I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \cdots$

Cor. 1.3.4. $\hat{S}(s) = \begin{bmatrix} \bar{\Gamma}(s) \\ \bar{X}(s)[I\otimes\bar{\Gamma}(s-\frac{1}{2})] \\ I\otimes[\bar{X}(s-\frac{1}{2})[I\otimes\bar{\Gamma}(s-\frac{3}{2})]] \\ \cdots \\ (I\otimes)^{2s-3}[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)]] \\ (I\otimes)^{2s-2}[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \hat{S}(s)\hat{S}^+(s) = \hat{S}^+(s)\hat{S}(s) = I_{4^s}$

Cor. 1.3.5. $\hat{S}^+(s) = [\Gamma(s), [I\otimes\Gamma(s)]X(s-\frac{1}{2}), I\otimes[[I\otimes\Gamma(s-\frac{1}{2})]X(s-\frac{3}{2})], \cdots, I\otimes\cdots I\otimes[[I\otimes\Gamma(\frac{1}{2})]X(1)]]$

Cor. 1.3.6. $\bar{\Gamma}(s)\Gamma(s) = I_{2s+1}, \bar{\Gamma}(s)\cdot I_{4^k}\otimes\{[I\otimes\Gamma(s-\frac{1}{2}-k)]X(s-k)\} = 0; k = 0, \frac{1}{2}, 1, \cdots, s-1$

Cor. 1.3.7. $\hat{S}(s)[\sigma(\frac{1}{2})\otimes I_{2^{2s-1}}]\hat{S}^+(s) = \frac{1}{2s}\begin{bmatrix} \sigma(s) & O^+(s) & 0 \\ O(s) & -\sigma(s-1) & 0 \\ 0 & 0 & 2s\sigma\otimes I_{(2^{2s-1}-2s)} \end{bmatrix}$

1.3.1 Several specific representations of representation transformation matrix $\hat{S}(s)$

Cor. 1.3.8. $\hat{S}(\frac{1}{2}) = I, \hat{S}^+(\frac{1}{2}) = I$

Cor. 1.3.9. $\hat{S}(1) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \hat{S}^+(1) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \\ 0 & \sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}$

Cor. 1.3.10. $\hat{S}(\frac{3}{2}) = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & -\sqrt{4} & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{1} & \sqrt{4} & 0 \\ 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 \end{bmatrix}, \hat{S}^+(\frac{3}{2}) = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{4} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{4} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \end{bmatrix}$

Cor. 1.3.11. $\hat{S}(2)=\frac{1}{\sqrt{12}}\begin{bmatrix}\sqrt{12}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&\sqrt{3}&\sqrt{3}&0&\sqrt{3}&0&0&0&\sqrt{3}&0&0&0&0&0&0&0\\0&0&0&\sqrt{2}&0&\sqrt{2}&\sqrt{2}&0&0&\sqrt{2}&\sqrt{2}&0&\sqrt{2}&0&0&0\\0&0&0&0&0&0&0&\sqrt{3}&0&0&0&\sqrt{3}&0&\sqrt{3}&\sqrt{3}&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\sqrt{12}\\0&-\sqrt{9}&\sqrt{1}&0&\sqrt{1}&0&0&0&\sqrt{1}&0&0&0&0&0&0&0\\0&0&0&-\sqrt{2}&0&-\sqrt{2}&\sqrt{2}&0&0&-\sqrt{2}&\sqrt{2}&0&\sqrt{2}&0&0&0\\0&0&0&0&0&0&0&-\sqrt{1}&0&0&0&-\sqrt{1}&0&-\sqrt{1}&\sqrt{9}&0\\0&0&-\sqrt{8}&0&\sqrt{2}&0&0&0&\sqrt{2}&0&0&0&0&0&0&0\\0&0&0&-\sqrt{8}&0&\sqrt{2}&0&0&0&\sqrt{2}&0&0&0&0&0&0\\0&0&0&0&0&0&-\sqrt{2}&0&0&0&-\sqrt{2}&0&\sqrt{8}&0&0&0\\0&0&0&0&0&0&0&-\sqrt{2}&0&0&0&-\sqrt{2}&0&\sqrt{8}&0&0\\0&0&0&0&-\sqrt{6}&0&0&0&\sqrt{6}&0&0&0&0&0&0&0\\0&0&0&0&0&-\sqrt{6}&0&0&0&\sqrt{6}&0&0&0&0&0&0\\0&0&0&0&0&0&-\sqrt{6}&0&0&0&\sqrt{6}&0&0&0&0&0\\0&0&0&0&0&0&0&-\sqrt{6}&0&0&0&\sqrt{6}&0&0&0&0\end{bmatrix}$

Cor. 1.3.12. $\hat{S}^{+}(2)=\frac{1}{\sqrt{12}}\begin{bmatrix}\sqrt{12}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&\sqrt{3}&0&0&0&-\sqrt{9}&0&0&0&0&0&0&0&0&0&0\\0&\sqrt{3}&0&0&0&\sqrt{1}&0&0&-\sqrt{8}&0&0&0&0&0&0&0\\0&0&\sqrt{2}&0&0&0&-\sqrt{2}&0&0&-\sqrt{8}&0&0&0&0&0&0\\0&\sqrt{3}&0&0&0&\sqrt{1}&0&0&\sqrt{2}&0&0&0&-\sqrt{6}&0&0&0\\0&0&\sqrt{2}&0&0&0&-\sqrt{2}&0&0&\sqrt{2}&0&0&0&-\sqrt{6}&0&0\\0&0&\sqrt{2}&0&0&0&\sqrt{2}&0&0&0&-\sqrt{2}&0&0&0&-\sqrt{6}&0\\0&0&0&\sqrt{3}&0&0&0&-\sqrt{1}&0&0&0&-\sqrt{2}&0&0&0&-\sqrt{6}\\0&\sqrt{3}&0&0&0&\sqrt{1}&0&0&\sqrt{2}&0&0&0&\sqrt{6}&0&0&0\\0&0&\sqrt{2}&0&0&0&-\sqrt{2}&0&0&\sqrt{2}&0&0&0&\sqrt{6}&0&0\\0&0&\sqrt{2}&0&0&0&\sqrt{2}&0&0&0&-\sqrt{2}&0&0&0&\sqrt{6}&0\\0&0&0&\sqrt{3}&0&0&0&-\sqrt{1}&0&0&0&-\sqrt{2}&0&0&0&\sqrt{6}\\0&0&\sqrt{2}&0&0&0&\sqrt{2}&0&0&0&\sqrt{8}&0&0&0&0&0\\0&0&0&\sqrt{3}&0&0&0&-\sqrt{1}&0&0&0&\sqrt{8}&0&0&0&0\\0&0&0&\sqrt{3}&0&0&0&\sqrt{9}&0&0&0&0&0&0&0&0\\0&0&0&0&\sqrt{12}&0&0&0&0&0&0&0&0&0&0&0\end{bmatrix}$

Cor. 1.3.13. $\bar{X}(s=1,\frac{3}{2},2)\bar{\Gamma}(s=1,\frac{3}{2},2,\cdots)$
$=\frac{1}{\sqrt{2}}\begin{bmatrix}0&-\sqrt{1}&\sqrt{1}&0\end{bmatrix},\frac{1}{\sqrt{6}}\begin{bmatrix}0&-2&1&0&1&0&0&0\\0&0&0&-1&0&-1&2&0\end{bmatrix},\frac{1}{\sqrt{12}}\begin{bmatrix}0&-3&\sqrt{1}&0&\sqrt{1}&0&0&0&\sqrt{1}&0&0&0&0&0&0&0\\0&0&0&-\sqrt{2}&0&-\sqrt{2}&\sqrt{2}&0&0&-\sqrt{2}&\sqrt{2}&0&\sqrt{2}&0&0&0\\0&0&0&0&0&0&0&-\sqrt{1}&0&0&0&-\sqrt{1}&0&-\sqrt{1}&3&0\end{bmatrix}$

1.4 An important theorem and its proof on representation transformation matrix $\hat{S}(s)$

Def. 1.4.1. $\pi(s,s'):=\Omega(s')\otimes I_{2(s-s')-1}+I_{4^{s'}}\otimes\sigma(s-s'-1); s'\geq 0, s-s'\geq 1$

Lem. 1.4.1. $\hat{S}(s)\Omega(s=1,\frac{3}{2},2)\hat{S}^{+}(s)=\begin{bmatrix}\sigma(1)&0\\0&\sigma(0)\end{bmatrix},\begin{bmatrix}\sigma(\frac{3}{2})&0&0\\0&\sigma(\frac{1}{2})&0\\0&0&\sigma(\frac{1}{2})\end{bmatrix},\begin{bmatrix}\sigma(2)&0&0&0\\0&\sigma(1)&0&0\\0&0&\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})&0\\0&0&0&\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})\end{bmatrix}$

Thm. 1.4.1.
$$\hat{S}(s)\Omega(s)=\begin{bmatrix}\sigma(s)&0&0&0&0&0\\0&\pi(s,0)&0&0&0&0\\0&0&\pi(s,\frac{1}{2})&0&0&0\\0&0&0&\cdots&0&0\\0&0&0&0&\pi(s,s-\frac{3}{2})&0\\0&0&0&0&0&\pi(s,s-1)\end{bmatrix}\hat{S}(s)[\Leftrightarrow]\Omega(s)\hat{S}^{+}(s)=\hat{S}^{+}(s)\begin{bmatrix}\sigma(s)&0&0&0&0&0\\0&\pi(s,0)&0&0&0&0\\0&0&\pi(s,\frac{1}{2})&0&0&0\\0&0&0&\cdots&0&0\\0&0&0&0&\pi(s,s-\frac{3}{2})&0\\0&0&0&0&0&\pi(s,s-1)\end{bmatrix}$$

Proof: Use mathematical induction to prove this theorem.
Step 1: When $s'=1$, the following is established: $\hat{S}(1)\Omega(1)=\begin{bmatrix}\sigma(1)&0\\0&\sigma(0)\end{bmatrix}\hat{S}(1)$
Step 2: Assume when $s'=s-\frac{1}{2}$, the following is established:
$$\hat{S}(s-\tfrac{1}{2})\Omega(s-\tfrac{1}{2})=\begin{bmatrix}\sigma(s-\frac{1}{2})&0&0&0&0&0\\0&\pi(s-\frac{1}{2},0)&0&0&0&0\\0&0&\pi(s-\frac{1}{2},\frac{1}{2})&0&0&0\\0&0&0&\cdots&0&0\\0&0&0&0&\pi(s-\frac{1}{2},s-\frac{1}{2}-\frac{3}{2})&0\\0&0&0&0&0&\pi(s-\frac{1}{2},s-\frac{1}{2}-1)\end{bmatrix}\hat{S}(s-\tfrac{1}{2})$$
Step 3: When $s'=s$, $\hat{S}(s)\Omega(s)=\begin{bmatrix}\tilde{S}(s)&0\\0&I_{4^s-4s}\end{bmatrix}[I\otimes\hat{S}(s-\frac{1}{2})][\sigma(\frac{1}{2})\otimes I_{2^{2s-1}}+I\otimes\Omega(s-\frac{1}{2})]$
$=\begin{bmatrix}\tilde{S}(s)&0\\0&I_{4^s-4s}\end{bmatrix}\{\sigma(\frac{1}{2})\otimes[I_{2^{2s-1}}\hat{S}(s)]+\{I\otimes[\hat{S}(s-\frac{1}{2})\Omega(s-\frac{1}{2})]\}\}$
$$=\begin{bmatrix}\tilde{S}(s)&0\\0&I_{4^s-4s}\end{bmatrix}[\sigma(\tfrac{1}{2})\otimes I_{2^{2s-1}}+I\otimes\begin{bmatrix}\sigma(s-\frac{1}{2})&0&0&0&0&0\\0&\pi(s-\frac{1}{2},0)&0&0&0&0\\0&0&\pi(s-\frac{1}{2},\frac{1}{2})&0&0&0\\0&0&0&\cdots&0&0\\0&0&0&0&\pi(s-\frac{1}{2},s-\frac{1}{2}-\frac{3}{2})&0\\0&0&0&0&0&\pi(s-\frac{1}{2},s-\frac{1}{2}-1)\end{bmatrix}][I\otimes\hat{S}(s-\tfrac{1}{2})]$$

$$= \begin{bmatrix} \tilde{S}(s) & 0 \\ 0 & I_{4^s-4s} \end{bmatrix} [\begin{bmatrix} \sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s,\frac{1}{2}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s,\frac{3}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s,s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s,s-1) \end{bmatrix}][I\otimes\hat{S}(s-\tfrac{1}{2})]$$

$$= \begin{bmatrix} \tilde{S}(s)[\sigma(\frac{1}{2})\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})] & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s,\frac{1}{2}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s,\frac{3}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s,s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s,s-1) \end{bmatrix} [I\otimes\hat{S}(s-\tfrac{1}{2})]$$

$$= \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s,0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s,\frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s,s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s,s-1) \end{bmatrix} \begin{bmatrix} \tilde{S}(s) & 0 \\ 0 & I_{4^s-4s} \end{bmatrix} [I\otimes\hat{S}(s-\tfrac{1}{2})]$$

$$= \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s,0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s,\frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s,s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s,s-1) \end{bmatrix} \hat{S}(s)$$

This step proves that when $s' = s$ it is established.

Step 4: Based on the above inductive reasoning, the proposition is established and the theorem is proved. □

Cor. 1.4.1. $\hat{S}(s)\Omega(s)\hat{S}^{+}(s) = \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s,0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s,\frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s,s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s,s-1) \end{bmatrix}$

Finally, the above conclusion has been strictly proved, and the previous complex properties of some constant invariant tensors can be easily obtained, as follows:

Cor. 1.4.2.

$$\begin{cases} \bar{\Gamma}(s)\Omega(s) = \sigma(s)\bar{\Gamma}(s), \Omega(s)\Gamma(s) = \Gamma(s)\sigma(s) \\ \bar{X}(s)[I\otimes\bar{\Gamma}(s-\frac{1}{2})]\Omega(s) = \sigma(s-1)\bar{X}(s)[I\otimes\bar{\Gamma}(s-\frac{1}{2})], \Omega(s)[I\otimes\Gamma(s-\frac{1}{2})]X(s) = [I\otimes\Gamma(s-\frac{1}{2})]X(s)\sigma(s-1) \end{cases}$$

Cor. 1.4.3.

$$\begin{cases} I_{4^{s-1}}\otimes\{\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]\}\Omega(s) = \Omega(s-1)I_{4^{s-1}}\otimes\{\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]\} \\ \Omega(s)I_{4^{s-1}}\otimes\{[I\otimes\Gamma(\frac{1}{2})]X(1)\} = I_{4^{s-1}}\otimes\{[I\otimes\Gamma(\frac{1}{2})]X(1)\}\Omega(s-1) \end{cases}$$

Cor. 1.4.4.

$$\begin{cases} I_{4^k}\otimes\{\bar{X}(s-k)[I\otimes\bar{\Gamma}(s-\frac{1}{2}-k)]\}\Omega(s) = \pi(s,k)I_{4^k}\otimes\{\bar{X}(s-k)[I\otimes\bar{\Gamma}(s-\frac{1}{2}-k)]\} \\ \Omega(s)I_{4^k}\otimes\{[I\otimes\Gamma(s-\frac{1}{2}-k)]X(s-k)\} = I_{4^k}\otimes\{[I\otimes\Gamma(s-\frac{1}{2}-k)]X(s-k)\}\pi(s,k) \end{cases}$$

Cor. 1.4.5.

$$\begin{cases} \sigma(s) = \bar{\Gamma}(s)\Omega(s)\Gamma(s) \\ \sigma(s-1) = \bar{X}(s)[I\otimes\bar{\Gamma}(s-\frac{1}{2})]\Omega(s)[I\otimes\Gamma(s-\frac{1}{2})]X(s) \\ \Omega(s-1) = I_{4^{s-1}}\otimes\{\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]\}\Omega(s)I_{4^{s-1}}\otimes\{[I\otimes\Gamma(\frac{1}{2})]X(1)\} \\ \pi(s,k) = I_{4^k}\otimes\{\bar{X}(s-k)[I\otimes\bar{\Gamma}(s-\frac{1}{2}-k)]\}\Omega(s)I_{4^k}\otimes\{[I\otimes\Gamma(s-\frac{1}{2}-k)]X(s-k)\} \end{cases}$$

1.5 Representation transformation of constant matrix $\pi(s,s')$

Cor. 1.5.1. $\pi(s,s') = \Omega(s'-\frac{1}{2})\otimes I_{4(s-s')-2} + I_{2^{2s'-1}}\otimes[\sigma(\frac{1}{2})\otimes I_{2(s-s')-1} + I\otimes\sigma(s-s'-1)]$

Proof: $\pi(s,s') := \Omega(s')\otimes I_{2(s-s')-1} + I_{4^{s'}}\otimes\sigma(s-s'-1)$

$= [\Omega(s'-\frac{1}{2})\otimes I + I_{2^{2s'-1}}\otimes\sigma(\frac{1}{2})]\otimes I_{2(s-s')-1} + I_{4^{s'}}\otimes\sigma(s-s'-1)$

$= \Omega(s'-\frac{1}{2})\otimes I_{4(s-s')-2} + I_{2^{2s'-1}}\otimes[\sigma(\frac{1}{2})\otimes I_{2(s-s')-1} + I\otimes\sigma(s-s'-1)]$ □

Cor. 1.5.2. $[I_{2^{2s'-1}}\otimes\tilde{S}(s-s'-\frac{1}{2})]\pi(s,s') = \begin{bmatrix} \pi(s,s'-\frac{1}{2}) & 0 \\ 0 & \pi(s-1,s'-\frac{1}{2}) \end{bmatrix} [I_{2^{2s'-1}}\otimes\tilde{S}(s-s'-\frac{1}{2})]; s'\geq\frac{1}{2}, s-s'\geq\frac{3}{2}$

Proof: $[I_{2^{2s'-1}}\otimes\tilde{S}(s-s'-\frac{1}{2})]\pi(s,s')$

$= [I_{2^{2s'-1}}\otimes\tilde{S}(s-s'-\frac{1}{2})]\{\Omega(s'-\frac{1}{2})\otimes I_{4(s-s')-2} + I_{2^{2s'-1}}\otimes[\sigma(\frac{1}{2})\otimes I_{2(s-s')-1} + I\otimes\sigma(s-s'-1)]\}$

$$= \{\Omega(s'-\tfrac{1}{2})\otimes I_{4(s-s')-2} + I_{2^{2s'-1}} \otimes \begin{bmatrix} \sigma(s-s'-\frac{1}{2}) & 0 \\ 0 & \sigma(s-s'-\frac{3}{2}) \end{bmatrix}\}[I_{2^{2s'-1}} \otimes \tilde{S}(s-s'-\tfrac{1}{2})]$$

$$= \begin{bmatrix} \Omega(s'-\frac{1}{2})\otimes I_{2(s-s')}+I_{2^{2s'-1}}\otimes\sigma(s-s'-\frac{1}{2}) & 0 \\ 0 & \Omega(s'-\frac{1}{2})\otimes I_{2(s-s')-2}+I_{2^{2s'-1}}\otimes\sigma(s-s'-\frac{3}{2}) \end{bmatrix} [I_{2^{2s'-1}} \otimes \tilde{S}(s-s'-\tfrac{1}{2})]$$

$$= \begin{bmatrix} \pi(s,s'-\frac{1}{2}) & 0 \\ 0 & \pi(s-1,s'-\frac{1}{2}) \end{bmatrix} [I_{2^{2s'-1}} \otimes \tilde{S}(s-s'-\tfrac{1}{2})] \qquad \square$$

Using the above reasoning and iterating repeatedly, the following corollary can be obtained.

Cor. 1.5.3. $I_{2^{2s'-2}} \otimes \{\begin{bmatrix} \tilde{S}(s-s') & 0 \\ 0 & \tilde{S}(s-s'-1) \end{bmatrix} [I\otimes \tilde{S}(s-s'-\frac{1}{2})]\}\pi(s,s'); s'\geq 1, s-s'\geq 2$

$$= \begin{bmatrix} \pi(s,s'-1) & 0 & 0 & 0 \\ 0 & \pi(s-1,s'-1) & 0 & 0 \\ 0 & 0 & \pi(s-1,s'-1) & 0 \\ 0 & 0 & 0 & \pi(s-2,s'-1) \end{bmatrix} I_{2^{2s'-2}} \otimes \{\begin{bmatrix} \tilde{S}(s-s') & 0 \\ 0 & \tilde{S}(s-s'-1) \end{bmatrix} [I\otimes \tilde{S}(s-s'-\tfrac{1}{2})]\}$$

Cor. 1.5.4.

$$I_{2^{2s'-3}} \otimes \{\begin{bmatrix} \begin{bmatrix} \tilde{S}(s-s'+\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{1}{2}) \end{bmatrix}[I\otimes\tilde{S}(s-s')] & 0 \\ 0 & \begin{bmatrix} \tilde{S}(s-s'-\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{3}{2}) \end{bmatrix}[I\otimes\tilde{S}(s-s'-1)] \end{bmatrix} [I\otimes I\otimes \tilde{S}(s-s'-\tfrac{1}{2})]\}\pi(s,s')$$

$$= \begin{bmatrix} \begin{bmatrix} \pi(s,s'-\frac{3}{2}) & 0 \\ 0 & \pi(s-1,s'-\frac{3}{2}) \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} \pi(s-1,s'-\frac{3}{2}) & 0 \\ 0 & \pi(s-2,s'-\frac{3}{2}) \end{bmatrix} & 0 & 0 \\ 0 & 0 & \begin{bmatrix} \pi(s-1,s'-\frac{3}{2}) & 0 \\ 0 & \pi(s-2,s'-\frac{3}{2}) \end{bmatrix} & 0 \\ 0 & 0 & 0 & \begin{bmatrix} \pi(s-2,s'-\frac{3}{2}) & 0 \\ 0 & \pi(s-3,s'-\frac{3}{2}) \end{bmatrix} \end{bmatrix}$$

$$I_{2^{2s'-3}} \otimes \{\begin{bmatrix} \begin{bmatrix} \tilde{S}(s-s'+\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{1}{2}) \end{bmatrix}[I\otimes\tilde{S}(s-s')] & 0 \\ 0 & \begin{bmatrix} \tilde{S}(s-s'-\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{3}{2}) \end{bmatrix}[I\otimes\tilde{S}(s-s'-1)] \end{bmatrix} [I\otimes I\otimes \tilde{S}(s-s'-\tfrac{1}{2})]\}$$

$; s'\geq \frac{3}{2}, s-s'\geq \frac{5}{2}$

1.6 General form of representation transformation for constant matrix $\pi(s,s')$

Cor. 1.6.1. $\{I_{2^{2s'-1}} \otimes \begin{bmatrix} [I_{2^0}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{X}(s-s'-\frac{1}{2})] \end{bmatrix}\}\pi(s,s') = [\pi(s,s'-\frac{1}{2})\oplus\pi(s-1,s'-\frac{1}{2})]\{I_{2^{2s'-1}} \otimes \begin{bmatrix} [I_{2^0}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{X}(s-s'-\frac{1}{2})] \end{bmatrix}\}$
$; s'\geq \frac{1}{2}, s-s'\geq \frac{3}{2}$

Cor. 1.6.2. $\{I_{2^{2s'-2}} \otimes \begin{bmatrix} [I_{2^0}\bar{N}(s-s')][I_{2^1}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\bar{X}(s-s')][I_{2^1}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\bar{N}(s-s'-1)][I_{2^1}\otimes\bar{X}(s-s'-\frac{1}{2})] \\ [I_{2^0}\bar{X}(s-s'-1)][I_{2^1}\otimes\bar{X}(s-s'-\frac{1}{2})] \end{bmatrix}\}\pi(s,s'); s'\geq 1, s-s'\geq 2$

$$= \{[\pi(s,s'-1)\oplus\pi(s-1,s'-1)]\oplus[\pi(s-1,s'-1)\oplus\pi(s-2,s'-1)]\}\{I_{2^{2s'-2}} \otimes \begin{bmatrix} [I_{2^0}\bar{N}(s-s')][I_{2^1}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\bar{X}(s-s')][I_{2^1}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\bar{N}(s-s'-1)][I_{2^1}\otimes\bar{X}(s-s'-\frac{1}{2})] \\ [I_{2^0}\bar{X}(s-s'-1)][I_{2^1}\otimes\bar{X}(s-s'-\frac{1}{2})] \end{bmatrix}\}$$

Cor. 1.6.3.

$$\{I_{2^{2s'-3}} \otimes \begin{bmatrix} [I_{2^0}\otimes\bar{N}(s-s'+\frac{1}{2})][I_{2^1}\otimes\bar{N}(s-s')][I_{2^2}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{X}(s-s'+\frac{1}{2})][I_{2^1}\otimes\bar{N}(s-s')][I_{2^2}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{N}(s-s'-\frac{1}{2})][I_{2^1}\otimes\bar{X}(s-s')][I_{2^2}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{X}(s-s'-\frac{1}{2})][I_{2^1}\otimes\bar{X}(s-s')][I_{2^2}\otimes\bar{N}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{N}(s-s'-\frac{1}{2})][I_{2^1}\otimes\bar{N}(s-s'-1)][I_{2^2}\otimes\bar{X}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{X}(s-s'-\frac{1}{2})][I_{2^1}\otimes\bar{N}(s-s'-1)][I_{2^2}\otimes\bar{X}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{N}(s-s'-\frac{3}{2})][I_{2^1}\otimes\bar{X}(s-s'-1)][I_{2^2}\otimes\bar{X}(s-s'-\frac{1}{2})] \\ [I_{2^0}\otimes\bar{X}(s-s'-\frac{3}{2})][I_{2^1}\otimes\bar{X}(s-s'-1)][I_{2^2}\otimes\bar{X}(s-s'-\frac{1}{2})] \end{bmatrix}\}\pi(s,s'); s'\geq \tfrac{3}{2}, s-s'\geq \tfrac{5}{2}$$

$$= \{[\pi(s,s'-\tfrac{3}{2})\oplus\pi(s-1,s'-\tfrac{3}{2})]\oplus[\pi(s-1,s'-\tfrac{3}{2})\oplus\pi(s-2,s'-\tfrac{3}{2})]\otimes I\oplus[\pi(s-2,s'-\tfrac{3}{2})\oplus\pi(s-3,s'-\tfrac{3}{2})]\}$$

$$\{I_{2^{2s'-3}}\otimes\begin{bmatrix}[I_{2^0}\otimes\bar N(s-s'+\frac12)][I_{2^1}\otimes\bar N(s-s')][I_{2^2}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'+\frac12)][I_{2^1}\otimes\bar N(s-s')][I_{2^2}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar N(s-s'-\frac12)][I_{2^1}\otimes\bar X(s-s')][I_{2^2}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12)][I_{2^1}\otimes\bar X(s-s')][I_{2^2}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar N(s-s'-\frac12)][I_{2^1}\otimes\bar N(s-s'-1)][I_{2^2}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12)][I_{2^1}\otimes\bar N(s-s'-1)][I_{2^2}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar N(s-s'-\frac32)][I_{2^1}\otimes\bar X(s-s'-1)][I_{2^2}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac32)][I_{2^1}\otimes\bar X(s-s'-1)][I_{2^2}\otimes\bar X(s-s'-\frac12)]\end{bmatrix}\}$$

Cor. 1.6.4. $s'\geq\frac{l+1}{2}, s-s'\geq\frac{l+3}{2}$

$$\{I_{2^{2s'-l-1}}\otimes\begin{bmatrix}[I_{2^0}\otimes\bar N(s-s'-\frac12+\frac l2)][I_{2^1}\otimes\bar N(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12+\frac l2)][I_{2^1}\otimes\bar N(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar N(s-s'-\frac12+\frac{l-2}{2})][I_{2^1}\otimes\bar X(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12+\frac{l-2}{2})][I_{2^1}\otimes\bar X(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ \cdots\cdots\\ [I_{2^0}\otimes\bar N(s-s'-\frac12-\frac{l-2}{2})][I_{2^1}\otimes\bar N(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12-\frac{l-2}{2})][I_{2^1}\otimes\bar N(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar N(s-s'-\frac12-\frac l2)][I_{2^1}\otimes\bar X(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12-\frac l2)][I_{2^1}\otimes\bar X(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\end{bmatrix}\}\pi(s,s')$$

$$=\{[\pi(s,s'-\tfrac{l+1}{2})\oplus\pi(s-1,s'-\tfrac{l+1}{2})]\oplus[\pi(s-1,s'-\tfrac{l+1}{2})\oplus\pi(s-2,s'-\tfrac{l+1}{2})]\otimes I\oplus[\pi(s-l,s'-\tfrac{l+1}{2})\oplus\pi(s-l-1,s'-\tfrac{l+1}{2})]\}$$

$$\{I_{2^{2s'-l-1}}\otimes\begin{bmatrix}[I_{2^0}\otimes\bar N(s-s'-\frac12+\frac l2)][I_{2^1}\otimes\bar N(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12+\frac l2)][I_{2^1}\otimes\bar N(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar N(s-s'-\frac12+\frac{l-2}{2})][I_{2^1}\otimes\bar X(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12+\frac{l-2}{2})][I_{2^1}\otimes\bar X(s-s'+\frac{l-2}{2})]\cdot\cdot[I_{2^{l-1}}\otimes\bar N(s-s')][I_{2^l}\otimes\bar N(s-s'-\frac12)]\\ \cdots\cdots\\ [I_{2^0}\otimes\bar N(s-s'-\frac12-\frac{l-2}{2})][I_{2^1}\otimes\bar N(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12-\frac{l-2}{2})][I_{2^1}\otimes\bar N(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar N(s-s'-\frac12-\frac l2)][I_{2^1}\otimes\bar X(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\\ [I_{2^0}\otimes\bar X(s-s'-\frac12-\frac l2)][I_{2^1}\otimes\bar X(s-s'-\frac l2)]\cdot\cdot[I_{2^{l-1}}\otimes\bar X(s-s'-1)][I_{2^l}\otimes\bar X(s-s'-\frac12)]\end{bmatrix}\}$$

1.7 Introduction and properties of representation transformation matrix $S(s)$

Cor. 1.7.1. $$\hat S(s)=\begin{bmatrix}\bar\Gamma(s)\\ \bar X(s)[I\otimes\bar\Gamma(s-\frac12)]\\ I\otimes[\bar X(s-\frac12)[I\otimes\bar\Gamma(s-1)]]\\ \cdots\\ (I\otimes)^{2s-3}[\bar X(\frac32)[I\otimes\bar\Gamma(1)]]\\ (I\otimes)^{2s-2}[\bar X(1)[I\otimes\bar\Gamma(\frac12)]]\end{bmatrix}=\begin{bmatrix}\tilde S(s)I\otimes\bar\Gamma(s-\frac12)\\ I\otimes[\bar X(s-\frac12)[I\otimes\bar\Gamma(s-1)]]\\ \cdots\\ (I\otimes)^{2s-3}[\bar X(\frac32)[I\otimes\bar\Gamma(1)]]\\ (I\otimes)^{2s-2}[\bar X(1)[I\otimes\bar\Gamma(\frac12)]]\end{bmatrix}$$

Cor. 1.7.2. $$S(s)=\begin{bmatrix}I\otimes\bar\Gamma(s-\frac12)\\ I\otimes[\bar X(s-\frac12)[I\otimes\bar\Gamma(s-1)]]\\ \cdots\\ (I\otimes)^{2s-3}[\bar X(\frac32)[I\otimes\bar\Gamma(1)]]\\ (I\otimes)^{2s-2}[\bar X(1)[I\otimes\bar\Gamma(\frac12)]]\end{bmatrix}=I\otimes\hat S(s-\tfrac12), S(s)S^+(s)=S^+(s)S(s)=I_{4^s}$$

Cor. 1.7.3. $S(s)\Omega(s)S^+(s)=$

$$\begin{bmatrix}\sigma(\frac12)\otimes I_{2s}+I\otimes\sigma(s-\frac12) & 0 & 0 & 0 & 0 & 0\\ 0 & \Omega(\frac12)\otimes I_{2s-2}+I\otimes\sigma(s-\frac32) & 0 & 0 & 0 & 0\\ 0 & 0 & \Omega(1)\otimes I_{2s-3}+[I\otimes]^2\sigma(s-2) & 0 & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 0 & \Omega(s-\frac32)\otimes I_2+[I\otimes]^{2s-1}\sigma(\frac12) & 0\\ 0 & 0 & 0 & 0 & 0 & \Omega(s-1)+[I\otimes]^{2s-2}\sigma(0)\end{bmatrix}$$

Def. 1.7.1. $\pi(s,s'):=\Omega(s')\otimes I_{2(s-s')-1}+I_{4^{s'}}\otimes\sigma(s-s'-1); s'\geq 0, s-s'\geq 1$

Cor. 1.7.4. $S(s)(\sigma(\frac12)\otimes I_{2^{2s-1}})S^+(s)=\sigma(\frac12)\otimes I_{2^{2s-1}}$

Cor. 1.7.5. $(\sigma\otimes I_{2^{2s-1}},-i\varsigma)^a\partial_a\hat\varphi(s,\varsigma)=i\hat K(s,\varsigma)\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)^a\partial_a S(s)\hat\varphi(s,\varsigma)=iS(s)\hat K(s,\varsigma)$

Cor. 1.7.6. $(\sigma\otimes I_{2^{2s-1}},-i\varsigma)^a\partial_a\hat\psi(s,\varsigma)=i\hat J(s,\varsigma)\Leftrightarrow\begin{cases}(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\tilde\psi(s,\varsigma)=i\tilde J(s,\varsigma)\\ (\sigma\otimes I_{2^{2s-1}-2s},-i\varsigma)^a\partial_a o(s,\varsigma)=io(s,\varsigma)\end{cases}$

Cor. 1.7.7. $(\sigma\otimes I_{2^{2s-1}},-i\varsigma)^a\partial_a\hat\psi(s,\varsigma)=i\hat J(s,\varsigma)\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\tilde\psi(s,\varsigma)=i\tilde J(s,\varsigma)$

1.8 Representation transformation of graviton

Cor. 1.8.1. $\hat{S}_0(2) = \begin{bmatrix} \bar{N}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ \bar{X}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ \bar{N}(1)\{I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)]\} \\ \bar{X}(1)\{I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)]\} \\ \bar{N}(1)\{I\otimes I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]]\} \\ \bar{X}(1)\{I\otimes I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]]\} \end{bmatrix} = \begin{bmatrix} \tilde{S}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ \tilde{S}(1)\{I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)]\} \\ \tilde{S}(1)\{I\otimes I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]]\} \end{bmatrix} = \begin{bmatrix} \tilde{S}(2) & 0 & 0 \\ 0 & \tilde{S}(1) & 0 \\ 0 & 0 & \tilde{S}(1) \end{bmatrix} [I\otimes\hat{S}(\frac{3}{2})]$

Thm. 1.8.1. $\hat{S}_0(2)\Omega(2)\hat{S}_0^+(2) = \begin{bmatrix} \sigma(2) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma(0) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma(0) \end{bmatrix}, \hat{S}_0(s)[\sigma(\frac{1}{2})\otimes I_8]\hat{S}_0^+(s) = \begin{bmatrix} \Sigma(2) & 0 & 0 \\ 0 & \Sigma(1) & 0 \\ 0 & 0 & \Sigma(1) \end{bmatrix}$

Cor. 1.8.2. $[\sigma\otimes I_8, -i\varsigma]^a\partial_a\hat{\varphi}(2,\varsigma) = i\hat{K}(2,\varsigma) \Leftrightarrow \begin{cases} [2\Sigma(2), -i\varsigma]^a\partial_a\tilde{\varphi}(2,\varsigma) = i\tilde{K}(2,\varsigma) \\ [2\Sigma(1), -i\varsigma]^a\partial_a\tilde{\varphi}(1,\varsigma) = i\tilde{K}(1,\varsigma) \\ [2\Sigma(1'), -i\varsigma]^a\partial_a\tilde{\varphi}(1',\varsigma) = i\tilde{K}(1',\varsigma) \end{cases}$

Thm. 1.8.2. $S_0(2) = \begin{bmatrix} \bar{N}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ \bar{X}(2)[I\otimes\bar{\Gamma}(\frac{3}{2})] \\ \bar{N}(1)\{I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)]\} \\ \bar{N}(1)\{I\otimes I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]]\} \\ \bar{X}(1)\{I\otimes[\bar{X}(\frac{3}{2})[I\otimes\bar{\Gamma}(1)]\} \\ \bar{X}(1)\{I\otimes I\otimes[\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]]\} \end{bmatrix}, S_0(2)\Omega(2)S_0^+(2) = \begin{bmatrix} \sigma(2) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma(1) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(0) & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma(0) \end{bmatrix}$

1.9 In-depth analysis of constant matrix $\pi(s,s')$

Cor. 1.9.1.

$$\begin{cases} \pi(s,0) := \sigma(s-1), s\geq 1; \pi(1,0)=0 \\ \pi(s,\frac{1}{2}) := \sigma(\frac{1}{2})\otimes I_{2(s-1)} + I\otimes\sigma(s-1-\frac{1}{2}), s\geq\frac{3}{2}; \pi(\frac{3}{2},\frac{1}{2}) = \sigma(\frac{1}{2}), \pi(2,\frac{1}{2}) = \pi(2,1) = \Omega(1) \\ \pi(s,s-1) := \Omega(s-1), s\geq 1 \\ \pi(s,s') := \phi, s-s'\leq\frac{1}{2} \\ \Omega(s) = \pi(s+1,s) = \pi(s+1,s-\frac{1}{2}), s\geq\frac{1}{2} \end{cases}$$

Cor. 1.9.2. $\Omega(s) = \pi(s+1,s) = \pi(s+1,s-\frac{1}{2}), s\geq\frac{1}{2}$
$\to [\pi(s+1,s-1)\oplus\pi(s,s-1)], s\geq 1$
$\to [\pi(s+1,s-\frac{3}{2})\oplus\pi(s,s-\frac{3}{2})]\oplus[\pi(s,s-\frac{3}{2})], s\geq\frac{3}{2}$
$\to [\pi(s+1,s-2)\oplus\pi(s,s-2)]\oplus[\pi(s,s-2)\oplus\pi(s-1,s-2)]^2, s\geq 2$
$\to [\pi(s+1,s-\frac{5}{2})\oplus\pi(s,s-\frac{5}{2})]\oplus[\pi(s,s-\frac{5}{2})\oplus\pi(s-1,s-\frac{5}{2})]^3\oplus[\pi(s-1,s-\frac{5}{2})]^2, s\geq\frac{5}{2}$
$\to [\pi(s+1,s-3)\oplus\pi(s,s-3)]\oplus[\pi(s,s-3)\oplus\pi(s-1,s-3)]^4\oplus[\pi(s-1,s-3)\oplus\pi(s-2,s-3)]^5, s\geq 3$
$\to \cdots\cdots$

Self comment: As long as the above method is followed, any $\Omega(s)$ can be concretely decomposed into the direct sum of multiple single spin states through representation transformation. In principle, this problem has been completely solved. In practical application, some calculations need to be made to explicitly write out the concrete representation transformation for use. At the same time, it is also constructively proved that $\Omega(s)$is indeed composed entirely of single spin states and there is no redundant components. $\Omega(s)$ does not contain both Bose and Fermi spin states, but rather represents a Bose or Fermi multiple state. And it traverses high and low boson or Fermi spin states. Generally, except for the highest spin state, other spin states have multiple redundant states.

1.10 Multiple state spin equation

Def. 1.10.1. $[(S^+\sqrt{[S\Omega(s)S^+]^2+\frac{1}{4}}S-\frac{1}{2})\partial_a + iS_{ab}(\Omega(s),\varsigma)\partial^b]\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0;$
$\Omega(s)\times\Omega(s) = i\Omega(s), S\Omega(s)S^+ = \sigma(s)\oplus\sigma(s-1)\oplus\cdots\oplus\sigma(\frac{1}{2})|\sigma(0)$

Cor. 1.10.1.

$$\begin{cases} \{\tilde{S}^+(s)[sI_{2s+1}\oplus(s-1)I_{2s-1}]\tilde{S}(s)\partial_a + iS_{ab}(\pi(s+1,\frac{1}{2}),\varsigma)\partial^b\}\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0 \\ \{[s-1+N(s)\bar{N}(s)]\partial_a + iS_{ab}(\pi(s+1,\frac{1}{2}),\varsigma)\partial^b\}\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0 \\ \{[s-X(s)\bar{X}(s)]\partial_a + iS_{ab}(\pi(s+1,\frac{1}{2}),\varsigma)\partial^b\}\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0 \end{cases}$$

1.11 Representation transformation property 1 for 2-spin

Cor. 1.11.1.

$$S(1\otimes 1)=\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{6}&0&0&0&0&0&0&0&0\\0&\sqrt{3}&0&\sqrt{3}&0&0&0&0&0\\0&0&\sqrt{1}&0&\sqrt{4}&0&\sqrt{1}&0&0\\0&0&0&0&0&\sqrt{3}&0&\sqrt{3}&0\\0&0&0&0&0&0&0&0&\sqrt{6}\\0&-\sqrt{3}&0&\sqrt{3}&0&0&0&0&0\\0&0&-\sqrt{3}&0&0&0&\sqrt{3}&0&0\\0&0&0&0&0&-\sqrt{3}&0&\sqrt{3}&0\\0&0&\sqrt{2}&0&-\sqrt{2}&0&\sqrt{2}&0&0\end{bmatrix},S^{+}(1\otimes 1)=\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{6}&0&0&0&0&0&0&0&0\\0&\sqrt{3}&0&0&0&-\sqrt{3}&0&0&0\\0&0&\sqrt{1}&0&0&0&-\sqrt{3}&0&\sqrt{2}\\0&\sqrt{3}&0&0&0&\sqrt{3}&0&0&0\\0&0&\sqrt{4}&0&0&0&0&0&-\sqrt{2}\\0&0&0&\sqrt{3}&0&0&0&-\sqrt{3}&0\\0&0&\sqrt{1}&0&0&0&\sqrt{3}&0&\sqrt{2}\\0&0&0&\sqrt{3}&0&0&0&\sqrt{3}&0\\0&0&0&0&\sqrt{6}&0&0&0&0\end{bmatrix}$$

Cor. 1.11.2. $\frac{1}{\sqrt{3}}[I_3\otimes\begin{bmatrix}-\sqrt{2}&-1\\-1&\sqrt{2}\end{bmatrix}][\sigma(1)\otimes I]\frac{1}{\sqrt{3}}[I_3\otimes\begin{bmatrix}-\sqrt{2}&-1\\-1&\sqrt{2}\end{bmatrix}]=\sigma(1)\otimes I$

Thm. 1.11.1. $S(1\otimes 1)[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]S^{+}(1\otimes 1)=\begin{bmatrix}\sigma(2)&0&0\\0&\sigma(1)&0\\0&0&\sigma(0)\end{bmatrix}$

Cor. 1.11.3. $$\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{6}&0&0&0&0&0&0&0&0\\0&\sqrt{3}&0&\sqrt{3}&0&0&0&0&0\\0&0&\sqrt{1}&0&\sqrt{4}&0&\sqrt{1}&0&0\\0&0&0&0&0&\sqrt{3}&0&\sqrt{3}&0\\0&0&0&0&0&0&0&0&\sqrt{6}\end{bmatrix}[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{6}&0&0&0&0\\0&\sqrt{3}&0&0&0\\0&0&\sqrt{1}&0&0\\0&\sqrt{3}&0&0&0\\0&0&\sqrt{4}&0&0\\0&0&0&\sqrt{3}&0\\0&0&\sqrt{1}&0&0\\0&0&0&\sqrt{3}&0\\0&0&0&0&\sqrt{6}\end{bmatrix}=\sigma(2)$$

Cor. 1.11.4. $$\frac{1}{\sqrt{6}}\begin{bmatrix}0&-\sqrt{3}&0&\sqrt{3}&0&0&0&0&0\\0&0&-\sqrt{3}&0&0&0&\sqrt{3}&0&0\\0&0&0&0&0&-\sqrt{3}&0&\sqrt{3}&0\end{bmatrix}[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]\frac{1}{\sqrt{6}}\begin{bmatrix}0&0&0\\-\sqrt{3}&0&0\\0&-\sqrt{3}&0\\\sqrt{3}&0&0\\0&0&0\\0&0&-\sqrt{3}\\0&\sqrt{3}&0\\0&0&\sqrt{3}\\0&0&0\end{bmatrix}=\sigma(1)$$

Cor. 1.11.5. $$\frac{1}{\sqrt{6}}\begin{bmatrix}0&0&\sqrt{2}&0&-\sqrt{2}&0&\sqrt{2}&0&0\end{bmatrix}[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]\frac{1}{\sqrt{6}}\begin{bmatrix}0\\0\\\sqrt{2}\\0\\-\sqrt{2}\\0\\\sqrt{2}\\0\\0\end{bmatrix}=\sigma(0)$$

1.12 Representation transformation property 2 for 2-spin

Thm. 1.12.1. $S(1\otimes 1)[\sigma(1)\otimes I_3]S^{+}(1\otimes 1)=\frac{1}{2}\begin{bmatrix}\sigma(2)&\frac{1}{\sqrt{3}}O^{+}(2)&0\\\frac{1}{\sqrt{3}}O(2)&\sigma(1)&\frac{2}{\sqrt{3}}0^{+}(2)\\0&\frac{2}{\sqrt{3}}0(2)&\sigma(0)\end{bmatrix},0(2)=\{\begin{bmatrix}-1&0&1\end{bmatrix},i\begin{bmatrix}-1&0&-1\end{bmatrix},\begin{bmatrix}0&\sqrt{2}&0\end{bmatrix}\}$

1.13 More general representation transformation properties (guess)

Def. 1.13.1. $S(s_1\otimes s_2\cdots\otimes s_n)[\sigma(s_1)\otimes I_*+I_{2s_1+1}\otimes\sigma(s_2)\otimes I_*+\cdots]S^{+}(s_1\otimes s_2\cdots\otimes s_n)=?$

Cor. 1.13.1. $\tilde{S}(s):=S[\frac{1}{2}\otimes(s-\frac{1}{2})],\hat{S}(s)?:=S[(\frac{1}{2})_1\otimes(\frac{1}{2})_2\cdots\otimes(\frac{1}{2})_{2s}]$

2 Physical application of advanced representation transformation

2.1 General new coupling theory

2.1.1 New coupling theory for s-spin particles

Cor. 2.1.1.

$$\begin{cases}(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\varphi(s,\varsigma)=iJ(s,\varsigma)\\\psi(s,\varsigma)=\bar{N}(s)\varphi(s,\varsigma)\\\psi(s-1,\varsigma)=\bar{X}(s)\varphi(s,\varsigma)\end{cases}\overset{S(s)}{\Leftrightarrow}\begin{cases}[\frac{1}{s}\sigma(s),-i\varsigma]^a\partial_a\psi(s,\varsigma)=-\frac{1}{s}O^{+}(s)\cdot\nabla\psi(s-1,\varsigma)+i\bar{N}(s)J(s,\varsigma)\\{}[\frac{1}{s}\sigma(s-1),i\varsigma]^a\partial_a\psi(s-1,\varsigma)=\frac{1}{s}O(s)\cdot\nabla\psi(s,\varsigma)-i\bar{X}(s)J(s,\varsigma)\end{cases}$$

Cor. 2.1.2.

$$\begin{cases}(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\varphi(s,\varsigma)=iJ(s,\varsigma)\\\psi(s,\varsigma)=\bar{N}(s)\varphi(s,\varsigma)=0\\\psi(s-1,\varsigma)=\bar{X}(s)\varphi(s,\varsigma)\end{cases}\overset{S(s)}{\Leftrightarrow}\begin{cases}\frac{1}{s}O^{+}(s)\cdot\nabla\psi(s-1,\varsigma)=i\bar{N}(s)J(s,\varsigma)\\{}[\frac{1}{s}\sigma(s-1),i\varsigma]^a\partial_a\psi(s-1,\varsigma)=-i\bar{X}(s)J(s,\varsigma)\end{cases}$$

Cor. 2.1.3.

$$\begin{cases}(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a\varphi(s,\varsigma)=iJ(s,\varsigma)\\\psi(s,\varsigma)=\bar{N}(s)\varphi(s,\varsigma)\\\psi(s-1,\varsigma)=\bar{X}(s)\varphi(s,\varsigma)=0\end{cases}\overset{S(s)}{\Leftrightarrow}\begin{cases}[\frac{1}{s}\sigma(s),-i\varsigma]^a\partial_a\psi(s,\varsigma)=i\bar{N}(s)J(s,\varsigma)\\\frac{1}{s}O(s)\cdot\nabla\psi(s,\varsigma)=i\bar{X}(s)J(s,\varsigma)\end{cases}$$

2.1.2 New coupling theory for s-spin particles with lower first derivatives

Cor. 2.1.4.

$$\begin{cases}(\sigma\otimes I_4,-i\varsigma)_a\varphi^{ab}(s,\varsigma)=iJ(s,\varsigma)\\ \psi(s,\varsigma)=\bar{N}(s)\varphi(s,\varsigma)\\ \psi(s-1,\varsigma)=\bar{X}(s)\varphi(s,\varsigma)\end{cases}\overset{S(s)}{\Leftrightarrow}\begin{cases}[\frac{1}{s}\sigma(s),-i\varsigma]_a\psi^{ab}(s,\varsigma)=-\frac{1}{s}O_i^+(s)\psi^{ib}(s-1,\varsigma)+i\bar{N}(s)J(s,\varsigma)\\ [\frac{1}{s}\sigma(s-1),i\varsigma]_a\psi^{ab}(s-1,\varsigma)=\frac{1}{s}O_i(s)\psi^{ib}(s,\varsigma)-i\bar{X}(s)J(s,\varsigma)\end{cases}$$

Cor. 2.1.5.

$$\begin{cases}(\sigma\otimes I_4,-i\varsigma)_a\varphi^{ab}(s,\varsigma)=iJ(s,\varsigma)\\ \psi(s,\varsigma)=\bar{N}(s)\varphi(s,\varsigma)=0\\ \psi(s-1,\varsigma)=\bar{X}(s)\varphi(s,\varsigma)\end{cases}\overset{S(s)}{\Leftrightarrow}\begin{cases}\frac{1}{s}O_i^+(s)\psi^{ib}(s-1,\varsigma)=i\bar{N}(s)J(s,\varsigma)\\ [\frac{1}{s}\sigma(s-1),i\varsigma]_a\psi^{ab}(s-1,\varsigma)=-i\bar{X}(s)J(s,\varsigma)\end{cases}$$

Cor. 2.1.6.

$$\begin{cases}(\sigma\otimes I_4,-i\varsigma)_a\varphi^{ab}(s,\varsigma)=iJ(s,\varsigma)\\ \psi(s,\varsigma)=\bar{N}(s)\varphi(s,\varsigma)\\ \psi(s-1,\varsigma)=\bar{X}(s)\varphi(s,\varsigma)=0\end{cases}\overset{S(s)}{\Leftrightarrow}\begin{cases}[\frac{1}{s}\sigma(s),-i\varsigma]_a\psi^{ab}(s,\varsigma)=i\bar{N}(s)J(s,\varsigma)\\ \frac{1}{s}O_i(s)\psi^{ib}(s,\varsigma)=i\bar{X}(s)J(s,\varsigma)\end{cases}$$

2.2 Concrete new coupling theory

2.2.1 New coupling theory for gravitino and neutrino

Cor. 2.2.1.

$$\begin{cases}[\frac{2}{3}\sigma(\frac{3}{2}),-i\varsigma]^a\partial_a\psi(\frac{3}{2},\varsigma)=-\frac{2}{3}O^+(\frac{3}{2})\cdot\nabla\psi(\frac{1}{2},\varsigma)\\ [-\frac{2}{3}\sigma(\frac{1}{2}),-i\varsigma]^a\partial_a\psi(\frac{1}{2},\varsigma)=-\frac{2}{3}O(\frac{3}{2})\cdot\nabla\psi(\frac{3}{2},\varsigma)\\ [2\sigma(\frac{1}{2}),-i\varsigma]^a\partial_a\psi(\frac{1}{2},\varsigma)=0\end{cases}\Leftrightarrow\begin{cases}i\varsigma\partial_\pi\psi(\frac{1}{2},\varsigma)=\frac{1}{2}O(\frac{3}{2})\cdot\nabla\psi(\frac{3}{2},\varsigma)\\ i\varsigma\partial_\pi\psi(\frac{3}{2},\varsigma)=\frac{2}{3}\sigma(\frac{3}{2})\cdot\nabla\psi(\frac{3}{2},\varsigma)+\frac{2}{3}O^+(\frac{3}{2})\cdot\nabla\psi(\frac{1}{2},\varsigma)\\ 4\sigma(\frac{1}{2})\cdot\nabla\psi(\frac{1}{2},\varsigma)-O(\frac{3}{2})\cdot\nabla\psi(\frac{3}{2},\varsigma)=0\end{cases}$$

2.2.2 New coupling theory for Graviton, photon and scalar field

Cor. 2.2.2.

$$\begin{cases}[\frac{1}{2}\sigma(2),-i\varsigma]^a\partial_a\psi(2,\varsigma)=-\frac{1}{2}O^+(2)\cdot\nabla\psi(1,\varsigma)+i\bar{N}(2)J(2,\varsigma)\\ [-\frac{1}{2}\sigma(1),-i\varsigma]^a\partial_a\psi(1,\varsigma)=-\frac{1}{2}O(2)\cdot\nabla\psi(2,\varsigma)+i\bar{X}(2)J(2,\varsigma)\\ [\sigma(1),-i\varsigma]^a\partial_a\psi(1,\varsigma)=-O^+(1)\cdot\nabla\phi+i\bar{N}(1)J(1,\varsigma)\\ [-\sigma(0),-i\varsigma]^a\partial_a\phi=-O(1)\cdot\nabla\psi(1,\varsigma)+i\bar{X}(1)J(1,\varsigma)\end{cases}$$

Cor. 2.2.3.

$$\begin{cases}[\frac{1}{2}\sigma(2),-i\varsigma]^a\partial_a\psi(2,\varsigma)=-\frac{1}{2}O^+(2)\cdot\nabla\psi(1,\varsigma)\\ [-\frac{1}{2}\sigma(1),-i\varsigma]^a\partial_a\psi(1,\varsigma)=-\frac{1}{2}O(2)\cdot\nabla\psi(2,\varsigma)\\ [\sigma(1),-i\varsigma]^a\partial_a\psi(1,\varsigma)=-O^+(1)\cdot\nabla\phi\\ [-\sigma(0),-i\varsigma]^a\partial_a\phi=-O(1)\cdot\nabla\psi(1,\varsigma)\end{cases}\Leftrightarrow\begin{cases}i\varsigma\partial_\pi\phi=O(1)\cdot\nabla\psi(1,\varsigma)\\ i\varsigma\partial_\pi\psi(1,\varsigma)=\frac{1}{3}O^+(1)\cdot\nabla\phi+\frac{1}{3}O(2)\cdot\nabla\psi(2,\varsigma)\\ i\varsigma\partial_\pi\psi(2,\varsigma)=\frac{1}{2}\sigma(2)\cdot\nabla\psi(2,\varsigma)+\frac{1}{2}O^+(2)\cdot\nabla\psi(1,\varsigma)\\ 2O^+(1)\cdot\nabla\phi+3\sigma(1)\cdot\nabla\psi(1,\varsigma)-O(2)\cdot\nabla\psi(2,\varsigma)=0\end{cases}$$

2.2.3 Spin equation of new coupling theory for graviton, photon and scalar field

Cor. 2.2.4. $\{\partial_a+iS_{ab}[\sigma(1),\varsigma]\partial^b\}\otimes I_3\psi(1\otimes 1,\varsigma)=0$

$$\Leftrightarrow\begin{cases}\{2\partial_a+iS_{ab}[\sigma(2),\varsigma]\partial^b\}\psi(2,\varsigma)+iS_{ab}[\frac{1}{\sqrt{3}}O^+(2),\varsigma]\partial^b\psi(1,\varsigma)=0\\ iS_{ab}[\frac{1}{\sqrt{3}}O(2),\varsigma]\partial^b\psi(2,\varsigma)+\{2\partial_a+iS_{ab}[\sigma(1),\varsigma]\partial^b\}\psi(1,\varsigma)+iS_{ab}[\frac{2}{\sqrt{3}}0^+(2),\varsigma]\partial^b\psi(0,\varsigma)=0\\ iS_{ab}[\frac{2}{\sqrt{3}}0(2),\varsigma]\partial^b\psi(1,\varsigma)+\{2\partial_a+iS_{ab}[\sigma(0),\varsigma]\partial^b\}\psi(0,\varsigma)=0\end{cases}$$

2.2.4 A theory of bound photons

Cor. 2.2.5. *No plane wave solution (Z-axis)*

$$\begin{cases}(\sigma\otimes I_4,-i\varsigma)^a\partial_a\varphi(2,\varsigma)=0\\ \psi(2,\varsigma)=0\end{cases}\Leftrightarrow\begin{cases}O^+(2)\cdot\nabla\psi(1,\varsigma)=0\\ [\sigma(1),2i\varsigma]^a\partial_a\psi(1,\varsigma)=0\end{cases}\Leftrightarrow\begin{cases}O^+(2)S_m^+(1)\cdot\nabla\Psi(1,\varsigma)=0\\ (\gamma,2i\varsigma)^a\partial_a\Psi(1,\varsigma)=0\end{cases}$$

2.2.5 A generalized new theory of gravity

Cor. 2.2.6.

$$\begin{cases}(\sigma\otimes I_4,-i\varsigma)_a\varphi^{ab}(2,\varsigma)=iJ(2,\varsigma)\\ \psi(2,\varsigma)=\bar{N}(2)\varphi(2,\varsigma)\\ \psi(1,\varsigma)=\bar{X}(2)\varphi(2,\varsigma)\end{cases}\overset{S(2)}{\Leftrightarrow}\begin{cases}[\frac{1}{2}\sigma(2),-i\varsigma]_a\psi^{ab}(2,\varsigma)=-\frac{1}{2}O_i^+(2)\psi^{ib}(1,\varsigma)+i\bar{N}(2)J(2,\varsigma)\\ [\frac{1}{2}\sigma(1),i\varsigma]_a\psi^{ab}(1,\varsigma)=\frac{1}{2}O_i(2)\psi^{ib}(2,\varsigma)-i\bar{X}(2)J(2,\varsigma)\end{cases}$$

Chapter14 Deep Analysis of Lorentz Transformation

1 Lorentz group representation in 3+1 dimensional space-time [8,12]

1.1 Poincare group representation [8]

Commutative relations of Poincare group generators M_{ab}, p_a:

$$M_{ab} = L_{ab} + S_{ab}, L_{ab} = x_a p_b - x_b p_a, g_{ab} = \delta_{ab} \tag{14.1}$$

$$\begin{cases} [M_{ab}, M_{cd}] = -i(g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac}) \\ [M_{ab}, p_c] = -i(g_{bc}p_a - g_{ac}p_b), [p_a, p_b] = 0 \end{cases} \tag{14.2}$$

Commutative relations of Poincare group generators L_{ab}, S_{ab}, p_a:

$$\begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [L_{ab}, p_c] = -i(g_{bc}p_a - g_{ac}p_b), [p_a, p_b] = 0 \end{cases} \tag{14.3}$$

$$[S_{ab}, S_{cd}] = -i(g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}) \tag{14.4}$$

$$[S_{ab}, L_{cd}] = 0, [S_{ab}, p_c] = 0 \tag{14.5}$$

1.2 Extracting vectors $\vec{X}, \vec{Y}, \vec{a}, \vec{b}$ from spin tensors

Def. 1.2.1. $X^i \equiv \frac{1}{2}\varepsilon^{ijk}S_{jk}, Y_i \equiv S_{\pi i}, a_i \equiv \frac{1}{2}(X_i + Y_i), b_i \equiv \frac{1}{2}(X_i - Y_i), g_{ab} := \delta_{ab}$

Pro. 1.2.1. $\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}$
$\varepsilon_{ijk}\varepsilon^{k}{}_{lm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \varepsilon_{ijk}\varepsilon^{jk}{}_{l} = 2\delta_{il}$

Cor. 1.2.1. $X^i = \frac{1}{2}\varepsilon^{ijk}S_{jk} \Leftrightarrow S_{ij} = \varepsilon_{ijk}X^k$

1.3 Positive proof of propositions for Lorentz group representation relation

Thm. 1.3.1. $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Rightarrow [X^i, X^l] = i\varepsilon^{il}{}_{k}X^k$

Proof: $[X^i, X^l] = [\frac{1}{2}\varepsilon^{ijk}S_{jk}, \frac{1}{2}\varepsilon^{lmn}S_{mn}]$
$= \frac{1}{4}\varepsilon^{ijk}\varepsilon^{lmn}[S_{jk}, S_{mn}]$
$= -i\frac{1}{4}\varepsilon^{ijk}\varepsilon^{lmn}(g_{jn}S_{km} - g_{jm}S_{kn} + g_{km}S_{jn} - g_{kn}S_{jm})$
$= -i\frac{1}{4}\varepsilon^{ijk}\varepsilon^{lmn}(2g_{jn}S_{km} + 2g_{km}S_{jn})$
$= -\frac{i}{2}(\varepsilon^{ijk}\varepsilon^{lm}{}_{j}S_{km} - \varepsilon^{ijk}\varepsilon^{ln}{}_{k}S_{jn})$
$= -\frac{i}{2}(\varepsilon^{ikj}\varepsilon^{ln}{}_{k}S_{jn} - \varepsilon^{ijk}\varepsilon^{ln}{}_{k}S_{jn})$
$= i\varepsilon^{ijk}\varepsilon^{ln}{}_{k}S_{jn}$
$= (\delta^{il}\delta^{jn} - \delta^{in}\delta^{jl})S_{jn}$
$= iS^{il} = i\varepsilon^{il}{}_{k}X^k$ □

Thm. 1.3.2. $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Rightarrow [Y_i, Y_j] = i\varepsilon_{ij}{}^{k}X_k$

Proof: $[Y_i, Y_j] = [S_{\pi i}, S_{\pi j}]$
$= [S_{i\pi}, S_{j\pi}]$
$= -i(g_{i\pi}S_{\pi j} - g_{ij}S_{\pi\pi} + g_{\pi j}S_{i\pi} - g_{\pi\pi}S_{ij})$
$= iS_{ij} = i\varepsilon_{ij}{}^{k}X_k$ □

Thm. 1.3.3. $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Rightarrow [X^i, Y_l] = i\varepsilon^{i}{}_{l}{}^{k}Y_k$

Proof: $[X^i, Y_l] = [\frac{1}{2}\varepsilon^{ijk}S_{jk}, S_{\pi l}]$
$= -\frac{1}{2}\varepsilon^{ijk}[S_{jk}, S_{l\pi}]$
$= \frac{i}{2}\varepsilon^{ijk}(g_{j\pi}S_{kl} - g_{jl}S_{k\pi} + g_{kl}S_{j\pi} - g_{k\pi}S_{jl})$
$= \frac{i}{2}\varepsilon^{ijk}(-g_{jl}S_{k\pi} + g_{kl}S_{j\pi})$
$= i\varepsilon^{ijk}g_{kl}S_{j\pi} = i\varepsilon^{i}{}_{l}{}^{k}Y_k$ □

Cor. 1.3.1. $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}$
$\Rightarrow [X_i, X_j] = i\varepsilon_{ij}{}^{k}X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^{k}X_k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^{k}Y_k$

1.4 Reverse proof of propositions for Lorentz group representation relation

Thm. 1.4.1. $[X^i, X^l] = i\varepsilon^{il}{}_k X^k \Rightarrow i[S_{ij}, S_{lm}] = g_{im}S_{jl} - g_{il}S_{jm} + g_{jl}S_{im} - g_{jm}S_{il}$

Proof: $i[S_{ij}, S_{lm}] = i[\varepsilon_{ijk}X^k, \varepsilon_{lmn}X^n]$
$= i\varepsilon_{ijk}\varepsilon_{lmn}[X^k, X^n] = -\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon^{kn}{}_h X^h = -\varepsilon_{ijk}\varepsilon_{lmn}S^{kn}$
$= -(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl})S^{kn}$
$= -(\delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl})S^{kn}$
$= -(\delta_{jl}S_{mi} + \delta_{im}S^{lj} - \delta_{il}S^{mj} - \delta_{jm}S^{li})$
$= \delta_{im}S_{jl} - \delta_{il}S_{jm} + \delta_{jl}S_{im} - \delta_{jm}S_{il}$
$= g_{im}S_{jl} - g_{il}S_{jm} + g_{jl}S_{im} - g_{jm}S_{il}$ ☐

Thm. 1.4.2. $[X_i, Y_j] = i\varepsilon_{ij}{}^k Y_k \Rightarrow i[S_{ij}, S_{\pi l}] = g_{il}S_{j\pi} - g_{i\pi}S_{jl} + g_{j\pi}S_{il} - g_{jl}S_{i\pi}$

Proof: $i[S_{ij}, S_{\pi l}] = i[\varepsilon_{ijk}X^k, Y_l]$
$= i\varepsilon_{ijk}[X^k, Y^l] = -\varepsilon_{ijk}\varepsilon^k{}_{lm}Y^m$
$= -(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})S^{\pi m} = \delta_{il}S_{j\pi} - \delta_{jl}S_{i\pi}$
$= \delta_{il}S_{j\pi} - \delta_{i\pi}S_{jl} + \delta_{j\pi}S_{il} - \delta_{jl}S_{i\pi}$
$= g_{il}S_{j\pi} - g_{i\pi}S_{jl} + g_{j\pi}S_{il} - g_{jl}S_{i\pi}$ ☐

Thm. 1.4.3. $[Y_i, Y_j] = i\varepsilon_{ij}{}^k X_k \Rightarrow i[S_{\pi i}, S_{\pi j}] = g_{\pi j}S_{i\pi} - g_{\pi\pi}S_{ij} + g_{i\pi}S_{\pi j} - g_{ij}S_{\pi\pi}$

Proof: $i[S_{\pi i}, S_{\pi j}] = i[Y_i, Y_j]$
$= -\varepsilon_{ijk}X^k = -S_{ij}$
$= \delta_{\pi j}S_{i\pi} - \delta_{\pi\pi}S_{ij} + \delta_{i\pi}S_{\pi j} - \delta_{ij}S_{\pi\pi}$
$= g_{\pi j}S_{i\pi} - g_{\pi\pi}S_{ij} + g_{i\pi}S_{\pi j} - g_{ij}S_{\pi\pi}$ ☐

Cor. 1.4.1. $[X_i, X_j] = i\varepsilon_{ij}{}^k X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^k X_k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^k Y_k$
$\Rightarrow i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}$

1.5 Comprehensive conclusion of propositions for Lorentz group representation relation

Cor. 1.5.1. $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}$
$\Leftrightarrow [X_i, X_j] = i\varepsilon_{ij}{}^k X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^k X_k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^k Y_k$

Cor. 1.5.2. $[X_i, X_j] = i\varepsilon_{ij}{}^k X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^k X_k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^k Y_k$
$\Leftrightarrow \vec{X} \times \vec{X} = i\vec{X}, \vec{Y} \times \vec{Y} = i\vec{X}, \vec{X} \times \vec{Y} = i\vec{Y}, [X_i, Y_i] = 0$
$\Leftrightarrow \vec{a} \times \vec{a} = i\vec{a}, \vec{b} \times \vec{b} = i\vec{b}, [a_i, b_j] = 0$

Cor. 1.5.3. $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Leftrightarrow \vec{a} \times \vec{a} = i\vec{a}, \vec{b} \times \vec{b} = i\vec{b}, [a_i, b_j] = 0$

Lorentz state transformation decomposition:

Cor. 1.5.4. $e^{\frac{i}{2}\varepsilon^{ab}S_{ab}} = e^{i\omega\cdot\vec{X} + \epsilon\cdot\vec{Y}} = e^{(i\omega+\epsilon)\cdot\vec{a}}e^{(i\omega-\epsilon)\cdot\vec{b}}$

2 Relativistic Lorentz boost transformation of a single particle

2.1 Lorentz transformation of coordinates [22–24]

Convention: The speed of O is $\vec{v}, v \neq 1$ in O'. The speed of O' is $-\vec{v}$ in O. The benefit of this convention is that it can visually describe moving particles. The general form of the relativistic Lorentz boost transformation of coordinates and their coordinate differentials:

Def. 2.1.1. $\begin{cases} \nabla' = \nabla - \gamma_v\vec{v}\partial_t + (\gamma_v - 1)\vec{v}/v^2(\vec{v}\cdot\nabla) \\ \partial_{t'} = \gamma_v(\partial_t - \vec{v}\cdot\nabla), \gamma_v \equiv (1-v^2)^{-\frac{1}{2}} \end{cases}$

Def. 2.1.2. $\begin{cases} \vec{r'} = \vec{r} + \gamma_v\vec{v}t + (\gamma_v - 1)(\vec{v}\cdot\vec{r})\vec{v}/v^2 \\ t' = \gamma_v(t + \vec{v}\cdot\vec{r}), \gamma_v \equiv (1-v^2)^{-\frac{1}{2}} \end{cases}$ $\begin{cases} d\vec{r'} = d\vec{r} + \gamma_v\vec{v}dt + (\gamma_v - 1)(\vec{v}\cdot d\vec{r})\vec{v}/v^2 \\ dt' = \gamma_v(dt + \vec{v}\cdot d\vec{r}) \end{cases}$

The above transformation is an important foundation for the entire theory of special relativity, and another important transformation is the vector rotation transformation.

Cor. 2.1.1. $\vec{r'}^2 - t'^2 = \vec{r}^2 - t^2 = invariant, d\vec{r'}^2 - dt'^2 = d\vec{r}^2 - dt^2 = invariant$

Def. 2.1.3. $L_{\vec{v}} := \begin{bmatrix} 1 & 0 & 0 & -i\gamma_v v_x \\ 0 & 1 & 0 & -i\gamma_v v_y \\ 0 & 0 & 1 & -i\gamma_v v_z \\ i\gamma_v v_x & i\gamma_v v_y & i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma_v - 1}{v^2}\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z & 0 \\ v_yv_x & v_yv_y & v_yv_z & 0 \\ v_zv_x & v_zv_y & v_zv_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, L_{\vec{v}}L_{-\vec{v}} = L_{-\vec{v}}L_{\vec{v}} = I$

Cor. 2.1.2. $L_{\vec{v}}\begin{bmatrix}\vec{0}\\i\end{bmatrix}=e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L}\begin{bmatrix}\vec{0}\\i\end{bmatrix}=\begin{bmatrix}\gamma_v\vec{v}\\i\gamma_v\end{bmatrix},L_{\vec{v}}\begin{bmatrix}\vec{0}\\im\end{bmatrix}=e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L}\begin{bmatrix}\vec{0}\\im\end{bmatrix}=\begin{bmatrix}\vec{p}\\iE\end{bmatrix}$

Lem. 2.1.1. $(\vec{v}\cdot R)^2+(\vec{v}\cdot L)^2=\vec{v}^2$

Cor. 2.1.3.
$L_{\vec{v}}=e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L}=1-\gamma_v(\vec{v}\cdot L)+\frac{\gamma_v-1}{v^2}(\vec{v}\cdot L)^2=\gamma_v(1-\vec{v}\cdot L)-\frac{\gamma_v-1}{v^2}(\vec{v}\cdot R)^2,L_{\vec{v}}L_{-\vec{v}}=L_{-\vec{v}}L_{\vec{v}}=I$

Cor. 2.1.4. $X'=L_{\vec{v}}dX,X\equiv\begin{bmatrix}\vec{r}\\it\end{bmatrix},X'\equiv\begin{bmatrix}\vec{r'}\\it'\end{bmatrix};dX'=L_{\vec{v}}dX,dX\equiv\begin{bmatrix}d\vec{r}\\idt\end{bmatrix},dX'\equiv\begin{bmatrix}d\vec{r'}\\idt'\end{bmatrix}$

2.2 Velocity synthesis formula

Cor. 2.2.1. $\vec{u}'=[\vec{u}+\gamma_v\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{u})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\vec{u})]$

Cor. 2.2.2. $1-\vec{u}'^2=\frac{(1-\vec{u}^2)(1-\vec{v}^2)}{(1+\vec{v}\cdot\vec{u})^2}$

Cor. 2.2.3. $\begin{cases}\gamma_{u'}\vec{u}'=\gamma_u\vec{u}+\gamma_v\vec{v}\gamma_u+(\gamma_v-1)[\vec{v}\cdot(\gamma_u\vec{u})]\vec{v}/v^2\\\gamma_{u'}=\gamma_v[\gamma_u+\vec{v}\cdot(\gamma_u\vec{u})]\end{cases}\Leftrightarrow\begin{bmatrix}\gamma_{u'}\vec{u'}\\i\gamma_{u'}\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\gamma_u\vec{u}\\i\gamma_u\end{bmatrix}$

2.3 Lorentz boost transformation of four momentum particles with mass

Massive particles: $m_0\neq0,u\neq1,u'\neq1$

Def. 2.3.1. $E\equiv m_0(1-u^2)^{-\frac{1}{2}},E'\equiv m_0(1-u'^2)^{-\frac{1}{2}},\vec{p}\equiv E\vec{u},\vec{p'}\equiv E'\vec{u'}$

The following Lorentz boost transformation of energy and momentum can be derived from the Lorentz boost transformation of coordinates.

Cor. 2.3.1. $\begin{cases}\vec{p'}=\vec{p}+\gamma_vE\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{p})\vec{v}/v^2\\E'=\gamma_v(E+\vec{v}\cdot\vec{p})\end{cases}\Leftrightarrow\begin{bmatrix}\vec{p'}\\iE'\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\vec{p}\\iE\end{bmatrix}$

Cor. 2.3.2. $\vec{p'}^2-E'^2=\vec{p}^2-E^2=-m_0^2=invariant$

2.4 Temperature lorentz transform conjecture between different velocity reference frames

Ass. 2.4.1.
$\begin{cases}\textit{Kinetic energy of motion system - translational kinetic energy of particle system}=E'_k-E'_{k0}=\sum\limits_i(\gamma_vE_i-m_0)-(\gamma_v-1)\sum\limits_i\\\textit{Kinetic energy of a stationary system - translational kinetic energy of particle system}=E_k-E_{k0}=\sum\limits_i(E_i-m_0)-0=\frac{3}{2}Nk\end{cases}$
$T'=T$

2.5 Lorentz boost transformation of four momentum for massless particles

massless particles: $m_0=0,u=1,u'=1$

Def. 2.5.1. $\vec{p}\equiv E\vec{u},\vec{p'}\equiv E'\vec{u'}$

Starting from the Lorentz push transformation of coordinates for massless particles, it is not strictly possible to derive the Lorentz push transformation of energy and momentum. But it can be obtained by making the mass infinitely close to zero.

$$\begin{cases}\vec{p'}=\vec{p}+\gamma_vE\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{p})\vec{v}/v^2\\E'=\gamma_v(E+\vec{v}\cdot\vec{p})\end{cases},\begin{bmatrix}\vec{p'}\\iE'\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\vec{p}\\iE\end{bmatrix}\quad(14.6)$$

Cor. 2.5.1. $\vec{p'}^2-E'^2=\vec{p}^2-E^2=0=invariant$

2.6 Lorentz boost transformation of a single particle external force

Def. 2.6.1. $\vec{F}\equiv\frac{dp}{dt},\vec{F'}\equiv\frac{dp'}{dt'},\vec{f}\equiv\frac{\vec{F}}{\sqrt{1-u^2}},\vec{f'}\equiv\frac{\vec{F'}}{\sqrt{1-u'^2}}$

Cor. 2.6.1. $\vec{a'}=[\vec{a}+(\gamma_v-1)(\vec{v}\cdot\vec{a})\vec{v}/v^2]/[\gamma_v^2(1+\vec{v}\cdot\vec{u})^2]-[\vec{u}+\gamma_v\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{u})\vec{v}/v^2](\vec{v}\cdot\vec{a})/[\gamma_v^2(1+\vec{v}\cdot\vec{u})^3]$

Cor. 2.6.2. $\vec{F'}=[\vec{F}+\gamma_v(\vec{u}\cdot\vec{F})\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{F})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\vec{u})]$

Cor. 2.6.3. $\vec{u'}\cdot\vec{F'}=\gamma_v(\vec{u}\cdot\vec{F}+\vec{v}\cdot\vec{F})/[\gamma_v(1+\vec{v}\cdot\vec{u})]=\frac{\vec{v}+\vec{u}}{1+\vec{v}\cdot\vec{u}}\cdot\vec{F}$

Cor. 2.6.4. $\begin{cases}\vec{f'}=\vec{f}+\gamma_v(\vec{u}\cdot\vec{f})\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{f})\vec{v}/v^2\\\vec{u'}\cdot\vec{f'}=\gamma_v(\vec{u}\cdot\vec{f}+\vec{v}\cdot\vec{f})=\gamma_v(\vec{u}+\vec{v})\cdot\vec{f}\end{cases}\Leftrightarrow\begin{bmatrix}\vec{f'}\\i\vec{u'}\cdot\vec{f'}\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\vec{f}\\i\vec{u}\cdot\vec{f}\end{bmatrix}$

Cor. 2.6.5. $\vec{f'}^2-(\vec{u'}\cdot\vec{f'})^2=\vec{f}^2-(\vec{u}\cdot\vec{f})^2=invariant$

2.7 General relativity transformation hypothesis of single particle external force

Def. 2.7.1. $\vec{a}\equiv\frac{d\vec{u}}{dt},\vec{a'}\equiv\frac{d\vec{u'}}{dt'},\vec{g}\equiv\frac{d\vec{v}}{dt}$

Def. 2.7.2.
$$\begin{cases}\vec{a'}=[\vec{a}+(\gamma_v-1)(\vec{v}\cdot\vec{a})\vec{v}/v^2]/[\gamma_v^2(1+\vec{v}\cdot\vec{u})^2]-[\vec{u}+\gamma_v\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{u})\vec{v}/v^2](\vec{v}\cdot\vec{a})/[\gamma_v^2(1+\vec{v}\cdot\vec{u})^3]+\\ [\gamma_v\vec{g}+(\gamma_v-1)(\vec{g}\cdot\vec{u})\vec{v}/v^2+(\gamma_v-1)(\vec{v}\cdot\vec{u})\vec{g}/v^2-2(\gamma_v-1)(\vec{v}\cdot\vec{u})(\vec{v}\cdot\vec{g})\vec{v}/v^4+\gamma_v^3(\vec{v}\cdot\vec{g})(\vec{v}\cdot\vec{u})\vec{v}/v^2]/[\gamma_v^2(1+\vec{v}\cdot\vec{u})^2]\\ -[\vec{u}+\gamma_v\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{u})\vec{v}/v^2][(\vec{g}\cdot\vec{u})+\gamma_v^2(1+\vec{v}\cdot\vec{u})(\vec{v}\cdot\vec{g})]/[\gamma_v^2(1+\vec{v}\cdot\vec{u})^3]\\ \vec{F'}=[\vec{F}+\gamma_v(\vec{u}\cdot\vec{F})\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{F})\vec{v}/v^2\\ +\gamma_vE\vec{g}+\gamma_v^3(\vec{v}\cdot\vec{g})E\vec{v}+(\gamma_v-1)(\vec{g}\cdot\vec{p})\vec{v}/v^2+(\gamma_v-1)(\vec{v}\cdot\vec{p})\vec{g}/v^2\\ +\gamma_v^3(\vec{v}\cdot\vec{g})(\vec{v}\cdot\vec{p})\vec{v}/v^2-2(\gamma_v-1)(\vec{v}\cdot\vec{g})(\vec{v}\cdot\vec{p})\vec{v}/v^4]/[\gamma_v(1+\vec{v}\cdot\vec{u})]\end{cases}$$

3 Relativistic Lorentz transformation of multiparticle particle system

3.1 Lorentz boost transformation of multiparticle particle system

$$\begin{cases}\vec{P}(v)=\sum\limits_i[\vec{p}_i+\gamma_vE_i\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{p}_i)\vec{v}/v^2]\\ H(\vec{v})=\sum\limits_i\gamma_v(E_i+\vec{v}\cdot\vec{p}_i)\end{cases}\Leftrightarrow\begin{bmatrix}\vec{P}(v)\\ iH(\vec{v})\end{bmatrix}=\sum_i L_{\vec{v}}\begin{bmatrix}\vec{p}_i\\ iE_i\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\sum\limits_i\vec{p}_i\\ i\sum\limits_i E_i\end{bmatrix}\tag{14.7}$$

3.2 Lorentz boost transformation of particle system in different velocity reference frames

Lorentz boost transformation between particle systems in different velocity reference frames:

$$\begin{cases}\vec{u'}=[\vec{u}+\gamma_v\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{u})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\vec{u})]\\ \vec{P}(\vec{u'})=\vec{P}(\vec{u})+\gamma_vH(\vec{u})\vec{v}+(\gamma_v-1)[\vec{v}\cdot\vec{P}(\vec{u})]\vec{v}/v^2\\ H(\vec{u'})=\gamma_v[H(\vec{u})+\vec{v}\cdot\vec{P}(\vec{u})]\end{cases}\Leftrightarrow\begin{cases}\begin{bmatrix}\gamma_{u'}\vec{u'}\\ i\gamma_{u'}\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\gamma_u\vec{u}\\ i\gamma_u\end{bmatrix}\\ \begin{bmatrix}\vec{P}(u')\\ iH(\vec{u'})\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\vec{P}(u)\\ iH(\vec{u})\end{bmatrix}\end{cases}\tag{14.8}$$

Cor. 3.2.1. $\vec{P}^2(\vec{u'})-H^2(\vec{u'})=\vec{P}^2(\vec{u})-H^2(\vec{u})=-M_0^2=invariant$

Lorentz boost transformation of external forces between particle systems with different centroid velocities:

Def. 3.2.1. $\vec{f}(\vec{u})\equiv\frac{\vec{F}(\vec{u})}{\sqrt{1-u^2}},\vec{f'}(\vec{u'})\equiv\frac{\vec{F'}(\vec{u'})}{\sqrt{1-u'^2}},\vec{f}(\vec{u})=\frac{dP(\vec{u})}{d\tau},\vec{f'}(\vec{u'})=\frac{dP'(\vec{u'})}{d\tau}$

Lem. 3.2.1. $\frac{dH(\vec{u})}{d\tau}\equiv\vec{u}\cdot\frac{dP(\vec{u})}{d\tau}$

Cor. 3.2.2. $\begin{cases}\vec{f'}(\vec{u'})=\vec{f}(\vec{u})+\gamma_v[\vec{u}\cdot\vec{f}(\vec{u})]\vec{v}+(\gamma_v-1)[\vec{v}\cdot\vec{f}(\vec{u})]\vec{v}/v^2\\ \vec{u'}\cdot\vec{f'}(\vec{u'})=\gamma_v[\vec{u}\cdot\vec{f}(\vec{u})+\vec{v}\cdot\vec{f}(\vec{u})]=\gamma_v(\vec{u}+\vec{v})\cdot\vec{f}(\vec{u})\end{cases}\Leftrightarrow\begin{bmatrix}\vec{f'}(\vec{u'})\\ i\vec{u'}\cdot\vec{f'}(\vec{u'})\end{bmatrix}=L_{\vec{v}}\begin{bmatrix}\vec{f}(\vec{u})\\ i\vec{u}\cdot\vec{f}(\vec{u})\end{bmatrix}$

Cor. 3.2.3. $\vec{f'}^2(\vec{u'})-[\vec{u'}\cdot\vec{f'}(\vec{u'})]^2=\vec{f}^2(\vec{u})-[\vec{u}\cdot\vec{f}(\vec{u})]^2=invariant$

3.3 Moving particle system boost transform to static particle system

Lem. 3.3.1. $|\sum\limits_i\vec{p}_i/\sum\limits_i E_i|\le 1$, *The equal sign exists and only if* $\vec{p}_i=E_i\vec{1}$ *is established.*

Def. 3.3.1. *Moving particle system:* $|\sum\limits_i\vec{p}_i/\sum\limits_i E_i|\neq 0$, *Static particle system:* $|\sum\limits_i\vec{p}_i/\sum\limits_i E_i|=0$

Lorentz boost transformation from a massive moving particle system to a static particle system:

$$\vec{v}=-\sum_i\vec{p}_i/\sum_i E_i\neq\vec{1}\Rightarrow\begin{cases}\vec{P}(\vec{v})=\sum\limits_i[\vec{p}_i+\gamma_vE_i\vec{v}+(\gamma_v-1)(\vec{v}\cdot\vec{p}_i)\vec{v}/v^2]=0\\ M_0=H(\vec{v})=\sum\limits_i\gamma_v(E_i+\vec{v}\cdot\vec{p}_i)=\sum\limits_i E_i/\gamma_v\end{cases}\tag{14.9}$$

3.4 Static particle system boost transform to moving particle system

Static particle system boost transform to moving particle system:

$$\sum_i\vec{p}_i/\sum_i E_i=0\Rightarrow\begin{cases}H(\vec{v})=\gamma_v\sum\limits_i E_i=M,M\equiv\gamma_vM_0,M_0\equiv\sum\limits_i E_i\\ \vec{P}(\vec{v})=\gamma_v\vec{v}\sum\limits_i E_i=M\vec{v}\end{cases}\tag{14.10}$$

The physical meaning of the above relationship is: You can equate a particle system to a particle. When the particle system moves, it can be equivalent to the motion of a particle. And it conforms to the laws of relativity just like particles. When the massive center of the particle system is static, the total energy of the particle system is just the equivalent static mass of the particle system. The

total energy of the moving particle system is just the equivalent relativistic moving mass. Therefore, a particle system can be completely equivalent to a particle. And conversely, a fundamental particle can also be considered to be a particle system. The difficulty is whether there is such a conclusion for the system of particles with interaction? Can we apply a constraint to the interaction and obtain new physics based on this clue?

3.5 Lorentz boost transformation of unidirectional multiphoton system

Lorentz boost transformation of unidirectional multiphoton system:

$$\vec{p}_i = E_i\vec{1}, \vec{1}' = [\vec{1} + \gamma_v\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{1})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\vec{1})] \tag{14.11}$$

$$\Rightarrow \begin{cases} \vec{P}(\vec{v}) = \sum\limits_i [E_i\vec{1} + \gamma_v E_i\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{1})E_i\vec{v}/v^2] = \sum\limits_i \gamma_v E_i(1+\vec{v}\cdot\vec{1})\vec{1}' \\ H(\vec{v}) = \sum\limits_i \gamma_v E_i(1+\vec{v}\cdot\vec{1}) \\ \vec{P}^2(\vec{v}) - H^2(\vec{v}) = -M_0^2, M_0 = \sum\limits_i E_i\sqrt{1-1'^2} = 0 \end{cases} \tag{14.12}$$

3.6 Universal static mass formula for particle systems

Based on the above conclusions, the following universal static mass formula can be obtained.

$$M_0 = \sum_i E_i\sqrt{1 - (\sum_i \vec{p}_i/\sum_i E_i)^2} = \sqrt{(\sum_i E_i)^2 - (\sum_i \vec{p}_i)^2} = \sqrt{\sum_{i,j}(E_iE_j - \vec{p}_i\cdot\vec{p}_j)} \tag{14.13}$$

Mass formula for simplified marking: $M_0 = \sqrt{\sum(E_iE_j - \vec{p}_i\cdot\vec{p}_j)}$

3.7 Lorentz boost transform hypothesis for potential energy of interacting particle system

Lorentz boost transform hypothesis for potential energy of interacting particle system:

$$\begin{cases} \vec{P}(\vec{0}) = (\sum\limits_k \vec{p}_k/\sum\limits_k E_k)\frac{1}{2}\sum\limits_{i\neq j} V_{ij} \\ H(\vec{0}) = \frac{1}{2}\sum\limits_{i\neq j} V_{ij} \end{cases} \tag{14.14}$$

Lorentz boost transformation for the potential energy of interacting particle system:

$$\begin{cases} \vec{P}(\vec{v}) = \vec{P}(\vec{0}) + \gamma_v H(\vec{0})\vec{v} + (\gamma_v - 1)[\vec{v}\cdot\vec{P}(\vec{0})]\vec{v}/v^2 = \frac{\sum_{i\neq j} V_{ij}}{\sum_k 2E_k}\sum\limits_i [\vec{p}_i + \gamma_v E_i\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{p}_i)\vec{v}/v^2] \\ H(\vec{v}) = \gamma_v[H(\vec{0}) + \vec{v}\cdot\vec{P}(\vec{0})] = \frac{\sum_{i\neq j} V_{ij}}{\sum_k 2E_k}\sum\limits_i \gamma_v(E_i + \vec{v}\cdot\vec{p}_i) \\ \vec{P}^2(\vec{v}) - H^2(\vec{v}) = \vec{P}^2(\vec{0}) - H^2(\vec{0}) = -M_{V0}^2 \end{cases} \tag{14.15}$$

Lorentz boost transformation for the potential energy of interacting particle system:

$$\vec{v} = -\sum_i \vec{p}_i/\sum_i E_i \neq \vec{1} \Rightarrow \begin{cases} \vec{P}(\vec{v}) = \vec{0} \\ H(\vec{v}) = \sum\limits_{i\neq j} V_{ij}\sqrt{1-v^2} \equiv M_{V0} \end{cases} \tag{14.16}$$

The mass formula for the potential energy of the interacting particle system:

$M_{V0} = (\frac{1}{2}\sum\limits_{i\neq j} V_{ij})\sqrt{\sum\limits_{i,j}(E_iE_j - \vec{p}_i\cdot\vec{p}_j)/\sum\limits_{i,j}(E_iE_j)}$

3.8 Lorentz boost transformation of multi particles interacting particle system

Lorentz boost transformation of interacting particle system:

Cor. 3.8.1. $\begin{cases} \vec{P}(v) = \frac{1}{\sum_k 2E_k}(\sum\limits_k 2E_k + \sum\limits_{i\neq j} V_{ij})\sum\limits_i [\vec{p}_i + \gamma_v E_i\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{p}_i)\vec{v}/v^2] \\ H(\vec{v}) = \frac{1}{\sum_k 2E_k}(\sum\limits_k 2E_k + \sum\limits_{i\neq j} V_{ij})\sum\limits_i \gamma_v(E_i + \vec{v}\cdot\vec{p}_i) \end{cases}$

Proof:

$\vec{P}(v) = \sum\limits_i [\vec{p}_i + \gamma_v E_i\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{p}_i)\vec{v}/v^2] + \frac{1}{2}\sum\limits_{i,j}\{V_{ij}(\sum\limits_k \vec{p}_k/\sum\limits_k E_k) + \gamma_v V_{ij}\vec{v} + (\gamma_v - 1)[\vec{v}\cdot(V_{ij}\sum\limits_k \vec{p}_k/\sum\limits_k E_k)]\vec{v}/v^2\}$

$= \frac{1}{\sum_k 2E_k}\{\sum\limits_k 2E_k\sum\limits_i [\vec{p}_i + \gamma_v E_i\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{p}_i)\vec{v}/v^2] + \sum\limits_{i\neq j} V_{ij}\sum\limits_k \{\vec{p}_k + \gamma_v E_k\vec{v} + (\gamma_v - 1)[\vec{v}\cdot(\vec{p}_k)]\vec{v}/v^2\}\}$

$= \frac{1}{\sum_k 2E_k}(\sum\limits_k 2E_k + \sum\limits_{i\neq j} V_{ij})\sum\limits_i [\vec{p}_i + \gamma_v E_i\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{p}_i)\vec{v}/v^2]$

$H(\vec{v}) = \sum\limits_i \gamma_v(E_i + \vec{v}\cdot\vec{p}_i) + \frac{1}{2}\sum\limits_{i,j}\gamma_v[V_{ij} + \vec{v}\cdot(V_{ij}\sum\limits_k \vec{p}_k/\sum\limits_k E_k)] = \frac{1}{\sum_k 2E_k}(\sum\limits_k 2E_k + \sum\limits_{i\neq j} V_{ij})\sum\limits_i \gamma_v(E_i + \vec{v}\cdot\vec{p}_i)$ □

Lorentz boost transform from interacting moving particle system to static particle system:

$$\vec{v} = -\sum_i \vec{p}_i / \sum_i E_i \neq \vec{1} \Rightarrow \begin{cases} \vec{P}(\vec{v}) = \vec{0} \\ H(\vec{v}) = (\sum\limits_k E_k + \frac{1}{2}\sum\limits_{i\neq j} V_{ij})\sqrt{1-v^2} \equiv M_0 \end{cases} \tag{14.17}$$

The mass formula of the interacting particle system:

$M_0 = (\sum\limits_i E_i + \frac{1}{2}\sum\limits_{i\neq j} V_{ij})\sqrt{\sum\limits_{i,j}(E_iE_j - \vec{p}_i\cdot\vec{p}_j)/\sum\limits_{i,j}(E_iE_j)}$

3.9 Lorentz boost transform of interact particle system with different centroid velocity

Lorentz boost transformation between interacting particle systems:

$$\begin{cases} \vec{u'} = [\vec{u} + \gamma_v\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{u})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\vec{u})] \\ H(\vec{u'}) = \gamma_v[H(\vec{u}) + \vec{v}\cdot\vec{P}(\vec{u})] \\ \vec{P}(\vec{u'}) = \vec{P}(\vec{u}) + \gamma_v H(\vec{u})\vec{v} + (\gamma_v-1)[\vec{v}\cdot\vec{P}(\vec{u})]\vec{v}/v^2 \\ \vec{P}^2(\vec{u'}) - H^2(\vec{u'}) = \vec{P}^2(\vec{u}) - H^2(\vec{u}) = -M_0^2 \end{cases} \tag{14.18}$$

Lorentz boost transformation of external force between interacting particle systems:

$$\begin{cases} \vec{f}'(\vec{u'}) = \vec{f}(\vec{u}) + \gamma_v[\vec{u}\cdot\vec{f}(\vec{u})]\vec{v} + (\gamma_v-1)[\vec{v}\cdot\vec{f}(\vec{u})]\vec{v}/v^2 \\ \vec{u'}\cdot\vec{f}'(\vec{u'}) = \gamma_v[\vec{u}\cdot\vec{f}(\vec{u}) + \vec{v}\cdot\vec{f}(\vec{u})] = \gamma_v(\vec{u}+\vec{v})\cdot\vec{f}(\vec{u}) \end{cases} \tag{14.19}$$

3.10 Universal static mass formula for interacting particle systems

Based on the above conclusions, the following universal static mass formula can be obtained.

$M_0 = (\sum\limits_i E_i + \frac{1}{2}\sum\limits_{i\neq j} V_{ij})\sqrt{\sum\limits_{i,j}(E_iE_j - \vec{p}_i\cdot\vec{p}_j)/\sum\limits_{i,j}(E_iE_j)}$

3.11 Universal static mass formula for hydrogen atoms (centroid system)???

$M_0 = \frac{[\sqrt{M^2+\vec{p}_M^2}+\sqrt{m^2+\vec{p}_m^2}+V(\vec{r}_M,\vec{r}_m)]\sqrt{(\sqrt{M^2+\vec{p}_M^2}+\sqrt{m^2+\vec{p}_m^2})^2-(\vec{p}_M+\vec{p}_m)^2}}{\sqrt{M^2+\vec{p}_M^2}+\sqrt{m^2+\vec{p}_m^2}}$

$M_0 = [M + m + V(\vec{r}_{M0}, \vec{r}_{m0})]$

$M + m = [\sqrt{M^2+\vec{p}^2} + \sqrt{m^2+\vec{p}^2} + V(\vec{r}_M, \vec{r}_m)]$

$V(\vec{r}_M, \vec{r}_m) = M + m - (\sqrt{M^2+\vec{p}^2} + \sqrt{m^2+\vec{p}^2})$

4 Lorentz boost transformation of various spinors

4.1 Lorentz transformation law of antisymmetric tensor and electromagnetic spinor [22–24]

Lorentz transformation law of angular momentum tensor and electromagnetic tensor of a single particle:

Thm. 4.1.1.

$$\begin{cases} F^{ab} = -F^{ba}, F' = L_{\vec{v}}FL_{\vec{v}}^T, F = \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix} \\ F' = \gamma_v \begin{bmatrix} 0 & (B_z+\vec{v}\times\vec{E})_z & -(B_y+\vec{v}\times\vec{E})_y & -i(\vec{E}-\vec{v}\times\vec{B})_x \\ -(B_z+\vec{v}\times\vec{E})_z & 0 & (B_x+\vec{v}\times\vec{E})_x & -i(\vec{E}-\vec{v}\times\vec{B})_y \\ (B_y+\vec{v}\times\vec{E})_y & -(B_x+\vec{v}\times\vec{E})_x & 0 & -i(\vec{E}-\vec{v}\times\vec{B})_z \\ i(\vec{E}-\vec{v}\times\vec{B})_x & i(\vec{E}-\vec{v}\times\vec{B})_y & i(\vec{E}-\vec{v}\times\vec{B})_z & 0 \end{bmatrix} - \frac{\gamma_v-1}{v^2}\begin{bmatrix} 0 & (\vec{v}\cdot\vec{B})v_z & -(\vec{v}\cdot\vec{B})v_y & -i(\vec{v}\cdot\vec{E})v_x \\ -(\vec{v}\cdot\vec{B})v_z & 0 & (\vec{v}\cdot\vec{B})v_x & -i(\vec{v}\cdot\vec{E})v_y \\ (\vec{v}\cdot\vec{B})v_y & -(\vec{v}\cdot\vec{B})v_x & 0 & -i(\vec{v}\cdot\vec{E})v_z \\ i(\vec{v}\cdot\vec{E})v_x & i(\vec{v}\cdot\vec{E})v_y & i(\vec{v}\cdot\vec{E})v_z & 0 \end{bmatrix} \end{cases}$$

Proof:

$$L_{\vec{v}}F = (\begin{bmatrix} 1 & 0 & 0 & -i\gamma_v v_x \\ 0 & 1 & 0 & -i\gamma_v v_y \\ 0 & 0 & 1 & -i\gamma_v v_z \\ i\gamma_v v_x & i\gamma_v v_y & i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma_v-1}{v^2}\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z & 0 \\ v_yv_x & v_yv_y & v_yv_z & 0 \\ v_zv_x & v_zv_y & v_zv_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix})\begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_v v_x E_x & B_z+\gamma_v v_x E_y & -B_y+\gamma_v v_x E_z & -iE_x \\ -B_z+\gamma_v v_y E_x & \gamma_v v_y E_y & B_x+\gamma_v v_y E_z & -iE_y \\ B_y+\gamma_v v_z E_x & -B_x+\gamma_v v_z E_y & \gamma_v v_z E_z & -iE_z \\ i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_x & i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_y & i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_z & \gamma_v\vec{v}\cdot\vec{E} \end{bmatrix} + \frac{\gamma_v-1}{v^2}\begin{bmatrix} -v_x(\vec{v}\times\vec{B})_x & -v_x(\vec{v}\times\vec{B})_y & -v_x(\vec{v}\times\vec{B})_z & -iv_x\vec{v}\cdot\vec{E} \\ -v_y(\vec{v}\times\vec{B})_x & -v_y(\vec{v}\times\vec{B})_y & -v_y(\vec{v}\times\vec{B})_z & -iv_y\vec{v}\cdot\vec{E} \\ -v_z(\vec{v}\times\vec{B})_x & -v_z(\vec{v}\times\vec{B})_y & -v_z(\vec{v}\times\vec{B})_z & -iv_z\vec{v}\cdot\vec{E} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$F' = L_{\vec{v}}FL_{\vec{v}}^T$

$$= (\begin{bmatrix} \gamma_v v_x E_x & B_z+\gamma_v v_x E_y & -B_y+\gamma_v v_x E_z & -iE_x \\ -B_z+\gamma_v v_y E_x & \gamma_v v_y E_y & B_x+\gamma_v v_y E_z & -iE_y \\ B_y+\gamma_v v_z E_x & -B_x+\gamma_v v_z E_y & \gamma_v v_z E_z & -iE_z \\ i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_x & i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_y & i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_z & \gamma_v\vec{v}\cdot\vec{E} \end{bmatrix} + \frac{\gamma_v-1}{v^2}\begin{bmatrix} -v_x(\vec{v}\times\vec{B})_x & -v_x(\vec{v}\times\vec{B})_y & -v_x(\vec{v}\times\vec{B})_z & -iv_x\vec{v}\cdot\vec{E} \\ -v_y(\vec{v}\times\vec{B})_x & -v_y(\vec{v}\times\vec{B})_y & -v_y(\vec{v}\times\vec{B})_z & -iv_y\vec{v}\cdot\vec{E} \\ -v_z(\vec{v}\times\vec{B})_x & -v_z(\vec{v}\times\vec{B})_y & -v_z(\vec{v}\times\vec{B})_z & -iv_z\vec{v}\cdot\vec{E} \\ 0 & 0 & 0 & 0 \end{bmatrix})$$

$$(\begin{bmatrix} 1 & 0 & 0 & i\gamma_v v_x \\ 0 & 1 & 0 & i\gamma_v v_y \\ 0 & 0 & 1 & i\gamma_v v_z \\ -i\gamma_v v_x & -i\gamma_v v_y & -i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma_v-1}{v^2}\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z & 0 \\ v_yv_x & v_yv_y & v_yv_z & 0 \\ v_zv_x & v_zv_y & v_zv_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix})$$

$$= \begin{bmatrix} 0 & B_z+\gamma_v(\vec{v}\times\vec{E})_z & -B_y-\gamma_v(\vec{v}\times\vec{E})_y & i\gamma_v^2 v_x\vec{v}\cdot\vec{E}-i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_x \\ -B_z-\gamma_v(\vec{v}\times\vec{E})_z & 0 & B_x+\gamma_v(\vec{v}\times\vec{E})_x & i\gamma_v^2 v_y\vec{v}\cdot\vec{E}-i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_y \\ B_y+\gamma_v(\vec{v}\times\vec{E})_y & -B_x-\gamma_v(\vec{v}\times\vec{E})_x & 0 & i\gamma_v^2 v_z\vec{v}\cdot\vec{E}-i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_z \\ -i\gamma_v^2 v_x\vec{v}\cdot\vec{E}+i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_x & -i\gamma_v^2 v_y\vec{v}\cdot\vec{E}+i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_y & -i\gamma_v^2 v_z\vec{v}\cdot\vec{E}+i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_z & \gamma_v^2\vec{v}\cdot(\vec{v}\times\vec{B})=0 \end{bmatrix}$$

$$+\frac{\gamma_v-1}{v^2}\begin{bmatrix} v_x(\vec{v}\times\vec{B})_x+\gamma_v v_x v_x \vec{v}\cdot\vec{E} & v_y(\vec{v}\times\vec{B})_x+\gamma_v v_x v_y \vec{v}\cdot\vec{E} & v_z(\vec{v}\times\vec{B})_x+\gamma_v v_x v_z \vec{v}\cdot\vec{E} & 0 \\ v_x(\vec{v}\times\vec{B})_y+\gamma_v v_y v_x \vec{v}\cdot\vec{E} & v_y(\vec{v}\times\vec{B})_y+\gamma_v v_y v_y \vec{v}\cdot\vec{E} & v_z(\vec{v}\times\vec{B})_y+\gamma_v v_y v_z \vec{v}\cdot\vec{E} & 0 \\ v_x(\vec{v}\times\vec{B})_z+\gamma_v v_z v_x \vec{v}\cdot\vec{E} & v_y(\vec{v}\times\vec{B})_z+\gamma_v v_z v_y \vec{v}\cdot\vec{E} & v_z(\vec{v}\times\vec{B})_z+\gamma_v v_z v_z \vec{v}\cdot\vec{E} & 0 \\ i\gamma_v v_x \vec{v}\cdot\vec{E} & i\gamma_v v_y \vec{v}\cdot\vec{E} & i\gamma_v v_z \vec{v}\cdot\vec{E} & 0 \end{bmatrix}$$

$$+\frac{\gamma_v-1}{v^2}\begin{bmatrix} -v_x(\vec{v}\times\vec{B})_x-\gamma_v v_x v_x \vec{v}\cdot\vec{E} & -v_x(\vec{v}\times\vec{B})_y-\gamma_v v_x v_y \vec{v}\cdot\vec{E} & -v_x(\vec{v}\times\vec{B})_z-\gamma_v v_x v_z \vec{v}\cdot\vec{E} & -i\gamma_v v_x \vec{v}\cdot\vec{E} \\ -v_y(\vec{v}\times\vec{B})_x-\gamma_v v_y v_x \vec{v}\cdot\vec{E} & -v_y(\vec{v}\times\vec{B})_y-\gamma_v v_y v_y \vec{v}\cdot\vec{E} & -v_y(\vec{v}\times\vec{B})_z-\gamma_v v_y v_z \vec{v}\cdot\vec{E} & -i\gamma_v v_y \vec{v}\cdot\vec{E} \\ -v_z(\vec{v}\times\vec{B})_x-\gamma_v v_z v_x \vec{v}\cdot\vec{E} & -v_z(\vec{v}\times\vec{B})_y-\gamma_v v_z v_y \vec{v}\cdot\vec{E} & -v_z(\vec{v}\times\vec{B})_z-\gamma_v v_z v_z \vec{v}\cdot\vec{E} & -i\gamma_v v_z \vec{v}\cdot\vec{E} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+\left(\frac{\gamma_v-1}{v^2}\right)^2\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & B_z+\gamma_v(\vec{v}\times\vec{E})_z & -B_y-\gamma_v(\vec{v}\times\vec{E})_y & i\gamma_v^2 v_x\vec{v}\cdot\vec{E}-i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_x \\ -B_z-\gamma_v(\vec{v}\times\vec{E})_z & 0 & B_x+\gamma_v(\vec{v}\times\vec{E})_x & i\gamma_v^2 v_y\vec{v}\cdot\vec{E}-i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_y \\ B_y+\gamma_v(\vec{v}\times\vec{E})_y & -B_x-\gamma_v(\vec{v}\times\vec{E})_x & 0 & i\gamma_v^2 v_z\vec{v}\cdot\vec{E}-i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_z \\ -i\gamma_v^2 v_x\vec{v}\cdot\vec{E}+i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_x & -i\gamma_v^2 v_y\vec{v}\cdot\vec{E}+i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_y & -i\gamma_v^2 v_z\vec{v}\cdot\vec{E}+i\gamma_v(\vec{E}-\vec{v}\times\vec{B})_z & \gamma_v^2\vec{v}\cdot(\vec{v}\times\vec{B})=0 \end{bmatrix}$$

$$-\frac{\gamma_v-1}{v^2}\begin{bmatrix} 0 & \vec{v}\times(\vec{v}\times\vec{B})_z & -\vec{v}\times(\vec{v}\times\vec{B})_y & i\gamma_v v_x\vec{v}\cdot\vec{E} \\ -\vec{v}\times(\vec{v}\times\vec{B})_z & 0 & \vec{v}\times(\vec{v}\times\vec{B})_x & i\gamma_v v_y\vec{v}\cdot\vec{E} \\ \vec{v}\times(\vec{v}\times\vec{B})_y & -\vec{v}\times(\vec{v}\times\vec{B})_x & 0 & i\gamma_v v_z\vec{v}\cdot\vec{E} \\ -i\gamma_v v_x\vec{v}\cdot\vec{E} & -i\gamma_v v_y\vec{v}\cdot\vec{E} & -i\gamma_v v_z\vec{v}\cdot\vec{E} & 0 \end{bmatrix}$$

$$=\gamma_v\begin{bmatrix} 0 & (B_z+\vec{v}\times\vec{E})_z & -(B_y+\vec{v}\times\vec{E})_y & -i(\vec{E}-\vec{v}\times\vec{B})_x \\ -(B_z+\vec{v}\times\vec{E})_z & 0 & (B_x+\vec{v}\times\vec{E})_x & -i(\vec{E}-\vec{v}\times\vec{B})_y \\ (B_y+\vec{v}\times\vec{E})_y & -(B_x+\vec{v}\times\vec{E})_x & 0 & -i(\vec{E}-\vec{v}\times\vec{B})_z \\ i(\vec{E}-\vec{v}\times\vec{B})_x & i(\vec{E}-\vec{v}\times\vec{B})_y & i(\vec{E}-\vec{v}\times\vec{B})_z & 0 \end{bmatrix}-\frac{\gamma_v-1}{v^2}\begin{bmatrix} 0 & (\vec{v}\cdot\vec{B})v_z & -(\vec{v}\cdot\vec{B})v_y & -i(\vec{v}\cdot\vec{E})v_x \\ -(\vec{v}\cdot\vec{B})v_z & 0 & (\vec{v}\cdot\vec{B})v_x & -i(\vec{v}\cdot\vec{E})v_y \\ (\vec{v}\cdot\vec{B})v_y & -(\vec{v}\cdot\vec{B})v_x & 0 & -i(\vec{v}\cdot\vec{E})v_z \\ i(\vec{v}\cdot\vec{E})v_x & i(\vec{v}\cdot\vec{E})v_y & i(\vec{v}\cdot\vec{E})v_z & 0 \end{bmatrix}$$

□

Cor. 4.1.1. $\vec{E}'=\gamma_v(\vec{E}-\vec{v}\times\vec{B})-(\gamma_v-1)(\vec{v}\cdot\vec{E})\vec{v}/v^2, \vec{B}'=\gamma_v(\vec{B}+\vec{v}\times\vec{E})-(\gamma_v-1)(\vec{v}\cdot\vec{B})\vec{v}/v^2$

Cor. 4.1.2. $\vec{\varphi}'_\varsigma=\gamma_v(\vec{\varphi}_\varsigma-i\varsigma\vec{v}\times\vec{\varphi}_\varsigma)-(\gamma_v-1)(\vec{v}\cdot\vec{\varphi}_\varsigma)\vec{v}/v^2, \psi^{\alpha_\varsigma}:=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}F^{ab}=-i\varsigma(\vec{E}-i\varsigma\vec{B}):=\vec{\varphi}_\varsigma$

Cor. 4.1.3. $\vec{\varphi}'_\varsigma=R_{\varsigma\vec{v}}\vec{\varphi}_\varsigma, R_{\varsigma\vec{v}}\equiv\gamma_v-\varsigma\gamma_v\begin{bmatrix} 0 & -iv_z & iv_y \\ iv_z & 0 & -iv_x \\ -iv_y & iv_x & 0 \end{bmatrix}-\frac{\gamma_v-1}{v^2}\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z \\ v_yv_x & v_yv_y & v_yv_z \\ v_zv_x & v_zv_y & v_zv_z \end{bmatrix}$

Cor. 4.1.4. $R_{\varsigma\vec{v}}=1-\varsigma\gamma_v\vec{v}\cdot\gamma+\frac{\gamma_v-1}{v^2}(\vec{v}\cdot\gamma)^2, R_{\varsigma\vec{v}}R_{-\varsigma\vec{v}}=R_{-\varsigma\vec{v}}R_{\varsigma\vec{v}}=I$

4.2 Angular momentum transformation law

Pro. 4.2.1. $M^{ab}=x^ap^b-x^bp^a, M'=L_{\vec{v}}ML^T_{\vec{v}}, \vec{J}=\vec{B}=\vec{r}\times\vec{p}, \vec{W}=\vec{E}=t\vec{p}-\vec{r}E$

Pro. 4.2.2. $M^{ab}=x^ap^b-x^bp^a, M'=L_{\vec{v}}ML^T_{\vec{v}}, \vec{J}=\vec{B}=\vec{r}\times\vec{p}, \vec{W}=\vec{E}=r\vec{p}-\vec{r}E$

Cor. 4.2.1. $\begin{cases}\vec{W}'=\gamma_v(\vec{W}-\vec{v}\times\vec{J})-(\gamma_v-1)(\vec{v}\cdot\vec{W})\vec{v}/v^2 \\ \vec{J}'=\gamma_v(\vec{J}+\vec{v}\times\vec{W})-(\gamma_v-1)(\vec{v}\cdot\vec{J})\vec{v}/v^2\end{cases}$

4.3 Guessing spinor transformation law from photon spinor transformation Law

Cor. 4.3.1. $\begin{cases}\vec{\varphi}'_\varsigma=\gamma_v(\vec{\varphi}_\varsigma-i\varsigma\vec{v}\times\vec{\varphi}_\varsigma)-(\gamma_v-1)(\vec{v}\cdot\vec{\varphi}_\varsigma)\vec{v}/v^2 \\ (\vec{\varphi}_\varsigma,0)=S_{em}(\kappa)\psi_\varsigma\otimes\psi_\varsigma\to\vec{\varphi}_\varsigma=\frac{1}{\sqrt{2}}(\psi^2_{\varsigma1}-\psi^2_{\varsigma2}, i\psi^2_{\varsigma1}+i\psi^2_{\varsigma2}, -2\psi_{\varsigma1}\psi_{\varsigma2})\end{cases}$

Cor. 4.3.2. $\Lambda(\psi^2_{\varsigma1},\psi^2_{\varsigma2},\psi_{\varsigma1}\psi_{\varsigma2})=\frac{1}{2}\begin{bmatrix} 1 & -i & 0 \\ -1 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix}\left(\gamma_v-\varsigma\gamma_v\begin{bmatrix} 0 & -iv_z & iv_y \\ iv_z & 0 & -iv_x \\ -iv_y & iv_x & 0 \end{bmatrix}-\frac{\gamma_v-1}{v^2}\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z \\ v_yv_x & v_yv_y & v_yv_z \\ v_zv_x & v_zv_y & v_zv_z \end{bmatrix}\right)\begin{bmatrix} 1 & -1 & 0 \\ i & i & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Guessing+Reasoning:

Cor. 4.3.3. $\Lambda(\psi^2_{\varsigma1},\psi^2_{\varsigma2},\psi_{\varsigma1}\psi_{\varsigma2})=\begin{bmatrix} (\gamma+1-\varsigma\gamma v_z)^2/[2(\gamma+1)] & [\gamma(v_x-iv_y)]^2/[2(\gamma+1)] & -i\varsigma v_y \\ [\gamma(v_x+iv_y)]^2/[2(\gamma+1)] & (\gamma+1+\varsigma\gamma v_z)^2/[2(\gamma+1)] & i\varsigma v_x \\ i\varsigma v_y & -i\varsigma v_x & 1 \end{bmatrix}$

$\Leftarrow\Lambda_{\varsigma\vec{v}}(\psi_{\varsigma1},\psi_{\varsigma2})=\frac{1}{\sqrt{2(\gamma+1)}}\begin{bmatrix} \gamma+1-\varsigma\gamma v_z & -\gamma\varsigma(v_x-iv_y) \\ -\varsigma\gamma(v_x+iv_y) & \gamma+1+\varsigma\gamma v_z \end{bmatrix}=\frac{1}{\sqrt{2(\gamma+1)}}(1+\gamma-\varsigma\gamma\vec{v}\cdot\sigma)$

4.4 Derive photon spinor transformation law from spinor transformation Law

Cor. 4.4.1. $\Lambda_{\varsigma\vec{v}}\otimes\Lambda_{\varsigma\vec{v}}=\frac{\gamma+1}{2}-\frac{1}{2}\varsigma\gamma\vec{v}\cdot(\sigma\otimes I+I\otimes\sigma)+\frac{\gamma_v-1}{2v^2}(\vec{v}\cdot\sigma)\otimes(\vec{v}\cdot\sigma)$

Cor. 4.4.2. $R_{\varsigma\vec{v}}=S_{em}(\kappa)\Lambda_{\varsigma\vec{v}}\otimes\Lambda_{\varsigma\vec{v}}S^+_{em}(\kappa)=\frac{\gamma_v+1}{2}-\varsigma\gamma_v\vec{v}\cdot R+\frac{\gamma_v-1}{2v^2}(\vec{v}\cdot\sigma_+)(\vec{v}\cdot\sigma_-)$

$=1-\varsigma\gamma_v\vec{v}\cdot R+\frac{\gamma_v-1}{v^2}(\vec{v}\cdot R)^2=\begin{bmatrix} \gamma_v & 0 & 0 & 0 \\ 0 & \gamma_v & 0 & 0 \\ 0 & 0 & \gamma_v & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}-\varsigma\gamma_v\begin{bmatrix} 0 & -iv_z & iv_y & 0 \\ iv_z & 0 & -iv_x & 0 \\ -iv_y & iv_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}-\frac{\gamma_v-1}{v^2}\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z & 0 \\ v_yv_x & v_yv_y & v_yv_z & 0 \\ v_zv_x & v_zv_y & v_zv_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Cor. 4.4.3. $(\vec{v}\cdot\sigma_+)(\vec{v}\cdot\sigma_-)=-2\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z & 0 \\ v_yv_x & v_yv_y & v_yv_z & 0 \\ v_zv_x & v_zv_y & v_zv_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}+v^2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Cor. 4.4.4. $(\vec{v}\cdot R)^2=-\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z & 0 \\ v_yv_x & v_yv_y & v_yv_z & 0 \\ v_zv_x & v_zv_y & v_zv_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}+v^2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, (\vec{v}\cdot\gamma)^2=-\begin{bmatrix} v_xv_x & v_xv_y & v_xv_z \\ v_yv_x & v_yv_y & v_yv_z \\ v_zv_x & v_zv_y & v_zv_z \end{bmatrix}+v^2$

Cor. 4.4.5. $(\vec{v}\cdot\sigma_+)(\vec{v}\cdot\sigma_-)=2(\vec{v}\cdot R)^2-v^2$

4.5 Derive vector transformation law from spinor transformation Law

Def. 4.5.1. $L_{\vec{v}} \equiv \begin{bmatrix} 1 & 0 & 0 & -i\gamma_v v_x \\ 0 & 1 & 0 & -i\gamma_v v_y \\ 0 & 0 & 1 & -i\gamma_v v_z \\ i\gamma_v v_x & i\gamma_v v_y & i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \gamma_v(1 - \vec{v} \cdot L) - \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot R)^2$

Cor. 4.5.1.
$\Lambda_{\varsigma\vec{v}} \otimes \Lambda_{-\varsigma\vec{v}} = \frac{1}{2(\gamma_v+1)}(1 + \gamma_v - \varsigma\gamma_v\vec{v} \cdot \sigma) \otimes (1 + \gamma_v + \varsigma\gamma_v\vec{v} \cdot \sigma) = \frac{\gamma_v+1}{2} - \frac{1}{2}\varsigma\gamma_v\vec{v} \cdot (\sigma \otimes I - I \otimes \sigma) - \frac{\gamma_v-1}{2v^2}(\vec{v} \cdot \sigma) \otimes (\vec{v} \cdot \sigma)$

Cor. 4.5.2.
$L_{-\kappa\varsigma\vec{v}} = S_{em}(\kappa)\Lambda_{\varsigma\vec{v}} \otimes \Lambda_{-\varsigma\vec{v}}S^+_{em}(\kappa) = \frac{\gamma_v+1}{2} + \kappa\varsigma\gamma_v\vec{v} \cdot L - \frac{\gamma_v-1}{2v^2}(\vec{v} \cdot \sigma_+)(\vec{v} \cdot \sigma_-) = \gamma_v(1 + \kappa\varsigma\vec{v} \cdot L) - \frac{\gamma_v-1}{v^2}(\vec{v} \cdot R)^2$

4.6 Summary of Lorentz boost transformation

Lorentz boost transformation of spinor:

Cor. 4.6.1. $\Lambda_{\varsigma\vec{v}} = e^{-\frac{1}{2}\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \varsigma\gamma_v\vec{v} \cdot \sigma), \epsilon \sim -v, {}^{A_\varsigma} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s)}$

Lorentz boost transformation of Dirac spinor:

Cor. 4.6.2. $D_{\varsigma\vec{v}} = e^{-\frac{1}{2}\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma\otimes\sigma_z} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \varsigma\gamma_v\vec{v} \cdot \sigma \otimes \sigma_z), D_{\varsigma\vec{v}} = \Lambda_{\varsigma\vec{v}} \oplus \Lambda_{-\varsigma\vec{v}}$

Lorentz boost transformation of vector:

Cor. 4.6.3. $L_{-\kappa\varsigma\vec{v}} = \gamma_v(1 + \kappa\varsigma\vec{v} \cdot L) - \frac{\gamma_v-1}{v^2}(\vec{v} \cdot R)^2, L_{-\kappa\varsigma\vec{v}} = S_{em}(\kappa)\Lambda_{\varsigma\vec{v}} \otimes \Lambda_{-\varsigma\vec{v}}S^+_{em}(\kappa)$

Lorentz boost transformation of electromagnetic spinor and angular momentum:

Cor. 4.6.4. $R_{\varsigma\vec{v}} = 1 - \varsigma\gamma_v\vec{v} \cdot R + \frac{\gamma_v-1}{v^2}(\vec{v} \cdot R)^2, R_{\varsigma\vec{v}} = S_{em}(\kappa)\Lambda_{\varsigma\vec{v}} \otimes \Lambda_{\varsigma\vec{v}}S^+_{em}(\kappa)$

Lorentz boost transformation of s-spinor:

Cor. 4.6.5. $\Lambda_{\varsigma\vec{v}}(s) = \bar{\mathcal{P}}(s + \frac{1}{2})\overbrace{\Lambda_{\varsigma\vec{v}} \otimes \cdots \otimes \Lambda_{\varsigma\vec{v}}}^{2s}\mathcal{P}(s + \frac{1}{2}), \Lambda_{\varsigma\vec{v}} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \varsigma\gamma_v\vec{v} \cdot \sigma)$

5 Polynomial representation of Lorentz transformation for various spin particles

The above method is tedious, intuitive, and speculative. Below, a more analytical, rigorous, organized, systematic analysis and derivation method will be used. It will get a more general and universal conclusion.

5.1 Mathematical preparation

5.1.1 Definition

Def. 5.1.1. $e(s, n, \sigma) \equiv (\overbrace{I \otimes \cdots \otimes I \otimes}^{n-1} \sigma \overbrace{\otimes I \otimes \cdots \otimes I}^{2s-n})$

Def. 5.1.2. $\hat{\Omega}(s) \equiv \hat{\Omega}(s, 1, \sigma) \equiv \sum_{n=1}^{2s} e(s, n, \sigma), \Omega(s) \equiv \frac{1}{2}\hat{\Omega}(s, 1, \sigma)$

$\hat{\Omega}(s, 1, \vec{\vartheta} \cdot \sigma) \equiv \sum_{n=1}^{2s} e(s, n, \vec{\vartheta} \cdot \sigma) = \vec{\vartheta} \cdot \hat{\Omega}(s, 1, \sigma)$

$\hat{\Omega}(s, 2, \vec{\vartheta} \cdot \sigma) \equiv \frac{1}{2!}\sum_{i\neq j}^{1,2s} e(s, i, \vec{\vartheta} \cdot \sigma)e(s, j, \vec{\vartheta} \cdot \sigma)$

$\hat{\Omega}(s, n, \vec{\vartheta} \cdot \sigma) \equiv \frac{1}{n!}\sum_{i_1\neq i_2\cdots\neq i_n}^{1,2s} e(s, i_1, \vec{\vartheta} \cdot \sigma)e(s, i_2, \vec{\vartheta} \cdot \sigma) \cdots e(s, i_n, \vec{\vartheta} \cdot \sigma),$

5.1.2 Important properties

Pro. 5.1.1. $\hat{\Omega}(s, 2, \vec{\vartheta} \cdot \sigma) = \frac{1}{2}\hat{\Omega}^2(s, 1, \vec{\vartheta} \cdot \sigma) - s\vec{\vartheta}^2$

Pro. 5.1.2. $\hat{\Omega}(s \leq 2, n, \vec{\vartheta} \cdot \sigma) = \frac{1}{n!}\hat{\Omega}^n(s, 1, \vec{\vartheta} \cdot \sigma)u(n - 1)$
$- \frac{1}{n!}(\frac{2sC^{n-2}_{2s-1}}{C^{n-2}_{2s}})[C^2_n(n-2)!]\vec{\vartheta}^2\hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma)u(n-2) - \frac{1}{n!}(\frac{2sC^{n-3}_{2s-1}}{C^{n-2}_{2s}})[C^3_n(n-3)!]\vec{\vartheta}^2\hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma)u(n-3) + \cdots$
$= \frac{1}{n!}\hat{\Omega}^n(s, 1, \vec{\vartheta} \cdot \sigma)u(n-1) - [s - \frac{1}{2}(n-2)]\vec{\vartheta}^2\hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma)u(n-2) - \frac{1}{6}(n-2)\vec{\vartheta}^2\hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma)u(n-3) - \frac{5}{3}\vec{\vartheta}^4\delta_{n,4}$

5.1.3 Properties of Lorentz generator matrix [22]

Pro. 5.1.3. $\vec{\vartheta}^2 = 0 \Rightarrow (\vec{\vartheta} \cdot \sigma)^2 = 0, (\vec{\vartheta} \cdot \gamma)^2 = 0, (\vec{\vartheta} \cdot R)^2 = 0, (\vec{\vartheta} \cdot L)^2 = 0$

Pro. 5.1.4. $\vec{\vartheta}^2 = 0 \Rightarrow [\vec{\vartheta} \cdot (\sigma \otimes I + I \otimes \sigma)]^2 = 0, (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) = 0$

Pro. 5.1.5. $\vec{\vartheta}^2 = 0 \Rightarrow [\vec{\vartheta} \cdot (\sigma \otimes I \otimes I + I \otimes \sigma \otimes I + I \otimes I \otimes \sigma)]^2 = 0$

Pro. 5.1.6. $\vec{\vartheta}^2 = 0 \Rightarrow [\vec{\vartheta} \cdot \hat{\Omega}(s)]^2 = 0, [\vec{\vartheta} \cdot \Omega(s)]^2 = 0, [\vec{\vartheta} \cdot \sigma(s)]^2 = 0$

Pro. 5.1.7. $\vec{\vartheta}^2 = 1 \Rightarrow (\vec{\vartheta} \cdot \sigma)^3 = \vec{\vartheta} \cdot \sigma, (\vec{\vartheta} \cdot \gamma)^3 = \vec{\vartheta} \cdot \gamma, (\vec{\vartheta} \cdot R)^3 = \vec{\vartheta} \cdot R, (\vec{\vartheta} \cdot L)^3 = \vec{\vartheta} \cdot L$

5.2 Polynomial expansion method

Thm. 5.2.1. $(a_1+a_2+\cdots+a_m)^n=\sum\limits_{(\sum\limits_k i_k)=n}\frac{n!}{i_1!i_2!\cdots i_m!}a_1^{i_1}a_2^{i_2}\cdots a_m^{i_m}$

Def. 5.2.1. $<k_1,k_2,\cdots,k_m>=\sum a_{i_1}^{k_1}a_{i_2}^{k_2}\cdots a_{i_m}^{k_m}$
$i_1\neq i_2\neq\cdots\neq i_m,k_1\geq k_2\geq\cdots\geq k_m\geq 1,k_1+k_2+\cdots+k_m=n$

Def. 5.2.2. $<k_1,k_2,\cdots,k_l>:=P_m^{-l}\sum\limits_{i_1i_2\cdots i_l=}^{P^l_{(1,\cdots,m)}}a_{i_1}^{k_1}a_{i_2}^{k_2}\cdots a_{i_l}^{k_l}$
$i_1\neq i_2\neq\cdots\neq i_l,k_1\geq k_2\geq\cdots\geq k_l\geq 1,k_1+k_2+\cdots+k_l=n$

Def. 5.2.3. $<(n_1;l_1),(n_2;l_2),\cdots,(n_k;l_k)>:=<\overbrace{n_1,\cdots,n_1}^{l_1},\overbrace{n_2,\cdots,n_2}^{l_2},\cdots,\overbrace{n_k,\cdots,n_k}^{l_k}>$
$n_1>n_2>\cdots>n_k;l_1,l_2,\cdots,l_k\geq 1;l_1+l_2+\cdots+l_k\leq m;n_1l_1+n_2l_2+\cdots+n_kl_k=n$

Ass. 5.2.1.
$$\begin{cases}(a_1+a_2+\cdots+a_m)^n=\sum\limits_{i_1\cdots+i_m=n}\frac{n!}{i_1!i_2!\cdots i_m!}a_1^{i_1}a_2^{i_2}\cdots a_m^{i_m}\\ =\sum\frac{n!}{(n_1!)^{l_1}(n_2!)^{l_2}\cdots(n_k!)^{l_k}}C_m^{l_1}C_{m-l_1}^{l_2}\cdots C_{m-(l_1+\cdots+l_{k-1})}^{l_k}<(n_1;l_1),(n_2;l_2),\cdots,(n_k;l_k)>\\ (a_1+a_2+\cdots+a_m)^n=\sum\limits_{i_1\cdots+i_m=n}\frac{n!}{i_1!i_2!\cdots i_m!}a_1^{i_1}a_2^{i_2}\cdots a_m^{i_m}\\ =\sum\frac{n!}{(n_1!)^{l_1}(n_2!)^{l_2}\cdots(n_k!)^{l_k}}\frac{m!}{l_1!l_2!\cdots l_k!(m-l_1-\cdots-l_k)!}<(n_1;l_1),(n_2;l_2),\cdots,(n_k;l_k)>\end{cases}$$

5.2.1 Example: Binomial expansion

Pro. 5.2.1. $(a_1+a_2)^2=\sum\frac{2!}{i_1!i_2!}a_1^{i_1}a_2^{i_2}$
$=\frac{2!}{2!0!}(a_1^2a_2^0+a_1^0a_2^2)+\frac{2!}{1!1!}(a_1^1a_2^1)$
$=\frac{2!}{2!0!}\frac{2!}{1!1!}<2,0>+\frac{2!}{1!1!}\frac{2!}{2!}<1,1>,<2,0>:=\frac{1!1!}{2!}(a_1^2a_2^0+a_1^0a_2^2),<1,1>:=\frac{2!}{2!}(a_1^1a_2^1)$
$=2<2,0>+2<1,1>$

Pro. 5.2.2. $(a_1+a_2)^3=\sum\frac{3!}{i_1!i_2!}a_1^{i_1}a_2^{i_2}$
$=\frac{3!}{3!0!}\frac{2!}{1!1!}<3,0>+\frac{3!}{2!1!}\frac{2!}{1!1!}<2,1>,<3,0>:=\frac{1!}{2!}(a_1^3a_2^0+a_1^0a_2^3),<2,1>:=\frac{1!1!}{2!}(a_1^2a_2^1+a_1^1a_2^2)$
$=2<3,0>+6<2,1>$

Pro. 5.2.3. $(a_1+a_2)^4=\sum\frac{4!}{i_1!i_2!}a_1^{i_1}a_2^{i_2}$
$=\frac{4!}{4!0!}\frac{2!}{1!1!}<4,0>+\frac{4!}{3!1!}\frac{2!}{1!1!}<3,1>+\frac{4!}{2!2!}\frac{2!}{2!}<2,2>$
$=2<4,0>+8<3,1>+6<2,2>$

5.2.2 Example: Trinomial expansion

Pro. 5.2.4. $(a_1+a_2+a_3)^2=\sum\frac{2!}{i_1!i_2!i_3!}a_1^{i_1}a_2^{i_2}a_3^{i_3}$
$=\frac{2!}{2!0!0!}\frac{3!}{1!2!}<2,0,0>+\frac{2!}{1!1!0!}\frac{3!}{2!1!}<1,1,0>$
$=3<2,0,0>+6<1,1,0>$

Pro. 5.2.5. $(a_1+a_2+a_3)^3=\sum\frac{3!}{i_1!i_2!i_3!}a_1^{i_1}a_2^{i_2}a_3^{i_3}$
$=\frac{3!}{3!0!0!}\frac{3!}{1!2!}<3,0,0>+\frac{3!}{2!1!0!}\frac{3!}{1!1!1!}<2,1,0>+\frac{3!}{1!1!1!}\frac{3!}{3!}<1,1,1>$
$=3<3,0,0>+18<2,1,0>+6<1,1,1>$

Pro. 5.2.6. $(a_1+a_2+a_3)^4=\sum\frac{4!}{i_1!i_2!i_3!}a_1^{i_1}a_2^{i_2}a_3^{i_3}$
$=\frac{4!}{4!0!0!}\frac{3!}{1!2!}<4,0,0>+\frac{4!}{3!1!0!}\frac{3!}{1!1!1!}<3,1,0>+\frac{4!}{2!2!0!}\frac{3!}{2!1!}<2,2,0>+\frac{4!}{2!1!1!}\frac{3!}{1!2!}<2,1,1>$
$=3<4,0,0>+24<3,1,0>+18<2,2,0>+36<2,1,1>$

5.2.3 Example: Quadrennial expansion

Pro. 5.2.7. $(a_1+a_2+a_3+a_4)^2=\sum\frac{2!}{i_1!i_2!i_3!i_4!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}$
$=\frac{2!}{2!0!0!0!}\frac{4!}{1!3!}<2,0,0,0>+\frac{2!}{1!1!0!0!}\frac{4!}{2!2!}<1,1,0,0>$
$=4<2,0,0,0>+12<1,1,0,0>$

Pro. 5.2.8. $(a_1+a_2+a_3+a_4)^3=\sum\frac{3!}{i_1!i_2!i_3!i_4!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}$
$=\frac{3!}{3!0!0!0!}\frac{4!}{1!3!}<3,0,0,0>+\frac{3!}{2!1!0!0!}\frac{4!}{1!1!2!}<2,1,0,0>+\frac{3!}{1!1!1!0!}\frac{4!}{3!1!}<1,1,1,0>$
$=4<3,0,0,0>+36<2,1,0,0>+24<1,1,1,0>$

Pro. 5.2.9. $(a_1+a_2+a_3+a_4)^4=\sum\frac{4!}{i_1!i_2!i_3!i_4!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}$
$=\frac{4!}{4!0!0!0!}\frac{4!}{1!3!}<4,0,0,0>+\frac{4!}{3!1!0!0!}\frac{4!}{1!1!2!}<3,1,0,0>+\frac{4!}{2!2!0!0!}\frac{4!}{2!2!}<2,2,0,0>$
$+\frac{4!}{2!1!1!0!}\frac{4!}{1!2!1!}<2,1,1,0>+\frac{4!}{1!1!1!1!}\frac{4!}{4!}<1,1,1,1>$
$=4<4,0,0,0>+48<3,1,0,0>+36<2,2>+144<2,1,1,0>+24<1,1,1,1>$

Pro. 5.2.10. $(a_1+a_2+a_3+a_4)^5=\sum\frac{5!}{i_1!i_2!i_3!i_4!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}$
$=\frac{5!}{5!0!0!0!}\frac{4!}{1!3!}<5,0,0,0>+\frac{5!}{4!1!0!0!}\frac{4!}{1!1!2!}<4,1,0,0>+\frac{5!}{3!2!0!0!}\frac{4!}{1!1!2!}<3,2,0,0>$
$+\frac{5!}{3!1!1!0!}\frac{4!}{1!2!1!}<3,1,1,0>+\frac{5!}{2!2!1!0!}\frac{4!}{2!1!1!}<2,2,1,0>+\frac{5!}{2!1!1!1!}\frac{4!}{1!3!}<2,1,1,1>$
$=4<5,0,0,0>+60<4,1,0,0>+120<3,2,0,0>+240<3,1,1,0>+360<2,2,1,0>+240<2,1,1,1>$

5.2.4 Example: Quinomial expansion

Thm. 5.2.2. $(a_1+a_2+\cdots+a_m)^n$
$=\sum\frac{n!}{(n_1!)^{l_1}(n_2!)^{l_2}\cdots(n_k!)^{l_k}}\frac{m!}{l_1!l_2!\cdots l_k!(m-l_1-\cdots-l_k)!}<(n_1;l_1),(n_2;l_2),\cdots,(n_k;l_k)>$

Pro. 5.2.11. $(a_1+a_2+a_3+a_4+a_5)^2=\sum\frac{2!}{i_1!i_2!i_3!i_4!i_5!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}a_5^{i_5}$
$=\frac{2!}{2!0!^4}\frac{5!}{1!4!}<2,0,0,0,0>+\frac{2!}{1!^20!^3}\frac{5!}{2!3!}<1,1,0,0,0>$
$=5<2,0,0,0>+20<1,1,0,0,0>$

Pro. 5.2.12. $(a_1+a_2+a_3+a_4+a_5)^3=\sum\frac{3!}{i_1!i_2!i_3!i_4!i_5!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}a_5^{i_5}$
$=\frac{3!}{3!0!^4}\frac{5!}{1!4!}<3,0,0,0,0>+\frac{3!}{2!1!0!^3}\frac{5!}{1!1!3!}<2,1,0,0,0>+\frac{3!}{1!^30!^2}\frac{5!}{3!2!}<1,1,1,0>$
$=5<3,0,0,0,0>+60<2,1,0,0,0>+60<1,1,1,0,0>$

Pro. 5.2.13. $(a_1+a_2+a_3+a_4+a_5)^4=\sum\frac{4!}{i_1!i_2!i_3!i_4!i_5!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}a_5^{i_5}$
$=\frac{4!}{4!0!^4}\frac{5!}{1!4!}<4,0,0,0,0>+\frac{4!}{3!1!0!^3}\frac{5!}{1!1!3!}<3,1,0,0,0>+\frac{4!}{2!^20!^3}\frac{5!}{2!3!}<2,2,0,0,0>$
$+\frac{4!}{2!1!^20!^2}\frac{5!}{1!2!2!}<2,1,1,0,0>+\frac{4!}{1!^40!}\frac{5!}{4!1!}<1,1,1,1,0>$
$=5<4,0,0,0,0>+80<3,1,0,0,0>+60<2,2,0>+360<2,1,1,0,0>+120<1,1,1,1,0>$

Pro. 5.2.14. $(a_1+a_2+a_3+a_4+a_5)^5=\sum\frac{5!}{i_1!i_2!i_3!i_4!i_5!}a_1^{i_1}a_2^{i_2}a_3^{i_3}a_4^{i_4}a_5^{i_5}$
$=\frac{5!}{5!0!^4}\frac{5!}{1!4!}<5,0,0,0,0>+\frac{5!}{4!1!0!^3}\frac{5!}{1!1!3!}<4,1,0,0,0>+\frac{5!}{3!2!0!^3}\frac{5!}{1!1!3!}<3,2,0,0,0>$
$+\frac{5!}{3!1!^20!^2}\frac{5!}{1!2!2!}<3,1,1,0,0>+\frac{5!}{2!^21!0!^2}\frac{5!}{2!1!2!}<2,2,1,0,0>+\frac{5!}{2!1!^30!}\frac{5!}{1!3!1!}<2,1,1,1,0>+\frac{5!}{1!^5}\frac{5!}{5!}<1,1,1,1,1>$
$=5<5,0,0,0,0>+100<4,1,0,0,0>+200<3,2,0,0,0>+600<3,1,1,0,0>+900<2,2,1,0,0>+1200<2,1,1,1,0>+120<1,1,1,1,1>$

5.3 Polynomial expansion under normalization constraints

5.3.1 Binomial expansion under normalization constraints

Def. 5.3.1. $[a_i,a_j]=0,a_i^2=1,<1,0>=\frac{1}{2}(a_1+a_2)$

Pro. 5.3.1. $(a_1+a_2)^2=2<2,0>+2<1,1>=2<0,0>+2<1,1>=2+2<1,1>$

Cor. 5.3.1. $<1,1>=\frac{1}{2}(a_1+a_2)^2-1$

Pro. 5.3.2. $(a_1+a_2)^3=2<3,0>+6<2,1>=2<1,0>+6<0,1>=8<1,0>$

Cor. 5.3.2. $(a_1+a_2)^3=4(a_1+a_2),2^3=4\cdot2^1,0^3=4\cdot0^1$

Cor. 5.3.3. $[\frac{1}{2}(a_1+a_2)]^3=[\frac{1}{2}(a_1+a_2)]$

Pro. 5.3.3.
$(a_1+a_2)^4=2<4,0>+8<3,1>+6<2,2>=2<0,0>+8<1,1>+6<0,0>=8+8<1,1>$

Cor. 5.3.4. $(a_1+a_2)^4=4(a_1+a_2)^2$

5.3.2 Trinomial expansion under normalization constraints

Def. 5.3.2. $[a_i,a_j]=0,a_i^2=1,<1,0,0>=\frac{1}{3}(a_1+a_2+a_3)$

Cor. 5.3.5. $\frac{3!}{1!2!}<1,0,0>=(a_1+a_2+a_3)$

Pro. 5.3.4. $(a_1+a_2+a_3)^2=3<2,0,0>+6<1,1,0>=3<0,0,0>+6<1,1,0>=3+6<1,1,0>$

Cor. 5.3.6. $\frac{3!}{2!1!}<1,1,0>=\frac{1}{2}(a_1+a_2+a_3)^2-\frac{3}{2}$

Pro. 5.3.5. $(a_1+a_2+a_3)^3=3<3,0,0>+18<2,1,0>+6<1,1,1>$
$=3<1,0,0>+18<0,1,0>+6<1,1,1>=21<1,0,0>+6<1,1,1>$

Cor. 5.3.7. $\frac{3!}{3!}<1,1,1>=\frac{1}{6}(a_1+a_2+a_3)^3-\frac{7}{6}(a_1+a_2+a_3)$

Pro. 5.3.6. $(a_1+a_2+a_3)^4=3<4,0,0>+24<3,1,0>+18<2,2,0,0>+36<2,1,1>$
$=3<0,0,0>+24<1,1,0>+18<0,0,0>+36<0,1,1>=21+60<1,1,0>$

Cor. 5.3.8. $(a_1+a_2+a_3)^4=10(a_1+a_2+a_3)^2-9,3^4=10\cdot3^2-9,1^4=10\cdot1^2-9$

Cor. 5.3.9. $[\frac{1}{2}(a_1+a_2+a_3)]^4=\frac{5}{2}[\frac{1}{2}(a_1+a_2+a_3)]^2-\frac{9}{16}$

5.3.3 Quadrennial expansion under normalization constraints

Def. 5.3.3. $[a_i, a_j] = 0, a_i^2 = 1, < 1,0,0,0 >= \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$

Cor. 5.3.10. $\frac{4!}{1!3!} < 1,0,0,0 >= (a_1 + a_2 + a_3 + a_4)$

Pro. 5.3.7. $(a_1 + a_2 + a_3 + a_4)^2 = 4 < 2,0,0,0 > +12 < 1,1,0,0 >= 4 < 0,0,0,0 > +12 < 1,1,0,0 >= 4 + 12 < 1,1,0,0 >$

Cor. 5.3.11. $\frac{4!}{2!2!} < 1,1,0,0 >= \frac{1}{2}(a_1 + a_2 + a_3 + a_4)^2 - 2$

Pro. 5.3.8. $(a_1 + a_2 + a_3 + a_4)^3 = 4 < 3,0,0,0 > +36 < 2,1,0,0 > +24 < 1,1,1,0 >$
$= 4 < 1,0,0,0 > +36 < 0,1,0,0 > +24 < 1,1,1,0 >= 40 < 1,0,0,0 > +24 < 1,1,1,0 >$

Cor. 5.3.12. $\frac{4!}{3!1!} < 1,1,1,0 >= \frac{1}{6}(a_1 + a_2 + a_3 + a_4)^3 - \frac{5}{3}(a_1 + a_2 + a_3 + a_4)$

Pro. 5.3.9. $(a_1 + a_2 + a_3 + a_4)^4 = 4 < 4,0,0,0 > +48 < 3,1,0,0 > +36 < 2,2,0,0 > +144 < 2,1,1,0 > +24 < 1,1,1,1 >$
$= 4 < 0,0,0,0 > +48 < 1,1,0,0 > +36 < 0,0,0,0 > +144 < 0,1,1,0 > +24 < 1,1,1,1 >= 40 + 192 < 1,1 > +24 < 1,1,1,1 >$

Cor. 5.3.13. $\frac{4!}{4!} < 1,1,1,1 >= \frac{1}{24}(a_1 + a_2 + a_3 + a_4)^4 - \frac{2}{3}(a_1 + a_2 + a_3 + a_4)^2 + 1$

Pro. 5.3.10. $(a_1 + a_2 + a_3 + a_4)^5$
$= 4 < 5,0,0,0 > +60 < 4,1,0,0 > +120 < 3,2,0,0 > +240 < 3,1,1,0 > +360 < 2,2,1,0 > +240 < 2,1,1,1 >= 4 < 1,0,0,0 > +60 < 0,1,0,0 > +120 < 1,0,0,0 > +240 < 1,1,1,0 > +360 < 0,0,1,0 > +240 < 0,1,1,1 >= 544 < 1,0,0,0 > +480 < 1,1,1,0 >$

Pro. 5.3.11. $(a_1 + a_2 + a_3 + a_4)^5 = 20(a_1 + a_2 + a_3 + a_4)^3 - 64(a_1 + a_2 + a_3 + a_4)$
$4^5 = 20 \cdot 4^3 - 64 \cdot 4^1, 2^5 = 20 \cdot 2^3 - 64 \cdot 2^1, 0^5 = 20 \cdot 0^3 - 64 \cdot 0^1$

Cor. 5.3.14. $[\frac{1}{2}(a_1 + a_2 + a_3 + a_4)]^5 = 5[\frac{1}{2}(a_1 + a_2 + a_3 + a_4)]^3 - 4[\frac{1}{2}(a_1 + a_2 + a_3 + a_4)]$

5.3.4 Example of symbol simplification: quinomial expansion

Def. 5.3.4. $[a_i, a_j] = 0, a_i^2 = 1, < 1, 0_4 >= \frac{1}{5}(a_1 + a_2 + a_3 + a_4 + a_5)$

Cor. 5.3.15. $\frac{5!}{1!4!} < 1, 0_4 >= (a_1 + a_2 + a_3 + a_4 + a_5)$

Pro. 5.3.12. $(a_1 + a_2 + a_3 + \cdots + a_5)^2 = \sum \frac{2!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5}$
$= \frac{2!}{2!}\frac{5!}{1!4!} < 2, 0_4 > + \frac{2!}{1!1!}\frac{5!}{2!3!} < 1_2, 0_3 >$
$= 5 < 2, 0_4 > +5(5-1) < 1_2, 0_3 >$
$\stackrel{1}{=} 5 + 2! < 1_2, 0_3 >_+$

Cor. 5.3.16. $< 1_2, 0_3 >_+ = \frac{1}{2!}[< 1, 0_4 >_+^2 -5]$

Pro. 5.3.13. $(a_1 + a_2 + a_3 + \cdots + a_5)^3 = \sum \frac{3!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5}$
$= \frac{3!}{3!}\frac{5!}{1!4!} < 3, 0_4 > + \frac{3!}{2!1!}\frac{5!}{1!1!3!} < 2, 1, 0_3 > + \frac{3!}{1!1!1!}\frac{5!}{3!2!} < 1_3, 0_2 >$
$= 5 < 3, 0_4 > +60 < 2, 1, 0_3 > +60 < 1_3, 0_2 >$
$\stackrel{1}{=} 65 < 1, 0_4 > +60 < 1_3, 0_2 >$
$\stackrel{1}{=} 13 < 1, 0_4 >_+ +3! < 1_3, 0_2 >_+$

Cor. 5.3.17. $< 1_3, 0_2 >_+ = \frac{1}{3!}[< 1, 0_4 >_+^3 -13 < 1, 0_4 >_+]$

Pro. 5.3.14. $(a_1 + a_2 + a_3 + \cdots + a_5)^4 = \sum \frac{4!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5}$
$= \frac{4!}{4!}\frac{5!}{1!4!} < 4, 0_4 > + \frac{4!}{3!1!}\frac{5!}{1!1!3!} < 3, 1, 0_3 > + \frac{4!}{2!2!}\frac{5!}{2!3!} < 2_2, 0_3 >$
$+ \frac{4!}{2!1!1!}\frac{5!}{1!2!2!} < 2, 1_2, 0_2 > + \frac{4!}{1!1!1!1!}\frac{5!}{4!1!} < 1_4, 0 >$
$= 5 < 4, 0_4 > +80 < 3, 1, 0_3 > +60 < 2_2, 0_3 > +360 < 2, 1_2, 0_2 > +120 < 1_4, 0 >$
$\stackrel{1}{=} 65 + 440 < 1_2, 0_3 > +120 < 1_4, 0 >$
$\stackrel{1}{=} 65 + 44 < 1_2, 0_3 >_+ +4! < 1_4, 0 >_+$

Cor. 5.3.18. $< 1_4, 0 >_+ = \frac{1}{4!}[< 1, 0_4 >_+^4 -22 < 1, 0_4 >_+^2 +45]$

Pro. 5.3.15. $(a_1 + a_2 + a_3 + \cdots + a_5)^5 = \sum \frac{5!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5}$
$= \frac{5!}{5!}\frac{5!}{1!4!} < 5, 0_4 > + \frac{5!}{4!1!}\frac{5!}{1!1!3!} < 4, 1, 0_3 > + \frac{5!}{3!2!}\frac{5!}{1!1!3!} < 3, 2, 0_3 >$
$+ \frac{5!}{3!1!1!}\frac{5!}{1!2!2!} < 3, 1_2, 0_2 > + \frac{5!}{2!2!1!}\frac{5!}{2!1!2!} < 2_2, 1, 0_2 >$
$+ \frac{5!}{2!1!1!1!}\frac{5!}{1!3!1!} < 2, 1_3, 0 > + \frac{5!}{1!1!1!1!1!}\frac{5!}{5!} < 1_5 >$

$= 5 < 5, 0_4 > +100 < 4, 1, 0_3 > +200 < 3, 2, 0_3 > +600 < 3, 1_2, 0_2 > +900 < 2_2, 1, 0_2 > +1200 < 2, 1_3, 0 > +120 < 1_5 >$
$\overset{1}{=} 1205 < 1, 0_4 > +1800 < 1_3, 0_2 > +120 < 1_5 >$
$\overset{1}{=} 241 < 1, 0_4 >_+ +180 < 1_3, 0_2 >_+ +5! < 1_5 >_+$

Cor. 5.3.19. $< 1_5 >_+ = \frac{1}{5!}[< 1, 0_4 >_+^5 -30 < 1, 0_4 >_+^3 +149 < 1, 0_4 >_+]$

Pro. 5.3.16. $(a_1 + a_2 + a_3 + \cdots + a_5)^6 = \sum \frac{6!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5}$
$= \frac{6!}{6!}\frac{5!}{1!4!} < 6, 0_4 > + \frac{6!}{5!1!}\frac{5!}{1!1!3!} < 5, 1, 0_3 > + \frac{6!}{4!2!}\frac{5!}{1!1!3!} < 4, 2, 0_3 > + \frac{6!}{4!1!1!}\frac{5!}{1!2!2!} < 4, 1_2, 0_2 >$
$+ \frac{6!}{3!3!}\frac{5!}{2!3!} < 3_2, 0_3 > + \frac{6!}{3!2!1!}\frac{5!}{1!1!1!2!} < 3, 2, 1, 0_2 > + \frac{6!}{3!1!1!1!}\frac{5!}{1!3!1!} < 3, 1_3, 0 >$
$+ \frac{6!}{2!2!2!}\frac{5!}{3!2!} < 2_3, 0_2 > + \frac{6!}{2!2!1!1!}\frac{5!}{2!2!1!} < 2_2, 1_2, 0 > + \frac{6!}{2!1!1!1!1!}\frac{5!}{1!4!0!} < 2, 1_4 >$
$= 5 + 120 < 1_2, 0_3 > +300 + 900 < 1_2, 0_3 >$
$+ 200 < 1_2, 0_3 > +3600 < 1_2, 0_3 > +2400 < 1_4, 0 >$
$+ 900 + 5400 < 1_2, 0_3 > +1800 < 1_4, 0 >$
$= 1205 + 10220 < 1_2, 0_3 > +4200 < 1_4, 0 >$
$= 1205 + 10220\frac{1}{C_5^2} < 1_2, 0_3 >_+ +4200\frac{1}{C_5^4} < 1_4, 0 >_+$
$= 1205 + 1022 < 1_2, 0_3 >_+ +840 < 1_4, 0 >_+$
$= 1205 + 1022\frac{1}{2!}[< 1, 0_4 >_+^2 -5] + 840\frac{1}{4!}[< 1, 0_4 >_+^4 -22 < 1, 0_4 >_+^2 +45]$

Cor. 5.3.20.
$< 1, 0_4 >_+^6 = 35 < 1, 0_4 >_+^4 -259 < 1, 0_4 >_+^2 +225$
$5^6 = 35 \cdot 5^4 - 259 \cdot 5^2 + 225$
$3^6 = 35 \cdot 3^4 - 259 \cdot 3^2 + 225$
$1^6 = 35 \cdot 1^4 - 259 \cdot 1^2 + 225$
$< 1, 0_5 >_+^7 = 56 < 1, 0_5 >_+^5 -784 < 1, 0_5 >_+^3 +2304 < 1, 0_5 >_+^1$

5.3.5 Example: m-term expansion (further simplify symbol)

Pro. 5.3.17. $(a_1 + a_2 + a_3 + \cdots + a_m)^2 = \sum \frac{2!}{i_1!i_2!\cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$
$= \frac{2!}{2!}\frac{m!}{1!(m-1)!} < 2 > + \frac{2!}{1!1!}\frac{m!}{2!(m-2)!} < 1, 1 >$
$= m < 2 > +m(m-1) < 1, 1 >$
$\overset{1}{=} m + m(m-1) < 1, 1 > \overset{1}{=} m + 2! < 1, 1 >_+$

Cor. 5.3.21. $< 1, 1 >_+ = \frac{1}{2!}[\hat{\Omega}^2(m) - m]$

Pro. 5.3.18. $(a_1 + a_2 + a_3 + \cdots + a_m)^3 = \sum \frac{3!}{i_1!i_2!\cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$
$= \frac{3!}{3!}\frac{m!}{1!(m-1)!} < 3 > + \frac{3!}{2!1!}\frac{m!}{1!1!(m-2)!} < 2, 1 > + \frac{3!}{1!1!1!}\frac{m!}{3!(m-3)!} < 1, 1, 1 >$
$= m < 3 > +3m(m-1) < 2, 1 > +m(m-1)(m-2) < 1, 1, 1 >$
$\overset{1}{=} [m + 3m(m-1)] < 1 > +m(m-1)(m-2) < 1, 1, 1 >$
$\overset{1}{=} (3m-2) < 1 >_+ +3! < 1, 1, 1 >_+$

Cor. 5.3.22. $< 1, 1, 1 >_+ = \frac{1}{3!}[\hat{\Omega}^3(m) - (3m-2)\hat{\Omega}(m)]$

Pro. 5.3.19. $(a_1 + a_2 + a_3 + \cdots + a_m)^4 = \sum \frac{4!}{i_1!i_2!\cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$
$= \frac{4!}{4!}\frac{m!}{1!(m-1)!} < 4 > + \frac{4!}{3!1!}\frac{m!}{1!1!(m-2)!} < 3, 1 > + \frac{4!}{2!2!}\frac{m!}{2!(m-2)!} < 2, 2 >$
$+ \frac{4!}{2!1!1!}\frac{m!}{1!2!(m-3)!} < 2, 1, 1 > + \frac{4!}{1!1!1!1!}\frac{m!}{4!(m-4)!} < 1, 1, 1, 1 >$
$= m < 4 > +4m(m-1) < 3, 1 > +3m(m-1) < 2, 2 >$
$+ 6m(m-1)(m-2) < 2, 1, 1 > +m(m-1)(m-2)(m-3) < 1, 1, 1, 1 >$
$\overset{1}{=} [m + 3m(m-1)] + [4m(m-1) + 6m(m-1)(m-2)] < 1, 1 > +m(m-1)(m-2)(m-3) < 1, 1, 1, 1 >$
$\overset{1}{=} m(3m-2) + 4(3m-4) < 1, 1 >_+ +4! < 1, 1, 1, 1 >_+$

Cor. 5.3.23. $< 1, 1, 1, 1 >_+ = \frac{1}{4!}[\hat{\Omega}^4(m) - 2(3m-4)\hat{\Omega}^2(m) + 3m(m-2)]$

Pro. 5.3.20. $(a_1 + a_2 + a_3 + \cdots + a_m)^5 = \sum \frac{5!}{i_1!i_2!\cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$
$= \frac{5!}{5!}\frac{m!}{1!(m-1)!} < 5 > + \frac{5!}{4!1!}\frac{m!}{1!1!(m-2)!} < 4, 1 > + \frac{5!}{3!2!}\frac{m!}{1!1!(m-2)!} < 3, 2 >$
$+ \frac{5!}{3!1!1!}\frac{m!}{1!2!(m-3)!} < 3, 1, 1 > + \frac{5!}{2!2!1!}\frac{m!}{2!1!(m-3)!} < 2, 2, 1 >$
$+ \frac{5!}{2!1!1!1!}\frac{m!}{1!3!(m-4)!} < 2, 1, 1, 1 > + \frac{5!}{1!1!1!1!1!}\frac{m!}{5!(m-5)!} < 1, 1, 1, 1, 1 >$
$= m < 5 > +5m(m-1) < 4, 1 > +10m(m-1) < 3, 2 >$
$+ 10m(m-1)(m-2) < 3, 1, 1 > +15m(m-1)(m-2) < 2, 2, 1 >$
$+ 10m(m-1)(m-2)(m-3) < 2, 1, 1, 1 > +m(m-1)(m-2)(m-3)(m-4) < 1, 1, 1, 1, 1 >$
$\overset{1}{=} m[1 + 15(m-1)^2] < 1 > +10m(m-1)(m-2)^2 < 1, 1, 1 >$

$+ m(m-1)(m-2)(m-3)(m-4) < 1,1,1,1,1 >$
$\stackrel{1}{=} [1+15(m-1)^2] < 1 >_+ +3!10(m-2) < 1,1,1 >_+ +5! < 1,1,1,1,1 >_+$

Cor. 5.3.24. $< 1,1,1,1,1 >_+ = \frac{1}{5!}[\hat{\Omega}^5(m) - 10(m-2)\hat{\Omega}^3(m) + (15m^2 - 50m + 24)\hat{\Omega}(m)]$

5.3.6 Discussion

Cor. 5.3.25.
$$\begin{cases} < 1,0_{m-1} >_+ = \frac{1}{1!} < 1,0_{m-1} >_+ \simeq C_m^1 \\ < 1_2,0_{m-2} >_+ = \frac{1}{2!}[< 1,0_{m-1} >_+^2 -m] \simeq C_m^2 \\ < 1_3,0_{m-3} >_+ = \frac{1}{3!}[< 1,0_{m-1} >_+^3 -(3m-2) < 1,0_{m-1} >_+] \simeq C_m^3 \\ < 1_4,0_{m-4} >_+ = \frac{1}{4!}[< 1,0_{m-1} >_+^4 -2(3m-4) < 1,0_{m-1} >_+^2 +3m(m-2)] \simeq C_m^4 \\ < 1_5,0_{m-5} >_+ = \frac{1}{5!}[< 1,0_{m-1} >_+^5 -10(m-2) < 1,0_{m-1} >_+^3 +(15m^2-50m+24) < 1,0_{m-1} >_+] \simeq C_m^5 \end{cases}$$

Cor. 5.3.26.
$$\begin{cases} < 1,0_{m-1} >_+ = \frac{1}{1!}[C_1^0 < 1,0_{m-1} >_+ -(C_1^2 m - 2C_1^3)] \simeq C_m^1 \\ < 1_2,0_{m-2} >_+ = \frac{1}{2!}[C_2^0 < 1,0_{m-1} >_+^2 -(C_2^2 m - 2C_2^3)] \simeq C_m^2 \\ < 1_3,0_{m-3} >_+ = \frac{1}{3!}[C_3^0 < 1,0_{m-1} >_+^3 -(C_3^2 m - 2C_3^3) < 1,0_{m-1} >_+] \simeq C_m^3 \\ < 1_4,0_{m-4} >_+ = \frac{1}{4!}[C_4^0 < 1,0_{m-1} >_+^4 -(C_4^2 m - 2C_4^3) < 1,0_{m-1} >_+^2 +3m(m-2)] \simeq C_m^4 \\ < 1_5,0_{m-5} >_+ = \frac{1}{5!}[C_5^0 < 1,0_{m-1} >_+^5 -(C_5^2 m - 2C_5^3) < 1,0_{m-1} >_+^3 +(15m^2-50m+24) < 1,0_{m-1} >_+] \simeq C_m^5 \\ \textit{General formula???, This is a difficult problem to overcome in the next step. Let's put it down first, 2022.10.5} \end{cases}$$

Pro. 5.3.21.
$<>_+^1 = 0^1$
$< 1 >_+^2 = 1^2 < 1 >_+$
$< 1,0 >_+^3 = 2^2 < 1,0 >_+^1$
$< 1,0_2 >_+^4 = C_5^2 < 1,0_2 >_+^2 -(1^2 3^2)$
$< 1,0_3 >_+^5 = C_6^3 < 1,0_3 >_+^3 -(2^2 4^2) < 1,0_3 >_+^1$
$< 1,0_4 >_+^6 = C_7^4 < 1,0_4 >_+^4 -259 < 1,0_4 >_+^2 +(1\cdot 3\cdot 5)^2$
$< 1,0_5 >_+^7 = C_8^5 < 1,0_5 >_+^5 -784 < 1,0_5 >_+^3 +(2\cdot 4\cdot 6)^2 < 1,0_5 >_+^1$
$< 1,0_6 >_+^8 = C_9^6 < 1,0_6 >_+^6 -1974 < 1,0_6 >_+^4 +12916 < 1,0_6 >_+^2 -(1\cdot 3\cdot 5\cdot 7)^2$
$< 1,0_7 >_+^9 = C_{10}^7 < 1,0_7 >_+^7 - < 1,0_7 >_+^5 + < 1,0_7 >_+^3 -(2\cdot 4\cdot 6\cdot 8)^2 < 1,0_7 >_+^1$
$1974 = 1^2 3^2 + 3^2 5^2 + 5^2 7^2 + 7^2 1^2 + 1^2 5^2 + 3^2 7^2, 12916 = 3^2 5^2 7^2 + 5^2 7^2 1^2 + 1^2 3^2 5^2 + 1^2 3^2 7^2$

Pro. 5.3.22.
$<>_+^1 = 0^1$
$< 1 >_+^2 = 1^2 < 1 >_+$
$< 1,0 >_+^3 = 2^2 < 1,0 >_+^1$
$< 1,0_2 >_+^4 = (1^2+3^2) < 1,0_2 >_+^2 -(1^2 3^2)$
$< 1,0_3 >_+^5 = (2^2+4^2) < 1,0_3 >_+^3 -(2^2 4^2) < 1,0_3 >_+^1$
$< 1,0_4 >_+^6 = (1^2+3^2+5^2) < 1,0_4 >_+^4 -(1^2 3^2 + 3^2 5^2 + 5^2 1^2) < 1,0_4 >_+^2 +(1^2 3^2 5^2)$
$< 1,0_5 >_+^7 = (2^2+4^2+6^2) < 1,0_5 >_+^5 -(2^2 4^2 + 4^2 6^2 + 6^2 2^2) < 1,0_5 >_+^3 +(2^2 4^2 6^2) < 1,0_5 >_+^1$
$< 1,0_6 >_+^8 = C_{\{1^2,3^2,5^2,7^2\}}^1 < 1,0_6 >_+^6 -C_{\{1^2,3^2,5^2,7^2\}}^2 < 1,0_6 >_+^4 +C_{\{1^2,3^2,5^2,7^2\}}^3 < 1,0_6 >_+^2 -C_{\{1^2,3^2,5^2,7^2\}}^4$
$< 1,0_7 >_+^9 = C_{\{2^2,4^2,6^2,8^2\}}^1 < 1,0_7 >_+^7 -C_{\{2^2,4^2,6^2,8^2\}}^2 < 1,0_7 >_+^5 +C_{\{2^2,4^2,6^2,8^2\}}^3 < 1,0_7 >_+^3 -C_{\{2^2,4^2,6^2,8^2\}}^4 < 1,0_7 >_+^1$
$< 1,0_{m-1} >_+^{m+1} = \sum_{i=1}^{[(m+1)/2]} (-1)^{i-1} C_{\{m^2,(m-2)^2,\cdots,(m\%2)^2\}}^i < 1,0_{m-1} >_+^{m+1-2i}$

Thm. 5.3.1. $\sum_{i=0}^{[(m+1)/2]} (-1)^i C_{\{m^2,(m-2)^2,\cdots,(m\%2)^2\}}^i < 1,0_{m-1} >_+^{m+1-2i} = 0$

5.4 Natural number splitting

Def. 5.4.1. $< n-l,(l) > := < n-l, \geq (l) >$

Cor. 5.4.1. $(n) = \{< n >, < n-1,(1) >, < n-2,(2) >, \cdots, < 2,(n-2) >, < 1,(n-1) >\}$ *Unlimited relevance makes it difficult to*

Cor. 5.4.2.

$$\begin{cases} (1)=\{<1>\} \\ (2)=\{<2>;<1,(1)>\}=\{<2>;<1,1>\} \\ (3)=\{<3>;<2,(1)>;<1,(2)>\}=\{<3>;<2,1>;<1,1,1>\} \\ (4)=\{<4>;<3,(1)>;<2,(2)>;<1,(3)>\}=\{<4>,<3,1>;<2,2>,<2,1,1>;<1,1,1,1>\} \\ (5)=\{<5>;<4,(1)>;<3,(2)>;<2,(3)>;<1,(4)>\} \\ =\{<5>;<4,1>;<3,2>,<3,1,1>;<2,2,1>,<2,1,1,1>;<1,1,1,1,1>\} \\ (6)=\{<6>;<5,(1)>;<4,(2)>;<3,(3)>;<2,(4)>;<1,(5)>\} \\ =\{<6>;<5,1>;<4,2>,<4,1,1>; \\ <3,3>,<3,2,1>,<3,1,1,1>;<2,2,2>,<2,2,1,1>,<2,1,1,1,1>;<1,1,1,1,1,1>\} \\ (7)=\{<7>;<6,(1)>;<5,(2)>;<4,(3)>;<3,(4)>;<2,(5)>;<1,(6)>\} \\ =\{<7>;<6,1>;<5,2>,<5,1,1>;<4,3>,<4,2,1>,<4,1,1,1> \\ ;<3,3,1>,<3,2,2>,<3,2,1,1>,<3,1,1,1,1> \\ ;<2,2,2,1>,<2,2,1,1,1>,<2,1,1,1,1,1>;<1,1,1,1,1,1,1>\} \\ \cdots \end{cases}$$

5.4.1 Conjecture of natural numbers splitting for polynomial theorem

Ass. 5.4.1.
$(\sum_{i=1}^{m} a_i)^n = \sum_{i_1 \cdots + i_m = n} \frac{n!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} = \sum \frac{n!}{(n_1!)^{l_1} \cdots (n_k!)^{l_k}} \frac{m!}{l_1! \cdots l_k! (m-l_1-\cdots-l_k)!} < (n_1; l_1), \cdots, (n_k; l_k) >$

5.5 Polynomial expansion with square zero constraints

5.5.1 Binomial expansion with square zero constraints

Def. 5.5.1. $[a_i, a_j] = 0, a_i^2 = 0, <1,0>= \frac{1}{2}(a_1 + a_2)$

Pro. 5.5.1. $(a_1 + a_2)^2 = 2 <2,0> +2 <1,1>= 2 <1,1>$

Cor. 5.5.1. $<1,1>= \frac{1}{2}(a_1 + a_2)^2$

Pro. 5.5.2. $(a_1 + a_2)^3 = 2 <3,0> +6 <2,1>= 0$

Pro. 5.5.3. $(a_1 + a_2)^4 = 2 <4,0> +8 <3,1> +6 <2,2>= 0$

5.5.2 Trinomial expansion under square zero constraints

Def. 5.5.2. $[a_i, a_j] = 0, a_i^2 = 0, <1,0,0>= \frac{1}{3}(a_1 + a_2 + a_3)$

Cor. 5.5.2. $\frac{3!}{1!2!} <1,0,0>= (a_1 + a_2 + a_3)$

Pro. 5.5.4. $(a_1 + a_2 + a_3)^2 = 3 <2,0,0> +6 <1,1,0>= 6 <1,1,0>$

Cor. 5.5.3. $\frac{3!}{2!1!} <1,1,0>= \frac{1}{2}(a_1 + a_2 + a_3)^2$

Pro. 5.5.5. $(a_1 + a_2 + a_3)^3 = 3 <3,0,0> +18 <2,1,0> +6 <1,1,1>= 6 <1,1,1>$

Cor. 5.5.4. $\frac{3!}{3!} <1,1,1>= \frac{1}{6}(a_1 + a_2 + a_3)^3$

Pro. 5.5.6. $(a_1 + a_2 + a_3)^4 = 3 <4,0,0> +24 <3,1,0> +18 <2,2,0,0> +36 <2,1,1>= 0$

5.5.3 Quadrennial expansion with square zero constraints

Def. 5.5.3. $[a_i, a_j] = 0, a_i^2 = 0, <1,0,0,0>= \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$

Cor. 5.5.5. $\frac{4!}{1!3!} <1,0,0,0>= (a_1 + a_2 + a_3 + a_4)$

Pro. 5.5.7. $(a_1 + a_2 + a_3 + a_4)^2 = 4 <2,0,0,0> +12 <1,1,0,0>= 12 <1,1,0,0>$

Cor. 5.5.6. $\frac{4!}{2!2!} <1,1,0,0>= \frac{1}{2}(a_1 + a_2 + a_3 + a_4)^2$

Pro. 5.5.8. $(a_1 + a_2 + a_3 + a_4)^3 = 4 <3,0,0,0> +36 <2,1,0,0> +24 <1,1,1,0>= 24 <1,1,1,0>$

Cor. 5.5.7. $\frac{4!}{3!1!} <1,1,1,0>= \frac{1}{6}(a_1 + a_2 + a_3 + a_4)^3$

Pro. 5.5.9. $(a_1 + a_2 + a_3 + a_4)^4 = 4 <4,0,0,0> +48 <3,1,0,0> +36 <2,2,0,0> +144 <2,1,1,0> +24 <1,1,1,1>= 24 <1,1,1,1>$

Cor. 5.5.8. $\frac{4!}{4!} <1,1,1,1>= \frac{1}{24}(a_1 + a_2 + a_3 + a_4)^4$

Pro. 5.5.10. $(a_1 + a_2 + a_3 + a_4)^5 = 4 <5,0,0,0> +60 <4,1,0,0> +120 <3,2,0,0> +240 <3,1,1,0> +360 <2,2,1,0> +240 <2,1,1,1>= 0$

5.6 More concrete and direct solution

5.6.1 two-D properties

Pro. 5.6.1. $(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)=\frac{1}{2}[\vec{\vartheta}\cdot(\sigma\otimes I+I\otimes\sigma)]^2-\vec{\vartheta}^2$

Pro. 5.6.2. $(\vec{\vartheta}\cdot\sigma)\otimes(-\vec{\vartheta}^*\cdot\sigma)+(-\vec{\vartheta}^*\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)=[\vec{\vartheta}\cdot\sigma\otimes I+(-\vec{\vartheta}^*)\cdot I\otimes\sigma]^2-\vec{\vartheta}^2-(-\vec{\vartheta}^*)\cdot(-\vec{\vartheta}^*)$

Pro. 5.6.3. $(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes I+(\vec{\vartheta}\cdot\sigma)\otimes I\otimes(\vec{\vartheta}\cdot\sigma)+I\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)$
$=\frac{1}{2}[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I+I\otimes\sigma\otimes I+I\otimes I\otimes\sigma)]^2-\frac{3}{2}\vec{\vartheta}^2$

Pro. 5.6.4. $(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes I\otimes I+(\vec{\vartheta}\cdot\sigma)\otimes I\otimes(\vec{\vartheta}\cdot\sigma)\otimes I+I\otimes(\vec{\vartheta}\cdot\sigma)\otimes I\otimes(\vec{\vartheta}\cdot\sigma)$
$+I\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes I+I\otimes(\vec{\vartheta}\cdot\sigma)\otimes I\otimes(\vec{\vartheta}\cdot\sigma)+I\otimes I\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)$
$=\frac{1}{2}[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I\otimes I+I\otimes\sigma\otimes I\otimes I+I\otimes I\otimes\sigma\otimes I+I\otimes I\otimes I\otimes\sigma]^2-2\vec{\vartheta}^2$

5.6.2 three-D properties

Pro. 5.6.5. $(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)$
$=\frac{1}{6}[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I+I\otimes\sigma\otimes I+I\otimes I\otimes\sigma)]^3-\frac{7}{6}\vec{\vartheta}^2[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I+I\otimes\sigma\otimes I+I\otimes I\otimes\sigma)]$

5.6.3 four-D properties

Pro. 5.6.6. $(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes I+(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes I\otimes(\vec{\vartheta}\cdot\sigma)+(\vec{\vartheta}\cdot\sigma)\otimes I\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)+I\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)$
$=\frac{1}{6}[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I\otimes I+I\otimes\sigma\otimes I\otimes I+I\otimes I\otimes\sigma\otimes I+I\otimes I\otimes I\otimes\sigma)]^3$
$-\frac{5}{3}\vec{\vartheta}^2[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I\otimes I+I\otimes\sigma\otimes I\otimes I+I\otimes I\otimes\sigma\otimes I+I\otimes I\otimes I\otimes\sigma)]$

Pro. 5.6.7. $(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}\cdot\sigma)$
$=\frac{1}{24}[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I\otimes I+I\otimes\sigma\otimes I\otimes I+I\otimes I\otimes\sigma\otimes I+I\otimes I\otimes I\otimes\sigma)]^4$
$-\frac{2}{3}\vec{\vartheta}^2[\vec{\vartheta}\cdot(\sigma\otimes I\otimes I\otimes I+I\otimes\sigma\otimes I\otimes I+I\otimes I\otimes\sigma\otimes I+I\otimes I\otimes I\otimes\sigma)]^2+\vec{\vartheta}^4$

5.7 Lorentz transformation of neutrino spinors

5.7.1 Mathematical preparation

Def. 5.7.1. $cosh\theta:=\frac{e^\theta+e^{-\theta}}{2}\sim cos\theta, sinh\theta:=\frac{e^\theta-e^{-\theta}}{2}\sim isin\theta, tanh\theta=\frac{sinh\theta}{cosh\theta}\sim itan\theta$

Pro. 5.7.1.
$cosh^2-sinh^2\theta=1$
$cosh(-\theta)=cosh\theta, sinh(-\theta)=-sinh\theta$

Pro. 5.7.2.
$cosh(\alpha+\beta)=cosh\alpha cosh\beta+sinh\alpha sinh\beta$
$cosh(\alpha-\beta)=cosh\alpha cosh\beta-sinh\alpha sinh\beta$
$sinh(\alpha+\beta)=sinh\alpha cosh\beta+cosh\alpha sinh\beta$
$sinh(\alpha-\beta)=sinh\alpha cosh\beta-cosh\alpha sinh\beta$

Pro. 5.7.3.
$cosh\alpha+cosh\beta=2cosh\frac{\alpha+\beta}{2}cosh\frac{\alpha-\beta}{2}$
$cosh\alpha-cosh\beta=2sinh\frac{\alpha+\beta}{2}sinh\frac{\alpha-\beta}{2}$
$sinh\alpha+sinh\beta=2sinh\frac{\alpha+\beta}{2}cosh\frac{\alpha-\beta}{2}$
$sinh\alpha-sinh\beta=2cosh\frac{\alpha+\beta}{2}sinh\frac{\alpha-\beta}{2}$

Pro. 5.7.4.
$cosh(2\alpha)=2cosh^2\alpha-1, sinh(2\alpha)=2sinh\alpha cosh\alpha$
$cosh^2\frac{\alpha}{2}=\frac{cosh\alpha+1}{2}, sinh^2\frac{\alpha}{2}=\frac{cosh\alpha-1}{2}$

5.7.2 Lorentz transformation of neutrino spinors

Cor. 5.7.1. $\begin{cases} e^{\vec{\vartheta}\cdot\frac{\sigma}{2}}=cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}+\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\vec{\vartheta}\cdot\sigma, \vec{\vartheta}^2\neq 0 \\ e^{\vec{\vartheta}\cdot\frac{\sigma}{2}}=1+\vec{\vartheta}\cdot\frac{\sigma}{2}, \vec{\vartheta}^2=0, \vec{\vartheta}=i\vec{\omega}+\varsigma\vec{\epsilon} \end{cases}$

Def. 5.7.2. $v:=|\vec{v}|, c:=cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}, s:=\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}, c^2-s^2\vec{\vartheta}^2\equiv 1$

Cor. 5.7.2. $e^{\vec{\vartheta}\cdot\sigma(\frac{1}{2})}=cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}+\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\vec{\vartheta}\cdot\sigma\equiv c+s\vec{\vartheta}\cdot\sigma$

Cor. 5.7.3. $\Lambda_{\varsigma\vec{v}}=e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(\frac{1}{2})}=\frac{1}{\sqrt{2(\gamma_v+1)}}(1+\gamma_v-\varsigma\gamma_v\vec{v}\cdot\sigma), c=\frac{(1+\gamma_v)}{\sqrt{2(\gamma_v+1)}}, s=-\frac{\varsigma\gamma_v}{\sqrt{2(\gamma_v+1)}}$

Cor. 5.7.4. $\Lambda_{\varsigma\vec{v}}=e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(\frac{1}{2})}=\frac{1}{\sqrt{2(\gamma_v+1)}}[1+\gamma_v-2\varsigma\gamma_v v\hat{v}\cdot\sigma(\frac{1}{2})]$

5.7.3 Lorentz transformation of electron spinor

Cor. 5.7.5. $D_{\varsigma\vec{v}} = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)} = \frac{1}{\sqrt{2(\gamma_v+1)}}[1+\gamma_v - i\varsigma\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_4]$

5.8 Polynomial representation of Lorentz transformation for photon spinors

5.8.1 Polynomial representation of general Lorentz transformation for photon spinors

Thm. 5.8.1. $e^{\vec{\vartheta}\cdot\Omega(1)} = 1 + \frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[\vec{\vartheta}\cdot\Omega(1)] + \frac{cosh\sqrt{\vec{\vartheta}^2}-1}{\vec{\vartheta}^2}[\vec{\vartheta}\cdot\Omega(1)]^2$

Proof: $e^{\vec{\vartheta}\cdot\Omega(1)} = (c+s\vec{\vartheta}\cdot\sigma)\otimes(c+s\vec{\vartheta}\cdot\sigma)$
$= c^2 + cs[\vec{\vartheta}\cdot\hat{\Omega}(1)] + s^2[\hat{\Omega}(1,2,\vec{\vartheta}\cdot\sigma)]$
$= c^2 + cs[\vec{\vartheta}\cdot\hat{\Omega}(1)] + s^2\{\frac{1}{2}[\vec{\vartheta}\cdot\hat{\Omega}(1)]^2 - \vec{\vartheta}^2\}$
$= (c^2 - s^2\vec{\vartheta}^2) + cs[\vec{\vartheta}\cdot\hat{\Omega}(1)] + \frac{1}{2}s^2[\vec{\vartheta}\cdot\hat{\Omega}(1)]^2$
$= 1 + 2cs[\vec{\vartheta}\cdot\Omega(1)] + 2s^2[\vec{\vartheta}\cdot\Omega(1)]^2$
$= 1 + \frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[\vec{\vartheta}\cdot\Omega(1)] + \frac{cosh\sqrt{\vec{\vartheta}^2}-1}{\vec{\vartheta}^2}[\vec{\vartheta}\cdot\Omega(1)]^2$ □

Cor. 5.8.1.
$$\begin{cases} e^{\vec{\vartheta}\cdot\Omega(1)} = 1 + \frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[\vec{\vartheta}\cdot\Omega(1)] + \frac{cosh\sqrt{\vec{\vartheta}^2}-1}{\vec{\vartheta}^2}[\vec{\vartheta}\cdot\Omega(1)]^2 \\ e^{\vec{\vartheta}\cdot\sigma(1)} = 1 + \frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[\vec{\vartheta}\cdot\sigma(1)] + \frac{cosh\sqrt{\vec{\vartheta}^2}-1}{\vec{\vartheta}^2}[\vec{\vartheta}\cdot\sigma(1)]^2 \\ e^{\vec{\vartheta}\cdot R} = 1 + \frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}(\vec{\vartheta}\cdot R) + \frac{cosh\sqrt{\vec{\vartheta}^2}-1}{\vec{\vartheta}^2}(\vec{\vartheta}\cdot R)^2 \\ e^{\vec{\vartheta}\cdot\gamma} = 1 + \frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}(\vec{\vartheta}\cdot\gamma) + \frac{cosh\sqrt{\vec{\vartheta}^2}-1}{\vec{\vartheta}^2}(\vec{\vartheta}\cdot\gamma)^2 \end{cases}$$

5.8.2 Polynomial representation of Lorentz boost transformation for photon spinors

Cor. 5.8.2. $\epsilon = ln[\gamma_v(1+v)] \Leftrightarrow sinh\epsilon = \gamma_v v \Leftrightarrow cosh\epsilon = \gamma_v, sinh\epsilon = \gamma_v v$

Cor. 5.8.3.
$$R_{\varsigma\vec{v}} = \begin{cases} e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(1)} = 1 - \varsigma\gamma_v v[\hat{v}\cdot\Omega(1)] + (\gamma_v - 1)[\hat{v}\cdot\Omega(1)]^2 \\ e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(1)} = 1 - \varsigma\gamma_v v[\hat{v}\cdot\sigma(1)] + (\gamma_v - 1)[\hat{v}\cdot\sigma(1)]^2 \\ e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot R} = 1 - \varsigma\gamma_v v(\hat{v}\cdot R) + (\gamma_v - 1)(\hat{v}\cdot R)^2 \\ e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\gamma} = 1 - \varsigma\gamma_v v(\hat{v}\cdot\gamma) + (\gamma_v - 1)(\hat{v}\cdot\gamma)^2 \end{cases}$$

5.9 Polynomial representation of Lorentz transformation for gravitino spinors

5.9.1 Polynomial representation of general Lorentz transformation for gravitino spinors

Thm. 5.9.1. $e^{\vec{\vartheta}\cdot\Omega(\frac{3}{2})} = cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1 - \frac{1}{2}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}) + 2\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1 - \frac{1}{6}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}][\vec{\vartheta}\cdot\Omega(\frac{3}{2})]$
$+ 2cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^2 + \frac{4}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^3$

Proof: $e^{\vec{\vartheta}\cdot\Omega(\frac{3}{2})} = (c+s\vec{\vartheta}\cdot\sigma)\otimes(c+s\vec{\vartheta}\cdot\sigma)\otimes(c+s\vec{\vartheta}\cdot\sigma)$
$= c^3 + c^2s[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})] + cs^2[\hat{\Omega}(\frac{3}{2},2,\vec{\vartheta}\cdot\sigma)] + s^3[\hat{\Omega}(\frac{3}{2},3,\vec{\vartheta}\cdot\sigma)]$
$= c^3 + c^2s[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})] + cs^2\{\frac{1}{2}[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^2 - \frac{3}{2}\vec{\vartheta}^2\} + s^3\{\frac{1}{6}[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^3 - \frac{7}{6}\vec{\vartheta}^2[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]\}$
$= c(c^2 - \frac{3}{2}s^2\vec{\vartheta}^2) + s(c^2 - \frac{7}{6}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(s)] + \frac{1}{2}cs^2[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^2 + \frac{1}{6}cs^3[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^3$
$= c(1 - \frac{1}{2}s^2\vec{\vartheta}^2) + s(1 - \frac{1}{6}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})] + \frac{1}{2}cs^2[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^2 + \frac{1}{6}cs^3[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^3$
$= c(1 - \frac{1}{2}s^2\vec{\vartheta}^2) + 2s(1 - \frac{1}{6}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\Omega(\frac{3}{2})] + 2cs^2[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^2 + \frac{4}{3}cs^3[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^3$
$= cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1 - \frac{1}{2}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}) + 2\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1 - \frac{1}{6}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}][\vec{\vartheta}\cdot\Omega(\frac{3}{2})]$
$+ 2cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^2 + \frac{4}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^3$ □

Cor. 5.9.1. $e^{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]}$
$= cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1 - \frac{1}{2}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}) + 2\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1 - \frac{1}{6}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}]\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]\}$
$+ 2cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]\}^2 + \frac{4}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]\}^3$

Cor. 5.9.2. $e^{\vec{\vartheta}\cdot\sigma(\frac{3}{2})}=cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1-\frac{1}{2}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})+2\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1-\frac{1}{6}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}][\vec{\vartheta}\cdot\sigma(\frac{3}{2})]$
$+2cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\sigma(\frac{3}{2})]^2+\frac{4}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3[\vec{\vartheta}\cdot\sigma(\frac{3}{2})]^3$

5.9.2 Polynomial representation of Lorentz boost transformation for gravitino spinors

Cor. 5.9.3. $e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(\frac{3}{2})}=\frac{(\gamma_v+1)}{\sqrt{2(\gamma_v+1)}}(1-\frac{\gamma_v-1}{4})-\frac{2\varsigma\gamma_v v}{\sqrt{2(\gamma_v+1)}}(1-\frac{\gamma_v-1}{12})[\hat{v}\cdot\Omega(\frac{3}{2})]$
$+\frac{\gamma_v^2-1}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^2-\frac{1}{3}\frac{2\varsigma\gamma_v v(\gamma_v-1)}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^3$

Cor. 5.9.4.

$$\Lambda_{\varsigma\vec{v}}(\frac{3}{2})=\begin{cases}e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(\frac{3}{2})}=\frac{1}{\sqrt{2(\gamma_v+1)}}[1+\gamma_v-2\varsigma\gamma_v\vec{v}\cdot\Omega(\frac{3}{2})]+\frac{\gamma_v-1}{\sqrt{2(\gamma_v+1)}}[1+\gamma_v-\frac{2}{3}\varsigma\gamma_v\vec{v}\cdot\Omega(\frac{3}{2})]\{[\hat{v}\cdot\Omega(\frac{3}{2})]^2-\frac{1}{4}\}\\ e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(\frac{3}{2})}=\frac{1}{\sqrt{2(\gamma_v+1)}}[1+\gamma_v-2\varsigma\gamma_v\vec{v}\cdot\sigma(\frac{3}{2})]+\frac{\gamma_v-1}{\sqrt{2(\gamma_v+1)}}[1+\gamma_v-\frac{2}{3}\varsigma\gamma_v\vec{v}\cdot\sigma(\frac{3}{2})]\{[\hat{v}\cdot\sigma(\frac{3}{2})]^2-\frac{1}{4}\}\end{cases}$$

5.10 Polynomial representation of Lorentz transformation for graviton spinors

5.10.1 Polynomial representation of general Lorentz transformation for graviton spinors

Thm. 5.10.1. $e^{\vec{\vartheta}\cdot\Omega(2)}=1+(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(1-\frac{2}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\Omega(2)]+2(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2(1-\frac{1}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\Omega(2)]^2$
$+\frac{2}{3}(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\Omega(2)]^3+\frac{2}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^4[\vec{\vartheta}\cdot\Omega(2)]^4$

Proof: $e^{\vec{\vartheta}\cdot\Omega(2)}=(c+s\vec{\vartheta}\cdot\sigma)\otimes(c+s\vec{\vartheta}\cdot\sigma)\otimes(c+s\vec{\vartheta}\cdot\sigma)\otimes(c+s\vec{\vartheta}\cdot\sigma)$
$=c^4+c^3s[\vec{\vartheta}\cdot\hat{\Omega}(2)]+c^2s^2[\hat{\Omega}(2,2,\vec{\vartheta}\cdot\sigma)]+cs^3[\hat{\Omega}(2,3,\vec{\vartheta}\cdot\sigma)]+s^4[\hat{\Omega}(2,4,\vec{\vartheta}\cdot\sigma)]$
$=c^4+c^3s[\vec{\vartheta}\cdot\hat{\Omega}(2)]+c^2s^2\{\frac{1}{2}[\vec{\vartheta}\cdot\hat{\Omega}(2)]^2-2\vec{\vartheta}^2\}$
$+cs^3\{\frac{1}{6}[\vec{\vartheta}\cdot\hat{\Omega}(2)]^3-\frac{5}{3}\vec{\vartheta}^2[\vec{\vartheta}\cdot\hat{\Omega}(2)]\}+s^4\{\frac{1}{24}[\vec{\vartheta}\cdot\hat{\Omega}(2)]^4-\frac{2}{3}\vec{\vartheta}^2[\vec{\vartheta}\cdot\hat{\Omega}(2)]^2+\vec{\vartheta}^4\}$
$=(c^2-s^2\vec{\vartheta}^2)^2+cs(c^2-\frac{5}{3}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(2)]+\frac{1}{2}s^2(c^2-\frac{4}{3}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(2)]^2+\frac{1}{6}cs^3[\vec{\vartheta}\cdot\hat{\Omega}(2)]^3+\frac{1}{24}s^4[\vec{\vartheta}\cdot\hat{\Omega}(2)]^4$
$=1+cs(1-\frac{2}{3}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(2)]+\frac{1}{2}s^2(1-\frac{1}{3}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(2)]^2+\frac{1}{6}cs^3[\vec{\vartheta}\cdot\hat{\Omega}(2)]^3+\frac{1}{24}s^4[\vec{\vartheta}\cdot\hat{\Omega}(2)]^4$
$=1+2cs(1-\frac{2}{3}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\Omega(2)]+2s^2(1-\frac{1}{3}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\Omega(2)]^2+\frac{4}{3}cs^3[\vec{\vartheta}\cdot\Omega(2)]^3+\frac{2}{3}s^4[\vec{\vartheta}\cdot\Omega(2)]^4$
$=1+(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(1-\frac{2}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\Omega(2)]+2(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2(1-\frac{1}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\Omega(2)]^2$
$+\frac{2}{3}(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\Omega(2)]^3+\frac{2}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^4[\vec{\vartheta}\cdot\Omega(2)]^4$ □

Cor. 5.10.1. $e^{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_4+I\otimes\sigma(\frac{3}{2})]}=1+(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(1-\frac{2}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_4+I\otimes\sigma(\frac{3}{2})]\}$
$+2(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2(1-\frac{1}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_4+I\otimes\sigma(\frac{3}{2})]\}^2$
$+\frac{2}{3}(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_4+I\otimes\sigma(\frac{3}{2})]\}^3+\frac{2}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^4\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_4+I\otimes\sigma(\frac{3}{2})]\}^4$

Cor. 5.10.2. $e^{\vec{\vartheta}\cdot\sigma(2)}=1+(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(1-\frac{2}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\sigma(2)]+2(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2(1-\frac{1}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\sigma(2)]^2$
$+\frac{2}{3}(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\sigma(2)]^3+\frac{2}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^4[\vec{\vartheta}\cdot\sigma(2)]^4$

Cor. 5.10.3. $e^{\vec{\vartheta}\cdot G_m}=1+(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(1-\frac{2}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot G_m]+2(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2(1-\frac{1}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot G_m]^2$
$+\frac{2}{3}(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot G_m]^3+\frac{2}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^4[\vec{\vartheta}\cdot G_m]^4$

5.10.2 Polynomial representation of Lorentz boost transformation for graviton spinors

Cor. 5.10.4. $e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(2)}=1-\varsigma\gamma_v(1-\frac{\gamma_v-1}{3})[\vec{v}\cdot\Omega(2)]+\frac{\gamma_v-1}{v^2}(1-\frac{\gamma_v-1}{6})[\vec{v}\cdot\Omega(2)]^2$
$-\frac{1}{3}\frac{\varsigma\gamma_v(\gamma_v-1)}{v^2}[\vec{v}\cdot\Omega(2)]^3+\frac{1}{6}\frac{(\gamma_v-1)^2}{v^4}[\vec{v}\cdot\Omega(2)]^4$

Cor. 5.10.5.

$$\Lambda_{\varsigma\vec{v}}(2)=\begin{cases}e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(2)}=1-\varsigma\gamma_v v[\hat{v}\cdot\Omega(2)]+(\gamma_v-1)[\hat{v}\cdot\Omega(2)]^2\\ \qquad+\frac{1}{3}(\gamma_v-1)\{-\varsigma\gamma_v v[\hat{v}\cdot\Omega(2)]+\frac{1}{2}(\gamma_v-1)[\hat{v}\cdot\Omega(2)]^2\}\{[\hat{v}\cdot\Omega(2)]^2-1\}\\ e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(2)}=1-\varsigma\gamma_v v[\hat{v}\cdot\sigma(2)]+(\gamma_v-1)[\hat{v}\cdot\sigma(2)]^2\\ \qquad+\frac{1}{3}(\gamma_v-1)\{-\varsigma\gamma_v v[\hat{v}\cdot\sigma(2)]+\frac{1}{2}(\gamma_v-1)[\hat{v}\cdot\sigma(2)]^2\}\{[\hat{v}\cdot\sigma(2)]^2-1\}\end{cases}$$

Cor. 5.10.6. $R_{\varsigma\vec{v}}(2) = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot G_m} = 1 - \varsigma\gamma_v v[\hat{v}\cdot G_m] + (\gamma_v - 1)[\hat{v}\cdot G_m]^2 + \frac{1}{3}(\gamma_v - 1)\{-\varsigma\gamma_v v[\hat{v}\cdot G_m] + \frac{1}{2}(\gamma_v - 1)[\hat{v}\cdot G_m]^2\}\{[\hat{v}\cdot G_m]^2 - 1\}$

5.11 Unified polynomial representation of s-spinor Lorentz transformation

Cor. 5.11.1. $e^{\vec{\vartheta}\cdot\Omega(s)} = \overbrace{e^{\vec{\vartheta}\cdot\sigma(\frac{1}{2})}\otimes\cdots\otimes e^{\vec{\vartheta}\cdot\sigma(\frac{1}{2})}}^{2s}$

Cor. 5.11.2.
$$\begin{cases} e^{\vec{\vartheta}\cdot\sigma(s)} = \bar{\Gamma}(s)e^{\vec{\vartheta}\cdot\Omega(s)}\Gamma(s) \\ e^{\vec{\vartheta}\cdot\sigma(s-1)} = \bar{X}(s)[I\otimes\bar{\Gamma}(s-\frac{1}{2})]e^{\vec{\vartheta}\cdot\Omega(s)}[I\otimes\Gamma(s-\frac{1}{2})]X(s) \\ e^{\vec{\vartheta}\cdot[\sigma\frac{1}{2}\otimes I_{2s}+I\otimes\sigma(s-\frac{1}{2})]} = [I\otimes\bar{\Gamma}(s-\frac{1}{2})]e^{\vec{\vartheta}\cdot\Omega(s)}[I\otimes\Gamma(s-\frac{1}{2})] \\ e^{\vec{\vartheta}\cdot\Omega(s-1)} = I_{4^{s-1}}\otimes\{\bar{X}(1)[I\otimes\bar{\Gamma}(\frac{1}{2})]\}e^{\vec{\vartheta}\cdot\Omega(s)}I_{4^{s-1}}\otimes\{[I\otimes\Gamma(\frac{1}{2})]X(1)\} \\ e^{\vec{\vartheta}\cdot\pi(s,k)} = I_{4^k}\otimes\{\bar{X}(s-k)[I\otimes\bar{\Gamma}(s-k-\frac{1}{2})]\}e^{\vec{\vartheta}\cdot\Omega(s)}I_{4^k}\otimes\{[I\otimes\Gamma(s-k-\frac{1}{2})]X(s-k)\} \end{cases}$$

Cor. 5.11.3.
$$\begin{cases} [\vec{\vartheta}\cdot\Omega(s)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\Omega(s)]^{2s+1-2k}, [\vec{\vartheta}\cdot\sigma(s)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\sigma(s)]^{2s+1-2k} \\ [\vec{\vartheta}\cdot\Omega(s-1)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\Omega(s-1)]^{2s+1-2k}, [\vec{\vartheta}\cdot\sigma(s-1)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\sigma(s-1)]^{2s+1-2k} \\ [\vec{\vartheta}\cdot\Omega(s-2)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\Omega(s-2)]^{2s+1-2k}, [\vec{\vartheta}\cdot\sigma(s-2)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\sigma(s-2)]^{2s+1-2k} \\ \cdots \\ [\vec{\vartheta}\cdot\Omega(\frac{1}{2}|0)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\Omega(\frac{1}{2}|0)]^{2s+1-2k}, [\vec{\vartheta}\cdot\sigma(\frac{1}{2}|0)]^{2s+1} = \sum\limits_{k=1}^{[s+1/2]} X_k(s)[\vec{\vartheta}\cdot\sigma(\frac{1}{2}|0)]^{2s+1-2k} \end{cases}$$

Cor. 5.11.4.
$$\begin{cases} e^{\vec{\vartheta}\cdot\Omega(s)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\Omega(s)]^k, e^{\vec{\vartheta}\cdot\sigma(s)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\sigma(s)]^k \\ e^{\vec{\vartheta}\cdot\Omega(s-1)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\Omega(s-1)]^k, e^{\vec{\vartheta}\cdot\sigma(s-1)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\sigma(s-1)]^k \\ e^{\vec{\vartheta}\cdot\Omega(s-2)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\Omega(s-2)]^k, e^{\vec{\vartheta}\cdot\sigma(s-2)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\sigma(s-2)]^k \\ \cdots \\ e^{\vec{\vartheta}\cdot\Omega(\frac{1}{2}|0)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\Omega(\frac{1}{2}|0)]^k, e^{\vec{\vartheta}\cdot\sigma(\frac{1}{2}|0)} = \sum\limits_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\sigma(\frac{1}{2}|0)]^k \end{cases}$$

That is, roughly speaking, the following statement is essentially the meaning of the above inference, and the conclusion in this section is strictly proven.

Cor. 5.11.5.
$$\begin{cases} e^{\vec{\vartheta}\cdot\Omega(n)}, e^{\vec{\vartheta}\cdot\Omega(n-1)}, \cdots, e^{\vec{\vartheta}\cdot\Omega(1)}, e^{\vec{\vartheta}\cdot\Omega(0)} \\ e^{\vec{\vartheta}\cdot\sigma(n)}, e^{\vec{\vartheta}\cdot\sigma(n-1)}, \cdots, e^{\vec{\vartheta}\cdot\sigma(1)}, e^{\vec{\vartheta}\cdot\sigma(0)} \end{cases} \quad expansion\ coefficient = e^{\vec{\vartheta}\cdot\Omega(n)} expansion\ coefficient$$
$$\begin{cases} e^{\vec{\vartheta}\cdot\Omega(n+\frac{1}{2})}, e^{\vec{\vartheta}\cdot\Omega(n-\frac{1}{2})}, \cdots, e^{\vec{\vartheta}\cdot\Omega(\frac{3}{2})}, e^{\vec{\vartheta}\cdot\Omega(\frac{1}{2})} \\ e^{\vec{\vartheta}\cdot\sigma(n+\frac{1}{2})}, e^{\vec{\vartheta}\cdot\sigma(n-\frac{1}{2})}, \cdots, e^{\vec{\vartheta}\cdot\sigma(\frac{3}{2})}, e^{\vec{\vartheta}\cdot\sigma(\frac{1}{2})} \end{cases} \quad expansion\ coefficient = e^{\vec{\vartheta}\cdot\Omega(n+\frac{1}{2})} expansion\ coefficient$$

Cor. 5.11.6. $e^{i2\pi\hat{\omega}\cdot\sigma(s)} = (-1)^{2s}$

Cor. 5.11.7. $\vec{\vartheta}^2 = 0$
$$\Rightarrow [\vec{\vartheta}\cdot\Omega(s)]^{2s+1} = 0, e^{\vec{\vartheta}\cdot\Omega(s)} = \sum_{n=0}^{2s}\frac{1}{n!}[\vec{\vartheta}\cdot\Omega(s)]^n \Rightarrow [\vec{\vartheta}\cdot\sigma(s)]^{2s+1} = 0, e^{\vec{\vartheta}\cdot\sigma(s)} = \sum_{n=0}^{2s}\frac{1}{n!}[\vec{\vartheta}\cdot\sigma(s)]^n$$

Ass. 5.11.1. $e^{\vec{\vartheta}\cdot\sigma(s)}|_{\vec{\vartheta}^2=0} =< e^{\vec{\vartheta}\cdot\sigma(s)} >_{\vec{\vartheta}^2\to 0}$???

5.12 Polynomial representation of Lorentz boost transformation for s-spinor???

Cor. 5.12.1. $R_{\varsigma\vec{v}}(n) = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(n)} = \sum\limits_{k=0}^{2n} f_k(v)[\hat{v}\cdot\sigma(n)]^k$
$$\Rightarrow \begin{cases} R_{\varsigma\vec{v}}(l) = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(l)} = \sum\limits_{k=0}^{2n} f_k(v)[\hat{v}\cdot\sigma(l)]^k, f_0(v) = 1, 0\le l\le n \\ R_{\varsigma\vec{v}}(l+\frac{1}{2}) = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(l)} = \sum\limits_{k=0}^{2n} f_k(\frac{v}{2})[2\hat{v}\cdot\sigma(l+\frac{1}{2})]^k = \sum\limits_{k=0}^{2n} 2^k f_k(\frac{v}{2})[\hat{v}\cdot\sigma(l+\frac{1}{2})]^k, 0\le l+\frac{1}{2}\le n \end{cases}$$

Cor. 5.12.2. $R_{\varsigma\vec{v}}(n+\frac{1}{2}) = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(n+\frac{1}{2})} = \sum_{k=0}^{2n+1} g_k(v)[\hat{v}\cdot\sigma(n+\frac{1}{2})]^k$

$$\Rightarrow \begin{cases} R_{\varsigma\vec{v}}(l) = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(l)} = \sum_{k=0}^{2n+1} g_k(2v)[\frac{1}{2}\hat{v}\cdot\sigma(l)]^k = \sum_{k=0}^{2n+1} 2^{-k} g_k(2v)[\hat{v}\cdot\sigma(l)]^k, f_0(v)=1, 0\le l\le n+\frac{1}{2} \\ R_{\varsigma\vec{v}}(l+\frac{1}{2}) = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(l+\frac{1}{2})} = \sum_{k=0}^{2n+1} g_k(v)[\hat{v}\cdot\sigma(l+\frac{1}{2})]^k, 0\le l+\frac{1}{2}\le n+\frac{1}{2} \end{cases}$$

5.13 Polynomial representation of Lorentz transformation for vectors

Cor. 5.13.1. $\Lambda(1,\epsilon) = (c+s\epsilon\cdot\sigma)\otimes(c-s\epsilon\cdot\sigma)$
$= c^2 + cs[\epsilon\cdot(\sigma\otimes I - I\otimes\sigma)] - s^2[\epsilon\cdot()_2]$
$= c^2 + cs[\epsilon\cdot(\sigma\otimes I - I\otimes\sigma)] - s^2\{\frac{1}{2}[\epsilon\cdot\hat{\Omega}(1)]^2 - \epsilon\cdot\epsilon\}$
$= (c^2 + s^2\epsilon\cdot\epsilon) + cs[\epsilon\cdot(\sigma\otimes I - I\otimes\sigma)] - \frac{1}{2}s^2[\epsilon\cdot\hat{\Omega}(1)]^2$
$= (c^2 + s^2\epsilon\cdot\epsilon) + 2cs[\epsilon\cdot\frac{1}{2}(\sigma\otimes I - I\otimes\sigma)] - 2s^2[\epsilon\cdot\Omega(1)]^2$
$= cosh\sqrt{\epsilon\cdot\epsilon} + \frac{sinh\sqrt{\epsilon\cdot\epsilon}}{\sqrt{\epsilon\cdot\epsilon}}[\epsilon\cdot\frac{1}{2}(\sigma\otimes I - I\otimes\sigma)] - \frac{cosh\sqrt{\epsilon\cdot\epsilon}-1}{\epsilon\cdot\epsilon}[\epsilon\cdot\Omega(1)]^2$

Cor. 5.13.2. $L(\epsilon) = e^{\epsilon\cdot L} = cosh\sqrt{\epsilon\cdot\epsilon} + \frac{sinh\sqrt{\epsilon\cdot\epsilon}}{\sqrt{\epsilon\cdot\epsilon}}\epsilon\cdot L - \frac{cosh\sqrt{\epsilon\cdot\epsilon}-1}{\epsilon\cdot\epsilon}(\epsilon\cdot R)^2$

Cor. 5.13.3. $\Lambda(1,\vec{\vartheta}) = (c+s\vec{\vartheta}\cdot\sigma)\otimes(c^* - s^*\vec{\vartheta}^*\cdot\sigma)$
$= cc^* + c^*s\vec{\vartheta}\cdot\sigma\otimes I - cs^*\vec{\vartheta}^*\cdot I\otimes\sigma - ss^*[(\vec{\vartheta}\cdot\sigma)\otimes(\vec{\vartheta}^*\cdot\sigma)], \vec{\vartheta} = \epsilon + i\omega$

Cor. 5.13.4. $L(1,\vec{\vartheta}) = e^{(i\omega\cdot R + \epsilon\cdot L)} = (c+s\vec{\vartheta}\cdot\sigma_+)(c^* - s^*\vec{\vartheta}^*\cdot\sigma_-)$
$= cc^* + c^*s(\vec{\vartheta}\cdot\sigma_+) - cs^*(\vec{\vartheta}^*\cdot\sigma_-) - ss^*(\vec{\vartheta}\cdot\sigma_+)(\vec{\vartheta}^*\cdot\sigma_-)$
$= cc^* + (c^*s\vec{\vartheta} - cs^*\vec{\vartheta}^*)\cdot R + (c^*s\vec{\vartheta} + cs^*\vec{\vartheta}^*)\cdot L - ss^*[\vec{\vartheta}\cdot(R+L)][\vec{\vartheta}^*\cdot(R-L)]$

Cor. 5.13.5. $\Lambda(1,i\omega) = (c+is\omega\cdot\sigma)\otimes(c+is\omega\cdot\sigma)$
$= c^2 + cs[i\omega\cdot(\sigma\otimes I + I\otimes\sigma)] + s^2[i\omega\cdot()_2]$
$= c^2 + cs[i\omega\cdot(\sigma\otimes I + I\otimes\sigma)] + s^2\{\frac{1}{2}[i\omega\cdot\hat{\Omega}(1)]^2 - i\omega\cdot i\omega\}$
$= c^2 + cs[i\omega\cdot(\sigma\otimes I + I\otimes\sigma)] + s^2\{\frac{1}{2}[i\omega\cdot\hat{\Omega}(1)]^2 - i\omega\cdot i\omega\}$
$= 1 + \frac{sinh\sqrt{i\omega\cdot i\omega}}{\sqrt{i\omega\cdot i\omega}}[i\omega\cdot\frac{1}{2}(\sigma\otimes I + I\otimes\sigma)] + \frac{cosh\sqrt{i\omega\cdot i\omega}-1}{i\omega\cdot i\omega}[i\omega\cdot\Omega(1)]^2$
$= 1 + i\frac{sinh\sqrt{\omega\cdot\omega}}{\sqrt{\omega\cdot\omega}}[\omega\cdot\frac{1}{2}(\sigma\otimes I + I\otimes\sigma)] + \frac{cos\sqrt{\omega\cdot\omega}-1}{\omega\cdot\omega}[\omega\cdot\Omega(1)]^2$

Cor. 5.13.6. $R(i\omega) = L(i\omega) = e^{i\omega\cdot R} = 1 + i\frac{sin\sqrt{\omega\cdot\omega}}{\sqrt{\omega\cdot\omega}}(\omega\cdot R) + \frac{cos\sqrt{\omega\cdot\omega}-1}{\omega\cdot\omega}(\omega\cdot R)^2$

Cor. 5.13.7. $R_3(i\omega) = L_3(i\omega) = e^{i\omega\cdot\gamma} = 1 + i\frac{sin\sqrt{\omega\cdot\omega}}{\sqrt{\omega\cdot\omega}}(\omega\cdot\gamma) + \frac{cos\sqrt{\omega\cdot\omega}-1}{\omega\cdot\omega}(\omega\cdot\gamma)^2$

5.14 Electromagnetic field of arbitrary moving charge [22]

5.14.1 Spatial coordinates and delay potential

Cor. 5.14.1. $\vec{r'} = \vec{r} + \gamma_v\vec{v}r + (\gamma_v - 1)(\vec{v}\cdot\vec{r})\vec{v}/v^2 \Rightarrow r' = \gamma_v(r + \vec{v}\cdot\vec{r})$

Cor. 5.14.2. $\vec{r'} = \vec{r} + \gamma_v\vec{v}r + (\gamma_v - 1)(\vec{v}\cdot\vec{r})\vec{v}/v^2 \Rightarrow \hat{r}' = [\hat{r} + \gamma_v\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\hat{r})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\hat{r})]$

Cor. 5.14.3. $\vec{r'} = \vec{r} + \gamma_v\vec{v}r + (\gamma_v - 1)(\vec{v}\cdot\vec{r})\vec{v}/v^2 \Leftrightarrow \begin{cases} \vec{r'} = \vec{r} + \gamma_v\vec{v}r + (\gamma_v - 1)(\vec{v}\cdot\vec{r})\vec{v}/v^2 \\ r' = \gamma_v(r+\vec{v}\cdot\vec{r}), \vec{r'}^2 - r'^2 = \vec{r}^2 - r^2 = 0 \end{cases}$

Cor. 5.14.4. $\begin{cases} \vec{r} = \vec{r}_0 + \gamma_v\vec{v}r_0 + (\gamma_v - 1)(\vec{v}\cdot\vec{r}_0)\vec{v}/v^2 \\ r = \gamma_v(r_0 + \vec{v}\cdot\vec{r}_0) \end{cases}$ $\begin{cases} \vec{A} = \vec{A}_0 + \gamma_v\vec{v}\phi_0 + (\gamma_v - 1)(\vec{v}\cdot\vec{A}_0)\vec{v}/v^2 = \frac{e\gamma_v\vec{v}}{4\pi\varepsilon_0 r_0} = \frac{e\vec{v}}{4\pi\varepsilon_0(r-\vec{v}\cdot\vec{r})} \\ \phi = \gamma_v(\phi_0 + \vec{v}\cdot\vec{A}_0) = \frac{e\gamma_v}{4\pi\varepsilon_0 r_0} = \frac{e}{4\pi\varepsilon_0(r-\vec{v}\cdot\vec{r})} \end{cases}$

5.14.2 Partial derivative analysis

Cor. 5.14.5. $t = t' + R(t'), R(t') = \sqrt{[x'(t')-x]^2 + [y'(t')-y]^2 + [z'(t')-z]^2}$

Cor. 5.14.6. ∂_x *means: t, y, z are fixed and unchanging, x changes, x and t' satisfy relation $t = t' + R(t')$*

$$\Rightarrow \begin{cases} \partial_x = \frac{\partial t'}{\partial_x}\partial_{t'} \\ 0 = \frac{\partial t'}{\partial_x} + \frac{\partial R(t')}{\partial_{t'}}\frac{\partial t'}{\partial_x} \end{cases} \Rightarrow \begin{cases} \frac{\partial R(t')}{\partial_{t'}} = -\frac{\vec{v}(t')\cdot\vec{R}(t') - \frac{\partial x}{\partial_{t'}}R_x(t')}{R(t')} = -1 \\ \frac{\partial t'}{\partial_x} = -\frac{\hat{R}_x(t')}{1-\vec{v}(t')\cdot\hat{R}(t')}, \partial_x = -\frac{\hat{R}_x(t')}{1-\vec{v}(t')\cdot\hat{R}(t')}\partial_{t'}|_x \end{cases}$$

$$\Rightarrow \begin{cases} \vec{v}(t')\cdot\partial_{t'}\vec{R}(t') = -\frac{1-\vec{v}(t')\cdot\hat{R}(t')}{\hat{R}_x(t')}v_x - v^2(t') \\ \vec{v}(t')\cdot\partial_x\vec{R}(t') = v_x + \frac{v^2(t')\hat{R}_x(t')}{1-\vec{v}(t')\cdot\hat{R}(t')} \end{cases}$$

Cor. 5.14.7. ∂_y *means: t, z, x are fixed and unchanging, y changes, y and t' satisfy relation $t = t' + R(t')$*

$$\Rightarrow \begin{cases} \partial_y = \frac{\partial t'}{\partial_y}\partial_{t'} \\ 0 = \frac{\partial t'}{\partial_y} + \frac{\partial R(t')}{\partial_{t'}}\frac{\partial t'}{\partial_y} \end{cases} \Rightarrow \begin{cases} \frac{\partial R(t')}{\partial_{t'}} = -\frac{\vec{v}(t')\cdot\vec{R}(t') - \frac{\partial y}{\partial_{t'}}R_y(t')}{R(t')} = -1 \\ \frac{\partial t'}{\partial_y} = -\frac{\hat{R}_y(t')}{1-\vec{v}(t')\cdot\hat{R}(t')}, \partial_y = -\frac{\hat{R}_y(t')}{1-\vec{v}(t')\cdot\hat{R}(t')}\partial_{t'}|_y \end{cases}$$

$$\Rightarrow \begin{cases} \vec{v}(t')\cdot\partial_{t'}\vec{R}(t') = -\frac{1-\vec{v}(t')\cdot\hat{R}(t')}{\hat{R}_y(t')}v_y - v^2(t') \\ \vec{v}(t')\cdot\partial_y\vec{R}(t') = v_y + \frac{v^2(t')\hat{R}_y(t')}{1-\vec{v}(t')\cdot\hat{R}(t')} \end{cases}$$

Cor. 5.14.8. ∂_z *means: t, x, y are fixed and unchanging, z changes, z and t' satisfy relation $t = t' + R(t')$*

$$\Rightarrow \begin{cases} \partial_z = \frac{\partial t'}{\partial_z}\partial_{t'} \\ 0 = \frac{\partial t'}{\partial_z} + \frac{\partial R(t')}{\partial_{t'}}\frac{\partial t'}{\partial_z} \end{cases} \Rightarrow \begin{cases} \frac{\partial R(t')}{\partial_{t'}} = -\frac{\vec{v}(t')\cdot\vec{R}(t') - \frac{\partial z}{\partial_{t'}}R_z(t')}{R(t')} = -1 \\ \frac{\partial t'}{\partial_z} = -\frac{\hat{R}_z(t')}{1-\vec{v}(t')\cdot\hat{R}(t')}, \partial_z = -\frac{\hat{R}_z(t')}{1-\vec{v}(t')\cdot\hat{R}(t')}\partial_{t'}|_z \end{cases}$$

$$\Rightarrow \begin{cases} \vec{v}(t')\cdot\partial_{t'}\vec{R}(t') = -\frac{1-\vec{v}(t')\cdot\hat{R}(t')}{\hat{R}_z(t')}v_z - v^2(t') \\ \vec{v}(t')\cdot\partial_z\vec{R}(t') = v_z + \frac{v^2(t')\hat{R}_z(t')}{1-\vec{v}(t')\cdot\hat{R}(t')} \end{cases}$$

Cor. 5.14.9. ∂_t *means: x, y, z are fixed and unchanging, t changes, t and t' satisfy relation $t = t' + R(t')$*

$$\Rightarrow \begin{cases} \partial_t = \frac{\partial t'}{\partial_t}\partial_{t'} \\ 1 = \frac{\partial t'}{\partial_t} + \frac{\partial R(t')}{\partial_{t'}}\frac{\partial t'}{\partial_t} \end{cases} \Rightarrow \begin{cases} \frac{\partial t'}{\partial_t} = \frac{1}{1-\vec{v}(t')\cdot\hat{R}(t')}, \partial_t = \frac{1}{1-\vec{v}(t')\cdot\hat{R}(t')}\partial_{t'}|_t \\ \frac{\partial R(t')}{\partial_{t'}} = -\frac{\vec{v}(t')\cdot\vec{R}(t')}{R(t')} \end{cases}$$

$$\Rightarrow \begin{cases} \vec{v}(t')\cdot\partial_{t'}\vec{R}(t') = -v^2(t') \\ \vec{v}(t')\cdot\partial_t\vec{R}(t') = -\frac{v^2(t')}{1-\vec{v}(t')\cdot\hat{R}(t')} \end{cases}$$

Cor. 5.14.10. $\begin{cases} \vec{A}(t,\vec{r}) = \frac{e\vec{v}(t')}{4\pi\varepsilon_0[R(t')-\vec{v}\cdot\vec{R}(t')]} \\ \phi(t,\vec{r}) = \frac{e}{4\pi\varepsilon_0[R(t')-\vec{v}(t')\cdot\vec{R}(t')]} \end{cases}$

Cor. 5.14.11. $\begin{cases} \vec{E}(t,\vec{r}) = -\nabla\phi(t,\vec{r}) - \partial_t\vec{A}(t,\vec{r}) = \frac{e[\vec{R}(t')-R(t')\vec{v}(t')]}{4\pi\varepsilon_0\gamma_v^2[R(t')-\vec{v}(t')\cdot\vec{R}(t')]^3} + \frac{e\vec{R}(t')\times[\vec{R}(t')-R(t')\vec{v}(t')]\times\dot{\vec{v}}(t')}{4\pi\varepsilon_0[R(t')-\vec{v}(t')\cdot\vec{R}(t')]^3} \\ \vec{B}(t,\vec{r}) = \nabla\times\vec{A}(t,\vec{r}) = \frac{\vec{R}(t')\times\vec{E}(t,\vec{r})}{R(t')} \end{cases}$

Cor. 5.14.12. $\begin{cases} \vec{E}(t,\vec{r}) = \frac{e}{4\pi\varepsilon_0\gamma_v^2[R(t')-\vec{v}(t')\cdot\vec{R}(t')]^3}\{\frac{1}{\gamma_v^2}[\vec{R}(t') - R(t')\vec{v}(t')] + \vec{R}(t')\times[\vec{R}(t') - R(t')\vec{v}(t')]\times\dot{\vec{v}}(t')\} \\ \vec{B}(t,\vec{r}) = \frac{\vec{R}(t')\times\vec{E}(t,\vec{r})}{R(t')}, t = t' + R(t'), R(t') = \sqrt{[x'(t')-x]^2 + [y'(t')-y]^2 + [z'(t')-z]^2} \end{cases}$

5.14.3 Comparison with photon energy momentum and delay vector

Cor. 5.14.13. $\begin{cases} \vec{p'} = \vec{p} + \gamma_v\vec{v}p + (\gamma_v - 1)(\vec{v}\cdot\vec{p})\vec{v}/v^2 \\ p' = \gamma_v(p + \vec{v}\cdot\vec{p}), \vec{p'}^2 - p'^2 = \vec{p}_0^2 - p_0^2 = 0 \end{cases}$ $\begin{cases} \vec{r'} = \vec{r} + \gamma_v\vec{v}r + (\gamma_v - 1)(\vec{v}\cdot\vec{r})\vec{v}/v^2 \\ r' = \gamma_v(r + \vec{v}\cdot\vec{r}), \vec{r'}^2 - r'^2 = \vec{r}^2 - r^2 = 0 \end{cases}$

5.15 Transformation law of spin vector [22]

Cor. 5.15.1. $\begin{cases} \vec{S}(\vec{v}) = \vec{s} + (\gamma_v - 1)(\vec{v}\cdot\vec{s})\vec{v}/v^2 \\ S_0(\vec{v}) = \gamma_v(\vec{v}\cdot\vec{s}) \end{cases} \Rightarrow \vec{v}\cdot\vec{S}(\vec{v}) = \gamma_v\vec{v}\cdot\vec{s} = S_0(\vec{v})$

Cor. 5.15.2. $\begin{cases} \vec{S}(\vec{v}) = \vec{s} + (\gamma_v - 1)(\vec{v}\cdot\vec{s})\vec{v}/v^2 \\ S_0(\vec{v}) = \gamma_v(\vec{v}\cdot\vec{s}) \end{cases} \Leftrightarrow \begin{cases} \vec{S}(\vec{v}) = \vec{s} + (\gamma_v - 1)(\vec{v}\cdot\vec{s})\vec{v}/v^2 \\ \vec{v}\cdot\vec{S}(\vec{v}) = \gamma_v(\vec{v}\cdot\vec{s}) \end{cases}$

Cor. 5.15.3. $\begin{cases} \vec{u}' = [\vec{u} + \gamma_v\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{u})\vec{v}/v^2]/[\gamma_v(1 + \vec{v}\cdot\vec{u})] \\ \vec{S}(\vec{u}') = \vec{S}(\vec{u}) + \gamma_v\vec{v}[\vec{u}\cdot\vec{S}(\vec{u})] + (\gamma_v - 1)[\vec{v}\cdot\vec{S}(\vec{u})]\vec{v}/v^2 \\ \vec{u'}\cdot\vec{S}(\vec{u'}) = \gamma_v[\vec{u}\cdot\vec{S}(\vec{u}) + \vec{v}\cdot\vec{S}(\vec{u})] \end{cases} \Leftrightarrow \begin{bmatrix} \vec{S}(\vec{u}') \\ i\vec{u'}\cdot\vec{S}(\vec{u'}) \end{bmatrix} = L_{\vec{v}}\begin{bmatrix} \vec{S}(\vec{u}) \\ i\vec{u}\cdot\vec{S}(\vec{u}) \end{bmatrix}$

The transformation law of spin is similar to that of force.

5.16 Angular momentum transformation law of massless particles

Cor. 5.16.1. $\vec{r'} = \vec{r} + \gamma_v\vec{v}r + (\gamma_v - 1)(\vec{v}\cdot\vec{r})\vec{v}/v^2 \Rightarrow r' = \gamma_v(r + \vec{v}\cdot\vec{r})$

Cor. 5.16.2. $\vec{p'} = \vec{p} + \gamma_v\vec{v}p + (\gamma_v - 1)(\vec{v}\cdot\vec{p})\vec{v}/v^2 \Rightarrow p' = \gamma_v(p + \vec{v}\cdot\vec{p})$

Cor. 5.16.3. $M_{ab} = r_ap_b - r_bp_a = \begin{bmatrix} 0 & (\vec{r}\times\vec{p})_z & -(\vec{r}\times\vec{p})_y & -i(rp_x - xp) \\ -(\vec{r}\times\vec{p})_z & 0 & (\vec{r}\times\vec{p})_x & -i(rp_y - yp) \\ (\vec{r}\times\vec{p})_y & -(\vec{r}\times\vec{p})_x & 0 & -i(rp_z - zp) \\ i(xp - rp_x) & i(yp - rp_y) & i(zp - rp_z) & 0 \end{bmatrix}$

Cor. 5.16.4. $\vec{J} = \vec{r}\times\vec{p}, \vec{W} = r\vec{p} - p\vec{r}$

5.17 Angular momentum transformation law of particle system

Cor. 5.17.1. $M_{ab} = r_a p_b - r_b p_a = \begin{bmatrix} 0 & (\vec{r}\times\vec{p})_z & -(\vec{r}\times\vec{p})_y & -i(rp_x - xp) \\ -(\vec{r}\times\vec{p})_z & 0 & (\vec{r}\times\vec{p})_x & -i(rp_y - yp) \\ (\vec{r}\times\vec{p})_y & -(\vec{r}\times\vec{p})_x & 0 & -i(rp_z - zp) \\ i(xp - rp_x) & i(yp - rp_y) & i(zp - rp_z) & 0 \end{bmatrix}$

Cor. 5.17.2. $\vec{J} = \sum\limits_i (\vec{r}_i \times \vec{p}_i), \vec{W} = \sum\limits_i (r_i \vec{p}_i - p_i \vec{r}_i)$

Cor. 5.17.3. $\vec{u}' = [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v}\cdot\vec{u})\vec{v}/v^2]/[\gamma_v(1 + \vec{v}\cdot\vec{u})]$

Neutrino spin:

Cor. 5.17.4. $s(\nu) = \int \nu^+(\vec{0})\sigma_y \sigma \nu(\vec{0}) dx^4$

Photon spin:

Cor. 5.17.5. $s(\gamma) = \Psi(\vec{0})^T \gamma \Psi(\vec{0})$

Electron spin

Cor. 5.17.6. $s(e) = \bar{\psi}(\vec{0})\gamma_e \psi(\vec{0})$

5.18 Wigner little group [35]

5.18.1 Little group of particles with mass

Cor. 5.18.1. $L_{\vec{v}} \forall \Lambda[SO(3)] \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = \begin{bmatrix} \gamma m \vec{v} \\ i\gamma m \end{bmatrix} = \begin{bmatrix} \vec{p} \\ iE \end{bmatrix}$

Cor. 5.18.2. $L_p \equiv L_{\vec{v}} \forall \Lambda[SO(3)]$

Cor. 5.18.3. $L_p p_0 = p, L_{\Lambda p} p_0 = \Lambda p = \Lambda L_p p_0$

Cor. 5.18.4. $p_0 = L_{\Lambda p}^{-1} \Lambda L_p p_0$

Cor. 5.18.5. $W(\Lambda, p) \equiv L_{\Lambda p}^{-1} \Lambda L_p = \forall \Lambda[SO(3)]$

5.18.2 Little group of particles without mass

Cor. 5.18.6. $\Lambda \forall \Lambda[E(2)] \begin{bmatrix} 0 \\ 0 \\ p_0 \\ ip_0 \end{bmatrix} = \begin{bmatrix} \vec{p} \\ ip \end{bmatrix}$

Cor. 5.18.7. $L_p \equiv \Lambda \forall \Lambda[E(2)]$

Cor. 5.18.8. $L_p p_{std} = \begin{bmatrix} \vec{p} \\ ip \end{bmatrix}, L_{\Lambda p} p_{std} = \Lambda \begin{bmatrix} \vec{p} \\ ip \end{bmatrix} = \Lambda L_p p_{std}$

Cor. 5.18.9. $p_{std} = L_{\Lambda p}^{-1} \Lambda L_p p_{std}$

Cor. 5.18.10. $W(\Lambda, p) \equiv L_{\Lambda p}^{-1} \Lambda L_p = \forall \Lambda[E(2)]$

Chapter15 Mathematical Analysis of Helicity

Self comment: In order to further study the physics of various spin particles, I developed a mathematical analysis method of helicity in this chapter. It provides a powerful mathematical tool for studying various spin particles.

1 Spatial rotation transformation of unit vector

1.1 Spatial rotation transformation of 1-spin spinor

Pro. 1.1.1. $e^{\vec{\vartheta}\cdot\bar{\Omega}(1)} = 1 + \frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[\vec{\vartheta}\cdot\bar{\Omega}(1)] + \frac{cosh\sqrt{\vec{\vartheta}^2}-1}{\vec{\vartheta}^2}[\vec{\vartheta}\cdot\bar{\Omega}(1)]^2$

$\Rightarrow e^{i\vec{\omega}\cdot\gamma} = 1 + i\hat{\omega}\cdot\gamma sin\omega + (i\hat{\omega}\cdot\gamma)^2(1-cos\omega), \omega := |\vec{\omega}|$

Cor. 1.1.1. $e^{i\vec{\omega}\cdot\gamma} = 1 + sin\omega\begin{bmatrix} 0 & \hat{\omega}_z & -\hat{\omega}_y \\ -\hat{\omega}_z & 0 & \hat{\omega}_x \\ \hat{\omega}_y & -\hat{\omega}_x & 0 \end{bmatrix} + (1-cos\omega)\begin{bmatrix} \hat{\omega}_x^2-1 & \hat{\omega}_x\hat{\omega}_y & \hat{\omega}_x\hat{\omega}_z \\ \hat{\omega}_y\hat{\omega}_x & \hat{\omega}_y^2-1 & \hat{\omega}_y\hat{\omega}_z \\ \hat{\omega}_z\hat{\omega}_x & \hat{\omega}_z\hat{\omega}_y & \hat{\omega}_z^2-1 \end{bmatrix}$

Cor. 1.1.2. $e^{i\vec{\omega}\cdot\sigma(1)} = 1 + i\hat{\omega}\cdot\sigma(1)sin\omega + [i\hat{\omega}\cdot\sigma(1)]^2(1-cos\omega)$

1.2 Spatial rotation transformation of unit vector $\hat{p}$

Def. 1.2.1. $\hat{\omega}_+ := \frac{1}{\sqrt{2}}(\hat{\omega}_x + i\hat{\omega}_y), \hat{\omega}_- := \frac{1}{\sqrt{2}}(\hat{\omega}_x - i\hat{\omega}_y), \hat{p}_+ := \frac{1}{\sqrt{2}}(\hat{p}_x + i\hat{p}_y), \hat{p}_- := \frac{1}{\sqrt{2}}(\hat{p}_x - i\hat{p}_y)$

Thm. 1.2.1. $\hat{p} = exp\{i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\begin{bmatrix}0\\0\\1\end{bmatrix}, 0 \le arccos\hat{p}_z \le \pi$

Proof: $\hat{p} = e^{i\vec{\omega}\cdot\gamma}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} + sin\omega\begin{bmatrix}-\hat{\omega}_y\\ \hat{\omega}_x\\0\end{bmatrix} + (1-cos\omega)\begin{bmatrix}\hat{\omega}_x\hat{\omega}_z\\ \hat{\omega}_y\hat{\omega}_z\\ \hat{\omega}_z^2-1\end{bmatrix}$

$$\Leftrightarrow \begin{cases} \hat{p}_x = -\hat{\omega}_y sin\omega + \hat{\omega}_x\hat{\omega}_z(1-cos\omega) \\ \hat{p}_y = \hat{\omega}_x sin\omega + \hat{\omega}_y\hat{\omega}_z(1-cos\omega) \\ \hat{p}_z = 1 + (\hat{\omega}_z^2-1)(1-cos\omega) \end{cases} \overset{\hat{\omega}_z=0}{\Rightarrow} \begin{cases} \hat{p}_x = -\hat{\omega}_y sin\omega \\ \hat{p}_y = \hat{\omega}_x sin\omega \\ \hat{p}_z = cos\omega \end{cases} \Leftarrow \begin{cases} \hat{\omega}_x = \frac{\hat{p}_y}{\sqrt{1-\hat{p}_z^2}} \\ \hat{\omega}_y = \frac{-\hat{p}_x}{\sqrt{1-\hat{p}_z^2}} \\ \hat{\omega}_z = 0, 0 \le \omega = arccos\hat{p}_z \le \pi \end{cases}$$

$\Rightarrow \hat{p} = exp\{i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\begin{bmatrix}0\\0\\1\end{bmatrix}, 0 \le arccos\hat{p}_z \le \pi$ □

Pro. 1.2.1. $\begin{cases} \hat{\omega}\cdot\gamma \overset{\hat{\omega}_z=0}{=} \gamma_x\hat{\omega}_x + \gamma_y\hat{\omega}_y = \frac{\gamma_x\hat{p}_y - \gamma_y\hat{p}_x}{\sqrt{1-\hat{p}_z^2}} = \frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \\ \vec{\omega}\cdot\gamma \overset{\hat{\omega}_z=0}{=} \frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z, 0 \le arccos\hat{p}_z \le \pi \end{cases}$

Pro. 1.2.2. $e^{i\vec{\omega}\cdot\gamma} \overset{\hat{\omega}_z=0}{=} exp\{i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\} = 1 + i(\gamma\times\hat{p})_z - (\gamma\times\hat{p})_z^2/(1+\hat{p}_z), 0 \le arccos\hat{p}_z \le \pi$

1.3 Wigner SO(2) little group

Pro. 1.3.1. $\hat{p} = e^{i\omega_z\gamma_z}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

1.4 Spatial rotation transformation of unit vector $\hat{p}$ [24]

Cor. 1.4.1. $\hat{p}' = [\hat{p} + \gamma_v\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\hat{p})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\hat{p})]$

Cor. 1.4.2. $\hat{p} = [\begin{bmatrix}0\\0\\1\end{bmatrix} + \gamma_v\vec{v} + (\gamma_v-1)(\vec{v}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix})] = [1 + i(\gamma\times\hat{p})_z - (\gamma\times\hat{p})_z^2/(1+\hat{p}_z)]\begin{bmatrix}0\\0\\1\end{bmatrix}$

Cor. 1.4.3. $e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(s)}, exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}$

2 Analysis of helicity eigenfunctions $\sigma(\frac{1}{2})\cdot\hat{p}$

2.1 Concrete solution of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ [36] eigenfunctions

Def. 2.1.1. $\sigma(\frac{1}{2})\cdot\hat{p}\lambda(\hat{p},h) = h\lambda(\hat{p},h), h = -\frac{1}{2}, \frac{1}{2}$

Cor. 2.1.1. $e^{\vec{\vartheta}\cdot\frac{\sigma}{2}} = cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2} + \frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\vec{\vartheta}\cdot\sigma \Rightarrow e^{i\vec{\omega}\cdot\frac{\sigma}{2}} = cos\frac{1}{2}\omega + i\hat{\omega}\cdot\sigma sin\frac{1}{2}\omega = \frac{(1+\hat{p}_z)+i(\sigma\times\hat{p})_z}{\sqrt{2(1+\hat{p}_z)}}$

Cor. 2.1.2. $i\hat{\omega}\cdot\sigma = i\{\begin{bmatrix}0 & \hat{\omega}_x\\ \hat{\omega}_x & 0\end{bmatrix} + \begin{bmatrix}0 & -i\hat{\omega}_y\\ i\hat{\omega}_y & 0\end{bmatrix} + \begin{bmatrix}\hat{\omega}_z & 0\\ 0 & -\hat{\omega}_z\end{bmatrix}\} = i\begin{bmatrix}\hat{\omega}_z & \sqrt{2}\hat{\omega}_-\\ \sqrt{2}\hat{\omega}_+ & -\hat{\omega}_z\end{bmatrix} \overset{\hat{\omega}_z=0}{=} i\sqrt{2}\begin{bmatrix}0 & \hat{\omega}_-\\ \hat{\omega}_+ & 0\end{bmatrix}$

Cor. 2.1.3. $e^{i\vec{\omega}\cdot\frac{\sigma}{2}} \overset{\hat{\omega}_z=0}{=} cos\frac{1}{2}\omega + i\hat{\omega}\cdot\sigma sin\frac{1}{2}\omega = \begin{bmatrix}cos\frac{1}{2}\omega & i\sqrt{2}\hat{\omega}_- sin\frac{1}{2}\omega\\ i\sqrt{2}\hat{\omega}_+ sin\frac{1}{2}\omega & cos\frac{1}{2}\omega\end{bmatrix}$

Cor. 2.1.4. $\begin{cases}\lambda(\hat{p},\frac{1}{2}) = e^{i\vec{\omega}\cdot\sigma(\frac{1}{2})}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}cos\frac{1}{2}\omega\\ i\sqrt{2}\hat{\omega}_+ sin\frac{1}{2}\omega\end{bmatrix} = \frac{1}{\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_+\end{bmatrix}, \lambda(-\hat{p},\frac{1}{2}) = -\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}}\lambda(\hat{p},-\frac{1}{2})\\ \lambda(\hat{p},-\frac{1}{2}) = e^{i\vec{\omega}\cdot\sigma(\frac{1}{2})}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}i\sqrt{2}\hat{\omega}_- sin\frac{1}{2}\omega\\ cos\frac{1}{2}\omega\end{bmatrix} = \frac{1}{\sqrt{1+\hat{p}_z}}\begin{bmatrix}-\hat{p}_-\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}, \lambda(-\hat{p},-\frac{1}{2}) = \frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}}\lambda(\hat{p},\frac{1}{2})\\ \lambda(\hat{p},\frac{1}{2}) = i\sigma_y\lambda^*(\hat{p},-\frac{1}{2}), \lambda(\hat{p},-\frac{1}{2}) = -i\sigma_y\lambda^*(\hat{p},\frac{1}{2})\end{cases}$

Cor. 2.1.5. $\begin{cases}\lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2}) = \frac{1}{2}(\sigma\cdot\hat{p}+I) = \frac{1}{2}(\sigma,-i)^a\hat{p}_a, \hat{p}_a := (\hat{p},i)\\ \lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2}) = -\frac{1}{2}(\sigma\cdot\hat{p}-I) = -\frac{1}{2}(\sigma,i)^a\hat{p}_a\\ \lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2}) = \frac{1}{2}(\sigma\cdot\hat{p}+I)i\sigma_y = \frac{1}{2}(\sigma,i)^a\hat{p}_a i\sigma_y\\ \lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2}) = -\frac{1}{2}(\sigma\cdot\hat{p}+I)i\sigma_y = -\frac{1}{2}(\sigma,i)^a\hat{p}_a i\sigma_y\end{cases}$

2.2 Orthogonality and completeness of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Cor. 2.2.1. $\lambda^+(\hat{p},h)\lambda(\hat{p},h') = \delta_{hh'}, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}}\lambda(\hat{p},h)\lambda^+(\hat{p},h) = 1, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}}h\lambda(\hat{p},h)\lambda^+(\hat{p},h) = \sigma(\frac{1}{2})\cdot\hat{p}$

2.3 Raising and lowering operator of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Thm. 2.3.1.
$\begin{cases}e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_x e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = \sigma_x - \hat{p}_x\frac{(\sigma\cdot\hat{p}+\sigma_z)}{(1+\hat{p}_z)}\\ e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_y e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = \sigma_y - \hat{p}_y\frac{(\sigma\cdot\hat{p}+\sigma_z)}{(1+\hat{p}_z)}\\ e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_z e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = \sigma\cdot\hat{p}\end{cases}$

Proof: $e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_x e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (e^{-i\vec{\omega}\cdot\gamma})_x{}^k\sigma_k$
$= \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_x\frac{(1+\hat{p}_z)-i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}$
$= \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\frac{(1+\hat{p}_z)-i(\sigma_x\hat{p}_y+\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_x$
$= \frac{(1+\hat{p}_z)^2-2i(1+\hat{p}_z)\sigma_y\hat{p}_x+(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)(\sigma_x\hat{p}_y+\sigma_y\hat{p}_x)}{2(1+\hat{p}_z)}\sigma_x$
$= \frac{(1+\hat{p}_z)^2-2i(1+\hat{p}_z)\sigma_y\hat{p}_x+\hat{p}_y^2-\hat{p}_x^2+2i\sigma_z\hat{p}_x\hat{p}_y}{2(1+\hat{p}_z)}\sigma_x$
$= \frac{2(1+\hat{p}_z)-2i(1+\hat{p}_z)\sigma_y\hat{p}_x-2\hat{p}_x^2+2i\sigma_z\hat{p}_x\hat{p}_y}{2(1+\hat{p}_z)}\sigma_x$
$= \frac{(1+\hat{p}_z)(\sigma_x-\hat{p}_x\sigma_z)-\hat{p}_x(\hat{p}_x\sigma_x+\hat{p}_y\sigma_y)}{(1+\hat{p}_z)}$
$= \frac{\sigma_x+\hat{p}_z\sigma_x-\hat{p}_x\sigma_z-\hat{p}_x(\sigma\cdot\hat{p})}{(1+\hat{p}_z)}$
$= \sigma_x - \hat{p}_x\frac{(\sigma\cdot\hat{p}+\sigma_z)}{(1+\hat{p}_z)}$
$= \frac{\sigma_x-(\sigma\times\hat{p})_y-\hat{p}_x(\sigma\cdot\hat{p})}{(1+\hat{p}_z)}$ □

Proof: $e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_y e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (e^{-i\vec{\omega}\cdot\gamma})_y{}^k\sigma_k$
$= \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_y\frac{(1+\hat{p}_z)-i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}$
$= \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y+\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_y$
$= \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)\sigma_x\hat{p}_y-(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)(\sigma_x\hat{p}_y+\sigma_y\hat{p}_x)}{2(1+\hat{p}_z)}\sigma_y$
$= \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)\sigma_x\hat{p}_y+\hat{p}_x^2-\hat{p}_y^2-2i\sigma_z\hat{p}_x\hat{p}_y}{2(1+\hat{p}_z)}\sigma_y$
$= \frac{2(1+\hat{p}_z)+2i(1+\hat{p}_z)\sigma_x\hat{p}_y-2\hat{p}_y^2-2i\sigma_z\hat{p}_x\hat{p}_y}{2(1+\hat{p}_z)}\sigma_y$
$= \frac{(1+\hat{p}_z)(\sigma_y-\hat{p}_y\sigma_z)-\hat{p}_y(\hat{p}_x\sigma_x+\hat{p}_y\sigma_y)}{(1+\hat{p}_z)}$
$= \frac{\sigma_y+\hat{p}_z\sigma_y-\hat{p}_y\sigma_z-\hat{p}_y(\sigma\cdot\hat{p})}{(1+\hat{p}_z)}$
$= \sigma_y - \hat{p}_y\frac{(\sigma\cdot\hat{p}+\sigma_z)}{(1+\hat{p}_z)}$
$= \frac{\sigma_y+(\sigma\times\hat{p})_x-\hat{p}_y(\sigma\cdot\hat{p})}{(1+\hat{p}_z)}$ □

Proof: $e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_z e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (e^{-i\vec{\omega}\cdot\gamma})_z{}^k\sigma_k$
$= \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_z\frac{(1+\hat{p}_z)-i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}$
$= \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_z$
$= \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)-(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)^2}{2(1+\hat{p}_z)}\sigma_z$
$= \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)-(\hat{p}_x^2+\hat{p}_y^2)}{2(1+\hat{p}_z)}\sigma_z$
$= [\hat{p}_z + i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)]\sigma_z$
$= \sigma\cdot\hat{p}$ □

Cor. 2.3.1.

$$\begin{cases} e^{i\vec{\omega}\cdot\frac{\sigma}{2}}(\sigma_x+i\sigma_y)e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (\sigma_x+i\sigma_y) - \frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}(\sigma\cdot\hat{p}+\sigma_z) \\ e^{i\vec{\omega}\cdot\frac{\sigma}{2}}(\sigma_x-i\sigma_y)e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (\sigma_x-i\sigma_y) - \frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}(\sigma\cdot\hat{p}+\sigma_z) \\ e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_z e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = \sigma\cdot\hat{p} \end{cases}$$

Cor. 2.3.2.

$$\begin{cases} \{[\sigma_x(\frac{1}{2})+i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})]\}\lambda(\hat{p},\frac{1}{2}) = 0 \\ \{[\sigma_x(\frac{1}{2})+i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})]\}\lambda(\hat{p},-\frac{1}{2}) = \lambda(\hat{p},\frac{1}{2}) \\ \{[\sigma_x(\frac{1}{2})-i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})]\}\lambda(\hat{p},\frac{1}{2}) = \lambda(\hat{p},-\frac{1}{2}) \\ \{[\sigma_x(\frac{1}{2})-i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})]\}\lambda(\hat{p},-\frac{1}{2}) = 0 \end{cases}$$

2.4 Basic properties of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Pro. 2.4.1. $\lambda^*(\hat{p},-\frac{\varsigma}{2}) \equiv -i\varsigma\sigma_y\lambda(\hat{p},\frac{\varsigma}{2}), \lambda^+(\hat{p},-\frac{\varsigma}{2}) \equiv i\varsigma\lambda^T(\hat{p},\frac{\varsigma}{2})\sigma_y$

2.5 Complicated properties of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Pro. 2.5.1. $\lambda^+(\hat{p},-\frac{\varsigma}{2})(\sigma,-i\varsigma)_a\lambda(\hat{p},-\frac{\varsigma}{2}) = -\varsigma\hat{p}_a, \lambda^T(\hat{p},\frac{\varsigma}{2})\sigma_y(\sigma,-i\varsigma)_a\lambda(\hat{p},-\frac{\varsigma}{2}) = i\hat{p}_a$

Pro. 2.5.2. $\lambda^+(\hat{p},-\frac{\varsigma}{2})(\sigma,-i\varsigma)_a\lambda(\hat{p},\frac{\varsigma}{2}) = \begin{bmatrix} \frac{\hat{p}_x\hat{p}_z-i\varsigma\hat{p}_y}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_y\hat{p}_z+i\varsigma\hat{p}_x}{\hat{p}_x-i\varsigma\hat{p}_y} \\ -\hat{p}_x-i\varsigma\hat{p}_y \\ 0 \end{bmatrix}, \lambda^T(\hat{p},\frac{\varsigma}{2})(\sigma,1)\lambda(\hat{p},\frac{\varsigma}{2}) = \begin{bmatrix} \varsigma\hat{p}_x+i\hat{p}_y \\ 0 \\ \frac{\varsigma\hat{p}_x\hat{p}_z-i\hat{p}_y}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_x-i\varsigma\hat{p}_y\hat{p}_z}{\hat{p}_x-i\varsigma\hat{p}_y} \end{bmatrix}$

Pro. 2.5.3. $\lambda^+(\hat{p},-\frac{\varsigma}{2})\sigma_i\lambda(\hat{p},\frac{\varsigma}{2}) = \begin{bmatrix} \frac{\hat{p}_x\hat{p}_z-i\varsigma\hat{p}_y}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_y\hat{p}_z+i\varsigma\hat{p}_x}{\hat{p}_x-i\varsigma\hat{p}_y} \\ -\hat{p}_x-i\varsigma\hat{p}_y \end{bmatrix} = \begin{bmatrix} \frac{\hat{p}_x\hat{p}_z-i\varsigma\hat{p}_y\delta_{xx}+i\varsigma\hat{p}_x\delta_{xy}-\delta_{xz}}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_y\hat{p}_z-i\varsigma\hat{p}_y\delta_{yx}+i\varsigma\hat{p}_x\delta_{yy}-\delta_{yz}}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_z\hat{p}_z-i\varsigma\hat{p}_y\delta_{yx}+i\varsigma\hat{p}_x\delta_{yy}-\delta_{zz}}{\hat{p}_x-i\varsigma\hat{p}_y} \end{bmatrix} = \frac{(\hat{p}_i\hat{p}_z-\delta_{iz})-i\varsigma(\hat{p}_y\delta_{ix}-\hat{p}_x\delta_{iy})}{\hat{p}_x-i\varsigma\hat{p}_y}$

Pro. 2.5.4. $\lambda^T(\hat{p},\frac{\varsigma}{2})(\sigma,1)\lambda(\hat{p},-\frac{\varsigma}{2}) = \begin{bmatrix} \hat{p}_z \\ -i\varsigma \\ -\hat{p}_x \\ i\hat{p}_y \end{bmatrix}$

Pro. 2.5.5. $\lambda(\hat{p},-\frac{\varsigma}{2})\lambda^T(\hat{p},-\frac{\varsigma}{2}) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\hat{p}_- & -\frac{1}{2}(\varsigma-\hat{p}_z) \\ \frac{1}{2}(\varsigma+\hat{p}_z) & \frac{1}{\sqrt{2}}\hat{p}_+ \end{bmatrix}$

Pro. 2.5.6. $\lambda(\hat{p},-\frac{\varsigma}{2})\lambda^T(\hat{p},\frac{\varsigma}{2}) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\hat{p}_- & -\frac{1}{2}(\varsigma-\hat{p}_z) \\ \frac{1}{2}(\varsigma+\hat{p}_z) & \frac{1}{\sqrt{2}}\hat{p}_+ \end{bmatrix} = \frac{i}{2}(\sigma\cdot\hat{p}-\varsigma I)\sigma_y = \frac{i}{2}(\sigma,i\varsigma)^a\hat{p}_a\sigma_y$

Pro. 2.5.7. $\lambda(\hat{p},-\frac{\varsigma}{2})\lambda^+(\hat{p},-\frac{\varsigma}{2}) = -\frac{\varsigma}{2}(\sigma,i\varsigma)^a\hat{p}_a, \lambda(\hat{p},-\frac{\varsigma}{2})\lambda^T(\hat{p},\frac{\varsigma}{2}) = \frac{i}{2}(\sigma,i\varsigma)^a\hat{p}_a\sigma_y$

Pro. 2.5.8. $\begin{cases} \lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2}) + \lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2}) = I, \lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2}) - \lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2}) = \sigma\cdot\hat{p} \\ \lambda(\hat{p},\frac{1}{2})\lambda^T(\hat{p},-\frac{1}{2}) - \lambda(\hat{p},-\frac{1}{2})\lambda^T(\hat{p},\frac{1}{2}) = i\sigma_y, \lambda(\hat{p},\frac{1}{2})\lambda^T(\hat{p},-\frac{1}{2}) + \lambda(\hat{p},-\frac{1}{2})\lambda^T(\hat{p},\frac{1}{2}) = i\sigma\cdot\hat{p}\sigma_y \end{cases}$

2.6 Derivative properties of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

2.6.1 Basic derivative properties

Cor. 2.6.1.

$$\begin{cases} \tilde{\partial}_i p = \hat{p}_i \\ \tilde{\partial}_i\hat{p}_j = \frac{p^2\delta_{ij}-p_ip_j}{p^3} = \frac{\delta_{ij}-\hat{p}_i\hat{p}_j}{p} \\ \tilde{\partial}_i\hat{p}_+ = \frac{\frac{1}{\sqrt{2}}(\delta_{ix}+i\delta_{iy})-\hat{p}_i\hat{p}_+}{p}, \tilde{\partial}_i\hat{p}_- = \frac{\frac{1}{\sqrt{2}}(\delta_{ix}-i\delta_{iy})-\hat{p}_i\hat{p}_-}{p} \\ \tilde{\partial}_i\frac{\hat{p}_+}{\hat{p}_-} = \frac{i\hat{p}_x\delta_{iy}-i\hat{p}_y\delta_{ix}}{p\hat{p}_-^2}, \tilde{\partial}_i\frac{\hat{p}_-}{\hat{p}_+} = \frac{-i\hat{p}_x\delta_{iy}+i\hat{p}_y\delta_{ix}}{p\hat{p}_+^2} \\ \tilde{\partial}_i\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}} = \frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}}\frac{i\hat{p}_x\delta_{iy}-i\hat{p}_y\delta_{ix}}{2p\hat{p}_-^2}, \tilde{\partial}_i\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}} = \frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}}\frac{-i\hat{p}_x\delta_{iy}+i\hat{p}_y\delta_{ix}}{2p\hat{p}_+^2} \end{cases}$$

2.6.2 Derivative properties 1 of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Cor. 2.6.2. $\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\delta_{iz}+\hat{p}_i\\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$
$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+(\delta_{iz}-\hat{p}_i\hat{p}_z)+\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)+i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\end{bmatrix}$

Proof: $\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = \tilde{\partial}_i\frac{1}{\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_+\end{bmatrix} + \frac{1}{\sqrt{1+\hat{p}_z}}\tilde{\partial}_i\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_+\end{bmatrix}$
$= \frac{-\delta_{iz}+\hat{p}_i\hat{p}_z}{2p\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_+\end{bmatrix} + \frac{1}{p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ \frac{1}{\sqrt{2}}(\delta_{ix}+i\delta_{iy})-\hat{p}_i\hat{p}_+\end{bmatrix}$
$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3}\{\begin{bmatrix}-\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+(\delta_{iz}-\hat{p}_i\hat{p}_z)\end{bmatrix} + \begin{bmatrix}\frac{1}{\sqrt{2}}(\delta_{iz}-\hat{p}_i\hat{p}_z)2(1+\hat{p}_z)\\ [\frac{1}{\sqrt{2}}(\delta_{ix}+i\delta_{iy})-\hat{p}_i\hat{p}_+]2(1+\hat{p}_z)\end{bmatrix}\}$
$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+(\delta_{iz}-\hat{p}_i\hat{p}_z)+2(1+\hat{p}_z)[\frac{1}{\sqrt{2}}(\delta_{ix}+i\delta_{iy})-\hat{p}_i\hat{p}_+]\end{bmatrix}$
$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+(\delta_{iz}-\hat{p}_i\hat{p}_z)+\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)+i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\end{bmatrix}$
$= \frac{1}{2\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\tilde{\partial}_i\hat{p}_z\\ -\hat{p}_+\tilde{\partial}_i\hat{p}_z+2(1+\hat{p}_z)\tilde{\partial}_i\hat{p}_+\end{bmatrix}$
$= \frac{1}{2\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}\tilde{\partial}_i\hat{p}_z\\ -\frac{\hat{p}_+}{1+\hat{p}_z}\tilde{\partial}_i\hat{p}_z+2\tilde{\partial}_i\hat{p}_+\end{bmatrix}$ □

Proof: $\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = \frac{-\delta_{iz}+\hat{p}_i\hat{p}_z}{2p(1+\hat{p}_z)}\lambda(\hat{p},\frac{1}{2}) + \frac{1}{p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ \frac{1}{\sqrt{2}}(\delta_{ix}+i\delta_{iy})-\hat{p}_i\hat{p}_+\end{bmatrix}$
$= \frac{-\delta_{iz}+\hat{p}_i\hat{p}_z}{2p(1+\hat{p}_z)}\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\delta_{iz}+\hat{p}_i\\ \delta_{ix}+i\delta_{iy}\end{bmatrix} - \frac{\hat{p}_i}{p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_+\end{bmatrix}$
$= [\frac{-\delta_{iz}+\hat{p}_i\hat{p}_z}{2p(1+\hat{p}_z)} - \frac{\hat{p}_i}{p}]\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\delta_{iz}+\hat{p}_i\\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$
$= -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\delta_{iz}+\hat{p}_i\\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$ □

Cor. 2.6.3. $\tilde{\partial}_i\lambda(\hat{p},-\frac{1}{2}) = -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\lambda(\hat{p},-\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\begin{bmatrix}-\delta_{ix}+i\delta_{iy}\\ \delta_{iz}+\hat{p}_i\end{bmatrix}$
$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\hat{p}_-(\delta_{iz}-\hat{p}_i\hat{p}_z)-\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)-i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\end{bmatrix}$

Proof: $\tilde{\partial}_i\lambda(\hat{p},-\frac{1}{2}) = -i\sigma_y\tilde{\partial}_i\lambda^*(\hat{p},\frac{1}{2})$
$= -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}[-i\sigma_y\lambda^*(\hat{p},\frac{1}{2})] + \frac{-i\sigma_y}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\begin{bmatrix}\delta_{iz}+\hat{p}_i\\ \delta_{ix}-i\delta_{iy}\end{bmatrix}$
$= \frac{-i\sigma_y}{2p\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_-(\delta_{iz}-\hat{p}_i\hat{p}_z)+\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)-i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\end{bmatrix}$
$= -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\lambda(\hat{p},-\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\begin{bmatrix}-\delta_{ix}+i\delta_{iy}\\ \delta_{iz}+\hat{p}_i\end{bmatrix}$
$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3}\begin{bmatrix}\hat{p}_-(\delta_{iz}-\hat{p}_i\hat{p}_z)-\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)-i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\end{bmatrix}$ □

2.6.3 Derivative properties 2 of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Cor. 2.6.4. $\begin{cases}\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2}) = [\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2})]^*, \lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2}) = [\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})]^*\\ \lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2}) = -[\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})]^*, \lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2}) = -[\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2})]^*\end{cases}$

Pro. 2.6.1. $\begin{cases}\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2}) = \lambda^+(\hat{p},\frac{1}{2})\frac{[\sigma_k(\frac{1}{2}),\sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)}\lambda(\hat{p},\frac{1}{2}) = \frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2}) = \lambda^+(\hat{p},-\frac{1}{2})\frac{[\sigma_k(\frac{1}{2}),\sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)}\lambda(\hat{p},-\frac{1}{2}) = -\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}\\ \lambda^+(-\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(-\hat{p},\frac{1}{2}) = -\lambda^+(-\hat{p},\frac{1}{2})\frac{[\sigma_k(\frac{1}{2}),\sigma_z(\frac{1}{2})]}{p(1-\hat{p}_z)}\lambda(-\hat{p},\frac{1}{2}) = \frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1-\hat{p}_z)}\\ \lambda^+(-\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(-\hat{p},-\frac{1}{2}) = -\lambda^+(-\hat{p},-\frac{1}{2})\frac{[\sigma_k(\frac{1}{2}),\sigma_z(\frac{1}{2})]}{p(1-\hat{p}_z)}\lambda(-\hat{p},-\frac{1}{2}) = -\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1-\hat{p}_z)}\\ \lambda^+(\hat{p},h)\partial_z\lambda(\hat{p},h) = 0\end{cases}$

Proof: $\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},-\frac{1}{2})$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}-\hat{p}_+\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}^T\begin{bmatrix}\hat{p}_-(\delta_{iz}-\hat{p}_i\hat{p}_z)-\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)-i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\end{bmatrix}$

$= \lambda^+(\hat{p},-\frac{1}{2})\frac{[\sigma_i(\frac{1}{2}),\sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)}\lambda(\hat{p},-\frac{1}{2}) = (\frac{i\hat{p}_y}{2p},\frac{-i\hat{p}_x}{2p},0) = \frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}$ □

Proof: $\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_-\end{bmatrix}^T\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+(\delta_{iz}-\hat{p}_i\hat{p}_z)+\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)+i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\end{bmatrix}$

$= \lambda^+(\hat{p},\frac{1}{2})\frac{[\sigma_i(\frac{1}{2}),\sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)}\lambda(\hat{p},\frac{1}{2}) = (\frac{-i\hat{p}_y}{2p},\frac{i\hat{p}_x}{2p},0) = -\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}$ □

2.6.4 Derivative properties 3 of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Pro. 2.6.2. $\begin{cases}\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},-\frac{1}{2}) = \frac{\sqrt{2}\hat{p}_-\hat{p}_i+\sqrt{2}\hat{p}_-\delta_{iz}-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{2p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = \frac{-\sqrt{2}\hat{p}_+\hat{p}_i-\sqrt{2}\hat{p}_+\delta_{iz}+(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{2p(1+\hat{p}_z)}\\ \lambda^+(-\hat{p},-\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},-\frac{1}{2}) = \frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}}\frac{\sqrt{2}\hat{p}_-\hat{p}_i+\sqrt{2}\hat{p}_-\delta_{iz}-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{2p(1+\hat{p}_z)}\\ \lambda^+(-\hat{p},\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = -\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}}\frac{-\sqrt{2}\hat{p}_+\hat{p}_i-\sqrt{2}\hat{p}_+\delta_{iz}+(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{2p(1+\hat{p}_z)}\end{cases}$

Proof: $\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}-\hat{p}_+\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}^T\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+(\delta_{iz}-\hat{p}_i\hat{p}_z)+\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)+i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\end{bmatrix}$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}-\hat{p}_+\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}^T\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+(\delta_{iz}-\hat{p}_i\hat{p}_z)+\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}+i\delta_{iy})-\sqrt{2}\hat{p}_i\hat{p}_+]\end{bmatrix}$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}-\hat{p}_+\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}^T\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\\ -\hat{p}_+\delta_{iz}-2\hat{p}_i\hat{p}_+-\hat{p}_+\hat{p}_i\hat{p}_z+\sqrt{2}(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})\end{bmatrix}$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}^T\begin{bmatrix}-\hat{p}_+\delta_{iz}+\hat{p}_+\hat{p}_i\hat{p}_z\\ -\hat{p}_+\delta_{iz}-2\hat{p}_i\hat{p}_+-\hat{p}_+\hat{p}_i\hat{p}_z+\sqrt{2}(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})\end{bmatrix}$

$= \frac{-\sqrt{2}\hat{p}_+\hat{p}_i-\sqrt{2}\hat{p}_+\delta_{iz}+(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{2p(1+\hat{p}_z)}$ □

Proof: $\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},-\frac{1}{2})$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_-\end{bmatrix}^T\begin{bmatrix}\hat{p}_-(\delta_{iz}-\hat{p}_i\hat{p}_z)-\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-\hat{p}_i\hat{p}_x)-i(\delta_{iy}-\hat{p}_i\hat{p}_y)]\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\end{bmatrix}$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_-\end{bmatrix}^T\begin{bmatrix}\hat{p}_-(\delta_{iz}-\hat{p}_i\hat{p}_z)-\sqrt{2}(1+\hat{p}_z)[(\delta_{ix}-i\delta_{iy})-\sqrt{2}\hat{p}_i\hat{p}_-]\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\end{bmatrix}$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_-\end{bmatrix}^T\begin{bmatrix}\hat{p}_-\delta_{iz}+2\hat{p}_i\hat{p}_-+\hat{p}_i\hat{p}_-\hat{p}_z-\sqrt{2}(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz}-\hat{p}_i\hat{p}_z)\end{bmatrix}$

$= \frac{1}{2p(1+\hat{p}_z)^2}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}^T\begin{bmatrix}\hat{p}_-\delta_{iz}+2\hat{p}_i\hat{p}_-+\hat{p}_i\hat{p}_-\hat{p}_z-\sqrt{2}(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})\\ \hat{p}_-\delta_{iz}-\hat{p}_i\hat{p}_-\hat{p}_z\end{bmatrix}$

$= \frac{\sqrt{2}\hat{p}_-\hat{p}_i+\sqrt{2}\hat{p}_-\delta_{iz}-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{p(1+\hat{p}_z)}$ □

2.6.5 Derivative properties 4 of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Cor. 2.6.5. $\lambda^+(\hat{p},h)\lambda(\hat{p},h') = \delta_{hh'}, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}}\lambda(\hat{p},h)\lambda^+(\hat{p},h) = 1, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}}\lambda^+(\hat{p},h)\tilde{\partial}_k\lambda(\hat{p},h) = 0$

Cor. 2.6.6. $(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a = -2\varsigma|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})$

Proof: $(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a$

$= (\sigma\cdot\vec{p})_{A_\varsigma}{}^{B_\varsigma}[\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})+\lambda_{B_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]-\varsigma|\vec{p}|[\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})+\lambda_{A_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]$

$= [-\varsigma|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})+\varsigma|\vec{p}|\lambda_{A_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]-\varsigma|\vec{p}|[\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})+\lambda_{A_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]$

$= -2\varsigma|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})$ □

Cor. 2.6.7. $\lambda_{A_\varsigma}(\hat{p},h)\lambda^+_{A'_\varsigma}(\hat{p},h) = h\sigma^k_{A_\varsigma A'_\varsigma}\hat{p}_k+\frac{1}{2}\delta_{A_\varsigma A'_\varsigma}$

Cor. 2.6.8. $e^{i\vec{\omega}\cdot\frac{\sigma}{2}} = cos\frac{1}{2}\omega + i\hat{\omega}\cdot\sigma sin\frac{1}{2}\omega$
$= \frac{1}{\sqrt{2}}[p + p_z + i(\sigma_x p_y - \sigma_y p_x][p^2 + pp_z]^{-1/2}$
$= \frac{1}{\sqrt{2}}[1 + \hat{p}_z + i(\sigma \times \hat{p})_z][1 + \hat{p}_z]^{-1/2}$

Cor. 2.6.9. $\partial_z e^{i\vec{\omega}\cdot\frac{\sigma}{2}} = \frac{\sqrt{1+\hat{p}_z}}{\sqrt{2}p}[1 - \hat{p}_z - i(\sigma \times \hat{p})_z], e^{-i\vec{\omega}\cdot\frac{\sigma}{2}}\partial_z e^{i\vec{\omega}\cdot\frac{\sigma}{2}} = \frac{-i}{p}(\sigma \times \hat{p})_z$

Cor. 2.6.10. $\lambda^+(\hat{p}, h)\partial_z\lambda(\hat{p}, h) = 0$

2.6.6 Summary of derivative properties of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions

Pro. 2.6.3. $\begin{cases} \lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2}) = \frac{1}{2}\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)} \\ \lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},-\frac{1}{2}) = \frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x-i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{2p(1+\hat{p}_z)} \\ \lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = -\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x+i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{2p(1+\hat{p}_z)} \\ \lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2}) = -\frac{1}{2}\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)} \end{cases}$

2.7 Continued exploration 1 of derivative properties of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions???

Cor. 2.7.1. $\sigma_j(\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\sigma_j\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\sigma_j\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$

Proof: $\lambda^+(\hat{p},\frac{1}{2})\sigma_j(\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})$
$= -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\lambda^+(\hat{p},\frac{1}{2})\sigma_j(\frac{1}{2})\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\lambda^+(\hat{p},\frac{1}{2})\sigma_j(\frac{1}{2})\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$
$= -\frac{\delta_{iz}\hat{p}_j+\hat{p}_i\hat{p}_j(2+\hat{p}_z)}{4p(1+\hat{p}_z)} + \frac{1}{4p(1+\hat{p}_z)}\begin{bmatrix}1+\hat{p}_z \\ \hat{p}_x-i\hat{p}_y\end{bmatrix}^T\sigma_j\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$ □

Proof: $\lambda^+(\hat{p},\frac{1}{2})[\sigma_i(\frac{1}{2})\tilde{\partial}_j - \sigma_j(\frac{1}{2})\tilde{\partial}_i]\lambda(\hat{p},\frac{1}{2})$
$= \frac{\delta_{iz}\hat{p}_j-\delta_{jz}\hat{p}_i}{4p(1+\hat{p}_z)} + \frac{1}{4p(1+\hat{p}_z)}\begin{bmatrix}1+\hat{p}_z \\ \hat{p}_x-i\hat{p}_y\end{bmatrix}^T(\sigma_i\begin{bmatrix}\delta_{jz}+\hat{p}_j \\ \delta_{jx}+i\delta_{jy}\end{bmatrix} - \sigma_j\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix})$ □

2.8 Continued exploration 2 of derivative properties of helicity $\sigma(\frac{1}{2})\cdot\hat{p}$ eigenfunctions???

Cor. 2.8.1. $\sigma_j(\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2}) = -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\sigma_j\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\sigma_j\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$

Proof: $\lambda^+(\hat{p},-\frac{1}{2})\sigma_j(\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})$
$= -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)}\lambda^+(\hat{p},-\frac{1}{2})\sigma_j(\frac{1}{2})\lambda(\hat{p},\frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\lambda^+(\hat{p},-\frac{1}{2})\sigma_j(\frac{1}{2})\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$
$= -\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{4p(1+\hat{p}_z)}\frac{(\hat{p}_j\hat{p}_z-\delta_{jz})-i(\hat{p}_y\delta_{jx}-\hat{p}_x\delta_{jy})}{\hat{p}_x-i\hat{p}_y} + \frac{1}{4p(1+\hat{p}_z)}\begin{bmatrix}-\hat{p}_x-i\hat{p}_y \\ 1+\hat{p}_z\end{bmatrix}^T\sigma_j\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix}$ □

Pro. 2.8.1. $\lambda^+(\hat{p},-\frac{\varsigma}{2})\sigma_i\lambda(\hat{p},\frac{\varsigma}{2}) = \begin{bmatrix}\frac{\hat{p}_x\hat{p}_z-i\varsigma\hat{p}_y}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_y\hat{p}_z+i\varsigma\hat{p}_x}{\hat{p}_x-i\varsigma\hat{p}_y} \\ -\hat{p}_x-i\varsigma\hat{p}_y\end{bmatrix} = \begin{bmatrix}\frac{\hat{p}_x\hat{p}_z-i\varsigma\hat{p}_y\delta_{xx}+i\varsigma\hat{p}_x\delta_{xy}-\delta_{xz}}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_y\hat{p}_z-i\varsigma\hat{p}_y\delta_{yx}+i\varsigma\hat{p}_x\delta_{yy}-\delta_{yz}}{\hat{p}_x-i\varsigma\hat{p}_y} \\ \frac{\hat{p}_z\hat{p}_z-i\varsigma\hat{p}_y\delta_{yx}+i\varsigma\hat{p}_x\delta_{yy}-\delta_{zz}}{\hat{p}_x-i\varsigma\hat{p}_y}\end{bmatrix} = \frac{(\hat{p}_i\hat{p}_z-\delta_{iz})-i\varsigma(\hat{p}_y\delta_{ix}-\hat{p}_x\delta_{iy})}{\hat{p}_x-i\varsigma\hat{p}_y}$

Proof: $\lambda^+(\hat{p},\frac{1}{2})[\sigma_i(\frac{1}{2})\tilde{\partial}_j - \sigma_j(\frac{1}{2})\tilde{\partial}_i]\lambda(\hat{p},\frac{1}{2})$
$= \frac{\delta_{iz}\hat{p}_j-\delta_{jz}\hat{p}_i}{4p(1+\hat{p}_z)} + \frac{1}{4p(1+\hat{p}_z)}\begin{bmatrix}1+\hat{p}_z \\ \hat{p}_x-i\hat{p}_y\end{bmatrix}^T(\sigma_i\begin{bmatrix}\delta_{jz}+\hat{p}_j \\ \delta_{jx}+i\delta_{jy}\end{bmatrix} - \sigma_j\begin{bmatrix}\delta_{iz}+\hat{p}_i \\ \delta_{ix}+i\delta_{iy}\end{bmatrix})$ □

3 Analysis of helicity $\sigma(1)\cdot\hat{p}$ eigenfunctions

3.1 Concrete solution I of helicity $\sigma(1)\cdot\hat{p}$ eigenfunctions

Cor. 3.1.1. $\sigma(1)\cdot\hat{p}\lambda(\hat{p},h;1) = h\lambda(\hat{p},h;1), h = -1, 0, 1$

Cor. 3.1.2. $i\hat{\omega}\cdot\sigma(1) = i\{\frac{1}{\sqrt{2}}\begin{bmatrix}0 & \hat{\omega}_x & 0 \\ \hat{\omega}_x & 0 & \hat{\omega}_x \\ 0 & \hat{\omega}_x & 0\end{bmatrix} + \frac{i}{\sqrt{2}}\begin{bmatrix}0 & -\hat{\omega}_y & 0 \\ \hat{\omega}_y & 0 & -\hat{\omega}_y \\ 0 & \hat{\omega}_y & 0\end{bmatrix} + \begin{bmatrix}\hat{\omega}_z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hat{\omega}_z\end{bmatrix}\} = i\begin{bmatrix}\hat{\omega}_z & \frac{1}{\sqrt{2}}(\hat{\omega}_x-i\hat{\omega}_y) & 0 \\ \frac{1}{\sqrt{2}}(\hat{\omega}_x+i\hat{\omega}_y) & 0 & \frac{1}{\sqrt{2}}(\hat{\omega}_x-i\hat{\omega}_y) \\ 0 & \frac{1}{\sqrt{2}}(\hat{\omega}_x+i\hat{\omega}_y) & -\hat{\omega}_z\end{bmatrix}$

Cor. 3.1.3. $i\hat{\omega}\cdot\sigma(1) = i\begin{bmatrix}\hat{\omega}_z & \hat{\omega}_- & 0 \\ \hat{\omega}_+ & 0 & \hat{\omega}_- \\ 0 & \hat{\omega}_+ & -\hat{\omega}_z\end{bmatrix} \overset{\hat{\omega}_z=0}{=} i\begin{bmatrix}0 & \hat{\omega}_- & 0 \\ \hat{\omega}_+ & 0 & \hat{\omega}_- \\ 0 & \hat{\omega}_+ & 0\end{bmatrix}$

Cor. 3.1.4. $[i\hat{\omega}\cdot\sigma(1)]^2=-\begin{bmatrix}\hat{\omega}_z & \hat{\omega}_- & 0\\ \hat{\omega}_+ & 0 & \hat{\omega}_-\\ 0 & \hat{\omega}_+ & -\hat{\omega}_z\end{bmatrix}^2=-\begin{bmatrix}\frac{1}{2}(\hat{\omega}_z^2+1) & \hat{\omega}_z\hat{\omega}_- & \hat{\omega}_-^2\\ \hat{\omega}_z\hat{\omega}_+ & 2\hat{\omega}_+\hat{\omega}_- & -\hat{\omega}_z\hat{\omega}_-\\ \hat{\omega}_+^2 & -\hat{\omega}_z\hat{\omega}_+ & \frac{1}{2}(\hat{\omega}_z^2+1)\end{bmatrix}\overset{\hat{\omega}_z=0}{=}-\begin{bmatrix}\frac{1}{2} & 0 & \hat{\omega}_-^2\\ 0 & 1 & 0\\ \hat{\omega}_+^2 & 0 & \frac{1}{2}\end{bmatrix}$

Cor. 3.1.5. $e^{i\vec{\omega}\cdot\sigma(1)}\overset{\hat{\omega}_z=0}{=}1+i\hat{\omega}\cdot\sigma(1)sin\omega+[i\hat{\omega}\cdot\sigma(1)]^2(1-cos\omega)=1+isin\omega\begin{bmatrix}0 & \hat{\omega}_- & 0\\ \hat{\omega}_+ & 0 & \hat{\omega}_-\\ 0 & \hat{\omega}_+ & 0\end{bmatrix}-(1-cos\omega)\begin{bmatrix}\frac{1}{2} & 0 & \hat{\omega}_-^2\\ 0 & 1 & 0\\ \hat{\omega}_+^2 & 0 & \frac{1}{2}\end{bmatrix}$

Cor. 3.1.6. $e^{i\vec{\omega}\cdot\sigma(1)}\overset{\hat{\omega}_z=0}{=}1+isin\omega\begin{bmatrix}0 & \hat{\omega}_- & 0\\ \hat{\omega}_+ & 0 & \hat{\omega}_-\\ 0 & \hat{\omega}_+ & 0\end{bmatrix}-(1-cos\omega)\begin{bmatrix}\frac{1}{2} & 0 & \hat{\omega}_-^2\\ 0 & 1 & 0\\ \hat{\omega}_+^2 & 0 & \frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(1+cos\omega) & i\hat{\omega}_-sin\omega & -\hat{\omega}_-^2(1-cos\omega)\\ i\hat{\omega}_+sin\omega & cos\omega & i\hat{\omega}_-sin\omega\\ -\hat{\omega}_+^2(1-cos\omega) & i\hat{\omega}_+sin\omega & \frac{1}{2}(1+cos\omega)\end{bmatrix}$

Cor. 3.1.7. $e^{i\vec{\omega}\cdot\sigma(1)}=exp\{i\frac{[\sigma(1)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}=1+i[\sigma(1)\times\hat{p}]_z-[\sigma(1)\times\hat{p}]_z^2/(1+\hat{p}_z),0\le arccos\hat{p}_z\le\pi$

Cor. 3.1.8. $\sigma(1)\cdot\hat{p}=e^{i\vec{\omega}\cdot\sigma(1)}\sigma_z(1)e^{-i\vec{\omega}\cdot\sigma(1)}$

Cor. 3.1.9.

$$\begin{cases}\lambda(\hat{p},1;1)=e^{i\vec{\omega}\cdot\sigma(1)}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(1+cos\omega)\\ i\hat{\omega}_+sin\omega\\ -\hat{\omega}_+^2(1-cos\omega)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(1+\hat{p}_z)\\ \hat{p}_+\\ \hat{p}_+^2/(1+\hat{p}_z)\end{bmatrix}=\frac{1}{\hat{p}_-}\begin{bmatrix}\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\\ \hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z)\end{bmatrix},\lambda(-\hat{p},1;1)=\frac{\hat{p}_+}{\hat{p}_-}\lambda(\hat{p},-1;1)\\ \lambda(\hat{p},0;1)=e^{i\vec{\omega}\cdot\sigma(1)}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}i\hat{\omega}_-sin\omega\\ cos\omega\\ i\hat{\omega}_+sin\omega\end{bmatrix}=\begin{bmatrix}-\hat{p}_-\\ \hat{p}_z\\ \hat{p}_+\end{bmatrix},\lambda(-\hat{p},0;1)=-\lambda(\hat{p},0;1)\\ \lambda(\hat{p},-1;1)=e^{i\vec{\omega}\cdot\sigma(1)}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-\hat{\omega}_-^2(1-cos\omega)\\ i\hat{\omega}_-sin\omega\\ \frac{1}{2}(1+cos\omega)\end{bmatrix}=\begin{bmatrix}\hat{p}_-^2/(1+\hat{p}_z)\\ -\hat{p}_-\\ \frac{1}{2}(1+\hat{p}_z)\end{bmatrix}=\frac{1}{\hat{p}_+}\begin{bmatrix}\frac{1}{2}\hat{p}_-(1-\hat{p}_z)\\ -\hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1+\hat{p}_z)\end{bmatrix},\lambda(-\hat{p},-1;1)=\frac{\hat{p}_-}{\hat{p}_+}\lambda(\hat{p},1;1)\end{cases}$$

3.2 Concrete solution II of helicity $\sigma(1)\cdot\hat{p}$ eigenfunctions

Thm. 3.2.1. $\lambda(\hat{p},h;1)=\sqrt{C_2^{1-h}}\bar{\Gamma}(1)\overbrace{\lambda(\hat{p},\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{1}{2})}^{1+h}\otimes\overbrace{\lambda(\hat{p},-\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{1}{2})}^{1-h}$

3.3 Verification of orthogonality and completeness of helicity $\sigma(1)\cdot\hat{p}$ eigenfunctions

Cor. 3.3.1. $\lambda^+(\hat{p},h;1)\lambda(\hat{p},h';1)=\delta_{hh'}$

Cor. 3.3.2. $\lambda(\hat{p},1;1)\lambda^+(\hat{p},1;1)=\frac{1}{2}\begin{bmatrix}\frac{1}{2}(1+cos\omega)^2 & -i\hat{\omega}_-sin\omega(1+cos\omega) & -\hat{\omega}_-^2sin^2\omega\\ i\hat{\omega}_+sin\omega(1+cos\omega) & sin^2\omega & -i\hat{\omega}_-sin\omega(1-cos\omega)\\ -\hat{\omega}_+^2sin^2\omega & i\hat{\omega}_+sin\omega(1-cos\omega) & \frac{1}{2}(1-cos\omega)^2\end{bmatrix}$

Cor. 3.3.3. $\lambda(\hat{p},0;1)\lambda^+(\hat{p},0;1)=\begin{bmatrix}\frac{1}{2}sin^2\omega & i\hat{\omega}_-sin\omega cos\omega & \hat{\omega}_-^2sin^2\omega\\ -i\hat{\omega}_+sin\omega cos\omega & cos^2\omega & -i\hat{\omega}_-sin\omega cos\omega\\ \hat{\omega}_+^2sin^2\omega & i\hat{\omega}_+sin\omega cos\omega & \frac{1}{2}sin^2\omega\end{bmatrix}$

Cor. 3.3.4. $\lambda(\hat{p},-1;1)\lambda^+(\hat{p},-1;1)=\frac{1}{2}\begin{bmatrix}\frac{1}{2}(1-cos\omega)^2 & i\hat{\omega}_-sin\omega(1-cos\omega) & -\hat{\omega}_-^2sin^2\omega\\ -i\hat{\omega}_+sin\omega(1-cos\omega) & sin^2\omega & i\hat{\omega}_-sin\omega(1+cos\omega)\\ -\hat{\omega}_+^2sin^2\omega & -i\hat{\omega}_+sin\omega(1+cos\omega) & \frac{1}{2}(1+cos\omega)^2\end{bmatrix}$

Cor. 3.3.5. $\sum_{h=1}^{-1}\lambda(\hat{p},h;1)\lambda^+(\hat{p},h;1)=1$

Cor. 3.3.6. $\lambda_{\alpha_\varsigma}(\hat{p},-\varsigma;1)\lambda^+_{\alpha'_\varsigma}(\hat{p},-\varsigma;1)=\frac{1}{2}[(-1)^h(2-|h|)[S_m^+(1)\hat{p}\hat{p}^TS_m(1)]_{A_\varsigma A'_\varsigma}+h\sigma^k(1)_{A_\varsigma A'_\varsigma}\hat{p}_k+|h|\delta_{A_\varsigma A'_\varsigma}]$

3.4 Orthogonality and completeness of helicity $\sigma(1)\cdot\hat{p}$ eigenfunctions

Cor. 3.4.1. $\lambda^+(\hat{p},h;1)\lambda(\hat{p},h';1)=\delta_{hh'},\sum_{h=1}^{-1}\lambda(\hat{p},h;1)\lambda^+(\hat{p},h;1)=1,\sum_{h=1}^{-1}h\lambda(\hat{p},h;1)\lambda^+(\hat{p},h;1)=\sigma(1)\cdot\hat{p}$

3.5 Properties of helicity $\sigma(1)\cdot\hat{p}$ eigenfunctions derivative

Cor. 3.5.1. $\lambda(\hat{p},\frac{1}{2})=\frac{1}{\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_+\end{bmatrix},\lambda(-\hat{p},\frac{1}{2})=\frac{1}{\sqrt{1+\hat{p}_z}}\begin{bmatrix}-\hat{p}_-\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix}$

Lem. 3.5.1. $\lambda(\hat{p},1;1)=\bar{\Gamma}(1)\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})$

Thm. 3.5.1. $\tilde{\partial}_i\lambda(\hat{p},1;1)=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}\lambda(\hat{p},1;1)+\frac{1}{p(1+\hat{p}_z)}\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\ \frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\ (\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$

Proof: $\tilde{\partial}_i\lambda(\hat{p},1;1)=\bar{\Gamma}(1)[\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})+\lambda(\hat{p},\frac{1}{2})\otimes\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})]$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}\lambda(\hat{p},1;1)+\frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}}\bar{\Gamma}(1)[\begin{bmatrix}\delta_{iz}+\hat{p}_i\\\delta_{ix}+i\delta_{iy}\end{bmatrix}\otimes\lambda(\hat{p},\tfrac{1}{2})+\lambda(\hat{p},\tfrac{1}{2})\otimes\begin{bmatrix}\delta_{iz}+\hat{p}_i\\\delta_{ix}+i\delta_{iy}\end{bmatrix}]$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}\lambda(\hat{p},1;1)+\frac{\sqrt{2}}{p\sqrt{1+\hat{p}_z}}\bar{\Gamma}(1)\begin{bmatrix}\delta_{iz}+\hat{p}_i\\\delta_{ix}+i\delta_{iy}\end{bmatrix}\otimes\lambda(\hat{p},\tfrac{1}{2})$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}\lambda(\hat{p},1;1)+\frac{1}{p(1+\hat{p}_z)}\bar{\Gamma}(1)\begin{bmatrix}\delta_{iz}+\hat{p}_i\\\delta_{ix}+i\delta_{iy}\end{bmatrix}\otimes\begin{bmatrix}1+\hat{p}_z\\\hat{p}_x+i\hat{p}_y\end{bmatrix}$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}\lambda(\hat{p},1;1)+\frac{1}{p(1+\hat{p}_z)}\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&\sqrt{1}&0\\0&0&0&\sqrt{2}\end{bmatrix}\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)\\(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)\\(\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}\lambda(\hat{p},1;1)+\frac{1}{p(1+\hat{p}_z)}\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\\frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\(\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$ □

Cor. 3.5.2. $\lambda^+(\hat{p},-1;1)\tilde{\partial}_i\lambda(\hat{p},1;1)=0$

Proof: $\lambda^+(\hat{p},-1;1)\tilde{\partial}_i\lambda(\hat{p},1;1)$

$$=0+\frac{1}{\hat{p}_-}\begin{bmatrix}\frac{1}{2}\hat{p}_+(1-\hat{p}_z)\\-\hat{p}_+\hat{p}_-\\\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\end{bmatrix}^T\frac{1}{p(1+\hat{p}_z)}\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\\frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\(\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$=\frac{1}{2p\hat{p}_-(1+\hat{p}_z)}\begin{bmatrix}\hat{p}_+(1-\hat{p}_z)\\-2\hat{p}_+\hat{p}_-\\\hat{p}_-(1+\hat{p}_z)\end{bmatrix}^T\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\\frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\(\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$=\frac{1}{2p\hat{p}_-(1+\hat{p}_z)}\begin{bmatrix}1\\1\\1\end{bmatrix}^T\begin{bmatrix}2\hat{p}_+\hat{p}_-\hat{p}_+(\delta_{iz}+\hat{p}_i)\\-\sqrt{2}\hat{p}_+\hat{p}_-[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\\sqrt{2}\hat{p}_+\hat{p}_-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})\end{bmatrix}$$

$$=\frac{\hat{p}_+}{\sqrt{2}p(1+\hat{p}_z)}\begin{bmatrix}1\\1\\1\end{bmatrix}^T\begin{bmatrix}(\hat{p}_x+i\hat{p}_y)(\delta_{iz}+\hat{p}_i)\\-[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})\end{bmatrix}$$

$=0$ □

Cor. 3.5.3. $\lambda^+(\hat{p},1;1)\tilde{\partial}_k\lambda(\hat{p},1;1)=\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}$

Proof: $\lambda^+(\hat{p},1;1)\tilde{\partial}_i\lambda(\hat{p},1;1)$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}+\frac{1}{\hat{p}_+}\begin{bmatrix}\frac{1}{2}\hat{p}_+(1+\hat{p}_z)\\\hat{p}_+\hat{p}_-\\\frac{1}{2}\hat{p}_-(1-\hat{p}_z)\end{bmatrix}^T\frac{1}{p(1+\hat{p}_z)}\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\\frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\(\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}+\frac{1}{p\hat{p}_+(1+\hat{p}_z)}\begin{bmatrix}1\\1\\1\end{bmatrix}^T\begin{bmatrix}\frac{1}{2}\hat{p}_+(1+\hat{p}_z)(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\\frac{1}{\sqrt{2}}\hat{p}_+\hat{p}_-[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\\frac{1}{2}\hat{p}_-(1-\hat{p}_z)(\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}+\frac{1}{2p(1+\hat{p}_z)}\begin{bmatrix}1\\1\\1\end{bmatrix}^T\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)^2\\(\hat{p}_x-i\hat{p}_y)[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\(\hat{p}_x-i\hat{p}_y)(1-\hat{p}_z)(\delta_{ix}+i\delta_{iy})\end{bmatrix}$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}+\frac{1}{2p(1+\hat{p}_z)}\begin{bmatrix}1\\1\\1\end{bmatrix}^T\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)^2\\(\hat{p}_x-i\hat{p}_y)(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(1-\hat{p}_z^2)]\\(\hat{p}_x-i\hat{p}_y)(1-\hat{p}_z)(\delta_{ix}+i\delta_{iy})\end{bmatrix}$$

$$=-\frac{\delta_{iz}+\hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)}+\frac{1}{p(1+\hat{p}_z)}\begin{bmatrix}1\\1\\1\end{bmatrix}^T\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\0\\(\hat{p}_x-i\hat{p}_y)(\delta_{ix}+i\delta_{iy})\end{bmatrix}$$

$$=\frac{(\hat{p}_x-i\hat{p}_y)(\delta_{ix}+i\delta_{iy})+\delta_{iz}\hat{p}_z-\hat{p}_i}{p(1+\hat{p}_z)}=\frac{-i\hat{p}_y\delta_{ix}+i\hat{p}_x\delta_{iy}}{p(1+\hat{p}_z)}$$ □

3.6 Summary of helicity $\sigma(1)\cdot\hat{p}$ eigenfunctions derivative properties

Cor. 3.6.1.
$$\begin{cases}\lambda^{+}(\hat{p},1;1)\tilde{\partial}_k\lambda(\hat{p},1;1)=\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda^{+}(\hat{p},1;1)\tilde{\partial}_k\lambda(\hat{p},0;1)=\frac{\sqrt{2}\hat{p}_-\hat{p}_k+\sqrt{2}\hat{p}_-\delta_{kz}-(1+\hat{p}_z)(\delta_{kx}-i\delta_{ky})}{\sqrt{2}p(1+\hat{p}_z)}\\ \lambda^{+}(\hat{p},1;1)\tilde{\partial}_k\lambda(\hat{p},-1;1)=0\\ \lambda^{+}(\hat{p},0;1)\tilde{\partial}_k\lambda(\hat{p},0;1)=0\\ \lambda^{+}(\hat{p},-1;1)\tilde{\partial}_k\lambda(\hat{p},1;1)=0\\ \lambda^{+}(\hat{p},-1;1)\tilde{\partial}_k\lambda(\hat{p},0;1)=-\frac{\sqrt{2}\hat{p}_+\hat{p}_k+\sqrt{2}\hat{p}_+\delta_{kz}-(1+\hat{p}_z)(\delta_{kx}+i\delta_{ky})}{\sqrt{2}p(1+\hat{p}_z)}\\ \lambda^{+}(\hat{p},-1;1)\tilde{\partial}_k\lambda(\hat{p},-1;1)=-\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\end{cases}$$

4 Analysis of helicity $\gamma\cdot\hat{p}$ eigenfunctions

4.1 Helicity $\gamma\cdot\hat{p}$ eigenfunctions

Cor. 4.1.1. $S_m(1)=\frac{1}{\sqrt{2}}\begin{bmatrix} i & 0 & -i\\ -1 & 0 & -1\\ 0 & -i\sqrt{2} & 0\end{bmatrix}, S_m^{+}(1)=\frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 & 0\\ 0 & 0 & i\sqrt{2}\\ i & -1 & 0\end{bmatrix}, S_m(1)S_m^{+}(1)=S_m^{+}(1)S_m(1)=I_3$

Cor. 4.1.2.
$$\begin{cases}\lambda(\hat{p},1;1)=e^{i\vec{\omega}\cdot\sigma(1)}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(1+cos\omega)\\ i\hat{\omega}_+sin\omega\\ -\hat{\omega}_+^2(1-cos\omega)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(1+\hat{p}_z)\\ \hat{p}_+\\ \hat{p}_+^2/(1+\hat{p}_z)\end{bmatrix}=\frac{1}{\hat{p}_-}\begin{bmatrix}\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\\ \hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z)\end{bmatrix}, \lambda(-\hat{p},1;1)=\frac{\hat{p}_+}{\hat{p}_-}\lambda(\hat{p},-1;1)\\ \lambda(\hat{p},0;1)=e^{i\vec{\omega}\cdot\sigma(1)}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}i\hat{\omega}_-sin\omega\\ cos\omega\\ i\hat{\omega}_+sin\omega\end{bmatrix}=\begin{bmatrix}-\hat{p}_-\\ \hat{p}_z\\ \hat{p}_+\end{bmatrix}, \lambda(-\hat{p},0;1)=-\lambda(\hat{p},0;1)\\ \lambda(\hat{p},-1;1)=e^{i\vec{\omega}\cdot\sigma(1)}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}-\hat{\omega}_-^2(1-cos\omega)\\ i\hat{\omega}_-sin\omega\\ \frac{1}{2}(1+cos\omega)\end{bmatrix}=\begin{bmatrix}\hat{p}_-^2/(1+\hat{p}_z)\\ -\hat{p}_-\\ \frac{1}{2}(1+\hat{p}_z)\end{bmatrix}=\frac{1}{\hat{p}_+}\begin{bmatrix}\frac{1}{2}\hat{p}_-(1-\hat{p}_z)\\ -\hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1+\hat{p}_z)\end{bmatrix}, \lambda(-\hat{p},-1;1)=\frac{\hat{p}_-}{\hat{p}_+}\lambda(\hat{p},1;1)\end{cases}$$

Cor. 4.1.3. $\gamma\cdot\hat{p}\lambda_m(\hat{p},h;1)=h\lambda_m(\hat{p},h;1), \lambda_m(\hat{p},h;1)=S_m(1)\lambda(\hat{p},h;1), h=-1,0,1$

Cor. 4.1.4.
$$\begin{cases}\lambda_m(\hat{p},1;1)=S_m(1)\lambda(\hat{p},1;1)=e^{i\vec{\omega}\cdot\gamma}\frac{1}{\sqrt{2}}\begin{bmatrix}i\\-1\\0\end{bmatrix}=\frac{1}{2\hat{p}_-}\begin{bmatrix}i(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\ -1(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\ -2i(\hat{p}_+\hat{p}_-)\end{bmatrix}, \lambda_m(-\hat{p},1;1)=\frac{\hat{p}_+}{\hat{p}_-}\lambda_m(\hat{p},-1;1)\\ \lambda_m(\hat{p},0;1)=S_m(1)\lambda(\hat{p},0;1)=e^{i\vec{\omega}\cdot\gamma}\begin{bmatrix}0\\0\\-i\end{bmatrix}=-i\begin{bmatrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{bmatrix}=-i\hat{p}, \lambda_m(-\hat{p},0;1)=-\lambda_m(\hat{p},0;1)\\ \lambda_m(\hat{p},-1;1)=S_m(1)\lambda(\hat{p},-1;1)=e^{i\vec{\omega}\cdot\gamma}\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\-1\\0\end{bmatrix}=\frac{1}{2\hat{p}_+}\begin{bmatrix}-i(\hat{p}_x\hat{p}_z+i\hat{p}_y)\\ -1(\hat{p}_x+i\hat{p}_y\hat{p}_z)\\ 2i(\hat{p}_+\hat{p}_-)\end{bmatrix}, \lambda_m(-\hat{p},-1;1)=\frac{\hat{p}_-}{\hat{p}_+}\lambda_m(\hat{p},1;1)\end{cases}$$

Cor. 4.1.5. $\gamma\cdot\hat{p}=e^{i\vec{\omega}\cdot\gamma}\gamma_z e^{-i\vec{\omega}\cdot\gamma}$

Lem. 4.1.1. $\lambda_m^{+}(-\hat{p},1;1)\begin{bmatrix}-\hat{p}_z\\ -i\\ \hat{p}_x\end{bmatrix}=0, \lambda_m^{+}(-\hat{p},1;1)\begin{bmatrix}i\\ -\hat{p}_z\\ \hat{p}_y\end{bmatrix}=0, \lambda_m^{+}(-\hat{p},-1;1)\begin{bmatrix}-\hat{p}_z\\ i\\ \hat{p}_x\end{bmatrix}=0, \lambda_m^{+}(-\hat{p},-1;1)\begin{bmatrix}-i\\ -\hat{p}_z\\ \hat{p}_y\end{bmatrix}=0$

Cor. 4.1.6.
$$\begin{cases}\lambda_m(\hat{p},1;1)=\frac{1}{2\hat{p}_-}\{-i\hat{p}_x\begin{bmatrix}-\hat{p}_z\\ -i\\ \hat{p}_x\end{bmatrix}-i\hat{p}_y\begin{bmatrix}i\\ -\hat{p}_z\\ \hat{p}_y\end{bmatrix}\}\\ \lambda_m(\hat{p},0;1)=-i\begin{bmatrix}\hat{p}_x\\ \hat{p}_y\\ \hat{p}_z\end{bmatrix}=-i\hat{p}\\ \lambda_m(\hat{p},-1;1)=\frac{1}{2\hat{p}_+}\{i\hat{p}_x\begin{bmatrix}-\hat{p}_z\\ i\\ \hat{p}_x\end{bmatrix}+i\hat{p}_y\begin{bmatrix}-i\\ -\hat{p}_z\\ \hat{p}_y\end{bmatrix}\}\end{cases}$$

Cor. 4.1.7.
$$\begin{cases}\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1;1)=\frac{1}{\sqrt{2}}\begin{bmatrix}i\\-1\\0\end{bmatrix}\\ \lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},0;1)=\begin{bmatrix}0\\0\\-i\end{bmatrix}\\ \lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1;1)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\-1\\0\end{bmatrix}\end{cases}$$

4.2 Basic properties of helicity $\gamma\cdot\hat{p}$ eigenfunctions

Cor. 4.2.1. $\lambda_m(\hat{p},-1;1)=\lambda_m^{*}(\hat{p},1;1), \lambda_m(\hat{p},0;1)=-\lambda_m^{*}(\hat{p},0;1), \lambda_m(\hat{p},1;1)=\lambda_m^{*}(\hat{p},-1;1)$

Cor. 4.2.2.
$$\begin{cases}\lambda_m(\hat{p},-1;1)\times\lambda_m(\hat{p},0;1)=-\lambda_m(\hat{p},-1;1), \lambda_m(\hat{p},0;1)\times\lambda_m(\hat{p},1;1)=-\lambda_m(\hat{p},1;1), \lambda_m(\hat{p},1;1)\times\lambda_m(\hat{p},-1;1)=\lambda_m(\hat{p},0;1),\\ \lambda_m(\hat{p},\varsigma;1)\cdot\lambda_m(\hat{p},\varsigma;1)=0, \lambda_m(\hat{p},0;1)\cdot\lambda_m(\hat{p},\varsigma;1)=0, \lambda_m(\hat{p},h;1)\times\lambda_m(\hat{p},h;1)=0\\ \lambda_m(\hat{p},0;1)\cdot\lambda_m(\hat{p},0;1)=-1, \lambda_m(\hat{p},\varsigma;1)\cdot\lambda_m(\hat{p},-\varsigma;1)=1\end{cases}$$

4.3 Orthogonality and completeness ofhelicity $\gamma\cdot\hat{p}$ eigenfunctions

Cor. 4.3.1. $\lambda_m^{+}(\hat{p},h)\lambda_m(\hat{p},h')=\delta_{hh'}, \sum_{h=1}^{-1}\lambda_m(\hat{p},h)\lambda_m^{+}(\hat{p},h)=1, \sum_{h=1}^{-1}h\lambda_m(\hat{p},h)\lambda_m^{+}(\hat{p},h)=\gamma\cdot\hat{p}$

4.4 Complex properties of helicity $\gamma\cdot\hat{p}$ eigenfunctions

Cor. 4.4.1. $$\begin{cases}\gamma\lambda_m(\hat{p},1;1)=\frac{1}{2\hat{p}_-}\gamma\begin{bmatrix}i(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\-1(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\-2i(\hat{p}_+\hat{p}_-)\end{bmatrix}=\frac{1}{2\hat{p}_-}\{\begin{bmatrix}0\\-2(\hat{p}_+\hat{p}_-)\\-i(\hat{p}_x-i\hat{p}_y\hat{p}_z)\end{bmatrix},\begin{bmatrix}2(\hat{p}_+\hat{p}_-)\\0\\(\hat{p}_x\hat{p}_z-i\hat{p}_y)\end{bmatrix},\begin{bmatrix}i(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\-1(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\0\end{bmatrix}\}\\ \gamma\lambda_m(\hat{p},0;1)=-i\gamma\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\\hat{p}_z\end{bmatrix}=\{\begin{bmatrix}0\\-\hat{p}_z\\\hat{p}_y\end{bmatrix},\begin{bmatrix}\hat{p}_z\\0\\-\hat{p}_x\end{bmatrix},\begin{bmatrix}-\hat{p}_y\\\hat{p}_x\\0\end{bmatrix}\}\\ \gamma\lambda_m(\hat{p},-1;1)=\frac{1}{2\hat{p}_+}\gamma\begin{bmatrix}-i(\hat{p}_x\hat{p}_z+i\hat{p}_y)\\-1(\hat{p}_x+i\hat{p}_y\hat{p}_z)\\2i(\hat{p}_+\hat{p}_-)\end{bmatrix}=\frac{1}{2\hat{p}_+}\{\begin{bmatrix}0\\2(\hat{p}_+\hat{p}_-)\\-i(\hat{p}_x+i\hat{p}_y\hat{p}_z)\end{bmatrix},\begin{bmatrix}-2(\hat{p}_+\hat{p}_-)\\0\\-(\hat{p}_x\hat{p}_z+i\hat{p}_y)\end{bmatrix},\begin{bmatrix}i(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\1(\hat{p}_x\hat{p}_z+i\hat{p}_y)\\0\end{bmatrix}\}\end{cases}$$

Cor. 4.4.2. $$\begin{cases}\lambda_m^+(\hat{p},1;1)\gamma\lambda_m(\hat{p},1;1)=\{\hat{p}_x,\hat{p}_y,\hat{p}_z\}=1\cdot\hat{p}\\ \lambda_m^+(\hat{p},0;1)\gamma\lambda_m(\hat{p},0;1)=\{0,0,0\}=0\cdot\hat{p}\\ \lambda_m^+(\hat{p},-1;1)\gamma\lambda_m(\hat{p},-1;1)=\{-\hat{p}_x,-\hat{p}_y,-\hat{p}_z\}=-1\cdot\hat{p}\end{cases}\qquad\begin{cases}\lambda_m^+(-\hat{p},1;1)\gamma\lambda_m(\hat{p},1;1)=0\\ \lambda_m^+(-\hat{p},0;1)\gamma\lambda_m(\hat{p},0;1)=0\\ \lambda_m^+(-\hat{p},-1;1)\gamma\lambda_m(\hat{p},-1;1)=0\end{cases}$$

Cor. 4.4.3. $\lambda_m^+(\hat{p},h)\gamma\lambda_m(\hat{p},h)=h\{\hat{p}_x,\hat{p}_y,\hat{p}_z\}=h\hat{p},\lambda_m^+(-\hat{p},h)\gamma\lambda_m(\hat{p},h)=0,\lambda_m^+(\hat{p},-h)\gamma\lambda_m(\hat{p},h)=0$

Cor. 4.4.4. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b=-2|\vec{p}|^2\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma;1)$

Proof: $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b$
$=p_{\alpha_\varsigma}p_{\alpha'_\varsigma}+\varsigma\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k|\vec{p}|-\delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$=p_{\alpha_\varsigma}p_{\alpha'_\varsigma}+\varsigma|\vec{p}|\gamma^k{}_{\alpha_\varsigma}{}^{\beta_\varsigma}p_k\delta_{\beta_\varsigma\alpha'_\varsigma}-\delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$=\lambda_{m\alpha_\varsigma}(\hat{p},0;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},0;1)|\vec{p}|^2+\varsigma|\vec{p}|\gamma^k{}_{\alpha_\varsigma}{}^{\beta_\varsigma}p_k\sum_{h=1}^{-1}\lambda_{m\beta_\varsigma}(\hat{p},h)\lambda^+_{m\alpha'_\varsigma}(\hat{p},h)-\delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$=\lambda_{m\alpha_\varsigma}(\hat{p},0;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},0;1)|\vec{p}|^2+\varsigma|\vec{p}|[\varsigma|\vec{p}|\lambda_{m\beta_\varsigma}(\hat{p},\varsigma;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},\varsigma;1)-\varsigma|\vec{p}|\lambda_{m\beta_\varsigma}(\hat{p},-\varsigma;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma;1)]-\delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$=|\vec{p}|^2\sum_{h=1}^{-1}\lambda_{m\alpha_\varsigma}(\hat{p},h)\lambda^+_{m\alpha'_\varsigma}(\hat{p},h)-\delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2-2|\vec{p}|^2\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma;1)$
$=-2|\vec{p}|^2\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma;1)$ □

Cor. 4.4.5. $\lambda_{m\alpha_\varsigma}(\hat{p},h)\lambda^+_{m\alpha'_\varsigma}(\hat{p},h)=\frac{1}{2}[(-1)^h(2-|h|)\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma}+h\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}\hat{p}_k+|h|\delta_{\alpha_\varsigma\alpha'_\varsigma}]$

Cor. 4.4.6. $$\begin{cases}\lambda_{m\alpha_\varsigma}(\hat{p},1;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},1;1)=\frac{1}{2}(-\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma}+\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}\hat{p}_k+\delta_{\alpha_\varsigma\alpha'_\varsigma})\\ \lambda_{m\alpha_\varsigma}(\hat{p},0;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},0;1)=\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma}\\ \lambda_{m\alpha_\varsigma}(\hat{p},-1;1)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-1;1)=\frac{1}{2}(-\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma}-\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}\hat{p}_k+\delta_{\alpha_\varsigma\alpha'_\varsigma})\end{cases}$$

4.5 Derivative properties 1 of helicity $\gamma\cdot\hat{p}$ eigenfunctions

Lem. 4.5.1. $\lambda_m^+(-\hat{p},1;1)\begin{bmatrix}-\hat{p}_z\\-i\\\hat{p}_x\end{bmatrix}=0,\lambda_m^+(-\hat{p},1;1)\begin{bmatrix}i\\-\hat{p}_z\\\hat{p}_y\end{bmatrix}=0,\lambda_m^+(-\hat{p},-1;1)\begin{bmatrix}-\hat{p}_z\\i\\\hat{p}_x\end{bmatrix}=0,\lambda_m^+(-\hat{p},-1;1)\begin{bmatrix}-i\\-\hat{p}_z\\\hat{p}_y\end{bmatrix}=0$

Lem. 4.5.2. $$\begin{cases}\tilde{\partial}_k\lambda_m(\hat{p},1;1)=\frac{1}{2p_-}\begin{bmatrix}-i\hat{p}_x(\hat{p}_k\hat{p}_z-\delta_{kz})+i\delta_{kx}(\hat{p}_z-\frac{\hat{p}_x\hat{p}_z-i\hat{p}_y}{\hat{p}_x-i\hat{p}_y})+\delta_{ky}(1-\frac{\hat{p}_x\hat{p}_z-i\hat{p}_y}{\hat{p}_x-i\hat{p}_y})\\-i\hat{p}_y(\hat{p}_k\hat{p}_z-\delta_{kz})-\delta_{kx}(1-\frac{\hat{p}_x-i\hat{p}_y\hat{p}_z}{\hat{p}_x-i\hat{p}_y})+i\delta_{ky}(\hat{p}_z-\frac{\hat{p}_x-i\hat{p}_y\hat{p}_z}{\hat{p}_x-i\hat{p}_y})\\i\hat{p}_k(\hat{p}_x^2+\hat{p}_y^2)-i(\hat{p}_x-i\hat{p}_y)(\delta_{kx}+i\delta_{ky})\end{bmatrix}\\ \tilde{\partial}_k\lambda_m(\hat{p},-1;1)=\frac{1}{2p_+}\begin{bmatrix}i\hat{p}_x(\hat{p}_k\hat{p}_z-\delta_{kz})-i\delta_{kx}(\hat{p}_z-\frac{\hat{p}_x\hat{p}_z+i\hat{p}_y}{\hat{p}_x+i\hat{p}_y})+\delta_{ky}(1-\frac{\hat{p}_x\hat{p}_z+i\hat{p}_y}{\hat{p}_x+i\hat{p}_y})\\+i\hat{p}_y(\hat{p}_k\hat{p}_z-\delta_{kz})-\delta_{kx}(1-\frac{\hat{p}_x+i\hat{p}_y\hat{p}_z}{\hat{p}_x+i\hat{p}_y})-i\delta_{ky}(\hat{p}_z-\frac{\hat{p}_x+i\hat{p}_y\hat{p}_z}{\hat{p}_x+i\hat{p}_y})\\-i\hat{p}_k(\hat{p}_x^2+\hat{p}_y^2)+i(\hat{p}_x+i\hat{p}_y)(\delta_{kx}-i\delta_{ky})\end{bmatrix}\end{cases}$$

Proof: $\tilde{\partial}_k\lambda_m(\hat{p},1;1)=\tilde{\partial}_k\frac{1}{2pp_-}\begin{bmatrix}i(p_xp_z-ipp_y)\\-1(pp_x-ip_yp_z)\\-2i(p_+p_-)\end{bmatrix}$
$=(2pp_-\tilde{\partial}_k\frac{1}{2pp_-})\frac{1}{2pp_-}\begin{bmatrix}i(p_xp_z-ipp_y)\\-1(pp_x-ip_yp_z)\\-2i(p_+p_-)\end{bmatrix}+\frac{1}{2pp_-}\tilde{\partial}_k\begin{bmatrix}i(p_xp_z-ipp_y)\\-1(pp_x-ip_yp_z)\\-2i(p_+p_-)\end{bmatrix}$
$=-\frac{1}{2p_-}[2\hat{p}_k\hat{p}_-+\sqrt{2}(\delta_{kx}-i\delta_{ky})]\lambda_m(\hat{p},1;1)+\frac{1}{2p_-}\begin{bmatrix}i(\delta_{kx}\hat{p}_z+\hat{p}_x\delta_{kz}-i\delta_{ky}-i\hat{p}_k\hat{p}_y)\\-1(\hat{p}_k\hat{p}_x+\delta_{kx}-i\delta_{ky}\hat{p}_z-i\hat{p}_y\delta_{kz})\\-2i(\hat{p}_x\delta_{kx}+\hat{p}_y\delta_{ky})\end{bmatrix}$
$=-\frac{1}{2p_-}[2\hat{p}_k\hat{p}_-+\sqrt{2}(\delta_{kx}-i\delta_{ky})]\frac{1}{2\hat{p}_-}\begin{bmatrix}i(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\-1(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\-2i(\hat{p}_+\hat{p}_-)\end{bmatrix}+\frac{1}{2p_-}\begin{bmatrix}i(\delta_{kx}\hat{p}_z+\hat{p}_x\delta_{kz}-i\delta_{ky}-i\hat{p}_k\hat{p}_y)\\-1(\hat{p}_k\hat{p}_x+\delta_{kx}-i\delta_{ky}\hat{p}_z-i\hat{p}_y\delta_{kz})\\-2i(\hat{p}_x\delta_{kx}+\hat{p}_y\delta_{ky})\end{bmatrix}$
$=-\frac{1}{2p_-}[\hat{p}_k+\frac{1}{\sqrt{2}\hat{p}_-}(\delta_{kx}-i\delta_{ky})]\begin{bmatrix}i(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\-1(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\-2i(\hat{p}_+\hat{p}_-)\end{bmatrix}+\frac{1}{2p_-}\begin{bmatrix}i(\delta_{kx}\hat{p}_z+\hat{p}_x\delta_{kz}-i\delta_{ky}-i\hat{p}_k\hat{p}_y)\\-1(\hat{p}_k\hat{p}_x+\delta_{kx}-i\delta_{ky}\hat{p}_z-i\hat{p}_y\delta_{kz})\\-2i(\hat{p}_x\delta_{kx}+\hat{p}_y\delta_{ky})\end{bmatrix}$
$=\frac{1}{2p_-}\begin{bmatrix}-i\hat{p}_x(\hat{p}_k\hat{p}_z-\delta_{kz})+i\delta_{kx}(\hat{p}_z-\frac{\hat{p}_x\hat{p}_z-i\hat{p}_y}{\hat{p}_x-i\hat{p}_y})+\delta_{ky}(1-\frac{\hat{p}_x\hat{p}_z-i\hat{p}_y}{\hat{p}_x-i\hat{p}_y})\\-i\hat{p}_y(\hat{p}_k\hat{p}_z-\delta_{kz})-\delta_{kx}(1-\frac{\hat{p}_x-i\hat{p}_y\hat{p}_z}{\hat{p}_x-i\hat{p}_y})+i\delta_{ky}(\hat{p}_z-\frac{\hat{p}_x-i\hat{p}_y\hat{p}_z}{\hat{p}_x-i\hat{p}_y})\\i\hat{p}_k(\hat{p}_x^2+\hat{p}_y^2)-i(\hat{p}_x-i\hat{p}_y)(\delta_{kx}+i\delta_{ky})\end{bmatrix}$ □

Cor. 4.5.1. $\begin{cases}\tilde{\partial}_x\lambda_m(\hat{p},-\varsigma;1)=\frac{1}{2p_{+\varsigma}}\{i\varsigma\hat{p}_x\hat{p}_z\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\\hat{p}_z\end{bmatrix}+i\varsigma\begin{bmatrix}-\hat{p}_z\\i\varsigma\\\hat{p}_x\end{bmatrix}-\sqrt{2}\lambda_m(\hat{p},-\varsigma;1)\}\\ \tilde{\partial}_y\lambda_m(\hat{p},-\varsigma;1)=\frac{1}{2p_+}\{i\varsigma\hat{p}_y\hat{p}_z\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\\hat{p}_z\end{bmatrix}+i\varsigma\begin{bmatrix}-i\varsigma\\-\hat{p}_z\\\hat{p}_y\end{bmatrix}-i\varsigma\sqrt{2}\lambda_m(\hat{p},-\varsigma;1)\}\\ \tilde{\partial}_z\lambda_m(\hat{p},-\varsigma;1)=-i\varsigma p_{-\varsigma}\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\\hat{p}_z\end{bmatrix}\end{cases}$

Cor. 4.5.2. $\lambda_m^+(\hat{p},-\varsigma;1)=\frac{1}{2\hat{p}_{-\varsigma}}\begin{bmatrix}i\varsigma(\hat{p}_x\hat{p}_z-i\varsigma\hat{p}_y)\\-1(\hat{p}_x-i\varsigma\hat{p}_y\hat{p}_z)\\-2i\varsigma(\hat{p}_+\hat{p}_-)\end{bmatrix},\lambda_m(\hat{p},-\varsigma;1)=\frac{1}{2\hat{p}_{+\varsigma}}\begin{bmatrix}-i\varsigma(\hat{p}_x\hat{p}_z+i\varsigma\hat{p}_y)\\-1(\hat{p}_x+i\varsigma\hat{p}_y\hat{p}_z)\\2i\varsigma(\hat{p}_+\hat{p}_-)\end{bmatrix}$

Cor. 4.5.3. $\lambda_m^+(\hat{p},-\varsigma;1)\tilde{\partial}_x\lambda_m(\hat{p},-\varsigma;1)=\frac{i\varsigma\hat{p}_y}{p(1+\hat{p}_z)},\lambda_m^+(\hat{p},-\varsigma;1)\tilde{\partial}_y\lambda_m(\hat{p},-\varsigma;1)=\frac{-i\varsigma\hat{p}_x}{p(1+\hat{p}_z)},\lambda_m^+(\hat{p},-\varsigma;1)\tilde{\partial}_z\lambda_m(\hat{p},-\varsigma;1)=0$
$\lambda_m^+(\hat{p},-\varsigma;1)\hat{p}\cdot\tilde{\nabla}\lambda_m(\hat{p},-\varsigma;1)=0$

Cor. 4.5.4. $\begin{cases}\lambda_m^+(\hat{p},1;1)\tilde{\partial}_k\lambda_m(\hat{p},1;1)=\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda_m^+(\hat{p},-1;1)\tilde{\partial}_k\lambda_m(\hat{p},-1;1)=-\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda_m^+(\hat{p},0;1)\tilde{\partial}_k\lambda_m(\hat{p},0;1)=0\\ \lambda_m^+(\hat{p},-1;1)\tilde{\partial}_k\lambda_m(\hat{p},1;1)=0\\ \lambda_m^+(\hat{p},1;1)\tilde{\partial}_k\lambda_m(\hat{p},-1;1)=0\end{cases}$

4.6 Derivative properties 2 helicity $\gamma\cdot\hat{p}$ eigenfunctions?

Cor. 4.6.1. $\begin{cases}(\gamma_y\tilde{\partial}_z-\gamma_z\tilde{\partial}_y)\lambda_m(\hat{p},-\varsigma;1)=\frac{\varsigma}{2p_{+\varsigma}}\{\hat{p}_x\hat{p}_z\hat{p}-\begin{bmatrix}-\hat{p}_z\\i\varsigma\\\hat{p}_x\end{bmatrix}+i\sqrt{2}\gamma_z\lambda_m(\hat{p},-\varsigma;1)\}\\ (\gamma_z\tilde{\partial}_x-\gamma_x\tilde{\partial}_z)\lambda_m(\hat{p},-\varsigma;1)=\frac{\varsigma}{2p_{+\varsigma}}\{\hat{p}_y\hat{p}_z\hat{p}-\begin{bmatrix}-i\varsigma\\-\hat{p}_z\\\hat{p}_y\end{bmatrix}-\varsigma\sqrt{2}\gamma_z\lambda_m(\hat{p},-\varsigma;1)\}\\ (\gamma_x\tilde{\partial}_y-\gamma_y\tilde{\partial}_x)\lambda_m(\hat{p},-\varsigma;1)=\frac{\varsigma}{2p_{+\varsigma}}\{\hat{p}_z\hat{p}_z\hat{p}+\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\\hat{p}_z\end{bmatrix}-i\sqrt{2}(\gamma_x+i\varsigma\gamma_y)\lambda_m(\hat{p},-\varsigma;1)\}\end{cases}$

Cor. 4.6.2. $\begin{cases}\lambda_m^+(\hat{p},-\varsigma;1)(\gamma_x\tilde{\partial}_y-\gamma_y\tilde{\partial}_x)\lambda_m(\hat{p},-\varsigma;1)=\frac{i\hat{p}_z}{p(1+\hat{p}_z)}+\frac{i}{p(1+\hat{p}_z)}\\ \lambda_m^+(\hat{p},-\varsigma;1)(\gamma_y\tilde{\partial}_z-\gamma_z\tilde{\partial}_y)\lambda_m(\hat{p},-\varsigma;1)=\frac{i\hat{p}_x}{p(1+\hat{p}_z)}\\ \lambda_m^+(\hat{p},-\varsigma;1)(\gamma_z\tilde{\partial}_x-\gamma_x\tilde{\partial}_z)\lambda_m(-\hat{p},-\varsigma;1)=\frac{i\hat{p}_y}{p(1+\hat{p}_z)}\end{cases}$

Cor. 4.6.3. $\lambda_m^+(-\hat{p},-\varsigma;1)(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma;1)=0,\lambda_m^+(\hat{p},-\varsigma;1)(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\lambda_m(-\hat{p},-\varsigma;1)=0$

Cor. 4.6.4. $\begin{cases}\tilde{\partial}_k\lambda_m(\hat{p},1;1)|_{\hat{p}_z\to1}=0\end{cases}$

5 Analysis of helicity $\sigma(2)\cdot\hat{p}$ eigenfunctions

5.1 Spin-2 Lorentz transformation $e^{i\omega\cdot\sigma(2)}$

Cor. 5.1.1. $\sigma(2)=(\frac{1}{\sqrt{2}}\begin{bmatrix}0&\sqrt{2}&0&0&0\\\sqrt{2}&0&\sqrt{3}&0&0\\0&\sqrt{3}&0&\sqrt{3}&0\\0&0&\sqrt{3}&0&\sqrt{2}\\0&0&0&\sqrt{2}&0\end{bmatrix},\frac{i}{\sqrt{2}}\begin{bmatrix}0&-\sqrt{2}&0&0&0\\\sqrt{2}&0&-\sqrt{3}&0&0\\0&\sqrt{3}&0&-\sqrt{3}&0\\0&0&\sqrt{3}&0&-\sqrt{2}\\0&0&0&\sqrt{2}&0\end{bmatrix},\begin{bmatrix}2&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&0&0&0&-2\end{bmatrix})$

Cor. 5.1.2. $\sigma(2)\cdot\hat{p}\lambda(\hat{p},h)=h\lambda(\hat{p},s),h=-2,-1,0,1,2$

Cor. 5.1.3. $e^{\vec{\vartheta}\cdot\bar{\Omega}(2)}=1+(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(1-\frac{2}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\bar{\Omega}(2)]+2(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2(1-\frac{1}{3}sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2})[\vec{\vartheta}\cdot\bar{\Omega}(2)]^2$
$+\frac{2}{3}(\frac{sinh\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\bar{\Omega}(2)]^3+\frac{2}{3}(\frac{sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^4[\vec{\vartheta}\cdot\bar{\Omega}(2)]^4$

Cor. 5.1.4. $e^{i\omega\cdot\sigma(2)}=1+isin\omega(1+\frac{2}{3}sin^2\frac{\omega}{2})[\hat{\omega}\cdot\sigma(2)]-2sin^2\frac{\omega}{2}(1+\frac{1}{3}sin^2\frac{\omega}{2})[\hat{\omega}\cdot\sigma(2)]^2-\frac{2}{3}isin\omega sin^2\frac{\omega}{2}[\hat{\omega}\cdot\sigma(2)]^3$
$+\frac{2}{3}sin^4\frac{\omega}{2}[\hat{\omega}\cdot\sigma(2)]^4$

Cor. 5.1.5.
$e^{i\omega\cdot\sigma(2)}=1+isin\omega[\hat{\omega}\cdot\sigma(2)]-2sin^2\frac{\omega}{2}[\hat{\omega}\cdot\sigma(2)]^2+\frac{2}{3}sin^2\frac{\omega}{2}[isin\omega[\hat{\omega}\cdot\sigma(2)]-sin^2\frac{\omega}{2}[\hat{\omega}\cdot\sigma(2)]^2][1-[\hat{\omega}\cdot\sigma(2)]^2]$

Cor. 5.1.6.
$e^{i\omega\cdot\sigma(1)}=1+isin\omega[\hat{\omega}\cdot\sigma(1)]-2sin^2\frac{\omega}{2}[\hat{\omega}\cdot\sigma(1)]^2+\frac{2}{3}sin^2\frac{\omega}{2}[isin\omega[\hat{\omega}\cdot\sigma(1)]-sin^2\frac{\omega}{2}[\hat{\omega}\cdot\sigma(1)]^2][1-[\hat{\omega}\cdot\sigma(1)]^2]$
$=1+i\hat{\omega}\cdot\sigma(1)sin\omega+(1-cos\omega)[i\hat{\omega}\cdot\sigma(1)]^2$

Cor. 5.1.7.
$e^{i\omega\cdot\sigma}=1+isin\omega(\hat{\omega}\cdot\sigma)-2sin^2\frac{\omega}{2}(\hat{\omega}\cdot\sigma)^2+\frac{2}{3}sin^2\frac{\omega}{2}[isin\omega(\hat{\omega}\cdot\sigma)-sin^2\frac{\omega}{2}(\hat{\omega}\cdot\sigma)^2][1-(\hat{\omega}\cdot\sigma)^2]=cos\omega+isin\omega(\hat{\omega}\cdot\sigma)$

Cor. 5.1.8. $e^{i\omega\cdot\sigma(2)}=1+isin\omega[\hat{\omega}\cdot\sigma(2)]-(1-cos\omega)[\hat{\omega}\cdot\sigma(2)]^2+\frac{1}{6}(1-cos\omega)[2isin\omega[\hat{\omega}\cdot\sigma(2)]$
$-(1-cos\omega)[\hat{\omega}\cdot\sigma(2)]^2][1-[\hat{\omega}\cdot\sigma(2)]^2]$

Cor. 5.1.9. $e^{i\omega\cdot\sigma(2)}=1+[i\sigma(2)\times\hat{p}]_z+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}+\frac{1}{6}[2[i\sigma(2)\times\hat{p}]_z+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}][(1-\hat{p}_z)+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}]$

Cor. 5.1.10. $\sigma(2)\cdot\hat{p}=e^{i\vec{\omega}\cdot\sigma(2)}\sigma_z(2)e^{-i\vec{\omega}\cdot\sigma(2)}$

5.2 Concrete solution I of helicity $\sigma(2)\cdot\hat{p}$ eigenfunctions

Cor. 5.2.1.

$$[i\sigma(2)\times\hat{p}]_z=\begin{bmatrix}0&-\sqrt{2}\hat{p}_-&0&0&0\\\sqrt{2}\hat{p}_+&0&-\sqrt{3}\hat{p}_-&0&0\\0&\sqrt{3}\hat{p}_+&0&-\sqrt{3}\hat{p}_-&0\\0&0&\sqrt{3}\hat{p}_+&0&-\sqrt{2}\hat{p}_-\\0&0&0&\sqrt{2}\hat{p}_+&0\end{bmatrix},[i\sigma(2)\times\hat{p}]_z^2=\begin{bmatrix}-2\hat{p}_+\hat{p}_-&0&\sqrt{6}\hat{p}_-^2&0&0\\0&-5\hat{p}_+\hat{p}_-&0&3\hat{p}_-^2&0\\\sqrt{6}\hat{p}_+^2&0&-6\hat{p}_+\hat{p}_-&0&\sqrt{6}\hat{p}_-^2\\0&3\hat{p}_+^2&0&-5\hat{p}_+\hat{p}_-&0\\0&0&\sqrt{6}\hat{p}_+^2&0&-2\hat{p}_+\hat{p}_-\end{bmatrix}$$

Cor. 5.2.2. $$[i\sigma(2)\times\hat{p}]_z^4=\begin{bmatrix}10\hat{p}_+^2\hat{p}_-^2&0&-8\sqrt{6}\hat{p}_+\hat{p}_-^3&0&6\hat{p}_-^4\\0&34\hat{p}_+^2\hat{p}_-^2&0&-30\hat{p}_+\hat{p}_-^3&0\\-8\sqrt{6}\hat{p}_+^3\hat{p}_-&0&48\hat{p}_+^2\hat{p}_-^2&0&-8\sqrt{6}\hat{p}_+\hat{p}_-^3\\0&-30\hat{p}_+^3\hat{p}_-&0&34\hat{p}_+^2\hat{p}_-^2&0\\6\hat{p}_+^4&0&-8\sqrt{6}\hat{p}_+^3\hat{p}_-&0&10\hat{p}_+^2\hat{p}_-^2\end{bmatrix}$$

Cor. 5.2.3. $$[i\sigma(2)\times\hat{p}]_z+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}=\frac{1}{1+\hat{p}_z}\begin{bmatrix}-2\hat{p}_+\hat{p}_-&-\sqrt{2}\hat{p}_-(1+\hat{p}_z)&\sqrt{6}\hat{p}_-^2&0&0\\\sqrt{2}\hat{p}_+(1+\hat{p}_z)&-5\hat{p}_+\hat{p}_-&-\sqrt{3}\hat{p}_-(1+\hat{p}_z)&3\hat{p}_-^2&0\\\sqrt{6}\hat{p}_+^2&\sqrt{3}\hat{p}_+(1+\hat{p}_z)&-6\hat{p}_+\hat{p}_-&-\sqrt{3}\hat{p}_-(1+\hat{p}_z)&\sqrt{6}\hat{p}_-^2\\0&-3\hat{p}_+^2&\sqrt{3}\hat{p}_+(1+\hat{p}_z)&-5\hat{p}_+\hat{p}_-&-\sqrt{2}\hat{p}_-(1+\hat{p}_z)\\0&0&\sqrt{6}\hat{p}_+^2&\sqrt{2}\hat{p}_+(1+\hat{p}_z)&-2\hat{p}_+\hat{p}_-\end{bmatrix}$$

Cor. 5.2.4.

$$\frac{1}{6}[2[i\sigma(2)\times\hat{p}]_z+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}]=\frac{1}{6}\frac{1}{1+\hat{p}_z}\begin{bmatrix}-2\hat{p}_+\hat{p}_-&-2\sqrt{2}\hat{p}_-(1+\hat{p}_z)&\sqrt{6}\hat{p}_-^2&0&0\\2\sqrt{2}\hat{p}_+(1+\hat{p}_z)&-5\hat{p}_+\hat{p}_-&-2\sqrt{3}\hat{p}_-(1+\hat{p}_z)&-3\hat{p}_-^2&0\\\sqrt{6}\hat{p}_+^2&2\sqrt{3}\hat{p}_+(1+\hat{p}_z)&-6\hat{p}_+\hat{p}_-&-2\sqrt{3}\hat{p}_-(1+\hat{p}_z)&\sqrt{6}\hat{p}_-^2\\0&3\hat{p}_+^2&2\sqrt{3}\hat{p}_+(1+\hat{p}_z)&-5\hat{p}_+\hat{p}_-&-2\sqrt{2}\hat{p}_-(1+\hat{p}_z)\\0&0&\sqrt{6}\hat{p}_+^2&2\sqrt{2}\hat{p}_+(1+\hat{p}_z)&-2\hat{p}_+\hat{p}_-\end{bmatrix}$$

Cor. 5.2.5. $$[(1-\hat{p}_z)+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}]=\frac{1}{1+\hat{p}_z}\begin{bmatrix}0&0&\sqrt{6}\hat{p}_-^2&0&0\\0&-3\hat{p}_+\hat{p}_-&0&3\hat{p}_-^2&0\\\sqrt{6}\hat{p}_+^2&0&-4\hat{p}_+\hat{p}_-&0&\sqrt{6}\hat{p}_-^2\\0&3\hat{p}_+^2&0&-3\hat{p}_+\hat{p}_-&0\\0&0&\sqrt{6}\hat{p}_+^2&0&0\end{bmatrix}$$

Cor. 5.2.6. $\frac{1}{6}[2[i\sigma(2)\times\hat{p}]_z+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}][(1-\hat{p}_z)+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}]$

$$=\frac{1}{(1+\hat{p}_z)^2}\begin{bmatrix}\hat{p}_+^2\hat{p}_-^2&\sqrt{2}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z)&-\sqrt{6}\hat{p}_+\hat{p}_-^3&-2\sqrt{2}\hat{p}_-^3(1+\hat{p}_z)&\hat{p}_-^4\\-\sqrt{2}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z)&4\hat{p}_+^2\hat{p}_-^2&2\sqrt{3}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z)&-4\hat{p}_+\hat{p}_-^3&-\sqrt{2}\hat{p}_-^3(1+\hat{p}_z)\\-\sqrt{6}\hat{p}_+^3\hat{p}_-&-2\sqrt{3}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z)&6\hat{p}_+^2\hat{p}_-^2&2\sqrt{3}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z)&-\sqrt{6}\hat{p}_+\hat{p}_-^3\\\sqrt{2}\hat{p}_+^3(1+\hat{p}_z)&-4\hat{p}_+^3\hat{p}_-&-2\sqrt{3}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z)&4\hat{p}_+^2\hat{p}_-^2&\sqrt{2}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z)\\\hat{p}_+^4&\sqrt{2}\hat{p}_+^3(1+\hat{p}_z)&-\sqrt{6}\hat{p}_+^3\hat{p}_-&-\sqrt{2}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z)&\hat{p}_+^2\hat{p}_-^2\end{bmatrix}$$

Cor. 5.2.7. $e^{i\omega\cdot\sigma(2)}\overset{\omega_z=0}{=}1+[i\sigma(2)\times\hat{p}]_z+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}+\frac{1}{6}[2[i\sigma(2)\times\hat{p}]_z+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}][(1-\hat{p}_z)+\frac{[i\sigma(2)\times\hat{p}]_z^2}{1+\hat{p}_z}]$

$$=\begin{bmatrix}\frac{1}{4}(1+\hat{p}_z)^2&-\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z)&\frac{\sqrt{6}}{2}\hat{p}_-^2&-\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z)&\hat{p}_-^4/(1+\hat{p}_z)^2\\\frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z)&\frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1)&-\sqrt{3}\hat{p}_-\hat{p}_z&\hat{p}_-^2(2\hat{p}_z+1)/(1+\hat{p}_z)&-\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z)\\\frac{\sqrt{6}}{2}\hat{p}_+^2&\sqrt{3}\hat{p}_+\hat{p}_z&\frac{1}{2}(3\hat{p}_z^2-1)&-\sqrt{3}\hat{p}_-\hat{p}_z&\frac{\sqrt{6}}{2}\hat{p}_-^2\\\sqrt{2}\hat{p}_+^3/(1+\hat{p}_z)&\hat{p}_+^2(2\hat{p}_z+1)/(1+\hat{p}_z)&\sqrt{3}\hat{p}_+\hat{p}_z&\frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1)&-\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z)\\\hat{p}_+^4/(1+\hat{p}_z)^2&\sqrt{2}\hat{p}_+^3/(1+\hat{p}_z)&\frac{\sqrt{6}}{2}\hat{p}_+^2&\frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z)&\frac{1}{4}(1+\hat{p}_z)^2\end{bmatrix}$$

Cor. 5.2.8. $$e^{-i\omega\cdot\sigma(2)}\overset{\omega_z=0}{=}\begin{bmatrix}\frac{1}{4}(1+\hat{p}_z)^2&\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z)&\frac{\sqrt{6}}{2}\hat{p}_-^2&\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z)&\hat{p}_-^4/(1+\hat{p}_z)^2\\-\frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z)&\frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1)&\sqrt{3}\hat{p}_-\hat{p}_z&\hat{p}_-^2(2\hat{p}_z+1)/(1+\hat{p}_z)&\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z)\\\frac{\sqrt{6}}{2}\hat{p}_+^2&-\sqrt{3}\hat{p}_+\hat{p}_z&\frac{1}{2}(3\hat{p}_z^2-1)&\sqrt{3}\hat{p}_-\hat{p}_z&\frac{\sqrt{6}}{2}\hat{p}_-^2\\-\sqrt{2}\hat{p}_+^3/(1+\hat{p}_z)&\hat{p}_+^2(2\hat{p}_z+1)/(1+\hat{p}_z)&-\sqrt{3}\hat{p}_+\hat{p}_z&\frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1)&\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z)\\\hat{p}_+^4/(1+\hat{p}_z)^2&-\sqrt{2}\hat{p}_+^3/(1+\hat{p}_z)&\frac{\sqrt{6}}{2}\hat{p}_+^2&-\frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z)&\frac{1}{4}(1+\hat{p}_z)^2\end{bmatrix}$$

5.3 Helicity $\sigma(2)\cdot\hat{p}$ eigenfunctions

Cor. 5.3.1.

$$\lambda(\hat{p},2;2):=\begin{bmatrix}\frac{1}{4}(1+\hat{p}_z)^2\\ \frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z)\\ \frac{\sqrt{6}}{2}\hat{p}_+^2\\ \sqrt{2}\hat{p}_+^3/(1+\hat{p}_z)\\ \hat{p}_+^4/(1+\hat{p}_z)^2\end{bmatrix},\lambda(\hat{p},-2;2):=\begin{bmatrix}\hat{p}_-^4/(1+\hat{p}_z)^2\\ -\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z)\\ \frac{\sqrt{6}}{2}\hat{p}_-^2\\ -\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z)\\ \frac{1}{4}(1+\hat{p}_z)^2\end{bmatrix},\begin{cases}\lambda(\hat{p},2;2)=\frac{\hat{p}_+^2}{\hat{p}_-^2}\lambda(-\hat{p},-2)\\ \lambda(-\hat{p},2;2)=\frac{\hat{p}_+^2}{\hat{p}_-^2}\lambda(\hat{p},-2)\\ \lambda(\hat{p},-2;2)=\frac{\hat{p}_-^2}{\hat{p}_+^2}\lambda(-\hat{p},2)\\ \lambda(-\hat{p},-2;2)=\frac{\hat{p}_-^2}{\hat{p}_+^2}\lambda(\hat{p},2)\end{cases}$$

Cor. 5.3.2.

$$\lambda(\hat{p},1;2):=\begin{bmatrix}-\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z)\\ \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1)\\ \sqrt{3}\hat{p}_+\hat{p}_z\\ \hat{p}_+^2(2\hat{p}_z+1)/(1+\hat{p}_z)\\ \sqrt{2}\hat{p}_+^3/(1+\hat{p}_z)\end{bmatrix},\lambda(\hat{p},-1;2):=\begin{bmatrix}-\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z)\\ \hat{p}_-^2(2\hat{p}_z+1)/(1+\hat{p}_z)\\ -\sqrt{3}\hat{p}_-\hat{p}_z\\ \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1)\\ \frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z)\end{bmatrix},\begin{cases}\lambda(\hat{p},1;2)=-\frac{\hat{p}_+}{\hat{p}_-}\lambda(-\hat{p},-1;2)\\ \lambda(-\hat{p},1;2)=-\frac{\hat{p}_+}{\hat{p}_-}\lambda(\hat{p},-1;2)\\ \lambda(\hat{p},-1;2)=-\frac{\hat{p}_-}{\hat{p}_+}\lambda(-\hat{p},1;2)\\ \lambda(-\hat{p},-1;2)=-\frac{\hat{p}_-}{\hat{p}_+}\lambda(\hat{p},1;2)\end{cases}$$

Cor. 5.3.3.

$$\lambda(\hat{p},0;2):=\begin{bmatrix}\frac{\sqrt{6}}{2}\hat{p}_-^2\\ -\sqrt{3}\hat{p}_-\hat{p}_z\\ \frac{1}{2}(3\hat{p}_z^2-1)\\ \sqrt{3}\hat{p}_+\hat{p}_z\\ \frac{\sqrt{6}}{2}\hat{p}_+^2\end{bmatrix},\begin{cases}\lambda(\hat{p},0;2)=\lambda(-\hat{p},0;2)\\ \lambda(-\hat{p},0;2)=\lambda(\hat{p},0;2)\end{cases}$$

Cor. 5.3.4.

$$\lambda(\hat{p},2;2)\lambda^+(\hat{p},2;2)=\begin{bmatrix}\frac{1}{4}(1+\hat{p}_z)^2\\ \frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z)\\ \frac{\sqrt{6}}{2}\hat{p}_+^2\\ \sqrt{2}\hat{p}_+^3/(1+\hat{p}_z)\\ \hat{p}_+^4/(1+\hat{p}_z)^2\end{bmatrix}=\begin{bmatrix}\frac{1}{16}(1+\hat{p}_z)^4 & \frac{\sqrt{2}}{8}\hat{p}_-(1+\hat{p}_z)^3 & \frac{\sqrt{6}}{8}\hat{p}_-^2(1+\hat{p}_z)^2 & \frac{\sqrt{2}}{4}\hat{p}_-^3(1+\hat{p}_z) & \frac{1}{4}\hat{p}_-^4\\ 0 & \frac{1}{2}\hat{p}_+\hat{p}_-(1+\hat{p}_z)^2 & \frac{\sqrt{3}}{2}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z) & \hat{p}_+\hat{p}_-^3 & \frac{\sqrt{2}}{2}\hat{p}_+\hat{p}_-^4/(1+\hat{p}_z)\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{bmatrix}$$

5.4 Concrete solution II of helicity $\sigma(2)\cdot\hat{p}$ eigenfunctions

Thm. 5.4.1. $\lambda(\hat{p},h;2)=\sqrt{C_4^{2-h}}\bar{\Gamma}(\frac{5}{2})\overbrace{\lambda(\hat{p},\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{1}{2})}^{2+h}\otimes\overbrace{\lambda(\hat{p},-\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{1}{2})}^{2-h}$

5.5 Orthogonality and completeness of helicity $\sigma(2)\cdot\hat{p}$ eigenfunctions

Cor. 5.5.1. $\lambda^+(\hat{p},h;2)\lambda(\hat{p},h';2)=\delta_{hh'},\sum_{h=2}^{-2}\lambda(\hat{p},h;2)\lambda^+(\hat{p},h;2)=1,\sum_{h=2}^{-2}h\lambda(\hat{p},h;2)\lambda^+(\hat{p},h;2)=\sigma(2)\cdot\hat{p}$

5.6 Summary of derivative properties of helicity $\sigma(2)\cdot\hat{p}$ eigenfunctions

Cor. 5.6.1.

$$\begin{cases}\lambda^+(\hat{p},2;2)\tilde{\partial}_k\lambda(\hat{p},2;2)=2\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},1;2)\tilde{\partial}_k\lambda(\hat{p},1;2)=\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},0;2)\tilde{\partial}_k\lambda(\hat{p},0;2)=0\\ \lambda^+(\hat{p},-1;2)\tilde{\partial}_k\lambda(\hat{p},-1;2)=-\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-2;2)\tilde{\partial}_k\lambda(\hat{p},-2;2)=-2\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\end{cases}\quad\begin{cases}\lambda^+(\hat{p},-2;2)\tilde{\partial}_k\lambda(\hat{p},2;2)=0\\ \lambda^+(\hat{p},-1;2)\tilde{\partial}_k\lambda(\hat{p},1;2)=0\\ \lambda^+(\hat{p},0;2)\tilde{\partial}_k\lambda(\hat{p},0;2)=0\\ \lambda^+(\hat{p},1;2)\tilde{\partial}_k\lambda(\hat{p},-1;2)=0\\ \lambda^+(\hat{p},2;2)\tilde{\partial}_k\lambda(\hat{p},-2;2)=0\end{cases}$$

Cor. 5.6.2.

$$\begin{cases}\lambda^+(\hat{p},2;2)\tilde{\partial}_k\lambda(\hat{p},2;2)=2\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},1;2)\tilde{\partial}_k\lambda(\hat{p},2;2)\\ =-\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x+i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},0;2)\tilde{\partial}_k\lambda(\hat{p},2;2)=0\\ \lambda^+(\hat{p},-1;2)\tilde{\partial}_k\lambda(\hat{p},2;2)=0\\ \lambda^+(\hat{p},-2;2)\tilde{\partial}_k\lambda(\hat{p},2;2)=0\end{cases}\quad\begin{cases}\lambda^+(\hat{p},2;2)\tilde{\partial}_k\lambda(\hat{p},1;2)=\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x-i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},1;2)\tilde{\partial}_k\lambda(\hat{p},1;2)=\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},0;2)\tilde{\partial}_k\lambda(\hat{p},1;2)=-\frac{\sqrt{6}}{2}\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x+i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-1;2)\tilde{\partial}_k\lambda(\hat{p},1;2)=0\\ \lambda^+(\hat{p},-2;2)\tilde{\partial}_k\lambda(\hat{p},1;2)=0\end{cases}$$

Cor. 5.6.3.

$$\begin{cases}\lambda^+(\hat{p},2;2)\tilde{\partial}_k\lambda(\hat{p},0;2)=0\\ \lambda^+(\hat{p},1;2)\tilde{\partial}_k\lambda(\hat{p},0;2)=\frac{\sqrt{6}}{2}\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x-i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},0;2)\tilde{\partial}_k\lambda(\hat{p},0;2)=0\\ \lambda^+(\hat{p},-1;2)\tilde{\partial}_k\lambda(\hat{p},0;2)=-\frac{\sqrt{6}}{2}\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x+i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-2;2)\tilde{\partial}_k\lambda(\hat{p},0;2)=0\end{cases}$$

Cor. 5.6.4.

$$\begin{cases}\lambda^+(\hat{p},2;2)\tilde{\partial}_k\lambda(\hat{p},-1;2)=0\\ \lambda^+(\hat{p},1;2)\tilde{\partial}_k\lambda(\hat{p},-1;2)=0\\ \lambda^+(\hat{p},0;2)\tilde{\partial}_k\lambda(\hat{p},-1;2)=\frac{\sqrt{6}}{2}\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x-i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-1;2)\tilde{\partial}_k\lambda(\hat{p},-1;2)=-\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-2;2)\tilde{\partial}_k\lambda(\hat{p},-1;2)=-\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x+i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{p(1+\hat{p}_z)}\end{cases}$$

$$\begin{cases}\lambda^+(\hat{p},2;2)\tilde{\partial}_k\lambda(\hat{p},-2;2)=0\\ \lambda^+(\hat{p},1;2)\tilde{\partial}_k\lambda(\hat{p},-2;2)=0\\ \lambda^+(\hat{p},0;2)\tilde{\partial}_k\lambda(\hat{p},-2;2)=0\\ \lambda^+(\hat{p},-1;2)\tilde{\partial}_k\lambda(\hat{p},-2;2)\\ =\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x-i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{p(1+\hat{p}_z)}\\ \lambda^+(\hat{p},-2;2)\tilde{\partial}_k\lambda(\hat{p},-2;2)=-2\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}\end{cases}$$

6 Analysis of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions

6.1 Definition of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions

Def. 6.1.1. $\sigma(s)\cdot\hat{p}\lambda(\hat{p},h;s)=h\lambda(\hat{p},h;s), h=-s,\cdot\cdot,s$

6.2 Helicity $\sigma(s)\cdot\hat{p}$ z-direction eigenfunctions

Def. 6.2.1. $\sigma(s)\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=h\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

Cor. 6.2.1. $\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},s;s)=\begin{bmatrix}1\\0\\\cdot\cdot\\0\\0\end{bmatrix},\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},s-1;s)=\begin{bmatrix}0\\1\\\cdot\cdot\\0\\0\end{bmatrix},\cdot\cdot,\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},-s+1;s)=\begin{bmatrix}0\\0\\\cdot\cdot\\1\\0\end{bmatrix},\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},-s;s)=\begin{bmatrix}0\\0\\\cdot\cdot\\0\\1\end{bmatrix}$

Cor. 6.2.2.

$\lambda(e^{i\omega_z\gamma_z}\begin{bmatrix}0\\0\\1\end{bmatrix},s;s)=e^{is\omega_z}\begin{bmatrix}1\\0\\\cdot\cdot\\0\\0\end{bmatrix},\lambda(e^{i\omega_z\gamma_z}\begin{bmatrix}0\\0\\1\end{bmatrix},s-1;s)=e^{i(s-1)\omega_z}\begin{bmatrix}0\\1\\\cdot\cdot\\0\\0\end{bmatrix},\cdot\cdot,\lambda(e^{i\omega_z\gamma_z}\begin{bmatrix}0\\0\\1\end{bmatrix},-s;s)=e^{-is\omega_z}\begin{bmatrix}0\\0\\\cdot\cdot\\0\\1\end{bmatrix}$

Cor. 6.2.3.

$\lambda(e^{\epsilon_z\cdot L_z}\begin{bmatrix}0\\0\\1\\i\end{bmatrix},s;s)=e^{s\epsilon_z}\begin{bmatrix}1\\0\\\cdot\cdot\\0\\0\end{bmatrix},\lambda(e^{\epsilon_z\cdot L_z}\begin{bmatrix}0\\0\\1\end{bmatrix},s-1;s)=e^{(s-1)\epsilon_z}\begin{bmatrix}0\\1\\\cdot\cdot\\0\\0\end{bmatrix},\cdot\cdot,\lambda(e^{\epsilon_z\cdot L_z}\begin{bmatrix}0\\0\\1\end{bmatrix},-s;s)=e^{-s\epsilon_z}\begin{bmatrix}0\\0\\\cdot\cdot\\0\\1\end{bmatrix}$

6.3 Helicity $\sigma(s)\cdot\hat{p}$ general eigenfunctions

Pro. 6.3.1. $\begin{cases}\hat{\omega}\cdot\sigma(s)\overset{\hat{\omega}_z=0}{=}\sigma_x(s)\hat{\omega}_x+\sigma_y(s)\hat{\omega}_y=\frac{\sigma_x(s)\hat{p}_y-\sigma_y(s)\hat{p}_x}{\sqrt{1-\hat{p}_z^2}}=\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}\\ \vec{\omega}\cdot\sigma(s)\overset{\hat{\omega}_z=0}{=}\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\end{cases}$

Pro. 6.3.2. $e^{i\vec{\omega}\cdot\sigma(s)}\overset{\hat{\omega}_z=0}{=}exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}$

Thm. 6.3.1. $\lambda(\hat{p},h;s)=exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

Proof: $\sigma(s)\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=h\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$\Leftrightarrow exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\sigma(s)\cdot exp\{i\frac{[\gamma\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\begin{bmatrix}0\\0\\1\end{bmatrix}exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=h\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$\Leftrightarrow\sigma(s)\cdot\hat{p}exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=hexp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$\Rightarrow\lambda(\hat{p},h;s)=exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$ □

6.4 Orthogonality and completeness of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions

Cor. 6.4.1. $\lambda^+(\hat{p},h;s)\lambda(\hat{p},h';s)=\delta_{hh'},\sum_{h=s}^{-s}\lambda(\hat{p},h;s)\lambda^+(\hat{p},h;s)=1,\sum_{h=s}^{-s}h\lambda(\hat{p},h;s)\lambda^+(\hat{p},h;s)=\sigma(s)\cdot\hat{p}$

The above three corollaries can be easily proven.

6.5 Guess on properties of spin Lorentz transformation(It still needs to be tightened.)

Ass. 6.5.1. $\begin{cases}exp\{-i\pi\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}\}=(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)}\varepsilon(s)=(-1)^{2s}\varepsilon^+(s)(\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)}\\ exp\{i\pi\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}\}=\varepsilon^+(s)(\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)}=(-1)^{2s}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)}\varepsilon(s)\end{cases},e^{i2\pi\hat{\omega}\cdot\sigma(s)}=(-1)^{2s}$

Cor. 6.5.1. $\lambda(-\hat{p},h;s)=(-1)^{s+h}(\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h}\lambda(\hat{p},-h;s)=(-1)^{s+h}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{-2h}\lambda(\hat{p},-h;s)$

Proof: $\lambda(-\hat{p},h;s)$

$=exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos(-\hat{p}_z)\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$=exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}exp\{-i\pi\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$=(-1)^{s+h}(\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h}\lambda(\hat{p},-h;s)$ □

6.6 Properties of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions

Lem. 6.6.1. $\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)\sigma(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=h\begin{bmatrix}0\\0\\1\end{bmatrix}$

Thm. 6.6.1. $\lambda^{+}(\hat{p},h;s)\sigma(s)\lambda(\hat{p},h;s)=h\hat{p},h=-s,\cdot\cdot,s$

Proof: $\lambda^{+}(\hat{p},h;s)\sigma_k(s)\lambda(\hat{p},h;s)$
$=\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)e^{-i\vec{\omega}\cdot\sigma(s)}\sigma_k(s)e^{i\vec{\omega}\cdot\sigma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$=\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)e^{-i\vec{\omega}\cdot\sigma(s)}[e^{i\vec{\omega}\cdot\gamma}|_k{}^l e^{i\vec{\omega}\cdot\sigma(s)}\sigma_l(s)e^{-i\vec{\omega}\cdot\sigma(s)}]e^{i\vec{\omega}\cdot\sigma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$=e^{i\vec{\omega}\cdot\gamma}|_k{}^l\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)\sigma_l(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$=h\{e^{i\vec{\omega}\cdot\gamma}\begin{bmatrix}0\\0\\1\end{bmatrix}\}_k=h\hat{p}_k$ □

Proof: $\lambda^{+}(\hat{p},h;s)\sigma_i(s)\sigma_j(s)\lambda(\hat{p},h;s)$
$=\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)e^{-i\vec{\omega}\cdot\sigma(s)}\sigma_i(s)\sigma_j(s)e^{i\vec{\omega}\cdot\sigma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$=\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)e^{-i\vec{\omega}\cdot\sigma(s)}[e^{i\vec{\omega}\cdot\gamma}|_i{}^k e^{i\vec{\omega}\cdot\sigma(s)}\sigma_k(s)e^{-i\vec{\omega}\cdot\sigma(s)}][e^{i\vec{\omega}\cdot\gamma}|_j{}^l e^{i\vec{\omega}\cdot\sigma(s)}\sigma_l(s)e^{-i\vec{\omega}\cdot\sigma(s)}]e^{i\vec{\omega}\cdot\sigma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$=e^{i\vec{\omega}\cdot\gamma}|_i{}^k e^{i\vec{\omega}\cdot\gamma}|_j{}^l\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)\sigma_k(s)\sigma_l(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$=e^{i\vec{\omega}\cdot\gamma}|_i{}^k e^{i\vec{\omega}\cdot\gamma}|_j{}^l\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)\sigma_k(s)\sum_{h'}[\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h';s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h';s)]\sigma_l(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$=e^{i\vec{\omega}\cdot\gamma}|_i{}^k e^{i\vec{\omega}\cdot\gamma}|_j{}^l\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$\sigma_k(s)[\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h-1;s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h-1;s)+\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)+\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h+1;s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h+1;s)]\sigma_l(s)$
$\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$
$?=h^2\hat{p}_i\hat{p}_j$ □

Proof: $\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},s;s)\sigma_k(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},s-1;s)$
$=\begin{bmatrix}1\\0\\\cdot\cdot\\0\\0\end{bmatrix}^{+}\sigma_k(s)\begin{bmatrix}0\\1\\\cdot\cdot\\0\\0\end{bmatrix}$
$=\frac{1}{2}\begin{bmatrix}\sqrt{2s}\\-i\sqrt{2s}\\0\end{bmatrix}_k$ □

Proof: $\lambda^{+}(\hat{p},\varsigma s;s)\sigma_i(s)\sigma_j(s)\lambda(\hat{p},\varsigma s;s)$
$=e^{i\vec{\omega}\cdot\gamma}|_i{}^k e^{i\vec{\omega}\cdot\gamma}|_j{}^l\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma s;s)\sigma_k(s)[\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma(s-1);s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma(s-1);s)$
$+\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma s;s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma s;s)]\sigma_l(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma s;s)$
$=s^2\hat{p}_i\hat{p}_j+\frac{s}{2}(\delta_{ij}-\hat{p}_i\hat{p}_j+i\varsigma\varepsilon_{ij}{}^k\hat{p}_k)$
$=s^2\hat{p}_i\hat{p}_j-\frac{s}{2}\sigma_{ij}^{ab}\hat{p}_a\hat{p}_b$ □

Proof: $\lambda^{+}(\hat{p},\varsigma s;s)\sigma_i(s)\sigma_j(s)\lambda(\hat{p},-\varsigma s;s),s\geq\frac{3}{2}$
$=e^{i\vec{\omega}\cdot\gamma}|_i{}^k e^{i\vec{\omega}\cdot\gamma}|_j{}^l\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma s;s)\sigma_k(s)[\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma(s-1);s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma(s-1);s)$
$+\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma s;s)\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},\varsigma s;s)]\sigma_l(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},-\varsigma s;s)$
$=0$ □

Proof: $\lambda^{+}(\hat{p},-\varsigma s;s)\sigma_i(s)\sigma_j(s)[\sigma(s)\cdot\hat{p}]^n\lambda(\hat{p},-\varsigma s;s)=(-\varsigma)^n s^n s^2\hat{p}_i\hat{p}_j+(-\varsigma)^n s^n\frac{s}{2}(\delta_{ij}-\hat{p}_i\hat{p}_j+i\varsigma\varepsilon_{ij}{}^k\hat{p}_k)$ □

Cor. 6.6.1. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b=p_{\alpha_\varsigma}p_{\alpha'_\varsigma}-\delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2-i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k|\vec{p}|$

Cor. 6.6.2. $\lambda^{+}(\hat{p},h;s)[\sigma(s),ih]_a\lambda(\hat{p},h;s)=h(\hat{p},i)_a=h\hat{p}_a,h=-s,\cdot\cdot,s$

Lem. 6.6.2. $\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)\sigma(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h';s)=0,|h-h'|\geq2$

Thm. 6.6.2. $\lambda^{+}(\hat{p},h;s)\sigma(s)\lambda(\hat{p},h';s)=0,h,h'=-s,\cdot\cdot,s;|h-h'|\geq2$

Proof: $\lambda^{+}(\hat{p},h;s)\sigma_k(s)\lambda(\hat{p},h';s)$
$=\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)e^{-i\vec{\omega}\cdot\sigma(s)}\sigma_k(s)e^{i\vec{\omega}\cdot\sigma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h';s)$
$=\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)e^{-i\vec{\omega}\cdot\sigma(s)}[e^{i\vec{\omega}\cdot\gamma}|_k{}^l e^{i\vec{\omega}\cdot\sigma(s)}\sigma_l(s)e^{-i\vec{\omega}\cdot\sigma(s)}]e^{i\vec{\omega}\cdot\sigma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h';s)$
$=e^{i\vec{\omega}\cdot\gamma}|_k{}^l\lambda^{+}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)\sigma_l(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h';s),|h-h'|\geq2$

$= e^{i\varpi\cdot\gamma}|_k{}^l \cdot 0$
$= 0$ □

6.7 Eigenstate $\lambda(\hat{p}, -s\varsigma)$ of spin vector operators

Def. 6.7.1. $\lambda(\hat{p}, -s\varsigma) := \lambda(\hat{p}, -s\varsigma; s)$

Thm. 6.7.1. $[s\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_a + iS_{ab}(s,\varsigma)\begin{bmatrix}0\\0\\1\\i\end{bmatrix}^b]\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix}, -s\varsigma) \equiv 0[\Leftrightarrow][s\hat{p}_a + iS_{ab}(s,\varsigma)\hat{p}^b]\lambda(\hat{p}, -s\varsigma) \equiv 0$

Thm. 6.7.2. $[s\hat{p}_a + iS_{ab}(s,\varsigma)\hat{p}^b]\lambda(\hat{p}, -s\varsigma) = 0[\Leftrightarrow]\begin{cases}W_a(\hat{p},\varsigma;s)\lambda(\hat{p},-s\varsigma) = -s\varsigma\hat{p}_a\lambda(\hat{p},-s\varsigma)\\ W_a(\hat{p},\varsigma;s) := -i * S_{ab}(s,\varsigma)\hat{p}^b = i\varsigma S_{ab}(s,\varsigma)\hat{p}^b\end{cases}$

Pro. 6.7.1. $W_a(\hat{p},\varsigma;s) = (\hat{W}(\hat{p},\varsigma;s), i\sigma(s)\cdot\hat{p}), \hat{W}(\hat{p},\varsigma;s) = \sigma(s) - i\varsigma\sigma(s)\times\hat{p}$

Cor. 6.7.1. $\hat{W}(\hat{p},\varsigma;s)\lambda(\hat{p},-s\varsigma) = -s\varsigma\hat{p}\lambda(\hat{p},-s\varsigma)[\Rightarrow]\sigma(s)\cdot\hat{p}\lambda(\hat{p},-s\varsigma) = -s\varsigma\lambda(\hat{p},-s\varsigma)$

Cor. 6.7.2. $\hat{W}(\hat{p},\varsigma;s)\lambda(\hat{p},-s\varsigma) = -s\varsigma\hat{p}\lambda(\hat{p},-s\varsigma)[\Leftrightarrow]W_a(\hat{p},\varsigma;s)\lambda(\hat{p},-s\varsigma) = -s\varsigma\hat{p}_a\lambda(\hat{p},-s\varsigma)$

$\lambda(\hat{p},-s\varsigma)$ is a common eigenstate of helicity operator, 4D spin vector operator and spin vector operator. Essentially, the spin vector operator already fully encompasses the first two operators. So $\lambda(\hat{p},-s\varsigma)$ is essentially just the eigenstate of the spin vector operator $\hat{W}(\hat{p},\varsigma;s)$. The other two are just its deductions. And $\lambda(\hat{p},-s\varsigma)$ is just the eigenstate of a massless particle.

6.8 Properties of eigenstate $\lambda(\hat{p},-s\varsigma)$ of spin vector operators

Pro. 6.8.1. $\sigma(s)\times\hat{p} = [\sigma(s), i\sigma(s)\cdot\hat{p}]$

Pro. 6.8.2. $[sp_a + iS_{ab}(s,\varsigma)p^b]\lambda(\hat{p},-s\varsigma) = 0[\Leftrightarrow]\begin{cases}[\sigma(s) - i\varsigma\sigma(s)\times\hat{p}]\lambda(\hat{p},-s\varsigma) = -s\varsigma\hat{p}\lambda(\hat{p},-s\varsigma)\\ \sigma(s)\cdot\hat{p}\lambda(\hat{p},-s\varsigma) = -s\varsigma\lambda(\hat{p},-s\varsigma)\end{cases}$

Pro. 6.8.3. $[sp_a + iS_{ab}(s,\varsigma)p^b]\lambda(\hat{p},-s\varsigma) = 0[\Leftrightarrow]\sigma(\frac{1}{2})\otimes I_{2s}\cdot\hat{p}\begin{bmatrix}\lambda(\hat{p},-s\varsigma)\\0_{2s-1}\end{bmatrix} = -\frac{1}{2}\varsigma\begin{bmatrix}\lambda(\hat{p},-s\varsigma)\\0_{2s-1}\end{bmatrix}$

Pro. 6.8.4. $\begin{cases}[sp_a + iS_{ab}(s,\varsigma)p^b]\lambda(\hat{p},-s\varsigma) = 0[\Leftrightarrow]-\varsigma[\sigma(s)\cdot\hat{p} + \varsigma(s-1)]\sigma(s)\lambda(\hat{p},-s\varsigma) = -s\varsigma\hat{p}\lambda(\hat{p},-s\varsigma)\\ \qquad[\Updownarrow]\qquad\qquad\qquad\qquad[\Downarrow]\\ [\sigma(s) - i\varsigma\sigma(s)\times\hat{p}]\lambda(\hat{p},-s\varsigma) = -s\varsigma\hat{p}\lambda(\hat{p},-s\varsigma)[\Rightarrow][\sigma(s)\cdot\hat{p}]^n\lambda(\hat{p},-s\varsigma) = (-s\varsigma)^n\lambda(\hat{p},-s\varsigma)\end{cases}$

Cor. 6.8.1. $[sp_a + iS_{ab}(s,\varsigma)p^b]\lambda(\hat{p},-s\varsigma) = 0[\Leftrightarrow][\sigma(s)\cdot\hat{p}]\sigma(s)\lambda(\hat{p},-s\varsigma) = [s\hat{p} - \varsigma(s-1)\sigma(s)]\lambda(\hat{p},-s\varsigma)$
$[\Leftrightarrow][\sigma(s)\cdot\hat{p}]^n\sigma(s)\lambda(\hat{p},-s\varsigma) = \{(-\varsigma)^{n-1}s[s^n - (s-1)^n]\hat{p} + (-\varsigma)^n(s-1)^n\sigma(s)\}\lambda(\hat{p},-s\varsigma)$

Proof:
$[\sigma(s)\cdot\hat{p}]\sigma(s)\lambda(\hat{p},-s\varsigma) = [e_1\hat{p} + d_1\sigma(s)]\lambda(\hat{p},-s\varsigma), e_1 = s, d_1 = -\varsigma(s-1)$
$\cdot\cdot$
$[\sigma(s)\cdot\hat{p}]^{n-1}\sigma(s)\lambda(\hat{p},-s\varsigma) = [e_{n-1}\hat{p} + d_{n-1}\sigma(s)]\lambda(\hat{p},-s\varsigma)$
$[\sigma(s)\cdot\hat{p}]^n\sigma(s)\lambda(\hat{p},-s\varsigma) = [e_n\hat{p} + d_n\sigma(s)]\lambda(\hat{p},-s\varsigma)$
$\cdot\cdot$
$[\sigma(s)\cdot\hat{p}]^n\sigma(s)\lambda(\hat{p},-s\varsigma)$
$= [\sigma(s)\cdot\hat{p}][e_{n-1}\hat{p} + d_{n-1}\sigma(s)]\lambda(\hat{p},-s\varsigma) = [(-s\varsigma e_{n-1} + d_1^{n-1}e_1)\hat{p} + d_{n-1}d_1\sigma(s)]\lambda(\hat{p},-s\varsigma)$

$$\begin{cases}e_n = -s\varsigma e_{n-1} + d_1^{n-1}e_1\\ d_n = d_{n-1}d_1\\ e_1 = s, d_1 = -\varsigma(s-1)\end{cases} \Leftrightarrow \begin{cases}e_n = (-\varsigma)^{n-1}s[s^n - (s-1)^n]\\ d_n = d_1^n = (-\varsigma)^n(s-1)^n\end{cases}$$

$[\sigma(s)\cdot\hat{p}]^n\sigma(s)\lambda(\hat{p},-s\varsigma) = \{(-\varsigma)^{n-1}s[s^n - (s-1)^n]\hat{p} + (-\varsigma)^n(s-1)^n\sigma(s)\}\lambda(\hat{p},-s\varsigma)$ □

Cor. 6.8.2. $\lambda^+(\hat{p},-s\varsigma)\sigma_i(s)[\sigma(s)\cdot\hat{p}]^n\sigma_j(s)\lambda(\hat{p},-s\varsigma)$
$= \lambda^+(\hat{p},-s\varsigma)\sigma_i(s)\{(-\varsigma)^{n-1}s[s^n - (s-1)^n]\hat{p}_j + (-\varsigma)^n(s-1)^n\sigma_j(s)\}\lambda(\hat{p},-s\varsigma)$
$= (-\varsigma)^n s^2[s^n - (s-1)^n]\hat{p}_i\hat{p}_j + (-\varsigma)^n(s-1)^n\lambda^+(\hat{p},-s\varsigma)\sigma_i(s)\sigma_j(s)\lambda(\hat{p},-s\varsigma)$
$= (-\varsigma)^n s^2[s^n - (s-1)^n]\hat{p}_i\hat{p}_j + (-\varsigma)^n(s-1)^n[s^2\hat{p}_i\hat{p}_j + \frac{s}{2}(\delta_{ij} - \hat{p}_i\hat{p}_j - i\varsigma\varepsilon_{ij}{}^k\hat{p}_k)]$
$= (-\varsigma)^n s^2 s^n\hat{p}_i\hat{p}_j + (-\varsigma)^n(s-1)^n[\frac{s}{2}(\delta_{ij} - \hat{p}_i\hat{p}_j - i\varsigma\varepsilon_{ij}{}^k\hat{p}_k)]$

Cor. 6.8.3. $\lambda^+(\hat{p},-s\varsigma)[\sigma(s)\cdot\hat{p}]^n\sigma(s)\lambda(\hat{p},-s\varsigma) = (-\varsigma s)^{n+1}\hat{p} = \lambda^+(\hat{p},-s\varsigma)\sigma(s)[\sigma(s)\cdot\hat{p}]^n\lambda(\hat{p},-s\varsigma)$

6.9 Helicity $\sigma(s)\cdot\hat{p}$ eigenstate $\lambda(\hat{p},h;s)$ decompose into $\frac{1}{2}$-spin eigenstates

Thm. 6.9.1. $\lambda(\hat{p},h;s)=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$

Proof: $\lambda(\hat{p},h;s)=e^{i\vec{\omega}\cdot\sigma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$=e^{i\vec{\omega}\cdot\bar{\Gamma}(s)\bar{\Omega}(s)\Gamma(s)}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$=\bar{\Gamma}(s)e^{i\vec{\omega}\cdot\bar{\Omega}(s)}\Gamma(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

$=\bar{\Gamma}(s)e^{i\vec{\omega}\cdot\bar{\Omega}(s)}\Gamma(s)\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\begin{bmatrix}1\\0\end{bmatrix}\otimes\cdot\cdot\begin{bmatrix}1\\0\end{bmatrix}}^{s+h}\otimes\overbrace{\begin{bmatrix}0\\1\end{bmatrix}\otimes\cdot\cdot\otimes\begin{bmatrix}0\\1\end{bmatrix}}^{s-h}$

$=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\Gamma(s)\bar{\Gamma}(s)e^{i\vec{\omega}\cdot\bar{\Omega}(s)}\overbrace{\begin{bmatrix}1\\0\end{bmatrix}\otimes\cdot\cdot\begin{bmatrix}1\\0\end{bmatrix}}^{s+h}\otimes\overbrace{\begin{bmatrix}0\\1\end{bmatrix}\otimes\cdot\cdot\otimes\begin{bmatrix}0\\1\end{bmatrix}}^{s-h}$

$=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)e^{i\vec{\omega}\cdot\bar{\Omega}(s)}\overbrace{\begin{bmatrix}1\\0\end{bmatrix}\otimes\cdot\cdot\begin{bmatrix}1\\0\end{bmatrix}}^{s+h}\otimes\overbrace{\begin{bmatrix}0\\1\end{bmatrix}\otimes\cdot\cdot\otimes\begin{bmatrix}0\\1\end{bmatrix}}^{s-h}$

$=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{e^{i\vec{\omega}\cdot\sigma(\frac{1}{2})}\begin{bmatrix}1\\0\end{bmatrix}\otimes\cdot\cdot e^{i\vec{\omega}\cdot\sigma(\frac{1}{2})}\begin{bmatrix}1\\0\end{bmatrix}}^{s+h}\otimes\overbrace{e^{i\vec{\omega}\cdot\sigma(\frac{1}{2})}\begin{bmatrix}0\\1\end{bmatrix}\otimes\cdot\cdot\otimes e^{i\vec{\omega}\cdot\sigma(\frac{1}{2})}\begin{bmatrix}0\\1\end{bmatrix}}^{s-h}$

$=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$ □

Thm. 6.9.2. $\lambda(-\hat{p},h;s)=(-1)^{s+h}(\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h}\lambda(\hat{p},-h;s)$

Proof: $\lambda(-\hat{p},h;s)=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(-\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(-\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(-\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(-\hat{p},-\tfrac{1}{2})}^{s-h}$

$=(-\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{s+h}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{s-h}\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s-h}$

$=(-\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{s+h}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{s-h}\sqrt{C_{2s}^{s+h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s-h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s+h}$

$=(-\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{s+h}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{s-h}\lambda(\hat{p},-h;s)$

$=(-1)^{s+h}(\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h}\lambda(\hat{p},-h;s)$ □

Cor. 6.9.1. $\lambda(-\hat{p},-h;s)=(-1)^{s-h}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h}\lambda(\hat{p},h;s)$

Cor. 6.9.2.

$$\begin{cases}\lambda_{k_\varsigma}(\hat{p},h;s)=\sqrt{C_{2s}^{s-h}}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma\cdot\cdot}^{2s}}(s)\overbrace{\lambda_{A_\varsigma}(\hat{p},\tfrac{1}{2})\lambda_{B_\varsigma}(\hat{p},\tfrac{1}{2})\cdot\cdot}^{s+h}\otimes\overbrace{\lambda_{C_\varsigma}(\hat{p},-\tfrac{1}{2})\lambda_{D_\varsigma}(\hat{p},-\tfrac{1}{2})\cdot\cdot}^{s-h}\\ \frac{1}{(2s)!}\lambda_{\{}\overbrace{{}_{A_\varsigma}(\hat{p},\tfrac{1}{2})\lambda_{B_\varsigma}(\hat{p},\tfrac{1}{2})\cdot\cdot}^{s+h}\otimes\overbrace{\lambda_{C_\varsigma}(\hat{p},-\tfrac{1}{2})\lambda_{D_\varsigma}(\hat{p},-\tfrac{1}{2})\cdot\cdot}^{s-h}{}_{\}}=\sqrt{C_{2s}^{h-s}}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma\cdot\cdot}_{2s}}(s)\lambda_{k_\varsigma}(\hat{p},h;s)\end{cases}$$

Cor. 6.9.3.

$$\begin{cases}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s)\overbrace{\lambda_{A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdot\cdot}^{2s}\\ \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s)\lambda_{k_\varsigma}(\hat{p},-s\varsigma)=\underbrace{\lambda_{A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdot\cdot}_{2s}\end{cases}$$

Cor. 6.9.4.

$$\begin{cases}\lambda(\hat{p},-s\varsigma)=\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},-\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{\varsigma}{2})}^{2s}\\ \Gamma(s)\lambda(\hat{p},-s\varsigma)=\overbrace{\lambda(\hat{p},-\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{\varsigma}{2})}^{2s}\end{cases}$$

Thm. 6.9.3. $\lambda_{k_\varsigma}(\hat{p},-\varsigma h;s)\lambda^+_{k'_\varsigma}(\hat{p},-\varsigma h;s)=(-\frac{i}{2})^{2h}2^sC_{2s}^{s-h}(-\frac{i\varsigma}{\sqrt{2}})^{s+h}(\frac{i\varsigma}{\sqrt{2}})^{s-h}$

$\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s)\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s)\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\cdot\cdot(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}}^{s+h}\overbrace{(\sigma,-i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot(\sigma,-i\varsigma)^d_{D_\varsigma D'_\varsigma}}^{s-h}\hat{p}_a\cdot\cdot\hat{p}_b\hat{p}_c\cdot\cdot\hat{p}_d$

Proof: $\lambda(\hat{p},-\varsigma h;s)\lambda^+(\hat{p},-\varsigma h;s)=C_{2s}^{s-h}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{\varsigma}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{\varsigma}{2})}^{s-h}$
$\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{s+h}\otimes\overbrace{\lambda^+(\hat{p},\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\frac{\varsigma}{2})}^{s-h}\Gamma(s)$
$=C_{2s}^{s-h}\bar{\Gamma}(s)$
$\overbrace{\lambda(\hat{p},-\frac{\varsigma}{2})\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{\varsigma}{2})\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},\frac{\varsigma}{2})\lambda^+(\hat{p},\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{\varsigma}{2})\lambda^+(\hat{p},\frac{\varsigma}{2})}^{s-h}\Gamma(s)$
$=(-\frac{\varsigma}{2})^{2h}C_{2s}^{s-h}\bar{\Gamma}(s)\overbrace{(\sigma,i\varsigma)^a\hat{p}_a\otimes\cdot\cdot\otimes(\sigma,i\varsigma)^b\hat{p}_b}^{s+h}\otimes\overbrace{(\sigma,-i\varsigma)^c\hat{p}_c\otimes\cdot\cdot\otimes(\sigma,-i\varsigma)^d\hat{p}_d}^{s-h}\Gamma(s)$
$=(-\frac{\varsigma}{2})^{2h}C_{2s}^{s-h}\bar{\Gamma}(s)\overbrace{(\sigma,i\varsigma)^a\otimes\cdot\cdot\otimes(\sigma,i\varsigma)^b}^{s+h}\otimes\overbrace{(\sigma,-i\varsigma)^c\otimes\cdot\cdot\otimes(\sigma,-i\varsigma)^d}^{s-h}\Gamma(s)\hat{p}_a\cdot\cdot\hat{p}_b\hat{p}_c\cdot\cdot\hat{p}_d$
$=(-\frac{i}{2})^{2h}2^sC_{2s}^{s-h}(-\frac{i\varsigma}{\sqrt{2}})^{s+h}(\frac{i\varsigma}{\sqrt{2}})^{s-h}$
$\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s)\Gamma_{k'_\varsigma}^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}(s)\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\cdot\cdot(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}}^{s+h}\overbrace{(\sigma,-i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot(\sigma,-i\varsigma)^d_{D_\varsigma D'_\varsigma}}^{s-h}\hat{p}_a\cdot\cdot\hat{p}_b\hat{p}_c\cdot\cdot\hat{p}_d$ □

6.10 Properties of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions derivative

Thm. 6.10.1. $\lambda(\hat{p},h;s)\tilde{\partial}_k\lambda(\hat{p},h;s)=C_{2s}^{s-h}h\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)},\lambda(-\hat{p},h;s)\tilde{\partial}_k\lambda(-\hat{p},h;s)=C_{2s}^{s-h}h\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{p(1-\hat{p}_z)}$

Cor. 6.10.1.

$$\begin{cases}\lambda^+(\hat{p},h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)=\sqrt{C_{2s}^{s-h'}C_{2s}^{s-h}}[(h'+h)\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}+(h'-h)\frac{\sqrt{2}\hat{p}_+\hat{p}_i+\sqrt{2}\hat{p}_+\delta_{iz}-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{2p(1+\hat{p}_z)}],h'\le h\\ \lambda^+(\hat{p},h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)=\sqrt{C_{2s}^{s-h'}C_{2s}^{s-h}}[(h'+h)\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}+(h'-h)\frac{\sqrt{2}\hat{p}_-\hat{p}_i+\sqrt{2}\hat{p}_-\delta_{iz}-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{2p(1+\hat{p}_z)},h'\ge h\\ \lambda^+(-\hat{p},-h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)\\ =(-1)^{s-h'}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h'}\sqrt{C_{2s}^{s-h'}C_{2s}^{s-h}}[(h'+h)\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}+(h'-h)\frac{\sqrt{2}\hat{p}_+\hat{p}_i+\sqrt{2}\hat{p}_+\delta_{iz}-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{2p(1+\hat{p}_z)}],h'\le h\\ \lambda^+(-\hat{p},-h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)\\ =(-1)^{s-h'}(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h'}\sqrt{C_{2s}^{s-h'}C_{2s}^{s-h}}[(h'+h)\frac{-i\hat{p}_y\delta_{kx}+i\hat{p}_x\delta_{ky}}{2p(1+\hat{p}_z)}+(h'-h)\frac{\sqrt{2}\hat{p}_-\hat{p}_i+\sqrt{2}\hat{p}_-\delta_{iz}-(1+\hat{p}_z)(\delta_{ix}-i\delta_{iy})}{2p(1+\hat{p}_z)},h'\ge h\end{cases}$$

6.11 General solution 1 of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions derivative

Lem. 6.11.1. $\lambda(\hat{p},s;s)=\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{1}{2})}^{2s},\lambda(\hat{p},-s;s)=\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},-\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{1}{2})}^{2s}$

Thm. 6.11.1. $\lambda^+(\hat{p},-\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},-\varsigma s;s)=2s\lambda^+(\hat{p},-\frac{\varsigma}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{\varsigma}{2})$

Proof: $\lambda^+(\hat{p},-\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},-\varsigma s;s)$
$=\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{2s}\Gamma(s)\bar{\Gamma}(s)\tilde{\partial}_k\overbrace{\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{\varsigma}{2})}^{2s}$
$=2s\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{2s}\Gamma(s)\bar{\Gamma}(s)\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{\varsigma}{2})}^{2s}$
$=2s\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{2s}\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{\varsigma}{2})}^{2s}$
$=2s\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})\lambda(\hat{p},-\frac{\varsigma}{2})}^{2s}$
$=2s\lambda^+(\hat{p},-\frac{\varsigma}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{\varsigma}{2})$ □

Thm. 6.11.2. $\lambda^+(\hat{p},-s;s)\tilde{\partial}_k\lambda(\hat{p},s;s)=0[\Leftrightarrow]\lambda^+(\hat{p},s;s)\tilde{\partial}_k\lambda(\hat{p},-s;s)=0;s\ge1$

Proof: $\lambda^+(\hat{p},-\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},\varsigma s;s);s\ge1$
$=\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{2s}\Gamma(s)\bar{\Gamma}(s)\tilde{\partial}_k\overbrace{\lambda(\hat{p},\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{\varsigma}{2})}^{2s}$
$=2s\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{2s}\Gamma(s)\bar{\Gamma}(s)\overbrace{\tilde{\partial}_k\lambda(\hat{p},\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{\varsigma}{2})}^{2s}$
$=2s\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})}^{2s}\overbrace{\tilde{\partial}_k\lambda(\hat{p},\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{\varsigma}{2})}^{2s}$
$=2s\overbrace{\lambda^+(\hat{p},-\frac{\varsigma}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\frac{\varsigma}{2})\lambda(\hat{p},\frac{\varsigma}{2})}^{2s}$
$=0$ □

Proof: $\lambda^+(\hat{p},-\varsigma s;s)\lambda(\hat{p},\varsigma s;s)=0, \lambda^+(\hat{p},-\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},\varsigma s;s)=0$
$\Leftrightarrow \tilde{\partial}_k[\lambda^+(\hat{p},-\varsigma s;s)\lambda(\hat{p},\varsigma s;s)]=0, \lambda^+(\hat{p},-\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},\varsigma s;s)=0$
$\Leftrightarrow \tilde{\partial}_k\lambda^+(\hat{p},-\varsigma s;s)\lambda(\hat{p},\varsigma s;s)+\lambda^+(\hat{p},-\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},\varsigma s;s)=0, \lambda^+(\hat{p},-\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},\varsigma s;s)=0$
$\Rightarrow \tilde{\partial}_k\lambda^+(\hat{p},-\varsigma s;s)\lambda(\hat{p},\varsigma s;s)=0$
$\Leftrightarrow [\tilde{\partial}_k\lambda^+(\hat{p},-\varsigma s;s)\lambda(\hat{p},\varsigma s;s)]^+=0$
$\Leftrightarrow \lambda^+(\hat{p},\varsigma s;s)\tilde{\partial}_k\lambda(\hat{p},-\varsigma s;s)=0$ □

6.12 General solution 2 of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions derivative

Lem. 6.12.1. $\lambda(\hat{p},h;s)=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$

Thm. 6.12.1. $\lambda^+(\hat{p},h;s)\tilde{\partial}_k\lambda(\hat{p},h;s)=2h\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})$

Proof: $\lambda^+(\hat{p},h;s)\tilde{\partial}_k\lambda(\hat{p},h;s)$

$$=[\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}]^+$$
$$\tilde{\partial}_k\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$$
$$=C_{2s}^{s-h}\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})}^{s-h}\Gamma(s)\bar{\Gamma}(s)$$
$$[(s+h)\overbrace{\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}+(s-h)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}]$$
$$=(s+h)\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$$
$$+(s-h)\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$$
$$=(s+h)\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})+(s-h)\lambda^+(\hat{p},-\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})$$
$$=2h\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})$$ □

Thm. 6.12.2. $\lambda^+(\hat{p},-h;s)\tilde{\partial}_k\lambda(\hat{p},h;s)=0, |h|\geq 1$

Proof: $\lambda^+(\hat{p},-h;s)\tilde{\partial}_k\lambda(\hat{p},h;s)$

$$=[\sqrt{C_{2s}^{s+h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s-h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s+h}]^+$$
$$\tilde{\partial}_k\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$$
$$=C_{2s}^{s-h}\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})}^{s-h}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})}^{s+h}\Gamma(s)\bar{\Gamma}(s)$$
$$[(s+h)\overbrace{\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}+(s-h)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}]$$
$$=0+0=0$$ □

Thm. 6.12.3. $\lambda^+(\hat{p},0;n)\tilde{\partial}_k\lambda(\hat{p},0;n)=0$

Proof: $\lambda^+(\hat{p},0;n)\tilde{\partial}_k\lambda(\hat{p},0;n)$

$$=[\sqrt{C_{2n}^{n}}\bar{\Gamma}(n)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}]^+$$
$$\tilde{\partial}_k\sqrt{C_{2n}^{n}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}$$
$$=C_{2n}^{n}\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})}^{n}\Gamma(s)\bar{\Gamma}(s)$$
$$[n\overbrace{\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}+n\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}]$$
$$=n[\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})}^{n}$$
$$+\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})}^{n}]$$
$$=n\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})+n\lambda^+(\hat{p},-\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})$$
$$=0$$ □

Thm. 6.12.4. $\lambda^+(\hat{p},-\frac{1}{2};n+\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2};n+\frac{1}{2})=(n+1)\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})$

Proof: $\lambda^+(\hat{p},-\frac{1}{2};n+\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2};n+\frac{1}{2})$

$$=[\sqrt{C_{2n+1}^{n}}\bar{\Gamma}(n+\tfrac{1}{2})\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n+1}]^+$$
$$\tilde{\partial}_k\sqrt{C_{2n+1}^{n}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n+1}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}$$
$$=C_{2n+1}^{n}\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})}^{n+1}\Gamma(s)\bar{\Gamma}(s)$$
$$[(n+1)\overbrace{\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n+1}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}+n\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n+1}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}]$$
$$=(n+1)[\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})}^{n+1}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})}^{n}+0$$
$$=(n+1)\lambda^+(\hat{p},-\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})$$ □

Thm. 6.12.5. $\lambda^+(\hat{p},\frac{1}{2};n+\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2};n+\frac{1}{2})=(n+1)\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2})$

Proof: $\lambda^+(\hat{p},\frac{1}{2};n+\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2};n+\frac{1}{2})$

$$=[\sqrt{C_{2n+1}^{n}}\bar{\Gamma}(n+\tfrac{1}{2})\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n+1}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n}]^+$$
$$\tilde{\partial}_k\sqrt{C_{2n+1}^{n}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n+1}$$
$$=C_{2n+1}^{n}\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})}^{n+1}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})}^{n}\Gamma(s)\bar{\Gamma}(s)$$
$$[n\overbrace{\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n+1}+(n+1)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{n+1}]$$
$$=0+(n+1)\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})\lambda(\hat{p},\tfrac{1}{2})}^{n}\otimes\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})\lambda(\hat{p},-\tfrac{1}{2})}^{n+1}]$$
$$=(n+1)\lambda^+(\hat{p},\tfrac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})$$ □

6.13 General solution 3 of helicity $\sigma(s)\cdot\hat{p}$ eigenfunctions derivative

Thm. 6.13.1. $\lambda^+(\hat{p},h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)=0,|h'-h|\geq 2$

Proof: $\lambda^+(\hat{p},h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)$

$$=[\sqrt{C_{2s}^{s-h'}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h'}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h'}]^+$$
$$\tilde{\partial}_k\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$$
$$=\sqrt{C_{2s}^{s-h'}}\sqrt{C_{2s}^{s-h}}\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})}^{s+h'}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})}^{s-h'}\Gamma(s)\bar{\Gamma}(s)$$
$$[(s+h)\overbrace{\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}+(s-h)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}]$$
$$=0+0=0$$ □

Thm. 6.13.2. $\lambda^+(\hat{p},h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)=\sqrt{(s+h')(s-h)}\lambda^+(\hat{p},\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},-\frac{1}{2}),h'-h=1$

Proof: $\lambda^+(\hat{p},h';s)\tilde{\partial}_k\lambda(\hat{p},h;s)$

$$=[\sqrt{C_{2s}^{s-h'}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h'}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h'}]^+$$
$$\tilde{\partial}_k\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}$$
$$=\sqrt{C_{2s}^{s-h'}}\sqrt{C_{2s}^{s-h}}\overbrace{\lambda^+(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},\tfrac{1}{2})}^{s+h'}\otimes\overbrace{\lambda^+(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat{p},-\tfrac{1}{2})}^{s-h'}\Gamma(s)\bar{\Gamma}(s)$$
$$[(s+h)\overbrace{\tilde{\partial}_k\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}+(s-h)\overbrace{\lambda(\hat{p},\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\tfrac{1}{2})}^{s+h}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p},-\tfrac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\tfrac{1}{2})}^{s-h}]$$

$= \sqrt{C_{2s}^{s-h'}}\sqrt{C_{2s}^{s-h}}\overbrace{\lambda^+(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac12)}^{s+h'}\otimes\overbrace{\lambda^+(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,-\tfrac12)}^{s-h'}\Gamma(s)\bar\Gamma(s)$
$(s-h)\overbrace{\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,\tfrac12)}^{s+h}\otimes\overbrace{\tilde\partial_k\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac12)}^{s-h}$
$= \sqrt{C_{2s}^{s-h'}}\sqrt{C_{2s}^{s-h}}(C_{2s}^{s-h'})^{-1}(s-h)\overbrace{\lambda^+(\hat p,\tfrac12)\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac12)\lambda(\hat p,\tfrac12)}^{s+h}\otimes\overbrace{\lambda^+(\hat p,\tfrac12)\tilde\partial_k\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,-\tfrac12)\lambda(\hat p,-\tfrac12)}^{s-h}$
$= \sqrt{(s+h')(s-h)}\lambda^+(\hat p,\tfrac12)\tilde\partial_k\lambda(\hat p,-\tfrac12)$ □

Thm. 6.13.3. $\lambda^+(\hat p,h';s)\tilde\partial_k\lambda(\hat p,h;s) = \sqrt{(s-h')(s+h)}\lambda^+(\hat p,-\tfrac12)\tilde\partial_k\lambda(\hat p,\tfrac12), h'-h=-1$

Proof: $\lambda^+(\hat p,h';s)\tilde\partial_k\lambda(\hat p,h;s)$
$= [\sqrt{C_{2s}^{s-h'}}\bar\Gamma(s)\overbrace{\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,\tfrac12)}^{s+h'}\otimes\overbrace{\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac12)}^{s-h'}]^+$
$\tilde\partial_k\sqrt{C_{2s}^{s-h}}\bar\Gamma(s)\overbrace{\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,\tfrac12)}^{s+h}\otimes\overbrace{\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac12)}^{s-h}$
$= \sqrt{C_{2s}^{s-h'}}\sqrt{C_{2s}^{s-h}}\overbrace{\lambda^+(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac12)}^{s+h'}\otimes\overbrace{\lambda^+(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,-\tfrac12)}^{s-h'}\Gamma(s)\bar\Gamma(s)$
$[(s+h)\overbrace{\tilde\partial_k\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,\tfrac12)}^{s+h}\otimes\overbrace{\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac12)}^{s-h}+(s-h)\overbrace{\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,\tfrac12)}^{s+h}\otimes\overbrace{\tilde\partial_k\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac12)}^{s-h}]$
$= \sqrt{C_{2s}^{s-h'}}\sqrt{C_{2s}^{s-h}}\overbrace{\lambda^+(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac12)}^{s+h'}\otimes\overbrace{\lambda^+(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,-\tfrac12)}^{s-h'}\Gamma(s)\bar\Gamma(s)$
$(s+h)\overbrace{\tilde\partial_k\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,\tfrac12)}^{s+h}\otimes\overbrace{\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac12)}^{s-h}$
$= \sqrt{C_{2s}^{s-h'}}\sqrt{C_{2s}^{s-h}}(C_{2s}^{s-h'})^{-1}(s+h)$
$\overbrace{\lambda^+(\hat p,-\tfrac12)\tilde\partial_k\lambda(\hat p,\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac12)\lambda(\hat p,\tfrac12)}^{s+h}\otimes\overbrace{\lambda^+(\hat p,-\tfrac12)\lambda(\hat p,-\tfrac12)\otimes\cdot\cdot\otimes\lambda^+(\hat p,-\tfrac12)\lambda(\hat p,-\tfrac12)}^{s-h}$
$= \sqrt{(s-h')(s+h)}\lambda^+(\hat p,-\tfrac12)\tilde\partial_k\lambda(\hat p,\tfrac12)$ □

6.14 Summary of helicity $\sigma(s)\cdot\hat p$ eigenfunctions derivative properties

Thm. 6.14.1.
$$\begin{cases}\lambda^+(\hat p,h;s)\tilde\partial_k\lambda(\hat p,h;s) = 2h\lambda^+(\hat p,\tfrac12)\tilde\partial_k\lambda(\hat p,\tfrac12) = h\frac{-i\hat p_y\delta_{kx}+i\hat p_x\delta_{ky}}{p(1+\hat p_z)}\\ \lambda^+(\hat p,h';s)\tilde\partial_k\lambda(\hat p,h;s) = \sqrt{(s+\varsigma h')(s-\varsigma h)}\lambda^+(\hat p,\tfrac{\varsigma}{2})\tilde\partial_k\lambda(\hat p,-\tfrac{\varsigma}{2})\\ = \varsigma\sqrt{(s+\varsigma h')(s-\varsigma h)}\frac{(\hat p_i+\delta_{iz})(\hat p_x-i\varsigma\hat p_y)-(1+\hat p_z)(\delta_{ix}-i\varsigma\delta_{iy})}{2p(1+\hat p_z)}, h'-h=\varsigma\\ \lambda^+(\hat p,h';s)\tilde\partial_k\lambda(\hat p,h;s)=0, |h'-h|\ge 2\end{cases}$$

6.15 General solution IV of helicity $\sigma(s)\cdot\hat p$ eigenfunctions derivative

Lem. 6.15.1. $\Gamma(s;w)\bar\Gamma(s;w)\Omega(s;w)\Gamma(s;w) = \Omega(s;w)\Gamma(s;w)$

Thm. 6.15.1. $\lambda^+(\hat p,\varsigma s;s)\sigma(s)\tilde\partial_k\lambda(\hat p,-\varsigma s;s)=0, s\ge\frac32$

Proof: $\lambda^+(\hat p,\varsigma s;s)\sigma(s)\tilde\partial_k\lambda(\hat p,-\varsigma s;s)$
$= \overbrace{\lambda^+(\hat p,\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac{\varsigma}{2})}^{2s}\Gamma(s)[\bar\Gamma(s)\Omega(s)\Gamma(s)]\bar\Gamma(s)\tilde\partial_k\overbrace{\lambda(\hat p,-\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac{\varsigma}{2})}^{2s}$
$= 2s\overbrace{\lambda^+(\hat p,\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac{\varsigma}{2})}^{2s}\Gamma(s)\bar\Gamma(s)\Omega(s)\Gamma(s)\bar\Gamma(s)\overbrace{\tilde\partial_k\lambda(\hat p,-\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac{\varsigma}{2})}^{2s}$
$= 2s\overbrace{\lambda^+(\hat p,\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac{\varsigma}{2})}^{2s}\Gamma(s)\bar\Gamma(s)\Omega(s)\overbrace{\tilde\partial_k\lambda(\hat p,-\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac{\varsigma}{2})}^{2s}$
$= 2s\overbrace{\lambda^+(\hat p,\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda^+(\hat p,\tfrac{\varsigma}{2})}^{2s}\Omega(s)\overbrace{\tilde\partial_k\lambda(\hat p,-\tfrac{\varsigma}{2})\otimes\cdot\cdot\otimes\lambda(\hat p,-\tfrac{\varsigma}{2})}^{2s}$
$= 0$ □

Cor. 6.15.1. $\lambda^+(\hat p,\varsigma s;s)[\sigma_i(s)\tilde\partial_j-\sigma_j(s)\tilde\partial_i]\lambda(\hat p,-\varsigma s;s)=0, s\ge\frac32$

Thm. 6.15.2. $\lambda^+(\hat p,-1;1)\sigma(s)\tilde\partial_k\lambda(\hat p,1;1)=0$

Proof: $\lambda^+(\hat{p},-1;s)\sigma(s)\tilde{\partial}_k\lambda(\hat{p},1;1)$
$=\lambda^+(\hat{p},-\frac{1}{2})\otimes\lambda^+(\hat{p},-\frac{1}{2})\Gamma(1)\bar{\Gamma}(1)\Omega(1)\Gamma(1)\bar{\Gamma}(1)\tilde{\partial}_k[\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})]$
$=2s\lambda^+(\hat{p},-\frac{1}{2})\otimes\lambda^+(\hat{p},-\frac{1}{2})\Gamma(1)\bar{\Gamma}(1)\Omega(1)\Gamma(1)\bar{\Gamma}(1)\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})$
$=2\lambda^+(\hat{p},-\frac{1}{2})\otimes\lambda^+(\hat{p},-\frac{1}{2})\Gamma(1)\bar{\Gamma}(1)\Omega(1)\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})$
$=2\lambda^+(\hat{p},-\frac{1}{2})\otimes\lambda^+(\hat{p},-\frac{1}{2})\Omega(1)\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})$
$=\lambda^+(\hat{p},-\frac{1}{2})\otimes\lambda^+(\hat{p},-\frac{1}{2})(\sigma\otimes I+I\otimes\sigma)\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})$
$=[\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p},\frac{1}{2})][\lambda^+(\hat{p},-\frac{1}{2})\sigma\lambda(\hat{p},\frac{1}{2})]$ □

Pro. 6.15.1.
$\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})=-\frac{(\hat{p}_i+\delta_{iz})(\hat{p}_x+i\hat{p}_y)-(1+\hat{p}_z)(\delta_{ix}+i\delta_{iy})}{2p(1+\hat{p}_z)}$

$$=-\begin{bmatrix}\frac{\hat{p}_x(\hat{p}_x+i\hat{p}_y)-(1+\hat{p}_z)}{2p(1+\hat{p}_z)}\\ \frac{\hat{p}_y(\hat{p}_x+i\hat{p}_y)-i(1+\hat{p}_z)}{2p(1+\hat{p}_z)}\\ \frac{(1+\hat{p}_z)(\hat{p}_x+i\hat{p}_y)}{2p(1+\hat{p}_z)}\end{bmatrix}=-\frac{1}{2p(1+\hat{p}_z)}[(\hat{p}_x+i\hat{p}_y)\hat{p}+\begin{bmatrix}-(1+\hat{p}_z)\\ -i(1+\hat{p}_z)\\ (\hat{p}_x+i\hat{p}_y)\end{bmatrix}]$$

$$\lambda^+(\hat{p},-\frac{1}{2})(\sigma,-i\varsigma)_a\lambda(\hat{p},\frac{1}{2})=\begin{bmatrix}\frac{\hat{p}_x\hat{p}_z-i\hat{p}_y}{\hat{p}_x-i\hat{p}_y}\\ \frac{\hat{p}_y\hat{p}_z+i\hat{p}_x}{\hat{p}_x-i\hat{p}_y}\\ \frac{\hat{p}_z\hat{p}_z-1}{\hat{p}_x-i\hat{p}_y}\\ 0\end{bmatrix}=\frac{1}{\hat{p}_x-i\hat{p}_y}(\hat{p}_z\hat{p}+\begin{bmatrix}-i\hat{p}_y\\ i\hat{p}_x\\ -1\\ 0\end{bmatrix})$$

Cor. 6.15.2. $\lambda^+(\hat{p},-1;1)[\sigma_i(1)\tilde{\partial}_j-\sigma_j(1)\tilde{\partial}_i]\lambda(\hat{p},1;1)$
$=[\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_j\lambda(\hat{p},\frac{1}{2})][\lambda^+(\hat{p},-\frac{1}{2})\sigma_i\lambda(\hat{p},\frac{1}{2})]-[\lambda^+(\hat{p},-\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p},\frac{1}{2})][\lambda^+(\hat{p},-\frac{1}{2})\sigma_j\lambda(\hat{p},\frac{1}{2})]$
$=0$

Cor. 6.15.3. $\lambda^+(\hat{p},\varsigma s;s)[\sigma_i(s)\tilde{\partial}_j-\sigma_j(s)\tilde{\partial}_i]\lambda(\hat{p},-\varsigma s;s)=0, s\geq 1$

6.16 Special case: 1-spin eigenstate $\lambda(\hat{p},h;1)$ decomposes into $\frac{1}{2}$-spin eigenstates

Lem. 6.16.1. $\begin{cases}\Gamma_{\alpha_\varsigma}{}^{k_\varsigma}(1)\succ S_m(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&-i\\-1&0&-1\\0&-i\sqrt{2}&0\end{bmatrix}\\ \Gamma_{k_\varsigma}{}^{\alpha_\varsigma}(1)\succ S_m^+(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&-1&0\\0&0&i\sqrt{2}\\i&-1&0\end{bmatrix}\end{cases}$ $\begin{cases}\Gamma^{\alpha_\varsigma}{}_{k_\varsigma}(1)\succ S_m^*(1)\\ \Gamma^{k_\varsigma}{}_{\alpha_\varsigma}(1)\succ S_m^T(1)\end{cases}$

Lem. 6.16.2. $[S_m(1)\bar{\Gamma}(1)]_{\alpha_\varsigma}{}^{A_\varsigma\otimes B_\varsigma}=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&0&-i\\-1&0&0&-1\\0&-i&-i&0\end{bmatrix}, [S_m(1)\bar{\Gamma}(1)]_{\alpha_\varsigma}{}^{A_\varsigma B_\varsigma}=-\frac{1}{\sqrt{2}}\sigma_y\sigma=\frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}$

Thm. 6.16.1. $\lambda(\hat{p},h;1)=\sqrt{C_2^{1-h}}\bar{\Gamma}(1)\overbrace{\lambda(\hat{p},\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{1}{2})}^{1+h}\otimes\overbrace{\lambda(\hat{p},-\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{1}{2})}^{1-h}$

Cor. 6.16.1. $\begin{cases}\lambda(\hat{p},-\varsigma;1)=\bar{\Gamma}(1)\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},-\frac{\varsigma}{2})=\lambda^T(\hat{p},-\frac{\varsigma}{2})\Gamma\lambda(\hat{p},-\frac{\varsigma}{2})\\ \lambda(\hat{p},0;1)=\sqrt{C_2^1}\bar{\Gamma}(1)\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},-\frac{\varsigma}{2})=\sqrt{C_2^1}\lambda^T(\hat{p},\frac{\varsigma}{2})\Gamma\lambda(\hat{p},-\frac{\varsigma}{2})\end{cases}$

Cor. 6.16.2. $\begin{cases}\lambda_m(\hat{p},-\varsigma;1)=S_m(1)\bar{\Gamma}(1)\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},-\frac{\varsigma}{2})=-\frac{1}{\sqrt{2}}\lambda^T(\hat{p},-\frac{\varsigma}{2})\sigma_y\sigma\lambda(\hat{p},-\frac{\varsigma}{2})\\ \lambda_m(\hat{p},0;1)=\sqrt{C_2^1}S_m(1)\bar{\Gamma}(1)\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},-\frac{\varsigma}{2})=-\lambda^T(\hat{p},\frac{\varsigma}{2})\sigma_y\sigma\lambda(\hat{p},-\frac{\varsigma}{2})\end{cases}$

Cor. 6.16.3. $\begin{cases}\lambda_{k_\varsigma}(\hat{p},-\varsigma;1)=\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\\ \lambda_{k_\varsigma}(\hat{p},0;1)=\sqrt{C_2^1}\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p},\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\end{cases}$

Cor. 6.16.4. $\begin{cases}\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma;1)=\frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})=-\frac{1}{\sqrt{2}}(\sigma_y\sigma)_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\\ \lambda_{m\alpha_\varsigma}(\hat{p},0;1)=\sqrt{C_2^1}\frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p},\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})=-(\sigma_y\sigma)_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p},\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\end{cases}$

6.17 Raising and lowering of helicity

Def. 6.17.1.

$$\begin{cases}\hat{Q}(\hat{p},s):=exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\hat{Q}exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\},\hat{Q}(s):=\begin{bmatrix}0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\\0&0&\cdot\cdot&0&0\\0&0&0&1&0\end{bmatrix}_{(2s+1)\times(2s+1)}\\ \hat{Q}^+(\hat{p},s):=exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\hat{Q}^+(s)exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\},\hat{Q}^+(s):=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&\cdot\cdot&0\\0&0&0&0&1\\1&0&0&0&0\end{bmatrix}_{(2s+1)\times(2s+1)}\end{cases}$$

Cor. 6.17.1. $\hat{Q}(\hat{p},s)\hat{Q}^+(\hat{p},s)=\hat{Q}^+(\hat{p},s)\hat{Q}(\hat{p},s)=\hat{Q}(s)\hat{Q}^+(s)=\hat{Q}^+(s)\hat{Q}(s)=1$

Cor. 6.17.2.
$$\begin{cases}\hat{Q}(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h-1;s),\hat{Q}(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},-s;s)=\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},s;s),h=s,s-1,\cdot\cdot,-(s-1)\\ \hat{Q}^{+}(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h+1;s),\hat{Q}^{+}(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},s;s)=\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},-s;s),h=-s,-(s-1),\cdot\cdot,s-1\end{cases}$$

Cor. 6.17.3.
$$\begin{cases}\hat{Q}(\hat{p},s)\lambda(\hat{p},h;s)=\lambda(\hat{p},h-1;s),\hat{Q}(\hat{p},s)\lambda(\hat{p},-s;s)=\lambda(\hat{p},s;s),h=s,s-1,\cdot\cdot,-(s-1)\\ \hat{Q}^{+}(\hat{p},s)\lambda(\hat{p},h;s)=\lambda(\hat{p},h+1;s),\hat{Q}^{+}(\hat{p},s)\lambda(\hat{p},s;s)=\lambda(\hat{p},-s;s),h=-s,-(s-1),\cdot\cdot,s-1\end{cases}$$

Cor. 6.17.4.
$$\begin{cases}\sigma(s)\cdot\hat{p}\hat{Q}(\hat{p},s)\lambda(\hat{p},h;s)=(h-1)\lambda(\hat{p},h-1;s),\hat{Q}(\hat{p},s)\sigma(s)\cdot\hat{p}\lambda(\hat{p},h;s)=h\lambda(\hat{p},h-1;s)\\ \sigma(s)\cdot\hat{p}\hat{Q}(\hat{p},s)\lambda(\hat{p},-s;s)=s\lambda(\hat{p},s;s),\hat{Q}(\hat{p},s)\sigma(s)\cdot\hat{p}\lambda(\hat{p},-s;s)=-s\lambda(\hat{p},s;s)\\ h=-(s-1),\cdot\cdot,s-1,s\end{cases}$$

Cor. 6.17.5.
$$\begin{cases}\sigma(s)\cdot\hat{p}\hat{Q}^{+}(\hat{p},s)\lambda(\hat{p},h;s)=(h+1)\lambda(\hat{p},h+1;s),\hat{Q}^{+}(\hat{p},s)\sigma(s)\cdot\hat{p}\lambda(\hat{p},h;s)=h\lambda(\hat{p},h+1;s)\\ \sigma(s)\cdot\hat{p}\hat{Q}^{+}(\hat{p},s)\lambda(\hat{p},s;s)=-s\lambda(\hat{p},-s;s),\hat{Q}^{+}(\hat{p},s)\sigma(s)\cdot\hat{p}\lambda(\hat{p},s;s)=s\lambda(\hat{p},-s;s)\\ h=-s,-(s-1),\cdot\cdot,s-1\end{cases}$$

Cor. 6.17.6.
$$\begin{cases}[\sigma(s)\cdot\hat{p},\hat{Q}(\hat{p},s)]\lambda(\hat{p},h;s)=-\hat{Q}(\hat{p},s)\lambda(\hat{p},h;s),[\sigma(s)\cdot\hat{p},\hat{Q}(\hat{p},s)]\lambda(\hat{p},-s;s)=2s\hat{Q}(\hat{p},s)\lambda(\hat{p},-s;s)\\ \{\sigma(s)\cdot\hat{p},\hat{Q}(\hat{p},s)\}\lambda(\hat{p},h;s)=(2h-1)\hat{Q}(\hat{p},s)\lambda(\hat{p},h;s),\{\sigma(s)\cdot\hat{p},\hat{Q}(\hat{p},s)\}\lambda(\hat{p},-s;s)=0\\ h=-(s-1),\cdot\cdot,s-1,s\end{cases}$$

Cor. 6.17.7.
$$\begin{cases}[\sigma(s)\cdot\hat{p},\hat{Q}^{+}(\hat{p},s)]\lambda(\hat{p},h;s)=\hat{Q}^{+}(\hat{p},s)\lambda(\hat{p},h;s),[\sigma(s)\cdot\hat{p},\hat{Q}^{+}(\hat{p},s)]\lambda(\hat{p},s;s)=-2\hat{Q}^{+}(\hat{p},s)\lambda(\hat{p},s;s)\\ \{\sigma(s)\cdot\hat{p},\hat{Q}^{+}(\hat{p},s)\}\lambda(\hat{p},h;s)=(2h+1)\hat{Q}^{+}(\hat{p},s)\lambda(\hat{p},h;s),\{\sigma(s)\cdot\hat{p},\hat{Q}^{+}(\hat{p},s)\}\lambda(\hat{p},s;s)=0\\ h=-s,-(s-1),\cdot\cdot,s-1\end{cases}$$

6.18 Arithmetization of helicity eigenfunctions–New mathematical tools

Def. 6.18.1. $\lambda(\hat{\nabla},h;s):=\lambda(\hat{p},h;s)|_{\hat{p}\to\hat{\nabla}},\hat{\nabla}:=\frac{-i\nabla}{\sqrt{-\nabla^2}}$

Cor. 6.18.1. $\lambda(\hat{\nabla},h;s)=exp\{i\frac{[\sigma(s)\times\hat{\nabla}]_z}{\sqrt{1-\hat{\nabla}_z^2}}arccos\hat{\nabla}_z\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)$

Cor. 6.18.2. $\sigma(s)\cdot\hat{\nabla}\lambda(\hat{\nabla},h;s)=h\lambda(\hat{\nabla},s;s),h=-s,\cdot\cdot,s$

Cor. 6.18.3. $\lambda^{+}(\hat{\nabla},h;s)\lambda(\hat{\nabla},h';s)=\delta_{hh'},\sum\limits_{h=s}^{-s}\lambda(\hat{\nabla},h;s)\lambda^{+}(\hat{\nabla},h;s)=1$

Cor. 6.18.4. $\lambda(-\hat{\nabla},h;s)=(-1)^{s+|h|}(\frac{\hat{\nabla}_+}{\hat{\nabla}_-})^h\lambda(\hat{\nabla},-h;s)$

Cor. 6.18.5. $\lambda^{+}(\hat{\nabla},h;s)\sigma(s)\lambda(\hat{\nabla},h;s)=h\hat{\nabla},h=-s,\cdot\cdot,s$

Cor. 6.18.6. $\lambda^{+}(-\hat{\nabla},h;s)\sigma(s)\lambda(\hat{\nabla},h;s)=0,\lambda^{+}(\hat{\nabla},-h;s)\sigma(s)\lambda(\hat{\nabla},h;s)=0,h=-s,\cdot\cdot,s$

Cor. 6.18.7.
$$\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^{+}_{k'_\varsigma}(\hat{p},-s\varsigma)\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\hat{\nabla},-s\varsigma)\lambda^{+}_{k'_\varsigma}(\hat{\nabla},-s\varsigma)\end{cases}$$

Cor. 6.18.8. $\tilde{\partial}_k\{\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}$
$=-\{\frac{[\sigma(s)\times\hat{p}]_z}{1-\hat{p}_z^2}\}\tilde{\partial}_k\hat{p}_z+\{\frac{[\sigma(s)\times\hat{p}]_z\hat{p}_z}{(1-\hat{p}_z^2)^{3/2}}arccos\hat{p}_z\}\tilde{\partial}_k\hat{p}_z+\{\frac{arccos\hat{p}_z}{\sqrt{1-\hat{p}_z^2}}\}\tilde{\partial}_k[\sigma(s)\times\hat{p}]_z$

7 Analytical continuation of helicity $\sigma(s)\cdot\hat{p}$(It still needs to be tightened.)

7.1 Analysis of helicity $\sigma(\frac{1}{2})\cdot\hat{p},\hat{p}\in C$ eigenfunctions

Def. 7.1.1. $\tilde{\lambda}^T(\hat{p},\frac{1}{2}):=-i\lambda^T(\hat{p},-\frac{1}{2})\sigma_y,\tilde{\lambda}^T(\hat{p},-\frac{1}{2}):=i\lambda^T(\hat{p},\frac{1}{2})\sigma_y,\hat{p}=\frac{\vec{p}}{\sqrt{\vec{p}\cdot\vec{p}}}\in C$

Cor. 7.1.1. $\lambda(\hat{p},\frac{1}{2})=\frac{1}{\sqrt{1+\hat{p}_z}}\begin{bmatrix}\frac{1}{\sqrt{2}}(1+\hat{p}_z)\\ \hat{p}_+\end{bmatrix},\lambda(\hat{p},-\frac{1}{2})=\frac{1}{\sqrt{1+\hat{p}_z}}\begin{bmatrix}-\hat{p}_-\\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)\end{bmatrix},\hat{p}^2=1,\hat{p}\in C$

Cor. 7.1.2. $[\sigma(\frac{1}{2})\cdot\hat{p}]\lambda(\hat{p},\frac{1}{2})=\frac{1}{2}\lambda(\hat{p},\frac{1}{2}),[\sigma(\frac{1}{2})\cdot\hat{p}]\lambda(\hat{p},-\frac{1}{2})=-\frac{1}{2}\lambda(\hat{p},\frac{1}{2}),\hat{p}^2=1,\hat{p}\in C$

Pro. 7.1.1. $\tilde{\lambda}^T(\hat{p},\frac{1}{2})=\lambda^+(\hat{p},\frac{1}{2}),\tilde{\lambda}^T(\hat{p},-\frac{1}{2})=\lambda^+(\hat{p},-\frac{1}{2}),\hat{p}\in R$

Cor. 7.1.3. $\tilde{\lambda}^T(\hat{p},h)\lambda(\hat{p},h')=\delta_{hh'},\sum_{h=\frac{1}{2}}^{-\frac{1}{2}}\lambda(\hat{p},h)\tilde{\lambda}^T(\hat{p},h)=1,\sum_{h=\frac{1}{2}}^{-\frac{1}{2}}h\lambda(\hat{p},h)\tilde{\lambda}^T(\hat{p},h)=\sigma(\frac{1}{2})\cdot\hat{p},\hat{p}\in C$

7.2 Analysis of helicity $\sigma(s)\cdot\hat{p},\hat{p}\in C$ eigenfunctions

Def. 7.2.1. $\lambda(\hat{p},h;s):=\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p},\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},\frac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p},-\frac{1}{2})\otimes\cdot\cdot\otimes\lambda(\hat{p},-\frac{1}{2})}^{s-h},\hat{p}\in C$

Def. 7.2.2.

$$\tilde{\lambda}^T(\hat{p},h;s):=(-1)^h\sqrt{C_{2s}^{s-h}}\overbrace{\lambda^T(\hat{p},\frac{1}{2})\sigma_y\otimes\cdot\cdot\otimes\lambda^T(\hat{p},\frac{1}{2})\sigma_y}^{s+h}\otimes\overbrace{\lambda^T(\hat{p},-\frac{1}{2})\sigma_y\otimes\cdot\cdot\otimes\lambda^T(\hat{p},-\frac{1}{2})\sigma_y}^{s-h}\Gamma(s),\hat{p}\in C$$

Cor. 7.2.1. $[\sigma(s)\cdot\hat{p}]\lambda(\hat{p},h;s)=h\lambda(\hat{p},h;s),\hat{p}^2=1,\hat{p}\in C$

Pro. 7.2.1. $\tilde{\lambda}^T(\hat{p},h;s)=\lambda^+(\hat{p},h;s),\hat{p}\in R$

Cor. 7.2.2. $\tilde{\lambda}^T(\hat{p},h;s)\lambda(\hat{p},h';s)=\delta_{hh'},\sum_{h=s}^{-s}\lambda(\hat{p},h;s)\tilde{\lambda}^T(\hat{p},h;s)=1,\sum_{h=s}^{-s}h\lambda(\hat{p},h;s)\tilde{\lambda}^T(\hat{p},h;s)=\sigma(s)\cdot\hat{p},\hat{p}\in C$

7.3 Miscellaneous analysis of helicity $\sigma(s)\cdot\hat{p},\hat{p}\in C$

Thm. 7.3.1. $\tilde{\partial}_i\lambda(\hat{p},1;1)=\frac{1}{p(1+\hat{p}_z)}\{-[\delta_{iz}+\hat{p}_i(2+\hat{p}_z)]\lambda(\hat{p},1;1)+[(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)+(\hat{p}_x-i\hat{p}_y)(\delta_{ix}+i\delta_{iy})]\lambda(\hat{p},1;1)$

$$+\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\ \frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\ (\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}-[(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)+(\hat{p}_x-i\hat{p}_y)(\delta_{ix}+i\delta_{iy})]\lambda(\hat{p},1;1)\}$$

$$=\frac{1}{p(1+\hat{p}_z)}\{(-i\hat{p}_y\delta_{ix}+i\hat{p}_x\delta_{iy})\lambda(\hat{p},1;1)$$

$$+\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\ \frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\ (\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$-[(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)+(\hat{p}_x-i\hat{p}_y)(\delta_{ix}+i\delta_{iy})]\frac{1}{\hat{p}_-}\begin{bmatrix}\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\\ \hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z)\end{bmatrix}\}$$

$$=\frac{1}{p(1+\hat{p}_z)}\{(-i\hat{p}_y\delta_{ix}+i\hat{p}_x\delta_{iy})\lambda(\hat{p},1;1)$$

$$+\begin{bmatrix}(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\\ \frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\ (\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$-(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\frac{1}{\hat{p}_-}\begin{bmatrix}\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\\ \hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z)\end{bmatrix}-\sqrt{2}(\delta_{ix}+i\delta_{iy})\begin{bmatrix}\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\\ \hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z)\end{bmatrix}\}$$

$$=\frac{1}{p(1+\hat{p}_z)}\{(-i\hat{p}_y\delta_{ix}+i\hat{p}_x\delta_{iy})\lambda(\hat{p},1;1)$$

$$+\begin{bmatrix}\frac{1}{2}(\delta_{iz}+\hat{p}_i)(1-\hat{p}_z)^2\\ \frac{1}{\sqrt{2}}[(\delta_{ix}+i\delta_{iy})(1+\hat{p}_z)+(\delta_{iz}+\hat{p}_i)(\hat{p}_x+i\hat{p}_y)]\\ (\delta_{ix}+i\delta_{iy})(\hat{p}_x+i\hat{p}_y)\end{bmatrix}$$

$$-(\delta_{iz}+\hat{p}_i)(1+\hat{p}_z)\frac{1}{\hat{p}_-}\begin{bmatrix}\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\\ \hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z)\end{bmatrix}-\sqrt{2}(\delta_{ix}+i\delta_{iy})\begin{bmatrix}\frac{1}{2}\hat{p}_-(1+\hat{p}_z)\\ \hat{p}_+\hat{p}_-\\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z)\end{bmatrix}\}$$

Chapter16 Mathematical Analysis of Spin Algebra

Self comment: In order to further study the physical content of general spin particles, I have independently developed the following mathematical analysis of spin algebra in this chapter. It provides another new mathematical tool for studying various spin particles.

1 Basic algebraic properties of spin unit vector operator

1.1 Basic operator rules for $\sigma(s), \vec{p}$

Pro. 1.1.1. $$\begin{cases} \sigma(s)\times\vec{p}=[\sigma(s), i\sigma(s)\cdot\vec{p}]=i\{\sigma(s)[\sigma(s)\cdot\vec{p}]-[\sigma(s)\cdot\vec{p}]\sigma(s)\} \\ \vec{p}\times\sigma(s)=-[\sigma(s), i\sigma(s)\cdot\vec{p}]=i\{[\sigma(s)\cdot\vec{p}]\sigma(s)-\sigma(s)[\sigma(s)\cdot\vec{p}]\} \end{cases}$$

Pro. 1.1.2. $$\begin{cases} \sigma(s)\times\vec{p}=-\vec{p}\times\sigma(s) \\ \sigma(s)\times\sigma(s)=i\sigma(s) \end{cases} \quad \begin{cases} \sigma(s)\cdot\vec{p}=\vec{p}\cdot\sigma(s) \\ \sigma^2(s)=s(s+1) \end{cases} \quad \begin{cases} [\sigma(s)\times\vec{p}]\cdot\vec{p}=0 \\ \sigma(s)\cdot[\sigma(s)\times\vec{p}]=i\sigma(s)\cdot\vec{p} \\ [\vec{p}\times\sigma(s)]\cdot\sigma(s)=i\sigma(s)\cdot\vec{p} \end{cases}$$

Pro. 1.1.3. $$\begin{cases} [\sigma(s)\times\vec{p}]\times\vec{p}=[\sigma(s)\cdot\vec{p}]\vec{p}-\sigma(s)(\vec{p}\cdot\vec{p}) \\ \sigma(s)\times[\sigma(s)\times\vec{p}]=[\sigma(s)\cdot\vec{p}]\sigma(s)-\sigma^2(s)\vec{p} \\ [\vec{p}\times\sigma(s)]\times\sigma(s)=\sigma(s)[\sigma(s)\cdot\vec{p}]-\sigma^2(s)\vec{p} \\ \sigma(s)\times\{\sigma(s)\times[\sigma(s)\times\vec{p}]\}=\sigma(s)\times\{[\sigma(s)\cdot\vec{p}]\sigma(s)\}-\sigma(s)\times\vec{p}\sigma^2(s) \\ \sigma(s)\times\{\sigma(s)\times[\sigma(s)\times\vec{p}]\}=\sigma(s)\times\{[\sigma(s)\cdot\vec{p}]\sigma(s)\}-\sigma(s)\times\vec{p}\sigma^2(s) \end{cases}$$

Pro. 1.1.4. $$\begin{cases} i^{2k-1}\hat{p}\cdot\{\sigma(s)\times\{\sigma(s)[\times\hat{p}|]^{2k-1}\}\}=i\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\} \\ i^{2k}\hat{p}\cdot\{\sigma(s)\times\{\sigma(s)[\times\hat{p}|]^{2k}\}\}=i\sigma(s)\cdot\hat{p} \end{cases}$$

Pro. 1.1.5. $$\begin{cases} \{[\sigma(s)\times\vec{p}]\times\vec{p}\}\cdot\vec{p}=0 \\ \{\sigma(s)\times[\sigma(s)\times\vec{p}]\}\cdot\sigma(s)=0 \\ \sigma(s)\cdot\{[\vec{p}\times\sigma(s)]\times\sigma(s)\}=0 \end{cases} \quad \begin{cases} \vec{p}\cdot\{[\sigma(s)\times\vec{p}]\times\vec{p}\}=0 \\ \sigma(s)\cdot\{\sigma(s)\times[\sigma(s)\times\vec{p}]\}=-\sigma(s)\cdot\vec{p} \\ \{[\vec{p}\times\sigma(s)]\times\sigma(s)\}\cdot\sigma(s)=-\sigma(s)\cdot\vec{p} \end{cases}$$

Pro. 1.1.6. $$\begin{cases} \{[\sigma(s)\times\vec{p}]\times\vec{p}\}\cdot\sigma(s)=[\sigma(s)\cdot\vec{p}]^2-\sigma^2(s)\vec{p}^2 \\ \sigma(s)\cdot\{[\sigma(s)\times\vec{p}]\times\vec{p}\}=[\sigma(s)\cdot\vec{p}]^2-\sigma^2(s)\vec{p}^2 \\ \{\sigma(s)\times[\sigma(s)\times\vec{p}]\}\cdot\vec{p}=[\sigma(s)\cdot\vec{p}]^2-\sigma^2(s)\vec{p}^2 \\ \vec{p}\cdot\{[\vec{p}\times\sigma(s)]\times\sigma(s)\}=[\sigma(s)\cdot\vec{p}]^2-\sigma^2(s)\vec{p}^2 \\ -[\sigma(s)\times\vec{p}]\cdot[\sigma(s)\times\vec{p}]=[\sigma(s)\cdot\vec{p}]^2-\sigma^2(s)\vec{p}^2 \end{cases}$$

Pro. 1.1.7. $$\begin{cases} [\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]=i[\sigma(s)\cdot\vec{p}]\vec{p}=i\{[\sigma(s)\times\vec{p}]\times\vec{p}+\sigma(s)\} \\ \{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}\cdot\vec{p}=i[\sigma(s)\cdot\vec{p}]\vec{p}^2 \\ \sigma(s)\cdot\{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}=i[\sigma(s)\cdot\vec{p}]^2 \\ \{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}\cdot\sigma(s)=i[\sigma(s)\cdot\vec{p}]^2 \end{cases}$$

Pro. 1.1.8. $$\begin{cases} [\sigma(s)\times\vec{p}]\times\{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}=i\{[\sigma(s)\times\vec{p}]\times\vec{p}\}[\sigma(s)\cdot\vec{p}] \\ \{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}\times[\sigma(s)\times\vec{p}]=-i[\sigma(s)\cdot\vec{p}]\{[\sigma(s)\times\vec{p}]\times\vec{p}\} \\ [\sigma(s)\times\vec{p}]\cdot\{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}=0 \\ \{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}\cdot[\sigma(s)\times\vec{p}]=0 \end{cases}$$

Pro. 1.1.9. $$\begin{cases} [\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]=i[\sigma(s)\cdot\vec{p}]\vec{p}=i\{[\sigma(s)\times\vec{p}]\times\vec{p}+\sigma(s)\} \\ [\sigma(s)\times\vec{p}]\times\{[\sigma(s)\times\vec{p}]\times[\sigma(s)\times\vec{p}]\}=i\{[\sigma(s)\times\vec{p}]\times\vec{p}\}[\sigma(s)\cdot\vec{p}] \end{cases}$$

Pro. 1.1.10. $$\begin{cases} \{[\sigma(s)\times\hat{p}]\times\hat{p}\}\times\vec{p}=-\sigma(s)\times\hat{p} \\ \sigma(s)\times[\sigma(s)\times\vec{p}]=[\sigma(s)\cdot\vec{p}]\sigma(s)-\sigma^2(s)\vec{p} \\ [\vec{p}\times\sigma(s)]\times\sigma(s)=\sigma(s)[\sigma(s)\cdot\vec{p}]-\sigma^2(s)\vec{p} \end{cases}$$

Pro. 1.1.11. $\begin{cases}\{\hat{p}\}\times\hat{p}=\{\vec{p};\hat{0}\}\times\hat{p}=\{\vec{p};\hat{0}\}|| \\ \{\sigma(s)\}\times\hat{p}=\{\sigma(s);\sigma(s)\times\hat{p}\}\times\hat{p}=\cdots=\{\sigma(s);\pm\sigma(s)\times\hat{p},\pm[\sigma(s)\times\hat{p}]\times\hat{p}\}||\end{cases}$

Pro. 1.1.12. $\begin{cases}\{\sigma(s)\}\times\sigma(s)=\cdots=\{\pm\sigma(s),\pm i\sigma(s)\}|| \\ \{\hat{p}\}\times\sigma(s)=\{\hat{p};\hat{p}\times\sigma(s)\}\times\sigma(s)=\cdots\end{cases}$

Pro. 1.1.13. $\{\sigma[\sigma\cdot\hat{p}]\}\times\sigma=-2i\hat{p}$

1.2 Classification of spin unit vector operator cross multiplication algebras

Pro. 1.2.1.
$$\begin{cases}[\hat{p}\times||]^n\hat{p}=\hat{p}[||\times\hat{p}]^n=\vec{0} \\ <\hat{p};\times\hat{p}>=<\hat{p};\hat{p}\times>\prec(\{\hat{p},\vec{0}\},\times) \\ \textit{Meet closure; Satisfying commutative and associative laws; There are zero elements but no unit elements.}\end{cases}$$

Pro. 1.2.2.
$$\begin{cases}[\sigma(s)\times||]^n\sigma(s)=\sigma(s)[||\times\sigma(s)]^n=i^n\sigma(s) \\ <\sigma(s);\times\sigma(s)>=<\sigma(s);\sigma(s)\times>\prec(\{i^0\sigma(s),i^1\sigma(s),i^2\sigma(s),i^3\sigma(s)\},\times) \\ \textit{Meet closure; Satisfying commutative and associative laws; There are no zero elements and unit elements.}\end{cases}$$

Pro. 1.2.3.
$$\begin{cases}\sigma(s)[||\times\hat{p}]^n=i^{-n}\{1:i\sigma(s)\times\hat{p},2:-\{[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\}\} \\ <\sigma(s);\times\hat{p}>\prec(\{\sigma(s),\pm\sigma(s)\times\hat{p},\pm\{[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\}\},\times\hat{p}) \\ \textit{Meet closure; Not satisfying commutative and associative laws; There are no zero elements and unit elements.}\end{cases}$$

Pro. 1.2.4.
$$\begin{cases}[\hat{p}\times||]^n\sigma(s)=i^n\{1:-i\hat{p}\times\sigma(s),2:-\{[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\}\} \\ <\sigma(s);\hat{p}\times>\prec(\{\sigma(s),\pm\hat{p}\times\sigma(s),\pm\{[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\}\},\hat{p}\times) \\ \textit{Meet closure; Not satisfying commutative and associative laws; There are no zero elements and unit elements.}\end{cases}$$

Pro. 1.2.5.
$$\begin{cases}\hat{p}[||\times\sigma(s)]^n=i^n\{a_n[\sigma(s)\cdot\hat{p}]\sigma(s)+b_n\sigma(s)[\sigma(s)\cdot\hat{p}]-c_n\sigma^2(s)\hat{p}\} \\ <\hat{p};\times\sigma(s)>\prec(i^n\{a_n[\sigma(s)\cdot\hat{p}]\sigma(s)+b_n\sigma(s)[\sigma(s)\cdot\hat{p}]-c_n\sigma^2(s)\hat{p}|n\geq 0\},\times\sigma(s)) \\ \textit{Meet closure; Not satisfying commutative and associative laws; There are no zero elements and unit elements.}\end{cases}$$

Pro. 1.2.6.
$$\begin{cases}[\sigma(s)\times||]^n\hat{p}=i^n\{a_n\sigma(s)[\sigma(s)\cdot\hat{p}]+b_n[\sigma(s)\cdot\hat{p}]\sigma(s)-c_n\sigma^2(s)\hat{p}\} \\ <\hat{p};\sigma(s)\times>\prec(i^n\{a_n\sigma(s)[\sigma(s)\cdot\hat{p}]+b_n[\sigma(s)\cdot\hat{p}]\sigma(s)-c_n\sigma^2(s)\hat{p}|n\geq 0\},\sigma(s)\times) \\ \textit{Meet closure; Not satisfying commutative and associative laws; There are no zero elements and unit elements.}\end{cases}$$

Pro. 1.2.7.
$$\begin{cases}<\sigma(s)\times\hat{p};\sigma(s)\times\hat{p}\times>,\textit{Meet closure; Not satisfying commutative and associative laws; There are no zero elements and unit}\end{cases}$$

Maximum algebra of spin unit vector: Infinite discrete algebra

Pro. 1.2.8.
$$\begin{cases}<\sigma(s),\hat{p};\times>,\textit{Meet closure; Not satisfying commutative and associative laws; There are zero elements but no unit elements.} \\ <\sigma(s),\hat{p};\times>=\sum\limits_{i,j,k=0}^{2s}a_{ij}[\sigma(s)\cdot\hat{p}]^i\sigma(s)[\sigma(s)\cdot\hat{p}]^j+b_k\hat{p}[\sigma(s)\cdot\hat{p}]^k \\ \textit{Linear independence: }\sum\limits_{i,j,k=0}^{2s}a_{ij}h^{i+j+1}+b_kh^k=0,h=s,s-1,\cdots,-(s-1),-s\end{cases}$$

Scalar product algebra of spin unit vector: Finite discrete algebra

Pro. 1.2.9.
$$\begin{cases}<\sigma(s)\cdot\hat{p};\cdot>,\textit{Meet closure; Satisfying commutative and associative laws; There are no zero elements and unit elements.} \\ <\sigma(s)\cdot\hat{p};\cdot>=\sum\limits_{k=0}^{2s}c_k[\sigma(s)\cdot\hat{p}]^k \\ \textit{Linear independence: }\sum\limits_{k=0}^{2s}c_kh^k=0,h=s,s-1,\cdots,-(s-1),-s\end{cases}$$

Using 2s+1 spin helicity functions is expected to solve the linear independence problem of the above two algebras. And completeness can be proven.

2 Basic expansion

2.1 Two types of expansions of $\sigma(s)[|\times\hat{p}]^n$ and $i^n[\hat{p}\times|]^n\sigma(s)$

2.1.1 Simple expansion and general term formula of $i^n\sigma(s)[|\times\hat{p}]^n$ and $i^n[\hat{p}\times|]^n\sigma(s)$

Pro. 2.1.1. $\sigma(s)[\times\hat{p}|]^1=\sigma(s)\times\hat{p}\qquad \sigma(s)[\times\hat{p}|]^2=[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\qquad \sigma(s)[\times\hat{p}|]^3=-\sigma(s)\times\hat{p}$

Thm. 2.1.1.

$$i^n\sigma(s)[|\times\hat{p}]^n=\begin{cases}i\sigma(s)\times\hat{p},n=2k-1\\ \sigma(s)-[\sigma(s)\cdot\hat{p}]\hat{p},n=2k\end{cases},i^n[\hat{p}\times|]^n\sigma(s)=\begin{cases}i\hat{p}\times\sigma(s),n=2k-1\\ \sigma(s)-[\sigma(s)\cdot\hat{p}]\hat{p},n=2k\end{cases},k\geq 1$$

Cor. 2.1.1. $\sigma(s)[|\times\hat{p}]^n=(-1)^n[\hat{p}\times|]^n\sigma(s)$

Cor. 2.1.2. $i^n\sigma(s)\cdot\{\sigma(s)[|\times\hat{p}]^n\}=i^n\{[\hat{p}\times|]^n\sigma(s)\}\cdot\sigma(s)=\begin{cases}-\sigma(s)\cdot\hat{p},n=2k-1\\ -\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\},n=2k\end{cases},k\geq 1$

Cor. 2.1.3. $i^n\sigma(s)\cdot\{\sigma(s)[|\times\hat{p}]^n\}=i^n\{[\hat{p}\times|]^n\sigma(s)\}\cdot\sigma(s)=-\{[\sigma(s)\cdot\hat{p}]^{2-n\%2}-(1-n\%2)\sigma^2(s)\},n\geq 1$

Cor. 2.1.4. $i^n\sigma(s)[|\times\hat{p}]^n|\cdot\hat{p}=i^n\hat{p}\cdot|[\hat{p}\times|]^n\sigma(s)=0,n\geq 1$

Cor. 2.1.5.

$$\begin{cases}\sigma(s)[\times\hat{p}|]^{2k-1}=(-1)^{k+1}\sigma(s)\times\hat{p}\\ \sigma(s)[\times\hat{p}|]^{2k}=(-1)^{k+1}\{[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\}\end{cases}\Rightarrow\begin{cases}i^{2k-1}i\hat{p}\cdot\{\sigma(s)\times\{\sigma(s)[\times\hat{p}|]^{2k-1}\}\}=-\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\}\\ i^{2k}i\hat{p}\cdot\{\sigma(s)\times\{\sigma(s)[\times\hat{p}|]^{2k}\}\}=-\sigma(s)\cdot\hat{p}\end{cases}$$

2.1.2 Similar binomial expansion of $\sigma(s)[|\times\hat{p}]^n$ and $i^n[\hat{p}\times|]^n\sigma(s)$

Pro. 2.1.2. $\sigma(s)\times\hat{p}=[\sigma(s),i\sigma(s)\cdot\hat{p}]=i\{\sigma(s)[\sigma(s)\cdot\hat{p}]-[\sigma(s)\cdot\hat{p}]\sigma(s)\}=i\{\sigma(s)A+B\sigma(s)\}$

Pro. 2.1.3. $\sigma(s)\times\hat{p}|\times\hat{p}=i^2\{\sigma(s)[\sigma(s)\cdot\hat{p}]^2-2[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]+[\sigma(s)\cdot\hat{p}]^2\sigma(s)\}$
$=i^2[\sigma(s)A^2+2B\sigma(s)A+B^2\sigma(s)]\simeq i^2[\sigma^{\frac{1}{2}}(s)A+B\sigma^{\frac{1}{2}}(s)]^2_{B[]A}$

Thm. 2.1.2. $\sigma(s)[|\times\hat{p}]^n=i^n\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n-k},n\geq 1$

Proof: Using mathematical induction to prove:

Step 1: When $i=1$, the following is established. $\sigma(s)[|\times\hat{p}]^1=i^1\sum\limits_{k=0}^{1}C_1^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{1-k}$

Step 2: Assume when $i=n$, the following is established. $\sigma(s)[|\times\hat{p}]^n=i^n\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n-k}$

Step 3: when $i=n+1$,

$\sigma(s)[|\times\hat{p}]^{n+1}=\sigma(s)[|\times\hat{p}]^n|\times\hat{p}$
$=i^n\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^k[\sigma(s)\times\hat{p}][\sigma(s)\cdot\hat{p}]^{n-k}$
$=i^{n+1}\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^k\{\sigma(s)[\sigma(s)\cdot\hat{p}]-[\sigma(s)\cdot\hat{p}]\sigma(s)\}[\sigma(s)\cdot\hat{p}]^{n-k}$
$=i^{n+1}\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1-k}+i^{n+1}\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^{k+1}\sigma(s)[\sigma(s)\cdot\hat{p}]^{n-k}$
$=i^{n+1}\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1-k}+i^{n+1}\sum\limits_{k=1}^{n+1}C_n^{k-1}[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1-k}$
$=i^{n+1}\sum\limits_{k=0}^{n+1}C_n^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1-k}+i^{n+1}\sum\limits_{k=0}^{n+1}C_n^{k-1}[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1-k}$
$=i^{n+1}\sum\limits_{k=0}^{n+1}(C_n^k+C_n^{k-1})[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1-k}$
$=i^{n+1}\sum\limits_{k=0}^{n+1}C_{n+1}^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1-k}$

This step proves that when $i=n+1$, it is established.

Step 4: Reasoning according to the above inductive method, the proposition is established and the theorem is proved. □

Cor. 2.1.6.

$$\begin{cases}i^n\sigma(s)[|\times\hat{p}]^n=\sum\limits_{k=0}^{n}C_n^k[\sigma(s)\cdot\hat{p}]^k\sigma(s)[-\sigma(s)\cdot\hat{p}]^{n-k},n\geq 0\\ i^n[\hat{p}\times|]^n\sigma(s)=\sum\limits_{k=0}^{n}C_n^k[-\sigma(s)\cdot\hat{p}]^{n-k}\sigma(s)[\sigma(s)\cdot\hat{p}]^k,n\geq 0\end{cases}$$

2.2 Recursive formula for $\sigma(s)\cdot\hat{p}]^n\sigma(s)$ and $\sigma(s)[\sigma(s)\cdot\hat{p}]^n$

Thm. 2.2.1.
$$\begin{cases}[\sigma(s)\cdot\hat{p}]^n\sigma(s)=i^n\sigma(s)[|\times\hat{p}]^n-\sum\limits_{k=0}^{n-1}C_n^k[\sigma(s)\cdot\hat{p}]^k\sigma(s)[-\sigma(s)\cdot\hat{p}]^{n-k},n\geq 0\\ \sigma(s)[\sigma(s)\cdot\hat{p}]^n=i^n[\hat{p}\times|]^n\sigma(s)-\sum\limits_{k=0}^{n-1}C_n^k[-\sigma(s)\cdot\hat{p}]^{n-k}\sigma(s)[\sigma(s)\cdot\hat{p}]^k,n\geq 0\end{cases}$$

Cor. 2.2.1.
$$\begin{cases}\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)=i^n\sigma(s)\cdot\{\sigma(s)[|\times\hat{p}]^n\}-\sum\limits_{k=0}^{n-1}C_n^k\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^k\sigma_\alpha(s)[-\sigma(s)\cdot\hat{p}]^{n-k},n\geq 0\\ \sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)=i^n\{[\hat{p}\times|]^n\sigma(s)\}\cdot\sigma(s)-\sum\limits_{k=0}^{n-1}C_n^k[-\sigma(s)\cdot\hat{p}]^{n-k}\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^k\sigma_\alpha(s),n\geq 0\end{cases}$$

Cor. 2.2.2.
$$\begin{cases}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}=i^n\sigma(s)\times\{\sigma(s)[|\times\hat{p}]^n\}-\sum\limits_{k=0}^{n-1}C_n^k\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}[-\sigma(s)\cdot\hat{p}]^{n-k},n\geq 0\\ \{\sigma(s)[\sigma(s)\cdot\hat{p}]^n\}\times\sigma(s)=i^n\{[\hat{p}\times|]^n\sigma(s)\}\times\sigma(s)-\sum\limits_{k=0}^{n-1}C_n^k[-\sigma(s)\cdot\hat{p}]^{n-k}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^k\}\times\sigma(s),n\geq 0\end{cases}$$

Cor. 2.2.3. $X(n)=O(n)-\sum\limits_{k=0}^{n-1}C_n^kX(k)[-\sigma(s)\cdot\hat{p}]^{n-k},n\geq 0$

2.3 General term formula for $i^{-n}[\sigma(s)\times|]^n\hat{p}$ and $i^{-n}\hat{p}[|\times\sigma(s)]^n$

2.3.1 General term formula for $i^{-n}[\sigma(s)\times|]^n\hat{p}$

Pro. 2.3.1.
$$\begin{cases}i^{-0}[\sigma(s)\times|]^0\hat{p}=\hat{p}\\ i^{-1}[\sigma(s)\times|]^1\hat{p}=\{\sigma(s)[\sigma(s)\cdot\hat{p}]-[\sigma(s)\cdot\hat{p}]\sigma(s)\}\\ i^{-2}[\sigma(s)\times|]^2\hat{p}=-[\sigma(s)\cdot\hat{p}]\sigma(s)+\sigma^2(s)\hat{p}\\ i^{-3}[\sigma(s)\times|]^3\hat{p}=-[1-\sigma^2(s)]\sigma(s)[\sigma(s)\cdot\hat{p}]-[1+\sigma^2(s)][\sigma(s)\cdot\hat{p}]\sigma(s)+\sigma^2(s)\hat{p}\\ i^{-4}[\sigma(s)\times|]^4\hat{p}=-[2-\sigma^2(s)]\sigma(s)[\sigma(s)\cdot\hat{p}]-[1+2\sigma^2(s)][\sigma(s)\cdot\hat{p}]\sigma(s)+[1+\sigma^2(s)]\sigma^2(s)\hat{p}\\ i^{-5}[\sigma(s)\times|]^5\hat{p}=-[3-\sigma^4(s)]\sigma(s)[\sigma(s)\cdot\hat{p}]-[1+3\sigma^2(s)+\sigma^4(s)][\sigma(s)\cdot\hat{p}]\sigma(s)+[1+2\sigma^2(s)]\sigma^2(s)\hat{p}\end{cases}$$

Pro. 2.3.2.
$$\begin{cases}\hat{p}\cdot|[\sigma(s)\times|]^2\hat{p}=[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\\ \hat{p}\cdot|[\sigma(s)\times|]^3\hat{p}=2i[\sigma(s)\cdot\hat{p}]^2-i\sigma^2(s)\\ \hat{p}\cdot|[\sigma(s)\times|]^4\hat{p}=i^2[3+\sigma^2(s)][\sigma(s)\cdot\hat{p}]^2-i^2[1+\sigma^2(s)]\sigma^2(s)\end{cases}$$

Lem. 2.3.1. $-(2s+1)=[(s+1)^4-(-s)^4]-2[(s+1)^3-(-s)^3]$

Thm. 2.3.1.
$$\begin{cases}i^{-n}[\sigma(s)\times|]^n\hat{p}=a_n\sigma(s)[\sigma(s)\cdot\hat{p}]+b_n[\sigma(s)\cdot\hat{p}]\sigma(s)-c_n\sigma^2(s)\hat{p}\\ a_n=\frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}\\ b_n=-\frac{(s+1)^n-(-s)^n}{2s+1},c_n=-\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1},n\geq 0\end{cases}$$

Proof: Using mathematical induction to prove:
Step 1: When $i=0$, the following is established.
$$\begin{cases}i^{-0}[\sigma(s)\times|]^0\hat{p}=a_0\sigma(s)[\sigma(s)\cdot\hat{p}]+b_0[\sigma(s)\cdot\hat{p}]\sigma(s)-c_0\sigma^2(s)\hat{p}\\ a_0=0,b_0=0,c_0=-\sigma^{-2}(s)\end{cases}$$
Step 2: Assume when $i=n$, the following is established.
$i^{-n}[\sigma(s)\times|]^n\hat{p}=a_n\sigma(s)[\sigma(s)\cdot\hat{p}]+b_n[\sigma(s)\cdot\hat{p}]\sigma(s)-c_n\sigma^2(s)\hat{p}$
Step 3: When $i=n+1$,
$i^{-(n+1)}[\sigma(s)\times|]^{n+1}\hat{p}$
$=a_n\sigma(s)[\sigma(s)\cdot\hat{p}]-ib_n\sigma(s)\times|[\sigma(s)\cdot\hat{p}]\sigma(s)+ic_n\sigma(s)\times\sigma^2(s)\hat{p}$
$=a_n\sigma(s)[\sigma(s)\cdot\hat{p}]+b_n\{\sigma(s)[\sigma(s)\cdot\hat{p}]+[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\}-c_n\{\sigma(s)[\sigma(s)\cdot\hat{p}]-[\sigma(s)\cdot\hat{p}]\sigma(s)\}\sigma^2(s)$
$=(a_n+b_n-c_n\sigma^2(s))\sigma(s)[\sigma(s)\cdot\hat{p}]+(b_n+c_n\sigma^2(s))[\sigma(s)\cdot\hat{p}]\sigma(s)-b_n\sigma^2(s)\hat{p}$
$=a_{n+1}\sigma(s)[\sigma(s)\cdot\hat{p}]+b_{n+1}[\sigma(s)\cdot\hat{p}]\sigma(s)-c_{n+1}\sigma^2(s)\hat{p}$
This step proves that when $i=n+1$, it is established.
Step 4: The following recursive relationship is obtained.
$$\begin{cases}a_{n+1}=a_n+b_n-c_n\sigma^2(s),b_{n+1}=b_n+c_n\sigma^2(s),c_{n+1}=b_n\\ a_0=0,b_0=0,c_0=-\sigma^{-2}(s);a_1=1,b_1=-1,c_1=0\end{cases}$$
$\Rightarrow$

$$\begin{cases} a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1}, n \ge 0 \end{cases}$$

Step 5: Based on the above inductive reasoning, the proposition is established, and the theorem is proved. □

Cor. 2.3.1.

$$\begin{cases} i^{-n}[\sigma(s) \times |]^n \hat{p} = a_n\sigma(s)[\sigma(s)\cdot\hat{p}] + b_n[\sigma(s)\cdot\hat{p}]\sigma(s) - b_{n-1}\sigma^2(s)\hat{p}, n \ge 0 \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1} \end{cases}$$

Cor. 2.3.2.

$$\begin{cases} i^{-n}\hat{p}\cdot\{[\sigma(s) \times |]^n \hat{p}\} = -k_n[\sigma(s)\cdot\hat{p}]^2 + b_{n-1}\sigma^2(s), n \ge 0 \\ k_n = -(a_n+b_n) = \frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)}, b_{n-1} = -\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1} \end{cases}$$

2.3.2 General term formula for $i^{-n}\hat{p}[| \times \sigma(s)]^n$

Thm. 2.3.2.

$$\begin{cases} i^{-n}\hat{p}[| \times \sigma(s)]^n = a_n[\sigma(s)\cdot\hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s)\cdot\hat{p}] - c_n\sigma^2(s)\hat{p} \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1}, n \ge 0 \end{cases}$$

Proof: Using mathematical induction to prove:
Step 1: When $i = 0$, the following is established.

$$\begin{cases} i^{-0}\hat{p}[| \times \sigma(s)]^0 = a_0[\sigma(s)\cdot\hat{p}]\sigma(s) + b_0\sigma(s)[\sigma(s)\cdot\hat{p}] - c_0\sigma^2(s)\hat{p} \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s) \end{cases}$$

Step 2: Assume when $i = n$, the following is established.
$i^{-n}\hat{p}[| \times \sigma(s)]^n = a_n[\sigma(s)\cdot\hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s)\cdot\hat{p}] - c_n\sigma^2(s)\hat{p}$
Step 3: When $i = n+1$,

$$\begin{aligned} & i^{-(n+1)}\hat{p}[| \times \sigma(s)]^{n+1} \\ &= a_n[\sigma(s)\cdot\hat{p}]\sigma(s) - ib_n\sigma(s)[\sigma(s)\cdot\hat{p}]| \times \sigma(s) + ic_n\hat{p} \times \sigma(s)\sigma^2(s) \\ &= a_n[\sigma(s)\cdot\hat{p}]\sigma(s) + b_n\{[\sigma(s)\cdot\hat{p}]\sigma(s) + \sigma(s)[\sigma(s)\cdot\hat{p}] - \sigma^2(s)\hat{p}\} - c_n\{[\sigma(s)\cdot\hat{p}]\sigma(s) - \sigma(s)[\sigma(s)\cdot\hat{p}]\}\sigma^2(s) \\ &= (a_n + b_n - c_n\sigma^2(s))[\sigma(s)\cdot\hat{p}]\sigma(s) + (b_n + c_n\sigma^2(s))\sigma(s)[\sigma(s)\cdot\hat{p}] - b_n\sigma^2(s)\hat{p} \\ &= a_{n+1}[\sigma(s)\cdot\hat{p}]\sigma(s) + b_{n+1}\sigma(s)[\sigma(s)\cdot\hat{p}] - c_{n+1}\sigma^2(s)\hat{p} \end{aligned}$$

This step proves that when $i = n+1$, it is established.
Step 4: The following recursive relationship is obtained.

$$\begin{cases} a_{n+1} = a_n + b_n - c_n\sigma^2(s), b_{n+1} = b_n + c_n\sigma^2(s), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s); a_1 = 1, b_1 = -1, c_1 = 0 \end{cases}$$

$\Rightarrow$

$$\begin{cases} a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1}, n \ge 0 \end{cases}$$

Step 5: Based on the above inductive reasoning, the proposition is established, and the theorem is proved. □

Cor. 2.3.3.

$$\begin{cases} i^{-n}\hat{p}[| \times \sigma(s)]^n = a_n[\sigma(s)\cdot\hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s)\cdot\hat{p}] - b_{n-1}\sigma^2(s)\hat{p}, n \ge 0 \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1} \end{cases}$$

Cor. 2.3.4.

$$\begin{cases} i^{-n}\{\hat{p}[| \times \sigma(s)]^n\}\cdot\hat{p} = -k_n[\sigma(s)\cdot\hat{p}]^2 + b_{n-1}\sigma^2(s), n \ge 0 \\ k_n = -(a_n+b_n) = \frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)}, b_{n-1} = -\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1} \end{cases}$$

Cor. 2.3.5. $\hat{p}\cdot\{[\sigma(s) \times |]^n\hat{p}\} = \{\hat{p}[| \times \sigma(s)]^n\}\cdot\hat{p}, n \ge 0$

2.3.3 Parameters summary

Cor. 2.3.6.

$$\begin{cases} a_{n+1} = a_n + b_n - c_n\sigma^2(s), b_{n+1} = b_n + c_n\sigma^2(s), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s); a_1 = 1, b_1 = -1, c_1 = 0; a_2 = 0, b_2 = -1, c_2 = -1 \end{cases}$$

$$\Rightarrow \begin{cases} a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1} \\ k_n = -(a_n+b_n) = \frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)}, c_n - (a_n+b_n) = \frac{(s+1)^n-(-s)^n-(2s+1)}{s(2s+1)(s+1)}, n \ge 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1, \sigma^2(s)k_n = -b_{n+1} - 1, c_n = b_{n-1}, k_{n+1} = k_n - b_n \\ b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, \sigma^2(s)(c_n - a_n - b_n) = -b_n - 1, n \ge 0 \end{cases}$$

2.3.4 Parameters generalization

Thm. 2.3.3. $b_{n+1}=b_n+b_{n-1}\sigma^2(s),b_0=0,b_{-1}=-\sigma^{-2}(s)\Leftrightarrow b_n=-\frac{(s+1)^n-(-s)^n}{2s+1};n\in Z$

Proof: $b_{n+1}=b_n+b_{n-1}\sigma^2(s),b_0=0,b_{-1}=-\sigma^{-2}(s);n\in Z$
$\Leftrightarrow b_{n+1}+sb_n=(s+1)(b_n+sb_{n-1}),b_0=0,b_{-1}=-\sigma^{-2}(s);n\in Z$
$\Leftrightarrow b_n+sb_{n-1}=(s+1)^n(b_0+sb_{-1}),b_0=0,b_{-1}=-\sigma^{-2}(s);n\in Z$
$\Leftrightarrow b_n+sb_{n-1}=-(s+1)^{n-1},b_0=0,b_{-1}=-\sigma^{-2}(s);n\in Z$

$$\Leftrightarrow\begin{cases}b_n-(-s)b_{n-1}=-(s+1)^{n-1}(-s)^0\\(-s)b_{n-1}-(-s)^2b_{n-2}=-(s+1)^{n-2}(-s)\\(-s)^2b_{n-2}-(-s)^3b_{n-3}=-(s+1)^{n-3}(-s)^2\\\cdots\cdots\\(-s)^{n-1}b_1-(-s)^nb_0=-(s+1)^0(-s)^{n-1}\\(-s)^nb_0-(-s)^{n+1}b_{-1}=-(s+1)^{-1}(-s)^n\\(-s)^{n+1}b_{-1}-(-s)^{n+2}b_{-2}=-(s+1)^{-2}(-s)^{n+1}\\\cdots\cdots\\(-s)^{n-l-2}b_{l+2}-(-s)^{n-l-1}b_{l+1}=-(s+1)^{l+1}(-s)^{n-l-2}\\(-s)^{n-l-1}b_{l+1}-(-s)^{n-l}b_l=-(s+1)^l(-s)^{n-l-1}\end{cases},b_0=0,b_{-1}=-\sigma^{-2}(s);n\geq 0,l\leq -1$$

$$\Leftrightarrow\begin{cases}b_n-(-s)^nb_0=-(s+1)^{n-1}\sum\limits_{i=0}^{n-1}(\frac{-s}{s+1})^i\\(-s)^nb_0-(-s)^{n-l}b_l=-(s+1)^{n-1}(\frac{-s}{s+1})^n\sum\limits_{i=0}^{-l-1}(\frac{-s}{s+1})^i\end{cases},b_0=0;n\geq 1,l\leq -1$$

$$\Leftrightarrow\begin{cases}b_n-(-s)^nb_0=-(s+1)^{n-1}\sum\limits_{i=0}^{n-1}(\frac{-s}{s+1})^i\\(-s)^lb_0-b_l=-\frac{(-s)^l}{s+1}\sum\limits_{i=0}^{-l-1}(\frac{-s}{s+1})^i\end{cases},b_0=0;n\geq 1,l\leq -1$$

$\Leftrightarrow b_n=-\frac{(s+1)^n-(-s)^n}{2s+1},b_l=-\frac{(s+1)^l-(-s)^l}{2s+1},b_0=0;n\geq 1,l\leq -1$
$\Leftrightarrow b_n=-\frac{(s+1)^n-(-s)^n}{2s+1};n\in Z$ □

Thm. 2.3.4.

$$\begin{cases}a_{n+1}=a_n+b_n-b_{n-1}\sigma^2(s),a_0=0\\b_n=-\frac{(s+1)^n-(-s)^n}{2s+1}\end{cases}\Leftrightarrow\begin{cases}a_n=\frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}\\b_n=-\frac{(s+1)^n-(-s)^n}{2s+1}\end{cases};n\in Z$$

Proof: $\begin{cases}a_{n+1}=a_n+b_n-b_{n-1}\sigma^2(s),a_0=0\\b_n=-\frac{(s+1)^n-(-s)^n}{2s+1}\end{cases};n\in Z$

$$\Leftrightarrow\begin{cases}a_{n+1}=a_n+b_n-b_{n-1}\sigma^2(s),b_{n+1}=b_n+b_{n-1}\sigma^2(s)\\a_0=0,b_0=0,b_{-1}=-\sigma^{-2}(s)\end{cases};n\in Z$$

$$\Leftrightarrow\begin{cases}(a_{n+1}+b_{n+1})-(a_n+b_n)=b_n,b_{n+1}=b_n+b_{n-1}\sigma^2(s)\\a_0+b_0=0,b_0=0,b_{-1}=-\sigma^{-2}(s)\end{cases};n\in Z$$

$$\Leftrightarrow\begin{cases}a_n=\sum\limits_{i=0}^{n-1}b_i-b_n,a_l=\sum\limits_{i=l}^{0}b_i-b_l;n\geq 1,l\leq 0\\b_{n+1}=b_n+b_{n-1}\sigma^2(s),b_0=0,b_{-1}=-\sigma^{-2}(s);n\in Z\end{cases}$$

$$\Leftrightarrow\begin{cases}a_n=\sum\limits_{i=0}^{n-1}b_i-b_n,a_l=-\sum\limits_{i=l}^{0}b_i-b_l;n\geq 1,l\leq 0\\b_n=-\frac{(s+1)^n-(-s)^n}{2s+1};n\in Z\end{cases}$$

$$\Leftrightarrow\begin{cases}a_n=-\sum\limits_{i=0}^{n-1}\frac{(s+1)^i-(-s)^i}{2s+1}+\frac{(s+1)^n-(-s)^n}{2s+1};n\geq 1\\a_l=\sum\limits_{i=l}^{0}\frac{(s+1)^i-(-s)^i}{2s+1}+\frac{(s+1)^l-(-s)^l}{2s+1};l\leq 0\\b_n=-\frac{(s+1)^n-(-s)^n}{2s+1};n\in Z\end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = -\sum_{i=0}^{n-1} \frac{(s+1)^i}{2s+1} + \sum_{i=0}^{n-1} \frac{(-s)^i}{2s+1} + \frac{(s+1)^n-(-s)^n}{2s+1} = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}; n \geq 1 \\ a_l = \sum_{i=l}^{0} \frac{(s+1)^i}{2s+1} - \sum_{i=l}^{0} \frac{(-s)^i}{2s+1} + \frac{(s+1)^l-(-s)^l}{2s+1} = \frac{1}{2s+1} \frac{1-(\frac{1}{s+1})^{-l+1}}{1-\frac{1}{s+1}} - \frac{1}{2s+1} \frac{1-(\frac{1}{-s})^{-l+1}}{1-\frac{1}{-s}} + \frac{(s+1)^l-(-s)^l}{2s+1} \\ = \frac{s+1-(s+1)^l}{s(2s+1)} - \frac{s+(-s)^l}{(2s+1)(s+1)} + \frac{(s+1)^l-(-s)^l}{2s+1} = \frac{(s+1)^2-(s+1)^{l+1}}{s(2s+1)(s+1)} - \frac{s^2-(-s)^{l+1}}{s(2s+1)(s+1)} + \frac{s(s+1)^{l+1}+(s+1)(-s)^{l+1}}{s(2s+1)(s+1)} \\ = \frac{[(s+1)^{l+2}-(-s)^{l+2}]-2[(s+1)^{l+1}-(-s)^{l+1}]+(2s+1)}{s(2s+1)(s+1)}; l \leq 0 \\ b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}; n \in Z \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n-(-s)^n}{2s+1} \end{cases} ; n \in Z$$

$$\Leftrightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1 \\ b_n = -\frac{(s+1)^n-(-s)^n}{2s+1} \end{cases} ; n \in Z$$

$$\Leftrightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1 \\ b_{n+1} = b_n + b_{n-1}\sigma^2(s), b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases} ; n \in Z \qquad \square$$

Cor. 2.3.7.

$$\begin{cases} a_{n+1} = a_n + b_n - b_{n-1}\sigma^2(s) \\ b_{n+1} = b_n + b_{n-1}\sigma^2(s) \\ a_0 = 0, b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases} \Leftrightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1 \\ b_{n+1} = b_n + b_{n-1}\sigma^2(s) \\ b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases} ; n \in Z$$

2.3.5 General term formula for $i^{l-n}\{[\sigma(s) \times |]^n \hat{p}\}[| \times \hat{p}]^l$

Cor. 2.3.8. $$\begin{cases} i^{l-n}\{[\sigma(s) \times |]^n \hat{p}\}[| \times \hat{p}]^l = a_n i^l \sigma(s)[| \times \hat{p}]^l [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}] i^l \sigma(s)[| \times \hat{p}]^l \\ = \begin{cases} -a_n \sigma(s)[\sigma(s) \cdot \hat{p}]^2 + (a_n - b_n)[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] + b_n[\sigma(s) \cdot \hat{p}]^2 \sigma(s), l = 2k-1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^2 \hat{p} + a_n \sigma(s)[\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}]\sigma(s), l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

Cor. 2.3.9. $$\begin{cases} i^{l-n}\{[\sigma(s) \times |]^n \hat{p}\}[| \times \hat{p}]^l| \cdot \sigma(s) = \begin{cases} (2a_n + b_n)[\sigma(s) \cdot \hat{p}]^2 - a_n \sigma^2(s), l = 2k-1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - a_n][\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

Cor. 2.3.10. $$\begin{cases} i^{l-n}\sigma(s) \cdot |\{[\sigma(s) \times |]^n \hat{p}\}[| \times \hat{p}]^l = \begin{cases} -(a_n + 2b_n)[\sigma(s) \cdot \hat{p}]^2 + b_n \sigma^2(s), l = 2k-1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - b_n][\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

Cor. 2.3.11. $i^{l-n}\hat{p} \cdot |\{[\sigma(s) \times |]^n \hat{p}\}[| \times \hat{p}]^l = 0, n \geq 0, l \geq 1$

2.3.6 General term formula for $i^{l-n}[\hat{p} \times |]^l \{\hat{p}[| \times \sigma(s)]^n\}$

Cor. 2.3.12. $$\begin{cases} i^{l-n}[\hat{p} \times |]^l \{\hat{p}[| \times \sigma(s)]^n\} = a_n [\sigma(s) \cdot \hat{p}] i^l [\hat{p} \times |]^l \sigma(s) + b_n i^l [\hat{p} \times |]^l \sigma(s)[\sigma(s) \cdot \hat{p}] - c_n \sigma^2(s)\hat{p} \\ = \begin{cases} -a_n[\sigma(s) \cdot \hat{p}]^2 \sigma(s) + (a_n - b_n)[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] + b_n \sigma(s)[\sigma(s) \cdot \hat{p}]^2, l = 2k-1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^2 \hat{p} + a_n[\sigma(s) \cdot \hat{p}]\sigma(s) + b_n \sigma(s)[\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

Cor. 2.3.13. $$\begin{cases} i^{l-n}\sigma(s) \cdot |[\hat{p} \times |]^l \{\hat{p}[| \times \sigma(s)]^n\} = \begin{cases} (2a_n + b_n)[\sigma(s) \cdot \hat{p}]^2 - a_n \sigma^2(s), l = 2k-1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - a_n][\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

Cor. 2.3.14. $$\begin{cases} i^{l-n}[\hat{p} \times |]^l \{\hat{p}[| \times \sigma(s)]^n\}| \cdot \sigma(s) = \begin{cases} -(a_n + 2b_n)[\sigma(s) \cdot \hat{p}]^2 + b_n \sigma^2(s), l = 2k-1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - b_n][\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

Cor. 2.3.15. $i^{l-n}\hat{p} \cdot |[\hat{p} \times |]^l \{\hat{p}[| \times \sigma(s)]^n\} = 0, n \geq 0, l \geq 1$

Cor. 2.3.16. $$\begin{cases} i^{l-n}\{[\sigma(s) \times |]^n \hat{p}\}[| \times \hat{p}]^l| \cdot \sigma(s) = i^{l-n}\sigma(s) \cdot |[\hat{p} \times |]^l \{\hat{p}[| \times \sigma(s)]^n\} \\ i^{l-n}\sigma(s) \cdot |\{[\sigma(s) \times |]^n \hat{p}\}[| \times \hat{p}]^l = i^{l-n}[\hat{p} \times |]^l \{\hat{p}[| \times \sigma(s)]^n\}| \cdot \sigma(s) \end{cases}$$

3 General term formulas for two kinds of basic spin composite operators

3.1 General term formula for $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)$

3.1.1 Probing and guessing of general term formula for $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)$

Cor. 3.1.1.

$$\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\} = -\{[\sigma(s)\cdot\hat{p}]^{2-n\%2} - (1-n\%2)\sigma^2(s)\} - \sum_{k=0}^{n-1} C_n^k\{\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{n-k}$$

Pro. 3.1.1.

$$\begin{cases}\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^0\sigma(s)\} = \sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^1\sigma(s)\} = [\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^2\sigma(s)\} = [\sigma^2(s)-3][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^3\sigma(s)\} = [\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^3+[3\sigma^2(s)-1]\sigma(s)\cdot\hat{p}\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^4\sigma(s)\} = [\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^4+[6\sigma^2(s)-5][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^5\sigma(s)\} = [\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^5+[10\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^3+[5\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^6\sigma(s)\} = [\sigma^2(s)-21][\sigma(s)\cdot\hat{p}]^6+[15\sigma^2(s)-35][\sigma(s)\cdot\hat{p}]^4+[15\sigma^2(s)-7][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^7\sigma(s)\} = [\sigma^2(s)-28][\sigma(s)\cdot\hat{p}]^7+[21\sigma^2(s)-70][\sigma(s)\cdot\hat{p}]^5+[35\sigma^2(s)-28][\sigma(s)\cdot\hat{p}]^2+[7\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\end{cases}$$

Proof: $\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^4\sigma(s)\}$

$$= -\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\} - \sum_{k=0}^{3} C_4^k\{\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{4-k}$$
$$= -\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\} - \sigma^2(s)[\sigma(s)\cdot\hat{p}]^4 - \sum_{k=1}^{3} C_4^k\{\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{4-k}$$
$$= -\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\} - C_4^0\sigma^2(s)[\sigma(s)\cdot\hat{p}]^4 + C_4^1[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^4 - \sum_{k=2}^{3} C_4^k\{\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{4-k}$$
$$= -\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\} - C_4^0\sigma^2(s)[\sigma(s)\cdot\hat{p}]^4 + C_4^1[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^4 - C_4^2\{[\sigma^2(s)-3][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^2$$
$$- \sum_{k=3}^{3} C_4^k\{\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{4-k}$$
$$= -\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\} - C_4^0\sigma^2(s)[\sigma(s)\cdot\hat{p}]^4 + C_4^1[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^4 - C_4^2\{[\sigma^2(s)-3][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^2$$
$$+ C_4^3\{[\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^3+[3\sigma^2(s)-1]\sigma(s)\cdot\hat{p}\}[\sigma(s)\cdot\hat{p}]$$
$$= [\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^4+[6\sigma^2(s)-5][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)$$ □

Proof: $\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^5\sigma(s)\}$

$$= -[\sigma(s)\cdot\hat{p}] + C_5^0\sigma^2(s)[\sigma(s)\cdot\hat{p}]^5 - C_5^1[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^5 + C_5^2\{[\sigma^2(s)-3][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^3$$
$$- C_5^3\{[\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^3+[3\sigma^2(s)-1]\sigma(s)\cdot\hat{p}\}[\sigma(s)\cdot\hat{p}]^2$$
$$+ C_5^4\{[\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^4+[6\sigma^2(s)-5][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]$$
$$= [\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^5+[10\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^3+[5\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]$$ □

Proof: $\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^6\sigma(s)\}$

$$= -\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\} - C_6^0\sigma^2(s)[\sigma(s)\cdot\hat{p}]^6 + C_6^1[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^6$$
$$- C_6^2\{[\sigma^2(s)-3][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^4$$
$$+ C_6^3\{[\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^3+[3\sigma^2(s)-1]\sigma(s)\cdot\hat{p}\}[\sigma(s)\cdot\hat{p}]^3$$
$$- C_6^4\{[\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^4+[6\sigma^2(s)-5][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^2$$
$$+ C_6^5\{[\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^5+[10\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^3+[5\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]$$
$$= [\sigma^2(s)-21][\sigma(s)\cdot\hat{p}]^6+[15\sigma^2(s)-35][\sigma(s)\cdot\hat{p}]^4+[15\sigma^2(s)-7][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)$$ □

Proof: $\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^7\sigma(s)\}$

$$= -[\sigma(s)\cdot\hat{p}] + C_7^0\sigma^2(s)[\sigma(s)\cdot\hat{p}]^7 - C_7^1[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^7$$
$$+ C_7^2\{[\sigma^2(s)-3][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^5$$
$$- C_7^3\{[\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^3+[3\sigma^2(s)-1]\sigma(s)\cdot\hat{p}\}[\sigma(s)\cdot\hat{p}]^4$$
$$+ C_7^4\{[\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^4+[6\sigma^2(s)-5][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^3$$
$$- C_7^5\{[\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^5+[10\sigma^2(s)-15][\sigma(s)\cdot\hat{p}]^3+[5\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^2$$
$$+ C_7^6\{[\sigma^2(s)-21][\sigma(s)\cdot\hat{p}]^6+[15\sigma^2(s)-35][\sigma(s)\cdot\hat{p}]^4+[15\sigma^2(s)-7][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]$$
$$= [\sigma^2(s)-28][\sigma(s)\cdot\hat{p}]^7+[21\sigma^2(s)-70][\sigma(s)\cdot\hat{p}]^5+[35\sigma^2(s)-28][\sigma(s)\cdot\hat{p}]^2+[7\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]$$ □

Rearranged to:

Pro. 3.1.2.

$$\begin{cases}\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^0\sigma(s)\} = \sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^1\sigma(s)\} = [\sigma^2(s)-C_2^2][\sigma(s)\cdot\hat{p}]\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^2\sigma(s)\} = [\sigma^2(s)-C_3^2][\sigma(s)\cdot\hat{p}]^2+C_2^2\sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^3\sigma(s)\} = [\sigma^2(s)-C_4^2][\sigma(s)\cdot\hat{p}]^3+[C_3^2\sigma^2(s)-C_4^4]\sigma(s)\cdot\hat{p}\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^4\sigma(s)\} = [\sigma^2(s)-C_5^2][\sigma(s)\cdot\hat{p}]^4+[C_4^2\sigma^2(s)-C_5^4][\sigma(s)\cdot\hat{p}]^2+C_4^4\sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^5\sigma(s)\} = [\sigma^2(s)-C_6^2][\sigma(s)\cdot\hat{p}]^5+[C_5^2\sigma^2(s)-C_6^4][\sigma(s)\cdot\hat{p}]^3+[C_5^4\sigma^2(s)-C_6^6][\sigma(s)\cdot\hat{p}]\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^6\sigma(s)\} = [\sigma^2(s)-C_7^2][\sigma(s)\cdot\hat{p}]^6+[C_6^2\sigma^2(s)-C_7^4][\sigma(s)\cdot\hat{p}]^4+[C_6^4\sigma^2(s)-C_7^6][\sigma(s)\cdot\hat{p}]^2+C_6^6\sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^7\sigma(s)\} = [\sigma^2(s)-C_8^2][\sigma(s)\cdot\hat{p}]^7+[C_7^2\sigma^2(s)-C_8^4][\sigma(s)\cdot\hat{p}]^5+[C_7^4\sigma^2(s)-C_8^6][\sigma(s)\cdot\hat{p}]^3+[C_7^6\sigma^2(s)-C_8^8][\sigma(s)\cdot\end{cases}$$

Ass. 3.1.1. $\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\} = [C_n^0\sigma^2(s)-C_{n+1}^2][\sigma(s)\cdot\hat{p}]^n+[C_n^2\sigma^2(s)-C_{n+1}^4][\sigma(s)\cdot\hat{p}]^{n-2}$
$+[C_n^4\sigma^2(s)-C_{n+1}^6][\sigma(s)\cdot\hat{p}]^{n-4}+[C_n^6\sigma^2(s)-C_{n+1}^8][\sigma(s)\cdot\hat{p}]^{n-6}+[C_n^8\sigma^2(s)-C_{n+1}^{10}][\sigma(s)\cdot\hat{p}]^{n-8}+\cdots$

Ass. 3.1.2. $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s) = \sum_{k=0}^{[n/2]}[C_n^{2k}\sigma^2(s)-C_{n+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n-2k}, n\ge 0$

3.1.2 Lemma and proof of general term formula for $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)$

Lem. 3.1.1. $\sum_{l=0}^{n}\sum_{k=0}^{[l/2]}A(k,l) = \sum_{k=0}^{[n/2]}\sum_{l=2k}^{n}A(k,l)$

Lem. 3.1.2. $\sum_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}C_l^{2k} = C_{n+1}^{2k}$

Proof: $\sum_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}C_l^{2k} = C_{n+1}^{2k}$

$\Leftrightarrow \sum_{l=0}^{n-2k}(-1)^{l}C_{n+1}^{n-l}C_{n-l}^{2k} = C_{n+1}^{2k}$

$\Leftrightarrow \sum_{l=0}^{n-2k}(-1)^{l}\frac{1}{(l+1)!(n-l-2k)!} = \frac{1}{(n+1-2k)!}$

$\Leftrightarrow \sum_{l=0}^{n-2k}(-1)^{l}\frac{(n+1-2k)!}{(l+1)!(n-l-2k)!} = 1$

$\Leftrightarrow \sum_{l=0}^{n-2k}(-1)^{l}C_{n+1-2k}^{l+1} = 1$

$\Leftrightarrow \sum_{l=0}^{n+1-2k}(-1)^{l}C_{n+1-2k}^{l} = 0$

$\Leftrightarrow [1+(-1)]^{n+1-2k} = 0$ □

Lem. 3.1.3. $\sum_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)}, k<(n-1)/2$

Proof: $\sum_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)}, k<(n-1)/2$

$\Leftrightarrow \sum_{l=2k+1}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)}$

$\Leftrightarrow \sum_{l=0}^{n-1-2k}(-1)^{l}C_{n+1}^{n-l}C_{n+1-l}^{2(k+1)} = C_{n+2}^{2(k+1)}$

$\Leftrightarrow \sum_{l=0}^{n-1-2k}(-1)^{l}\frac{n+1-l}{(l+1)!(n-l-1-2k)!} = \frac{n+2}{(n-2k)!}$

$\Leftrightarrow \sum_{l=0}^{n-1-2k}(-1)^{l}\frac{(n+1-l)(n-2k)!}{(l+1)!(n-l-1-2k)!} = n+2$

$\Leftrightarrow \sum_{l=0}^{n-1-2k}(-1)^{l}(n+1-l)C_{n-2k}^{l+1} = n+2$

$\Leftrightarrow \sum_{l=0}^{n-1-2k}(-1)^{l+1}(l+1)C_{n-2k}^{l+1} = (n+2)\sum_{l=0}^{n-2k}(-1)^{l}C_{n-2k}^{l}$

$\Leftrightarrow \sum_{l=0}^{n-1-2k}(-1)^{l+1}(l+1)C_{n-2k}^{l+1} = 0$

$\Leftrightarrow \sum_{l=1}^{n-2k}(-1)^{l}lC_{n-2k}^{l} = 0$

$\Leftrightarrow (n-2k)\sum\limits_{l=1}^{n-2k}(-1)^l C_{n-1-2k}^{l-1}=0$

$\Leftrightarrow \sum\limits_{l=0}^{n-1-2k}(-1)^l C_{n-1-2k}^{l}=0$

$\Leftrightarrow [1+(-1)]^{n-1-2k}=0$ □

Lem. 3.1.4. $\sum\limits_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2(k+1)}=C_{n+2}^{2(k+1)}-1, k=(n-1)/2$

Proof: $\sum\limits_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2(k+1)}=C_{n+2}^{2(k+1)}-1, k=[(n-1)/2]=(n-1)/2, n\in odd$

$\Leftrightarrow \sum\limits_{l=2k+1}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2(k+1)}=C_{n+2}^{2(k+1)}-1$ □

Lem. 3.1.5. $\sum\limits_{k=0}^{[(n+1)/2]}[C_{n+1}^{2k}\sigma^2(s)-C_{n+2}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}$

$=-[\sigma(s)\cdot\hat{p}]^{2-(n+1)\%2}-[1-(n+1)\%2]\sigma^2(s)+\sum\limits_{k=0}^{[n/2]}\sum\limits_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}$

Proof: $\sum\limits_{k=0}^{[n/2]}\sum\limits_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}$

$=\sum\limits_{k=0}^{[n/2]}[C_{n+1}^{2k}\sigma^2(s)-C_{n+2}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}+[\sigma(s)\cdot\hat{p}]^{2-(n+1)\%2}$

$=\sum\limits_{k=0}^{[(n+1)/2]}[C_{n+1}^{2k}\sigma^2(s)-C_{n+2}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}+[\sigma(s)\cdot\hat{p}]^{2-(n+1)\%2}+[1-(n+1)\%2]\sigma^2(s)$ □

3.1.3 Mathematical induction proof of general term formula for $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)$

Thm. 3.1.1. $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)=\sum\limits_{k=0}^{[n/2]}[C_n^{2k}\sigma^2(s)-C_{n+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n-2k}, n\geq 0$

Proof:
Use mathematical induction to prove this theorem.
Step 1: When $i=0$, the following is established.

$\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^0\sigma_\alpha(s)=\sum\limits_{k=0}^{[0/2]}[C_0^{2k}\sigma^2(s)-C_{0+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{0-2k}$

Step 2: Assume when $i<=n$, the following is established.

$\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^i\sigma_\alpha(s)=\sum\limits_{k=0}^{[i/2]}[C_i^{2k}\sigma^2(s)-C_{i+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{i-2k}, 0\leq i\leq n$

Step 3: When $i=n+1$, $\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^{n+1}\sigma(s)\}$

$=-\{[\sigma(s)\cdot\hat{p}]^{2-(n+1)\%2}-[1-(n+1)\%2]\sigma^2(s)\}-\sum\limits_{l=0}^{n}C_{n+1}^{l}\{\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{n+1-l}$

$=-\{[\sigma(s)\cdot\hat{p}]^{2-(n+1)\%2}-[1-(n+1)\%2]\sigma^2(s)\}-\sum\limits_{l=0}^{n}C_{n+1}^{l}\sum\limits_{k=0}^{[l/2]}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{l-2k}[-\sigma(s)\cdot\hat{p}]^{n+1-l}$

$=-\{[\sigma(s)\cdot\hat{p}]^{2-(n+1)\%2}-[1-(n+1)\%2]\sigma^2(s)\}-\sum\limits_{l=0}^{n}\sum\limits_{k=0}^{[l/2]}(-1)^{n+1-l}C_{n+1}^{l}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}$

$=-\{[\sigma(s)\cdot\hat{p}]^{2-(n+1)\%2}-[1-(n+1)\%2]\sigma^2(s)\}+\sum\limits_{k=0}^{[n/2]}\sum\limits_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}$

$=\sum\limits_{k=0}^{[(n+1)/2]}[C_{n+1}^{2k}\sigma^2(s)-C_{n+2}^{2(k+1)}][\sigma(s)\cdot\hat{p}]^{n+1-2k}$

This step proves that when $i=n+1$, it is established.
Step 4: Based on the above inductive reasoning, the proposition is established, and the theorem is proved. □

3.1.4 Corollaries of general term formula for $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)$

Lem. 3.1.6. $\sum\limits_{k=0}^{[n/2]}s^{n-2k}[C_n^{2k}\sigma^2(s)-C_{n+1}^{2(k+1)}]=s[s^{n+1}+(s-1)^n]$

Cor. 3.1.2. $\sigma^\alpha(s)[\sigma(s)\cdot\hat{p}]^n\sigma_\alpha(s)\lambda(\hat{p},-s\varsigma)=s[s^{n+1}+(s-1)^n][-\varsigma]^n\lambda(\hat{p},-s\varsigma)$

Cor. 3.1.3. $[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^2\psi=\psi\Rightarrow\sigma^\alpha(s)[\sigma(s)\cdot\hat{\nabla}]^n\sigma_\alpha(s)\psi=s[s^{n+1}+(s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\psi$

3.2 General term formula for $i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}$

3.2.1 Probing of general term formula for $i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}$

Pro. 3.2.1. $\begin{cases}\sigma(s)\times\hat{p}=[\sigma(s),i\sigma(s)\cdot\hat{p}]=i\{\sigma(s)[\sigma(s)\cdot\hat{p}]-[\sigma(s)\cdot\hat{p}]\sigma(s)\}\\ \hat{p}\times\sigma(s)=-[\sigma(s),i\sigma(s)\cdot\hat{p}]=i\{[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]\}\end{cases}$

Pro. 3.2.2. $\begin{cases}i^n\sigma(s)[|\times\hat{p}]^n=\begin{cases}i\sigma(s)\times\hat{p},n=2k-1\\-\{[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\},n=2k\end{cases},k\geq 1\\ i^n[\hat{p}\times|]^n\sigma(s)=\begin{cases}i\hat{p}\times\sigma(s),n=2k-1\\-\{[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\},n=2k\end{cases},k\geq 1\end{cases}$

Pro. 3.2.3. $\begin{cases}i^n\sigma(s)\times\{\sigma(s)[\times\hat{p}|]^n\}=\begin{cases}i\{[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\},n=2k-1\\-[\sigma(s)\times\hat{p}][\sigma(s)\cdot\hat{p}]+i\sigma(s),n=2k\end{cases},k\geq 1\\ i^ni^{-1}\sigma(s)\times\{\sigma(s)[|\times\hat{p}]^n\}=\begin{cases}[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p},n=2k-1\\-\sigma(s)[\sigma(s)\cdot\hat{p}]^2+[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]+\sigma(s),n=2k\end{cases},k\geq 1\end{cases}$

Cor. 3.2.1.

$i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}=i^ni^{-1}\sigma(s)\times\{\sigma(s)[|\times\hat{p}]^n\}-\sum\limits_{k=0}^{n-1}C_n^k\{i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{n-k}$

$$\Rightarrow\begin{cases}i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^0\sigma(s)\}=\sigma(s)\\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^1\sigma(s)\}=\begin{cases}[\sigma(s)\cdot\hat{p}]\sigma(s)\\ \sigma(s)[\sigma(s)\cdot\hat{p}]-\sigma^2(s)\hat{p}\end{cases}\\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^2\sigma(s)\}=\begin{cases}3[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]\\-2\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]+\sigma(s)\end{cases}\\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^3\sigma(s)\}=\begin{cases}6[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]^2+[\sigma(s)\cdot\hat{p}]\sigma(s)\\-2\sigma(s)[\sigma(s)\cdot\hat{p}]^3-3\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]^2+3\sigma(s)[\sigma(s)\cdot\hat{p}]-\sigma^2(s)\hat{p}\end{cases}\\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^4\sigma(s)\}=\begin{cases}[\sigma(s)\cdot\hat{p}]\sigma(s)\{10[\sigma(s)\cdot\hat{p}]^2+5\}[\sigma(s)\cdot\hat{p}]\\+\sigma(s)\{-5[\sigma(s)\cdot\hat{p}]^4+5[\sigma(s)\cdot\hat{p}]^2+1\}-\sigma^2(s)\hat{p}\{4[\sigma(s)\cdot\hat{p}]^2+4\}[\sigma(s)\cdot\hat{p}]^1\end{cases}\end{cases}$$

Proof:
$i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^1\sigma(s)\}$
$=\{[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\}-i\{\sigma(s)\times\sigma(s)\}[\sigma(s)\cdot\hat{p}]$
$=\{\sigma(s)\cdot\hat{p},\sigma(s)\}-\sigma^2(s)\hat{p}$ □

Proof:
$i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^2\sigma(s)\}$
$=-[\sigma(s),\sigma(s)\cdot\hat{p}][\sigma(s)\cdot\hat{p}]+\sigma(s)+\sum\limits_{k=0}^{1}C_2^k\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{2-k}$
$=-[\sigma(s),\sigma(s)\cdot\hat{p}][\sigma(s)\cdot\hat{p}]+\sigma(s)-C_2^0\sigma(s)[\sigma(s)\cdot\hat{p}]^2+\sum\limits_{k=1}^{1}C_2^k\{i\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{2-k}$
$=-[\sigma(s),\sigma(s)\cdot\hat{p}][\sigma(s)\cdot\hat{p}]-\sigma(s)-C_2^0\sigma(s)[\sigma(s)\cdot\hat{p}]^2+C_2^1[\{\sigma(s),\sigma(s)\cdot\hat{p}\}-\sigma^2(s)\hat{p}][\sigma(s)\cdot\hat{p}]$
$=3[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]-2\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]+\sigma(s)$ □

Proof:
$i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^3\sigma(s)\}$
$=-i^3i\sigma(s)\times\{\sigma(s)[|\times\hat{p}]^3\}+\sum\limits_{k=0}^{2}C_3^k\{i\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{3-k}$
$=\{[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\}+C_3^0\sigma(s)[\sigma(s)\cdot\hat{p}]^3-\sum\limits_{k=1}^{2}C_3^k\{i\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{3-k}$
$=-\{[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\}-C_3^0\sigma(s)[\sigma(s)\cdot\hat{p}]^3+C_3^1\{-[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]+\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^2+\sum\limits_{k=2}^{2}C_3^k\{i\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{3-k}$
$=\{[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\}+C_3^0\sigma(s)[\sigma(s)\cdot\hat{p}]^3+C_3^1\{-[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]+\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^2$
$-C_3^2\{-3[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]+2\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]-\sigma(s)\}[\sigma(s)\cdot\hat{p}]$
$=6[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]^2-2\sigma(s)[\sigma(s)\cdot\hat{p}]^3-3\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]^2+[\sigma(s)\cdot\hat{p}]\sigma(s)+3\sigma(s)[\sigma(s)\cdot\hat{p}]-\sigma^2(s)\hat{p}$ □

Proof: $i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^4\sigma(s)\}$
$=i^4i^{-1}\sigma(s)\times\{\sigma(s)[|\times\hat{p}]^4\}-\sum\limits_{k=0}^{3}C_4^k\{i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{4-k}$

$= -\sigma(s)[\sigma(s)\cdot\hat{p}]^2 + [\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}] + \sigma(s)$
$- C_4^0\sigma(s)[\sigma(s)\cdot\hat{p}]^4$
$+ C_4^1\{[\sigma(s)\cdot\hat{p}]\sigma(s) + \sigma(s)[\sigma(s)\cdot\hat{p}] - \sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^3$
$- C_4^2\{3[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}] - 2\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}] + \sigma(s)\}[\sigma(s)\cdot\hat{p}]^2$
$+ C_4^3\{6[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]^2 + [\sigma(s)\cdot\hat{p}]\sigma(s) - 2\sigma(s)[\sigma(s)\cdot\hat{p}]^3 - 3\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]^2 + 3\sigma(s)[\sigma(s)\cdot\hat{p}] - \sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^1$
$= [\sigma(s)\cdot\hat{p}]\sigma(s)\{(1 + C_4^3)[\sigma(s)\cdot\hat{p}] + (C_4^1 - 3C_4^2 + 6C_4^3)[\sigma(s)\cdot\hat{p}]^3\}$
$+ \{(-C_4^0 + C_4^1 - 2C_4^3)\sigma(s)\}[\sigma(s)\cdot\hat{p}]^4 + \{-C_4^1 + 2C_4^2 - 3C_4^3\}\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]^3$
$+ \{-1 - C_4^2 + 3C_4^3\}\sigma(s)[\sigma(s)\cdot\hat{p}]^2$
$+ \{-C_4^3\}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^1$
$+ \sigma(s)$
$= [\sigma(s)\cdot\hat{p}]\sigma(s)\{(C_4^1 + C_4^2)[\sigma(s)\cdot\hat{p}]^2 + (C_4^3 + C_4^4)\}[\sigma(s)\cdot\hat{p}] + \sigma(s)\{(C_4^0 - C_4^2)[\sigma(s)\cdot\hat{p}]^4 + (C_4^2 - C_4^4)[\sigma(s)\cdot\hat{p}]^2 + (C_4^4 - C_4^6)\} - \sigma^2(s)\hat{p}\{C_4^1[\sigma(s)\cdot\hat{p}]^2 + C_4^3\}[\sigma(s)\cdot\hat{p}]^1$
$= [\sigma(s)\cdot\hat{p}]\sigma(s)\{10[\sigma(s)\cdot\hat{p}]^2 + 5\}[\sigma(s)\cdot\hat{p}] + \sigma(s)\{-5[\sigma(s)\cdot\hat{p}]^4 + 5\}[\sigma(s)\cdot\hat{p}]^2 + 1\} - \sigma^2(s)\hat{p}\{4[\sigma(s)\cdot\hat{p}]^2 + 4\}[\sigma(s)\cdot\hat{p}]^1$ □

Cor. 3.2.2.

$$\begin{cases} i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^0\sigma(s)\} = \sigma(s) \\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^1\sigma(s)\} = \begin{cases} (C_1^1 + C_1^2)[\sigma(s)\cdot\hat{p}]\sigma(s) \\ +(C_1^0 - C_1^2)\sigma(s)[\sigma(s)\cdot\hat{p}] \\ -C_1^1\sigma^2(s)\hat{p} \end{cases} \\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^2\sigma(s)\} = \begin{cases} (C_2^1 + C_2^2)[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}] \\ +(C_2^0 - C_2^2)\sigma(s)[\sigma(s)\cdot\hat{p}]^2 + (C_2^2 - C_2^4)\sigma(s) \\ -C_2^1\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}] \end{cases} \\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^3\sigma(s)\} = \begin{cases} [\sigma(s)\cdot\hat{p}]\sigma(s)\{(C_3^1 + C_3^2)[\sigma(s)\cdot\hat{p}]^2 + (C_3^3 + C_3^4)\} \\ +\sigma(s)\{(C_3^0 - C_3^2)[\sigma(s)\cdot\hat{p}]^2 + (C_3^2 - C_3^4)\}[\sigma(s)\cdot\hat{p}] \\ -\sigma^2(s)\hat{p}\{C_3^1[\sigma(s)\cdot\hat{p}]^2 + C_3^3\} \end{cases} \\ i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^4\sigma(s)\} = \begin{cases} [\sigma(s)\cdot\hat{p}]\sigma(s)\{(C_4^1 + C_4^2)[\sigma(s)\cdot\hat{p}]^2 + (C_4^3 + C_4^4)\}[\sigma(s)\cdot\hat{p}] \\ +\sigma(s)\{(C_4^0 - C_4^2)[\sigma(s)\cdot\hat{p}]^4 + (C_4^2 - C_4^4)[\sigma(s)\cdot\hat{p}]^2 + (C_4^4 - C_4^6)\} \\ -\sigma^2(s)\hat{p}\{C_4^1[\sigma(s)\cdot\hat{p}]^2 + C_4^3\}[\sigma(s)\cdot\hat{p}]^1 \end{cases} \end{cases}$$

Ass. 3.2.1. $i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}, n \geq 0$
$= [\sigma(s)\cdot\hat{p}]\sigma(s)\sum_{k=0}^{[(n-1)/2]}(C_n^{2k+1} + C_n^{2k+2})[\sigma(s)\cdot\hat{p}]^{n-2k-1} + \sigma(s)\sum_{k=0}^{[n/2]}(C_n^{2k} - C_n^{2k+2})[\sigma(s)\cdot\hat{p}]^{n-2k}$
$- \sigma^2(s)\hat{p}\sum_{k=0}^{[(n-1)/2]} C_n^{2k+1}[\sigma(s)\cdot\hat{p}]^{n-2k-1}$

3.2.2 Mathematical induction proof of general term formula for $i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}$

Thm. 3.2.1. $i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}, n \geq 0$
$= \sum_{k=0}^{[n/2]}\{(C_n^{2k+1} + C_n^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) + (C_n^{2k} - C_n^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] - C_n^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{n-2k-1}$

Proof:
Use mathematical induction to prove this theorem.
Step 1: When $i = 0$, the following is established.
$i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^0\sigma(s)\} = \sum_{k=0}^{[0/2]}\{(C_0^{2k+1} + C_0^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) + (C_0^{2k} - C_0^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] - C_0^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{0-2k-1}$
Step 2: Assume when $0 \leq l \leq n$, the following is established.
$i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\} = \sum_{k=0}^{[l/2]}\{(C_i^{2k+1} + C_i^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) + (C_i^{2k} - C_i^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] - C_i^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{i-2k-1}$
Step 3: When $i = n+1$,$i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^{n+1}\sigma(s)\}$
$= i^{n+1}i^{-1}\sigma(s)\times\{\sigma(s)[| \times\hat{p}]^{n+1}\} - \sum_{l=0}^{n} C_{n+1}^l\{i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{n+1-l}$
$= i^{n+1}i^{-1}\sigma(s)\times\{\sigma(s)[| \times\hat{p}]^{n+1}\}$
$- \sum_{l=0}^{n} C_{n+1}^l\sum_{k=0}^{[l/2]}\{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) + (C_l^{2k} - C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] - C_l^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}[-\sigma(s)\cdot\hat{p}]^{n+1-l}$
$= i^{n+1}i^{-1}\sigma(s)\times\{\sigma(s)[| \times\hat{p}]^{n+1}\}$
$+ \sum_{l=0}^{n}\sum_{k=0}^{[l/2]}(-1)^{n-l}C_{n+1}^l\{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) + (C_l^{2k} - C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] - C_l^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{n-2k}$
$= i^{n+1}i^{-1}\sigma(s)\times\{\sigma(s)[| \times\hat{p}]^{n+1}\}$

$+\sum_{k=0}^{[n/2]}\sum_{l=2k}^{n}(-1)^{n-l}C_{n+1}^{l}\{(C_l^{2k+1}+C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)+(C_l^{2k}-C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]-C_l^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{n-2k}$

$=\sum_{k=0}^{[(n+1)/2]}\{(C_{n+1}^{2k+1}+C_{n+1}^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)+(C_{n+1}^{2k}-C_{n+1}^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]-C_{n+1}^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{n-2k}$

This step proves that when $i=n+1$, it is established.

Step 4: Based on the above inductive reasoning, the proposition is established, and the theorem is proved. □

3.2.3 Induction proof of general term formula for $i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^n\}\times\sigma(s)$

Thm. 3.2.2. $i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^n\}\times\sigma(s), n\geq 0$

$=\{\sum_{k=0}^{[(n-1)/2]}(C_n^{2k+1}+C_n^{2k+2})[\sigma(s)\cdot\hat{p}]^{n-2k-1}\}\sigma(s)[\sigma(s)\cdot\hat{p}]+\{\sum_{k=0}^{[n/2]}(C_n^{2k}-C_n^{2k+2})[\sigma(s)\cdot\hat{p}]^{n-2k}\}\sigma(s)$

$-\{\sum_{k=0}^{[(n-1)/2]}C_n^{2k+1}[\sigma(s)\cdot\hat{p}]^{n-2k-1}\}\sigma^2(s)\hat{p}$

Thm. 3.2.3. $i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^n\}\times\sigma(s), n\geq 0$

$=\sum_{k=0}^{[n/2]}[\sigma(s)\cdot\hat{p}]^{n-2k-1}\{(C_n^{2k+1}+C_n^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]+(C_n^{2k}-C_n^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)-C_n^{2k+1}\sigma^2(s)\hat{p}\}$

Proof:

Use mathematical induction to prove this theorem.

Step 1: When $i=0$, the following is established.

$i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^0\}\times\sigma(s)=\sum_{k=0}^{[0/2]}[\sigma(s)\cdot\hat{p}]^{0-2k-1}\{(C_0^{2k+1}+C_0^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]+(C_0^{2k}-C_0^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)-C_0^{2k+1}\sigma^2(s)\hat{p}\}$

Step 2: Assume when $0\leq l\leq n$, the following is established.

$i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}\times\sigma(s)=\sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{i-2k-1}\{(C_i^{2k+1}+C_i^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]+(C_i^{2k}-C_i^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)-C_i^{2k+1}\sigma^2(s)\hat{p}\}$

Step 3: When $i=n+1$,$i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^{n+1}\}\times\sigma(s)=i^{n+1}i^{-1}\{[\hat{p}\times|]^{n+1}\sigma(s)\}\times\sigma(s)-\sum_{l=0}^{n}C_{n+1}^{l}[-\sigma(s)\cdot\hat{p}]^{n+1-l}\{i^{-1}\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}\times\sigma(s)$

$=i^{n+1}i^{-1}\{[\hat{p}\times|]^{n+1}\sigma(s)\}\times\sigma(s)$

$-\sum_{l=0}^{n}C_{n+1}^{l}\sum_{k=0}^{[l/2]}[-\sigma(s)\cdot\hat{p}]^{n+1-l}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{(C_l^{2k+1}+C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]+(C_l^{2k}-C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)-C_l^{2k+1}\sigma^2(s)\hat{p}\}$

$=i^{n+1}i^{-1}\{[\hat{p}\times|]^{n+1}\sigma(s)\}\times\sigma(s)$

$+\sum_{l=0}^{n}\sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{n-2k}(-1)^{n-l}C_{n+1}^{l}\{(C_l^{2k+1}+C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]+(C_l^{2k}-C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)-C_l^{2k+1}\sigma^2(s)\hat{p}\}$

$=i^{n+1}i^{-1}\{[\hat{p}\times|]^{n+1}\sigma(s)\}\times\sigma(s)$

$+\sum_{k=0}^{[n/2]}\sum_{l=2k}^{n}[\sigma(s)\cdot\hat{p}]^{n-2k}(-1)^{n-l}C_{n+1}^{l}\{(C_l^{2k+1}+C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]+(C_l^{2k}-C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)-C_l^{2k+1}\sigma^2(s)\hat{p}\}$

$=\sum_{k=0}^{[(n+1)/2]}[\sigma(s)\cdot\hat{p}]^{n-2k}\{(C_{n+1}^{2k+1}+C_{n+1}^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]+(C_{n+1}^{2k}-C_{n+1}^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)-C_{n+1}^{2k+1}\sigma^2(s)\hat{p}\}$

This step proves that when $i=n+1$, it is established.

Step 4: Based on the above inductive reasoning, the proposition is established, and the theorem is proved. □

3.2.4 General term formula for $i^{-1}\hat{p}\cdot|\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}$

Cor. 3.2.3. $i^{-1}\hat{p}\cdot|\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}=\sum_{k=0}^{(n+1)/2}[C_{n+1}^{2k+1}-C_n^{2k-1}\sigma^2(s)][\sigma(s)\cdot\hat{p}]^{n+1-2k}, n\geq 0$

Proof: $i^{-1}\hat{p}\cdot|\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}=i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^n\}\times\sigma(s)|\cdot\hat{p}$

$=\sum_{k=0}^{[n/2]}\{(C_n^{2k+1}+C_n^{2k+2})[\sigma(s)\cdot\hat{p}]^2+(C_n^{2k}-C_n^{2k+2})[\sigma(s)\cdot\hat{p}]^2-C_n^{2k+1}\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^{n-2k-1}$

$=\sum_{k=0}^{[n/2]}\{(C_n^{2k+1}+C_n^{2k+2})+(C_n^{2k}-C_n^{2k+2})\}[\sigma(s)\cdot\hat{p}]^{n+1-2k}-\sum_{k=0}^{[n/2]}C_n^{2k+1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{n-1-2k}$

$=\sum_{k=0}^{[n/2]}C_{n+1}^{2k+1}[\sigma(s)\cdot\hat{p}]^{n+1-2k}-\sum_{k=1}^{[n/2]+1}C_n^{2k-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{n+1-2k}$

$=\sum_{k=0}^{[(n+1)/2]}C_{n+1}^{2k+1}[\sigma(s)\cdot\hat{p}]^{n+1-2k}-\sum_{k=0}^{[(n-1)/2]+1}C_n^{2k-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{n+1-2k}$

$=\sum_{k=0}^{[(n+1)/2]}[C_{n+1}^{2k+1}-C_n^{2k-1}\sigma^2(s)][\sigma(s)\cdot\hat{p}]^{n+1-2k}$ □

4 General term formula of complex spin composite operator

4.1 General term formula for $i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)$ and $i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n$

4.1.1 General term formula for $i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)$

Pro. 4.1.1.
$$\begin{cases} i^{-0}[\sigma(s)\times|]^0\{[\sigma(s)\cdot\hat{p}]\sigma(s)\}=[\sigma(s)\cdot\hat{p}]\sigma(s)\\ i^{-1}[\sigma(s)\times|]^1\{[\sigma(s)\cdot\hat{p}]\sigma(s)\}=\sigma(s)[\sigma(s)\cdot\hat{p}]+[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\\ i^{-2}[\sigma(s)\times|]^2[\sigma(s)\cdot\hat{p}]\sigma(s)=[2-\sigma^2(s)]\sigma(s)[\sigma(s)\cdot\hat{p}]+[1+\sigma^2(s)][\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma^2(s)\hat{p}\\ i^{-3}[\sigma(s)\times|]^3[\sigma(s)\cdot\hat{p}]\sigma(s)=[3-\sigma^2(s)]\sigma(s)[\sigma(s)\cdot\hat{p}]+[1+2\sigma^2(s)][\sigma(s)\cdot\hat{p}]\sigma(s)-[1+\sigma^2(s)]\sigma^2(s)\hat{p}\\ i^{-4}[\sigma(s)\times|]^4[\sigma(s)\cdot\hat{p}]\sigma(s)=[4-\sigma^4(s)]\sigma(s)[\sigma(s)\cdot\hat{p}]+[1+3\sigma^2(s)+\sigma^4(s)][\sigma(s)\cdot\hat{p}]\sigma(s)-[1+2\sigma^2(s)]\sigma^2(s)\hat{p} \end{cases}$$

Pro. 4.1.2.
$$\begin{cases} i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=\sigma(s)[\sigma(s)\cdot\hat{p}]-i^{-(n+1)}[\sigma(s)\times|]^{n+1}\hat{p}\\ i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=[\sigma(s)\cdot\hat{p}]^2-i^{-(n+1)}\hat{p}\cdot|[\sigma(s)\times|]^{n+1}\hat{p} \end{cases}$$

Proof: $i^{-(n+1)}[\sigma(s)\times|]^{n+1}\hat{p}$
$=i^{-n}[\sigma(s)\times|]^n i^{-1}\sigma(s)\times\hat{p}$
$=i^{-n}[\sigma(s)\times|]^n\{\sigma(s)[\sigma(s)\cdot\hat{p}]-[\sigma(s)\cdot\hat{p}]\sigma(s)\}$
$=\sigma(s)[\sigma(s)\cdot\hat{p}]-i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)$
$\Rightarrow i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=\sigma(s)[\sigma(s)\cdot\hat{p}]-i^{-(n+1)}[\sigma(s)\times|]^{n+1}\hat{p}$ □

Cor. 4.1.1.
$$\begin{cases} i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=(1-a_{n+1})\sigma(s)[\sigma(s)\cdot\hat{p}]-b_{n+1}[\sigma(s)\cdot\hat{p}]\sigma(s)+c_{n+1}\sigma^2(s)\hat{p}\\ a_{n+1}=\frac{[(s+1)^{n+3}-(-s)^{n+3}]-2[(s+1)^{n+2}-(-s)^{n+2}]+(2s+1)}{s(2s+1)(s+1)}\\ b_{n+1}=-\frac{(s+1)^{n+1}-(-s)^{n+1}}{2s+1},c_{n+1}=-\frac{(s+1)^n-(-s)^n}{2s+1},n\geq 0 \end{cases}$$

Cor. 4.1.2.
$$\begin{cases} i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=(1-a_{n+1})\sigma(s)[\sigma(s)\cdot\hat{p}]-b_{n+1}[\sigma(s)\cdot\hat{p}]\sigma(s)+b_n\sigma^2(s)\hat{p},n\geq 0\\ a_{n+1}=\frac{[(s+1)^{n+3}-(-s)^{n+3}]-2[(s+1)^{n+2}-(-s)^{n+2}]+(2s+1)}{s(2s+1)(s+1)},b_{n+1}=-\frac{(s+1)^{n+1}-(-s)^{n+1}}{2s+1} \end{cases}$$

Cor. 4.1.3.
$$\begin{cases} i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=(1-a_{n+1}-b_{n+1})[\sigma(s)\cdot\hat{p}]^2+c_{n+1}\sigma^2(s)\hat{p}\\ a_{n+1}=\frac{[(s+1)^{n+3}-(-s)^{n+3}]-2[(s+1)^{n+2}-(-s)^{n+2}]+(2s+1)}{s(2s+1)(s+1)}\\ b_{n+1}=-\frac{(s+1)^{n+1}-(-s)^{n+1}}{2s+1},c_{n+1}=-\frac{(s+1)^n-(-s)^n}{2s+1},n\geq 0 \end{cases}$$

Cor. 4.1.4.
$$\begin{cases} i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=(1-a_n-2b_n)[\sigma(s)\cdot\hat{p}]^2+b_n\sigma^2(s)\hat{p}\\ a_n=\frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)},b_n=-\frac{(s+1)^n-(-s)^n}{2s+1},n\geq 0 \end{cases}$$

Cor. 4.1.5. $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=[1+\frac{(s+1)^{n+2}-(-s)^{n+2}-(2s+1)}{s(2s+1)(s+1)}][\sigma(s)\cdot\hat{p}]^2-\frac{(s+1)^n-(-s)^n}{2s+1}\sigma^2(s),n\geq 0$

4.1.2 General term formula for $i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n$

Pro. 4.1.3.
$$\begin{cases} i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n=[\sigma(s)\cdot\hat{p}]\sigma(s)-i^{-(n+1)}\hat{p}[|\times\sigma(s)]^{n+1}\\ i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n=[\sigma(s)\cdot\hat{p}]^2-i^{-(n+1)}\hat{p}[|\times\sigma(s)]^{n+1}|\cdot\hat{p} \end{cases}$$

Proof: $i^{-(n+1)}\hat{p}[|\times\sigma(s)]^{n+1}$
$=i^{-1}\hat{p}\times\sigma(s)][|\times\sigma(s)]^n i^{-n}$
$=\{[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]\}[|\times\sigma(s)]^n i^{-n}$
$=[\sigma(s)\cdot\hat{p}]\sigma(s)-i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n$
$\Rightarrow i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n=[\sigma(s)\cdot\hat{p}]\sigma(s)-i^{-(n+1)}\hat{p}[|\times\sigma(s)]^{n+1}$ □

Cor. 4.1.6.
$$\begin{cases} i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=(1-a_{n+1})[\sigma(s)\cdot\hat{p}]\sigma(s)-b_{n+1}\sigma(s)[\sigma(s)\cdot\hat{p}]+c_{n+1}\sigma^2(s)\hat{p}\\ a_{n+1}=\frac{[(s+1)^{n+3}-(-s)^{n+3}]-2[(s+1)^{n+2}-(-s)^{n+2}]+(2s+1)}{s(2s+1)(s+1)}\\ b_{n+1}=-\frac{(s+1)^{n+1}-(-s)^{n+1}}{2s+1},c_{n+1}=-\frac{(s+1)^n-(-s)^n}{2s+1},n\geq 0 \end{cases}$$

Cor. 4.1.7.
$$\begin{cases} i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=(1-a_{n+1})[\sigma(s)\cdot\hat{p}]\sigma(s)-b_{n+1}\sigma(s)[\sigma(s)\cdot\hat{p}]+b_n\sigma^2(s)\hat{p},n\geq 0\\ a_{n+1}=\frac{[(s+1)^{n+3}-(-s)^{n+3}]-2[(s+1)^{n+2}-(-s)^{n+2}]+(2s+1)}{s(2s+1)(s+1)},b_{n+1}=-\frac{(s+1)^{n+1}-(-s)^{n+1}}{2s+1} \end{cases}$$

Cor. 4.1.8.
$$\begin{cases} i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n=(1-a_{n+1}-b_{n+1})[\sigma(s)\cdot\hat{p}]^2+c_{n+1}\sigma^2(s)\hat{p} \\ a_{n+1}=\frac{[(s+1)^{n+3}-(-s)^{n+3}]-2[(s+1)^{n+2}-(-s)^{n+2}]+(2s+1)}{s(2s+1)(s+1)} \\ b_{n+1}=-\frac{(s+1)^{n+1}-(-s)^{n+1}}{2s+1},c_{n+1}=-\frac{(s+1)^n-(-s)^n}{2s+1},n\geq 0 \end{cases}$$

Cor. 4.1.9.
$$\begin{cases} i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n=(1-a_n-2b_n)[\sigma(s)\cdot\hat{p}]^2+b_n\sigma^2(s)\hat{p} \\ a_n=\frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)},b_n=-\frac{(s+1)^n-(-s)^n}{2s+1},n\geq 0 \end{cases}$$

Cor. 4.1.10. $i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n=[1+\frac{(s+1)^{n+2}-(-s)^{n+2}-(2s+1)}{s(2s+1)(s+1)}][\sigma(s)\cdot\hat{p}]^2-\frac{(s+1)^n-(-s)^n}{2s+1}\sigma^2(s),n\geq 0$

Cor. 4.1.11. $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]\sigma(s)=i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}][|\times\sigma(s)]^n,n\geq 0$

4.2 General term formula for $i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s)$ and $i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n$

4.2.1 General term formula for $i^{-n}[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s)$

Thm. 4.2.1. $i^{-n}[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\},n\geq 1,l\geq 0$
$$=\sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}]-[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s)$$
$$+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}]\hat{p}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$

Proof: $i^{-n}[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}$
$$=i^{-(n-1)}[\sigma(s)\times|]^{n-1}i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}$$
$$=i^{-(n-1)}[\sigma(s)\times|]^{n-1}\sum_{k=0}^{[l/2]}\{(C_l^{2k+1}+C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)+(C_l^{2k}-C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]-C_l^{2k+1}\sigma^2(s)\hat{p}\}$$
$$[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{(C_l^{2k+1}+C_l^{2k+2})\{\sigma(s)[\sigma(s)\cdot\hat{p}]-i^{-n}[\sigma(s)\times|]^n\hat{p}\}+(C_l^{2k}-C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}]$$
$$-C_l^{2k+1}\sigma^2(s)i^{-(n-1)}[\sigma(s)\times|]^{n-1}\hat{p}\}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{-C_{l+1}^{2k+2}i^{-n}[\sigma(s)\times|]^n\hat{p}+C_{l+1}^{2k+1}\sigma(s)[\sigma(s)\cdot\hat{p}]-C_l^{2k+1}\sigma^2(s)i^{-(n-1)}[\sigma(s)\times|]^{n-1}\hat{p}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{-C_{l+1}^{2k+2}\{a_n\sigma(s)[\sigma(s)\cdot\hat{p}]+b_n[\sigma(s)\cdot\hat{p}]\sigma(s)-c_n\sigma^2(s)\hat{p}\}+C_{l+1}^{2k+1}\sigma(s)[\sigma(s)\cdot\hat{p}]$$
$$-C_l^{2k+1}\sigma^2(s)\{a_{n-1}\sigma(s)[\sigma(s)\cdot\hat{p}]+b_{n-1}[\sigma(s)\cdot\hat{p}]\sigma(s)-c_{n-1}\sigma^2(s)\hat{p}\}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{[-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}+C_{l+1}^{2k+1}]\sigma(s)[\sigma(s)\cdot\hat{p}]+[-C_{l+1}^{2k+2}b_n-C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s)$$
$$+[C_{l+1}^{2k+2}c_n+C_l^{2k+1}\sigma^2(s)c_{n-1}]\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}]-[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s)$$
$$+[C_{l+1}^{2k+2}\sigma^2(s)c_n+C_l^{2k+1}\sigma^4(s)c_{n-1}]\hat{p}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}]-[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s)$$
$$+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}]\hat{p}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
□

Cor. 4.2.1. $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\},n\geq 1,l\geq 0$
$$=\sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}(a_n+b_n)-C_l^{2k+1}\sigma^2(s)(a_{n-1}+b_{n-1})][\sigma(s)\cdot\hat{p}]^2+[C_{l+1}^{2k+2}\sigma^2(s)c_n+C_l^{2k+1}\sigma^4(s)c_{n-1}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}(a_n+b_n)-C_l^{2k+1}(1+b_n)][\sigma(s)\cdot\hat{p}]^2+[C_{l+1}^{2k+2}\sigma^2(s)c_n+C_l^{2k+1}\sigma^4(s)c_{n-1}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$
$$=\sum_{k=0}^{[l/2]}\{[C_l^{2k}-C_{l+1}^{2k+2}(a_n+b_n)-C_l^{2k+1}b_n][\sigma(s)\cdot\hat{p}]^2+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$

Cor. 4.2.2. $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\},n\geq 1,l\geq 0$
$$=\sum_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1}+C_l^{2k+2}k_n+(C_l^{2k-1}b_n+C_l^{2k}b_{n-1})\sigma^2(s)+C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k}-b_{n-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}$$

Proof: $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\},n\geq 1,l\geq 0$
$$=\sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}(a_n+b_n)-C_l^{2k+1}\sigma^2(s)(a_{n-1}+b_{n-1})][\sigma(s)\cdot\hat{p}]^2+[C_{l+1}^{2k+2}\sigma^2(s)c_n+C_l^{2k+1}\sigma^4(s)c_{n-1}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$$

$= \sum_{k=0}^{[(l+1)/2]}[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n+b_n) - C_l^{2k+1}\sigma^2(s)(a_{n-1}+b_{n-1})][\sigma(s)\cdot\hat{p}]^2[\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$+ \sum_{k=0}^{[(l-1)/2]}[C_{l+1}^{2k+2}\sigma^2(s)c_n + C_l^{2k+1}\sigma^4(s)c_{n-1}][\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$= \sum_{k=0}^{[(l+1)/2]}[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n+b_n) - C_l^{2k+1}\sigma^2(s)(a_{n-1}+b_{n-1})][\sigma(s)\cdot\hat{p}]^{l+1-2k}$
$+ \sum_{k=1}^{[(l+1)/2]}[C_{l+1}^{2k}\sigma^2(s)c_n + C_l^{2k-1}\sigma^4(s)c_{n-1}][\sigma(s)\cdot\hat{p}]^{l+1-2k}$
$= \sum_{k=0}^{[(l+1)/2]}[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n+b_n) - C_l^{2k+1}\sigma^2(s)(a_{n-1}+b_{n-1})][\sigma(s)\cdot\hat{p}]^{l+1-2k}$
$+ \sum_{k=0}^{[(l+1)/2]}[C_{l+1}^{2k}\sigma^2(s)c_n + C_l^{2k-1}\sigma^4(s)c_{n-1}][\sigma(s)\cdot\hat{p}]^{l+1-2k} - \sigma^2(s)c_n[\sigma(s)\cdot\hat{p}]^{l+1}$
$= \sum_{k=0}^{[(l+1)/2]}[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n+b_n) - C_l^{2k+1}\sigma^2(s)(a_{n-1}+b_{n-1}) + C_{l+1}^{2k}\sigma^2(s)c_n + C_l^{2k-1}\sigma^4(s)c_{n-1}][\sigma(s)\cdot\hat{p}]^{l+1-2k}$
$- \sigma^2(s)c_n[\sigma(s)\cdot\hat{p}]^{l+1}$
$= \sum_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}c_{n+1} + C_l^{2k}c_n)\sigma^2(s) + C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k} - c_n\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}$
$= \sum_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}b_n + C_l^{2k}b_{n-1})\sigma^2(s) + C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k} - b_{n-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}$ □

Cor. 4.2.3. $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}, n\geq 0, l\geq 0$
$= \sum_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}b_n + C_l^{2k}b_{n-1})\sigma^2(s) + C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k} - b_{n-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}$

4.2.2 General term formula for $i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n$

Thm. 4.2.2. $i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n, n\geq 1, l\geq 0$
$= \sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s) - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}] +$
$[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}]\hat{p}\}$

Proof: $i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n, n\geq 1, l\geq 0$
$= i^{-(n-1)}i^{-1}\sigma(s)[\sigma(s)\cdot\hat{p}]^l\times\sigma(s)[|\times\sigma(s)]^{n-1}$
$= i^{-(n-1)}\sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{(C_l^{2k+1}+C_l^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] + (C_l^{2k}-C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) - C_l^{2k+1}\sigma^2(s)\hat{p}\}[|\times\sigma(s)]^{n-1}$
$= \sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{(C_l^{2k+1}+C_l^{2k+2})\{[\sigma(s)\cdot\hat{p}]\sigma(s) - i^{-n}\hat{p}[|\times\sigma(s)]^n\} + (C_l^{2k}-C_l^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s)$
$- C_l^{2k+1}\sigma^2(s)i^{-(n-1)}\hat{p}[|\times\sigma(s)]^{n-1}\}\}$
$= \sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{-C_{l+1}^{2k+2}i^{-n}\hat{p}[|\times\sigma(s)]^n + C_{l+1}^{2k+1}[\sigma(s)\cdot\hat{p}]\sigma(s) - C_l^{2k+1}\sigma^2(s)i^{-(n-1)}\hat{p}[|\times\sigma(s)]^{n-1}\}$
$= \sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{-C_{l+1}^{2k+2}\{a_n[\sigma(s)\cdot\hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s)\cdot\hat{p}] - c_n\sigma^2(s)\hat{p}\} + C_{l+1}^{2k+1}[\sigma(s)\cdot\hat{p}]\sigma(s)$
$- C_l^{2k+1}\sigma^2(s)\{a_{n-1}[\sigma(s)\cdot\hat{p}]\sigma(s) + b_{n-1}\sigma(s)[\sigma(s)\cdot\hat{p}] - c_{n-1}\sigma^2(s)\hat{p}\}\}$
$= \sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s) - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}] +$
$[C_{l+1}^{2k+2}\sigma^2(s)c_n + C_l^{2k+1}\sigma^4(s)c_{n-1}]\hat{p}\}$
$= \sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s) - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}] +$
$[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}]\hat{p}\}$ □

Cor. 4.2.4. $i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n, n\geq 1, l\geq 0$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n+b_n) - C_l^{2k+1}\sigma^2(s)(a_{n-1}+b_{n-1})][\sigma(s)\cdot\hat{p}]^2 + [C_{l+1}^{2k+2}\sigma^2(s)c_n + C_l^{2k+1}\sigma^4(s)c_{n-1}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n+b_n) - C_l^{2k+1}(1+b_n)][\sigma(s)\cdot\hat{p}]^2 + [C_{l+1}^{2k+2}\sigma^2(s)c_n + C_l^{2k+1}\sigma^4(s)c_{n-1}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$= \sum_{k=0}^{[l/2]}\{[C_l^{2k} - C_{l+1}^{2k+2}(a_n+b_n) - C_l^{2k+1}b_n][\sigma(s)\cdot\hat{p}]^2 + [C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$

Cor. 4.2.5. $i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n, n\geq 0, l\geq 0$
$= \sum_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1}+C_l^{2k+2}k_n+(C_l^{2k-1}b_n+C_l^{2k}b_{n-1})\sigma^2(s)+C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k}-b_{n-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}$

Cor. 4.2.6. $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\} = i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n, n\geq 0, l\geq 0$

4.2.3 General term formula for $i^{-n}\sigma(s)\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}, i^{-n}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n|\cdot\sigma(s)$

Thm. 4.2.3. $i^{-n}\sigma(s)\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\} = i^{-n}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n|\cdot\sigma(s)$
$= \sum_{k=0}^{[l/2]}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2k+2}][\sigma(s)\cdot\hat{p}]^{l-2k}, n\geq 1, l\geq 0$

Proof: $i^{-n}\sigma(s)\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\} = i^{-n}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n|\cdot\sigma(s), n\geq 1, l\geq 0$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma^2(s)[\sigma(s)\cdot\hat{p}]-[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma(s)\cdot[\sigma(s)\cdot\hat{p}]\sigma(s)$
$+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma^2(s)[\sigma(s)\cdot\hat{p}]-[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]$
$+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}-C_{l+1}^{2k+2}b_n-C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma^2(s)$
$+[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}]+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}]\}[\sigma(s)\cdot\hat{p}]^{l-2k}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}+C_{l+1}^{2k+2}k_n+C_l^{2k+1}\sigma^2(s)k_{n-1}]\sigma^2(s)+[C_{l+1}^{2k+2}b_{n+1}+C_l^{2k+1}\sigma^2(s)b_n]\}[\sigma(s)\cdot\hat{p}]^{l-2k}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}\sigma^{-2}(s)(b_{n+1}+1)-C_l^{2k+1}(b_n+1)]\sigma^2(s)+[C_{l+1}^{2k+2}b_{n+1}+C_l^{2k+1}\sigma^2(s)b_n]\}[\sigma(s)\cdot\hat{p}]^{l-2k}$
$= \sum_{k=0}^{[l/2]}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2k+2}][\sigma(s)\cdot\hat{p}]^{l-2k}$ □

Cor. 4.2.7. $i^{-n}\sigma(s)\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\} = i^{-n}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n|\cdot\sigma(s) = \sigma(s)\cdot[\sigma(s)\cdot\hat{p}]^l\sigma(s)$
$= \sum_{k=0}^{[l/2]}[C_l^{2k}\sigma^2(s)-C_{l+1}^{2k+2}][\sigma(s)\cdot\hat{p}]^{l-2k}, n\geq 0, l\geq 0$

4.2.4 General term formula for $i^{-n}[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}|\cdot\sigma(s), i^{-n}\sigma(s)\cdot|\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n$

Thm. 4.2.4. $i^{-n}[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}|\cdot\sigma(s) = i^{-n}\sigma(s)\cdot|\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n$
$= \sum_{k=0}^{[l/2]}[C_l^{2k}\sigma^2(s)-C_{l+2}^{2k+2}+C_{l+1}^{2k+2}(a_n-b_n)+C_l^{2k+1}\sigma^2(s)(a_{n-1}-b_{n-1})][\sigma(s)\cdot\hat{p}]^{l-2k}, n\geq 1, l\geq 0$

Proof: $i^{-n}[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}|\cdot\sigma(s) = i^{-n}\sigma(s)\cdot|\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n, n\geq 1, l\geq 0$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}]\cdot\sigma(s)-[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma(s)\cdot\hat{p}]\sigma^2(s)$
$+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}][\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]-[C_{l+1}^{2k+2}b_n+C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma^2(s)[\sigma(s)\cdot\hat{p}]$
$+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^{l-2k-1}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}a_n-C_l^{2k+1}\sigma^2(s)a_{n-1}-C_{l+1}^{2k+2}b_n-C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma^2(s)$
$+[-C_{l+1}^{2k+1}+C_{l+1}^{2k+2}a_n+C_l^{2k+1}\sigma^2(s)a_{n-1}]+[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1}+C_l^{2k+1}\sigma^4(s)b_{n-2}]\}[\sigma(s)\cdot\hat{p}]^{l-2k}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}+C_{l+1}^{2k+2}k_n+C_l^{2k+1}\sigma^2(s)k_{n-1}]\sigma^2(s)+[C_{l+1}^{2k+2}b_{n+1}+C_l^{2k+1}\sigma^2(s)b_n]$
$+[-C_{l+1}^{2k+1}+C_{l+1}^{2k+2}(a_n-b_n)+C_l^{2k+1}\sigma^2(s)(a_{n-1}-b_{n-1})]\}[\sigma(s)\cdot\hat{p}]^{l-2k}$
$= \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1}-C_{l+1}^{2k+2}\sigma^{-2}(s)(b_{n+1}+1)-C_l^{2k+1}(b_n+1)]\sigma^2(s)+[C_{l+1}^{2k+2}b_{n+1}+C_l^{2k+1}\sigma^2(s)b_n]$
$+[-C_{l+1}^{2k+1}+C_{l+1}^{2k+2}(a_n-b_n)+C_l^{2k+1}\sigma^2(s)(a_{n-1}-b_{n-1})]\}[\sigma(s)\cdot\hat{p}]^{l-2k}$
$= \sum_{k=0}^{[l/2]}\{C_l^{2k}\sigma^2(s)-C_{l+1}^{2k+2}+[-C_{l+1}^{2k+1}+C_{l+1}^{2k+2}(a_n-b_n)+C_l^{2k+1}\sigma^2(s)(a_{n-1}-b_{n-1})]\}[\sigma(s)\cdot\hat{p}]^{l-2k}$
$= \sum_{k=0}^{[l/2]}[C_l^{2k}\sigma^2(s)-C_{l+2}^{2k+2}+C_{l+1}^{2k+2}(a_n-b_n)+C_l^{2k+1}\sigma^2(s)(a_{n-1}-b_{n-1})][\sigma(s)\cdot\hat{p}]^{l-2k}$ □

5 Another independent method for solving general term formula

5.1 Another method for solving general term formula of $i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}\}$

5.1.1 Probing and guessing of general term formula for $i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}\}$

Def. 5.1.1. $A(1,n):=i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}\}, A(1,0)=\sigma(s)\cdot\hat{p}$

Cor. 5.1.1.
$$\begin{cases}\sigma(s)[\times\hat{p}|]^{2k-1}=(-1)^{k+1}\sigma(s)\times\hat{p}\\ \sigma(s)[\times\hat{p}|]^{2k}=(-1)^{k+1}[\sigma(s)\cdot\hat{p}]\hat{p}-\sigma(s)\end{cases}\Rightarrow\begin{cases}i^{2k-1}i^{-1}\hat{p}\cdot\{\sigma(s)\times\{\sigma(s)[\times\hat{p}|]^{2k-1}\}\}=\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\}\\ i^{2k}i^{-1}\hat{p}\cdot\{\sigma(s)\times\{\sigma(s)[\times\hat{p}|]^{2k}\}\}=\sigma(s)\cdot\hat{p}\end{cases}$$

Cor. 5.1.2. $i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}\}$
$$=i^{-1}i^n\hat{p}\cdot\{\sigma(s)\times\{\sigma(s)[|\times\hat{p}]^n\}\}-\sum_{k=0}^{n-1}C_n^k i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^k\sigma(s)\}\}[-\sigma(s)\cdot\hat{p}]^{n-k}$$

Cor. 5.1.3. $A(1,n)=\{[\sigma(s)\cdot\hat{p}]^{1+n\%2}-(n\%2)\sigma^2(s)\}-\sum\limits_{k=0}^{n-1}C_n^kA(1,k)[-\sigma(s)\cdot\hat{p}]^{n-k}, A(1,0)=-\sigma(s)\cdot\hat{p}$

Pro. 5.1.1.
$$\begin{cases}A(1,0)=1[\sigma(s)\cdot\hat{p}]\\ A(1,1)=2[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\\ A(1,2)=3[\sigma(s)\cdot\hat{p}]^3-[2\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\\ A(1,3)=4[\sigma(s)\cdot\hat{p}]^4-[3\sigma^2(s)-4][\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\\ A(1,4)=5[\sigma(s)\cdot\hat{p}]^5-[4\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^3-[4\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^1\\ A(1,5)=6[\sigma(s)\cdot\hat{p}]^6-[5\sigma^2(s)-20][\sigma(s)\cdot\hat{p}]^4-[10\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\end{cases}$$

Proof: $-A(1,1)$
$=-\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\}+C_1^0A(1,0)[\sigma(s)\cdot\hat{p}]$
$=-2[\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)$ □

Proof: $-A(1,2)$
$=-[\sigma(s)\cdot\hat{p}]-\sum\limits_{k=0}^{1}C_2^kA(1,k)[-\sigma(s)\cdot\hat{p}]^{2-k}$
$=-[\sigma(s)\cdot\hat{p}]-C_2^0A(1,0)[\sigma(s)\cdot\hat{p}]^2-\sum\limits_{k=1}^{1}C_2^kA(1,k)[-\sigma(s)\cdot\hat{p}]^{2-k}$
$=-[\sigma(s)\cdot\hat{p}]+C_2^0[\sigma(s)\cdot\hat{p}]^3+C_2^1A_1(1)[\sigma(s)\cdot\hat{p}]$
$=-[\sigma(s)\cdot\hat{p}]+C_2^0[\sigma(s)\cdot\hat{p}]^3+C_2^1\{-2[\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]$
$=-3[\sigma(s)\cdot\hat{p}]^3+[2\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]$ □

Proof: $-A(1,3)$
$=-\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\}+C_3^0[-\sigma(s)\cdot\hat{p}][\sigma(s)\cdot\hat{p}]^3$
$-C_3^1\{-2[\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^2$
$+C_3^2\{-3[\sigma(s)\cdot\hat{p}]^3+[2\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^1$
$=-4[\sigma(s)\cdot\hat{p}]^4+[3\sigma^2(s)-4][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)$ □

Proof: $-A(1,4)$
$=-[\sigma(s)\cdot\hat{p}]-C_4^0[-\sigma(s)\cdot\hat{p}][\sigma(s)\cdot\hat{p}]^4$
$+C_4^1\{-2[\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^3$
$-C_4^2\{-3[\sigma(s)\cdot\hat{p}]^3+[2\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^2$
$+C_4^3\{-4[\sigma(s)\cdot\hat{p}]^4+[3\sigma^2(s)-4][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^1$
$=-5[\sigma(s)\cdot\hat{p}]^5+[4\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^3+[4\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^1$ □

Proof: $-A(1,5)$
$=-\{[\sigma(s)\cdot\hat{p}]^2-\sigma^2(s)\}+C_5^0[-\sigma(s)\cdot\hat{p}][\sigma(s)\cdot\hat{p}]^5$
$-C_5^1\{-2[\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^4$
$+C_5^2\{-3[\sigma(s)\cdot\hat{p}]^3+[2\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^3$
$-C_5^3\{-4[\sigma(s)\cdot\hat{p}]^4+[3\sigma^2(s)-4][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^2$
$+C_5^4\{-5[\sigma(s)\cdot\hat{p}]^5+[4\sigma^2(s)-10][\sigma(s)\cdot\hat{p}]^3+[4\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]^1\}[\sigma(s)\cdot\hat{p}]^1$
$=-6[\sigma(s)\cdot\hat{p}]^6+[5\sigma^2(s)-20][\sigma(s)\cdot\hat{p}]^4+[10\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)$ □

Reorganize to get:

Pro. 5.1.2. $\begin{cases} A(1,0)=C_1^1[\sigma(s)\cdot\hat{p}] \\ A(1,1)=C_2^1[\sigma(s)\cdot\hat{p}]^2-C_1^1\sigma^2(s) \\ A(1,2)=C_3^1[\sigma(s)\cdot\hat{p}]^3-[C_2^1\sigma^2(s)-C_3^3][\sigma(s)\cdot\hat{p}] \\ A(1,3)=C_4^1[\sigma(s)\cdot\hat{p}]^4-[C_3^1\sigma^2(s)-C_4^3][\sigma(s)\cdot\hat{p}]^2-C_3^3\sigma^2(s) \\ A(1,4)=C_5^1[\sigma(s)\cdot\hat{p}]^5-[C_4^1\sigma^2(s)-C_5^3][\sigma(s)\cdot\hat{p}]^3-[C_4^3\sigma^2(s)-C_5^5][\sigma(s)\cdot\hat{p}]^1 \\ A(1,5)=C_6^1[\sigma(s)\cdot\hat{p}]^6-[C_5^1\sigma^2(s)-C_6^3][\sigma(s)\cdot\hat{p}]^4-[C_5^3\sigma^2(s)-C_6^5][\sigma(s)\cdot\hat{p}]^2-C_5^5\sigma^2(s) \end{cases}$

Ass. 5.1.1. $A(1,n)=C_{n+1}^1[\sigma(s)\cdot\hat{p}]^{n+1}-[C_n^1\sigma^2(s)-C_{n+1}^3][\sigma(s)\cdot\hat{p}]^{n-1}-[C_n^3\sigma^2(s)-C_{n+1}^5][\sigma(s)\cdot\hat{p}]^{n-3}+\cdots$

Ass. 5.1.2. $i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}\}=\sum\limits_{k=0}^{[(n+1)/2]}[C_{n+1}^{2k+1}-C_n^{2k-1}\sigma^2(s)][\sigma(s)\cdot\hat{p}]^{n+1-2k}, n\geq 0$

The following strictly proves the above conjecture by using mathematical induction.

5.1.2 Relevant lemmas of general term formula for $i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}\}$

Cor. 5.1.4. $A(1,n)=\{[\sigma(s)\cdot\hat{p}]^{1+n\%2}-(n\%2)\sigma^2(s)\}-\sum\limits_{l=0}^{n-1}C_n^lA(1,l)[-\sigma(s)\cdot\hat{p}]^{n-l}, A(1,0)=-\sigma(s)\cdot\hat{p}$

Lem. 5.1.1. $\sum\limits_{l=0}^{n}\sum\limits_{k=0}^{[(l+1)/2]}A(k,l)=\sum\limits_{k=0}^{[(n+1)/2]}\sum\limits_{l=2k-1|0}^{n}A(k,l)$

Lem. 5.1.2. $\sum\limits_{l=2k-1|0}^{n}(-1)^{n-l}C_{n+1}^lC_l^{2k-1}=C_{n+1}^{2k-1}$

Lem. 5.1.3. $\sum\limits_{l=r}^{n}(-1)^{n-l}C_{n+1}^lC_l^r=C_{n+1}^r$

Proof: $\sum\limits_{l=r}^{n}(-1)^{n-l}C_{n+1}^lC_l^r=C_{n+1}^r$

$\Leftrightarrow\sum\limits_{l=0}^{n-r}(-1)^lC_{n+1}^{n-l}C_{n-l}^r=C_{n+1}^r$

$\Leftrightarrow\sum\limits_{l=0}^{n-r}(-1)^l\frac{1}{(l+1)!(n-l-r)!}=\frac{1}{(n+1-r)!}$

$\Leftrightarrow\sum\limits_{l=0}^{n-r}(-1)^l\frac{(n+1-r)!}{(l+1)!(n-l-r)!}=1$

$\Leftrightarrow\sum\limits_{l=0}^{n-r}(-1)^lC_{n+1-r}^{l+1}=1$

$\Leftrightarrow\sum\limits_{l=0}^{n+1-r}(-1)^lC_{n+1-r}^l=0$

$\Leftrightarrow[1+(-1)]^{n+1-r}=0$ □

Cor. 5.1.5. $\sum\limits_{l=2k-1}^{n}(-1)^{n-l}C_{n+1}^lC_l^{2k-1}=C_{n+1}^{2k-1}, k\geq 0$

Lem. 5.1.4. $\begin{cases}\sum\limits_{l=r}^{n}(-1)^{n-l}C_{n+1}^lC_{l+1}^{r+2}=C_{n+2}^{r+2}, r<n-1 \\ \sum\limits_{l=r}^{n}(-1)^{n-l}C_{n+1}^lC_{l+1}^{r+2}=C_{n+2}^{r+2}-1, r=n-1\end{cases}$

Proof: $\sum\limits_{l=r}^{n}(-1)^{n-l}C_{n+1}^lC_{l+1}^{r+2}=C_{n+2}^{r+2}, r<n-1$

$\Leftrightarrow\sum\limits_{l=r+1}^{n}(-1)^{n-l}C_{n+1}^lC_{l+1}^{r+2}=C_{n+2}^{r+2}$

$\Leftrightarrow\sum\limits_{l=0}^{n-1-r}(-1)^lC_{n+1}^{n-l}C_{n+1-l}^{r+2}=C_{n+2}^{r+2}$

$\Leftrightarrow\sum\limits_{l=0}^{n-1-r}(-1)^l\frac{n+1-l}{(l+1)!(n-l-1-r)!}=\frac{n+2}{(n-r)!}$

$\Leftrightarrow\sum\limits_{l=0}^{n-1-r}(-1)^l\frac{(n+1-l)(n-r)!}{(l+1)!(n-l-1-r)!}=n+2$

$\Leftrightarrow\sum\limits_{l=0}^{n-1-r}(-1)^l(n+1-l)C_{n-r}^{l+1}=n+2$

$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l+1}(l+1)C_{n-r}^{l+1}=(n+2)\sum\limits_{l=0}^{n-r}(-1)^{l}C_{n-r}^{l}$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l+1}(l+1)C_{n-r}^{l+1}=0$
$\Leftrightarrow \sum\limits_{l=1}^{n-r}(-1)^{l}lC_{n-r}^{l}=0$
$\Leftrightarrow (n-r)\sum\limits_{l=1}^{n-r}(-1)^{l}C_{n-1-r}^{l-1}=0$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l}C_{n-1-r}^{l}=0$
$\Leftrightarrow [1+(-1)]^{n-1-r}=0$ □

Proof: $\sum\limits_{l=r}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{r+2}=C_{n+2}^{r+2}, r<n-1$
$\Leftrightarrow \sum\limits_{l=r+1}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{r+2}=C_{n+2}^{r+2}$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l}C_{n+1}^{n-l}C_{n+1-l}^{r+2}=C_{n+2}^{r+2}$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l}\frac{n+1-l}{(l+1)!(n-l-1-r)!}=\frac{n+2}{(n-r)!}$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l}\frac{(n+1-l)(n-r)!}{(l+1)!(n-l-1-r)!}=n+2$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l}(n+1-l)C_{n-r}^{l+1}=n+2$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l+1}(l+1)C_{n-r}^{l+1}=(n+2)\sum\limits_{l=0}^{n-r}(-1)^{l}C_{n-r}^{l}$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l+1}(l+1)C_{n-r}^{l+1}=0$
$\Leftrightarrow \sum\limits_{l=1}^{n-r}(-1)^{l}lC_{n-r}^{l}=0$
$\Leftrightarrow (n-r)\sum\limits_{l=1}^{n-r}(-1)^{l}C_{n-1-r}^{l-1}=0$
$\Leftrightarrow \sum\limits_{l=0}^{n-1-r}(-1)^{l}C_{n-1-r}^{l}=0$
$\Leftrightarrow [1+(-1)]^{n-1-r}=0$ □

Proof: $\sum\limits_{l=r}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{r+2}=C_{n+2}^{r+2}-1, r=n-1$
$\Leftrightarrow \sum\limits_{l=r+1}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{r+2}=C_{n+2}^{r+2}-1$ □

Cor. 5.1.6. $\begin{cases}\sum\limits_{l=2k-1}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2k+1}=C_{n+2}^{2k+1}, 2k-1<n-1\\ \sum\limits_{l=2k-1}^{n}(-1)^{n-l}C_{n+1}^{l}C_{l+1}^{2k+1}=C_{n+2}^{2k+1}-1, 2k-1=n-1\end{cases}$

Lem. 5.1.5.
$\{[\sigma(s)\cdot\hat{p}]^{1+(n+1)\%2}-(n+1)\%2\sigma^{2}(s)\}-\sum\limits_{k=0}^{[(n+1)/2]}[\sigma(s)\cdot\hat{p}]^{n+2-2k}\sum\limits_{l=2k-1|0}^{n}(-1)^{n-l}C_{n+1}^{l}[C_{l}^{2k-1}\sigma^{2}(s)-C_{l+1}^{2k+1}]$
$=\sum\limits_{k=0}^{[(n+2)/2]}[C_{n+2}^{2k+1}-C_{n+1}^{2k-1}\sigma^{2}(s)][\sigma(s)\cdot\hat{p}]^{n+2-2k}$

5.1.3 Mathematical induction proof of general term formula for $i^{-1}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^{n}\sigma(s)\}\}$

Thm. 5.1.1. $A(1,n)=\sum\limits_{k=0}^{[(n+1)/2]}[C_{n+1}^{2k+1}-C_{n}^{2k-1}\sigma^{2}(s)][\sigma(s)\cdot\hat{p}]^{n+1-2k}, n\geq 0$

Proof: $A(1,n+1)=\{[\sigma(s)\cdot\hat{p}]^{1+(n+1)\%2}-(n+1)\%2\sigma^{2}(s)\}-\sum\limits_{l=0}^{n}C_{n+1}^{l}A(1,l)[-\sigma(s)\cdot\hat{p}]^{n+1-l}$
$=\{[\sigma(s)\cdot\hat{p}]^{1+(n+1)\%2}-(n+1)\%2\sigma^{2}(s)\}-\sum\limits_{l=0}^{n}C_{n+1}^{l}\sum\limits_{k=0}^{[(l+1)/2]}[C_{l+1}^{2k+1}-C_{l}^{2k-1}\sigma^{2}(s)][\sigma(s)\cdot\hat{p}]^{l+1-2k}[-\sigma(s)\cdot\hat{p}]^{n+1-l}$

$$= \{[\sigma(s)\cdot\hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{l=0}^{n}\sum_{k=0}^{[(l+1)/2]}(-1)^{n+1-l}C_{n+1}^{l}[C_{l+1}^{2k+1} - C_{l}^{2k-1}\sigma^2(s)][\sigma(s)\cdot\hat{p}]^{n+2-2k}$$
$$= \{[\sigma(s)\cdot\hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{k=0}^{[(n+1)/2]}\sum_{l=2k-1|0}^{n}(-1)^{n+1-l}C_{n+1}^{l}[C_{l+1}^{2k+1} - C_{l}^{2k-1}\sigma^2(s)][\sigma(s)\cdot\hat{p}]^{n+2-2k}$$
$$= \{[\sigma(s)\cdot\hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{k=0}^{[(n+1)/2]}[\sigma(s)\cdot\hat{p}]^{n+2-2k}\sum_{l=2k-1|0}^{n}(-1)^{n-l}C_{n+1}^{l}[C_{l}^{2k-1}\sigma^2(s) - C_{l+1}^{2k+1}]$$
$$= \sum_{k=0}^{[(n+1)/2]}[C_{n+1}^{2k+1} - C_{n}^{2k-1}\sigma^2(s)][\sigma(s)\cdot\hat{p}]^{n+1-2k} \qquad \square$$

5.2 Another solution method for $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s), i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n$

5.2.1 General term formula for $i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s)$

Def. 5.2.1. $A_L(n,l) := i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s), A_L(n,0) = \sigma(s)\cdot\hat{p}$

Cor. 5.2.1.

$$i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s) = i^{-n}i^l\hat{p}\cdot|[\sigma(s)\times|]^n\sigma(s)[|\times\hat{p}]^l - \sum_{k=0}^{l-1}C_l^k i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^k\sigma(s)[-\sigma(s)\cdot\hat{p}]^{l-k}$$

Cor. 5.2.2.

$$\begin{cases} i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n i^{2k-1}\sigma(s)[\times\hat{p}|]^{2k-1} \\ = -i^{-(n+1)}\hat{p}\cdot|[\sigma(s)\times|]^n\sigma(s)\times\hat{p} = -(a_{n+1}+b_{n+1})[\sigma(s)\cdot\hat{p}]^2 + c_{n+1}\sigma^2(s) \\ i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n i^{2k}\sigma(s)[\times\hat{p}|]^{2k} \\ = -i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]\hat{p} - \sigma(s)\} = -(a_n+b_n)[\sigma(s)\cdot\hat{p}]^3 + [c_n\sigma^2(s)+1][\sigma(s)\cdot\hat{p}] \end{cases}$$

Cor. 5.2.3.

$$i^{l-n}[\sigma(s)\times|]^n\sigma(s)[|\times\hat{p}]^l = \begin{cases} -a_{n+1}\sigma(s)[\sigma(s)\cdot\hat{p}] - b_{n+1}[\sigma(s)\cdot\hat{p}]\sigma(s) + c_{n+1}\sigma^2(s)\hat{p}, l = 2k-1 \\ -a_n\sigma(s)[\sigma(s)\cdot\hat{p}]^2 - b_n[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}] + c_n\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}] + \sigma(s), l = 2k \\ i^{-n}[\sigma(s)\times|]^n\hat{p} = a_n\sigma(s)[\sigma(s)\cdot\hat{p}] + b_n[\sigma(s)\cdot\hat{p}]\sigma(s) - c_n\sigma^2(s)\hat{p} \end{cases}$$

Cor. 5.2.4.

$$i^{l-n}\hat{p}\cdot|[\sigma(s)\times|]^n\sigma(s)[|\times\hat{p}]^l = \begin{cases} -(a_{n+1}+b_{n+1})[\sigma(s)\cdot\hat{p}]^2 + c_{n+1}\sigma^2(s), l = 2k-1 \\ -(a_n+b_n)[\sigma(s)\cdot\hat{p}]^3 + [1+c_n\sigma^2(s)][\sigma(s)\cdot\hat{p}], l = 2k \end{cases}$$

Thm. 5.2.1.

$$\begin{cases} A_L(n,l) = i^{l-n}\hat{p}\cdot|[\sigma(s)\times|]^n\sigma(s)[|\times\hat{p}]^l - \sum_{k=0}^{l-1}C_l^k A_L(n,k)[-\sigma(s)\cdot\hat{p}]^{l-k}, A_L(n,0) = \sigma(s)\cdot\hat{p} \\ i^{l-n}\hat{p}\cdot|[\sigma(s)\times|]^n\sigma(s)[|\times\hat{p}]^l = \begin{cases} k_{n+1}[\sigma(s)\cdot\hat{p}]^2 + c_{n+1}\sigma^2(s), l = 2k-1; k_n = -(a_n+b_n) \\ k_n[\sigma(s)\cdot\hat{p}]^3 + [1+c_n\sigma^2(s)][\sigma(s)\cdot\hat{p}], l = 2k \end{cases} \end{cases}$$

Cor. 5.2.5.

$$\begin{cases} A_L(n,0) = C_0^0\sigma(s)\cdot\hat{p} \\ A_L(n,1) = (C_1^1k_{n+1} + C_1^0)[\sigma(s)\cdot\hat{p}]^2 + C_1^1c_{n+1}\sigma^2(s), k_{n+1} = -(a_{n+1}+b_{n+1}) \\ A_L(n,2) = [C_2^1k_{n+1} + C_2^2k_n + C_2^0][\sigma(s)\cdot\hat{p}]^3 + [(C_2^1c_{n+1} + C_2^2c_n)\sigma^2(s) + C_2^2][\sigma(s)\cdot\hat{p}] \\ A_L(n,3) = [C_3^1k_{n+1} + C_3^2k_n + C_3^0][\sigma(s)\cdot\hat{p}]^4 + [C_3^3k_{n+1} + (C_3^1c_{n+1} + C_3^2c_n)\sigma^2(s) + C_3^2][\sigma(s)\cdot\hat{p}]^2 \\ +C_3^3c_{n+1}\sigma^2(s) \\ A_L(n,4) = [C_4^1k_{n+1} + C_4^2k_n + C_4^0][\sigma(s)\cdot\hat{p}]^5 + [C_4^3k_{n+1} + C_4^4k_n + (C_4^1c_{n+1} + C_4^2c_n)\sigma^2(s) + C_4^2][\sigma(s)\cdot\hat{p}]^3 \\ +[(C_4^3c_{n+1} + C_4^4c_n)\sigma^2(s) + C_4^4][\sigma(s)\cdot\hat{p}]^1 \end{cases}$$

Proof: $A_L(n,1) = k_{n+1}[\sigma(s)\cdot\hat{p}]^2 + c_{n+1}\sigma^2(s) + C_1^0A_L(n,0)[\sigma(s)\cdot\hat{p}]^1$
$= (C_1^1k_{n+1} + C_1^0)[\sigma(s)\cdot\hat{p}]^2 + C_1^1c_{n+1}\sigma^2(s)$ □

Proof: $A_L(n,2) = k_n[\sigma(s)\cdot\hat{p}]^3 + [1+c_n\sigma^2(s)][\sigma(s)\cdot\hat{p}] - C_2^0A_L(n,0)[\sigma(s)\cdot\hat{p}]^2 + C_2^1A_L(n,1)[\sigma(s)\cdot\hat{p}]$
$= [C_2^1k_{n+1} + C_2^2k_n + C_2^0][\sigma(s)\cdot\hat{p}]^3 + [(C_2^1c_{n+1} + C_2^2c_n)\sigma^2(s) + C_2^2][\sigma(s)\cdot\hat{p}]$ □

Proof: $A_L(n,3) = k_{n+1}[\sigma(s)\cdot\hat{p}]^2 + c_{n+1}\sigma^2(s)$
$+ C_3^0A_L(n,0)[\sigma(s)\cdot\hat{p}]^3 - C_3^1A_L(n,1)[\sigma(s)\cdot\hat{p}]^2 + C_3^2A_L(n,2)[\sigma(s)\cdot\hat{p}]^1$
$= k_{n+1}[\sigma(s)\cdot\hat{p}]^2 + c_{n+1}\sigma^2(s)$
$+ C_3^0[\sigma(s)\cdot\hat{p}]^4 - C_3^1\{(C_1^1k_{n+1} + C_1^0)[\sigma(s)\cdot\hat{p}]^2 + C_1^1c_{n+1}\sigma^2(s)\}[\sigma(s)\cdot\hat{p}]^2$
$+ C_3^2\{[C_2^1k_{n+1} + C_2^2k_n + C_2^0][\sigma(s)\cdot\hat{p}]^3 + [(C_2^1c_{n+1} + C_2^2c_n)\sigma^2(s) + C_2^2][\sigma(s)\cdot\hat{p}]\}[\sigma(s)\cdot\hat{p}]^1$
$= [C_3^1k_{n+1} + C_3^2k_n + C_3^0][\sigma(s)\cdot\hat{p}]^4 + [C_3^3k_{n+1} + (C_3^1c_{n+1} + C_3^2c_n)\sigma^2(s) + C_3^2][\sigma(s)\cdot\hat{p}]^2 + C_3^3c_{n+1}\sigma^2(s)$ □

Proof: $A_L(n,4)=(k_n-1)[\sigma(s)\cdot\hat{p}]^3+[1+c_n\sigma^2(s)][\sigma(s)\cdot\hat{p}]$
$-C_4^0A_L(n,0)[\sigma(s)\cdot\hat{p}]^4+C_4^1A_L(n,1)[\sigma(s)\cdot\hat{p}]^3-C_4^2A_L(n,2)[\sigma(s)\cdot\hat{p}]^2+C_4^3A_L(n,3)[\sigma(s)\cdot\hat{p}]^1$
$=[C_4^1k_{n+1}+C_4^2k_n+C_4^0][\sigma(s)\cdot\hat{p}]^5+[C_4^3k_{n+1}+C_3^3k_n+(C_4^1c_{n+1}+C_4^2c_n)\sigma^2(s)+C_4^2][\sigma(s)\cdot\hat{p}]^3$
$+[(C_4^3c_{n+1}+C_4^4c_n)\sigma^2(s)+C_4^4][\sigma(s)\cdot\hat{p}]^1$ □

Thm. 5.2.2.
$$\begin{cases}A_L(n,l)=i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s)\\ =\sum\limits_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1}+C_l^{2k+2}k_n+(C_l^{2k-1}c_{n+1}+C_l^{2k}c_n)\sigma^2(s)+C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k}-c_n\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}\\ k_n=-(a_n+b_n)=\frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)},c_n=-\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1},n\geq0,l\geq0\end{cases}$$

Cor. 5.2.6.
$$\begin{cases}A_L(n,l)=i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s)\\ =\sum\limits_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1}+C_l^{2k+2}k_n+(C_l^{2k-1}b_n+C_l^{2k}b_{n-1})\sigma^2(s)+C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k}-b_{n-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}\\ k_n=-(a_n+b_n)=\frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)},b_n=-\frac{(s+1)^{n}-(-s)^{n}}{2s+1},n\geq0,l\geq0\end{cases}$$

The above theorem can be strictly proved by using mathematical induction, and supplement it when I have time.

5.2.2 General term formula for $i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n$

Def. 5.2.2. $A_R(n,l):=i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n,A_R(n,0)=\sigma(s)\cdot\hat{p}$

Pro. 5.2.1. $\sigma(s)[\sigma(s)\cdot\hat{p}]^l=i^l[\hat{p}\times|]^l\sigma(s)-\sum\limits_{k=0}^{l-1}C_n^k[-\sigma(s)\cdot\hat{p}]^{l-k}\sigma(s)[\sigma(s)\cdot\hat{p}]^k$

Cor. 5.2.7.
$$i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n=i^{l-n}\hat{p}\cdot|[\hat{p}\times|]^l\sigma(s)[|\times\sigma(s)]^n-\sum\limits_{k=0}^{l-1}C_n^k[-\sigma(s)\cdot\hat{p}]^{l-k}i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^k[|\times\sigma(s)]^n$$

Cor. 5.2.8.
$$i^{l-n}[\hat{p}\times|]^l\sigma(s)[|\times\sigma(s)]^n=\begin{cases}-a_{n+1}[\sigma(s)\cdot\hat{p}]\sigma(s)-b_{n+1}\sigma(s)[\sigma(s)\cdot\hat{p}]+c_{n+1}\sigma^2(s)\hat{p},l=2k-1\\ -a_n[\sigma(s)\cdot\hat{p}]^2\sigma(s)-b_n[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]+c_n\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]+\sigma(s),l=2k\\ i^{-n}\hat{p}[|\times\sigma(s)]^n=a_n[\sigma(s)\cdot\hat{p}]\sigma(s)+b_n\sigma(s)[\sigma(s)\cdot\hat{p}]-c_n\sigma^2(s)\hat{p}\end{cases}$$

Cor. 5.2.9.
$$i^{l-n}\hat{p}\cdot|[\hat{p}\times|]^l\sigma(s)[|\times\sigma(s)]^n=\begin{cases}-(a_{n+1}+b_{n+1})[\sigma(s)\cdot\hat{p}]^2+c_{n+1}\sigma^2(s),l=2k-1\\ -(a_n+b_n)[\sigma(s)\cdot\hat{p}]^3+[1+c_n\sigma^2(s)][\sigma(s)\cdot\hat{p}],l=2k\end{cases}$$

Thm. 5.2.3.
$$\begin{cases}A_R(n,l)=i^{l-n}\hat{p}\cdot|[\hat{p}\times|]^l\sigma(s)[|\times\sigma(s)]^n-\sum\limits_{k=0}^{l-1}C_n^k[-\sigma(s)\cdot\hat{p}]^{l-k}A_R(n,k),A_R(n,0)=\sigma(s)\cdot\hat{p}\\ i^{l-n}\hat{p}\cdot|[\hat{p}\times|]^l\sigma(s)[|\times\sigma(s)]^n=\begin{cases}k_{n+1}[\sigma(s)\cdot\hat{p}]^2+c_{n+1}\sigma^2(s),l=2k-1;k_n=-(a_n+b_n)\\ k_n[\sigma(s)\cdot\hat{p}]^3+[1+c_n\sigma^2(s)][\sigma(s)\cdot\hat{p}],l=2k\end{cases}\end{cases}$$

It is completely equivalent to the discrete equation and initial conditions in the previous section, so it has the same solution as the following.

Thm. 5.2.4.
$$\begin{cases}A_R(n,l)=i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n\\ =\sum\limits_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1}+C_l^{2k+2}k_n+(C_l^{2k-1}c_{n+1}+C_l^{2k}c_n)\sigma^2(s)+C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k}-c_n\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}\\ k_n=-(a_n+b_n)=\frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)},c_n=-\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1},n\geq0,l\geq0\end{cases}$$

Cor. 5.2.10.
$$\begin{cases}A_R(n,l)=i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n\\ =\sum\limits_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1}+C_l^{2k+2}k_n+(C_l^{2k-1}b_n+C_l^{2k}b_{n-1})\sigma^2(s)+C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k}-b_{n-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1}\\ k_n=-(a_n+b_n)=\frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)},b_n=-\frac{(s+1)^{n}-(-s)^{n}}{2s+1},n\geq0,l\geq0\end{cases}$$

Cor. 5.2.11. $A_L(n,l)=A_R(n,l),\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s)=\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n,n\geq0$

6 Summary of various general term formulas

6.1 Summary of basic general term formulas

Thm. 6.1.1.

$$\begin{cases} i^n\sigma(s)[| \times \hat{p}]^n = \sum_{k=0}^{n} C_n^k[\sigma(s)\cdot\hat{p}]^k\sigma(s)[-\sigma(s)\cdot\hat{p}]^{n-k}, n\geq 0 = \begin{cases} i\sigma(s)\times\hat{p}, n=2k-1, k\geq 1 \\ \sigma(s)-[\sigma(s)\cdot\hat{p}]\hat{p}, n=2k \end{cases} \\ i^n[\hat{p}\times|]^n\sigma(s) = \sum_{k=0}^{n} C_n^k[-\sigma(s)\cdot\hat{p}]^{n-k}\sigma(s)[\sigma(s)\cdot\hat{p}]^k, n\geq 0 = \begin{cases} i\hat{p}\times\sigma(s), n=2k-1, k\geq 1 \\ \sigma(s)-[\sigma(s)\cdot\hat{p}]\hat{p}, n=2k \end{cases} \end{cases}$$

Thm. 6.1.2.

$$\begin{cases} i^{-n}[\sigma(s)\times|]^n\hat{p} = a_n\sigma(s)[\sigma(s)\cdot\hat{p}] + b_n[\sigma(s)\cdot\hat{p}]\sigma(s) - b_{n-1}\sigma^2(s)\hat{p}, n\geq 0 \\ i^{-n}\hat{p}[|\times\sigma(s)]^n = a_n[\sigma(s)\cdot\hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s)\cdot\hat{p}] - b_{n-1}\sigma^2(s)\hat{p}, n\geq 0 \\ a_n = \frac{[(s+1)^{n+2}-(-s)^{n+2}]-2[(s+1)^{n+1}-(-s)^{n+1}]+(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1} \end{cases}$$

Thm. 6.1.3.

$$\begin{cases} [\sigma(s)\cdot\hat{p}]^n\sigma(s) = i^n\sigma(s)[|\times\hat{p}]^n - \sum_{k=0}^{n-1} C_n^k[\sigma(s)\cdot\hat{p}]^k\sigma(s)[-\sigma(s)\cdot\hat{p}]^{n-k}, n\geq 0 \\ \sigma(s)[\sigma(s)\cdot\hat{p}]^n = i^n[\hat{p}\times|]^n\sigma(s) - \sum_{k=0}^{n-1} C_n^k[-\sigma(s)\cdot\hat{p}]^{n-k}\sigma(s)[\sigma(s)\cdot\hat{p}]^k, n\geq 0 \end{cases}$$

6.2 Summary of general term formulas for basic cross multiplication type

Thm. 6.2.1.

$$\begin{cases} i^{-1}\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^n\sigma(s)\}, n\geq 0 \\ = \sum_{k=0}^{[n/2]}\{(C_n^{2k+1}+C_n^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) + (C_n^{2k}-C_n^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] - C_n^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]^{n-2k-1} \\ i^{-1}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^n\}\times\sigma(s), n\geq 0 \\ = \sum_{k=0}^{[n/2]}[\sigma(s)\cdot\hat{p}]^{n-2k-1}\{(C_n^{2k+1}+C_n^{2k+2})\sigma(s)[\sigma(s)\cdot\hat{p}] + (C_n^{2k}-C_n^{2k+2})[\sigma(s)\cdot\hat{p}]\sigma(s) - C_n^{2k+1}\sigma^2(s)\hat{p}\} \end{cases}$$

6.3 Summary of basic extended general term formulas for cross multiplication type

Thm. 6.3.1.

$$\begin{cases} i^{-n}[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\}, n\geq 1, l\geq 0 \\ = \sum_{k=0}^{[l/2]}\{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}] - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s) \\ +[C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}]\hat{p}\}[\sigma(s)\cdot\hat{p}]^{l-2k-1} \\ i^{-n}\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n, n\geq 1, l\geq 0 \\ = \sum_{k=0}^{[l/2]}[\sigma(s)\cdot\hat{p}]^{l-2k-1}\{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1}][\sigma(s)\cdot\hat{p}]\sigma(s) \\ -[C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma(s)[\sigma(s)\cdot\hat{p}] + [C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}]\hat{p}\} \end{cases}$$

6.4 Summary of basic general term formulas for scalar product type

Thm. 6.4.1. $\sigma(s)\cdot[\sigma(s)\cdot\hat{p}]^n\sigma(s) = \sigma(s)[\sigma(s)\cdot\hat{p}]^n\cdot\sigma(s) = \sum_{k=0}^{[n/2]}[C_n^{2k}\sigma^2(s) - C_{n+1}^{2k+2}][\sigma(s)\cdot\hat{p}]^{n-2k}, n\geq 0$

6.5 Summary of basic extended general term formulas for scalar product type

Thm. 6.5.1. $i^n\sigma(s)[|\times\hat{p}]^n|\cdot\hat{p} = i^n\hat{p}\cdot|[\hat{p}\times|]^n\sigma(s) = 0, n\geq 1$

Thm. 6.5.2.

$$\begin{cases} i^{-n}\hat{p}\cdot\{[\sigma(s)\times|]^n\hat{p}\} = i^{-n}\{\hat{p}[|\times\sigma(s)]^n\}\cdot\hat{p} = -k_n[\sigma(s)\cdot\hat{p}]^2 + b_{n-1}\sigma^2(s) \\ k_n = -(a_n+b_n) = \frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)}, b_{n-1} = -\frac{(s+1)^{n-1}-(-s)^{n-1}}{2s+1}, n\geq 0 \end{cases}$$

Thm. 6.5.3.

$$\begin{cases} A_L(n,l) = i^{-n}\hat{p}\cdot|[\sigma(s)\times|]^n[\sigma(s)\cdot\hat{p}]^l\sigma(s) = i^{-n}\hat{p}\cdot|\sigma(s)[\sigma(s)\cdot\hat{p}]^l[|\times\sigma(s)]^n = A_R(n,l) \\ = \sum_{k=0}^{[(l+1)/2]}[C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}b_n + C_l^{2k}b_{n-1})\sigma^2(s) + C_l^{2k}][\sigma(s)\cdot\hat{p}]^{l+1-2k} - b_{n-1}\sigma^2(s)[\sigma(s)\cdot\hat{p}]^{l+1} \\ k_n = -(a_n+b_n) = \frac{(s+1)^{n+1}-(-s)^{n+1}-(2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n-(-s)^n}{2s+1}, n\geq 0, l\geq 0 \end{cases}$$

Thm. 6.5.4.

$$\begin{cases} i^{-n}\sigma(s)\cdot|[\sigma(s)\times|]^n\{[\sigma(s)\cdot\hat{p}]^l\sigma(s)\} = i^{-n}\{\sigma(s)[\sigma(s)\cdot\hat{p}]^l\}[|\times\sigma(s)]^n|\cdot\sigma(s) \\ = \sigma(s)\cdot[\sigma(s)\cdot\hat{p}]^l\sigma(s) = \sum_{k=0}^{[l/2]}[C_l^{2k}\sigma^2(s) - C_{l+1}^{2k+2}][\sigma(s)\cdot\hat{p}]^{l-2k}, n\geq 0, l\geq 0 \end{cases}$$

6.6 Discussion on more general various general term formulas

By using the above basic general term formulas, it is relatively easy to strictly derive more and more complex general term formulas. There are no longer difficulties in deriving in principle.

7 Special general term formulas

7.1 A special general term formula

Pro. 7.1.1. $\begin{cases}\hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^1\sigma(s)\}\}=2i[\sigma(s)\cdot\hat{p}]^2-i\sigma^2(s)\\ \hat{p}\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{p}]^2\sigma(s)\}\}=3i[\sigma(s)\cdot\hat{p}]^2-i[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\end{cases}$

Thm. 7.1.1. $[\sigma(s)\cdot\hat{p}]\hat{p}=-i[\sigma(s)\times\hat{p}]\times[\sigma(s)\times\hat{p}]=\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma(s)+2[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]$

Proof: $[\sigma(s)\cdot\hat{p}]\hat{p}=-i[\sigma(s)\times\hat{p}]\times[\sigma(s)\times\hat{p}]$
$=i\sigma(s)\times|[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]+[\sigma(s)\cdot\hat{p}]i\sigma(s)[\sigma(s)\cdot\hat{p}]|\times\sigma(s)$
$-i\sigma(s)\times|[\sigma(s)\cdot\hat{p}]^2\sigma(s)+[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]$
$=\{-[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]+\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]+[\sigma(s)\cdot\hat{p}]\{-[\sigma(s)\cdot\hat{p}]\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]+\sigma^2(s)\hat{p}\}$
$+3[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]-2\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]+\sigma(s)+[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]$
$=-\{[\sigma(s)\cdot\hat{p}]\sigma(s)+\sigma(s)[\sigma(s)\cdot\hat{p}]-\sigma^2(s)\hat{p}\}[\sigma(s)\cdot\hat{p}]-[\sigma(s)\cdot\hat{p}]\{[\sigma(s)\cdot\hat{p}]\sigma(s)+\sigma(s)[\sigma(s)\cdot\hat{p}]-\sigma^2(s)\hat{p}\}$
$+3[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]-2\sigma^2(s)\hat{p}[\sigma(s)\cdot\hat{p}]+\sigma(s)+[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]$
$=\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma(s)+2[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]$ □

7.2 Deduction of the special general term formula

Cor. 7.2.1. $[1-(h-h')^2]\lambda^+(\hat{p},h;s)\sigma(s)\lambda(\hat{p},h';s)=\delta_{hh'}h\hat{p}$

Proof: $\lambda^+(\hat{p},h;s)[\sigma(s)\cdot\hat{p}]\hat{p}\lambda(\hat{p},h';s)$
$=\lambda^+(\hat{p},h;s)\{\sigma(s)-\sigma(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma(s)+2[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]\}\lambda(\hat{p},h';s)$
$\Leftrightarrow[1-(h-h')^2]\lambda^+(\hat{p},h;s)\sigma(s)\lambda(\hat{p},h';s)=\delta_{hh'}h\hat{p}$ □

Cor. 7.2.2. $[\sigma(s)\cdot\hat{p}]^n\hat{p}_{i_1}\hat{p}_{i_2}\cdot\cdot\hat{p}_{i_n}=$
$\{\sigma_{i_1}(s)-\sigma_{i_1}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_1}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_1}(s)[\sigma(s)\cdot\hat{p}]\}$
$\{\sigma_{i_2}(s)-\sigma_{i_2}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_2}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_2}(s)[\sigma(s)\cdot\hat{p}]\}$
$\cdots\cdots$
$\{\sigma_{i_n}(s)-\sigma_{i_n}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_n}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_n}(s)[\sigma(s)\cdot\hat{p}]\}$

Cor. 7.2.3. $[\sigma\cdot\hat{p}]^n\hat{p}_{i_1}\hat{p}_{i_2}\cdot\cdot\hat{p}_{i_n}=\{-\sigma_{i_1}+2[\sigma\cdot\hat{p}]\sigma_{i_1}[\sigma\cdot\hat{p}]\}\{-\sigma_{i_2}+2[\sigma\cdot\hat{p}]\sigma_{i_2}[\sigma\cdot\hat{p}]\}\cdot\cdot\{-\sigma_{i_n}+2[\sigma\cdot\hat{p}]\sigma_{i_n}[\sigma(s)\cdot\hat{p}]\}$

Cor. 7.2.4. $\frac{1}{4}\hat{p}_i\hat{p}_j$
$=\{\frac{1}{4}\sigma_i+\frac{1}{4}[\sigma\cdot\hat{p}]\sigma_i[\sigma\cdot\hat{p}]\}\{\frac{1}{4}\sigma_j+\frac{1}{4}[\sigma\cdot\hat{p}]\sigma_j[\sigma\cdot\hat{p}]\}$
$=\frac{1}{16}\{\sigma_i\sigma_j+[\sigma\cdot\hat{p}]\sigma_i[\sigma\cdot\hat{p}]\sigma_j+\sigma_i[\sigma\cdot\hat{p}]\sigma_j[\sigma\cdot\hat{p}]+[\sigma\cdot\hat{p}]\sigma_i\sigma_j[\sigma\cdot\hat{p}]\}$

Proof: $\lambda^+(\hat{p},h;s)[\sigma(s)\cdot\hat{p}]^n\hat{p}_{i_1}\hat{p}_{i_2}\cdot\cdot\hat{p}_{i_n}\lambda(\hat{p},h';s)$
$=\lambda^+(\hat{p},h;s)$
$\{\sigma_{i_1}(s)-\sigma_{i_1}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_1}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_1}(s)[\sigma(s)\cdot\hat{p}]\}$
$\{\sigma_{i_2}(s)-\sigma_{i_2}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_2}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_2}(s)[\sigma(s)\cdot\hat{p}]\}\lambda(\hat{p},h';s)$
$\cdots\cdots$
$\{\sigma_{i_n}(s)-\sigma_{i_n}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_n}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_n}(s)[\sigma(s)\cdot\hat{p}]\}$
$\lambda(\hat{p},h';s)$
$=\lambda^+(\hat{p},h;s)$
$\{\sigma_{i_1}(s)-\sigma_{i_1}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_1}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_1}(s)[\sigma(s)\cdot\hat{p}]\}\lambda(\hat{p},h';s)$
$\sum_{h_1=s}^{-s}\lambda(\hat{p},h_1;s)\lambda^+(\hat{p},h_1;s)$
$\{\sigma_{i_2}(s)-\sigma_{i_2}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_2}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_2}(s)[\sigma(s)\cdot\hat{p}]\}$
$\sum_{h_2=s}^{-s}\lambda(\hat{p},h_2;s)\lambda^+(\hat{p},h_2;s)$
$\cdots\cdots$
$\{\sigma_{i_n}(s)-\sigma_{i_n}(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma_{i_n}(s)+2[\sigma(s)\cdot\hat{p}]\sigma_{i_n}(s)[\sigma(s)\cdot\hat{p}]\}$
$\lambda(\hat{p},h';s)$ □

Thm. 7.2.1. $[\sigma(s)\cdot\hat{p}]\hat{p}$
$=\sigma(s)[\sigma(s)\cdot\hat{p}]^2-[\sigma(s)\cdot\hat{p}]^2\sigma(s)+2[\sigma(s)\cdot\hat{p}]\sigma(s)[\sigma(s)\cdot\hat{p}]$
$=\sigma(s)-\sum_{k=0}^{2}C_2^k[-\sigma(s)\cdot\hat{p}]^k\sigma(s)[\sigma(s)\cdot\hat{p}]^{2-k}$

Cor. 7.2.5. $[\sigma(s)\cdot\hat{p}]^n\hat{p}_{i_1}\hat{p}_{i_2}\cdot\cdot\hat{p}_{i_n}=$
$\{\sigma_{i_1}(s)-\sum\limits_{k=0}^{2}C_2^k[-\sigma(s)\cdot\hat{p}]^k\sigma_{i_1}(s)[\sigma(s)\cdot\hat{p}]^{2-k}\}$
$\{\sigma_{i_2}(s)-\sum\limits_{k=0}^{2}C_2^k[-\sigma(s)\cdot\hat{p}]^k\sigma_{i_2}(s)[\sigma(s)\cdot\hat{p}]^{2-k}\}$
$\cdots\cdots$
$\{\sigma_{i_n}(s)-\sum\limits_{k=0}^{2}C_2^k[-\sigma(s)\cdot\hat{p}]^k\sigma_{i_n}(s)[\sigma(s)\cdot\hat{p}]^{2-k}\}$

Cor. 7.2.6. $\lambda^+(\hat{p},-s\varsigma)\sigma_i(s)[\sigma(s)\cdot\hat{p}]^n\sigma_j(s)\lambda(\hat{p},-s\varsigma)=(-\varsigma)^n s^2 s^n\hat{p}_i\hat{p}_j+(-\varsigma)^n(s-1)^n[\frac{s}{2}(\delta_{ij}-\hat{p}_i\hat{p}_j-i\varsigma\varepsilon_{ij}{}^k\hat{p}_k)]$

Cor. 7.2.7. $\lambda^+(\hat{p},h;s)\sigma_i(s)[\sigma(s)\cdot\hat{p}]^n\sigma_j(s)\lambda(\hat{p},h';s)=???$

8 Lower order expansion of $[\sigma(s)\cdot\hat{p}]^{2s+1}$

8.1 Lower order expansion conjecture of $[\sigma(s)\cdot\hat{p}]^{2s+1}$

Ass. 8.1.1. $[\sigma(s)\cdot\hat{p}]^{2s+1}=\sum\limits_{k=1}^{[s+\frac{1}{2}]}X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

The above conjecture can be fully proved through the natural number splitting conjecture of the polynomial theorem. Let it go for a moment, and I'll do it later when I have time.

8.2 Strict proof of lower order expansion coefficients equation for $[\sigma(s)\cdot\hat{p}]^{2s+1}$

Thm. 8.2.1. $[\sigma(s)\cdot\hat{p}]^{2s+1}=\sum\limits_{k=1}^{[s+\frac{1}{2}]}X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}\Rightarrow h^{2s+1}=\sum\limits_{k=1}^{[s+\frac{1}{2}]}X_k(s)h^{2s+1-2k},h=s,(s-1),\cdots,1|2$

Proof: $[\sigma(s)\cdot\hat{p}]^{2s+1}=\sum\limits_{k=1}^{[s+\frac{1}{2}]}X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

$\Rightarrow\lambda^+(s,h)[\sigma(s)\cdot\hat{p}]^{2s+1}\lambda(s,h)=\sum\limits_{k=1}^{[s+\frac{1}{2}]}X_k(s)\lambda^+(s,h)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}\lambda(s,h),h=s,(s-1),\cdots,-(s-1),-s$

$\Rightarrow h^{2s+1}=\sum\limits_{k=1}^{[s+\frac{1}{2}]}X_k(s)h^{2s+1-2k},h=s,(s-1),\cdots,-(s-1),-s$

$\Rightarrow h^{2s+1}=\sum\limits_{k=1}^{[s+\frac{1}{2}]}X_k(s)h^{2s+1-2k},h=s,(s-1),\cdots,\frac{1}{2}|1$ □

Thm. 8.2.2. $[\sigma(n-\frac{1}{2})\cdot\hat{p}]^{2n}=\sum\limits_{k=0}^{n-1}\bar{X}_k(n-\frac{1}{2})[\sigma(n-\frac{1}{2})\cdot\hat{p}]^{2k},[\sigma(n)\cdot\hat{p}]^{2n+1}=\sum\limits_{k=0}^{n-1}\bar{X}_k(n)[\sigma(n)\cdot\hat{p}]^{2k+1}$

8.3 Solution for low order expansion of $[\sigma(s)\cdot\hat{p}]^{2s+1}$

Def. 8.3.1. $\begin{cases}C^k_{\{a_1,a_2,\cdots,a_n\}}:=\textit{Multiply } k\ a_i \textit{ according to combination rule and add up all product terms}\\ C^0_{\{a_1,a_2,\cdots,a_n\}}:=1\end{cases}$

Cor. 8.3.1. $C^k_{\{1_1,1_2,\cdots,1_n\}}=C^k_n$

Ass. 8.3.1.

$$\begin{cases}\begin{bmatrix}(2n-1)^{2n-2} & (2n-1)^{2n-4} & \cdots & (2n-1)^2 & (2n-1)^0\\ (2n-3)^{2n-2} & (2n-3)^{2n-4} & \cdots & (2n-3)^2 & (2n-3)^0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 3^{2n-2} & 3^{2n-4} & \cdots & 3^2 & 3^0\\ 1^{2n-2} & 1^{2n-4} & \cdots & 1^2 & 1^0\end{bmatrix}Y(n-\frac{1}{2})=\begin{bmatrix}(2n-1)^{2n}\\ (2n-3)^{2n}\\ \cdots\\ 3^{2n}\\ 1^{2n}\end{bmatrix},Y(n-\frac{1}{2})=\begin{bmatrix}(-1)^0[C^1_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-1)^1[C^2_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ \cdots\\ (-1)^{n-2}[C^{n-1}_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-1)^{n-1}[C^n_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\end{bmatrix}\\ \begin{bmatrix}(n-\frac{1}{2})^{2n-2} & (n-\frac{1}{2})^{2n-4} & \cdots & (n-\frac{1}{2})^2 & (n-\frac{1}{2})^0\\ (n-\frac{3}{2})^{2n-2} & (n-\frac{3}{2})^{2n-4} & \cdots & (n-\frac{3}{2})^2 & (n-\frac{3}{2})^0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ (\frac{3}{2})^{2n-2} & (\frac{3}{2})^{2n-4} & \cdots & (\frac{3}{2})^2 & (\frac{3}{2})^0\\ (\frac{1}{2})^{2n-2} & (\frac{1}{2})^{2n-4} & \cdots & (\frac{1}{2})^2 & (\frac{1}{2})^0\end{bmatrix}X(n-\frac{1}{2})=\begin{bmatrix}(n-\frac{1}{2})^{2n}\\ (n-\frac{3}{2})^{2n}\\ \cdots\\ (\frac{3}{2})^{2n}\\ (\frac{1}{2})^{2n}\end{bmatrix},X(n-\frac{1}{2})=-\begin{bmatrix}(-4)^{-1}[C^1_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-4)^{-2}[C^2_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ \cdots\\ (-4)^{-(n-1)}[C^{n-1}_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-4)^{-n}[C^n_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\end{bmatrix}\\ \begin{bmatrix}(n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \cdots & (n-\frac{1}{2})^{2n-4} & (n-\frac{1}{2})^{2n-2}\\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \cdots & (n-\frac{3}{2})^{2n-4} & (n-\frac{3}{2})^{2n-2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ (\frac{3}{2})^0 & (\frac{3}{2})^2 & \cdots & (\frac{3}{2})^{2n-4} & (\frac{3}{2})^{2n-2}\\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \cdots & (\frac{1}{2})^{2n-4} & (\frac{1}{2})^{2n-2}\end{bmatrix}\bar{X}(n-\frac{1}{2})=\begin{bmatrix}(n-\frac{1}{2})^{2n}\\ (n-\frac{3}{2})^{2n}\\ \cdots\\ (\frac{3}{2})^{2n}\\ (\frac{1}{2})^{2n}\end{bmatrix},\bar{X}(n-\frac{1}{2})=-\begin{bmatrix}(-4)^{-n}[C^n_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-4)^{-(n-1)}[C^{n-1}_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ \cdots\\ (-4)^{-2}[C^2_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-4)^{-1}[C^1_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\end{bmatrix}\end{cases}$$

Ass. 8.3.2.

$$\begin{cases}\begin{bmatrix}(2n)^{2n-2} & (2n)^{2n-4} & \cdots & (2n)^2 & (2n)^0\\ (2n-2)^{2n-2} & (2n-2)^{2n-4} & \cdots & (2n-2)^2 & (2n-2)^0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 4^{2n-2} & 4^{2n-4} & \cdots & 4^2 & 4^0\\ 2^{2n-2} & 2^{2n-4} & \cdots & 2^2 & 2^0\end{bmatrix} Y(n) = \begin{bmatrix}(2n)^{2n}\\ (2n-2)^{2n}\\ \cdots\\ 4^{2n}\\ 2^{2n}\end{bmatrix}, Y(n) = \begin{bmatrix}(-1)^0[C^1_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-1)^1[C^2_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ \cdots\\ (-1)^{n-2}[C^{n-1}_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-1)^{n-1}[C^n_{\{2^2,4^2,\cdots,(2n)^2\}}]\end{bmatrix}\\ \begin{bmatrix}n^{2n-2} & n^{2n-4} & \cdots & n^2 & n^0\\ (n-1)^{2n-2} & (n-1)^{2n-4} & \cdots & (n-1)^2 & (n-1)^0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 2^{2n-2} & 2^{2n-4} & \cdots & 2^2 & 2^0\\ 1^{2n-2} & 1^{2n-4} & \cdots & 1^2 & 1^0\end{bmatrix} X(n) = \begin{bmatrix}n^{2n}\\ (n-1)^{2n}\\ \cdots\\ 2^{2n}\\ 1^{2n}\end{bmatrix}, X(n) = -\begin{bmatrix}(-4)^{-1}[C^1_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-4)^{-2}[C^2_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ \cdots\\ (-4)^{-(n-1)}[C^{n-1}_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-4)^{-n}[C^n_{\{2^2,4^2,\cdots,(2n)^2\}}]\end{bmatrix}\\ \begin{bmatrix}n^0 & n^2 & \cdots & n^{2n-4} & n^{2n-2}\\ (n-1)^0 & (n-1)^2 & \cdots & (n-1)^{2n-4} & (n-1)^{2n-2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 2^0 & 2^2 & \cdots & 2^{2n-4} & 2^{2n-2}\\ 1^0 & 1^2 & \cdots & 1^{2n-4} & 1^{2n-2}\end{bmatrix} \bar{X}(n) = \begin{bmatrix}n^{2n}\\ (n-1)^{2n}\\ \cdots\\ 2^{2n}\\ 1^{2n}\end{bmatrix}, \bar{X}(n) = -\begin{bmatrix}(-4)^{-n}[C^n_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-4)^{-(n-1)}[C^{n-1}_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ \cdots\\ (-4)^{-2}[C^2_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-4)^{-1}[C^1_{\{2^2,4^2,\cdots,(2n)^2\}}]\end{bmatrix}\end{cases}$$

Cor. 8.3.2.

$$\begin{cases}X(n-\frac{1}{2}) = \begin{bmatrix}(4)^{-1}Y_1(n-\frac{1}{2})\\ (4)^{-2}Y_2(n-\frac{1}{2})\\ \cdots\\ (4)^{-n}Y_n(n-\frac{1}{2})\end{bmatrix}, X(n) = \begin{bmatrix}(4)^{-1}Y_1(n)\\ (4)^{-2}Y_2(n)\\ \cdots\\ (4)^{-n}Y_n(n)\end{bmatrix}\\ Y(n-\frac{1}{2}) = \begin{bmatrix}(4)^{1}X_1(n-\frac{1}{2})\\ (4)^{2}X_2(n-\frac{1}{2})\\ \cdots\\ (4)^{n}X_n(n-\frac{1}{2})\end{bmatrix}, Y(n) = \begin{bmatrix}(4)^{1}X_1(n)\\ (4)^{2}X_2(n)\\ \cdots\\ (4)^{n}X_n(n)\end{bmatrix}\end{cases}$$

Cor. 8.3.3.

$$\begin{cases}[\sigma(s)\cdot\hat{p}]^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k} = \sum_{k=1}^{[s+\frac{1}{2}]} 4^{-k}Y_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}\\ \Omega^{2s+1}(s) = \sum_{k=1}^{[s+\frac{1}{2}]} Y_k(s)\Omega^{2s+1-2k}(s) = \sum_{k=1}^{[s+\frac{1}{2}]} 4^{k}X_k(s)\Omega^{2s+1-2k}(s), [\sigma(s)\cdot\hat{p}] \leftrightarrow \frac{1}{2}\Omega(s)\end{cases}$$

Ass. 8.3.3.

$$\begin{cases}\begin{bmatrix}(n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \cdots & (n-\frac{1}{2})^{2n-4} & (n-\frac{1}{2})^{2n-2}\\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \cdots & (n-\frac{3}{2})^{2n-4} & (n-\frac{3}{2})^{2n-2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ (\frac{3}{2})^0 & (\frac{3}{2})^2 & \cdots & (\frac{3}{2})^{2n-4} & (\frac{3}{2})^{2n-2}\\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \cdots & (\frac{1}{2})^{2n-4} & (\frac{1}{2})^{2n-2}\end{bmatrix} \bar{X}(n-\frac{1}{2}) = \begin{bmatrix}(n-\frac{1}{2})^{2n}\\ (n-\frac{3}{2})^{2n}\\ \cdots\\ (\frac{3}{2})^{2n}\\ (\frac{1}{2})^{2n}\end{bmatrix}, \bar{X}(n-\frac{1}{2}) = -\begin{bmatrix}(-4)^{-n}[C^n_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-4)^{-(n-1)}[C^{n-1}_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ \cdots\\ (-4)^{-2}[C^2_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\\ (-4)^{-1}[C^1_{\{1^2,3^2,\cdots,(2n-1)^2\}}]\end{bmatrix}\\ \begin{bmatrix}n^0 & n^2 & \cdots & n^{2n-4} & n^{2n-2}\\ (n-1)^0 & (n-1)^2 & \cdots & (n-1)^{2n-4} & (n-1)^{2n-2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 2^0 & 2^2 & \cdots & 2^{2n-4} & 2^{2n-2}\\ 1^0 & 1^2 & \cdots & 1^{2n-4} & 1^{2n-2}\end{bmatrix} \bar{X}(n) = \begin{bmatrix}n^{2n}\\ (n-1)^{2n}\\ \cdots\\ 2^{2n}\\ 1^{2n}\end{bmatrix}, \bar{X}(n) = -\begin{bmatrix}(-4)^{-n}[C^n_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-4)^{-(n-1)}[C^{n-1}_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ \cdots\\ (-4)^{-2}[C^2_{\{2^2,4^2,\cdots,(2n)^2\}}]\\ (-4)^{-1}[C^1_{\{2^2,4^2,\cdots,(2n)^2\}}]\end{bmatrix}\\ \begin{bmatrix}s^0 & s^2 & \cdots & s^{2[s-\frac{1}{2}]}\\ (s-1)^0 & (s-1)^2 & \cdots & (s-1)^{2[s-\frac{1}{2}]}\\ \cdots & \cdots & \cdots & \cdots\\ (\frac{1}{2}|1)^0 & (\frac{1}{2}|1)^2 & \cdots & (\frac{1}{2}|1)^{2[s-\frac{1}{2}]}\end{bmatrix} \bar{X}(s) = \begin{bmatrix}s^{2[s+\frac{1}{2}]}\\ (s-1)^{2[s+\frac{1}{2}]}\\ \cdots\\ (\frac{1}{2}|1)^{2[s+\frac{1}{2}]}\end{bmatrix}, \bar{X}(s) = -\begin{bmatrix}(-4)^{-[s+\frac{1}{2}]}[C^{[s+\frac{1}{2}]}_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}]\\ (-4)^{-[s-\frac{1}{2}]}[C^{[s-\frac{1}{2}]}_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}]\\ \cdots\\ (-4)^{-2}[C^2_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}]\\ (-4)^{-1}[C^1_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}]\end{bmatrix}\end{cases}$$

Ass. 8.3.4.

$$\begin{cases}[\sigma(s)\cdot\hat{p}]^{2s+1} = -\sum_{k=1}^{[s+\frac{1}{2}]}(-4)^{-k}C^k_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}[\sigma(s)\cdot\hat{p}]^{2s+1-2k}\\ \Omega^{2s+1}(s) = -\sum_{k=1}^{[s+\frac{1}{2}]}(-1)^{k}C^k_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}\Omega^{2s+1-2k}(s)\end{cases}$$

Ass. 8.3.5.

$$\begin{cases}\sum_{k=0}^{[s+\frac{1}{2}]}(-4)^{-k}C^k_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}[\sigma(s)\cdot\hat{p}]^{2s+1-2k} = 0\\ \sum_{k=0}^{[s+\frac{1}{2}]}(-1)^{k}C^k_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}\Omega^{2s+1-2k}(s) = 0\\ \sum_{k=0}^{[s+\frac{1}{2}]}(-4h^2)^{-k}C^k_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}} = 0, h = s,(s-1),\cdots,\frac{1}{2}|1\end{cases}$$

8.4 Properties of $C^k_{\{a_1,a_2,\cdots,a_n\}}$

Pro. 8.4.1. $C^k_{\{a_1,a_2,\cdots,a_n\}} = C^1_{\{a_1\}}C^{k-1}_{\{a_2,\cdots,a_n\}} + C^0_{\{a_1\}}C^k_{\{a_2,\cdots,a_n\}}$

Pro. 8.4.2. $C^k_{\{a_1,a_2,\cdots,a_n\}} = C^2_{\{a_1,a_2\}} C^{k-2}_{\{a_3,\cdots,a_n\}} + C^1_{\{a_1,a_2\}} C^{k-1}_{\{a_3,\cdots,a_n\}}) + C^0_{\{a_1,a_2\}} C^k_{\{a_3,\cdots,a_n\}}$

Pro. 8.4.3. $C^k_{\{a_1,a_2,\cdots,a_n\}} = \sum\limits_{i=0}^{l} C^i_{\{a_1,a_2,\cdots,a_l\}} C^{k-i}_{\{a_{l+1},\cdots,a_n\}}$

Pro. 8.4.4. $C^{i_0}_{\{a_1,a_2,\cdots,a_{l_0}\}} = \sum\limits_{i_1=0}^{l_1}\sum\limits_{i_2=0}^{l_2} C^{i_2}_{\{a_1,a_2,\cdots,a_{l_2}\}} C^{i_1-i_2}_{\{a_{l_2+1},\cdots,a_{l_1}\}} C^{i_0-i_1}_{\{a_{l_1+1},\cdots,a_{l_0}\}}$

Pro. 8.4.5. $C^{i_0}_{\{a_1,a_2,\cdots,a_{l_0}\}} = \sum\limits_{i_1=0}^{l_1}\sum\limits_{i_2=0}^{l_2}\cdots\sum\limits_{i_3=0}^{l_3} C^{i_3}_{\{a_1,a_2,\cdots,a_{l_3}\}} C^{i_2-i_3}_{\{a_{l_3+1},a_2,\cdots,a_{l_2}\}} C^{i_1-i_2}_{\{a_{l_2+1},\cdots,a_{l_1}\}} C^{i_0-i_1}_{\{a_{l_1+1},\cdots,a_{l_0}\}}$

Pro. 8.4.6. $C^{i_0}_{\{a_1,a_2,\cdots,a_{l_0}\}} = \sum\limits_{i_1=0}^{l_1}\sum\limits_{i_2=0}^{l_2}\cdots\sum\limits_{i_r=0}^{l_r} C^{i_r}_{\{a_1,a_2,\cdots,a_{l_r}\}} \cdots C^{i_2-i_3}_{\{a_{l_3+1},a_2,\cdots,a_{l_2}\}} C^{i_1-i_2}_{\{a_{l_2+1},\cdots,a_{l_1}\}} C^{i_0-i_1}_{\{a_{l_1+1},\cdots,a_{l_0}\}}$

8.5 Special case verification of low order expansions of $[\sigma(s)\cdot\hat{p}]^{2s+1}$

Def. 8.5.1. $a_i^2 = 1$

Pro. 8.5.1. $(a_1)^2 = 1, [\sigma(\frac{1}{2})\cdot\hat{p}]^4 = \frac{1}{4}$

Pro. 8.5.2. $(a_1+a_2)^3 = 4(a_1+a_2), [\sigma(1)\cdot\hat{p}]^3 = [\sigma(1)\cdot\hat{p}]$

Cor. 8.5.1.
$\begin{bmatrix} 3^2 & 3^0 \\ 1^2 & 1^0 \end{bmatrix} Y = \begin{bmatrix} 3^4 \\ 1^4 \end{bmatrix} \Leftrightarrow Y = \frac{1}{8}\begin{bmatrix} 1^0 & -1^2 \\ -3^0 & 3^2 \end{bmatrix}\begin{bmatrix} 3^4 \\ 1^4 \end{bmatrix} = \begin{bmatrix} 10 \\ -9 \end{bmatrix} = \begin{bmatrix} 1^2+3^2 \\ -1^2 3^2 \end{bmatrix}$

Pro. 8.5.3. $(a_1+a_2+a_3)^4 = 10(a_1+a_2+a_3)^2 - 9, [\sigma(\frac{3}{2})\cdot\hat{p}]^4 = \frac{5}{2}[\sigma(\frac{3}{2})\cdot\hat{p}]^2 - \frac{9}{16}$

Cor. 8.5.2.
$\begin{bmatrix} 4^2 & 4^0 \\ 2^2 & 2^0 \end{bmatrix} Y = \begin{bmatrix} 4^4 \\ 2^4 \end{bmatrix} \Leftrightarrow Y = \frac{1}{12}\begin{bmatrix} 2^0 & 2^2 \\ -4^0 & 4^2 \end{bmatrix}\begin{bmatrix} 4^4 \\ 2^4 \end{bmatrix} = \begin{bmatrix} 20 \\ -64 \end{bmatrix} = \begin{bmatrix} 2^2+4^2 \\ -2^2 4^2 \end{bmatrix}$

Pro. 8.5.4. $(a_1+a_2+a_3+a_4)^5 = 20(a_1+a_2+a_3+a_4)^3 - 64(a_1+a_2+a_3+a_4), [\sigma(2)\cdot\hat{p}]^5 = 5[\sigma(2)\cdot\hat{p}]^3 - 4[\sigma(2)\cdot\hat{p}]$

Cor. 8.5.3.
$\begin{bmatrix} 5^4 & 5^2 & 5^0 \\ 3^4 & 3^2 & 3^0 \\ 1^4 & 1^2 & 1^0 \end{bmatrix} Y = \begin{bmatrix} 5^6 \\ 3^6 \\ 1^6 \end{bmatrix} \Leftrightarrow Y = \frac{1}{384}\begin{bmatrix} 1 & -3 & 2 \\ -10 & 78 & -68 \\ 9 & -75 & 450 \end{bmatrix}\begin{bmatrix} 5^6 \\ 3^6 \\ 1^6 \end{bmatrix} = \begin{bmatrix} 35 \\ -259 \\ 225 \end{bmatrix} = \begin{bmatrix} 1^2+3^2+5^2 \\ -(1^2 3^2+3^2 5^2+5^2 1^2) \\ 1^2 3^2 5^2 \end{bmatrix}$

Pro. 8.5.5. $\begin{cases} (a_1+a_2+\cdots+a_5)^6 = 35(a_1+a_2+\cdots+a_5)^4 - 259(a_1+a_2+\cdots+a_5)^2 + 225 \\ [\sigma(\frac{5}{2})\cdot\hat{p}]^6 = \frac{35}{4}[\sigma(\frac{5}{2})\cdot\hat{p}]^4 - \frac{259}{16}[\sigma(\frac{5}{2})\cdot\hat{p}]^2 + \frac{225}{64} \end{cases}$

Cor. 8.5.4.
$\begin{bmatrix} 6^4 & 6^2 & 6^0 \\ 4^4 & 4^2 & 4^0 \\ 2^4 & 2^2 & 2^0 \end{bmatrix} Y = \begin{bmatrix} 6^6 \\ 4^6 \\ 2^6 \end{bmatrix} \Leftrightarrow Y = \frac{1}{7680}\begin{bmatrix} 12 & -32 & 20 \\ -240 & 1280 & -1040 \\ 768 & -4608 & 11520 \end{bmatrix}\begin{bmatrix} 6^6 \\ 4^6 \\ 2^6 \end{bmatrix} = \begin{bmatrix} 56 \\ -784 \\ 48^2 \end{bmatrix} = \begin{bmatrix} 2^2+4^2+6^2 \\ -(2^2 4^2+4^2 6^2+6^2 2^2) \\ 2^2 4^2 6^2 \end{bmatrix}$

Pro. 8.5.6. $\begin{cases} (a_1+a_2+\cdots+a_6)^7 = 56(a_1+a_2+\cdots+a_5)^5 - 784(a_1+a_2+\cdots+a_5)^3 + 2304(a_1+a_2+\cdots+a_5) \\ [\sigma(3)\cdot\hat{p}]^7 = 14[\sigma(3)\cdot\hat{p}]^5 - 49[\sigma(3)\cdot\hat{p}]^3 + 36[\sigma(3)\cdot\hat{p}] \end{cases}$

Cor. 8.5.5.
$\begin{bmatrix} 3^2 & 3^0 \\ 1^2 & 1^0 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} 1^0 & -3^0 \\ -1^2 & 3^2 \end{bmatrix}, \begin{bmatrix} 4^2 & 4^0 \\ 2^2 & 2^0 \end{bmatrix}^{-1} = \frac{1}{12}\begin{bmatrix} 2^0 & -4^0 \\ -2^2 & 4^2 \end{bmatrix}$
$\begin{bmatrix} 5^4 & 5^2 & 5^0 \\ 3^4 & 3^2 & 3^0 \\ 1^4 & 1^2 & 1^0 \end{bmatrix}^{-1} = \frac{1}{384}\begin{bmatrix} 1 & -3 & 2 \\ -10 & 78 & -68 \\ 9 & -75 & 450 \end{bmatrix}, \begin{bmatrix} 6^4 & 6^2 & 6^0 \\ 4^4 & 4^2 & 4^0 \\ 2^4 & 2^2 & 2^0 \end{bmatrix}^{-1} = \frac{1}{7680}\begin{bmatrix} 12 & -32 & 20 \\ -240 & 1280 & -1040 \\ 768 & -4608 & 11520 \end{bmatrix}$

Pro. 8.5.7. $(a_1+a_2+\cdots+a_7)^8 = 84(a_1+a_2+\cdots+a_5)^6 - 1974(a_1+a_2+\cdots+a_5)^4 + 12916(a_1+a_2+\cdots+a_5)^2 - 11025$
$84 = C_9^6 = 1^2+3^2+5^2+7^2, 1974 = 1^2 3^2 + 3^2 5^2 + 5^2 7^2 + 7^2 1^2 + 1^2 5^2 + 3^2 7^2$
$12916 = 3^2 5^2 7^2 + 5^2 7^2 1^2 + 1^2 3^2 5^2 + 1^2 3^2 7^2, 11025 = 1^2 3^2 5^2 7^2$
$[\sigma(\frac{7}{2})\cdot\hat{p}]^8 = 21[\sigma(\frac{7}{2})\cdot\hat{p}]^6 - \frac{987}{8}[\sigma(\frac{7}{2})\cdot\hat{p}]^4 + \frac{3229}{16}[\sigma(\frac{7}{2})\cdot\hat{p}]^2 - \frac{11025}{256}$

Pro. 8.5.8. $(a_1+a_2+\cdots+a_8)^9$
$= 120(a_1+a_2+\cdots+a_8)^7 - 4368(a_1+a_2+\cdots+a_8)^5 + 52480(a_1+a_2+\cdots+a_8)^3 - 147456(a_1+a_2+\cdots+a_8)$
$120 = C_{10}^7 = 2^2+4^2+6^2+8^2, 4368 = 2^2 4^2 + 4^2 6^2 + 6^2 8^2 + 8^2 2^2 + 2^2 6^2 + 4^2 8^2$
$52480 = 4^2 6^2 8^2 + 6^2 8^2 2^2 + 2^2 4^2 6^2 + 2^2 4^2 8^2, 147456 = 2^2 4^2 6^2 8^2$
$[\sigma(4)\cdot\hat{p}]^9 = 30[\sigma(4)\cdot\hat{p}]^7 - 273[\sigma(4)\cdot\hat{p}]^5 + 820[\sigma(4)\cdot\hat{p}]^3 - 576[\sigma(4)\cdot\hat{p}]$

8.6 A more rigorous proof of lower order expansion coefficients equation for $[\sigma(s)\cdot\hat{p}]^{2s+1}$

Thm. 8.6.1. $[\sigma(s)\cdot\hat{p}]^{2s+1}=\sum\limits_{k=0}^{2s}B_k(s)[\sigma(s)\cdot\hat{p}]^k \Rightarrow h^{2s+1}=\sum\limits_{k=0}^{2s}B_k(s)h^k, h=s,(s-1),\cdots,-(s-1),-s$

Proof: $[\sigma(s)\cdot\hat{p}]^{2s+1}=\sum\limits_{k=0}^{2s}B_k(s)[\sigma(s)\cdot\hat{p}]^k$

$\Rightarrow \lambda^{+}(s,h)[\sigma(s)\cdot\hat{p}]^{2s+1}\lambda(s,h)=\sum\limits_{k=0}^{2s}B_k(s)\lambda^{+}(s,h)[\sigma(s)\cdot\hat{p}]^k\lambda(s,h), h=s,(s-1),\cdots,-(s-1),-s$

$\Rightarrow h^{2s+1}=\sum\limits_{k=0}^{2s}B_k(s)h^k, h=s,(s-1),\cdots,-(s-1),-s$

$$\Rightarrow \begin{bmatrix} s^0 & s^1 & \cdots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \cdots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \cdots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix}$$

$\Rightarrow h^{2s+1}=\sum\limits_{k=0}^{2s}B_k(s)h^k, h=s,(s-1),\cdots,-(s-1),-s$

$$\Rightarrow \begin{bmatrix} B_0(s) \\ B_1(s) \\ \cdots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} s^0 & s^1 & \cdots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \cdots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix} \quad \square$$

It can be verified that the above $s=\frac{1}{2},1,\frac{3}{2},2$ solutions are correct, and the following are more concrete solutions.

Thm. 8.6.2. $[\sigma(s)\cdot\hat{p}]^{2s+1}=-\frac{1}{2}\sum\limits_{k=0}^{2s}[1-(-1)^{2s-k}](-\frac{1}{4})^{[s+\frac{1}{2}]-[\frac{k}{2}]}[C_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}^{[s+\frac{1}{2}]-[\frac{k}{2}]}][\sigma(s)\cdot\hat{p}]^k$

Proof:

$$\begin{bmatrix} s^0 & s^1 & \cdots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \cdots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \cdots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix}$$

$$\Leftrightarrow \frac{1}{2}\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & -1 & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} \begin{bmatrix} s^0 & s^1 & \cdots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \cdots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & -1 & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \cdots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} s^0 & 0 & s^2 & 0 & \cdots & \frac{1}{2}[1+(-1)^{2s}]s^{2s} \\ (s-1)^0 & 0 & (s-1)^2 & 0 & \cdots & \frac{1}{2}[1+(-1)^{2s}](s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & (s-1)^1 & 0 & (s-1)^3 & \cdots & \frac{1}{2}[1-(-1)^{2s}](s-1)^{2s} \\ 0 & s^1 & 0 & s^3 & \cdots & \frac{1}{2}[1-(-1)^{2s}]s^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \cdots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[1+(-1)^{2s+1}]s^{2s+1} \\ \frac{1}{2}[1+(-1)^{2s+1}](s-1)^{2s+1} \\ \cdots \\ \frac{1}{2}[1-(-1)^{2s+1}](s-1)^{2s+1} \\ \frac{1}{2}[1-(-1)^{2s+1}]s^{2s+1} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} s^0 & s^2 & \cdots & s^{2[s]} & 0 & 0 & \cdots & 0 \\ (s-1)^0 & (s-1)^2 & \cdots & (s-1)^{2[s]} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & (s-1)^1 & (s-1)^3 & \cdots & (s-1)^{2[s-\frac{1}{2}]+1} \\ 0 & 0 & \cdots & 0 & s^1 & s^3 & \cdots & s^{2[s-\frac{1}{2}]+1} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_2(s) \\ \cdots \\ B_{2[s]}(s) \\ B_1(s) \\ B_3(s) \\ \cdots \\ B_{2[s-1/2]+1}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[1+(-1)^{2s+1}]s^{2s+1} \\ \frac{1}{2}[1+(-1)^{2s+1}](s-1)^{2s+1} \\ \cdots \\ \frac{1}{2}[1-(-1)^{2s+1}](s-1)^{2s+1} \\ \frac{1}{2}[1-(-1)^{2s+1}]s^{2s+1} \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} n^0 & n^2 & \cdots & n^{2n} & 0 & 0 & \cdots & 0 \\ (n-1)^0 & (n-1)^2 & \cdots & (n-1)^{2n} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1^0 & 1^2 & \cdots & 1^{2n} & 0 & 0 & \cdots & 0 \\ 0^0=1 & 0^2 & \cdots & 0^{2n} & 0^1 & 0^3 & \cdots & 0^{2n-1} \\ 0 & 0 & \cdots & 0 & 1^1 & 1^3 & \cdots & 1^{2n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & (n-1)^1 & (n-1)^3 & \cdots & (n-1)^{2n-1} \\ 0 & 0 & \cdots & 0 & n^1 & n^3 & \cdots & n^{2n-1} \end{bmatrix} \begin{bmatrix} B_0(n) \\ B_2(n) \\ \cdots \\ B_{2n}(n) \\ B_1(n) \\ B_3(n) \\ \cdots \\ B_{2n-1}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0^{2n+1} \\ 1^{2n+1} \\ \cdots \\ (n-1)^{2n+1} \\ n^{2n+1} \end{bmatrix} \\ \begin{bmatrix} (n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \cdots & (n-\frac{1}{2})^{2n-2} & 0 & 0 & \cdots & 0 \\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \cdots & (n-\frac{3}{2})^{2n-2} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \cdots & (\frac{1}{2})^{2n-2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (\frac{1}{2})^1 & (\frac{1}{2})^3 & \cdots & (\frac{1}{2})^{2n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & (n-\frac{3}{2})^1 & (n-\frac{3}{2})^3 & \cdots & (n-\frac{3}{2})^{2n-1} \\ 0 & 0 & \cdots & 0 & (n-\frac{1}{2})^1 & (n-\frac{1}{2})^3 & \cdots & (n-\frac{1}{2})^{2n-1} \end{bmatrix} \begin{bmatrix} B_0(n-\frac{1}{2}) \\ B_2(n-\frac{1}{2}) \\ \cdots \\ B_{2n-2}(n-\frac{1}{2}) \\ B_1(n-\frac{1}{2}) \\ B_3(n-\frac{1}{2}) \\ \cdots \\ B_{2n-1}(n-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} (n-\frac{1}{2})^{2n} \\ (n-\frac{3}{2})^{2n} \\ \cdots \\ (\frac{1}{2})^{2n} \\ 0 \\ \cdots \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

$$\Leftrightarrow \left\{ \begin{array}{l} \begin{bmatrix} (n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \cdots & (n-\frac{1}{2})^{2n-2} & \\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \cdots & (n-\frac{3}{2})^{2n-2} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\frac{3}{2})^0 & (\frac{3}{2})^2 & \cdots & (\frac{3}{2})^{2n-2} & \\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \cdots & (\frac{1}{2})^{2n-2} & \end{bmatrix} \begin{bmatrix} B_0(n-\frac{1}{2}) \\ B_2(n-\frac{1}{2}) \\ \cdots \\ B_{2n-4}(n-\frac{1}{2}) \\ B_{2n-2}(n-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} (n-\frac{1}{2})^{2n} \\ (n-\frac{3}{2})^{2n} \\ \cdots \\ (\frac{3}{2})^{2n} \\ (\frac{1}{2})^{2n} \end{bmatrix} \\ \begin{bmatrix} n^0 & n^2 & \cdots & n^{2n-2} \\ (n-1)^0 & (n-1)^2 & \cdots & (n-1)^{2n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 2^0 & 2^2 & \cdots & 2^{2n-2} \\ 1^0 & 1^2 & \cdots & 1^{2n-2} \end{bmatrix} \begin{bmatrix} B_1(n) \\ B_3(n) \\ \cdots \\ B_{2n-3}(s) \\ B_{2n-1}(s) \end{bmatrix} = \begin{bmatrix} n^{2n} \\ (n-1)^{2n} \\ \cdots \\ 2^{2n} \\ 1^{2n} \end{bmatrix} \\ \begin{bmatrix} B_0(n-\frac{1}{2}) \\ B_1(n-\frac{1}{2}) \\ B_2(n-\frac{1}{2}) \\ B_3(n-\frac{1}{2}) \\ \cdots \\ B_{2n-1}(n-\frac{1}{2}) \\ B_{2n}(n-\frac{1}{2}) \end{bmatrix} = - \begin{bmatrix} (-4)^{-n}[C^n_{\{1^2,3^2,\cdots,(2n-1)^2\}}] \\ 0 \\ (-4)^{-(n-1)}[C^{n-1}_{\{1^2,3^2,\cdots,(2n-1)^2\}}] \\ 0 \\ \cdots \\ (-4)^{-2}[C^2_{\{1^2,3^2,\cdots,(2n-1)^2\}}] \\ 0 \\ (-4)^{-1}[C^1_{\{1^2,3^2,\cdots,(2n-1)^2\}}] \\ 0 \end{bmatrix}, \begin{bmatrix} B_0(n) \\ B_1(n) \\ B_2(n) \\ B_3(n) \\ \cdots \\ B_{2n-3}(n) \\ B_{2n-2}(n) \\ B_{2n-1}(n) \\ B_{2n}(n) \end{bmatrix} = - \begin{bmatrix} 0 \\ (-4)^{-n}[C^n_{\{2^2,4^2,\cdots,(2n)^2\}}] \\ 0 \\ (-4)^{-(n-1)}[C^{n-1}_{\{2^2,4^2,\cdots,(2n)^2\}}] \\ 0 \\ \cdots \\ (-4)^{-2}[C^2_{\{2^2,4^2,\cdots,(2n)^2\}}] \\ 0 \\ (-4)^{-1}[C^1_{\{2^2,4^2,\cdots,(2n)^2\}}] \\ 0 \end{bmatrix} \end{array} \right.$$

$$\Leftrightarrow \begin{bmatrix} B_0(s) \\ B_1(s) \\ \cdots \\ B_k(s) \\ \cdots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} [1-(-1)^{2s}](-4)^{-[s+\frac{1}{2}]}[C^{[s+\frac{1}{2}]}_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}] \\ [1-(-1)^{2s-1}](-4)^{-[s+\frac{1}{2}]}[C^{[s+\frac{1}{2}]}_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}] \\ \cdots \\ [1-(-1)^{2s-k}](-4)^{[\frac{k}{2}]-[s+\frac{1}{2}]}[C^{[s+\frac{1}{2}]-[\frac{k}{2}]}_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}] \\ \cdots \\ [1-(-1)^1](-4)^{-1}[C^1_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}] \\ [1-(-1)^0](-4)^{-1}[C^1_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}] \end{bmatrix}$$

$$\Rightarrow [\sigma(s)\cdot\hat{p}]^{2s+1} = -\frac{1}{2}\sum_{k=0}^{2s}[1-(-1)^{2s-k}](-\frac{1}{4})^{[s+\frac{1}{2}]-[\frac{k}{2}]}[C^{[s+\frac{1}{2}]-[\frac{k}{2}]}_{\{(1|2)^2,\cdots,(2s-2)^2,(2s)^2\}}][\sigma(s)\cdot\hat{p}]^k \qquad \square$$

8.7 Low order expansion coefficients of $[\sigma(s)\cdot\vec{p}]^{2s+1+m}$

Pro. 8.7.1. $[\sigma(s)\cdot\hat{p}]^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}, n := [s+\frac{1}{2}]$

Pro. 8.7.2. $[\sigma(s)\cdot\hat{p}]^{2s+2} = \sum_{k=1}^{n} X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+2-2k}$

Pro. 8.7.3. $[\sigma(s)\cdot\hat{p}]^{2s+3} = \sum_{k=1}^{n} X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+3-2k}$

$= X_1(s)[\sigma(s)\cdot\hat{p}]^{2s+1} + \sum_{k=1}^{n-1} X_{k+1}(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

$= \sum_{k=1}^{n} X_1(s)X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k} + \sum_{k=1}^{n-1} X_{k+1}(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

$= \sum_{k=1}^{n} [X_1(s)X_k(s) + X_{k+1}(s)][\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

Pro. 8.7.4. $[\sigma(s)\cdot\hat{p}]^{2n+5}$

$= \sum_{k=1}^{n} [X_1(s)X_k(s) + X_{k+1}(s)][\sigma(s)\cdot\hat{p}]^{2s+3-2k}$

$= [X_1^2(s) + X_2(s)][\sigma(s)\cdot\hat{p}]^{2s+1} + \sum_{k=1}^{n} [X_1(s)X_{k+1}(s) + X_{k+2}(s)][\sigma(s)\cdot\hat{p}]^{2s+3-2k}$

$= \sum_{k=1}^{[s+\frac{1}{2}]} [X_1^2(s) + X_2(s)]X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k} + \sum_{k=1}^{n} [X_1(s)X_{k+1}(s) + X_{k+2}(s)][\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

$= \sum_{k=1}^{n} \{[X_1^2(s) + X_2(s)]X_k(s) + X_1(s)X_{k+1}(s) + X_{k+2}(s)\}[\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

Pro. 8.7.5. $[\sigma(s)\cdot\hat{p}]^{2n+7}$

$= \sum_{k=1}^{n} [\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

$\{[X_1^3(s) + 2X_1(s)X_2(s) + X_3(s)]X_k(s) + [X_1^2(s) + X_2(s)]X_{k+1}(s) + X_1(s)X_{k+2}(s) + X_{k+3}(n)\}$

Pro. 8.7.6. $[\sigma(s)\cdot\hat{p}]^{2n+9}$

$= \sum_{k=1}^{n} [\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

$\{[X_1^4(s) + 3X_1^2(s)X_2(s) + 2X_1(s)X_3(s) + X_2^2(s) + X_4(s)]X_k(s)$

$+ [X_1^3(s) + 2X_1(s)X_2(s) + X_3(s)]X_{k+1}(s) + [X_1^2(s) + X_2(s)]X_{k+2}(s) + X_1(s)X_{k+3}(s) + X_{k+4}(n)\}$

Pro. 8.7.7. $[\sigma(s)\cdot\hat{p}]^{2s+1+2m}$
$= \sum_{k=1}^{n}\sum_{l=0}^{m}\{(l+1-m)X_1^{m-l}(s) + \sum_{\sum_{i=1}^{n} ir_i}^{=m-l}\sum_{i=1}^{n} r_i[X_1^{r_1}(s)X_2^{r_1}(s)\cdot\cdot X_n^{r_n}(s)]\}u(n-k-l)X_{k+l}(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}$
$= \sum_{k=1}^{n}\sum_{l=0}^{m|(n-k)}\{(l+1-m)X_1^{m-l}(s) + \sum_{\sum_{i=1}^{n} ir_i}^{=m-l}\sum_{i=1}^{n} r_i[X_1^{r_1}(s)X_2^{r_1}(s)\cdot\cdot X_n^{r_n}(s)]\}X_{k+l}(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

Pro. 8.7.8. $[\sigma(s)\cdot\vec{p}]^{2s+1+2m}$
$= \sum_{k=1}^{n}\sum_{l=0}^{m|(n-k)}\{(l+1-m)X_1^{m-l}(s) + \sum_{\sum_{j=1}^{n} jr_j}^{=m-l}\sum_{i=1}^{n} r_i[X_1^{r_1}(s)X_2^{r_1}(s)\cdot\cdot X_n^{r_n}(s)]\}X_{k+l}(s)(\vec{p}^2)^{k+m}[\sigma(s)\cdot\vec{p}]^{2s+1-2k}$

Pro. 8.7.9. $[\sigma(s)\cdot\vec{p}]^{2s+2+2m}$
$= \sum_{k=1}^{n}\sum_{l=0}^{m|(n-k)}\{(l+1-m)X_1^{m-l}(s) + \sum_{\sum_{j=1}^{n} jr_j}^{=m-l}\sum_{i=1}^{n} r_i[X_1^{r_1}(s)X_2^{r_1}(s)\cdot\cdot X_n^{r_n}(s)]\}X_{k+l}(s)(\vec{p}^2)^{k+m}[\sigma(s)\cdot\vec{p}]^{2s+2-2k}$

Pro. 8.7.10. $[\sigma(\frac{1}{2})\cdot\vec{p}]^{2+2m} = X_1^{1+m}(\frac{1}{2})(\vec{p}^2)^{1+m}[\sigma(\frac{1}{2})\cdot\vec{p}]^0, [\sigma(\frac{1}{2})\cdot\vec{p}]^{3+2m} = X_1^{1+m}(\frac{1}{2})(\vec{p}^2)^{1+m}[\sigma(\frac{1}{2})\cdot\vec{p}]^1$

Pro. 8.7.11. $[\sigma(1)\cdot\vec{p}]^{3+2m} = X_1^{1+m}(1)(\vec{p}^2)^{1+m}[\sigma(1)\cdot\vec{p}]^1, [\sigma(1)\cdot\vec{p}]^{4+2m} = X_1^{1+m}(1)(\vec{p}^2)^{1+m}[\sigma(1)\cdot\vec{p}]^2$

9 Polynomial expansion of $e^{\vec{\vartheta}\cdot\sigma(s)}$

9.1 Solution for polynomial expansion coefficients of $e^{\vec{\vartheta}\cdot\sigma(s)}$(solved in principle)

Thm. 9.1.1. $e^{\vec{\vartheta}\cdot\sigma(s)} = \sum_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\sigma(s)]^k, \vec{\vartheta} = \sqrt{\vec{\vartheta}^2}\hat{\vartheta}, \hat{\vartheta}^2 = 1 \Rightarrow e^{h\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2s} A_k(s)[h\sqrt{\vec{\vartheta}^2}]^k, h = s,\cdots,-s$

Proof: $e^{\vec{\vartheta}\cdot\sigma(s)} = \sum_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\sigma(s)]^k, \vec{\vartheta} = \sqrt{\vec{\vartheta}^2}\hat{\vartheta}, \hat{\vartheta}^2 = 1(\sqrt{\vec{\vartheta}^2}$ There are ± two values, either of which can be taken, and the con

$\Rightarrow \tilde{\lambda}^T(s,h)e^{\vec{\vartheta}\cdot\sigma(s)}\lambda(s,h) = \sum_{k=0}^{2s} A_k(s)\tilde{\lambda}^T(s,h)[\vec{\vartheta}\cdot\sigma(s)]^k\lambda(s,h), h = s,\cdots,-s$

$\Rightarrow e^{h\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2s} A_k(s)[h\sqrt{\vec{\vartheta}^2}]^k, h = s,\cdots,-s$

$$\Rightarrow \begin{bmatrix} s^0 & s^1 & \cdots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} (\sqrt{\vec{\vartheta}^2})^0 A_0(s) \\ (\sqrt{\vec{\vartheta}^2})^1 A_1(s) \\ \cdots \\ (\sqrt{\vec{\vartheta}^2})^{2s-1} A_{2s-1}(s) \\ (\sqrt{\vec{\vartheta}^2})^{2s} A_{2s}(s) \end{bmatrix} = \begin{bmatrix} e^{s\sqrt{\vec{\vartheta}^2}} \\ e^{(s-1)\sqrt{\vec{\vartheta}^2}} \\ \cdots \\ e^{(1-s)\sqrt{\vec{\vartheta}^2}} \\ e^{(-s)\sqrt{\vec{\vartheta}^2}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_0(s) \\ A_1(s) \\ \cdots \\ A_{2s-1}(s) \\ A_{2s}(s) \end{bmatrix} = \begin{bmatrix} (\sqrt{\vec{\vartheta}^2})^{-0} & 0 & \cdots & 0 & 0 \\ 0 & (\sqrt{\vec{\vartheta}^2})^{-1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (\sqrt{\vec{\vartheta}^2})^{1-2s} & 0 \\ 0 & 0 & \cdots & 0 & (\sqrt{\vec{\vartheta}^2})^{-2s} \end{bmatrix} \begin{bmatrix} s^0 & s^1 & \cdots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} \begin{bmatrix} e^{s\sqrt{\vec{\vartheta}^2}} \\ e^{(s-1)\sqrt{\vec{\vartheta}^2}} \\ \cdots \\ e^{(1-s)\sqrt{\vec{\vartheta}^2}} \\ e^{(-s)\sqrt{\vec{\vartheta}^2}} \end{bmatrix} \quad \square$$

It can be verified that the above $s = \frac{1}{2}, 1, \frac{3}{2}, 2$ solutions are correct.

9.2 Key I of solving polynomial expansion coefficients

Cor. 9.2.1. $\begin{bmatrix} (\frac{1}{2})^0 & (\frac{1}{2})^1 \\ (-\frac{1}{2})^0 & (-\frac{1}{2})^1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

Cor. 9.2.2. $\begin{bmatrix} 1^0 & 1^1 & 1^2 \\ 0^0 & 0^1 & 0^2 \\ (-1)^0 & (-1)^1 & (-1)^2 \end{bmatrix}^{-1} = \frac{1}{2!}\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

Cor. 9.2.3. $\begin{bmatrix} (\frac{3}{2})^0 & (\frac{3}{2})^1 & (\frac{3}{2})^2 & (\frac{3}{2})^3 \\ (\frac{1}{2})^0 & (\frac{1}{2})^1 & (\frac{1}{2})^2 & (\frac{1}{2})^3 \\ (-\frac{1}{2})^0 & (-\frac{1}{2})^1 & (-\frac{1}{2})^2 & (-\frac{1}{2})^3 \\ (-\frac{3}{2})^0 & (-\frac{3}{2})^1 & (-\frac{3}{2})^2 & (-\frac{3}{2})^3 \end{bmatrix}^{-1} = \frac{1}{48}\begin{bmatrix} -3 & 27 & 27 & -3 \\ -2 & 18 & -18 & 2 \\ 12 & -12 & -12 & 12 \\ 8 & -8 & 8 & -8 \end{bmatrix} = \begin{bmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \\ -\frac{1}{24} & \frac{9}{24} & -\frac{9}{24} & \frac{1}{24} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix}$

Cor. 9.2.4. $\begin{bmatrix} 2^0 & 2^1 & 2^2 & 2^3 & 2^4 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 \\ (-1)^0 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 \\ (-2)^0 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 \end{bmatrix}^{-1} = \frac{1}{4!}\begin{bmatrix} 0 & 0 & 24 & 0 & 0 \\ -2 & 16 & 0 & -16 & 2 \\ -1 & 16 & -30 & 16 & -1 \\ 2 & -4 & 0 & 4 & -2 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{12} & \frac{2}{3} & 0 & -\frac{2}{3} & \frac{1}{12} \\ -\frac{1}{24} & \frac{2}{3} & -\frac{5}{4} & \frac{2}{3} & -\frac{1}{24} \\ \frac{1}{12} & -\frac{1}{6} & 0 & \frac{1}{6} & -\frac{1}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{bmatrix}$

Cor. 9.2.5. $\begin{bmatrix} (\frac{5}{2})^0 & (\frac{5}{2})^1 & (\frac{5}{2})^2 & (\frac{5}{2})^3 & (\frac{5}{2})^4 & (\frac{5}{2})^5 \\ (\frac{3}{2})^0 & (\frac{3}{2})^1 & (\frac{3}{2})^2 & (\frac{3}{2})^3 & (\frac{3}{2})^4 & (\frac{3}{2})^5 \\ (\frac{1}{2})^0 & (\frac{1}{2})^1 & (\frac{1}{2})^2 & (\frac{1}{2})^3 & (\frac{1}{2})^4 & (\frac{1}{2})^5 \\ (-\frac{1}{2})^0 & (-\frac{1}{2})^1 & (-\frac{1}{2})^2 & (-\frac{1}{2})^3 & (-\frac{1}{2})^4 & (-\frac{1}{2})^5 \\ (-\frac{3}{2})^0 & (-\frac{3}{2})^1 & (-\frac{3}{2})^2 & (-\frac{3}{2})^3 & (-\frac{3}{2})^4 & (-\frac{3}{2})^5 \\ (-\frac{5}{2})^0 & (-\frac{5}{2})^1 & (-\frac{5}{2})^2 & (-\frac{5}{2})^3 & (-\frac{5}{2})^4 & (-\frac{5}{2})^5 \end{bmatrix}^{-1} = \frac{1}{96}\begin{bmatrix} \frac{9}{8} & -\frac{75}{8} & \frac{225}{4} & \frac{225}{4} & -\frac{75}{8} & \frac{9}{8} \\ \frac{9}{20} & -\frac{25}{4} & \frac{225}{2} & -\frac{225}{2} & \frac{25}{4} & -\frac{9}{20} \\ -5 & 39 & -34 & -34 & 39 & -5 \\ -2 & 26 & -68 & 68 & -26 & 2 \\ 2 & -6 & 4 & 4 & -6 & 2 \\ \frac{4}{5} & -4 & -8 & 8 & 4 & -\frac{4}{5} \end{bmatrix}$

Thm. 9.2.1.

$$\begin{bmatrix} s^0 & s^1 & \cdots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \cdots & (s-1)^{2s-1} & (s-1)^{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^1 & \cdots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \cdots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} =? \begin{bmatrix} s^0 & s^2 & \cdots & s^{2[s+1/2]} \\ (s-1)^0 & (s-1)^2 & \cdots & (s-1)^{2[s+1/2]} \\ \cdots & \cdots & \cdots & \cdots \\ (1-s)^0 & (1-s)^2 & \cdots & (1-s)^{2[s+1/2]} \\ (-s)^0 & (-s)^2 & \cdots & (-s)^{2[s+1/2]} \end{bmatrix}^{-1} =? \begin{bmatrix} s^1 & s^3 & \cdots & s^{2[s+1/2]-1} \\ (s-1)^1 & (s-1)^3 & \cdots & (s-1)^{2[s+1/2]-1} \\ \cdots & \cdots & \cdots & \cdots \\ (1-s)^1 & (1-s)^3 & \cdots & (1-s)^{2[s+1/2]-1} \\ (-s)^1 & (-s)^3 & \cdots & (-s)^{2[s+1/2]-1} \end{bmatrix}^{-1} =?$$

Cor. 9.2.6. $\begin{bmatrix} s^0 & s^2 \\ (s-1)^0 & (s-1)^2 \end{bmatrix}^{-1} = -\frac{1}{2s-1}\begin{bmatrix} (s-1)^2 & -s^2 \\ -1 & 1 \end{bmatrix}$

Cor. 9.2.7. $\det\begin{bmatrix} s^0 & s^2 \\ (s-1)^0 & (s-1)^2 \end{bmatrix} = -(2s-1), \det\begin{bmatrix} s^0 & s^2 & s^4 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 \end{bmatrix} = -2(2s-1)(2s-2)(2s-3)$

$$\det\begin{bmatrix} s^0 & s^2 & s^4 & s^6 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 & (s-1)^6 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 & (s-2)^6 \\ (s-3)^0 & (s-3)^2 & (s-3)^4 & (s-3)^6 \end{bmatrix} = 12(2s-3)(2s-1)(2s-2)(2s-3)(2s-4)(2s-5)$$

Cor. 9.2.8.

$$\begin{bmatrix} s^0 & s^2 & s^4 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 \end{bmatrix}^{-1} = -\frac{1}{2(2s-1)(2s-2)(2s-3)}\begin{bmatrix} -(s-1)^2(s-2)^2(2s-3) & s^2(s-2)^2 2(2s-2) & -s^2(s-1)^2(2s-1) \\ [(s-1)^2+(s-2)^2](2s-3) & -[s^2+(s-2)^2]2(2s-2) & [s^2+(s-1)^2](2s-1) \\ -(2s-3) & 2(2s-2) & -(2s-1) \end{bmatrix}$$

Cor. 9.2.9.

$$\begin{bmatrix} s^k & s^{k+2} & s^{k+4} \\ (s-1)^k & (s-1)^{k+2} & (s-1)^{k+4} \\ (s-2)^k & (s-2)^{k+2} & (s-2)^{k+4} \end{bmatrix}^{-1} = \begin{bmatrix} s^0 & s^2 & s^4 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 \end{bmatrix}^{-1}\begin{bmatrix} s^{-k} & 0 & 0 \\ 0 & (s-1)^{-k} & 0 \\ 0 & 0 & (s-2)^{-k} \end{bmatrix}$$

Cor. 9.2.10.

$$\begin{bmatrix} s^0 & s^2 & s^4 & s^6 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 & (s-1)^6 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 & (s-2)^6 \\ (s-3)^0 & (s-3)^2 & (s-3)^4 & (s-3)^6 \end{bmatrix}^{-1} = \frac{1}{12(2s-3)(2s-1)(2s-2)(2s-3)(2s-4)(2s-5)}$$

$\Big[-2(s-1)^2(s-2)^2(s-3)^2(2s-3)(2s-4)(2s-5),\ 6s^2(s-2)^2(s-3)^2(2s-2)(2s-3)(2s-5),\ -6s^2(s-1)^2(s-3)^2(2s-1)(2s-3)(2s-4),\ 2s^2(s-1)^2(s-2)^2(2s-1)(2s-2$

9.3 Key II of solving polynomial expansion coefficients

Cor. 9.3.1. $\begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ (-1)^0 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 & (-1)^5 & (-1)^6 \\ (-2)^0 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 & (-2)^5 & (-2)^6 \\ (-3)^0 & (-3)^1 & (-3)^2 & (-3)^3 & (-3)^4 & (-3)^5 & (-3)^6 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{60} & -\frac{9}{60} & \frac{3}{4} & 0 & -\frac{3}{4} & \frac{9}{60} & -\frac{1}{60} \\ \frac{1}{80} & -\frac{7}{60} & \frac{41}{48} & -\frac{3}{2} & \frac{41}{48} & -\frac{7}{60} & \frac{1}{80} \\ -\frac{1}{48} & \frac{1}{6} & -\frac{13}{48} & 0 & \frac{13}{48} & -\frac{1}{6} & \frac{1}{48} \\ -\frac{1}{72} & \frac{1}{8} & -\frac{3}{8} & \frac{19}{36} & -\frac{3}{8} & \frac{1}{8} & -\frac{1}{72} \\ \frac{1}{240} & -\frac{1}{60} & \frac{1}{48} & 0 & -\frac{1}{48} & \frac{1}{60} & -\frac{1}{240} \\ \frac{1}{720} & -\frac{1}{120} & \frac{1}{48} & -\frac{1}{36} & \frac{1}{48} & -\frac{1}{120} & \frac{1}{720} \end{bmatrix}$

Proof:

$$= \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ (-1)^0 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 & (-1)^5 & (-1)^6 \\ (-2)^0 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 & (-2)^5 & (-2)^6 \\ (-3)^0 & (-3)^1 & (-3)^2 & (-3)^3 & (-3)^4 & (-3)^5 & (-3)^6 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix} = \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ 2*1^0 & 0 & 2*1^2 & 0 & 2*1^4 & 0 & 2*1^6 \\ 2*2^0 & 0 & 2*2^2 & 0 & 2*2^4 & 0 & 2*2^6 \\ 2*3^0 & 0 & 2*3^2 & 0 & 2*3^4 & 0 & 2*3^6 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&1&0&1&0&0 \\ 0&1&0&0&0&1&0 \\ 1&0&0&0&0&0&1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ 1^0 & 0 & 1^2 & 0 & 1^4 & 0 & 1^6 \\ 2^0 & 0 & 2^2 & 0 & 2^4 & 0 & 2^6 \\ 3^0 & 0 & 3^2 & 0 & 3^4 & 0 & 3^6 \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&\frac{1}{2}&0&\frac{1}{2}&0&0 \\ 0&\frac{1}{2}&0&0&0&\frac{1}{2}&0 \\ \frac{1}{2}&0&0&0&0&0&\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 3^1 & 0 & 3^3 & 0 & 3^5 & 0 \\ 0 & 2^1 & 0 & 2^3 & 0 & 2^5 & 0 \\ 0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1^0 & 0 & 1^2 & 0 & 1^4 & 0 & 1^6 \\ 2^0 & 0 & 2^2 & 0 & 2^4 & 0 & 2^6 \\ 3^0 & 0 & 3^2 & 0 & 3^4 & 0 & 3^6 \end{bmatrix}\begin{bmatrix} \frac{1}{2}&0&0&0&0&0&-\frac{1}{2} \\ 0&\frac{1}{2}&0&0&0&-\frac{1}{2}&0 \\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&\frac{1}{2}&0&\frac{1}{2}&0&0 \\ 0&\frac{1}{2}&0&0&0&\frac{1}{2}&0 \\ \frac{1}{2}&0&0&0&0&0&\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3^1 & 0 & 3^3 & 0 & 3^5 & 0 \\ 0 & 2^1 & 0 & 2^3 & 0 & 2^5 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2^2 & 0 & 2^4 & 0 & 2^6 \\ 0 & 0 & 3^2 & 0 & 3^4 & 0 & 3^6 \end{bmatrix}\begin{bmatrix} \frac{1}{2}&0&0&0&0&0&-\frac{1}{2} \\ 0&\frac{1}{2}&0&0&0&-\frac{1}{2}&0 \\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0 \\ 0&\frac{1}{2}&0&-1&0&\frac{1}{2}&0 \\ \frac{1}{2}&0&0&-1&0&0&\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0&0&0&3^3-3^1&0&3^5-3^1&0 \\ 0&0&0&2^3-2^1&0&2^5-2^1&0 \\ 0&1&0&1&0&1&0 \\ 1&0&0&0&0&0&0 \\ 0&0&1&0&1&0&1 \\ 0&0&0&0&2^4-2^2&0&2^6-2^2 \\ 0&0&0&0&3^4-3^2&0&3^6-3^2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}&0&-\frac{3}{2}&0&\frac{3}{2}&0&-\frac{1}{2} \\ 0&\frac{1}{2}&-1&0&1&-\frac{1}{2}&0 \\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0 \\ 0&\frac{1}{2}&-2&3&-2&\frac{1}{2}&0 \\ \frac{1}{2}&0&-\frac{9}{2}&8&-\frac{9}{2}&0&\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0&3^3-3^1&0&3^5-3^1&0\\ 0&0&0&2^3-2^1&0&2^5-2^1&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&0&2^3-2^1&0&2^5-2^1\\ 0&0&0&0&3^3-3^1&0&3^5-3^1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}&0&-\frac{3}{2}&0&\frac{3}{2}&0&-\frac{1}{2}\\ 0&\frac{1}{2}&-1&0&1&-\frac{1}{2}&0\\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\ 0&0&0&1&0&0&0\\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0\\ 0&\frac{1}{4}&-1&\frac{3}{2}&-1&\frac{1}{4}&0\\ \frac{1}{6}&0&-\frac{3}{2}&\frac{8}{3}&-\frac{3}{2}&0&\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0&0&0&24&0&240&0\\ 0&0&0&6&0&30&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&0&6&0&30\\ 0&0&0&0&24&0&240 \end{bmatrix} \begin{bmatrix} \frac{1}{2}&0&-\frac{3}{2}&0&\frac{3}{2}&0&-\frac{1}{2}\\ 0&\frac{1}{2}&-1&0&1&-\frac{1}{2}&0\\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\ 0&0&0&1&0&0&0\\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0\\ 0&\frac{1}{4}&-1&\frac{3}{2}&-1&\frac{1}{4}&0\\ \frac{1}{6}&0&-\frac{3}{2}&\frac{8}{3}&-\frac{3}{2}&0&\frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0&1&0&10&0\\ 0&0&0&1&0&5&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&0&1&0&5\\ 0&0&0&0&1&0&10 \end{bmatrix} \begin{bmatrix} \frac{1}{48}&0&-\frac{1}{16}&0&\frac{1}{16}&0&-\frac{1}{48}\\ 0&\frac{1}{12}&-\frac{1}{6}&0&\frac{1}{6}&-\frac{1}{12}&0\\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\ 0&0&0&1&0&0&0\\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0\\ 0&\frac{1}{24}&-\frac{1}{6}&\frac{1}{4}&-\frac{1}{6}&\frac{1}{24}&0\\ \frac{1}{144}&0&-\frac{1}{16}&\frac{1}{9}&-\frac{1}{16}&0&\frac{1}{144} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&5&0\\ 0&0&0&1&0&5&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&0&1&0&5\\ 0&0&0&0&0&0&5 \end{bmatrix} \begin{bmatrix} \frac{1}{48}&-\frac{1}{12}&\frac{5}{48}&0&-\frac{5}{48}&\frac{1}{12}&-\frac{1}{48}\\ 0&\frac{1}{12}&-\frac{1}{6}&0&\frac{1}{6}&-\frac{1}{12}&0\\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\ 0&0&0&1&0&0&0\\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0\\ 0&\frac{1}{24}&-\frac{1}{6}&\frac{1}{4}&-\frac{1}{6}&\frac{1}{24}&0\\ \frac{1}{144}&-\frac{1}{24}&\frac{5}{48}&-\frac{5}{36}&\frac{5}{48}&-\frac{1}{24}&\frac{1}{144} \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0&0&0&5&0\\ 0&0&0&1&0&5&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&0&1&0&5\\ 0&0&0&0&0&0&5 \end{bmatrix} \begin{bmatrix} \frac{1}{48}&-\frac{1}{12}&\frac{5}{48}&0&-\frac{5}{48}&\frac{1}{12}&-\frac{1}{48}\\ 0&\frac{1}{12}&-\frac{1}{6}&0&\frac{1}{6}&-\frac{1}{12}&0\\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\ 0&0&0&1&0&0&0\\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0\\ -\frac{1}{144}&\frac{1}{12}&-\frac{13}{48}&\frac{7}{18}&-\frac{13}{48}&\frac{1}{12}&-\frac{1}{144}\\ \frac{1}{144}&-\frac{1}{24}&\frac{5}{48}&-\frac{5}{36}&\frac{5}{48}&-\frac{1}{24}&\frac{1}{144} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&5&0\\ 0&0&0&1&0&0&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&5 \end{bmatrix} \begin{bmatrix} \frac{1}{48}&-\frac{1}{12}&\frac{5}{48}&0&-\frac{5}{48}&\frac{1}{12}&-\frac{1}{48}\\ -\frac{1}{48}&\frac{1}{6}&-\frac{13}{48}&0&\frac{13}{48}&-\frac{1}{6}&\frac{1}{48}\\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\ 0&0&0&1&0&0&0\\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0\\ -\frac{1}{72}&\frac{1}{8}&-\frac{3}{8}&\frac{19}{36}&-\frac{3}{8}&\frac{1}{8}&-\frac{1}{72}\\ \frac{1}{144}&-\frac{1}{24}&\frac{5}{48}&-\frac{5}{36}&\frac{5}{48}&-\frac{1}{24}&\frac{1}{144} \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&1&0&1&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \frac{1}{240}&-\frac{1}{60}&\frac{1}{48}&0&-\frac{1}{48}&\frac{1}{60}&-\frac{1}{240}\\ -\frac{1}{48}&\frac{1}{6}&-\frac{13}{48}&0&\frac{13}{48}&-\frac{1}{6}&\frac{1}{48}\\ 0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0\\ 0&0&0&1&0&0&0\\ 0&0&\frac{1}{2}&-1&\frac{1}{2}&0&0\\ -\frac{1}{72}&\frac{1}{8}&-\frac{3}{8}&\frac{19}{36}&-\frac{3}{8}&\frac{1}{8}&-\frac{1}{72}\\ \frac{1}{720}&-\frac{1}{120}&\frac{1}{48}&-\frac{1}{36}&\frac{1}{48}&-\frac{1}{120}&\frac{1}{720} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \frac{1}{240}&-\frac{1}{60}&\frac{1}{48}&0&-\frac{1}{48}&\frac{1}{60}&-\frac{1}{240}\\ -\frac{1}{48}&\frac{1}{6}&-\frac{13}{48}&0&\frac{13}{48}&-\frac{1}{6}&\frac{1}{48}\\ \frac{1}{60}&-\frac{9}{60}&\frac{3}{4}&0&-\frac{3}{4}&\frac{9}{60}&-\frac{1}{60}\\ 0&0&0&1&0&0&0\\ \frac{1}{80}&-\frac{7}{60}&\frac{41}{48}&-\frac{3}{2}&\frac{41}{48}&-\frac{7}{60}&\frac{1}{80}\\ -\frac{1}{72}&\frac{1}{8}&-\frac{3}{8}&\frac{19}{36}&-\frac{3}{8}&\frac{1}{8}&-\frac{1}{72}\\ \frac{1}{720}&-\frac{1}{120}&\frac{1}{48}&-\frac{1}{36}&\frac{1}{48}&-\frac{1}{120}&\frac{1}{720} \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&0&0&1&0&0&0\\ \frac{1}{60}&-\frac{9}{60}&\frac{3}{4}&0&-\frac{3}{4}&\frac{9}{60}&-\frac{1}{60}\\ \frac{1}{80}&-\frac{7}{60}&\frac{41}{48}&-\frac{3}{2}&\frac{41}{48}&-\frac{7}{60}&\frac{1}{80}\\ -\frac{1}{48}&\frac{1}{6}&-\frac{13}{48}&0&\frac{13}{48}&-\frac{1}{6}&\frac{1}{48}\\ -\frac{1}{72}&\frac{1}{8}&-\frac{3}{8}&\frac{19}{36}&-\frac{3}{8}&\frac{1}{8}&-\frac{1}{72}\\ \frac{1}{240}&-\frac{1}{60}&\frac{1}{48}&0&-\frac{1}{48}&\frac{1}{60}&-\frac{1}{240}\\ \frac{1}{720}&-\frac{1}{120}&\frac{1}{48}&-\frac{1}{36}&\frac{1}{48}&-\frac{1}{120}&\frac{1}{720} \end{bmatrix}$$ □

Proof:

$$= \begin{bmatrix} n^0&n^1&n^2&n^3&n^4&n^5&n^6&n^7&n^8&\cdots&n^{2n-1}&n^{2n}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots\\ 4^0&4^1&4^2&4^3&4^4&4^5&4^6&4^7&4^8&\cdots&4^{2n-1}&4^{2n}\\ 3^0&3^1&3^2&3^3&3^4&3^5&3^6&3^7&3^8&\cdots&3^{2n-1}&3^{2n}\\ 2^0&2^1&2^2&2^3&2^4&2^5&2^6&2^7&2^8&\cdots&2^{2n-1}&2^{2n}\\ 1^0&1^1&1^2&1^3&1^4&1^5&1^6&1^7&1^8&\cdots&1^{2n-1}&1^{2n}\\ 0^0&0^1&0^2&0^3&0^4&0^5&0^6&0^7&0^8&\cdots&0^{2n-1}&0^{2n}\\ (-1)^0&(-1)^1&(-1)^2&(-1)^3&(-1)^4&(-1)^5&(-1)^6&(-1)^7&(-1)^8&\cdots&(-1)^{2n-1}&(-1)^{2n}\\ (-2)^0&(-2)^1&(-2)^2&(-2)^3&(-2)^4&(-2)^5&(-2)^6&(-2)^7&(-2)^8&\cdots&(-2)^{2n-1}&(-2)^{2n}\\ (-3)^0&(-3)^1&(-3)^2&(-3)^3&(-3)^4&(-3)^5&(-3)^6&(-3)^7&(-3)^8&\cdots&(-3)^{2n-1}&(-3)^{2n}\\ (-4)^0&(-4)^1&(-4)^2&(-4)^3&(-4)^4&(-4)^5&(-4)^6&(-4)^7&(-4)^8&\cdots&(-4)^{2n-1}&(-4)^{2n}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots\\ (-n)^0&(-n)^1&(-n)^2&(-n)^3&(-n)^4&(-n)^5&(-n)^6&(-n)^7&(-n)^8&\cdots&(-n)^{2n-1}&(-n)^{2n} \end{bmatrix} \begin{bmatrix} 1&\cdots&0&0&0&0&0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\cdots&1&0&0&0&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&1&0&0&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&1&0&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&0&1&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&0&0&1&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&0&0&0&1&0&0&0&\cdots&0\\ 0&\cdots&0&0&0&0&0&0&1&0&0&\cdots&0\\ 0&\cdots&0&0&0&0&0&0&0&1&0&\cdots&0\\ 0&\cdots&0&0&0&0&0&0&0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\cdots&0&0&0&0&0&0&0&0&0&\cdots&1 \end{bmatrix}$$

$$= \begin{bmatrix} n^0&n^1&n^2&n^3&n^4&n^5&n^6&n^7&n^8&\cdots&n^{2n-1}&n^{2n}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots\\ 4^0&4^1&4^2&4^3&4^4&4^5&4^6&4^7&4^8&\cdots&4^{2n-1}&4^{2n}\\ 3^0&3^1&3^2&3^3&3^4&3^5&3^6&3^7&3^8&\cdots&3^{2n-1}&3^{2n}\\ 2^0&2^1&2^2&2^3&2^4&2^5&2^6&2^7&2^8&\cdots&2^{2n-1}&2^{2n}\\ 1^0&1^1&1^2&1^3&1^4&1^5&1^6&1^7&1^8&\cdots&1^{2n-1}&1^{2n}\\ 0^0&0^1&0^2&0^3&0^4&0^5&0^6&0^7&0^8&\cdots&0^{2n-1}&0^{2n}\\ 1^0&0&1^2&0&1^4&0&1^6&0&1^8&\cdots&0&1^{2n}\\ 2^0&0&2^2&0&2^4&0&2^6&0&2^8&\cdots&0&2^{2n}\\ 3^0&0&3^2&0&3^4&0&3^6&0&3^8&\cdots&0&3^{2n}\\ 4^0&0&4^2&0&4^4&0&4^6&0&4^8&\cdots&0&4^{2n}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots&\vdots\\ n^0&0&n^2&0&n^4&0&n^6&0&n^8&\cdots&0&n^{2n} \end{bmatrix} \begin{bmatrix} 1&\cdots&0&0&0&0&0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 0&\cdots&1&0&0&0&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&1&0&0&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&1&0&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&0&1&0&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&0&0&1&0&0&0&0&\cdots&0\\ 0&\cdots&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&\cdots&0\\ 0&\cdots&0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0&\cdots&0\\ 0&\cdots&0&\frac{1}{2}&0&0&0&0&0&\frac{1}{2}&0&\cdots&0\\ 0&\cdots&\frac{1}{2}&0&0&0&0&0&0&0&\frac{1}{2}&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \frac{1}{2}&\cdots&0&0&0&0&0&0&0&0&0&\cdots&\frac{1}{2} \end{bmatrix}$$

$$
=\begin{bmatrix}
0 & n^1 & 0 & n^3 & 0 & n^5 & 0 & n^7 & 0 & \cdots\cdots & n^{2n-1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
0 & 4^1 & 0 & 4^3 & 0 & 4^5 & 0 & 4^7 & 0 & \cdots\cdots & 4^{2n-1} & 0\\
0 & 3^1 & 0 & 3^3 & 0 & 3^5 & 0 & 3^7 & 0 & \cdots\cdots & 3^{2n-1} & 0\\
0 & 2^1 & 0 & 2^3 & 0 & 2^5 & 0 & 2^7 & 0 & \cdots\cdots & 2^{2n-1} & 0\\
0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 & 1^7 & 0 & \cdots\cdots & 1^{2n-1} & 0\\
0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 & 0^7 & 0^8 & \cdots\cdots & 0^{2n-1} & 0^{2n}\\
1^0 & 0 & 1^2 & 0 & 1^4 & 0 & 1^6 & 0 & 1^8 & \cdots\cdots & 0 & 1^{2n}\\
2^0 & 0 & 2^2 & 0 & 2^4 & 0 & 2^6 & 0 & 2^8 & \cdots\cdots & 0 & 2^{2n}\\
3^0 & 0 & 3^2 & 0 & 3^4 & 0 & 3^6 & 0 & 3^8 & \cdots\cdots & 0 & 3^{2n}\\
4^0 & 0 & 4^2 & 0 & 4^4 & 0 & 4^6 & 0 & 4^8 & \cdots\cdots & 0 & 4^{2n}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
n^0 & 0 & n^2 & 0 & n^4 & 0 & n^6 & 0 & n^8 & \cdots\cdots & 0 & n^{2n}
\end{bmatrix}
\begin{bmatrix}
\frac12 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac12\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \frac12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & \cdots & 0\\
0 & \cdots & 0 & \frac12 & 0 & 0 & 0 & 0 & 0 & -\frac12 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac12 & 0 & 0 & 0 & -\frac12 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac12 & 0 & -\frac12 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac12 & 0 & \frac12 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac12 & 0 & 0 & 0 & \frac12 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & \frac12 & 0 & 0 & 0 & 0 & 0 & \frac12 & 0 & \cdots & 0\\
0 & \cdots & \frac12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac12 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac12 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac12
\end{bmatrix}
$$

$$
=\begin{bmatrix}
0 & n^1 & 0 & n^3 & 0 & n^5 & 0 & n^7 & 0 & \cdots\cdots & n^{2n-1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
0 & 4^1 & 0 & 4^3 & 0 & 4^5 & 0 & 4^7 & 0 & \cdots\cdots & 4^{2n-1} & 0\\
0 & 3^1 & 0 & 3^3 & 0 & 3^5 & 0 & 3^7 & 0 & \cdots\cdots & 3^{2n-1} & 0\\
0 & 2^1 & 0 & 2^3 & 0 & 2^5 & 0 & 2^7 & 0 & \cdots\cdots & 2^{2n-1} & 0\\
0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 & 1^7 & 0 & \cdots\cdots & 1^{2n-1} & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0\\
0 & 0 & 1^2 & 0 & 1^4 & 0 & 1^6 & 0 & 1^8 & \cdots\cdots & 0 & 1^{2n}\\
0 & 0 & 2^2 & 0 & 2^4 & 0 & 2^6 & 0 & 2^8 & \cdots\cdots & 0 & 2^{2n}\\
0 & 0 & 3^2 & 0 & 3^4 & 0 & 3^6 & 0 & 3^8 & \cdots\cdots & 0 & 3^{2n}\\
0 & 0 & 4^2 & 0 & 4^4 & 0 & 4^6 & 0 & 4^8 & \cdots\cdots & 0 & 4^{2n}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
0 & 0 & n^2 & 0 & n^4 & 0 & n^6 & 0 & n^8 & \cdots\cdots & 0 & n^{2n}
\end{bmatrix}
\begin{bmatrix}
\frac12 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac12\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \frac12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & \cdots & 0\\
0 & \cdots & 0 & \frac12 & 0 & 0 & 0 & 0 & 0 & -\frac12 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac12 & 0 & 0 & 0 & -\frac12 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac12 & 0 & -\frac12 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac12 & -1 & \frac12 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac12 & 0 & -1 & 0 & \frac12 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & \frac12 & 0 & 0 & -1 & 0 & 0 & \frac12 & 0 & \cdots & 0\\
0 & \cdots & \frac12 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \frac12 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac12 & \cdots & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & \frac12
\end{bmatrix}
$$

$$
=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0\\
0 & n^1 & 0 & n^3 & 0 & n^5 & 0 & n^7 & 0 & \cdots\cdots & n^{2n-1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
0 & 4^1 & 0 & 4^3 & 0 & 4^5 & 0 & 4^7 & 0 & \cdots\cdots & 4^{2n-1} & 0\\
0 & 3^1 & 0 & 3^3 & 0 & 3^5 & 0 & 3^7 & 0 & \cdots\cdots & 3^{2n-1} & 0\\
0 & 2^1 & 0 & 2^3 & 0 & 2^5 & 0 & 2^7 & 0 & \cdots\cdots & 2^{2n-1} & 0\\
0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 & 1^7 & 0 & \cdots\cdots & 1^{2n-1} & 0\\
0 & 0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 & 1^7 & \cdots\cdots & 0 & 1^{2n-1}\\
0 & 0 & 2^1 & 0 & 2^3 & 0 & 2^5 & 0 & 2^7 & \cdots\cdots & 0 & 2^{2n-1}\\
0 & 0 & 3^1 & 0 & 3^3 & 0 & 3^5 & 0 & 3^7 & \cdots\cdots & 0 & 3^{2n-1}\\
0 & 0 & 4^1 & 0 & 4^3 & 0 & 4^5 & 0 & 4^7 & \cdots\cdots & 0 & 4^{2n-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
0 & 0 & n^1 & 0 & n^3 & 0 & n^5 & 0 & n^7 & \cdots\cdots & 0 & n^{2n-1}
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac12 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac12\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \frac12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac12 & \cdots & 0\\
0 & \cdots & 0 & \frac12 & 0 & 0 & 0 & 0 & 0 & -\frac12 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac12 & 0 & 0 & 0 & -\frac12 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac12 & 0 & -\frac12 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac{1}{2\cdot1} & -\frac11 & \frac{1}{2\cdot1} & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac{1}{2\cdot2} & 0 & -\frac12 & 0 & \frac{1}{2\cdot2} & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & \frac{1}{2\cdot3} & 0 & 0 & -\frac13 & 0 & 0 & \frac{1}{2\cdot3} & 0 & \cdots & 0\\
0 & \cdots & \frac{1}{2\cdot4} & 0 & 0 & 0 & -\frac14 & 0 & 0 & 0 & \frac{1}{2\cdot4} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{1}{2\cdot n} & \cdots & 0 & 0 & 0 & 0 & -\frac1n & 0 & 0 & 0 & 0 & \cdots & \frac{1}{2\cdot n}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0\\
0 & 0 & 0 & n^3-n^1 1^3 & 0 & n^5-n^1 1^5 & 0 & n^7-n^1 1^7 & 0 & \cdots\cdots & n^{2n-1}-n^1 1^{2n-1} & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
0 & 0 & 0 & 4^3-4^1 1^3 & 0 & 4^5-4^1 1^5 & 0 & 4^7-4^1 1^7 & 0 & \cdots\cdots & 4^{2n-1}-4^1 1^{2n-1} & 0\\
0 & 0 & 0 & 3^3-3^1 1^3 & 0 & 3^5-3^1 1^5 & 0 & 3^7-3^1 1^7 & 0 & \cdots\cdots & 3^{2n-1}-3^1 1^{2n-1} & 0\\
0 & 0 & 0 & 2^3-2^1 1^3 & 0 & 2^5-2^1 1^5 & 0 & 2^7-2^1 1^7 & 0 & \cdots\cdots & 2^{2n-1}-2^1 1^{2n-1} & 0\\
0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 & 1^7 & 0 & \cdots\cdots & 1^{2n-1} & 0\\
0 & 0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 & 1^7 & \cdots\cdots & 0 & 1^{2n-1}\\
0 & 0 & 0 & 0 & 2^3-2^1 1^3 & 0 & 2^5-2^1 1^5 & 0 & 2^7-2^1 1^7 & \cdots\cdots & 0 & 2^{2n-1}-2^1 1^{2n-1}\\
0 & 0 & 0 & 0 & 3^3-3^1 1^3 & 0 & 3^5-3^1 1^5 & 0 & 3^7-3^1 1^7 & \cdots\cdots & 0 & 3^{2n-1}-3^1 1^{2n-1}\\
0 & 0 & 0 & 0 & 4^3-4^1 1^3 & 0 & 4^5-4^1 1^5 & 0 & 4^7-4^1 1^7 & \cdots\cdots & 0 & 4^{2n-1}-4^1 1^{2n-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots\\
0 & 0 & n^1 & 0 & n^3-n^1 1^3 & 0 & n^5-n^1 1^5 & 0 & n^7-n^1 1^7 & \cdots\cdots & 0 & n^{2n-1}-n^1 1^{2n-1}
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac12 & \cdots & 0 & 0 & 0 & -\frac{n^1}{2} & 0 & \frac{n^1}{2} & 0 & 0 & 0 & \cdots & -\frac12\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \frac12 & 0 & 0 & -\frac{4^1}{2} & 0 & \frac{4^1}{2} & 0 & 0 & -\frac12 & \cdots & 0\\
0 & \cdots & 0 & \frac12 & 0 & -\frac{3^1}{2} & 0 & \frac{3^1}{2} & 0 & -\frac12 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac12 & -\frac{2^1}{2} & 0 & \frac{2^1}{2} & -\frac12 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac12 & 0 & -\frac12 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & \frac{1}{2\cdot1} & -\frac11 & \frac{1}{2\cdot1} & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & \frac{1}{2\cdot2} & -\frac{2^1}{2} & 2^1-\frac12 & -\frac{2^1}{2} & \frac{1}{2\cdot2} & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & \frac{1}{2\cdot3} & 0 & -\frac{3^1}{2} & 3^1-\frac13 & -\frac{3^1}{2} & 0 & \frac{1}{2\cdot3} & 0 & \cdots & 0\\
0 & \cdots & \frac{1}{2\cdot4} & 0 & 0 & -\frac{4^1}{2} & 4^1-\frac14 & -\frac{4^1}{2} & 0 & 0 & \frac{1}{2\cdot4} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{1}{2\cdot n} & \cdots & 0 & 0 & 0 & -\frac{n^1}{2} & n^1-\frac1n & -\frac{n^1}{2} & 0 & 0 & 0 & \cdots & \frac{1}{2\cdot n}
\end{bmatrix}
$$

□

10 Isomorphism of lower order expansion coefficients for $[\sigma(s)\cdot\hat{p}]^{2s+1}$

10.1 An important recursive relationship

Lem. 10.1.1. $[\sigma(s)\cdot\hat{p}]^{2s+3}=\sum_{k=1}^{n}[X_1(s)X_k(s)+X_{k+1}(s)][\sigma(s)\cdot\hat{p}]^{2s+1-2k}$

Thm. 10.1.1.
$[\sigma(s-1)\cdot\hat{p}]^{2s+1}=\sum_{k=1}^{[s+1/2]}X_k(s)[\sigma(s-1)\cdot\hat{p}]^{2s+1-2k}\Leftrightarrow X_{k+1}(s)=X_{k+1}(s-1)-s^2X_k(s-1)$
$;[\sigma(s)\cdot\hat{p}]^{2s+1}=\sum_{k=1}^{[s+1/2]}X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k},X_1(s)=\frac{1}{4}C_{2s+2}^{3},X_{[s+1/2]}(s-1):=0,k=1,\cdot\cdot,[s-1/2]$

Proof: $[\sigma(s-1)\cdot\hat{p}]^{2s+1}=\sum_{k=1}^{[s+1/2]}X_k(s)[\sigma(s-1)\cdot\hat{p}]^{2s+1-2k}$

$\Leftrightarrow[\sigma(s-1)\cdot\hat{p}]^{2s+1}=X_1(s)[\sigma(s-1)\cdot\hat{p}]^{2s-1}+\sum_{k=2}^{[s+1/2]}X_k(s)[\sigma(s-1)\cdot\hat{p}]^{2s+1-2k}$

$\Leftrightarrow[\sigma(s-1)\cdot\hat{p}]^{2s+1}=X_1(s)[\sigma(s-1)\cdot\hat{p}]^{2s-1}+\sum_{k=1}^{[s-1/2]}X_{k+1}(s)[\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}$

$\Leftrightarrow\sum_{k=1}^{[s-1/2]}X_{k+1}(s)[\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}=[\sigma(s-1)\cdot\hat{p}]^{2s+1}-X_1(s)[\sigma(s-1)\cdot\hat{p}]^{2s-1}$

$\Leftrightarrow\sum_{k=1}^{[s-1/2]}X_{k+1}(s)[\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}$
$=\sum_{k=1}^{[s-1/2]}[X_1(s-1)X_k(s-1)+X_{k+1}(s-1)][\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}-X_1(s)\sum_{k=1}^{[s-1/2]}X_k(s-1)[\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}$

$\Leftrightarrow\sum_{k=1}^{[s-1/2]}X_{k+1}(s)[\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}$
$=[X_1(s-1)-X_1(s)]\sum_{k=1}^{[s-1/2]}X_k(s-1)[\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}+\sum_{k=1}^{[s-1/2]}X_{k+1}(s-1)[\sigma(s-1)\cdot\hat{p}]^{2s-1-2k}$
$\Leftrightarrow X_{k+1}(s)=[X_1(s-1)-X_1(s)]X_k(s-1)+X_{k+1}(s-1),k=1,\cdot\cdot,[s-1/2]$
$\Leftrightarrow X_{k+1}(s)=X_{k+1}(s-1)-s^2X_k(s-1),k=1,\cdot\cdot,[s-1/2]$ □

Thm. 10.1.2. $\begin{cases}X_{k+1}(n+1)=X_{k+1}(n)-(n+1)^2X_k(n)\\X_1(n+1)=\frac{1}{4}C_{2n+4}^{3},X_{n+1}(n):=0,k=1,\cdot\cdot,n\end{cases}\Rightarrow\begin{cases}X_k(n)=-(-1)^kC_{\{1^2,2^2,\cdots,n^2\}}^{k}\\k=1,\cdot\cdot,n\end{cases}$

Proof: Use mathematical induction to prove this theorem.
Step 1: When $i=n$, the flowing is established.
$X_1(1)=-(-1)^1C_{\{1^2\}}^{1}$
Step 2: Assume when $i=n$, the flowing is established.
$X_k(n)=-(-1)^kC_{\{1^2,2^2,\cdots,n^2\}}^{k},k=1,\cdot\cdot,n$
Step 3: When $i=n+1$,
$X_{k+1}(n+1)=X_{k+1}(n)-(n+1)^2X_k(n)=-(-1)^{k+1}C_{\{1^2,2^2,\cdots,n^2\}}^{k+1}+(n+1)^2(-1)^kC_{\{1^2,2^2,\cdots,n^2\}}^{k},k=1,\cdot\cdot,n$
$\Leftrightarrow X_{k+1}(n+1)=-(-1)^{k+1}[C_{\{1^2,2^2,\cdots,n^2\}}^{k+1}+(n+1)^2C_{\{1^2,2^2,\cdots,n^2\}}^{k}],k=1,\cdot\cdot,n$
$\Leftrightarrow X_{k+1}(n+1)=-(-1)^{k+1}C_{\{1^2,2^2,\cdots,n^2,(n+1)^2\}}^{k+1},k=1,\cdot\cdot,n$
$\Rightarrow X_k(n+1)=-(-1)^kC_{\{1^2,2^2,\cdots,n^2,(n+1)^2\}}^{k+1},k=1,\cdot\cdot,n+1$
This step proves that when $i=n+1$, it is established.
Step 4: Based on the above inductive reasoning, the proposition is established, and the theorem is proved. □

Thm. 10.1.3.
$\begin{cases}X_{k+1}(n+\frac{1}{2})=X_{k+1}(n-\frac{1}{2})-(n+\frac{1}{2})^2X_k(n-\frac{1}{2})\\X_1(n+\frac{1}{2})=\frac{1}{4}C_{2n+3}^{3},X_{n+1}(n-\frac{1}{2}):=0,k=1,\cdot\cdot,n-1\end{cases}\Rightarrow\begin{cases}X_k(n-\frac{1}{2})=-(-1)^kC_{\{(1/2)^2,\cdots,(n-1/2)^2\}}^{k}\\k=1,\cdot\cdot,n\end{cases}$

Proof: Use mathematical induction to prove this theorem.
Step 1: When $i=n$, the flowing is established.
$X_1(\frac{1}{2})=-(-1)^1C_{\{(1/2)^2\}}^{1}$
Step 2: Assume when $i=n$, the flowing is established.
$X_k(n-\frac{1}{2})=-(-1)^kC_{\{(1/2)^2,\cdots,(n-1/2)^2\}}^{k},k=1,\cdot\cdot,n$
Step 3: When $i=n+1$,
$X_{k+1}(n+\frac{1}{2})=X_{k+1}(n-\frac{1}{2})-(n+\frac{1}{2})^2X_k(n-\frac{1}{2})$
$=-(-1)^{k+1}C_{\{(1/2)^2,\cdots,(n-1/2)^2\}}^{k+1}+(n+1)^2(-1)^kC_{\{(1/2)^2,\cdots,(n-1/2)^2\}}^{k},k=1,\cdot\cdot,n$

$\Leftrightarrow X_{k+1}(n+\frac{1}{2}) = -(-1)^{k+1}[C^{k+1}_{\{(1/2)^2,\cdots,(n-1/2)^2\}} + (n+\frac{1}{2})^2 C^k_{\{(1/2)^2,\cdots,(n-1/2)^2\}}], k=1,\cdot\cdot,n$
$\Leftrightarrow X_{k+1}(n+\frac{1}{2}) = -(-1)^{k+1}C^{k+1}_{\{(1/2)^2,\cdots,(n-1/2)^2,(n+1/2)^2\}}, k=1,\cdot\cdot,n$
$\Rightarrow X_k(n+\frac{1}{2}) = -(-1)^{k}C^{k+1}_{\{(1/2)^2,\cdots,(n-1/2)^2,(n+1/2)^2\}}, k=1,\cdot\cdot,n+1$
This step proves that when $i=n+1$, it is established.
Step 4: Based on the above inductive reasoning, the proposition is established, and the theorem is proved. □

Cor. 10.1.1. $\begin{cases} X_{k+1}(s) = X_{k+1}(s-1) - s^2 X_k(s-1) \\ X_1(s) = \frac{1}{4}C^3_{2s+2}, X_{[s+1/2]}(s-1) := 0, k=1,\cdot\cdot,[s-1] \end{cases} \Leftrightarrow \begin{cases} X_k(s) = (-1)^{k+1}C^k_{\{(1/2|1)^2,\cdots,s^2\}} \\ k=1,\cdot\cdot,[s+\frac{1}{2}] \end{cases}$

Cor. 10.1.2. $X_k(s) = (-1)^{k+1}C^k_{\{(1/2|1)^2,\cdots,s^2\}}, k=1,\cdot\cdot,[s+\frac{1}{2}] [\Leftrightarrow] X(s) = -\begin{bmatrix} (-4)^{-1}[C^1_{\{(1|2)^2,\cdots,(2s)^2\}}] \\ (-4)^{-2}[C^2_{\{(1|2)^2,\cdots,(2s)^2\}}] \\ \cdots \\ (-4)^{-[s+1/2]}[C^{[s+1/2]}_{\{(1|2)^2,\cdots,(2s)^2\}}] \end{bmatrix}$

Cor. 10.1.3.
$$\begin{cases} [\sigma(s-1)\cdot\hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s-1)\cdot\hat{p}]^{2s+1-2k} \\ [\sigma(s)\cdot\hat{p}]^{2s+1} := \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}, X_1(s) := \frac{1}{4}C^3_{2s+2} \end{cases} \Leftrightarrow \begin{cases} X_k(s) = (-1)^{k+1}C^k_{\{(1/2|1)^2,\cdots,s^2\}} \\ k=1,\cdot\cdot,[s+\frac{1}{2}] \end{cases}$$

Cor. 10.1.4. $[\sigma(s)\cdot\hat{p}]^{2s+1} := \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s)\cdot\hat{p}]^{2s+1-2k}, X_1(s) := \frac{1}{4}C^3_{2s+2} \Leftrightarrow \begin{cases} X_k(s) = (-1)^{k+1}C^k_{\{(1/2|1)^2,\cdots,s^2\}} \\ k=1,\cdot\cdot,[s+\frac{1}{2}] \end{cases}$

10.2 Isomorphism of expansion coefficients for $[\sigma(s)\cdot\hat{p}]^{2s+1}$

Def. 10.2.1. $X_k(s) := (-1)^{k+1}C^k_{\{(1/2|1)^2,\cdots,s^2\}}, k=1,2,\cdot\cdot,[s-1]$

Lem. 10.2.1. $z^{2s-1} = \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-1-2k} \Rightarrow z^{2s+1} = \sum_{k=1}^{[s-1/2]} [X_1(s-1)X_k(s-1) + X_{k+1}(s-1)]z^{2s-1-2k}$

Proof: $z^{2s+1} = \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s+1-2k}$
$= X_1(s-1)z^{2s-1} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-1-2k}$
$= \sum_{k=1}^{[s-1/2]} X_1(s-1)X_k(s-1)z^{2s-1-2k} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-1-2k}$
$= \sum_{k=1}^{[s-1/2]} [X_1(s-1)X_k(s-1) + X_{k+1}(s-1)]z^{2s-1-2k}$ □

Thm. 10.2.1. $z^{2s-1} = \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-1-2k} \Rightarrow z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k}$

Proof: $X_{k+1}(s) = X_{k+1}(s-1) - s^2X_k(s-1), k=1,\cdot\cdot,[s-1]$
$\Leftrightarrow X_{k+1}(s) = [X_1(s-1) - X_1(s)]X_k(s-1) + X_{k+1}(s-1), k=1,\cdot\cdot,[s-1]$
$\Rightarrow \sum_{k=1}^{[s-1/2]} X_{k+1}(s)z^{2s-1-2k} = [X_1(s-1) - X_1(s)] \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-1-2k} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-1-2k}$
$\Leftrightarrow z^{2s+1} = X_1(s)z^{2s-1} + \sum_{k=1}^{[s-1/2]} X_{k+1}(s)z^{2s-1-2k}$
$\Leftrightarrow z^{2s+1} = X_1(s)z^{2s-1} + \sum_{k=2}^{[s+1/2]} X_k(s)z^{2s+1-2k}$
$\Leftrightarrow z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k}$ □

Cor. 10.2.1. $z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k} \Rightarrow z^{2(s+l)+1} = \sum_{k=1}^{[(s+l)+1/2]} X_k(s+l)z^{2(s+l)+1-2k}, l \geq 0$

Cor. 10.2.2. $\begin{cases} [\sigma(\frac{1}{2})\cdot\hat{p}]^2 = \frac{1}{4} \Rightarrow [\sigma(\frac{1}{2})\cdot\hat{p}]^{2n} = \sum_{k=1}^{n} X_k(n-\frac{1}{2})[\sigma(\frac{1}{2})\cdot\hat{p}]^{2n-2k}, n\geq 1 \\ [\sigma(1)\cdot\hat{p}]^3 = [\sigma(1)\cdot\hat{p}] \Rightarrow [\sigma(1)\cdot\hat{p}]^{2n+1} = \sum_{k=1}^{n} X_k(n)[\sigma(1)\cdot\hat{p}]^{2n+1-2k}, n\geq 1 \\ [2\sigma(\frac{1}{2})\cdot\hat{p}]^3 = [2\sigma(\frac{1}{2})\cdot\hat{p}] \Rightarrow [2\sigma(\frac{1}{2})\cdot\hat{p}]^{2n+1} = \sum_{k=1}^{n} X_k(n)[2\sigma(\frac{1}{2})\cdot\hat{p}]^{2n+1-2k}, n\geq 1 \end{cases}$

Cor. 10.2.3. $\begin{cases}(\frac{1}{2})^2=\frac{1}{4}\Rightarrow(\frac{1}{2})^{2n}=\sum\limits_{k=1}^{n}X_k(n-\frac{1}{2})(\frac{1}{2})^{2n-2k},n\geq1\Leftrightarrow\sum\limits_{k=1}^{n}4^kX_k(n-\frac{1}{2})=1,n\geq1\\1^3=1\Rightarrow1^{2n+1}=\sum\limits_{k=1}^{n}X_k(n)1^{2n+1-2k},n\geq1\Leftrightarrow\sum\limits_{k=1}^{n}X_k(n)=1,n\geq1\end{cases}$

10.3 Isomorphism of expansion coefficients for $[\sigma(s)\cdot\hat{p}]^{2s+2}$

Def. 10.3.1. $X_k(s):=(-1)^{k+1}C^k_{\{(1/2|1)^2,\cdots,s^2\}},k=1,2,\cdot\cdot,[s-1]$

Lem. 10.3.1. $z^{2s}=\sum\limits_{k=1}^{[s-1/2]}X_k(s-1)z^{2s-2k}\Rightarrow z^{2s+2}=\sum\limits_{k=1}^{[s-1/2]}[X_1(s-1)X_k(s-1)+X_{k+1}(s-1)]z^{2s-2k}$

Proof: $z^{2s+2}=\sum\limits_{k=1}^{[s-1/2]}X_k(s-1)z^{2s+2-2k}$

$=X_1(s-1)z^{2s}+\sum\limits_{k=1}^{[s-3/2]}X_{k+1}(s-1)z^{2s-2k}$

$=\sum\limits_{k=1}^{[s-1/2]}X_1(s-1)X_k(s-1)z^{2s-2k}+\sum\limits_{k=1}^{[s-3/2]}X_{k+1}(s-1)z^{2s-2k}$

$=\sum\limits_{k=1}^{[s-1/2]}[X_1(s-1)X_k(s-1)+X_{k+1}(s-1)]z^{2s-2k}$ □

Thm. 10.3.1. $z^{2s}=\sum\limits_{k=1}^{[s-1/2]}X_k(s-1)z^{2s-2k}\Rightarrow z^{2s+2}=\sum\limits_{k=1}^{[s+1/2]}X_k(s)z^{2s+2-2k}$

Proof: $X_{k+1}(s)=X_{k+1}(s-1)-s^2X_k(s-1),k=1,\cdot\cdot,[s-1/2]$

$\Leftrightarrow X_{k+1}(s)=[X_1(s-1)-X_1(s)]X_k(s-1)+X_{k+1}(s-1),k=1,\cdot\cdot,[s-1/2]$

$\Rightarrow\sum\limits_{k=1}^{[s-1/2]}X_{k+1}(s)z^{2s-2k}=[X_1(s-1)-X_1(s)]\sum\limits_{k=1}^{[s-1/2]}X_k(s-1)z^{2s-2k}+\sum\limits_{k=1}^{[s-3/2]}X_{k+1}(s-1)z^{2s-2k}$

$\Leftrightarrow z^{2s+2}=X_1(s)z^{2s}+\sum\limits_{k=1}^{[s-1/2]}X_{k+1}(s)z^{2s-2k}$

$\Leftrightarrow z^{2s+2}=X_1(s)z^{2s}+\sum\limits_{k=2}^{[s+1/2]}X_k(s)z^{2s+2-2k}$

$\Leftrightarrow z^{2s+2}=\sum\limits_{k=1}^{[s+1/2]}X_k(s)z^{2s+2-2k}$ □

Cor. 10.3.1. $z^{2s+2}=\sum\limits_{k=1}^{[s+1/2]}X_k(s)z^{2s+2-2k}\Rightarrow z^{2(s+l)+2}=\sum\limits_{k=1}^{[(s+l)+1/2]}X_k(s+l)z^{2(s+l)+2-2k},l\geq0$

Cor. 10.3.2. $\begin{cases}[\sigma(\frac{1}{2})\cdot\hat{p}]^3=\frac{1}{4}[\sigma(\frac{1}{2})\cdot\hat{p}]\Rightarrow[\sigma(\frac{1}{2})\cdot\hat{p}]^{2n+1}=\sum\limits_{k=1}^{n}X_k(n-\frac{1}{2})[\sigma(\frac{1}{2})\cdot\hat{p}]^{2n+1-2k},n\geq1\\ [\frac{1}{2}\sigma(1)\cdot\hat{p}]^3=\frac{1}{4}[\frac{1}{2}\sigma(1)\cdot\hat{p}]\Rightarrow[\frac{1}{2}\sigma(1)\cdot\hat{p}]^{2n+1}=\sum\limits_{k=1}^{n}X_k(n-\frac{1}{2})[\frac{1}{2}\sigma(1)\cdot\hat{p}]^{2n+1-2k},n\geq1\\ [\sigma(1)\cdot\hat{p}]^4=[\sigma(1)\cdot\hat{p}]^2\Rightarrow[\sigma(1)\cdot\hat{p}]^{2n+2}=\sum\limits_{k=1}^{n}X_k(n)[\sigma(1)\cdot\hat{p}]^{2n+2-2k},n\geq1\\ [2\sigma(\frac{1}{2})\cdot\hat{p}]^4=[2\sigma(\frac{1}{2})\cdot\hat{p}]^2\Rightarrow[2\sigma(\frac{1}{2})\cdot\hat{p}]^{2n+2}=\sum\limits_{k=1}^{n}X_k(n)[2\sigma(\frac{1}{2})\cdot\hat{p}]^{2n+2-2k},n\geq1\end{cases}$

11 Isomorphism of lower order expansion coefficients for $e^{\vec{\vartheta}\cdot\sigma(s)}$

11.1 An important theorem and its corollaries

Thm. 11.1.1.

$$z^{2s+1}=\sum\limits_{k=1}^{[s+1/2]}X_k(s)z^{2s+1-2k}\Leftrightarrow\begin{cases}z^{2s+1}=\sum\limits_{k=1}^{[s+1/2]}X_k(s)z^{2s+1-2k}\\z^l=\sum\limits_{k=0}^{2s}c(l,k;s)z^k,l\geq2s+1\end{cases}\Rightarrow\begin{cases}e^{\rho z}=\sum\limits_{k=0}^{2s}A_k(s)(\rho z)^k\\A_k(s):=\frac{1}{k!}+\sum\limits_{l=2s+1}^{+\infty}\frac{\rho^{l-k}}{l!}c(l,k;s)\end{cases}$$

Proof: $z^{2s+1}=\sum\limits_{k=1}^{[s+1/2]}X_k(s)z^{2s+1-2k},z^l=\sum\limits_{k=0}^{2s}c(l,k;s)z^k,l\geq2s+1$

$\Rightarrow e^{\rho z}=\sum\limits_{k=0}^{+\infty}\frac{\rho^k}{k!}z^k$

$=\sum\limits_{k=0}^{2s}\frac{\rho^k}{k!}z^k+\sum\limits_{k=2s+1}^{+\infty}\frac{\rho^k}{k!}z^k$

$= \sum_{k=0}^{2s} \frac{\rho^k}{k!} z^k + \sum_{l=2s+1}^{+\infty} \frac{\rho^l}{l!} \sum_{k=0}^{2s} c(l,k;s) z^k$

$= \sum_{k=0}^{2s} [\frac{\rho^k}{k!} + \sum_{l=2s+1}^{+\infty} \frac{\rho^l}{l!} c(l,k;s)] z^k$

$= \sum_{k=0}^{2s} A_k(s)(\rho z)^k, A_k(s) := \frac{1}{k!} + \sum_{l=2s+1}^{+\infty} \frac{\rho^{l-k}}{l!} c(l,k;s)$ □

Thm. 11.1.2. $z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) z^{2s+1-2k} \Rightarrow e^{\rho z} = \sum_{k=0}^{2(s+l)} A_k(s+l)(\rho z)^k = \sum_{k=0}^{+\infty} \frac{1}{k!}(\rho z)^k$

11.2 An important conjecture and its corollaries

Ass. 11.2.1. $[\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s) \cdot \hat{p}]^{2s+1-2k}$

Cor. 11.2.1. $[\sigma(s-l) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s-l) \cdot \hat{p}]^{2s+1-2k}, l = 0, 1, \cdots, [s+1/2]$

Cor. 11.2.2. $e^{\vec{\vartheta} \cdot \sigma(s-l)} = \sum_{k=0}^{2s} A_k(s)[\vec{\vartheta} \cdot \sigma(s-l)]^k, l = 0, 1, \cdots, [s+1/2]$

11.3 Conjecture corollary: Relationship between polynomial expansion coefficients of $e^{\vec{\vartheta} \cdot \sigma(n-\frac{1}{2})}$

Thm. 11.3.1. $e^{\vec{\vartheta} \cdot \sigma(\frac{1}{2})} = \sum_{k=0}^{2n-1} A_k(n-\frac{1}{2})[\vec{\vartheta} \cdot \sigma(\frac{1}{2})]^k$

Cor. 11.3.1. $\begin{cases} e^{\frac{1}{2}\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n-1} \frac{A_k(n-\frac{1}{2})}{2^k}[\sqrt{\vec{\vartheta}^2}]^k \\ e^{-\frac{1}{2}\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n-1} \frac{A_k(n-\frac{1}{2})}{2^k}[-\sqrt{\vec{\vartheta}^2}]^k \end{cases} \Leftrightarrow \begin{cases} cosh\frac{\sqrt{\vec{\vartheta}^2}}{2} = \sum_{i=0}^{n-1} \frac{A_{2i}(n-\frac{1}{2})}{2^{2i}}[\sqrt{\vec{\vartheta}^2}]^{2i} \\ sinh\frac{\sqrt{\vec{\vartheta}^2}}{2} = \sum_{i=0}^{n-1} \frac{A_{2i+1}(n-\frac{1}{2})}{2^{2i+1}}[\sqrt{\vec{\vartheta}^2}]^{2i+1} \end{cases}$

11.4 Conjecture corollary: Relationship between polynomial expansion coefficients of $e^{\vec{\vartheta} \cdot \sigma(n)}$

Thm. 11.4.1. $e^{\vec{\vartheta} \cdot \sigma(1)} = \sum_{k=0}^{2n} A_k(n)[\vec{\vartheta} \cdot \sigma(1)]^k, e^{\vec{\vartheta} \cdot \sigma} = \sum_{k=0}^{2s} A_k(n)(\vec{\vartheta} \cdot \sigma)^k$

Cor. 11.4.1. $\begin{cases} e^{\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n)[\sqrt{\vec{\vartheta}^2}]^k \\ A_0(n) = 1 \\ e^{-\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n)[-\sqrt{\vec{\vartheta}^2}]^k \end{cases} \Leftrightarrow \begin{cases} cosh\sqrt{\vec{\vartheta}^2} = \sum_{i=0}^{n} A_{2i}(n)[\sqrt{\vec{\vartheta}^2}]^{2i} \\ A_0(n) = 1 \\ sinh\sqrt{\vec{\vartheta}^2} = \sum_{i=0}^{n-1} A_{2i+1}(n)[\sqrt{\vec{\vartheta}^2}]^{2i+1} \end{cases}$

Cor. 11.4.2. $\begin{cases} e^{\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n)[\sqrt{\vec{\vartheta}^2}]^k \\ e^{-\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n)[-\sqrt{\vec{\vartheta}^2}]^k \end{cases} \Leftrightarrow \begin{cases} cosh\sqrt{\vec{\vartheta}^2} = \sum_{i=0}^{n} A_{2i}(n)[\sqrt{\vec{\vartheta}^2}]^{2i} \\ sinh\sqrt{\vec{\vartheta}^2} = \sum_{i=0}^{n-1} A_{2i+1}(n)[\sqrt{\vec{\vartheta}^2}]^{2i+1} \end{cases}$

11.5 Equality of Taylor expansion coefficients for $e^{\vec{\vartheta} \cdot \sigma(s)}$

Thm. 11.5.1. $\lim_{s \to +\infty} A_k(s) = \frac{1}{k!}, e^{\vec{\vartheta} \cdot \sigma(s)} = \sum_{k=0}^{+\infty} \frac{1}{k!}[\vec{\vartheta} \cdot \sigma(s)]^k$

11.6 Sorting out the reasoning process

First of all, there is the following conjecture: It can be proved by using polynomial expansion theorem and natural number splitting method. However, only in the lower order case it has strictly proved, and the general case is still a conjecture. This conjecture has not been strictly proved and others can be strictly proved.

Ass. 11.6.1. $[\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s) \cdot \hat{p}]^{2s+1-2k}$

With the above conjectures, one is that $X_k(s)$ can be directly obtained through linear algebraic methods. However, it is only strictly solved in the lower order case. And in general, it is still guessed. The other is to strictly obtain the recurrence relationship through the above reasoning. And the

coefficients can be completely and strictly solved. The following two corollaries of isomorphism: One is that it can be completely inferred from the above conjectures and coefficients. Second, it can be inferred from advanced representation transformation technology. However, the coefficients cannot be derived concretely.

Cor. 11.6.1. $[\sigma(s-l)\cdot\hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s-l)\cdot\hat{p}]^{2s+1-2k}, l = 0, 1, \cdots, [s+1/2]$

Cor. 11.6.2. $e^{\vec{\vartheta}\cdot\sigma(s-l)} = \sum_{k=0}^{2s} A_k(s)[\vec{\vartheta}\cdot\sigma(s-l)]^k, l = 0, 1, \cdots, [s+1/2]$

Chapter17 Quasidifferential Operators and Matrix Continuous Products

1 Establishment of new mathematical tools

1.1 Introduction of special quasidifferential operators

Plane wave solution hypothesis: Assume that all plane wave solutions satisfying the massless particle physics equation do not contain zero frequency solutions. Therefore, the constant solution is not a plane wave solution of the massless particle equation. And it should be treated separately.

Def. 1.1.1. $f(\vec{r},t) := \int\limits_{\vec{p}\neq 0} f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}$

Def. 1.1.2. $\begin{cases} \frac{1}{\sqrt{m^2-\nabla^2}}f(\vec{r},t) := \frac{1}{(2\pi)^3}\int \frac{1}{E}f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}, \forall f(\vec{r},t) \\ \sqrt{m^2-\nabla^2}f(\vec{r},t) := \frac{1}{(2\pi)^3}\int Ef(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}, \forall f(\vec{r},t) \end{cases}$ $\sqrt{m^2-\nabla^2} \longleftrightarrow E = \sqrt{m^2+\vec{p}^2}$

Def. 1.1.3. $\begin{cases} \frac{1}{\sqrt{-\nabla^2}}f(\vec{r},t) := \frac{1}{(2\pi)^3}\int \frac{1}{|\vec{p}|}f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}, \forall f(\vec{r},t) \\ \sqrt{-\nabla^2}f(\vec{r},t) := \frac{1}{(2\pi)^3}\int |\vec{p}|f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}, \forall f(\vec{r},t) \end{cases}$ $\sqrt{-\nabla^2} \longleftrightarrow |\vec{p}|$

1.2 Basic properties of special quasidifferential operators with mass

Pro. 1.2.1. $\begin{cases} (\sqrt{m^2-\nabla^2})^2 = m^2-\nabla^2, (\frac{1}{\sqrt{m^2-\nabla^2}})^2 = \frac{1}{m^2-\nabla^2} \\ \sqrt{m^2-\nabla^2}\frac{1}{\sqrt{m^2-\nabla^2}} = \frac{1}{\sqrt{m^2-\nabla^2}}\sqrt{m^2-\nabla^2} = 1 \\ [\sqrt{m^2-\nabla^2}]^* = \sqrt{m^2-\nabla^2}, [\frac{1}{\sqrt{m^2-\nabla^2}}]^* = \frac{1}{\sqrt{m^2-\nabla^2}} \end{cases}$

Proof: $(\sqrt{m^2-\nabla^2})^* f(\vec{r},t)$
$= [\sqrt{m^2-\nabla^2}f^*(\vec{r},t)]^*$
$= [\frac{1}{(2\pi)^3}\int \sqrt{m^2+\vec{p}^2}f^*(-\hat{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}]^*$
$= \frac{1}{(2\pi)^3}\int \sqrt{m^2+\vec{p}^2}f(-\hat{p},t)e^{-i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int \sqrt{m^2+\vec{p}^2}f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$= \sqrt{m^2-\nabla^2}f(\vec{r},t)$ □

Proof: $(\frac{1}{\sqrt{m^2-\nabla^2}})^* f(\vec{r},t)$
$= [\frac{1}{\sqrt{m^2-\nabla^2}}f^*(\vec{r},t)]^*$
$= [\frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{m^2+\vec{p}^2}}f^*(-\hat{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}]^*$
$= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{m^2+\vec{p}^2}}f(-\hat{p},t)e^{-i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{m^2+\vec{p}^2}}f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$= \frac{1}{\sqrt{m^2-\nabla^2}}f(\vec{r},t)$ □

Pro. 1.2.2. $(\sqrt{m^2-\nabla^2})^n f(\vec{r},t) = \int \sqrt{m^2+\vec{p}^2}^n f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}, n \in Z$

Pro. 1.2.3. $\int \sqrt{m^2-\nabla^2}f(\vec{r},t)d^3\vec{r} = mf(\vec{p}=0,t), \int \frac{1}{\sqrt{m^2-\nabla^2}}f(\vec{r},t)d^3\vec{r} = \frac{1}{m}f(\vec{p}=0,t)$

Pro. 1.2.4. $(\sqrt{m^2-\nabla^2})^n f(\vec{r},t) = \int \sqrt{m^2+\vec{p}^2}^n f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}, n \in Z$

Pro. 1.2.5. $\int f(\vec{r},t)(\sqrt{m^2-\nabla^2})^n g(\vec{r},t)d^3\vec{r} = \int [(\sqrt{m^2-\nabla^2})^n f(\vec{r},t)]g(\vec{r},t)d^3\vec{r}$

Proof: $\int f(\vec{r},t)(\sqrt{m^2-\nabla^2})^n g(\vec{r},t)d^3\vec{r}$
$= \frac{1}{(2\pi)^3}\int f(\vec{p}',t)e^{i\vec{p}'\cdot\vec{r}}d^3\vec{p}'\frac{1}{(2\pi)^3}\int \sqrt{m^2+\vec{p}^2}^n g(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}d^3\vec{r}$
$= \frac{1}{(2\pi)^3}\int \sqrt{m^2+\vec{p}^2}^n f(\vec{p}',t)g(\vec{p},t)\delta^3(\vec{p}'+\vec{p})d^3\vec{p}'d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int \sqrt{m^2+\vec{p}^2}^n f(-\hat{p},t)g(\vec{p},t)d^3\vec{p}$ □

Proof: $\int[(\sqrt{m^2-\nabla^2})^n f(\vec{r},t)]g(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int\sqrt{m^2+\vec{p}'^2}^n f(\vec{p}',t)e^{i\vec{p}'\cdot\vec{r}}d^3\vec{p}'\frac{1}{(2\pi)^3}\int g(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int\sqrt{m^2+\vec{p}'^2}^n f(\vec{p}',t)g(\vec{p},t)\delta^3(\vec{p}'+\vec{p})d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\sqrt{m^2+\vec{p}^2}^n f(-\hat{p},t)g(\vec{p},t)d^3\vec{p}$ □

Pro. 1.2.6. $(\sqrt{m^2-\nabla^2})^n\delta^3(\vec{r})=\frac{1}{(2\pi)^3}\int\sqrt{m^2+\vec{p}^2}^n e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}=(\sqrt{m^2-\nabla^2})^n\delta^3(-\vec{r})$

Pro. 1.2.7. $\int f(\vec{r}',t)(\sqrt{-\nabla'^2})^n\delta^3(\vec{r}-\vec{r}')d^3\vec{r}'=(\sqrt{m^2-\nabla^2})^n f(\vec{r},t)$

1.3 Basic properties of special quasidifferential operators without mass

Pro. 1.3.1. $\begin{cases}(\sqrt{-\nabla^2})^2=-\nabla^2,(\frac{1}{\sqrt{-\nabla^2}})^2=\frac{1}{-\nabla^2}\\ \sqrt{-\nabla^2}\frac{1}{\sqrt{-\nabla^2}}=\frac{1}{\sqrt{-\nabla^2}}\sqrt{-\nabla^2}=1\\ [\sqrt{-\nabla^2}]^*=\sqrt{-\nabla^2},[\frac{1}{\sqrt{-\nabla^2}}]^*=\frac{1}{\sqrt{-\nabla^2}}\end{cases}$

Proof: $(\sqrt{-\nabla^2})^*f(\vec{r},t)$
$=[\sqrt{-\nabla^2}f^*(\vec{r},t)]^*$
$=[\frac{1}{(2\pi)^3}\int|\vec{p}|f^*(-\hat{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}]^*$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|f(-\hat{p},t)e^{-i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$=\sqrt{-\nabla^2}f(\vec{r},t)$ □

Proof: $(\frac{1}{\sqrt{-\nabla^2}})^*f(\vec{r},t)$
$=[\frac{1}{\sqrt{-\nabla^2}}f^*(\vec{r},t)]^*$
$=[\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|}f^*(-\hat{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}]^*$
$=\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|}f(-\hat{p},t)e^{-i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|}f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}$
$=\frac{1}{\sqrt{-\nabla^2}}f(\vec{r},t)$ □

Pro. 1.3.2. $(\sqrt{-\nabla^2})^n f(\vec{r},t)=\int|\vec{p}|^n f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p},n\in Z$

Pro. 1.3.3. $\int\sqrt{-\nabla^2}f(\vec{r},t)d^3\vec{r}=0,\int\frac{1}{\sqrt{-\nabla^2}}f(\vec{r},t)d^3\vec{r}=$

Pro. 1.3.4. $(\sqrt{-\nabla^2})^n f(\vec{r},t)=\int|\vec{p}|^n f(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p},n\in Z$

Pro. 1.3.5. $\int f(\vec{r},t)(\sqrt{-\nabla^2})^n g(\vec{r},t)d^3\vec{r}=\int[(\sqrt{-\nabla^2})^n f(\vec{r},t)]g(\vec{r},t)d^3\vec{r}$

Proof: $\int f(\vec{r},t)(\sqrt{-\nabla^2})^n g(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int f(\vec{p}',t)e^{i\vec{p}'\cdot\vec{r}}d^3\vec{p}'\frac{1}{(2\pi)^3}\int|\vec{p}|^n g(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|^n f(\vec{p}',t)g(\vec{p},t)\delta^3(\vec{p}'+\vec{p})d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|^n f(-\hat{p},t)g(\vec{p},t)d^3\vec{p}$ □

Proof: $\int[(\sqrt{-\nabla^2})^n f(\vec{r},t)]g(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int|\vec{p}'|^n f(\vec{p}',t)e^{i\vec{p}'\cdot\vec{r}}d^3\vec{p}'\frac{1}{(2\pi)^3}\int g(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int|\vec{p}'|^n f(\vec{p}',t)g(\vec{p},t)\delta^3(\vec{p}'+\vec{p})d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|^n f(-\hat{p},t)g(\vec{p},t)d^3\vec{p}$ □

Pro. 1.3.6. $(\sqrt{-\nabla^2})^n\delta^3(\vec{r})=\frac{1}{(2\pi)^3}\int|\vec{p}|^n e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}=(\sqrt{-\nabla^2})^n\delta^3(-\vec{r})$

Pro. 1.3.7. $\int f(\vec{r}',t)(\sqrt{-\nabla'^2})^n\delta^3(\vec{r}-\vec{r}')d^3\vec{r}'=(\sqrt{-\nabla^2})^n f(\vec{r},t)$

2 Matrices continuous multiplication trace in four dimensional space-time

2.1 Properties of spin matrices continuous multiplication trace $tr[\sigma_{\alpha_1}(s)\cdots\sigma_{\alpha_n}(s)]$

Cor. 2.1.1.
$tr[\sigma_{\alpha'_\varsigma}(s)]=0,tr[\sigma^{\alpha_\varsigma}(s)]=0$
$tr[\sigma_{\alpha'_\varsigma}(s)\sigma_{\beta'_\varsigma}(s)]=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta_{\alpha'_\varsigma\beta'_\varsigma},tr[\sigma^{\alpha_\varsigma}(s)\sigma^{\beta_\varsigma}(s)]=\frac{2}{3}s(s+\frac{1}{2})(s+1)\delta^{\alpha_\varsigma\beta_\varsigma}$

2.2 Properties of Pauli matrices continuous multiplication trace $tr[\sigma_{\alpha_1}\cdots\sigma_{\alpha_n}]$

Def. 2.2.1. $A_{\alpha_1\cdots\alpha_n} := tr[\sigma_{\alpha_1}\cdots\sigma_{\alpha_n}]$

Pro. 2.2.1.
$A_{\alpha_1} = 0$
$A_{\alpha_1\alpha_2} = 2\delta_{\alpha_1\alpha_2}$
$A_{\alpha_1\alpha_2\alpha_3} = 2i\varepsilon_{\alpha_1\alpha_2\alpha_3}$
$A_{\alpha_1\alpha_2\alpha_3\alpha_4} = 2[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} - S_{\alpha_1\alpha_2\alpha_3\alpha_4}]$
$A_{\alpha_1\alpha_2\alpha_3\alpha_4} = 2[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} - \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} + \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}]$
$A_{\alpha_1\cdots\alpha_5} = 2i[\varepsilon_{\alpha_1\alpha_2\alpha_3}\delta_{\alpha_4\alpha_5} - \varepsilon_{\alpha_1\alpha_2\alpha_4}\delta_{\alpha_3\alpha_5} + \varepsilon_{\alpha_1\alpha_2\alpha_5}\delta_{\alpha_3\alpha_4} + \varepsilon_{\alpha_3\alpha_4\alpha_5}\delta_{\alpha_1\alpha_2}]$

Thm. 2.2.1. $A_{\alpha_1\cdots\alpha_n} = i\varepsilon_{\alpha_1\alpha_2}{}^{\alpha}A_{\alpha\alpha_3\cdots\alpha_n} + \delta_{\alpha_1\alpha_2}A_{\alpha_3\cdots\alpha_n}$

2.3 General properties of Dirac matrices continuous multiplication trace $tr[\gamma_{a_1}\cdots\gamma_{a_n}]$

Def. 2.3.1. $B_{a_1\cdots a_n} := tr[\gamma_{a_1}\cdots\gamma_{a_n}], B^5_{a_1\cdots a_n} := tr[\gamma^5\gamma_{a_1}\cdots\gamma_{a_n}]$

Pro. 2.3.1.
$B_{a_1} = 0, B^5_{a_1} = 0$
$B_{a_1a_2} = 4\delta_{a_1a_2}, B^5_{a_1a_2} = 0$

Thm. 2.3.1.
$$\begin{cases} B_{a_1\cdots a_n} = \varepsilon_{a_1a_2a_3}{}^{a}B^5_{aa_4\cdots a_n} + \delta_{a_1a_2}B_{a_3\cdots a_n} + \delta_{a_3[a_2}B_{a_1]a_4\cdots a_n} \\ B^5_{a_1\cdots a_n} = \varepsilon_{a_1a_2a_3}{}^{a}B_{aa_4\cdots a_n} + \delta_{a_1a_2}B^5_{a_3\cdots a_n} + \delta_{a_3[a_2}B^5_{a_1]a_4\cdots a_n} \end{cases}$$

2.4 Concrete properties of Dirac matrices continuous multiplication trace $tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\cdots]$

$$tr[\gamma_a(\varsigma)] = 0 \qquad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \qquad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \tag{17.1}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)] = 0 \qquad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \qquad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \tag{17.2}$$
$$tr[\gamma_5(\varsigma)] = 0 \qquad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \qquad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \tag{17.3}$$
$$tr[S_{ab}(e,\varsigma)] = 0 \qquad tr[\gamma_c(\varsigma)S_{ab}(e,\varsigma)] = 0 \qquad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \tag{17.4}$$
$$tr[\gamma_5(\varsigma)S_{ab}(e,\varsigma)] = 0 \qquad tr[\gamma_5(\varsigma)\gamma_c(\varsigma)S_{ab}(e,\varsigma)] = 0 \qquad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \tag{17.5}$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)] = 4\delta_{ab} \qquad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4[\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b}] \tag{17.6}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \qquad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4\varepsilon_{abcd} \tag{17.7}$$
$$tr[S_{ab}(e,\varsigma)S_{cd}(e,\varsigma)] = S_{abcd} = \delta_{ac}\delta_{db} - \delta_{ad}\delta_{cb} \qquad tr[\gamma_5 S_{ab}(e,\varsigma)S_{cd}(e,\varsigma)] = -\varepsilon_{abcd} \tag{17.8}$$
$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e,\varsigma)] = 2iS_{abcd} \qquad tr[\gamma_5\gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e,\varsigma)] = -2i\varepsilon_{abcd} \tag{17.9}$$
$$tr[S_{ab}(e,\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 2iS_{abcd} \qquad tr[\gamma_5 S_{ab}(e,\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = -2i\varepsilon_{abcd} \tag{17.10}$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = 2i\{\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef}\} \tag{17.11}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = -2i\{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \tag{17.12}$$
$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{(\delta_{ab}\delta_{cd} - S_{abcd})\delta_{ef} - (\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef})\} \tag{17.13}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{\varepsilon_{abcd}\delta_{ef} + \delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \tag{17.14}$$

$$tr[\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = \delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef} - \delta_{bc}S_{adef} \tag{17.15}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = \delta_{bc}\varepsilon_{adef} - \{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \tag{17.16}$$

$$tr[\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = \delta_{ad}S_{bcef} + \delta_{a[b}S_{c]def} + \delta_{d[b}S_{c]aef} \tag{17.17}$$
$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)] = -\{\delta_{ad}\varepsilon_{bcef} + \delta_{a[b}\varepsilon_{c]def} + \delta_{d[b}\varepsilon_{c]aef}\} \tag{17.18}$$

3 Dirac matrices continuous multiplication trace in n=N+1 dimensional space-time

3.1 First conjecture of Dirac matrices continuous multiplication trace in n=N+1-D

Def. 3.1.1.
$$\begin{cases} \frac{1}{2!}\langle\delta_{ab}\gamma_{[c}\gamma_{d]}, \frac{4!}{1!1!|2!2!}\rangle := \frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]} - \delta_{ac}\gamma_{[b}\gamma_{d]} + \delta_{ad}\gamma_{[b}\gamma_{c]} + \gamma_{[a}\gamma_{b]}\delta_{cd} - \gamma_{[a}\gamma_{c]}\delta_{bd} + \gamma_{[a}\gamma_{d]}\delta_{bc}) \\ \frac{1}{0!}\langle\delta_{ab}\delta_{cd}, \frac{4!}{2!|2!2!}\rangle := \frac{1}{0!}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \\ \frac{1}{1!}\langle\delta_{ab}\gamma_c, \frac{3!}{1!1!|2!1!}\rangle := \frac{1}{1!}(\delta_{ab}\gamma_c + \delta_{bc}\gamma_a - \delta_{ac}\gamma_b) \\ \frac{1}{0!}\langle\delta_{ab}, \frac{2!}{1!|2!}\rangle := \frac{1}{0!}\delta_{ab} \end{cases}$$

Pro. 3.1.1.

$$\begin{cases}\langle\delta_{ab}\gamma_{[c}\gamma_{d]},\frac{4!}{1!1!|2!2!}\rangle=\frac{1}{2!}(\langle\delta_{\{ab\}}\gamma_{[c}\gamma_{d]},\frac{4!}{1!1!|2!2!}\rangle+\langle\delta_{[ab]}\gamma_{[c}\gamma_{d]},\frac{4!}{1!1!|2!2!}\rangle)\\ \langle\delta_{ab}\delta_{cd},\frac{4!}{2!|2!2!}\rangle=\frac{1}{2!}(\langle\delta_{\{ab\}}\delta_{cd},\frac{4!}{2!|2!2!}\rangle+\langle\delta_{[ab]}\delta_{cd},\frac{4!}{2!|2!2!}\rangle)\\ \langle\delta_{ab}\gamma_c,\frac{3!}{1!1!|2!1!}\rangle=\frac{1}{2!}(\langle\delta_{\{ab\}}\gamma_c,\frac{3!}{1!1!|2!1!}\rangle+\langle\delta_{[ab]}\gamma_c,\frac{3!}{1!1!|2!1!}\rangle)\\ \langle\delta_{ab},\frac{2!}{1!|2!}\rangle=\frac{1}{2!}(\langle\delta_{\{ab\}},\frac{2!}{1!|2!}\rangle+\langle\delta_{[ab]},\frac{2!}{1!|2!}\rangle)\end{cases}$$

Ass. 3.1.1.

$$\begin{cases}\gamma_a=\frac{1}{1!}\gamma_a\\ \gamma_a\gamma_b=\frac{1}{2!}\gamma_{[a}\gamma_{b]}+\frac{1}{0!}\langle\delta_{ab},\frac{2!}{1!|2!}\rangle\\ \gamma_a\gamma_b\gamma_c=\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{1!}\langle\delta_{ab}\gamma_c,\frac{3!}{1!1!|2!1!}\rangle\\ \gamma_a\gamma_b\gamma_c\gamma_d=\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}+\frac{1}{2!}\langle\delta_{ab}\gamma_{[c}\gamma_{d]},\frac{4!}{1!1!|2!2!}\rangle+\frac{1}{0!}\langle\delta_{ab}\delta_{cd},\frac{4!}{2!|2!2!}\rangle\\ \gamma_a\gamma_b\gamma_c\gamma_d\gamma_e=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\langle\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]},\frac{5!}{1!1!|2!3!}\rangle+\frac{1}{1!}\langle\delta_{ad}\delta_{bc}\gamma_e,\frac{5!}{2!1!|2!2!1!}\rangle\\ \gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f=\frac{1}{6!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_{f]}+\frac{1}{4!}\langle\delta_{ab}\gamma_{[c}\gamma_d\gamma_e\gamma_{f]},\frac{6!}{1!1!|2!4!}\rangle+\frac{1}{2!}\langle\delta_{ab}\delta_{cd}\gamma_{[e}\gamma_{f]},\frac{6!}{2!1!|2!2!2!}\rangle+\frac{1}{0!}\langle\delta_{ab}\delta_{cd}\delta_{ef},\frac{6!}{3!|2!2!2!}\rangle\\ \gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_g=\frac{1}{7!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_{g]}+\frac{1}{5!}\langle\delta_{ab}\gamma_{[c}\gamma_d\gamma_e\gamma_f\gamma_{g]},\frac{7!}{1!1!|2!5!}\rangle+\frac{1}{3!}\langle\delta_{ab}\delta_{cd}\gamma_{[e}\gamma_f\gamma_{g]},\frac{7!}{2!1!|2!2!3!}\rangle\\ +\frac{1}{1!}\langle\delta_{ab}\delta_{cd}\delta_{ef}\gamma_g,\frac{7!}{3!1!|2!2!2!1!}\rangle\\ \gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_g\gamma_h=\frac{1}{8!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_g\gamma_{h]}+\frac{1}{6!}\langle\delta_{ab}\gamma_{[c}\gamma_d\gamma_e\gamma_f\gamma_g\gamma_{h]},\frac{8!}{1!1!|2!6!}\rangle\\ +\frac{1}{4!}\langle\delta_{ab}\delta_{cd}\gamma_{[e}\gamma_f\gamma_g\gamma_{h]},\frac{8!}{2!1!|2!2!4!}\rangle+\frac{1}{2!}\langle\delta_{ab}\delta_{cd}\delta_{ef}\gamma_{[g}\gamma_{h]},\frac{8!}{3!1!|2!2!2!2!}\rangle+\frac{1}{0!}\langle\delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh},\frac{8!}{4!|2!2!2!2!}\rangle\\ \cdots\end{cases}$$

Prop. 3.1.1.

$$\begin{cases}\gamma_a=\frac{1}{1!}\gamma_a\\ \gamma_{\{a}\gamma_{b\}}=\langle\{\delta_{ab},\frac{2!}{1!|2!}\}\rangle=\delta_{\{ab\}}\\ \gamma_{\{a}\gamma_b\gamma_{c\}}=\langle\{\delta_{ab}\gamma_c,\frac{3!}{1!1!|2!1!}\}\rangle=\delta_{\{ab}\gamma_{c\}}\\ \gamma_{\{a}\gamma_b\gamma_c\gamma_{d\}}=\langle\{\delta_{ab}\delta_{cd},\frac{4!}{2!|2!2!}\}\rangle=\delta_{\{ab}\delta_{cd\}}\\ \gamma_{\{a}\gamma_b\gamma_c\gamma_d\gamma_{e\}}=\langle\{\delta_{ad}\delta_{bc}\gamma_e,\frac{5!}{2!1!|2!2!1!}\}\rangle=\delta_{\{ad}\delta_{bc}\gamma_{e\}}\\ \gamma_{\{a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_{f\}}=\langle\{\delta_{ab}\delta_{cd}\delta_{ef},\frac{6!}{3!|2!2!2!}\}\rangle=\delta_{\{ab}\delta_{cd}\delta_{ef\}}\\ \gamma_{\{a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_{g\}}=\langle\{\delta_{ab}\delta_{cd}\delta_{ef}\gamma_g,\frac{7!}{3!1!|2!2!2!1!}\}\rangle=\delta_{\{ab}\delta_{cd}\delta_{ef}\gamma_{g\}}\\ \gamma_{\{a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_g\gamma_{h\}}=\langle\{\delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh},\frac{8!}{4!|2!2!2!2!}\}\rangle=\delta_{\{ab}\delta_{cd}\delta_{ef}\delta_{gh\}}\\ \cdots\end{cases}$$

3.2 Second conjecture of Dirac matrices continuous multiplication trace in n=N+1-D

It is easy to derive the second guess from the first guess, but it is somewhat difficult to conversely derive it. But in essence, they can be derived from each other, so the two conjectures are equivalent.

Pro. 3.2.1.

$$\begin{cases}\gamma_a=\frac{1}{1!}\gamma_a\\ \frac{1}{1!}\gamma_a\gamma_b=\frac{1}{2!}\gamma_{[a}\gamma_{b]}+\frac{1}{0!}\delta_{ab}\\ \frac{1}{2!}\gamma_a\gamma_{[b}\gamma_{c]}=\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{1!}\delta_{a[b}\gamma_{c]},\frac{1}{2!}\gamma_{[a}\gamma_{b]}\gamma_c=\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{1!}\gamma_{[a}\delta_{b]c}\\ \frac{1}{3!}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]}=\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}+\frac{1}{2!}\delta_{a[b}\gamma_c\gamma_{d]},\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}\gamma_d=\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}+\frac{1}{2!}\gamma_{[a}\gamma_b\delta_{c]d}\\ \frac{1}{4!}\gamma_a\gamma_{[b}\gamma_c\gamma_d\gamma_{e]}=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\delta_{a[b}\gamma_c\gamma_d\gamma_{e]},\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\gamma_e=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_c\delta_{d]e}\\ \frac{1}{5!}\gamma_a\gamma_{[b}\gamma_c\gamma_d\gamma_{e]}\gamma_f=\frac{1}{6!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_{f]}+\frac{1}{4!}\delta_{a[b}\gamma_c\gamma_d\gamma_e\gamma_{f]},\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\gamma_e\gamma_f=\frac{1}{6!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_{f]}+\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\delta_{e]f}\end{cases}$$

Ass. 3.2.1.

$$\begin{cases}\frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_l]}\\ \frac{1}{(l-1)!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}]}\gamma_{a_l}=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l-2}}\delta_{a_{l-1}]a_l}\end{cases}$$

Prop. 3.2.1. $\frac{1}{(l-2)!}\gamma_{a_1}\gamma_{a_2}\gamma_{[a_3}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_l]}+\frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_l]}$

Proof: $\frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_l]}$

$\Rightarrow\gamma_{a_1}[\frac{1}{(l-2)!}\gamma_{a_2}\gamma_{[a_3}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}-\frac{1}{(l-3)!}\delta_{a_2[a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_l]}]=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_l]}$

$\Leftrightarrow\frac{1}{(l-2)!}\gamma_{a_1}\gamma_{a_2}\gamma_{[a_3}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_l]}+\frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_l]}$

$\Leftrightarrow\frac{1}{(l-2)!}\gamma_{a_1}\gamma_{a_2}\gamma_{[a_3}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_l]}+\frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_l]}$ □

Prop. 3.2.2. $\frac{1}{(l-k)!}\gamma_{a_1}\cdot\cdot\gamma_{a_k}\gamma_{[a_{k+1}}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}$

$=\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]}+\frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_l]}+\frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_l]}+\cdot\cdot+\frac{1}{(l-k-1)!}\gamma_{a_1}\cdot\cdot\gamma_{a_{k-1}}\delta_{a_k[a_{k+1}}\gamma_{a_{k+2}}\cdot\cdot\gamma_{a_l]}$

Prop. 3.2.3.

$$\begin{cases}\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\gamma_e=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_c\delta_{d]e}\\ \gamma_a\gamma_b\gamma_c\gamma_d=\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}+\frac{1}{2!}\langle\delta_{ab}\gamma_{[c}\gamma_{d]},\frac{4!}{1!1!|2!2!}\rangle+\frac{1}{0!}\langle\delta_{ab}\delta_{cd},\frac{4!}{2!|2!2!}\rangle\end{cases}$$

$$\Rightarrow\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\langle\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]},\frac{5!}{1!1!|2!3!}\rangle+\frac{1}{1!}\langle\delta_{ad}\delta_{bc}\gamma_e,\frac{5!}{2!1!|2!2!1!}\rangle$$

Proof: $\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e$

$$=\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\gamma_e+\frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]}-\delta_{ac}\gamma_{[b}\gamma_{d]}+\delta_{ad}\gamma_{[b}\gamma_{c]}+\gamma_{[a}\gamma_{b]}\delta_{cd}-\gamma_{[a}\gamma_{c]}\delta_{bd}+\gamma_{[a}\gamma_{d]}\delta_{bc})\gamma_e$$
$$+\frac{1}{0!}(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\gamma_e$$
$$=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_c\delta_{d]e}$$
$$+\frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]}-\delta_{ac}\gamma_{[b}\gamma_{d]}+\delta_{ad}\gamma_{[b}\gamma_{c]}+\gamma_{[a}\gamma_{b]}\delta_{cd}-\gamma_{[a}\gamma_{c]}\delta_{bd}+\gamma_{[a}\gamma_{d]}\delta_{bc})\gamma_e$$
$$+\frac{1}{0!}(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\gamma_e$$
$$=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_c\delta_{d]e}$$
$$+\delta_{ab}(\frac{1}{3!}\gamma_{[c}\gamma_d\gamma_{e]}+\frac{1}{1!}\gamma_{[c}\delta_{d]e})-\delta_{ac}(\frac{1}{3!}\gamma_{[b}\gamma_d\gamma_{e]}+\frac{1}{1!}\gamma_{[b}\delta_{d]e})+\delta_{ad}(\frac{1}{3!}\gamma_{[b}\gamma_c\gamma_{e]}+\frac{1}{1!}\gamma_{[b}\delta_{c]e})+(\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{e]}+\frac{1}{1!}\gamma_{[a}\delta_{b]e})\delta_{cd}$$
$$-(\frac{1}{3!}\gamma_{[a}\gamma_c\gamma_{e]}+\frac{1}{1!}\gamma_{[a}\delta_{c]e})\delta_{bd}+(\frac{1}{3!}\gamma_{[a}\gamma_d\gamma_{e]}+\frac{1}{1!}\gamma_{[a}\delta_{d]e})\delta_{bc}+\frac{1}{0!}(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\gamma_e$$
$$=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}$$
$$+\frac{1}{3!}(\gamma_{[a}\gamma_b\gamma_{c]}\delta_{de}-\gamma_{[b}\gamma_c\gamma_{d]}\delta_{ae}+\gamma_{[c}\gamma_d\gamma_{a]}\delta_{be}-\gamma_{[d}\gamma_a\gamma_{b]}\delta_{ce})$$
$$+\frac{1}{3!}(\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]}-\delta_{ac}\gamma_{[b}\gamma_d\gamma_{e]}+\delta_{ad}\gamma_{[b}\gamma_c\gamma_{e]}+\gamma_{[a}\gamma_b\gamma_{e]}\delta_{cd})-\gamma_{[a}\gamma_c\gamma_{e]}\delta_{bd}+\gamma_{[a}\gamma_d\gamma_{e]}\delta_{bc}$$
$$+\frac{1}{1!}(\delta_{ab}\gamma_{[c}\delta_{d]e}-\delta_{ac}\gamma_{[b}\delta_{d]e}+\delta_{ad}\gamma_{[b}\delta_{c]e}+\gamma_{[a}\delta_{b]e}\delta_{cd}-\gamma_{[a}\delta_{c]e}\delta_{bd}+\gamma_{[a}\delta_{d]e}\delta_{bc})+\frac{1}{1!}(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\gamma_e$$
$$=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}$$
$$+\frac{1}{3!}[(\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]}-\delta_{ac}\gamma_{[b}\gamma_d\gamma_{e]}+\delta_{ad}\gamma_{[b}\gamma_c\gamma_{e]}-\delta_{ae}\gamma_{[b}\gamma_c\gamma_{d]})$$
$$+(\delta_{bc}\gamma_{[a}\gamma_d\gamma_{e]}-\delta_{bd}\gamma_{[a}\gamma_c\gamma_{e]}+\gamma_{[c}\gamma_d\gamma_{a]}\delta_{be})+(\delta_{cd}\gamma_{[a}\gamma_b\gamma_{e]}-\delta_{ce}\gamma_{[d}\gamma_a\gamma_{b]})+(\delta_{de}\gamma_{[a}\gamma_b\gamma_{c]})]$$
$$+\frac{1}{1!}[(\delta_{be}\delta_{cd}-\delta_{bd}\delta_{ce}+\delta_{bc}\delta_{de})\gamma_a+(-\delta_{ac}\delta_{de}+\delta_{ad}\delta_{ce}-\delta_{ae}\delta_{cd})\gamma_b$$
$$+(\delta_{ab}\delta_{de}-\delta_{ad}\delta_{be}+\delta_{ae}\delta_{bd})\gamma_c+(-\delta_{ab}\delta_{ce}+\delta_{ac}\delta_{be}-\delta_{ae}\delta_{bc})\gamma_d+(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\gamma_e]$$
$$=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\langle\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]},C_5^3\rangle+\langle\frac{1}{1!}\delta_{ad}\delta_{bc}\gamma_e,C_5^1C_4^2/2!\rangle$$
$$=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}\langle\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]},\frac{5!}{1!1!|2!3!}\rangle+\langle\frac{1}{1!}\delta_{ad}\delta_{bc}\gamma_e,\frac{5!}{2!1!|2!2!1!}\rangle$$ □

3.3 Verification for conjecture of Dirac matrices continuous multiplication trace in n=N+1-D

Ass. 3.3.1.

$$\begin{cases}\gamma_a=\frac{1}{1!}\gamma_a\\ \gamma_a\gamma_b=\frac{1}{2!}\gamma_{[a}\gamma_{b]}+\frac{1}{0!}\delta_{ab}\\ \gamma_a\gamma_b\gamma_c=\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{1!}(\delta_{ab}\gamma_c+\delta_{bc}\gamma_a-\delta_{ac}\gamma_b)\\ \gamma_a\gamma_b\gamma_c\gamma_d=\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}+\frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]}+\cdot\cdot)+\frac{1}{0!}(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\\ \gamma_a\gamma_b\gamma_c\gamma_d\gamma_e=\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}(\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]}+\cdot\cdot)+\frac{1}{1!}(\delta_{ab}\delta_{cd}\gamma_e+\cdot\cdot)\\ \gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f=\frac{1}{6!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_{f]}+\frac{1}{4!}(\delta_{ab}\gamma_{[c}\gamma_d\gamma_e\gamma_{f]}+\cdot\cdot)+\frac{1}{2!}(\delta_{ab}\delta_{cd}\gamma_{[e}\gamma_{f]}+\cdot\cdot)+\frac{1}{0!}(\delta_{ab}\delta_{cd}\delta_{ef}+\cdot\cdot)\\ \gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_g=\frac{1}{7!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_e\gamma_f\gamma_{g]}+\frac{1}{5!}(\delta_{ab}\gamma_{[c}\gamma_d\gamma_e\gamma_f\gamma_{g]}+\cdot\cdot)+\frac{1}{3!}(\delta_{ab}\delta_{cd}\gamma_{[e}\gamma_f\gamma_{g]}+\cdot\cdot)+\frac{1}{1!}(\delta_{ab}\delta_{cd}\delta_{ef}\gamma_g+\cdot\cdot)\\ \dots\end{cases}$$

Proof: $\gamma_a\gamma_b=\frac{1}{2!}\gamma_{[a}\gamma_{b]}+\frac{1}{0!}\delta_{ab}$

$$\Leftrightarrow\gamma_{a_1}\gamma_{a_2}=\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{0!}\delta_{a_1a_2}$$
$$\Rightarrow\gamma^{a_1}\gamma^{a_2}\gamma_{a_1'}\gamma_{a_2'}=(\frac{1}{2!}\gamma^{[a_1}\gamma^{a_2]}+\frac{1}{0!}\delta^{a_1a_2})(\frac{1}{2!}\gamma_{[a_1'}\gamma_{a_2']}+\frac{1}{0!}\delta_{a_1'a_2'})$$
$$\Rightarrow tr\{\gamma^{a_1}\gamma^{a_2}\gamma_{a_1'}\gamma_{a_2'}\}=\frac{2^{[\frac{n}{2}]}(1!)^2}{2!}\delta^{[a_1}_{[a_1'}\delta^{a_2]}_{a_2']}+\frac{2^{[\frac{n}{2}]}(1!)^2}{0!}\delta^{a_1a_2}\delta_{a_1'a_2'}=2^{[\frac{n}{2}]}(1!)^2(\frac{1}{2!}\delta^{[a_1}_{[a_1'}\delta^{a_2]}_{a_2']}+\frac{1}{0!}\delta^{a_1a_2}\delta_{a_1'a_2'})$$ □

Proof: $\gamma_a\gamma_b\gamma_c$

$$=\frac{1}{2}(\gamma_a\gamma_b\gamma_c-\gamma_a\gamma_c\gamma_b+2\gamma_a\delta_{bc})$$
$$=\frac{1}{4}(\gamma_a\gamma_b\gamma_c-\gamma_b\gamma_a\gamma_c+\gamma_c\gamma_a\gamma_b-\gamma_a\gamma_c\gamma_b+2\delta_{ab}\gamma_c-2\delta_{ac}\gamma_b+4\gamma_a\delta_{bc})$$
$$=\frac{1}{8}(\gamma_a\gamma_{[b}\gamma_{c]}+\gamma_b\gamma_{[c}\gamma_{a]}+\gamma_c\gamma_{[a}\gamma_{b]}+6\delta_{ab}\gamma_c-6\delta_{ac}\gamma_b+6\gamma_a\delta_{bc}+2\gamma_a\gamma_b\gamma_c)$$
$$\Leftrightarrow\gamma_a\gamma_b\gamma_c=\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+(\delta_{ab}\gamma_c+\delta_{bc}\gamma_a-\delta_{ac}\gamma_b)$$
$$\Leftrightarrow\gamma_a\gamma_b\gamma_c=\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+(\delta_{a[b}\gamma_{c]}+\gamma_a\delta_{bc})$$ □

Proof: $\gamma_a\gamma_b\gamma_c=\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+(\delta_{a[b}\gamma_{c]}+\gamma_a\delta_{bc})\Rightarrow\begin{cases}\gamma_a\gamma_{[b}\gamma_{c]}=\frac{1}{3}\gamma_{[a}\gamma_b\gamma_{c]}+2\delta_{a[b}\gamma_{c]}\\ \gamma_{[a}\gamma_{b]}\gamma_c=\frac{1}{3}\gamma_{[a}\gamma_b\gamma_{c]}+2\gamma_{[a}\delta_{b]c}\end{cases}$ □

Proof: $\begin{cases}\gamma_a\gamma_{[b}\gamma_{c]}=\frac{1}{3}\gamma_{[a}\gamma_b\gamma_{c]}+2\delta_{a[b}\gamma_{c]}\\ \gamma_{[a}\gamma_{b]}\gamma_c=\frac{1}{3}\gamma_{[a}\gamma_b\gamma_{c]}+2\gamma_{[a}\delta_{b]c}\end{cases}$

$$\Rightarrow\gamma^{a_1}\gamma^{[a_2}\gamma^{a_3]}\gamma_{[a_1'}\gamma_{a_2']}\gamma_{a_3'}=(\frac{1}{3}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3]}+2\delta^{a_1[a_2}\gamma^{a_3]})(\frac{1}{3}\gamma_{[a_1'}\gamma_{a_2'}\gamma_{a_3']}+2\gamma_{[a_1'}\delta_{a_2']a_3'})$$
$$\Rightarrow tr\{\gamma^{a_1}\gamma^{[a_2}\gamma^{a_3]}\gamma_{[a_1'}\gamma_{a_2']}\gamma_{a_3'}\}$$

$= \frac{2^{[\frac{n}{2}]}(2!)^2}{3!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3]}_{a'_3]} + 4tr\{\delta^{a_1[a_2}\gamma^{a_3]}\gamma_{[a'_1}\delta_{a'_2]a'_3}\}$
$= \frac{2^{[\frac{n}{2}]}(2!)^2}{3!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3]}_{a'_3]} + \frac{2^{[\frac{n}{2}]}(2!)^2}{1!}\delta^{a_1[a_2}\delta^{a_3]}_{[a'_1}\delta_{a'_2]a'_3}$
$= 2^{[\frac{n}{2}]}(2!)^2(\frac{1}{3!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3]}_{a'_3]} + \frac{1}{1!}\delta^{a_1[a_2}\delta^{a_3]}_{[a'_1}\delta_{a'_2]a'_3})$ □

Proof: $\gamma_a\gamma_b\gamma_c\gamma_d = \frac{1}{3!}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]} + \gamma_a(\delta_{b[c}\gamma_{d]} + \gamma_b\delta_{cd})$
$= \frac{1}{3!}\gamma_a(\gamma_b\gamma_{[c}\gamma_{d]} + \gamma_c\gamma_{[d}\gamma_{b]} + \gamma_d\gamma_{[b}\gamma_{c]}) + \gamma_a(\delta_{b[c}\gamma_{d]} + \gamma_b\delta_{cd})$
$= \frac{1}{3!2}[(\gamma_a\gamma_b - \gamma_b\gamma_a)\gamma_{[c}\gamma_{d]} + (\gamma_a\gamma_c - \gamma_c\gamma_a)\gamma_{[d}\gamma_{b]} + (\gamma_a\gamma_d - \gamma_d\gamma_a)\gamma_{[b}\gamma_{c]})]$
$+ \frac{1}{3!}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]}) + \gamma_a(\delta_{b[c}\gamma_{d]} + \gamma_b\delta_{cd})$
$= \frac{1}{3!2}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]} - \frac{1}{3!2}(\gamma_b\gamma_a\gamma_{[c}\gamma_{d]} + \gamma_c\gamma_a\gamma_{[d}\gamma_{b]} + \gamma_d\gamma_a\gamma_{[b}\gamma_{c]})$
$+ \frac{1}{3!}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]}) + \gamma_a(\delta_{b[c}\gamma_{d]} + \gamma_b\delta_{cd})$
$= \frac{1}{3!3!}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]} - \frac{1}{3!}[\gamma_b(\frac{1}{3!}\gamma_{[a}\gamma_c\gamma_{d]} + \delta_{a[c}\gamma_{d]}) + \gamma_c(\frac{1}{3!}\gamma_{[a}\gamma_d\gamma_{b]} + \delta_{a[d}\gamma_{b]}) + \gamma_d(\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]} + \delta_{a[b}\gamma_{c]})]$
$+ \frac{1}{3!3}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]} + \frac{1}{3!}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]}) + \gamma_a(\delta_{b[c}\gamma_{d]} + \gamma_b\delta_{cd})$
$= \frac{1}{3!3!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} - \frac{1}{3!}(\gamma_b\delta_{a[c}\gamma_{d]} + \gamma_c\delta_{a[d}\gamma_{b]} + \gamma_d\delta_{a[b}\gamma_{c]})$
$+ \frac{1}{3!3}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]} + \frac{1}{3!}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]}) + \gamma_a(\delta_{b[c}\gamma_{d]} + \gamma_b\delta_{cd})$
$= \frac{1}{3!3!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + \frac{1}{3}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]}) + \frac{2}{3}\gamma_a(\delta_{b[c}\gamma_{d]} + \gamma_b\delta_{cd}) + \frac{1}{3}\gamma_a\gamma_b\gamma_c\gamma_d$
$= \frac{1}{3!3!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + \frac{1}{3}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]}) + \frac{1}{3}(\gamma_{[a}\gamma_{b]}\delta_{cd} + \gamma_{[c}\gamma_{a]}\delta_{bd} + \gamma_{[a}\gamma_{d]}\delta_{bc})$
$+ \frac{2}{3}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{1}{3}\gamma_a\gamma_b\gamma_c\gamma_d$
$\Leftrightarrow \gamma_a\gamma_b\gamma_c\gamma_d = \frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + \frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]} + \gamma_{[a}\gamma_{b]}\delta_{cd} + \gamma_{[c}\gamma_{a]}\delta_{bd} + \gamma_{[a}\gamma_{d]}\delta_{bc})$
$+ (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$
$\Leftrightarrow \gamma_a\gamma_b\gamma_c\gamma_d = \frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + \frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \cdot\cdot) + (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$ □

Proof: $\gamma_a\gamma_b\gamma_c\gamma_d = \frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + \frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]} + \delta_{ac}\gamma_{[d}\gamma_{b]} + \delta_{ad}\gamma_{[b}\gamma_{c]} + \gamma_{[a}\gamma_{b]}\delta_{cd} + \gamma_{[c}\gamma_{a]}\delta_{bd} + \gamma_{[a}\gamma_{d]}\delta_{bc})$
$+ (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$
$\Rightarrow \begin{cases}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]} = \frac{1}{4}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + 3\delta_{a[b}\gamma_c\gamma_{d]}\\ \gamma_{[a}\gamma_b\gamma_{c]}\gamma_d = \frac{1}{4}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + 3\gamma_{[a}\gamma_b\delta_{c]d}\end{cases}$ □

Proof:
$\begin{cases}\gamma_a\gamma_{[b}\gamma_c\gamma_{d]} = \frac{1}{4}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + 3\delta_{a[b}\gamma_c\gamma_{d]}\\ \gamma_{[a}\gamma_b\gamma_{c]}\gamma_d = \frac{1}{4}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]} + 3\gamma_{[a}\gamma_b\delta_{c]d}\end{cases}$
$\Rightarrow \gamma_{a_1}\gamma_{[a_2}\gamma_{a_3}\gamma_{a_4]}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3]}\gamma_{a'_4} = (\frac{1}{4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]} + 3\delta_{a_1[a_2}\gamma_{a_3}\gamma_{a_4]})(\frac{1}{4}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4]} + 3\gamma_{[a'_1}\gamma_{a'_2}\delta_{a'_3]a'_4})$
$\Rightarrow tr\{\gamma^{a_1}\gamma^{[a_2}\gamma^{a_3}\gamma^{a_4]}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3]}\gamma_{a'_4}\} = (\frac{1}{4}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4]} + 3\delta^{a_1[a_2}\gamma^{a_3}\gamma^{a_4]})(\frac{1}{4}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4]} + 3\gamma_{[a'_1}\gamma_{a'_2}\delta_{a'_3]a'_4})$
$= \frac{2^{[\frac{n}{2}]}(3!)^2}{4!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3}_{a'_3}\delta^{a_4]}_{a'_4]} + 9tr\{\delta^{a_1[a_2}\gamma^{a_3}\gamma^{a_4]}\gamma_{[a'_1}\gamma_{a'_2}\delta_{a'_3]a'_4}\}$
$= \frac{2^{[\frac{n}{2}]}(3!)^2}{4!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3}_{a'_3}\delta^{a_4]}_{a'_4]} + \frac{9}{4}tr\{\delta^{a_1[a_2}\gamma^{[a_3}\gamma^{a_4]]}\gamma_{[[a'_1}\gamma_{a'_2]}\delta_{a'_3]a'_4}\}$
$= \frac{2^{[\frac{n}{2}]}(3!)^2}{4!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3}_{a'_3}\delta^{a_4]}_{a'_4]} + 9\frac{2^{[\frac{n}{2}]}}{2!}\delta^{a_1[a_2}\delta^{[a_3}_{[[a'_1}\delta^{a_4]]}_{a'_2]}\delta_{a'_3]a'_4}$
$= \frac{2^{[\frac{n}{2}]}(3!)^2}{4!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3}_{a'_3}\delta^{a_4]}_{a'_4]} + \frac{2^{[\frac{n}{2}]}(3!)^2}{2!}\delta^{a_1[a_2}\delta^{a_3}_{[a'_1}\delta^{a_4]}_{a'_2}\delta_{a'_3]a'_4}$
$= 2^{[\frac{n}{2}]}(3!)^2(\frac{1}{4!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\delta^{a_3}_{a'_3}\delta^{a_4]}_{a'_4]} + \frac{1}{2!}\delta^{a_1[a_2}\delta^{a_3}_{[a'_1}\delta^{a_4]}_{a'_2}\delta_{a'_3]a'_4})$ □

Self comment: The above Dirac matrix continuous product expansion is based on the specific calculation results of the previous items. And then it is summed up and reasonably guessed out. Essentially, it has not been strictly proven, and there will be time to supplement it later. Although the above can be written strictly, concretely, and completely step by step, the writing method is not compact enough. It is not easy to conveniently use. We must think of a good way to express it, and then we can use it conveniently.

3.4 Concrete calculation of Dirac matrices continuous multiplication trace in n=N+1-D

Lem. 3.4.1. $\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}$ *constant terms*
$= \delta_{a_1a_2}\delta_{a_3a_4} - \delta_{a_1a_3}\delta_{a_2a_4} + \delta_{a_1a_4}\delta_{a_2a_3}$
$= \delta_{a_1a_2}\delta_{a_3a_4} - \delta_{a_1[a_3}\delta_{a_4]a_2}$
$= \delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_2[a_3}\delta_{a_4]a_1}$

Lem. 3.4.2. $\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6}$ *constant terms*
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5}) - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})$
$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5}) - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})$
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3[a_5}\delta_{a_6]a_4}) - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2[a_5}\delta_{a_6]a_4})$

$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2[a_5}\delta_{a_6]a_3}) - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2[a_4}\delta_{a_6]a_3})$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2[a_4}\delta_{a_5]a_3})$

Lem. 3.4.3. $\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6}\gamma_{a_7}\gamma_{a_8}$ *constant terms*
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\delta_{a_7a_8} + \cdot\cdot - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})\delta_{a_7a_8} + \cdot\cdot$
$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5})\delta_{a_7a_8} + \cdot\cdot - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})\delta_{a_7a_8} + \cdot\cdot$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\delta_{a_7a_8} + \cdot\cdot - \delta_{a_1a_7}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\delta_{a_6a_8} + \cdot\cdot$
$+ \delta_{a_1a_8}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\delta_{a_6a_7} + \cdot\cdot$
$= \delta_{a_1a_2}[\delta_{a_3a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_3a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_3a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]$
$- \delta_{a_1a_3}[\delta_{a_2a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]$
$+ \delta_{a_1a_4}[\delta_{a_2a_3}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})]$
$- \delta_{a_1a_5}[\delta_{a_2a_3}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7}) - \delta_{a_2a_4}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7}]$
$+ \delta_{a_1a_6}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7})]$
$- \delta_{a_1a_7}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_8} - \delta_{a_4a_6}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_6}]$
$+ \delta_{a_1a_8}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_7} - \delta_{a_4a_6}\delta_{a_5a_7} + \delta_{a_4a_7}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_5a_7} + \delta_{a_3a_7}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_4a_7} + \delta_{a_3a_7}\delta_{a_4a_6})]$

Although the above can also be written strictly, concretely and completely step by step. The writing method is not compact and concise. It is not convenient for use. We must think of a good way to express it in order to use it conveniently.

3.5 Conjecture of Dirac matrices continuous multiplication trace in n=N+1-D
(Can be established through construction)

Ass. 3.5.1. $tr\{\frac{1}{l!}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}\} = 0, l \le [n/2](2 - n\%2)$

Ass. 3.5.2. $tr\{\frac{1}{1!}\gamma^{b_1}\frac{1}{1!}\gamma_{a_1}\} = 2^{[\frac{n}{2}]}\delta^{b_1}_{a_1}$
$tr\{\frac{1}{2!}\gamma^{[b_1}\gamma^{b_2]}\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]}\} = -2^{[\frac{n}{2}]}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2]}, tr\{\frac{1}{3!}\gamma^{[b_1}\cdot\cdot\gamma^{b_3]}\frac{1}{3!}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\} = -2^{[\frac{n}{2}]}\delta^{b_1}_{[a_1}\cdot\cdot\delta^{b_3}_{a_3]}$
$tr\{\frac{1}{4!}\gamma^{[b_1}\cdot\cdot\gamma^{b_4]}\frac{1}{4!}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\} = 2^{[\frac{n}{2}]}\delta^{b_1}_{[a_1}\cdot\cdot\delta^{b_4}_{a_4]}, tr\{\frac{1}{5!}\gamma^{[b_1}\cdot\cdot\gamma^{b_5]}\frac{1}{5!}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\} = 2^{[\frac{n}{2}]}\delta^{b_1}_{[a_1}\cdot\cdot\delta^{b_5}_{a_5]}$
$\cdots$
$tr\{\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\frac{1}{l!}\gamma_{[b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{l-1}}\gamma_{b_l]}\} = i^{l(l-1)}\frac{2^{[\frac{n}{2}]}}{l!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4}_{b_4}\cdot\cdot\delta^{a_{l-1}}_{b_{l-1}}\delta^{a_l]}_{b_l]}$
$tr\{\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\gamma^0\frac{1}{l!}\gamma_{[b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{l-1}}\gamma_{b_l]}\gamma_0\} = i^{l(l+1)}\frac{2^{[\frac{n}{2}]}}{l!}\eta^{[a_1}_{[b_1}\eta^{a_2}_{b_2}\eta^{a_3}_{b_3}\eta^{a_4}_{b_4}\cdot\cdot\eta^{a_{l-1}}_{b_{l-1}}\eta^{a_l]}_{b_l]}$

Ass. 3.5.3.
$tr\{(\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]})^+\frac{1}{l!}\gamma_{[b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{l-1}}\gamma_{b_l]}\} = \frac{2^{[\frac{n}{2}]}}{l!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4}_{b_4}\cdot\cdot\delta^{a_{l-1}}_{b_{l-1}}\delta^{a_l]}_{b_l]}$
$tr\{(\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]})^+\gamma^0\frac{1}{l!}\gamma_{[b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{l-1}}\gamma_{b_l]}\gamma_0\} = \frac{2^{[\frac{n}{2}]}}{l!}\eta^{[a_1}_{[b_1}\eta^{a_2}_{b_2}\eta^{a_3}_{b_3}\eta^{a_4}_{b_4}\cdot\cdot\eta^{a_{l-1}}_{b_{l-1}}\eta^{a_l]}_{b_l]}$
$tr\{(\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\gamma^0)^+\frac{1}{l!}\gamma_{[b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{l-1}}\gamma_{b_l]}\gamma_0\} = \frac{2^{[\frac{n}{2}]}}{l!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4}_{b_4}\cdot\cdot\delta^{a_{l-1}}_{b_{l-1}}\delta^{a_l]}_{b_l]}$

Ass. 3.5.4.
$tr\{\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\frac{1}{l'!}\gamma_{[b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{l'-1}}\gamma_{b_{l'}]}\} = i^{l(l-1)}\frac{2^{[\frac{n}{2}]}}{l!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4}_{b_4}\cdot\cdot\delta^{a_{l-1}}_{b_{l-1}}\delta^{a_l]}_{b_l]}\delta_{ll'}$
$tr\{\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\gamma^0\frac{1}{l'!}\gamma_{[b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{l'-1}}\gamma_{b_{l'}]}\gamma_0\} = i^{l(l+1)}\frac{2^{[\frac{n}{2}]}}{l!}\eta^{[a_1}_{[b_1}\eta^{a_2}_{b_2}\eta^{a_3}_{b_3}\eta^{a_4}_{b_4}\cdot\cdot\eta^{a_{l-1}}_{b_{l-1}}\eta^{a_l]}_{b_l]}\delta_{ll'}$

Ass. 3.5.5.
$$\sum_{l=0}^{[n/2](2-n\%2)}\{(\frac{1}{l!}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]})^+{}_\lambda{}^\mu(\frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_{l-1}}\gamma_{a_l]})_\eta{}^\xi\} = 4^{[\frac{n}{2}]}\delta_\lambda{}^\xi\delta_\eta{}^\mu$$

3.6 Relational conjecture of Dirac matrices continuous multiplication trace in n=N+1-D

Ass. 3.6.1.
$tr\{\gamma^{[a_1}\gamma^{a_2}\cdot\cdot\gamma^{a_{l-1}]}\gamma^{a_l}\gamma_{[a'_1}\gamma_{a'_2}\cdot\cdot\gamma_{a'_{l-1}]}\gamma_{a'_l}\} = i^{l(l-1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\cdot\cdot\delta^{a_l]}_{a'_l]} - \frac{1}{(l-2)!}\delta^{[a_1}_{[a'_1}\cdot\cdot\delta^{a_{l-2}}_{a'_{l-2}]}\delta^{a_{l-1}]a_l}\delta_{a'_{l-1}]a'_l}\}$
$tr\{\gamma^{[a_1}\cdot\cdot\gamma^{a_{l-1}]}\gamma^{a_l}\gamma^0\gamma_{[a'_1}\cdot\cdot\gamma_{a'_{l-1}]}\gamma_{a'_l}\gamma_0\} = i^{l(l+1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_l]}_{a'_l]} - \frac{1}{(l-2)!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_{l-2}}_{a'_{l-2}}\delta^{a_{l-1}]a_l}\delta_{a'_{l-1}]a'_l}\}$

Ass. 3.6.2.
$tr\{\gamma^{a_1}\gamma^{[a_2}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\gamma_{a'_1}\gamma_{[a'_2}\cdot\cdot\gamma_{a'_{l-1}}\gamma_{a'_l]}\} = i^{l(l-1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\cdot\cdot\delta^{a_l]}_{a'_l]} - \frac{1}{(l-2)!}\delta^{a_1[a_2}\delta_{a'_1[a'_2}\delta^{a_3}_{a'_3}\cdot\cdot\delta^{a_l]}_{a'_l]}\}$
$tr\{\gamma^0\gamma^{a_1}\gamma^{[a_2}\cdot\cdot\gamma^{a_l]}\gamma_0\gamma_{a'_1}\gamma_{[a'_2}\cdot\cdot\gamma_{a'_l]}\} = i^{l(l+1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\cdot\cdot\delta^{a_l]}_{a'_l]} - \frac{1}{(l-2)!}\delta^{a_1[a_2}\delta_{a'_1[a'_2}\delta^{a_3}_{a'_3}\cdot\cdot\delta^{a_l]}_{a'_l]}\}$

Ass. 3.6.3.
$tr\{\gamma^{a_1}\gamma^{[a_2}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\gamma_{[a'_1}\gamma_{a'_2}\cdot\cdot\gamma_{a'_{l-1}]}\gamma_{a'_l}\}=i^{l(l-1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\cdot\cdot\delta^{a_l]}_{a'_l]}-\frac{1}{(l-2)!}\delta^{a_1[a_2}\delta^{a_3}_{[a'_1}\cdot\cdot\delta^{a_l]}_{a'_{l-2}}\delta_{a'_{l-1}]a'_l}\}$
$tr\{\gamma^0\gamma^{a_1}\gamma^{[a_2}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\gamma_0\gamma_{[a'_1}\gamma_{a'_2}\cdot\cdot\gamma_{a'_{l-1}]}\gamma_{a'_l}\}=i^{l(l+1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_l]}_{a'_l]}-\frac{1}{(l-2)!}\delta^{a_1[a_2}\eta^{a_3}_{[a'_1}\cdot\cdot\eta^{a_l]}_{a'_{l-2}}\delta_{a'_{l-1}]a'_l}\}$

Ass. 3.6.4.
$tr\{\gamma^{[a_1}\gamma^{a_2}\cdot\cdot\gamma^{a_{l-1}]}\gamma^{a_l}\gamma_{a'_1}\gamma_{[a'_2}\cdot\cdot\gamma_{a'_{l-1}}\gamma_{a'_l]}\}=i^{l(l-1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\delta^{[a_1}_{[a'_1}\delta^{a_2}_{a'_2}\cdot\cdot\delta^{a_l]}_{a'_l]}-\frac{1}{(l-2)!}\delta_{a'_1[a'_2}\delta^{[a_1}_{a'_3}\cdot\cdot\delta^{a_{l-2}}_{a'_l]}\delta^{a_{l-1}]a_l}\}$
$tr\{\gamma^0\gamma^{a_1}\gamma^{[a_2}\cdot\cdot\gamma^{a_{l-1}}\gamma^{a_l]}\gamma_0\gamma_{[a'_1}\gamma_{a'_2}\cdot\cdot\gamma_{a'_{l-1}]}\gamma_{a'_l}\}=i^{l(l+1)}2^{[\frac{n}{2}]}[(l-1)!]^2\{\frac{1}{l!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_l]}_{a'_l]}-\frac{1}{(l-2)!}\delta_{a'_1[a'_2}\eta^{[a_1}_{a'_3}\cdot\cdot\eta^{a_{l-2}}_{a'_l]}\delta^{a_{l-1}]a_l}\}$

Self comment: The above Dirac matrix continuous multiplication trace formula was proved through the conjecture of Dirac matrix continuous multiplication expansion, and in essence it has not been strictly proved.

4 Properties of product sum for δ functions in N+1 dimensional space-time

4.1 Indices monotonic cyclic summation rule of product sum for δ functions in n=N+1-D

Lem. 4.1.1.

$$\begin{cases}\frac{1}{2}\sum_b^a\{\delta^{a_1a_2}\delta^{[a_3}_{[b_1}\delta^{a_4]}_{b_2]}\delta_{b_3b_4}\}=\sum_b^a\{\delta^{a_1a_2}\delta^{[a_3}_{b_1}\delta^{a_4]}_{b_2}\delta_{b_3b_4}\}=\sum_b^a\{\delta^{a_1a_2}\delta^{a_3}_{[b_1}\delta^{a_4}_{b_2]}\delta_{b_3b_4}\}\\ =\frac{1}{2}(\delta^{a_1[a_2}\delta^{a_3}_{[b_1}\delta^{a_4]}_{b_2}\delta_{b_3]b_4}+\delta^{a_1[a_2}\delta^{a_3}_{[b_1}\delta^{a_4]}_{b_4]}\delta_{b_2b_3}-\delta^{a_1[a_2}\delta^{a_3}_{[b_1}\delta^{a_4]}_{b_3]}\delta_{b_2b_4}+\delta^{a_1[a_2}\delta^{a_3}_{[b_3}\delta^{a_4]}_{b_4]}\delta_{b_1b_2})\\ +\frac{1}{2}(\delta^{a_2a_3}\delta^{[a_1}_{[b_1}\delta^{a_4]}_{b_2}\delta_{b_3]b_4}+\delta^{a_2a_3}\delta^{[a_1}_{[b_1}\delta^{a_4]}_{b_4]}\delta_{b_2b_3}-\delta^{a_2a_3}\delta^{[a_1}_{[b_1}\delta^{a_4]}_{b_3]}\delta_{b_2b_4}+\delta^{a_2a_3}\delta^{[a_1}_{[b_3}\delta^{a_4]}_{b_4]}\delta_{b_1b_2})\\ -\frac{1}{2}(\delta^{a_2a_4}\delta^{[a_1}_{[b_1}\delta^{a_3]}_{b_2}\delta_{b_3]b_4}+\delta^{a_2a_4}\delta^{[a_1}_{[b_1}\delta^{a_3]}_{b_4]}\delta_{b_2b_3}-\delta^{a_2a_4}\delta^{[a_1}_{[b_1}\delta^{a_3]}_{b_3]}\delta_{b_2b_4}+\delta^{a_2a_4}\delta^{[a_1}_{[b_3}\delta^{a_3]}_{b_4]}\delta_{b_1b_2})\\ +\frac{1}{2}(\delta^{a_3a_4}\delta^{[a_1}_{[b_1}\delta^{a_2]}_{b_2}\delta_{b_3]b_4}+\delta^{a_3a_4}\delta^{[a_1}_{[b_1}\delta^{a_2]}_{b_4]}\delta_{b_2b_3}-\delta^{a_3a_4}\delta^{[a_1}_{[b_1}\delta^{a_2]}_{b_3]}\delta_{b_2b_4}+\delta^{a_3a_4}\delta^{[a_1}_{[b_3}\delta^{a_2]}_{b_4]}\delta_{b_1b_2})\end{cases}$$

Lem. 4.1.2.

$$\begin{cases}\sum_{ab}\{\delta^{a_1a_2}\delta_{b_1b_2}\}=\delta^{a_1a_2}\delta_{b_1b_2}-\delta^{a_1}_{[b_1}\delta^{a_2}_{b_2]},\sum_a\{\delta_{a_1a_2}\delta_{a_3a_4}\}=\delta_{a_1a_2}\delta_{a_3a_4}-\delta_{a_1[a_3}\delta_{a_4]a_2}\\ \sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\delta_{b_1b_2}\delta_{b_3b_4}\}=(\delta^{a_1a_2}\delta^{a_3a_4}-\delta^{a_1[a_3}\delta^{a_4]a_2})(\delta_{b_1b_2}\delta_{b_3b_4}-\delta_{b_1[b_3}\delta_{b_4]b_2})\\ \sum_{ab}\{\delta^{a_1a_2}\delta^{a_3a_4}\delta_{b_1b_2}\delta_{b_3b_4}\}=\frac{1}{0!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\delta_{b_1b_2}\delta_{b_3b_4}\}+\frac{1}{2!}\sum_b^a\{\delta^{a_1a_2}\delta^{[a_3}_{[b_1}\delta^{a_4]}_{b_2]}\delta_{b_3b_4}\}+\frac{1}{4!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4]}_{b_4]}\end{cases}$$

Lem. 4.1.3. $\gamma^{a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\gamma^{a_5}\gamma^{a_6}\gamma^{a_7}\gamma^{a_8}\gamma_{b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\gamma_{b_5}\gamma_{b_6}\gamma_{b_7}\gamma_{b_8}$

$$=\cdots+\frac{1}{0!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\delta^{a_5a_6}\delta^{a_7a_8}\delta_{b_1b_2}\delta_{b_3b_4}\delta_{b_5b_6}\delta_{b_7b_8}\}+\frac{1}{2!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\delta^{a_5a_6}\delta^{[a_7}_{[b_1}\delta^{a_8]}_{b_2]}\delta_{b_3b_4}\delta_{b_5b_6}\delta_{b_7b_8}\}$$
$$+\frac{1}{4!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\delta^{[a_5}_{[b_1}\delta^{a_6}_{b_2}\delta^{a_7}_{b_3}\delta^{a_8]}_{b_4]}\delta_{b_5b_6}\delta_{b_7b_8}\}+\frac{1}{6!}\sum_b^a\{\delta^{a_1a_2}\delta^{[a_3}_{[b_1}\delta^{a_4}_{b_2}\delta^{a_5}_{b_3}\delta^{a_6}_{b_4}\delta^{a_7}_{b_5}\delta^{a_8]}_{b_6]}\delta_{b_7b_8}\}+\frac{1}{8!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4}_{b_4}\delta^{a_5}_{b_5}\delta^{a_6}_{b_6}\delta^{a_7}_{b_7}\delta^{a_8]}_{b_8]}$$

Lem. 4.1.4. $\gamma^{a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{2l-3}}\gamma^{a_{2l-2}}\gamma^{a_{2l-1}}\gamma^{a_{2l}}\gamma_{b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{2l-3}}\gamma_{b_{2l-2}}\gamma_{b_{2l-1}}\gamma_{b_{2l}}$

$$=\cdots$$
$$+\frac{1}{0!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\cdot\cdot\delta^{a_{2l-3}a_{2l-2}}\delta^{a_{2l-1}a_{2l}}\delta_{b_1b_2}\delta_{b_3b_4}\cdot\cdot\delta_{b_{2l-3}b_{2l-2}}\delta_{b_{2l-1}b_{2l}}\}$$
$$+\frac{1}{2!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\cdot\cdot\delta^{a_{2l-3}a_{2l-2}}\delta^{[a_{2l-1}}_{[b_1}\delta^{a_{2l}]}_{b_2]}\delta_{b_3b_4}\cdot\cdot\delta_{b_{2l-3}b_{2l-2}}\delta_{b_{2l-1}b_{2l}}\}$$
$$+\frac{1}{4!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\cdot\cdot\delta^{[a_{2l-3}}_{[b_1}\delta^{a_{2l-2}}_{b_2}\delta^{a_{2l-1}}_{b_3}\delta^{a_{2l}]}_{b_4]}\cdot\cdot\delta_{b_{2l-3}b_{2l-2}}\delta_{b_{2l-1}b_{2l}}\}+\cdot\cdot$$
$$+\frac{1}{(2l-2)!}\sum_b^a\{\delta^{a_1a_2}\delta^{[a_3}_{[b_1}\delta^{a_4}_{b_2}\delta^{a_5}_{b_3}\delta^{a_6}_{b_4}\cdot\cdot\delta^{a_{2l-3}}_{b_{2l-5}}\delta^{a_{2l-2}}_{b_{2l-4}}\delta^{a_{2l-1}}_{b_{2l-3}}\delta^{a_{2l}]}_{b_{2l-2}]}\delta_{b_{2l-1}b_{2l}}\}$$
$$+\frac{1}{(2l)!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4}_{b_4}\cdot\cdot\delta^{a_{2l-3}}_{b_{2l-3}}\delta^{a_{2l-2}}_{b_{2l-2}}\delta^{a_{2l-1}}_{b_{2l-1}}\delta^{a_{2l}]}_{b_{2l}]}$$

Lem. 4.1.5. $\gamma^{a_1}\gamma^{a_2}\gamma^{a_3}\gamma^{a_4}\cdot\cdot\gamma^{a_{2l-3}}\gamma^{a_{2l-2}}\gamma^{a_{2l-1}}\gamma^{a_{2l}}\gamma^{a_{2l+1}}\gamma_{b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{2l-3}}\gamma_{b_{2l-2}}\gamma_{b_{2l-1}}\gamma_{b_{2l}}\gamma^{b_{2l+1}}$

$$=\cdots$$
$$+\frac{1}{1!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\cdot\cdot\delta^{a_{2l-3}a_{2l-2}}\delta^{a_{2l-1}a_{2l}}\delta^{a_{2l+1}}_{b_1}\delta_{b_2b_3}\delta_{b_4b_5}\cdot\cdot\delta_{b_{2l-2}b_{2l-1}}\delta_{b_{2l}b_{2l+1}}\}$$
$$+\frac{1}{3!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\cdot\cdot\delta^{a_{2l-3}a_{2l-2}}\delta^{[a_{2l-1}}_{[b_1}\delta^{a_{2l}}_{b_2}\delta^{a_{2l+1}]}_{b_3]}\delta_{b_4b_5}\cdot\cdot\delta_{b_{2l-2}b_{2l-1}}\delta_{b_{2l}b_{2l+1}}\}$$
$$+\frac{1}{5!}\sum_b^a\{\delta^{a_1a_2}\delta^{a_3a_4}\cdot\cdot\delta^{[a_{2l-3}}_{[b_1}\delta^{a_{2l-2}}_{b_2}\delta^{a_{2l-1}}_{b_3}\delta^{a_{2l}}_{b_4}\delta^{a_{2l+1}]}_{b_5]}\cdot\cdot\delta_{b_{2l-2}b_{2l-1}}\delta_{b_{2l}b_{2l+1}}\}+\cdot\cdot$$
$$+\frac{1}{(2l-1)!}\sum_b^a\{\delta^{a_1a_2}\delta^{[a_3}_{[b_1}\delta^{a_4}_{b_2}\delta^{a_5}_{b_3}\delta^{a_6}_{b_4}\cdot\cdot\delta^{a_{2l-1}}_{b_{2l-3}}\delta^{a_{2l}}_{b_{2l-2}}\delta^{a_{2l+1}]}_{b_{2l-1}]}\delta_{b_{2l}b_{2l+1}}\}$$
$$+\frac{1}{(2l+1)!}\delta^{[a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{a_4}_{b_4}\cdot\cdot\delta^{a_{2l-3}}_{b_{2l-3}}\delta^{a_{2l-2}}_{b_{2l-2}}\delta^{a_{2l-1}}_{b_{2l-1}}\delta^{a_{2l}}_{b_{2l}}\delta^{a_{2l+1}]}_{b_{2l+1}]}$$

Lem. 4.1.6. $\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_{2l-3}}\gamma_{a_{2l-2}}\gamma_{a_{2l-1}}\gamma_{a_{2l}}\gamma_{b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{2l-3}}\gamma_{b_{2l-2}}\gamma_{b_{2l-1}}\gamma_{b_{2l}}$

$$=\cdots$$

$+\frac{1}{0!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{a_3a_4}\cdot\cdot\delta_{a_{2l-3}a_{2l-2}}\delta_{a_{2l-1}a_{2l}}\delta_{b_1b_2}\delta_{b_3b_4}\cdot\cdot\delta_{b_{2l-3}b_{2l-2}}\delta_{b_{2l-1}b_{2l}}\}$
$+\frac{1}{2!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{a_3a_4}\cdot\cdot\delta_{a_{2l-3}a_{2l-2}}\delta_{[a_{2l-1}\langle b_1}\delta_{a_{2l}]b_2\rangle}\delta_{b_3b_4}\cdot\cdot\delta_{b_{2l-3}b_{2l-2}}\delta_{b_{2l-1}b_{2l}}\}$
$+\frac{1}{4!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{a_3a_4}\cdot\cdot\delta_{[a_{2l-3}\langle b_1}\delta_{a_{2l-2}b_2}\delta_{a_{2l-1}b_3}\delta_{a_{2l}]b_4\rangle}\cdot\cdot\delta_{b_{2l-3}b_{2l-2}}\delta_{b_{2l-1}b_{2l}}\}+\cdot\cdot$
$+\frac{1}{(2l-2)!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{[a_3\langle b_1}\delta_{a_4b_2}\delta_{a_5b_3}\delta_{a_6b_4}\cdot\cdot\delta_{a_{2l-3}b_{2l-5}}\delta_{a_{2l-2}b_{2l-4}}\delta_{a_{2l-1}b_{2l-3}}\delta_{a_{2l}]b_{2l-2}\rangle}\delta_{b_{2l-1}b_{2l}}\}$
$+\frac{1}{(2l)!}\delta_{[a_1\langle b_1}\delta_{a_2b_2}\delta_{a_3b_3}\delta_{a_4b_4}\cdot\cdot\delta_{a_{2l-3}b_{2l-3}}\delta_{a_{2l-2}b_{2l-2}}\delta_{a_{2l-1}b_{2l-1}}\delta_{a_{2l}]b_{2l}\rangle}$

Lem. 4.1.7. $\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\cdot\cdot\gamma_{a_{2l-3}}\gamma_{a_{2l-2}}\gamma_{a_{2l-1}}\gamma_{a_{2l}}\gamma_{a_{2l+1}}\gamma_{b_1}\gamma_{b_2}\gamma_{b_3}\gamma_{b_4}\cdot\cdot\gamma_{b_{2l-3}}\gamma_{b_{2l-2}}\gamma_{b_{2l-1}}\gamma_{b_{2l}}\gamma_{b_{2l+1}}$
$=\cdots$
$+\frac{1}{1!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{a_3a_4}\cdot\cdot\delta_{a_{2l-3}a_{2l-2}}\delta_{a_{2l-1}a_{2l}}\delta_{a_{2l+1}b_1}\delta_{b_2b_3}\delta_{b_4b_5}\cdot\cdot\delta_{b_{2l-2}b_{2l-1}}\delta_{b_{2l}b_{2l+1}}\}$
$+\frac{1}{3!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{a_3a_4}\cdot\cdot\delta_{a_{2l-3}a_{2l-2}}\delta_{[a_{2l-1}\langle b_1}\delta_{a_{2l}b_2}\delta_{a_{2l+1}]b_3\rangle}\delta_{b_4b_5}\cdot\cdot\delta_{b_{2l-2}b_{2l-1}}\delta_{b_{2l}b_{2l+1}}\}$
$+\frac{1}{5!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{a_3a_4}\cdot\cdot\delta_{[a_{2l-3}\langle b_1}\delta_{a_{2l-2}b_2}\delta_{a_{2l-1}b_3}\delta_{a_{2l}b_4}\delta_{a_{2l+1}]b_5\rangle}\cdot\cdot\delta_{b_{2l-2}b_{2l-1}}\delta_{b_{2l}b_{2l+1}}\}+\cdot\cdot$
$+\frac{1}{(2l-1)!}\sum_{b}^{a}\{\delta_{a_1a_2}\delta_{[a_3\langle b_1}\delta_{a_4b_2}\delta_{a_5b_3}\delta_{a_6b_4}\cdot\cdot\delta_{a_{2l-1}b_{2l-3}}\delta_{a_{2l}b_{2l-2}}\delta_{a_{2l+1}]b_{2l-1}\rangle}\delta_{b_{2l}b_{2l+1}}\}$
$+\frac{1}{(2l+1)!}\delta_{[a_1\langle b_1}\delta_{a_2b_2}\delta_{a_3b_3}\delta_{a_4b_4}\cdot\cdot\delta_{a_{2l-3}b_{2l-3}}\delta_{a_{2l-2}b_{2l-2}}\delta_{a_{2l-1}b_{2l-1}}\delta_{a_{2l}b_{2l}}\delta_{a_{2l+1}]b_{2l+1}\rangle}$

Although the above method of writing has become more compact, it can be strictly, concretely, and completely written step by step. However, it is still not concise enough to be easily used. We must think of a better way to express it for conveniently using.

4.2 Expansion and contraction of fully symmetric tensors with $w+1$ order

Pro. 4.2.1. $A_{(a_1a_2a_3a_4\cdot\cdot a_{2s})}=A_{a_1(a_2a_3a_4\cdot\cdot a_{2s})}+A_{a_2(a_1a_3a_4\cdot\cdot a_{2s})}+A_{a_3(a_1a_2a_4\cdot\cdot a_{2s})}+\cdot\cdot$

Pro. 4.2.2. $A_{(a_1a_2a_3a_4\cdot\cdot a_{2s})}=\langle A_{(a_1\cdot\cdot a_l)\{a_{l+1}\cdot\cdot a_{2s}\}},\frac{(2s)!}{l!(2s-l)!}\rangle$
$=\langle A_{\underbrace{(a_1\cdot\cdot a_{l_1})}_{l_1}\underbrace{(a_{l_1+1}\cdot\cdot a_{l_1+l_2})}_{l_2}\cdot\cdot\underbrace{(a_{l_1+\cdot\cdot+l_{n-1}+1}\cdot\cdot a_{l_1+\cdot\cdot+l_n})}_{l_n}},\frac{(2s)!}{l_1!l_2!\cdot\cdot l_n!}\rangle, l_1+l_2+\cdot\cdot+l_n=2s$

Pro. 4.2.3. $\Gamma^{k_\varsigma}_{A_{1\varsigma}A_{2\varsigma}\cdot\cdot A_{2s\varsigma}}(s;w)\Gamma^{B_{1\varsigma}B_{2\varsigma}\cdot\cdot B_{2s\varsigma}}_{k_\varsigma}(s;w)=\frac{1}{(2s)!}\delta^{(B_{1\varsigma}}_{A_{1\varsigma}}\delta^{B_{2\varsigma}}_{A_{2\varsigma}}\cdot\cdot\delta^{B_{2s\varsigma})}_{A_{2s\varsigma}}=\frac{1}{(2s)!}\delta^{B_{1\varsigma}}_{(A_{1\varsigma}}\delta^{B_{2\varsigma}}_{A_{2\varsigma}}\cdot\cdot\delta^{B_{2s\varsigma}}_{A_{2s\varsigma})}$

Pro. 4.2.4. $\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}=\delta^{b_1}_{a_1}\delta^{b_2}_{(a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\delta^{b_1}_{a_2}\delta^{b_2}_{(a_1}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\delta^{b_1}_{a_3}\delta^{b_2}_{(a_1}\delta^{b_3}_{a_2}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\cdot\cdot$

Pro. 4.2.5. $\delta^{a_1}_{b_1}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}=(2s+w)\delta^{b_2}_{(a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}$

Proof: $\delta^{a_1}_{b_1}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}$
$=\delta^{a_1}_{b_1}\delta^{b_1}_{a_1}\delta^{b_2}_{(a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\delta^{a_1}_{b_1}\delta^{b_1}_{a_2}\delta^{b_2}_{(a_1}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\delta^{a_1}_{b_1}\delta^{b_1}_{a_3}\delta^{b_2}_{(a_1}\delta^{b_3}_{a_2}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\cdot\cdot$
$=n\delta^{b_2}_{(a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\delta^{b_2}_{(a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\delta^{b_2}_{(a_3}\delta^{b_3}_{a_2}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}+\cdot\cdot$
$=(2s+w)\delta^{b_2}_{(a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}$ □

Pro. 4.2.6.
$$\begin{cases}\delta^{a_1}_{b_1}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}=1!C^1_{2s+w}\delta^{b_2}_{(a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}, \quad \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}=2!C^2_{2s+w}\delta^{b_3}_{(a_3}\delta^{b_4}_{a_4}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}\\ \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}=3!C^3_{2s+w}\delta^{b_4}_{(a_4}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}\cdot\cdot\cdot\cdot\cdot\cdot\\ \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\cdot\cdot\delta^{a_{2s-1}}_{b_{2s-1}}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}=(2s-1)!C^{2s-1}_{2s+w}\delta^{b_{2s}}_{a_{2s}}, \quad \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\cdot\cdot\delta^{a_{2s}}_{b_{2s}}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}=(2s)!C^{2s}_{2s+w}\end{cases}$$

Pro. 4.2.7.
$$\begin{cases}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}\delta^{a_{2s}}_{b_{2s}}=1!C^1_{2s+w}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{2s-1}}_{a_{2s-1})}, \quad \delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}\delta^{a_{2s-1}}_{b_{2s-1}}\delta^{a_{2s}}_{b_{2s}}=2!C^2_{2s+w}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{2s-2}}_{a_{2s-2})}\\ \delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}\delta^{a_{2s-2}}_{b_{2s-2}}\cdot\cdot\delta^{a_{2s}}_{b_{2s}}=3!C^3_{2s+w}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{2s-3}}_{a_{2s-3})}\cdot\cdot\cdot\cdot\cdot\cdot\\ \delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}\delta^{a_{2s-1}}_{b_{2s-1}}\cdot\cdot\delta^{a_2}_{b_2}\delta^{a_1}_{b_1}=(2s-1)!C^{2s-1}_{2s+w}\delta^{b_{2s}}_{a_{2s}}, \quad \delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_{2s}}_{a_{2s})}\delta^{a_{2s}}_{b_{2s}}\cdot\cdot\delta^{a_2}_{b_2}\delta^{a_1}_{b_1}=(2s)!C^{2s}_{2s+w}\end{cases}$$

4.3 Expansion and contraction of antisymmetric tensors in n=N+1 dimensional space-time

Pro. 4.3.1. $A_{[a_1a_2a_3a_4\cdot\cdot a_n]}=A_{a_1[a_2a_3a_4\cdot\cdot a_n]}-A_{a_2[a_1a_3a_4\cdot\cdot a_n]}+A_{a_3[a_1a_2a_4\cdot\cdot a_n]}+\cdot\cdot$

Pro. 4.3.2. $A_{[a_1a_2a_3a_4\cdot\cdot a_{2s}]}=\langle A_{[a_1\cdot\cdot a_l][a_{l+1}\cdot\cdot a_{2s}]},\frac{(2s)!}{l!(2s-l)!}\rangle$
$=\langle A_{\underbrace{[a_1\cdot\cdot a_{l_1}]}_{l_1}\underbrace{[a_{l_1+1}\cdot\cdot a_{l_1+l_2}]}_{l_2}\cdot\cdot\underbrace{[a_{l_1+\cdot\cdot+l_{n-1}+1}\cdot\cdot a_{l_1+\cdot\cdot+l_n}]}_{l_n}},\frac{(2s)!}{l_1!l_2!\cdot\cdot l_n!}\rangle, l_1+l_2+\cdot\cdot+l_n=2s$

Pro. 4.3.3. $\varepsilon_{a_1a_2\cdot\cdot a_n}\varepsilon^{b_1b_2\cdot\cdot b_n}=\delta^{[b_1}_{a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_n]}_{a_n}=\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_n}_{a_n]}$

Pro. 4.3.4. $\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}=\delta^{b_1}_{a_1}\delta^{b_2}_{[a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}-\delta^{b_1}_{a_2}\delta^{b_2}_{[a_1}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}+\delta^{b_1}_{a_3}\delta^{b_2}_{[a_1}\delta^{b_3}_{a_2}\cdot\cdot\delta^{b_n}_{a_n]}+\cdot\cdot$

Pro. 4.3.5. $\delta^{a_1}_{b_1}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}=\delta^{b_2}_{[a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}$

Proof: $\delta^{a_1}_{b_1}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}$
$=\delta^{a_1}_{b_1}\delta^{b_1}_{a_1}\delta^{b_2}_{[a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}-\delta^{a_1}_{b_1}\delta^{b_1}_{a_2}\delta^{b_2}_{[a_1}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}+\delta^{a_1}_{b_1}\delta^{b_1}_{a_3}\delta^{b_2}_{[a_1}\delta^{b_3}_{a_2}\cdot\cdot\delta^{b_n}_{a_n]}+\cdot\cdot$
$=n\delta^{b_2}_{[a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}-\delta^{b_2}_{[a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}+\delta^{b_2}_{[a_3}\delta^{b_3}_{a_2}\cdot\cdot\delta^{b_n}_{a_n]}+\cdot\cdot$
$=\delta^{b_2}_{[a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}$ □

Pro. 4.3.6.
$$\begin{cases}\delta^{a_1}_{b_1}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}=1!\delta^{b_2}_{[a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}, \quad \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}=2!\delta^{b_3}_{[a_3}\delta^{b_4}_{a_4}\cdot\cdot\delta^{b_n}_{a_n]}\\ \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}=3!\delta^{b_4}_{[a_4}\cdot\cdot\delta^{b_n}_{a_n]}\cdots\cdots\\ \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\cdot\cdot\delta^{a_{n-1}}_{b_{n-1}}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}=(n-1)!\delta^{b_n}_{a_n}, \quad \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\cdot\cdot\delta^{a_n}_{b_n}\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}=n!\end{cases}$$

Pro. 4.3.7.
$$\begin{cases}\delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_n}_{a_n]}\delta^{a_n}_{b_n}=1!\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{n-1}}_{a_{n-1}]}, \quad \delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_n}_{a_n]}\delta^{a_{n-1}}_{b_{n-1}}\delta^{a_n}_{b_n}=2!\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{n-2}}_{a_{n-2}]}\\ \delta^{b_1}_{(a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}\delta^{a_{n-2}}_{b_{n-2}}\cdot\cdot\delta^{a_n}_{b_n}=3!\delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\cdot\cdot\delta^{b_{n-3}}_{a_{n-3}]}\cdots\cdots\\ \delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}\delta^{a_n}_{b_n}\cdot\cdot\delta^{a_2}_{b_2}=(n-1)!\delta^{b_1}_{a_1}, \quad \delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3}\cdot\cdot\delta^{b_n}_{a_n]}\delta^{a_n}_{b_n}\cdot\cdot\delta^{a_2}_{b_2}\delta^{a_1}_{b_1}=n!\end{cases}$$

5 Q product in n=N+1 dimensional space-time

5.1 Concrete calculation of product in n=N+1 dimensional space-time

Def. 5.1.1. $K:=(m-\gamma_a\partial^a)\gamma_0, \tilde{K}:=CK^T\bar{C}=-\gamma_0(m+\gamma_a\partial^a), Q:=(m-\gamma_a\partial^a), \tilde{Q}:=(m+\gamma_a\partial^a)$

Pro. 5.1.1. $\Gamma_0Q=\tilde{Q}\Gamma_0, Q\Gamma_0=\Gamma_0\tilde{Q}; \Gamma_0Q\Gamma_0=\tilde{Q}, \Gamma_0\tilde{Q}\Gamma_0=Q$

Pro. 5.1.2.
$$\begin{cases}\gamma_{a_1}Q\gamma_{a'_1}\tilde{Q}=-\Gamma_0\gamma_{a_1}\tilde{Q}\Gamma_0\gamma_{a'_1}\tilde{Q}=-\gamma_{a_1}\Gamma_0\tilde{Q}\gamma_{a'_1}\Gamma_0\tilde{Q}\\ \gamma_{a_1}Q\gamma_{a'_1}\tilde{Q}=-\gamma_{a_1}Q\Gamma_0\gamma_{a'_1}Q\Gamma_0=\gamma_{a_1}Q\gamma_{a'_1}\Gamma_0Q\Gamma_0\\ \gamma_{a_1}Q\gamma_{a'_1}\tilde{Q}\cdot\cdot\gamma_{a_l}Q\gamma_{a'_l}\tilde{Q}=(-1)^l\gamma_{a_1}\Gamma_0\tilde{Q}\gamma_{a'_1}\Gamma_0\tilde{Q}\cdot\cdot\gamma_{a_l}\Gamma_0\tilde{Q}\gamma_{a'_l}\Gamma_0\tilde{Q}\\ \gamma_{a_1}Q\gamma_{a'_1}\tilde{Q}\cdot\cdot\gamma_{a_l}Q\gamma_{a'_l}\tilde{Q}=(-1)^{l-1}\gamma_{a_1}Q\gamma_{a'_1}\Gamma_0Q\gamma_{a_2}\Gamma_0Q\gamma_{a'_2}\Gamma_0Q\cdot\cdot\gamma_{a_l}\Gamma_0Q\gamma_{a'_l}\Gamma_0Q\Gamma_0\end{cases}$$

Pro. 5.1.3.
$$\begin{cases}tr(\gamma_{a_1}Q\gamma_{a'_1}\tilde{Q})=-tr(\Gamma_0\gamma_{a_1}\tilde{Q}\Gamma_0\gamma_{a'_1}\tilde{Q})=-tr(\gamma_{a_1}\Gamma_0\tilde{Q}\gamma_{a'_1}\Gamma_0\tilde{Q})\\ tr(\gamma_{a_1}Q\gamma_{a'_1}\tilde{Q})=-tr(\Gamma_0\gamma_{a_1}Q\Gamma_0\gamma_{a'_1}Q)=-tr(\gamma_{a_1}\Gamma_0Q\gamma_{a'_1}\Gamma_0Q)\\ tr(\gamma_{a_1}Q\gamma_{a'_1}\tilde{Q}\cdot\cdot\gamma_{a_l}Q\gamma_{a'_l}\tilde{Q})=(-1)^ltr(\gamma_{a_1}\Gamma_0\tilde{Q}\gamma_{a'_1}\Gamma_0\tilde{Q}\cdot\cdot\gamma_{a_l}\Gamma_0\tilde{Q}\gamma_{a'_l}\Gamma_0\tilde{Q})\\ tr(\gamma_{a_1}Q\gamma_{a'_1}\tilde{Q}\cdot\cdot\gamma_{a_l}Q\gamma_{a'_l}\tilde{Q})=(-1)^ltr(\gamma_{a_1}\Gamma_0Q\gamma_{a'_1}\Gamma_0Q\cdot\cdot\gamma_{a_l}\Gamma_0Q\gamma_{a'_l}\Gamma_0Q)\end{cases}$$

Pro. 5.1.4.
$$\begin{cases}tr[\gamma_aQ\gamma_{a'}\tilde{Q}]=8(m^2\delta_{aa'}-\partial_a\partial_{a'})=8m^2(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\\ tr[\gamma_a\Gamma_0\tilde{Q}\gamma_{a'}\Gamma_0\tilde{Q}]=-8(m^2\delta_{aa'}-\partial_a\partial_{a'})=-8m^2(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\\ tr[\gamma_a\Gamma_0Q\gamma_{a'}\Gamma_0Q]=-8(m^2\delta_{aa'}-\partial_a\partial_{a'})=-8m^2(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\end{cases}$$

Pro. 5.1.5.
$$\begin{cases}tr[\gamma_a\tilde{Q}\gamma_{a'}\tilde{Q}]=tr[\gamma_aQ\gamma_{a'}Q]=8\partial_a\partial_{a'}\\ tr[\gamma_a\Gamma_0Q\gamma_{a'}\Gamma_0\tilde{Q}]=tr[\gamma_a\Gamma_0\tilde{Q}\gamma_{a'}\Gamma_0\tilde{Q}]=-8\partial_a\partial_{a'}\end{cases}$$

Proof: $tr[\gamma_aQ\gamma_{a'}\tilde{Q}]=tr[\gamma_a(m-\gamma_{a_1}\partial^{a_1})\gamma_{a'}(m+\gamma_{a_2}\partial^{a_2})]$
$=m^2tr(\gamma_a\gamma_{a'})-tr(\gamma_a\gamma_{a_1}\gamma_{a'}\gamma_{a_2})\partial^{a_1}\partial^{a_2}$
$=4m^2\delta_{aa'}-4(\delta_{aa_1}\delta_{a'a_2}-\delta_{aa'}\delta_{a_1a_2}+\delta_{aa_2}\delta_{a_1a'})\partial^{a_1}\partial^{a_2}$
$=4m^2\delta_{aa'}-4(2\partial_a\partial_{a'}-\delta_{aa'}m^2)$
$=8(m^2\delta_{aa'}-\partial_a\partial_{a'})$ □

Proof: $tr[\gamma_a\tilde{Q}\gamma_{a'}\tilde{Q}]=tr[\gamma_a(m+\gamma_{a_1}\partial^{a_1})\gamma_{a'}(m+\gamma_{a_2}\partial^{a_2})]$
$=m^2tr(\gamma_a\gamma_{a'})+tr(\gamma_a\gamma_{a_1}\gamma_{a'}\gamma_{a_2})\partial^{a_1}\partial^{a_2}$
$=4m^2\delta_{aa'}+4(\delta_{aa_1}\delta_{a'a_2}-\delta_{aa'}\delta_{a_1a_2}+\delta_{aa_2}\delta_{a_1a'})\partial^{a_1}\partial^{a_2}$
$=4m^2\delta_{aa'}+4(2\partial_a\partial_{a'}-\delta_{aa'}m^2)$
$=8\partial_a\partial_{a'}$ □

Proof: $tr[\gamma_aQ\gamma_{a'}Q]=tr[\gamma_a(m-\gamma_{a_1}\partial^{a_1})\gamma_{a'}(m-\gamma_{a_2}\partial^{a_2})]$
$=m^2tr(\gamma_a\gamma_{a'})+tr(\gamma_a\gamma_{a_1}\gamma_{a'}\gamma_{a_2})\partial^{a_1}\partial^{a_2}$
$=4m^2\delta_{aa'}+4(\delta_{aa_1}\delta_{a'a_2}-\delta_{aa'}\delta_{a_1a_2}+\delta_{aa_2}\delta_{a_1a'})\partial^{a_1}\partial^{a_2}$
$=4m^2\delta_{aa'}+4(2\partial_a\partial_{a'}-\delta_{aa'}m^2)$
$=8\partial_a\partial_{a'}$ □

The potential commutation rule can be calculated from the field commutation rule through the Q product. n principle, it can be used to strictly prove the Behrends-Frontsdal conjecture formula. But it is very cumbersome and difficult to use. In fact, it is still difficult to prove the Behrends-Frontsdal conjecture formula.

Chapter18 Quantization of Non Relativistic Particles

1 Fourier analysis technique and plane wave solutions expansion [37]

1.1 Fourier expansion of wave function

Basic idea: The wave function is completely and uniquely expanded according to Fourier analysis. The equations and constraints are considered as conditions for selecting wave functions. The final result is a complete plane wave solutions that conforms to the equations and constraints.

Def. 1.1.1. $k \equiv (\vec{k}, iE), x \equiv (\vec{r}, it), d^4x \equiv d^3\vec{r}dt, \omega_k \equiv \sqrt{\vec{k}^2 + m^2} > 0, p_a(\omega_k) \equiv (\vec{k}, i\omega_k)_a$

Def. 1.1.2. $\int\limits_{\vec{r}=-\infty}^{+\infty} \equiv \int\limits_{r_x=-\infty}^{+\infty} \int\limits_{r_y=-\infty}^{+\infty} \int\limits_{r_z=-\infty}^{+\infty}, \quad \int\limits_{x=-\infty}^{+\infty} \equiv \int\limits_{r_x=-\infty}^{+\infty} \int\limits_{r_y=-\infty}^{+\infty} \int\limits_{r_z=-\infty}^{+\infty} \int\limits_{t=-\infty}^{+\infty}$

Def. 1.1.3. $\int\limits_{\vec{k}=-\infty}^{+\infty} \equiv \int\limits_{p_x=-\infty}^{+\infty} \int\limits_{p_y=-\infty}^{+\infty} \int\limits_{p_z=-\infty}^{+\infty}, \quad \int\limits_{k=-\infty}^{+\infty} \equiv \int\limits_{p_x=-\infty}^{+\infty} \int\limits_{p_y=-\infty}^{+\infty} \int\limits_{p_z=-\infty}^{+\infty} \int\limits_{E=-\infty}^{+\infty}$

Fourier expansion of the wave function:

$$\phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{k}=-\infty}^{+\infty} \int\limits_{E=-\infty}^{+\infty} \Phi(\vec{k}, E) e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k}dE, \Phi(\vec{k}, E) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{r}=-\infty}^{+\infty} \int\limits_{t=-\infty}^{+\infty} \phi(\vec{r}, t) e^{-i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{r}dt$$

The Fourier expansion compact form of the wave function:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int\limits_{k=-\infty}^{+\infty} \Phi(k) e^{ik\cdot x} d^4k \Leftrightarrow \Phi(k) = \frac{1}{(2\pi)^{3/2}} \int\limits_{x=-\infty}^{+\infty} \phi(x) e^{-ik\cdot x} d^4x \tag{18.1}$$

1.2 Fourier expansion of wave function and Lorentz covariance

Def. 1.2.1. $\Phi(k) \equiv \frac{1}{(2\pi)^{3/2}} \int\limits_{x=-\infty}^{+\infty} \phi(x) e^{-ik\cdot x} d^4x, \Phi'(k') \equiv \frac{1}{(2\pi)^{3/2}} \int\limits_{x'=-\infty}^{+\infty} \phi'(x') e^{-ik'\cdot x'} d^4x'$

The above two expressions are mathematically just two arbitrary definitions. It can have arbitrary mathematical meanings, but it can be given explicit physical meanings. Think of the former as an expression in the reference system O and the latter as an expression in the reference system O'. Specifically, consider x as the expression of space-time coordinates in the reference system O. Consider k as an expression of four dimensional momentum in the reference system O. Consider $\phi(x)$ as an expression of the space-time wave function in the reference system O. Consider $\Phi(k)$ as an expression of the momentum phase spatial wave function in the reference system O. consider x' as the expression of space-time coordinates in the reference system O'. Consider k' as an expression of four dimensional momentum in the reference system O'. Consider $\phi(x')$ as an expression of the space-time wave function in the reference system O'. Consider $\Phi(k')$ as an expression of the momentum phase spatial wave function in the reference system O'. The physical quantity in the reference system O' is associated with the corresponding physical quantity in the reference system O through the Lorentz transformation. In this connection, $k \cdot x, d^4x$ are represented as scalars. ,$\phi(x), \Phi(k)$ are represented as covariates.

Thm. 1.2.1. $\phi'(e^{\varepsilon}x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\phi(x) \Leftrightarrow \Phi'(e^{\varepsilon}k) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k)$

Proof: $\phi'(e^{\varepsilon}x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\phi(x)$

$\Leftrightarrow \frac{i}{(2\pi)^{3/2}} \int\limits_{x=-\infty}^{+\infty} \phi'(e^{\varepsilon}x) e^{-ik\cdot x} d^4x = \frac{i}{(2\pi)^{3/2}} \int\limits_{x=-\infty}^{+\infty} e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\phi(x) e^{-ik\cdot x} d^4x$

$\Leftrightarrow \frac{i}{(2\pi)^{3/2}} \int\limits_{e^{-\varepsilon}x'=-\infty}^{+\infty} \phi'(x') e^{-ike^{-\varepsilon}\cdot x'} d^4x' = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k)$

$\Leftrightarrow \frac{i}{(2\pi)^{3/2}} \int\limits_{x'=-\infty}^{+\infty} \phi'(x') e^{-ike^{-\varepsilon}\cdot x'} d^4x' = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k)$

$\Leftrightarrow \Phi'(e^{\varepsilon}k) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k)$ □

The above theorem shows: If the spatiotemporal wave function is a covariate, then its Fourier expansion coefficients are all covariates of the same type, and vice versa. The above conclusion is mathematically equivalent to make a variable substitution $x' = e^{\varepsilon}x, k' = e^{\varepsilon}k$, and meet $\Phi'(e^{\varepsilon}x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(x)$. This transformation is just the Lorentz transformation.

1.3 Lorentz covariance of special functions

Cor. 1.3.1. $\int\limits_{\vec{k}=-\infty}^{+\infty} \frac{1}{\omega}d^3\vec{k} = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \frac{1}{2\omega}[\delta(E-\omega)+\delta(E+\omega)]d^3\vec{k}dE = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \delta(E^2-\omega^2)d^3\vec{k}dE(It's\ a\ scalar.)$

Get an important mathematical skill: $\frac{1}{\omega}d^3\vec{k} = \delta(E^2-\omega^2)d^3\vec{k}dE$ **is a scalar. Is there a more intuitive proof?**

Cor. 1.3.2. $\int\limits_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega}d^3\vec{k} = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \frac{1}{2\omega}\delta(E-\omega)d^3\vec{k}dE = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=0}^{+\infty} \delta(E^2-\omega^2)d^3\vec{k}dE(It's\ a\ scalar.)$

Cor. 1.3.3. $\int\limits_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega}d^3\vec{k} = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \frac{1}{2\omega}\delta(E+\omega)d^3\vec{k}dE = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{0} \delta(E^2-\omega^2)d^3\vec{k}dE(It's\ a\ scalar.)$

Cor. 1.3.4. $\delta(E^2-\omega^2) = \frac{1}{2\omega}[\delta(E-\omega)+\delta(E+\omega)]$

Cor. 1.3.5. $\delta(E^2-\omega^2)U(E) = \frac{1}{2\omega}\delta(E-\omega), \delta(E^2-\omega^2)U(-E) = \frac{1}{2\omega}\delta(E+\omega)$

Cor. 1.3.6. $\delta(E^2-\omega^2)U(E-\omega) = \frac{1}{2\omega}\delta(E-\omega), \delta(E^2-\omega^2)U(-E-\omega) = \frac{1}{2\omega}\delta(E+\omega)$

Cor. 1.3.7. $\delta(-k_ak^a-m^2)It's\ a\ scalar.$

Cor. 1.3.8. $\frac{1}{\omega}\delta(E-\omega), \frac{1}{\omega}\delta(E+\omega)It's\ a\ scalar.$

Cor. 1.3.9. $\delta^4(k'-k), \delta^4(x'-x)It's\ a\ scalar.$

Cor. 1.3.10. $d^4k, d^4x, \frac{1}{\omega}d^3\vec{k}It's\ a\ scalar.$

2 Particle conservation covariates

2.1 Conservation equations and conserved covariates of current sources

Cor. 2.1.1. $Q = \int\limits_{\vec{r}=-\infty}^{+\infty} \rho(\vec{r},t_0)d^3\vec{r}, \quad \partial_aJ^a(\vec{r},t) = 0 \Rightarrow Q = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \partial_a[J^a(\vec{r},t)U(t-t_0)]d^3\vec{r}dt$

Proof: $Q = \int\limits_{\vec{r}=-\infty}^{+\infty} \rho(\vec{r},t_0)d^3\vec{r} = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \rho(\vec{r},t)\delta(t-t_0)d^3\vec{r}dt$

$= \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \rho(\vec{r},t)\partial_tU(t-t_0)d^3\vec{r}dt = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} J^a(\vec{r},t)\partial_aU(t-t_0)d^3\vec{r}dt$

$\overset{\partial_aJ^a(\vec{r},t)=0}{=} \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \partial_a[J^a(\vec{r},t)U(t-t_0)]d^3\vec{r}dt$ □

Cor. 2.1.2. $P^a = \int\limits_{\vec{r}=-\infty}^{+\infty} T^{a0}(\vec{r},t_0)d^3\vec{r}, \quad \partial_bT^{ab}(\vec{r},t) = 0 \Rightarrow P^a = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \partial_b[T^{ab}(\vec{r},t)U(t-t_0)]d^3\vec{r}dt,$

Proof: $P^a = \int\limits_{\vec{r}=-\infty}^{+\infty} T^{a0}(\vec{r},t_0)d^3\vec{r} = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} T^{a0}(\vec{r},t)\delta(t-t_0)d^3\vec{r}dt$

$= \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} T^{a0}(\vec{r},t)\partial_tU(t-t_0)d^3\vec{r}dt = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} T^{ab}(\vec{r},t)\partial_bU(t-t_0)d^3\vec{r}dt$

$\overset{\partial_bT^{ab}(\vec{r},t)=0}{=} \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \partial_b[T^{ab}(\vec{r},t)U(t-t_0)]d^3\vec{r}dt,$ □

Cor. 2.1.3. $M^{ab} = \int\limits_{\vec{r}=-\infty}^{+\infty} J^{ab0}(\vec{r},t_0)d^3\vec{r}, \quad \partial_cJ^{abc}(\vec{r},t) = 0 \Rightarrow M^{ab} = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \partial_c[J^{abc}(\vec{r},t)U(t-t_0)]d^3\vec{r}dt$

Proof: $M^{ab} = \int\limits_{\vec{r}=-\infty}^{+\infty} J^{ab0}(\vec{r},t_0)d^3\vec{r} = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} J^{ab0}(\vec{r},t)\delta(t-t_0)d^3\vec{r}dt$

$= \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} J^{ab0}(\vec{r},t)\partial_tU(t-t_0)d^3\vec{r}dt = \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} J^{abc}(\vec{r},t)\partial_cU(t-t_0)d^3\vec{r}dt$

$\overset{\partial_cJ^{abc}(\vec{r},t)=0}{=} \int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \partial_c[J^{abc}(\vec{r},t)U(t-t_0)]d^3\vec{r}dt$ □

Math Skills: I only provide key skills here and not provide a complete proof. In Weinberg's book, using the conservation equation of current sources, the property of physical functions being zero at infinite distance in space, and the invariance of physical time under Lorentz transformation, Lorentz covariance of Q, P^a, M^{ab} can be proved from the above proposition.

3 Non relativistic particle

3.1 Plane wave solutions of complex scalar field equations

Complex scalar field equation: $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0 \Leftrightarrow (\nabla^2 - \partial_t^2 - m^2)\phi(\vec{r},t) = 0$ (18.2)

Thm. 3.1.1. $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0 \Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k},-\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$

Proof: $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0 \Rightarrow \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \phi(\vec{k},E)(-\vec{k}^2 + E^2 - m^2)e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE = 0$

$\Leftrightarrow \phi(\vec{k},E)(E^2 - \vec{k}^2 - m^2) = 0 \Leftrightarrow \phi(\vec{k},E) = a(\vec{k},E)\delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k},E)\delta_{E^2,\vec{k}^2+m^2}$

$\Rightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} [a(\vec{k},E)\delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k},E)\delta_{E^2,\vec{k}^2+m^2}]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} a(\vec{k},E)\delta(E^2 - \vec{k}^2 - m^2)e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$, Apparent Lorentz covariant

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \frac{1}{2\omega_k}a(\vec{k},E)[\delta(E-\omega_k) + \delta(E+\omega_k)]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \frac{1}{2\omega_k}a(\vec{k},E)[\delta(E-\omega_k) + \delta(E+\omega_k)]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(\vec{k},-\omega_k)e^{i(\vec{k}\cdot\vec{r}+\omega_k t)}]d^3\vec{k}$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k},-\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$ □

Here, a different approach is used than in ordinary books. Four-dimensional rather than three-dimensional Fourier expansion is used. Clearly showing the physical concepts of particles in and out of the shell. Lorentz covariance is also evident in it, and includes a new algebraic solution for Dirac function solutions. In the process of proof, we also saw the decomposition of positive and negative energy solutions. And the negative energy solution can be understood in two meanings: one is to understand the negative energy solution as a negative mass particle, and the other is to understand the negative energy solution as a positive mass particle. However, the negative energy solution should be understood as a reflected wave and the positive energy solution should be understood as an incident wave.

Cor. 3.1.1. $a'(e^{\varepsilon}[\vec{k},E])\delta(E^2 - \vec{k}^2 - m^2) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}a(\vec{k},E)\delta(E^2 - \vec{k}^2 - m^2)$
$\Rightarrow a'(e^{\varepsilon}[\vec{k},\omega_k]) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}a(\vec{k},\omega_k), a'(e^{\varepsilon}[\vec{k},-\omega_k]) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}a(\vec{k},-\omega_k)$

Cor. 3.1.2. $a(\vec{k},E)\delta(E^2 - \vec{k}^2 - m^2) = \frac{1}{2\omega_k}a(\vec{k},E)[\delta(E-\omega_k) + \delta(E+\omega_k)], |\vec{k}| << m$
$\approx \frac{1}{2(m+\frac{\vec{k}^2}{2m})}[a(\vec{k},m+\frac{\vec{k}^2}{2m})\delta(E-m-\frac{\vec{k}^2}{2m}) + a(\vec{k},-m-\frac{\vec{k}^2}{2m})\delta(E+m+\frac{\vec{k}^2}{2m})]$

Cor. 3.1.3. $\phi(\vec{r},t) \approx \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty} \frac{1}{2(m+\frac{\vec{k}^2}{2m})}[a(\vec{k},m+\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}e^{-imt} + a(-\vec{k},-m-\frac{\vec{k}^2}{2m})e^{-i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}e^{imt}]d^3\vec{k}$

From the above, it can be seen that under the non relativistic limit, the plane wave solutions of a complex scalar field is divided into two non relativistic positive and negative particles. They can exist simultaneously. This can be analyzed further. Can we prove that the positive and negative energy solutions are independently conserved?

3.2 Two non relativistic branches of complex scalar field equation

Thm. 3.2.1. $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0 \Rightarrow$ *Two branches:* $\begin{cases} \text{Positive energy solution:}(\frac{1}{2m}\nabla^2 + i\partial_t)\phi_+(\vec{r},t) = 0 \\ \text{Negative energy solution:}(\frac{1}{2m}\nabla^2 - i\partial_t)\phi_-(\vec{r},t) = 0 \end{cases}$

Thm. 3.2.2.
Positive energy solution: $(i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r},t) = 0 \Leftrightarrow$ *Negative energy solution:* $(-i\partial_t + \frac{1}{2m}\nabla^2)\phi^*(\vec{r},t) = 0$

3.3 Action of positive energy solution for Schrodinger equation and its Poisson bracket

Cor. 3.3.1. *Lagrangian density:* $\mathscr{L} = \frac{1}{2}[i\phi^*(\vec{r},t)\partial_t\phi(\vec{r},t) - i\phi(\vec{r},t)\partial_t\phi^*(\vec{r},t) - \frac{1}{m}\nabla\phi^*(\vec{r},t)\cdot\nabla\phi(\vec{r},t)]$
Lagrangian density: $\mathscr{L} = i\phi^*(\vec{r},t)\partial_t\phi(\vec{r},t) + \frac{1}{2m}\phi^*(\vec{r},t)\nabla^2\phi(\vec{r},t)$
Motion equation: $(i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r},t) = 0$

Cor. 3.3.2. *Canonical Variable:* $\pi(\vec{r},t) = \frac{\partial\mathscr{L}}{\partial\dot{\phi}} = i\phi^*(\vec{r},t)$

Cor. 3.3.3. *Hamiltonian density:* $\mathscr{H} = \frac{\partial\mathscr{L}}{\partial\dot{\phi}}\dot{\phi} - \mathscr{L} = -\frac{1}{2m}\phi^*(\vec{r},t)\nabla^2\phi(\vec{r},t) = \frac{i}{2m}\pi(\vec{r},t)\nabla^2\phi(\vec{r},t)$

Cor. 3.3.4. *Momentum density:* $\mathscr{P} = -\frac{\partial\mathscr{L}}{\partial\dot{\phi}}\nabla\phi = -i\phi^*(\vec{r},t)\nabla\phi(\vec{r},t) = -\pi(\vec{r},t)\nabla\phi(\vec{r},t)$

Cor. 3.3.5. *Lagrangian density:* $\mathscr{L}_H = \pi(\vec{r},t)\partial_t\phi(\vec{r},t) - \frac{i}{2m}\pi(\vec{r},t)\nabla^2\phi(\vec{r},t)$

The basic relationship between canonical variables:

Cor. 3.3.6. $\{\phi(\vec{r'},t),\phi(\vec{r},t)\}_p = 0, \{\pi(\vec{r'},t),\pi(\vec{r},t)\}_p = 0, \{\phi(\vec{r'},t),\pi(\vec{r},t)\}_p = \delta^3(\vec{r'}-\vec{r})$

Hamiltonian motion equation:

Cor. 3.3.7. $\begin{cases}\dot{\phi}(\vec{r},t) = \frac{i}{2m}\nabla^2\phi(\vec{r},t) = \{\phi(\vec{r},t),H\}_p \\ \dot{\pi}(\vec{r},t) = -\frac{i}{2m}\nabla^2\pi(\vec{r},t) = \{\pi(\vec{r},t),H\}_p\end{cases}$

Probability flow conservation equation and conservation charge:

Cor. 3.3.8. $i\partial_t[\phi^*(\vec{r},t)\phi(\vec{r},t)] + \frac{1}{2m}\nabla\cdot[\phi^*(\vec{r},t)\nabla\phi(\vec{r},t) - \phi(\vec{r},t)\nabla\phi^*(\vec{r},t)] = 0$
$\Rightarrow \dot{Q} = 0, Q = \int\limits_{\vec{r}=-\infty}^{+\infty}\phi^*(\vec{r},t)\phi(\vec{r},t)d^3\vec{r} \in R$

The existence of the above conserved quantities indicates the conservation of total probability. Probability interpretation has a mathematical foundation. But there can also be other explanations, such as electric charge. This is the connection and difference between mathematics and physics. There are several reasonable physical explanations for a clear mathematical conclusion.

3.4 Plane wave solutions for positive energy Schrodinger equation

The positive energy solution branch of Schrodinger equation:

Thm. 3.4.1. $(i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r},t) = 0 \Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}a(\vec{k},\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}d^3\vec{k}$

Proof: $(i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r},t) = 0 \Rightarrow \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\phi(\vec{k},E)(E-\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE = 0$

$\Leftrightarrow \phi(\vec{k},E)(E-\frac{\vec{k}^2}{2m}) = 0 \Leftrightarrow \phi(\vec{k},E) = a(\vec{k},E)\delta(E-\frac{\vec{k}^2}{2m}) + \phi_0(\vec{k},E)\delta_{E,\frac{\vec{k}^2}{2m}}$

$\Rightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}[a(\vec{k},E)\delta(E-\frac{\vec{k}^2}{2m}) + \phi_0(\vec{k},E)\delta_{E,\frac{\vec{k}^2}{2m}}]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}a(\vec{k},E)\delta(E-\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}a(\vec{k},\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}d^3\vec{k}$

$\Leftrightarrow a(\vec{k}) \equiv a(\vec{k},\frac{\vec{k}^2}{2m}) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{r}=-\infty}^{+\infty}\phi(\vec{r},t)e^{-i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}d^3\vec{r}$ □

Cor. 3.4.1. $\phi(-\vec{r},-t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}a(\vec{k})e^{-i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}d^3\vec{k}$

Cor. 3.4.2. $H = \int\limits_{\vec{k}=-\infty}^{+\infty}\frac{\vec{k}^2}{2m}a^*(\vec{k})a(\vec{k})d^3\vec{k}, \vec{P} = \int\limits_{\vec{k}=-\infty}^{+\infty}\vec{k}a^*(\vec{k})a(\vec{k})d^3\vec{k}, Q = \int\limits_{\vec{k}=-\infty}^{+\infty}a^*(\vec{k})a(\vec{k})d^3\vec{k}$

3.5 Action of negative energy solution for Schrodinger equation and its Poisson bracket

Cor. 3.5.1. *Lagrangian density:* $\mathscr{L}=\frac{1}{2}[-i\phi^*(\vec{r},t)\partial_t\phi(\vec{r},t)+i\phi(\vec{r},t)\partial_t\phi^*(\vec{r},t)-\frac{1}{m}\nabla\phi^*(\vec{r},t)\cdot\nabla\phi(\vec{r},t)]$
Lagrangian density: $\mathscr{L}=-i\phi^*(\vec{r},t)\partial_t\phi(\vec{r},t)+\frac{1}{2m}\phi^*(\vec{r},t)\nabla^2\phi(\vec{r},t)$
Motion equation: $(-i\partial_t+\frac{1}{2m}\nabla^2)\phi(\vec{r},t)=0$

Cor. 3.5.2. *Canonical Variable:* $\pi(\vec{r},t)=\frac{\partial\mathscr{L}}{\partial\dot{\phi}}=-i\phi^*(\vec{r},t)$

Cor. 3.5.3. *Hamiltonian density:* $\mathscr{H}=\frac{\partial\mathscr{L}}{\partial\dot{\phi}}\dot{\phi}-\mathscr{L}=-\frac{1}{2m}\phi^*(\vec{r},t)\nabla^2\phi(\vec{r},t)=-\frac{i}{2m}\pi(\vec{r},t)\nabla^2\phi(\vec{r},t)$

Cor. 3.5.4. *Momentum density:* $\mathscr{P}=-\frac{\partial\mathscr{L}}{\partial\dot{\phi}}\nabla\phi=i\phi^*(\vec{r},t)\nabla\phi(\vec{r},t)=-\pi(\vec{r},t)\nabla\phi(\vec{r},t)$

Cor. 3.5.5. *Lagrangian density:* $\mathscr{L}_H=\pi(\vec{r},t)\partial_t\phi(\vec{r},t)+\frac{i}{2m}\pi(\vec{r},t)\nabla^2\phi(\vec{r},t)$

The basic relationship between canonical variables:

Cor. 3.5.6. $\{\phi(\vec{r'},t),\phi(\vec{r},t)\}_p=0,\{\pi(\vec{r'},t),\pi(\vec{r},t)\}_p=0,\{\phi(\vec{r'},t),\pi(\vec{r},t)\}_p=\delta^3(\vec{r'}-\vec{r})$

Hamiltonian motion equation:

Cor. 3.5.7. $\begin{cases}\dot{\phi}(\vec{r},t)=-\frac{i}{2m}\nabla^2\phi(\vec{r},t)=\{\phi(\vec{r},t),H\}_p\\ \dot{\pi}(\vec{r},t)=\frac{i}{2m}\nabla^2\pi(\vec{r},t)=\{\pi(\vec{r},t),H\}_p\end{cases}$

Probability flow conservation equation and conservation charge:

Cor. 3.5.8. $-i\partial_t[\phi^*(\vec{r},t)\phi(\vec{r},t)]+\frac{1}{2m}\nabla\cdot[\phi^*(\vec{r},t)\nabla\phi(\vec{r},t)-\phi(\vec{r},t)\nabla\phi^*(\vec{r},t)]=0$
$\Rightarrow\dot{Q}=0,Q=\int\limits_{\vec{r}=-\infty}^{+\infty}\phi^*(\vec{r},t)\phi(\vec{r},t)d^3\vec{r}\in R$

The existence of the above conserved quantities indicates the conservation of total probability. Probability interpretation has a mathematical foundation. But there can also be other explanations, such as electric charge. This is the connection and difference between mathematics and physics. There are several reasonable physical explanations for a clear mathematical conclusion.

3.6 Plane wave solutions for negative energy Schrodinger equation

The negative energy solution branch of Schrodinger equation:

Thm. 3.6.1. $(-i\partial_t+\frac{1}{2m}\nabla^2)\phi(\vec{r},t)=0\Leftrightarrow\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}a(-\vec{k},-\frac{\vec{k}^2}{2m})e^{-i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}d^3\vec{k}$

Proof: $(-i\partial_t+\frac{1}{2m}\nabla^2)\phi(\vec{r},t)=0\Rightarrow\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\phi(\vec{k},E)(E+\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE=0$

$\Leftrightarrow\phi(\vec{k},E)(E+\frac{\vec{k}^2}{2m})=0\Leftrightarrow\phi(\vec{k},E)=a(\vec{k},E)\delta(E+\frac{\vec{k}^2}{2m})+\phi_0(\vec{k},E)\delta_{E,-\frac{\vec{k}^2}{2m}}$

$\Rightarrow\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}[a(\vec{k},E)\delta(E+\frac{\vec{k}^2}{2m})+\phi_0(\vec{k},E)\delta_{E,-\frac{\vec{k}^2}{2m}}]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}a(\vec{k},E)\delta(E+\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}a(\vec{k},-\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}+\frac{\vec{k}^2}{2m}t)}d^3\vec{k}$

$\Leftrightarrow\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}a(-\vec{k},-\frac{\vec{k}^2}{2m})e^{-i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}d^3\vec{k}$

$\Leftrightarrow a(\vec{k})\equiv a(-\vec{k},-\frac{\vec{k}^2}{2m})=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{r}=-\infty}^{+\infty}\phi(\vec{r},t)e^{i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}d^3\vec{r}$ □

Cor. 3.6.1. $H=\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{\vec{k}^2}{2m}a^*(\vec{k})a(\vec{k})d^3\vec{k},\vec{P}=\int\limits_{\vec{k}=-\infty}^{+\infty}\vec{k}a^*(\vec{k})a(\vec{k})d^3\vec{k},Q=\int\limits_{\vec{k}=-\infty}^{+\infty}a^*(\vec{k})a(\vec{k})d^3\vec{k}$

From the above, under the non relativistic limit, the positive and negative energy branchs for Schrodinger equation both describe a non relativistic particle. They can't describe two positive and negative particles simultaneously. If they appear in the same equation, they become zero. It makes no sense. And the conjugate solution of a positive energy branch is the solution of a negative energy

branch, and vice versa. The conjugation of positive particle characterizes an antiparticle. In addition from the proof of the above two theorems, it can be seen that the two branches can been uniformly expressed only by assuming that m can take a positive or negative value. Then the negative energy branch is same as a positive energy branch in form. $m > 0$ describes a positive branch, and $m < 0$ describes a negative branch. There are two understandings on negative energy solutions, one describe particles with a negative mass, and the other describe particles with a positive mass. Quantized solution with a negative mass is equivalent to positive energy solution.

4 Quadratic quantization of ± energy solutions for Schrodinger equation

4.1 Discussion on quadratic quantization of ± energy solutions for Schrodinger equation

Cor. 4.1.1. $H = \int\limits_{\vec{k}=-\infty}^{+\infty} \frac{\vec{k}^2}{2m} a^+(\vec{k})a(\vec{k})d^3\vec{k}, \vec{P} = \int\limits_{\vec{k}=-\infty}^{+\infty} \vec{k}a^+(\vec{k})a(\vec{k})d^3\vec{k}, Q = \hat{N} = \int\limits_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})d^3\vec{k}$

The above relationships do not rely on commutative or anti commutative relations, nor do rely on positive or negative energy solutions.

Def. 4.1.1. *Quantum equation:* $i\partial_t|\Psi >= H|\Psi >, |\Psi >$ *is a quantum wave function.*

Cor. 4.1.2. $-i\nabla|\Psi >= \vec{P}|\Psi >$

Understanding in terms of commutators: Describe non relativistic bosons.

Cor. 4.1.3. $\begin{cases} [\phi(\vec{r}', t), \phi(\vec{r}, t)] = 0 \\ [\phi^+(\vec{r}', t), \phi^+(\vec{r}, t)] = 0 \\ [\phi(\vec{r}', t), \phi^+(\vec{r}, t)] = \delta^3(\vec{r}' - \vec{r}) \end{cases} \Leftrightarrow \begin{cases} [a(\vec{k}'), a(\vec{k})] = 0 \\ [a^+(\vec{k}'), a^+(\vec{k})] = 0 \\ [a(\vec{k}'), a^+(\vec{k})] = \delta^3(\vec{k}' - \vec{k}) \end{cases}$

Understanding in terms of anti commutators: Describe non relativistic fermions.

Cor. 4.1.4. $\begin{cases} \{\phi(\vec{r}', t), \phi(\vec{r}, t)\} = 0 \\ \{\phi^+(\vec{r}', t), \phi^+(\vec{r}, t)\} = 0 \\ \{\phi(\vec{r}', t), \phi^+(\vec{r}, t)\} = \delta^3(\vec{r}' - \vec{r}) \end{cases} \Leftrightarrow \begin{cases} \{a(\vec{k}'), a(\vec{k})\} = 0 \\ \{a^+(\vec{k}'), a^+(\vec{k})\} = 0 \\ \{a(\vec{k}'), a^+(\vec{k})\} = \delta^3(\vec{k}' - \vec{k}) \end{cases}$

4.2 Quantum description of particles

Def. 4.2.1. $\hat{N} = \int\limits_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})d^3\vec{k}, P^a = \int\limits_{\vec{k}=-\infty}^{+\infty} k^a a^+(\vec{k})a(\vec{k})d^3\vec{k}$

Def. 4.2.2. *If* $|0 >\neq 0, a(\vec{k})|0 >= 0, \forall\vec{k}$*, then* $|0 >$ *is in a vacuum state or ground state.*

Cor. 4.2.1. $\hat{N}|0 >= 0$

Proof: $a(\vec{k})|0 >= 0, \forall\vec{k} \Rightarrow a^+(\vec{k})a(\vec{k})|0 >= 0, \forall\vec{k} \Rightarrow \int\limits_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})|0 > d^3\vec{k} = 0$

$\Rightarrow \int\limits_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})d^3\vec{k}|0 >= 0 \Rightarrow \hat{N}|0 >= 0$ □

Cor. 4.2.2. $[< 0|0 >] > 0,$ *normalization:* $< 0|0 >= 1$

5 Quantum description of bosons

5.1 Basic commutative relation of bosons

Def. 5.1.1. $[a(\vec{k}'), a(\vec{k})] = 0, [a^+(\vec{k}'), a^+(\vec{k})] = 0, [a(\vec{k}'), a^+(\vec{k})] = \delta^3(\vec{k}' - \vec{k})$

Def. 5.1.2. $\hat{N}(k) \equiv a^+(\vec{k})a(\vec{k}), k^a \equiv (\vec{k}, i\omega_k), \omega_k \equiv \frac{\vec{k}^2}{2m}$

5.2 Particle number operator properties of bosons

Cor. 5.2.1. $[\hat{N}, a(\vec{k})] = -a(\vec{k})$

Proof: $[\hat{N}, a(\vec{k})] = \int\limits_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}')a(\vec{k}')a(\vec{k}) - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}'$

$= \int\limits_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}')a(\vec{k})a(\vec{k}') - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}'$

$= \int\limits_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}'), a(\vec{k})]a(\vec{k}')d^3\vec{k}'$

$= \int\limits_{\vec{k}'=-\infty}^{+\infty} -\delta^3(\vec{k}' - \vec{k})a(\vec{k}')d^3\vec{k}' = -a(\vec{k})$ □

Cor. 5.2.2. $[\hat{N}, a(\vec{k})] = -a(\vec{k}) \Leftrightarrow [\hat{N}, a^{+}(\vec{k})] = a^{+}(\vec{k})$

Cor. 5.2.3. $[\hat{N}, a^{+}(\vec{k})a(\vec{k})] = 0, \forall \vec{k}$

Proof: $[\hat{N}, a^{+}(\vec{k})a(\vec{k})] = \hat{N}a^{+}(\vec{k})a(\vec{k}) - a^{+}(\vec{k})a(\vec{k})\hat{N}$
$= \hat{N}a^{+}(\vec{k})a(\vec{k}) - a^{+}(\vec{k})\hat{N}a(\vec{k}) + a^{+}(\vec{k})\hat{N}a(\vec{k}) - a^{+}(\vec{k})a(\vec{k})\hat{N}$
$= [\hat{N}, a^{+}(\vec{k})]a(\vec{k}) + a^{+}(\vec{k})[\hat{N}, a(\vec{k})] = a^{+}(\vec{k})a(\vec{k}) - a^{+}(\vec{k})a(\vec{k}) = 0$ □

The following conclusions can be proved by mathematical induction.

Cor. 5.2.4. $[\hat{N}, a^{n}(\vec{k})] = -na^{n}(\vec{k}), [\hat{N}, a^{+n}(\vec{k})] = na^{+n}(\vec{k})$

Cor. 5.2.5. $[\hat{N}, \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)] = (\sum_{i=1}^{\infty} n_i) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)$

Cor. 5.2.6. $[\hat{N}, a^{+n}(\vec{k})a^{n}(\vec{k})] = 0, [\hat{N}, a^{n}(\vec{k})a^{+n}(\vec{k})] = 0, [\hat{N}, \hat{N}^{n}(k)] = 0, [\hat{N}, [a(\vec{k})a^{+}(\vec{k})]^{n}] = 0$

5.3 Energy and momentum operator properties of bosons

Cor. 5.3.1. $[P^{a}, a(\vec{k})] = -k^{a}a(\vec{k})$

Proof: $[P^{a}, a(\vec{k})] = \int_{\vec{k}'=-\infty}^{+\infty} k'^{a}[a^{+}(\vec{k}')a(\vec{k}')a(\vec{k}) - a(\vec{k})a^{+}(\vec{k}')a(\vec{k}')]d^{3}\vec{k}'$

$= \int_{\vec{k}'=-\infty}^{+\infty} k'^{a}[a^{+}(\vec{k}')a(\vec{k})a(\vec{k}') - a(\vec{k})a^{+}(\vec{k}')a(\vec{k}')]d^{3}\vec{k}'$

$= \int_{\vec{k}'=-\infty}^{+\infty} k'^{a}[a^{+}(\vec{k}'), a(\vec{k})]a(\vec{k}')d^{3}\vec{k}'$

$= \int_{\vec{k}'=-\infty}^{+\infty} -k'^{a}\delta^{3}(\vec{k}' - \vec{k})a(\vec{k}')d^{3}\vec{k}' = -\vec{k}^{a}a(\vec{k})$ □

Cor. 5.3.2. $[P^{a}, a(\vec{k})] = -k^{a}a(\vec{k}) \Leftrightarrow [P^{a}, a^{+}(\vec{k})] = \vec{k}^{a}a^{+}(\vec{k})$

Cor. 5.3.3. $[P^{a}, a^{+}(\vec{k})a(\vec{k})] = 0, \forall \vec{k}$

Proof: $[P^{a}, a^{+}(\vec{k})a(\vec{k})] = P^{a}a^{+}(\vec{k})a(\vec{k}) - a^{+}(\vec{k})a(\vec{k})P^{a}$
$= P^{a}a^{+}(\vec{k})a(\vec{k}) - a^{+}(\vec{k})P^{a}a(\vec{k}) + a^{+}(\vec{k})P^{a}a(\vec{k}) - a^{+}(\vec{k})a(\vec{k})P^{a}$
$= [P^{a}, a^{+}(\vec{k})]a(\vec{k}) + a^{+}(\vec{k})[P^{a}, a(\vec{k})] = k^{a}a^{+}(\vec{k})a(\vec{k}) - k^{a}a^{+}(\vec{k})a(\vec{k}) = 0$ □

The following conclusions can be proved by mathematical induction.

Cor. 5.3.4. $[P^{a}, a^{n}(\vec{k})] = -nk^{a}a(\vec{k}) \Leftrightarrow [P^{a}, a^{+n}(\vec{k})] = nk^{a}a^{+n}(\vec{k})$

Cor. 5.3.5. $[P^{a}, \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)] = (\sum_{i=1}^{\infty} n_i k_i^{a}) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)$

Cor. 5.3.6. $[P^{a}, \hat{N}] = 0$

5.4 General construction of boson solutions

Def. 5.4.1. $a(\vec{k}, t) \equiv a(k)e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)}, a^{+}(k, t) \equiv a^{+}(k)e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)}$

Cor. 5.4.1. $\dot{a}(\vec{k}, t) = i[H, a(\vec{k}, t)], \dot{a}^{+}(\vec{k}, t) = i[H, a^{+}(\vec{k}, t)]$

Thm. 5.4.1. $i\partial_t|\Psi >= H|\Psi > \Rightarrow i\partial_t[a(\vec{k}, t)|\Psi >] = Ha(\vec{k}, t)|\Psi >$

Thm. 5.4.2. $i\partial_t|\Psi >= H|\Psi > \Rightarrow i\partial_t[a^{+}(\vec{k}, t)|\Psi >] = Ha^{+}(\vec{k}, t)|\Psi >$

5.5 Construction I of boson quantum states

Cor. 5.5.1. $i\partial_t|0 >= H|0 > \Rightarrow i\partial_t[\prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i, t)|0 >] = H\prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i, t)|0 >, n_i = 0, 1, 2, \cdots, \infty$

Def. 5.5.1. $|n_1, n_2, \cdots, n_\infty > \equiv \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)|0 >, n_i = 0, 1, 2, \cdots, \infty$

Def. 5.5.2. $|n_1, n_2, \cdots, n_\infty, t > \equiv \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i, t)|0 >= exp\{i\sum_{i=1}^{\infty} n_i(\vec{k}_i \cdot \vec{r} - \frac{\vec{k}_i^2}{2m}t)\}|n_1, n_2, \cdots, n_\infty >$

The meaning of this quantum state is as follows: every mathematical point in the momentum space $\vec{k}_i$ corresponds to an energy level. This energy level is filled with n_i particles with mass m and momentum $\vec{k}_i$. There are infinite similar energy levels. Because the total number of physical particles is limited, the number of particles at many energy levels is zero. This quantum state represents a distribution of multiple particles in the momentum space, and the total number of particles in each quantum state is variable. It is a common eigenstate of particle number operators and energy momentum operators.

Cor. 5.5.2. $\begin{cases} \hat{N}|n_1, n_2, \cdots, n_\infty, t>=\sum\limits_{i=1}^{\infty} n_i |n_1, n_2, \cdots, n_\infty, t> \\ H|n_1, n_2, \cdots, n_\infty, t>=\sum\limits_{i=1}^{\infty} n_i \frac{\vec{k}_i^2}{2m} |n_1, n_2, \cdots, n_\infty, t> \\ \vec{P}|n_1, n_2, \cdots, n_\infty, t>=\sum\limits_{i=1}^{\infty} n_i \vec{k}_i |n_1, n_2, \cdots, n_\infty, t> \end{cases}$

Cor. 5.5.3. $\phi(-\vec{r}, -t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{k}=-\infty}^{+\infty} a(\vec{k}) e^{-i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)} d^3\vec{k}, \phi^+(-\vec{r}, -t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k}) e^{i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)} d^3\vec{k}$

Cor. 5.5.4. $i\partial_t|\Psi>=H|\Psi>\Rightarrow i\partial_t[\hat{\phi}(-\vec{r}, -t)|\Psi>] = H[\hat{\phi}(-\vec{r}, -t)|\Psi>]$

Cor. 5.5.5. $i\partial_t|\Psi>=H|\Psi>\Rightarrow i\partial_t[\hat{\phi}^+(-\vec{r}, -t)|\Psi>] = H[\hat{\phi}^+(-\vec{r}, -t)|\Psi>]$

5.6 Construction II of boson quantum states

Def. 5.6.1.

$|n>=[n!\int\limits_{\vec{k}=-\infty}^{+\infty} |F(\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_n)|^2 d^n\vec{k}]^{-\frac{1}{2}} \int\limits_{\vec{k}=-\infty}^{+\infty} d^n\vec{k} F(\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_n) a^+(\vec{k}_1) a^+(\vec{k}_2) \cdots a^+(\vec{k}_n)|0>$

$F(\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_n)$ *Fully symmetric for bosons and fully antisymmetric for fermions.*

Def. 5.6.2. $|n, t>=exp\{i\sum\limits_{i=1}^{n}(\vec{k}_i\cdot\vec{r}-\frac{\vec{k}_i^2}{2m}t)\}|n>$

The meaning of this quantum state is as follows: n particles filled into a mixed momentum state of all possible distributions in momentum space. The total number of particles in this quantum state is fixed. The momentum of each particle may take any value. It is an eigenstate of the particle number operator, but not an eigenstate of the energy momentum operator.

Cor. 5.6.1. $i\partial_t|0>=H|0>\Rightarrow i\partial_t|n, t>=H|n, t>$

Cor. 5.6.2. $\hat{N}|n>=n|n>, \hat{N}|n, t>=n|n, t>, <n|n>=1$

5.7 Correspondence between boson coordinate space and momentum space

Cor. 5.7.1. $[P^a, a(\vec{k})] = -k^a a(\vec{k}) \Leftrightarrow [P^a, \phi(\vec{r}, t)] = i\partial^a\phi(\vec{r}, t)$

Cor. 5.7.2. $[P^a, a^+(\vec{k})] = k^a a^+(\vec{k}) \Leftrightarrow [P^a, \phi^+(\vec{r}, t)] = -i\partial^a\phi^+(\vec{r}, t)$

5.8 Existence of boson quantum states

Cor. 5.8.1. $<0|0>=1\Rightarrow|0>\neq 0$

Cor. 5.8.2. $a^n(\vec{k})a^{+n}(\vec{k}) = a^{+n}(\vec{k})a^n(\vec{k}) + n!\delta^n(0), n\geq 1$

Cor. 5.8.3. $\prod\limits_{i=1}^{\infty} a^{n_i}(\vec{k}_i)\prod\limits_{i=1}^{\infty} a^{+n_i}(\vec{k}_i) = \prod\limits_{i=1, n_i\geq 1}^{\infty}[a^{+n_i}(\vec{k})a^{n_i}(\vec{k}) + n_i!\delta^{n_i}(0)]$

Cor. 5.8.4. $\prod\limits_{i=1}^{\infty} a^{n_i}(\vec{k}_i)\prod\limits_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)|0>=\prod\limits_{i=1, n_i\geq 1}^{\infty} n_i!\delta^{n_i}(0)|0>\neq 0, \forall\vec{k}_1\neq\vec{k}_2\neq\cdots\neq\vec{k}_n$

6 Quantum description of fermions

6.1 Basic commutative relation of fermions

Def. 6.1.1. $\{a(\vec{k}'), a(\vec{k})\} = 0, \{a^+(\vec{k}'), a^+(\vec{k})\} = 0, \{a(\vec{k}'), a^+(\vec{k})\} = \delta^3(\vec{k}'-\vec{k})$

Cor. 6.1.1. $a(\vec{k}')a(\vec{k}) = 0, a^+(\vec{k}')a^+(\vec{k}) = 0$

6.2 Particle number operator properties of fermions

Cor. 6.2.1. $[\hat{N}, a(\vec{k})] = -a(\vec{k})$

Proof: $[\hat{N}, a(\vec{k})] = \int\limits_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}')a(\vec{k}')a(\vec{k}) - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}'$
$= \int\limits_{\vec{k}'=-\infty}^{+\infty} [-a^+(\vec{k}')a(\vec{k})a(\vec{k}') - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}'$
$= \int\limits_{\vec{k}'=-\infty}^{+\infty} -\{a^+(\vec{k}'), a(\vec{k})\}a(\vec{k}')d^3\vec{k}'$
$= \int\limits_{\vec{k}'=-\infty}^{+\infty} -\delta^3(\vec{k}' - \vec{k})a(\vec{k}')d^3\vec{k}' = -a(\vec{k})$ □

Cor. 6.2.2. $[\hat{N}, a(\vec{k})] = -a(\vec{k}) \Leftrightarrow [\hat{N}, a^+(\vec{k})] = a^+(\vec{k})$

Cor. 6.2.3. $[\hat{N}, a^+(\vec{k})a(\vec{k})] = 0, \forall \vec{k}$

Proof: $[\hat{N}, a^+(\vec{k})a(\vec{k})] = \hat{N}a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})a(\vec{k})\hat{N}$
$= \hat{N}a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})\hat{N}a(\vec{k}) + a^+(\vec{k})\hat{N}a(\vec{k}) - a^+(\vec{k})a(\vec{k})\hat{N}$
$= [\hat{N}, a^+(\vec{k})]a(\vec{k}) + a^+(\vec{k})[\hat{N}, a(\vec{k})] = a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})a(\vec{k}) = 0$ □

The following conclusions can be proved by mathematical induction.

Cor. 6.2.4. $\hat{N}a^+(\vec{k}_1)a^+(\vec{k}_2)\cdots a^+(\vec{k}_n)|0>= na^+(\vec{k}_1)a^+(\vec{k}_2)\cdots a^+(\vec{k}_n)|0>, \forall \vec{k}_1 \neq \vec{k}_2 \neq \cdots \neq \vec{k}_n$

Cor. 6.2.5. $\hat{N}a^+(\vec{k}_1)a^+(\vec{k}_2)\cdots a^+(\vec{k}_n)|0>= na^+(\vec{k}_1)a^+(\vec{k}_2)\cdots a^+(\vec{k}_n)|0>, \forall \vec{k}_1\vec{k}_2\cdots\vec{k}_n$

Cor. 6.2.6. $[\hat{N}, a^n(\vec{k})] = -na^n(\vec{k}), [\hat{N}, a^{+n}(\vec{k})] = na^{+n}(\vec{k}), n = 0, 1$

Cor. 6.2.7. $[\hat{N}, \prod\limits_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)] = (\sum\limits_{i=1}^{\infty} n_i) \prod\limits_{i=1}^{\infty} a^{+n_i}(\vec{k}_i), n_i = 0, 1$

Cor. 6.2.8. $[\hat{N}, a^{+n}(\vec{k})a^n(\vec{k})] = 0, [\hat{N}, a^n(\vec{k})a^{+n}(\vec{k})] = 0, [\hat{N}, \hat{N}^n(k)] = 0, [\hat{N}, [a(\vec{k})a^+(\vec{k})]^n] = 0$

6.3 Existence of fermion quantum states

Cor. 6.3.1. $<0|0>= 1 \Rightarrow |0>\neq 0$

Cor. 6.3.2. $a(\vec{k})a^+(\vec{k}) = -a^+(\vec{k})a(\vec{k}) + \delta(0), n = 0, 1$

Cor. 6.3.3. $\prod\limits_{i=1}^{\infty} a^{n_i}(\vec{k}_i) \prod\limits_{i=1}^{\infty} a^{+n_i}(\vec{k}_i) = \prod\limits_{i=1,n_i=1}^{\infty} [a^{+n_i}(\vec{k})a^{n_i}(\vec{k}) + n_i!\delta^{n_i}(0)]$

Cor. 6.3.4. $\prod\limits_{i=1}^{\infty} a^{n_i}(\vec{k}_i) \prod\limits_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)|0>= \pm \prod\limits_{i=1,n_i=1}^{\infty} n_i!\delta^{n_i}(0)|0>\neq 0, \forall \vec{k}_1 \neq \vec{k}_2 \neq \cdots \neq \vec{k}_n$

Cor. 6.3.5. $a^+(\vec{k}_1)a^+(\vec{k}_2)\cdots a^+(\vec{k}_n)|0>\neq 0, \forall \vec{k}_1 \neq \vec{k}_2 \neq \cdots \neq \vec{k}_n$

6.4 Occupancy representation

Def. 6.4.1. $|n_1n_2\cdots n_k\cdots>= |n_1>\otimes|n_2>\otimes\cdots\otimes|n_k>\otimes\cdots, <n_1n_2\cdots n_k\cdots| = |n_1n_2\cdots n_k\cdots>^+$

Cor. 6.4.1. $|n_1n_2\cdots n_k\cdots>= \frac{1}{n_1!n_2!\cdots n_k!\cdots}(a_1^+)^{n_1} \otimes (a_2^+)^{n_2}\cdots\otimes(a_k^+)^{n_k}\cdots|0_10_2\cdots0_k\cdots>$

Cor. 6.4.2. *Orthogonality:* $<n_1'n_2'\cdots n_k'\cdots|n_1n_2\cdots n_k\cdots>= \delta n_1', n_1\delta n_2', n_2\cdots\delta n_k', n_k\cdots$

Cor. 6.4.3. *Completeness:* $\sum|n_1n_2\cdots n_k\cdots><n_1n_2\cdots n_k\cdots| = 1, \sum|n_k><n_k| = 1$

Chapter19 Quantization of Majorana Particle and Neutrino

Self comment: Because most books on quantum field theory do not discuss the quantization of Majorana particles and neutrinos in detail and I have never found the corresponding content. In order to make up for this shortcoming, I decided to derive calculations by myself. In this chapter, I first give the quantization of Dirac particles by using Lorentz push transformation. And then I give detailed quantization details of Majorana particles and neutrinos.

1 Application of Lorentz boost transform: Solving plane waves of Dirac equation [25, 26]

1.1 Lorentz boost transformation of Dirac equation under general representation

Dirac equation:

Def. 1.1.1. $(\gamma^a \partial_a + m)\psi = 0, \gamma^a p_a = \gamma \cdot \vec{p} + \gamma_4 iE, E = \sqrt{\vec{p}^2 + m^2} > 0$

Dirac spinor boost transformation:

Cor. 1.1.1. $D_{\vec{v}} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)} = \frac{1+\gamma_v - i\gamma_v \vec{v}\cdot\vec{\gamma}\gamma_4}{\sqrt{2(\gamma_v+1)}} = \frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}} = \frac{m-i\gamma^a p_a \gamma_4}{\sqrt{2m(E+m)}}$

Properties of Dirac spinor Lorentz boost transformation factor:

Pro. 1.1.1. $(m - i\gamma^a p_a \gamma_4)^+ = (m - i\gamma^a p_a \gamma_4)$

Pro. 1.1.2. $(m - i\gamma^a p_a \gamma_4)^+ \gamma_4 (m - i\gamma^a p_a \gamma_4) = 2m(E + m)\gamma_4$

Pro. 1.1.3. $(E + m + i\vec{p} \cdot \vec{\gamma}\gamma_4)(E + m - i\vec{p} \cdot \vec{\gamma}\gamma_4) = 2m(E + m)$

Pro. 1.1.4. $(m - i\gamma^a p_a \gamma_4)^+ (m - i\gamma^a p_a \gamma_4) = 2(E + m)(E - i\vec{p} \cdot \vec{\gamma}\gamma_4)$

Pro. 1.1.5. $(m + i\gamma^a p_a \gamma_4)^+ (m - i\gamma^a p_a \gamma_4) = 2m^2 - 2E(E - i\vec{p} \cdot \vec{\gamma}\gamma_4)$

1.2 Static and kinematic solutions of Dirac equation under general representation

Static electron solution:

Cor. 1.2.1. $\partial_{t_0}\psi(\vec{0}) = -im\gamma_4\psi(\vec{0}) \Leftrightarrow \psi(\vec{0}) = e^{-i\gamma_4 m t_0}\psi_0, \forall\psi_0$

Momentum $\vec{p}$ electron solution:

Cor. 1.2.2. $\psi(\vec{p}) = \frac{m-i\gamma^a p_a\gamma_4}{\sqrt{2m(E+m)}} e^{i\gamma_4(\vec{p}\cdot\vec{r}-Et)}\psi_p = \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_4}{E+m})e^{i\gamma_4(\vec{p}\cdot\vec{r}-Et)}\psi_p, \bar{\psi}(\vec{p})\psi(\vec{p}) = \bar{\psi}_p\psi_p$

1.3 Dirac Lorentz boost transform and plane wave solutions under special representation

1.3.1 Lorentz boost transformation of Dirac equation under special representation

Special representation: $(\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z), -\varsigma I \otimes \sigma_x]$

Cor. 1.3.1. $\gamma^a p_a = i \begin{bmatrix} \varsigma E & -\sigma\cdot\vec{p} \\ \sigma\cdot\vec{p} & -\varsigma E \end{bmatrix}, E = \sqrt{\vec{p}^2 + m^2} > 0$

Cor. 1.3.2. $S_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
$S_y(\sigma_x, \sigma_y, \sigma_z)S_y^+ = (-\sigma_z, \sigma_y, \sigma_x), S_y^+(\sigma_x, \sigma_y, \sigma_z)S_y = (\sigma_z, \sigma_y, -\sigma_x)$
$I \otimes S_y[(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z), -\varsigma I \otimes \sigma_x]I \otimes S_y^+ = [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z]$
$I \otimes S_y^+[(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_x]I \otimes S_y = [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z), -\varsigma I \otimes \sigma_x]$

Dirac spinor boost transformation:

Cor. 1.3.3. $D_{\vec{v}} = \frac{m-i\gamma^a p_a\gamma_4}{\sqrt{2m(E+m)}} = \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}} = \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & \varsigma\sigma\cdot\vec{p} \\ \varsigma\sigma\cdot\vec{p} & E+m \end{bmatrix}$

1.3.2 Static and kinematic solutions of Dirac equation under special representation
Dirac equation:

Def. 1.3.1. $(\gamma^a\partial_a + m)\psi = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z)$

Static electron solution:

Cor. 1.3.4. $\partial_{t_0}\psi(\vec{0}) = -im\gamma_4\psi(\vec{0}) \Leftrightarrow \psi(\vec{0}) = e^{-i\gamma_4 m t_0}\psi_0 = \begin{bmatrix}\xi_0 e^{-i\varsigma m t_0}\\ \eta_0 e^{i\varsigma m t_0}\end{bmatrix}, \forall \xi_0, \eta_0$

Momentum $\vec{p}$ electron solution:

Cor. 1.3.5.
$\psi(\vec{p}) = \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_4}{E+m})e^{i\gamma_4(\vec{p}\cdot\vec{r}-Et)}\psi_p = \sqrt{\frac{E+m}{2m}}[\begin{bmatrix}\xi(\vec{p})\\ \frac{\varsigma\sigma\cdot\vec{p}}{E+m}\xi(\vec{p})\end{bmatrix}e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + \begin{bmatrix}\frac{\varsigma\sigma\cdot\vec{p}}{E+m}\eta(\vec{p})\\ \eta(\vec{p})\end{bmatrix}e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}], \forall\xi(\vec{p}), \eta(\vec{p})$

1.3.3 $\vec{p}$ plane wave solutions of Dirac equation along z-axis under special representation
$\vec{p}$-momentum plane wave solutions of Dirac equation expanded by z-spin eigenstates:

Cor. 1.3.6.
$\psi(\vec{p}) = \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\{[a_\varsigma(\vec{p},\frac{1}{2})\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + a_\varsigma(\vec{p},-\frac{1}{2})\begin{bmatrix}0\\1\\0\\0\end{bmatrix}]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + [b_\varsigma^+(\vec{p},\frac{1}{2})\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + b_\varsigma^+(\vec{p},-\frac{1}{2})\begin{bmatrix}0\\0\\0\\1\end{bmatrix}]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}\}$

Cor. 1.3.7. $\psi(\vec{p}) = \sum_h[a_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + b_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$*(It also holds true under general representation.)*

1.3.4 Spin basis of Dirac equation under special representation

Def. 1.3.2. $\xi_+ = \eta_+ := \begin{bmatrix}1\\0\end{bmatrix}, \xi_- = \eta_- := \begin{bmatrix}0\\1\end{bmatrix}$

Four spin bases

Def. 1.3.3. $u_\varsigma(\vec{p},\frac{1}{2}) \equiv \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{m-i\varsigma\gamma^a p_a}{\sqrt{2m(E+m)}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\xi_+\\ \frac{\varsigma\sigma\cdot\vec{p}}{E+m}\xi_+\end{bmatrix}$

Def. 1.3.4. $u_\varsigma(\vec{p},-\frac{1}{2}) = \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{m-i\varsigma\gamma^a p_a}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\xi_-\\ \frac{\varsigma\sigma\cdot\vec{p}}{E+m}\xi_-\end{bmatrix}$

Def. 1.3.5. $v_\varsigma(\vec{p},\frac{1}{2}) = \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \frac{m+i\varsigma\gamma^a p_a}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{\varsigma\sigma\cdot\vec{p}}{E+m}\eta_+\\ \eta_+\end{bmatrix}$

Def. 1.3.6. $v_\varsigma(\vec{p},-\frac{1}{2}) = \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \frac{m+i\varsigma\gamma^a p_a}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \sqrt{\frac{E+m}{2m}}\begin{bmatrix}\frac{\varsigma\sigma\cdot\vec{p}}{E+m}\eta_-\\ \eta_-\end{bmatrix}$

Cor. 1.3.8. $u_\varsigma(\vec{p},h) \equiv -\varsigma\gamma_5 v_\varsigma(\vec{p},h), u_\varsigma(\vec{p},h) \equiv i\gamma_2\gamma_4\gamma_5 u_\varsigma(\vec{p},-h), v_\varsigma(\vec{p},h) \equiv i\gamma_2\gamma_4\gamma_5 v_\varsigma(\vec{p},-h)$

Cor. 1.3.9. $(E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x)\left[\begin{smallmatrix}I&0\\0&0\end{smallmatrix}\right](E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x)^+\varsigma I\otimes\sigma_z = (E+m)(\varsigma m - i\gamma^a p_a)$

Cor. 1.3.10. $(E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x)\left[\begin{smallmatrix}0&0\\0&I\end{smallmatrix}\right](E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x)^+\varsigma I\otimes\sigma_z = (E+m)(-\varsigma m - i\gamma^a p_a)$

1.3.5 Spin basis properties of Dirac equation under general representation

Cor. 1.3.11. $u_\varsigma(\vec{p},h) = -\varsigma\gamma_5 v_\varsigma(\vec{p},h), v_\varsigma(\vec{p},h) = -\varsigma\gamma_5 u_\varsigma(\vec{p},h), h = -\frac{1}{2},\frac{1}{2}$

Cor. 1.3.12. $\bar{u}_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h') = \varsigma\delta_{hh'}, \bar{v}_\varsigma(\vec{p},h)v_\varsigma(\vec{p},h') = -\varsigma\delta_{hh'}, \bar{u}_\varsigma(\vec{p},h)v_\varsigma(\vec{p},h') = 0, \bar{v}_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h') = 0$

Cor. 1.3.13. $u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h') = \frac{E}{m}\delta_{hh'}, v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h') = \frac{E}{m}\delta_{hh'}, u_\varsigma^+(\vec{p},h)v_\varsigma(-\vec{p},h') = 0, v_\varsigma^+(\vec{p},h)u_\varsigma(-\vec{p},h') = 0$

Cor. 1.3.14. $\sum_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h) = \frac{\varsigma m - i\gamma^a p_a}{2m}, \sum_h v_\varsigma(\vec{p},h)\bar{v}_\varsigma(\vec{p},h) = \frac{-\varsigma m - i\gamma^a p_a}{2m}$

Cor. 1.3.15. $\begin{cases}\sum_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h) - v_\varsigma(\vec{p},h)\bar{v}_\varsigma(\vec{p},h)] = \varsigma\\ \sum_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h) + v_\varsigma(\vec{p},h)\bar{v}_\varsigma(\vec{p},h)] = \frac{-i\gamma^a p_a}{m}\\ \sum_h u_\varsigma(\vec{p},h)u_\varsigma^+(\vec{p},h) + v_\varsigma(-\vec{p},h)v_\varsigma^+(-\vec{p},h)] = \frac{E}{m}\end{cases}$

1.4 Plane wave solutions of Dirac equation under general representation

Cor. 1.4.1. $\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h}[a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 1.4.2. $\psi^+(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 1.4.3.

$$\begin{cases}a_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)\psi(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u^{+\lambda_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma}(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ b_\varsigma^+(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)\psi(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}v^{+\lambda_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma}(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

Cor. 1.4.4.

$$\begin{cases}a_\varsigma^+(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u^{\lambda'_\varsigma}(\vec{p},h)\psi_{\lambda'_\varsigma}^+(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ b_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}v^{\lambda'_\varsigma}(\vec{p},h)\psi_{\lambda'_\varsigma}^+(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

1.5 Spin basis and its properties of Dirac equation under general representation

Def. 1.5.1. $\tilde{a}_(\vec{p},h):=\begin{cases}a_\varsigma(\vec{p},h),\varsigma=1\\ b_\varsigma^+(\vec{p},h),\varsigma=-1\end{cases}$, $\tilde{b}_(\vec{p},h):=\begin{cases}b_\varsigma(\vec{p},h),\varsigma=1\\ a_\varsigma^+(\vec{p},h),\varsigma=-1\end{cases}$

Def. 1.5.2. $u_(\vec{p},h):=\begin{cases}u_+(\vec{p},h),\varsigma=1\\ v_-(\vec{p},h),\varsigma=-1\end{cases}$, $v_(\vec{p},h):=\begin{cases}v_+(\vec{p},h),\varsigma=1\\ u_-(\vec{p},h),\varsigma=-1\end{cases}$

Cor. 1.5.1.

$$\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h}[\tilde{a}(\vec{p},h)\sqrt{\frac{m}{E}}u(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+\tilde{b}^+(\vec{p},h)\sqrt{\frac{m}{E}}v(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\begin{cases}\tilde{a}(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u^+(\vec{p},h)\psi(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ \tilde{b}^+(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}v^+(\vec{p},h)\psi(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

Properties between spin bases (It also holds true under general representation.):

Cor. 1.5.2. $\bar{u}(\vec{p},h)u(\vec{p},h')=\delta_{hh'},\bar{v}(\vec{p},h)v(\vec{p},h')=-\delta_{hh'},\bar{u}(\vec{p},h)v(\vec{p},h')=0,\bar{v}(\vec{p},h)u(\vec{p},h')=0$

Cor. 1.5.3. $u^+(\vec{p},h)u(\vec{p},h')=\frac{E}{m}\delta_{hh'},v^+(\vec{p},h)v(\vec{p},h')=\frac{E}{m}\delta_{hh'},u^+(\vec{p},h)v(-\vec{p},h')=0,v^+(\vec{p},h)u(-\vec{p},h')=0$

Cor. 1.5.4. $\sum_{h}u(\vec{p},h)\bar{u}(\vec{p},h)=\frac{m-i\gamma^a p_a}{2m},\sum_{h}v(\vec{p},h)\bar{v}(\vec{p},h)=\frac{-m-i\gamma^a p_a}{2m}$

Cor. 1.5.5. $\begin{cases}\sum_{h}u(\vec{p},h)\bar{u}(\vec{p},h)-v(\vec{p},h)\bar{v}(\vec{p},h)]=1\\ \sum_{h}u(\vec{p},h)\bar{u}(\vec{p},h)+v(\vec{p},h)\bar{v}(\vec{p},h)]=\frac{-i\gamma^a p_a}{m}\\ \sum_{h}u(\vec{p},h)u^+(\vec{p},h)+v(-\vec{p},h)v^+(-\vec{p},h)]=\frac{E}{m}\end{cases}$

1.6 Isochronous quantization of Dirac equation under general representation

Cor. 1.6.1. $\begin{cases}\{\psi_{\lambda_\varsigma}(\vec{r},t),\psi_{\lambda'_\varsigma}^+(\vec{r}',t)\}=\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ \{\psi_{\lambda_\varsigma}(\vec{r},t),\psi_{\lambda'_\varsigma}(\vec{r}',t)\}=0\\ \{\psi_{\lambda_\varsigma}^+(\vec{r},t),\psi_{\lambda'_\varsigma}^+(\vec{r}',t)\}=0\end{cases}\Leftrightarrow\begin{cases}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\ \{a_\varsigma^+(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=0\end{cases}$

Proof: $\{\psi_{\lambda_\varsigma}(\vec{r},t),\psi_{\lambda'_\varsigma}^+(\vec{r}',t)\}$

$=\frac{1}{(2\pi)^3}\int\limits_{\vec{p},\vec{p}'=-\infty}^{+\infty}\frac{m}{E}\sum_{h,h'}[u_{\varsigma\lambda_\varsigma}(\vec{p},h)u_{\varsigma\lambda'_\varsigma}^*(\vec{p}',h')e^{i\varsigma(\vec{p}\cdot\vec{r}-Et-\vec{p}'\cdot\vec{r}'+E't)}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}$

$+v_{\varsigma\lambda_\varsigma}(\vec{p},h)v_{\varsigma\lambda'_\varsigma}^*(\vec{p}',h')e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et-\vec{p}'\cdot\vec{r}'+E't)}\{b_\varsigma^+(\vec{p},h),b_\varsigma(\vec{p}',h')\}]d^3\vec{p}d^3\vec{p}'$

$=\frac{1}{(2\pi)^3}\int\limits_{\vec{p},\vec{p}'=-\infty}^{+\infty}\frac{m}{E}\sum_{h,h'}[u_{\varsigma\lambda_\varsigma}(\vec{p},h)u_{\varsigma\lambda'_\varsigma}^*(\vec{p}',h')e^{i\varsigma(\vec{p}\cdot\vec{r}-Et-\vec{p}'\cdot\vec{r}'+E't)}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')$

$+ v_{\varsigma\lambda_\varsigma}(\vec{p},h)v^*_{\varsigma\lambda'_\varsigma}(\vec{p}',h')e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et-\vec{p}'\cdot\vec{r}'+E't)}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')]d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{m}{E}\sum\limits_h[u_{\varsigma\lambda_\varsigma}(\vec{p},h)u^*_{\varsigma\lambda'_\varsigma}(\vec{p},h)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + v_{\varsigma\lambda_\varsigma}(\vec{p},h)v^*_{\varsigma\lambda'_\varsigma}(\vec{p},h)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{m}{E}\sum\limits_h[u_{\varsigma\lambda_\varsigma}(\vec{p},h)u^*_{\varsigma\lambda'_\varsigma}(\vec{p},h) + v_{\varsigma\lambda_\varsigma}(-\vec{p},h)v^*_{\varsigma\lambda'_\varsigma}(-\vec{p},h)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \delta_{\lambda_\varsigma\lambda'_\varsigma}\frac{1}{(2\pi)^3}\int\limits_{\vec{p}=-\infty}^{+\infty}e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}')$ □

Proof: $\{a_\varsigma(\vec{p},h),a^+_\varsigma(\vec{p}',h')\}$
$= \frac{1}{(2\pi)^3}\frac{m}{E}\int\limits_{\vec{r},\vec{r}'=-\infty}^{+\infty}u^*_{\lambda_\varsigma}(\vec{p},h)u^{\lambda'_\varsigma}(\vec{p}',h')\{\psi^{\lambda_\varsigma}(\vec{r},t),\psi^+_{\lambda'_\varsigma}(\vec{r}',t)\}e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{m}{E}\int\limits_{\vec{r},\vec{r}'=-\infty}^{+\infty}u^*_{\lambda_\varsigma}(\vec{p},h)u^{\lambda'_\varsigma}(\vec{p}',h')\delta^{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}')e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$= \frac{m}{E}u^*_{\lambda_\varsigma}(\vec{p},h)u^{\lambda_\varsigma}(\vec{p}',h')e^{-i\varsigma(E't-Et)}\frac{1}{(2\pi)^3}\int\limits_{\vec{r}=-\infty}^{+\infty}e^{i\varsigma(\vec{p}'-\vec{p})\cdot\vec{r}'}d^3\vec{r}$
$= \frac{m}{E}u^+_\varsigma(\vec{p},h)u_\varsigma(\vec{p}',h')e^{-i\varsigma(E't-Et)}\delta^3(\vec{p}-\vec{p}')$
$= \delta_{hh'}\delta^3(\vec{p}-\vec{p}')$ □

Cor. 1.6.2. $\begin{cases} :P_u:=:\int -i\psi^+\partial_u\psi dr^3:\overset{\varsigma=1}{=}\int\sum\limits_h p_u[a^+_\varsigma(\vec{p},h)a_\varsigma(\vec{p},h)+b^+_\varsigma(\vec{p},h)b_\varsigma(\vec{p},h)]d^3\vec{p} \\ :Q:=:\int\psi^+\psi dr^3:=\int\sum\limits_h[a^+_\varsigma(\vec{p},h)a_\varsigma(\vec{p},h)-b^+_\varsigma(\vec{p},h)b_\varsigma(\vec{p},h)]d^3\vec{p}\overset{\varsigma=1}{=}0 \end{cases}$

1.7 Covariant anticommutative rule of Dirac equation under general representation

Cor. 1.7.1. $\begin{cases} \psi_{\lambda_\varsigma}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_h[a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u_{\varsigma\lambda_\varsigma}(\vec{p},h)e^{i\varsigma px}+b^+_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_{\varsigma\lambda_\varsigma}(\vec{p},h)e^{-i\varsigma px}]d^3\vec{p} \\ \bar{\psi}_{\lambda'_\varsigma}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_h[a^+_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}\bar{u}_{\varsigma\lambda'_\varsigma}(\vec{p},h)e^{-i\varsigma px}+b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}\bar{v}_{\varsigma\lambda'_\varsigma}(\vec{p},h)e^{i\varsigma px}]d^3\vec{p} \end{cases}$

Cor. 1.7.2. $\{\psi_{\lambda_\varsigma}(x),\bar{\psi}_{\lambda'_\varsigma}(x')\}=i(m-\gamma^a\partial_a)_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')$

Proof: $\{\psi_{\lambda_\varsigma}(x),\bar{\psi}_{\lambda'_\varsigma}(x')\}=\frac{1}{(2\pi)^3}\int\sum\limits_{h,h'}\sqrt{\frac{m}{E}}\sqrt{\frac{m}{E'}}$
$[\{a_\varsigma(\vec{p},h),a^+_\varsigma(\vec{p}',h')\}u_{\varsigma\lambda_\varsigma}(\vec{p},h)\bar{u}_{\varsigma\lambda'_\varsigma}(\vec{p}',h')e^{i\varsigma(px-p'x')}+\{b^+_\varsigma(\vec{p},h),b_\varsigma(\vec{p}',h')\}v_{\varsigma\lambda_\varsigma}(\vec{p},h)\bar{v}_{\varsigma\lambda'_\varsigma}(\vec{p}',h')e^{-i\varsigma(px-p'x')}]d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int\sum\limits_{h,h'}\sqrt{\frac{m}{E}}\sqrt{\frac{m}{E'}}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')[u_{\varsigma\lambda_\varsigma}(\vec{p},h)\bar{u}_{\varsigma\lambda'_\varsigma}(\vec{p}',h')e^{i\varsigma(px-p'x')}+v_{\varsigma\lambda_\varsigma}(\vec{p},h)\bar{v}_{\varsigma\lambda'_\varsigma}(\vec{p}',h')e^{-i\varsigma(px-p'x')}]d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int\sum\limits_h\frac{m}{E}[u_{\varsigma\lambda_\varsigma}(\vec{p},h)\bar{u}_{\varsigma\lambda'_\varsigma}(\vec{p},h)e^{i\varsigma p(x-x')}+v_{\varsigma\lambda_\varsigma}(\vec{p},h)\bar{v}_{\varsigma\lambda'_\varsigma}(\vec{p},h)e^{-i\varsigma p(x-x')}]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{m}{E}[\frac{(\varsigma m-i\gamma^a p_a)_{\lambda_\varsigma\lambda'_\varsigma}}{2m}e^{i\varsigma p(x-x')}+\frac{(-\varsigma m-i\gamma^a p_a)_{\lambda_\varsigma\lambda'_\varsigma}}{2m}e^{-i\varsigma p(x-x')}]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{1}{2E}[(\varsigma m-i\gamma^a p_a)_{\lambda_\varsigma\lambda'_\varsigma}e^{i\varsigma p(x-x')}+(-\varsigma m-i\gamma^a p_a)_{\lambda_\varsigma\lambda'_\varsigma}e^{-i\varsigma p(x-x')}]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{1}{2E}[\varsigma(m-\gamma^a\partial_a)_{\lambda_\varsigma\lambda'_\varsigma}e^{i\varsigma p(x-x')}-\varsigma(m-\gamma^a\partial_a)_{\lambda_\varsigma\lambda'_\varsigma}e^{-i\varsigma p(x-x')}]d^3\vec{p}$
$= i(m-\gamma^a\partial_a)_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')$ □

Cor. 1.7.3. $\{\psi_{\lambda_\varsigma}(x),\bar{\psi}_{\lambda'_\varsigma}(x')\}=i(m-\gamma^a\partial_a)_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')\Leftrightarrow\{\psi_{\lambda_\varsigma}(x),\psi^+_{\lambda'_\varsigma}(x')\}=i[(m-\gamma^a\partial_a)\gamma^4]_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')$

Cor. 1.7.4. $\{\psi_{\lambda_\varsigma}(\vec{r},t),\bar{\psi}_{\lambda'_\varsigma}(\vec{r}',t)\}=\gamma^4_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}')\Leftrightarrow\{\psi_{\lambda_\varsigma}(\vec{r},t),\psi^+_{\lambda'_\varsigma}(\vec{r}',t)\}=\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}')$

1.8 Conserved charge of Dirac equation under general representation

Cor. 1.8.1. $Q=\int\psi^+\psi dr^3=\int\sum\limits_h[a^+_\varsigma(\vec{p},h)a_\varsigma(\vec{p},h)+b_\varsigma(\vec{p},h)b^+_\varsigma(\vec{p},h)]d^3\vec{p}$

Proof: $Q=\int\psi^+\psi dr^3$
$= \frac{1}{(2\pi)^3}\int\sum\limits_{h,h'}[a^+_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u^+_\varsigma(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v^+_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}+b^+_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'d^3\vec{p}dr^3$
$= \int\sum\limits_{h,h'}\frac{m}{E}[a^+_\varsigma(\vec{p},h)a_\varsigma(\vec{p},h')u^+_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h')+b_\varsigma(\vec{p},h)b^+_\varsigma(\vec{p},h')v^+_\varsigma(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}'d^3\vec{p}$
$= \int\sum\limits_{h,h'}\frac{m}{E}[a^+_\varsigma(\vec{p},h)a_\varsigma(\vec{p},h')u^+_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h')+b_\varsigma(\vec{p},h)b^+_\varsigma(\vec{p},h')v^+_\varsigma(\vec{p},h)v_\varsigma(\vec{p},h')]d^3\vec{p}$
$= \int\sum\limits_h[a^+_\varsigma(\vec{p},h)a_\varsigma(\vec{p},h)+b_\varsigma(\vec{p},h)b^+_\varsigma(\vec{p},h)]d^3\vec{p}$ □

Cor. 1.8.2. $H = i\int \psi^+ \partial_t \psi dr^3 = \varsigma \int \sum_h E(\vec{p})[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h) - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^3\vec{p}$

Proof: $H = i\int \psi^+ \partial_t \psi dr^3$
$= i\frac{1}{(2\pi)^3}\int \sum_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} + b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$(-i\varsigma E')[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)} - b_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'d^3\vec{p}dr^3$
$= -i\int \sum_{h,h'}\frac{m}{E}(-i\varsigma E')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h') - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}'d^3\vec{p}$
$= -i\int \sum_{h,h'}\frac{m}{E}(-i\varsigma E')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h') - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]d^3\vec{p}$
$= \varsigma\int \sum_h E(\vec{p})[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h) - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^3\vec{p}$ □

Cor. 1.8.3. $\vec{P} = -i\int \psi^+ \nabla \psi dr^3 = \varsigma \int \sum_h \vec{p}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h) - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^3\vec{p}$

Proof: $\vec{P} = -i\int \psi^+ \nabla \psi dr^3$
$= -i\frac{1}{(2\pi)^3}\int \sum_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} + b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$(i\varsigma\vec{p}')[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)} - b_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'d^3\vec{p}dr^3$
$= -i\int \sum_{h,h'}\frac{m}{E}(i\varsigma\vec{p}')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h') - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}'d^3\vec{p}$
$= -i\int \sum_{h,h'}\frac{m}{E}(i\varsigma\vec{p})[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h') - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]d^3\vec{p}$
$= \varsigma\int \sum_h \vec{p}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h) - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^3\vec{p}$ □

Cor. 1.8.4. $P_u = -i\int \psi^+ \partial_u \psi dr^3 = \varsigma \int \sum_h p_u[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h) - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^3\vec{p}$

Proof: $P_u = -i\int \psi^+ \partial_u \psi dr^3$
$= i\frac{1}{(2\pi)^3}\int \sum_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} + b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$(i\varsigma p_u')[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)} - b_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'d^3\vec{p}dr^3$
$= -i\int \sum_{h,h'}\frac{m}{E}(i\varsigma p_u')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h') - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}'d^3\vec{p}$
$= -i\int \sum_{h,h'}\frac{m}{E}(i\varsigma p_u')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h') - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]d^3\vec{p}$
$= \varsigma\int \sum_h p_u[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h) - b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^3\vec{p}$ □

2 Plane wave solutions and quantization of Majorana equation under real representation

2.1 Strictly solving plane wave solutions of Majorana equation under real representation [25]

2.1.1 Relations between Majorana single momentum solutions under two representations

Majorana equations under real representation and Dirac representation:

Def. 2.1.1. $\begin{cases}(\gamma_s^a\partial_a + m)\psi_s = 0, \gamma_s^a = (\sigma_{-\varsigma}\sigma_{\varsigma y}, \varsigma\sigma_{\varsigma z}), \psi_s^* = \psi_s \\ (\gamma^a\partial_a + m)\psi = 0, \gamma^a = (\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_z), \psi^* = -e^{2i\theta}\sigma_y\otimes\sigma_y\psi, S_{em}^T(\varsigma)S_{em}(\varsigma) = -\sigma_y\otimes\sigma_y\end{cases}$

Cor. 2.1.1. $\begin{cases}\psi_s(\vec{p}) := e^{i\theta}S_{em}(\varsigma)\psi(\vec{p}), \psi_s(\vec{p}) = \psi_s^*(\vec{p}) \\ \psi_s(\vec{0}) := e^{i\theta}S_{em}(\varsigma)\psi(\vec{0}), \psi_s(\vec{0}) = \psi_s^*(\vec{0})\end{cases}$ $S_{em}(\varsigma) = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\varsigma & \varsigma & 0\end{bmatrix}$

Cor. 2.1.2. $\psi_s(\vec{p}) = \psi_s^*(\vec{p}) \Leftrightarrow \psi^*(\vec{p}) = -e^{2i\theta}\sigma_y\otimes\sigma_y\psi(\vec{p})$

Cor. 2.1.3. $\psi(\vec{0}) = \begin{bmatrix}\xi_0 e^{-i\varsigma mt_0} \\ \eta_0 e^{i\varsigma mt_0}\end{bmatrix}; \psi^*(\vec{0}) = -e^{2i\theta}\sigma_y\otimes\sigma_y\psi(\vec{0}) \Leftrightarrow \eta_0 = -ie^{-2i\theta}\sigma_y\xi_0^* \Leftrightarrow \psi(\vec{0}) = \begin{bmatrix}\xi_0 e^{-i\varsigma mt_0} \\ -ie^{-2i\theta}\sigma_y\xi_0^* e^{i\varsigma mt_0}\end{bmatrix}$

Cor. 2.1.4. $\psi(\vec{0}) = \begin{bmatrix}\xi e^{-i\varsigma mt_0} \\ -ie^{-2i\theta}\sigma_y\xi^* e^{i\varsigma mt_0}\end{bmatrix} \Leftrightarrow \psi_s(\vec{0}) = \frac{1}{\sqrt{2}}\begin{bmatrix} i(e^{i\theta}\xi_1 e^{-i\varsigma mt_0} - e^{-i\theta}\xi_1^* e^{i\varsigma mt_0}) \\ -(e^{i\theta}\xi_1 e^{-i\varsigma mt_0} + e^{-i\theta}\xi_1^* e^{i\varsigma mt_0}) \\ -i(e^{i\theta}\xi_2 e^{-i\varsigma mt_0} - e^{-i\theta}\xi_2^* e^{i\varsigma mt_0}) \\ -\varsigma(e^{i\theta}\xi_2 e^{-i\varsigma mt_0} + e^{-i\theta}\xi_2^* e^{i\varsigma mt_0})\end{bmatrix} \in R; \xi = \begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix}$

Cor. 2.1.5. $\begin{cases}\psi(\vec{p}) = \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\psi(\vec{0}) = \frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\begin{bmatrix}\xi_0 e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} \\ -ie^{-2i\theta}\sigma_y\xi_0^* e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}\end{bmatrix} \\ \psi_s(\vec{p}) = \frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}\psi_s(\vec{0}) = \frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}e^{i\gamma_{s4}(\vec{p}\cdot\vec{r}-Et)}\psi_{s0} = \psi_s^*(\vec{p}), \psi_{s0} = e^{i\theta}S_{em}(\varsigma)\psi_0\end{cases}$

2.1.2 Concrete single momentum solutions of Majorana equation under Dirac representation

Cor. 2.1.6. $\psi(\vec{p})=\sum_h[a_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}-e^{-2i\theta}\sigma_y\otimes\sigma_y a_\varsigma^+(\vec{p},h)u_\varsigma^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$

Proof: $\psi(\vec{p})=\frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\psi(\vec{0})=\frac{E+m+\varsigma\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}$

$\{[a_\varsigma(\vec{p},\frac{1}{2})\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+a_\varsigma(\vec{p},-\frac{1}{2})\begin{bmatrix}0\\1\\0\\0\end{bmatrix}]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+[-e^{-2i\theta}a_\varsigma^+(\vec{p},-\frac{1}{2})\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+e^{-2i\theta}a_\varsigma^+(\vec{p},\frac{1}{2})\begin{bmatrix}0\\0\\0\\1\end{bmatrix}]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}\}$

$=[a_\varsigma(\vec{p},\frac{1}{2})u_\varsigma(\vec{p},\frac{1}{2})+a_\varsigma(\vec{p},-\frac{1}{2})u_\varsigma(\vec{p},-\frac{1}{2})]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{-2i\theta}[-a_\varsigma^+(\vec{p},-\frac{1}{2})v_\varsigma(\vec{p},\frac{1}{2})+a_\varsigma^+(\vec{p},\frac{1}{2})v_\varsigma(\vec{p},-\frac{1}{2})]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}$

$=\sum_h[a_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}-e^{-2i\theta}\sigma_y\otimes\sigma_y a_\varsigma^+(\vec{p},h)u^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$

$=\sum_h[a_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+[e^{i\theta}S_{em}(\varsigma)]^+[e^{i\theta}S_{em}(\varsigma)]^*a_\varsigma^+(\vec{p},h)u^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$ □

Cor. 2.1.7. $\begin{cases}u^*(\vec{p},h)=(-1)^{s+\frac{1}{2}}\sigma_y\otimes\sigma_y v_\varsigma(\vec{p},-h)\\ v^*(\vec{p},h)=(-1)^{h-\frac{1}{2}}\sigma_y\otimes\sigma_y u_\varsigma(\vec{p},-h)\end{cases}$

Cor. 2.1.8. $\psi^+(\vec{p})=$
$[a_\varsigma^+(\vec{p},\frac{1}{2})u_\varsigma^+(\vec{p},\frac{1}{2})+a_\varsigma^+(\vec{p},-\frac{1}{2})u_\varsigma^+(\vec{p},-\frac{1}{2})]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{2i\theta}[-a_\varsigma(\vec{p},-\frac{1}{2})v_\varsigma^+(\vec{p},\frac{1}{2})+a_\varsigma(\vec{p},\frac{1}{2})v_\varsigma^+(\vec{p},-\frac{1}{2})]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}$

2.1.3 Concrete single momentum solutions of Majorana equation under real representation

Cor. 2.1.9. $\psi_s(\vec{p})=\sum_h[a_\varsigma(\vec{p},h)u_s(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_\varsigma^+(\vec{p},h)u_s^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$

Proof: $\psi_s(\vec{p})=\frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}\psi_s(\vec{0})=e^{i\theta}S_{em}(\varsigma)\psi(\vec{p})$

$=e^{i\theta}S_{em}(\varsigma)[a_\varsigma(\vec{p},\frac{1}{2})u_\varsigma(\vec{p},\frac{1}{2})+a_\varsigma(\vec{p},-\frac{1}{2})u_\varsigma(\vec{p},-\frac{1}{2})]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{-2i\theta}[-a_\varsigma^+(\vec{p},-\frac{1}{2})v_\varsigma(\vec{p},\frac{1}{2})+a_\varsigma^+(\vec{p},\frac{1}{2})v_\varsigma(\vec{p},-\frac{1}{2})]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}$

$=[a_\varsigma(\vec{p},\frac{1}{2})u_s(\vec{p},\frac{1}{2})+a_\varsigma(\vec{p},-\frac{1}{2})u_s(\vec{p},-\frac{1}{2})]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{-2i\theta}[-a_\varsigma^+(\vec{p},-\frac{1}{2})v_s(\vec{p},\frac{1}{2})+a_\varsigma^+(\vec{p},\frac{1}{2})v_s(\vec{p},-\frac{1}{2})]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}$

$=\frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}\{[a_\varsigma(\vec{p},\frac{1}{2})e^{i\theta}\begin{bmatrix}i\\-1\\0\\0\end{bmatrix}+a_\varsigma(\vec{p},-\frac{1}{2})e^{i\theta}\begin{bmatrix}0\\0\\-i\\-\varsigma\end{bmatrix}]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{-i\theta}[-a_\varsigma^+(\vec{p},-\frac{1}{2})\begin{bmatrix}0\\0\\-i\\\varsigma\end{bmatrix}+a_\varsigma^+(\vec{p},\frac{1}{2})\begin{bmatrix}-i\\-1\\0\\0\end{bmatrix}]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}\}$

$=\sum_h[a_\varsigma(\vec{p},h)u_s(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_\varsigma^+(\vec{p},h)u_s^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$ □

Cor. 2.1.10. $\psi_s^+(\vec{p})=\sum_h[a_\varsigma^+(\vec{p},h)u_s^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_\varsigma(\vec{p},h)u_s^T(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$

Cor. 2.1.11. $\bar{\psi}_s(\vec{p})=\sum_h[a_\varsigma^+(\vec{p},h)\bar{u}_s(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}-a_\varsigma(\vec{p},h)\bar{u}_s^*(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$

Cor. 2.1.12. $\begin{cases}u_s(\vec{p},\frac{1}{2})=e^{i\theta}S_{em}(\varsigma)u_\varsigma(\vec{p},\frac{1}{2})=\frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}e^{i\theta}\begin{bmatrix}i\\-1\\0\\0\end{bmatrix}=e^{2i\theta}v_s^*(\vec{p},-\frac{1}{2})\\ u_s(\vec{p},-\frac{1}{2})=e^{i\theta}S_{em}(\varsigma)u_\varsigma(\vec{p},-\frac{1}{2})=\frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}e^{i\theta}\begin{bmatrix}0\\0\\-i\\-\varsigma\end{bmatrix}=-e^{2i\theta}v_s^*(\vec{p},\frac{1}{2})\\ v_s(\vec{p},\frac{1}{2})=e^{i\theta}S_{em}(\varsigma)v_\varsigma(\vec{p},\frac{1}{2})=\frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}e^{i\theta}\begin{bmatrix}0\\0\\-i\\\varsigma\end{bmatrix}=-e^{2i\theta}u_s^*(\vec{p},-\frac{1}{2})\\ v_s(\vec{p},-\frac{1}{2})=e^{i\theta}S_{em}(\varsigma)v_\varsigma(\vec{p},-\frac{1}{2})=\frac{m-i\gamma_s^a p_a\gamma_{s4}}{\sqrt{2m(E+m)}}e^{i\theta}\begin{bmatrix}0\\0\\-i\\\varsigma\end{bmatrix}=e^{2i\theta}u_s^*(\vec{p},\frac{1}{2})\end{cases}$

Cor. 2.1.13. $\begin{cases}u_s^*(\vec{p},h)=(-1)^{h-\frac{1}{2}}e^{-2i\theta}v_s(\vec{p},-h)\\ v_s^*(\vec{p},h)=(-1)^{s+\frac{1}{2}}e^{2i\theta}u_s(\vec{p},-h)\end{cases}$

2.1.4 Relations between single momentum solutions of Majorana and neutrino equation

Cor. 2.1.14.
$(\gamma^a\partial_a+m)\psi(\vec{p})=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_z),\psi^*(\vec{p})=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi(\vec{p})$
$\psi(\vec{p})=\frac{1}{\sqrt{2m(E+m)}}\begin{bmatrix}(E+m)\xi_0e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}-\varsigma\vec{p}\cdot\sigma(ie^{-2i\theta}\sigma_y\xi_0^*)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}\\-(E+m)(ie^{-2i\theta}\sigma_y\xi_0^*)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+\varsigma\vec{p}\cdot\sigma\xi_0e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}\end{bmatrix}=\begin{bmatrix}\lambda(\vec{p})\\-ie^{-2i\theta}\sigma_y\lambda^*(\vec{p})\end{bmatrix}$

$[\Updownarrow]$

Cor. 2.1.15.
$(\sigma,-i\varsigma)_a\partial^a\nu(\vec{p})-me^{-2i\theta}\sigma_y\nu^*(\vec{p})=0$
$\nu(\vec{p})=\frac{1}{\sqrt{2}}[\lambda(\vec{p})+ie^{-2i\theta}\sigma_y\lambda^*(\vec{p})]=\frac{E+m-\varsigma\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}}\frac{1}{\sqrt{2}}(\xi_0e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+ie^{-2i\theta}\sigma_y\xi_0^*e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)})$

$[\Updownarrow]$

Cor. 2.1.16.
$(\sigma, i\varsigma)_a\partial^a[-ie^{-2i\theta}\sigma_y\nu^*(\vec{p})] - me^{-2i\theta}\sigma_y[-ie^{-2i\theta}\sigma_y\nu^*(\vec{p})]^* = 0$
$-ie^{-2i\theta}\sigma_y\nu^*(\vec{p}) = \frac{1}{\sqrt{2}}[\lambda(\vec{p}) - ie^{-2i\theta}\sigma_y\lambda^*(\vec{p})] = \frac{E+m+\varsigma\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}}\frac{1}{\sqrt{2}}(\xi_0 e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} - ie^{-2i\theta}\sigma_y\xi_0^* e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)})$

Plane wave solutions expanded by helicity:

Cor. 2.1.17. $\psi(\vec{p}) = [a(p,+)\begin{bmatrix}\lambda(p,+)\\ \varsigma\sqrt{\frac{E-m}{E+m}}\lambda(p,+)\end{bmatrix} + a(p,-)\begin{bmatrix}\lambda(p,-)\\ -\varsigma\sqrt{\frac{E-m}{E+m}}\lambda(p,-)\end{bmatrix}]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}$
$+ [b(p,+)\begin{bmatrix}\varsigma\sqrt{\frac{E-m}{E+m}}\lambda(p,+)\\ \lambda(p,+)\end{bmatrix} + b(p,-)\begin{bmatrix}-\varsigma\sqrt{\frac{E-m}{E+m}}\lambda(p,-)\\ \lambda(p,-)\end{bmatrix}]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}$, $\frac{\sigma\cdot\vec{p}}{|\vec{p}|}\lambda(+) = \lambda(+)$, $\frac{\sigma\cdot\vec{p}}{|\vec{p}|}\lambda(-) = -\lambda(-)$

2.1.5 Construct plane wave solutions from Dirac equation with special representation [25]

Cor. 2.1.18. $(\gamma^a\partial_a + m)\psi(\vec{p}) = 0, \gamma^a = (\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_z), \psi^*(\vec{p}) = -e^{2i\theta}\sigma_y\otimes\sigma_y\psi(\vec{p})$

Cor. 2.1.19. $\lambda(\vec{p}) = \psi_1(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}[(E+m)\xi_0 e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} - i\varsigma e^{-2i\theta}\sigma\cdot\vec{p}\sigma_y\xi_0^* e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$

Cor. 2.1.20. $\psi(\vec{p}) = \begin{bmatrix}\lambda(\vec{p})\\ -i\sigma_y e^{-2i\theta}\lambda^*(\vec{p})\end{bmatrix}, \psi_s(\vec{p}) = S_{em}(\varsigma)\begin{bmatrix}e^{i\theta}\lambda(\vec{p})\\ -i\sigma_y[e^{i\theta}\lambda(\vec{p})]^*\end{bmatrix}, \nu(\vec{p}) = \frac{1}{\sqrt{2}}[\lambda(\vec{p}) + ie^{-2i\theta}\sigma_y\lambda^*(\vec{p})]$

2.1.6 Construct plane wave solutions from neutrino equation

Cor. 2.1.21. $(\sigma, -i\varsigma)_a\partial^a\nu(\vec{p}) - me^{-2i\theta}\sigma_y\nu^*(\vec{p}) = 0$

Cor. 2.1.22. $\nu(\vec{p}) = \frac{E+m-\varsigma\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}}\frac{1}{\sqrt{2}}(\xi_0 e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + ie^{-2i\theta}\sigma_y\xi_0^* e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)})$

Cor. 2.1.23. $\psi(\vec{p}) = \frac{1}{\sqrt{2}}\begin{bmatrix}\nu(\vec{p}) - ie^{-2i\theta}\sigma_y\nu^*(\vec{p})\\ -\nu(\vec{p}) - ie^{-2i\theta}\sigma_y\nu^*(\vec{p})\end{bmatrix}, \psi_s(\vec{p}) = \frac{1}{\sqrt{2}}S_{em}(\varsigma)\begin{bmatrix}e^{i\theta}\nu(\vec{p}) - i\sigma_y[e^{i\theta}\nu(\vec{p})]^*\\ -e^{i\theta}\nu(\vec{p}) - i\sigma_y[e^{i\theta}\nu(\vec{p})]^*\end{bmatrix}$

2.2 Properties of spin basis of Majorana equation under real representation

Majorana equation: $(\gamma_s^a\partial_a + m)\psi = 0, \gamma_s^a = (\sigma_{-\varsigma}\sigma_{\varsigma y}, \varsigma\sigma_{\varsigma z}), \psi_s^* = \psi_s$
Properties of two spin bases under real representation:

Pro. 2.2.1. $\bar{u}_s(\vec{p},h)u_s(\vec{p},h') = \varsigma\delta_{hh'}, \bar{u}_s(\vec{p},h)u_s^*(\vec{p},h') = 0$

Pro. 2.2.2. $\sum_h u_s(\vec{p},h)\bar{u}_s(\vec{p},h) = \frac{\varsigma m - i\gamma_s^a p_a}{2m}$

Pro. 2.2.3. $\begin{cases}\sum_h u_s(\vec{p},h)\bar{u}_s(\vec{p},h) - [\sum_h u_s(\vec{p},h)\bar{u}_s(\vec{p},h)]^* = \varsigma\\ \sum_h u_s(\vec{p},h)\bar{u}_s(\vec{p},h) + [\sum_h u_s(\vec{p},h)\bar{u}_s(\vec{p},h)]^* = \frac{-i\gamma^a p_a}{m}\end{cases}$

Pro. 2.2.4. $u_s^+(\vec{p},h)u_s(\vec{p},h') = \frac{E}{m}\delta_{hh'}, u_s^+(\vec{p},h)u_s^*(-\vec{p},h') = 0$

Pro. 2.2.5. $\sum_h u_s(\vec{p},h)u_s^+(\vec{p},h) + [\sum_h u_s(-\vec{p},h)u_s^+(-\vec{p},h)]^* = \frac{E}{m}$

2.3 Plane wave solutions of Majorana equation under real representation

Cor. 2.3.1. $\psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_h[a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u_s(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_s^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 2.3.2. $\nabla\psi(\vec{r},t) = i\varsigma\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\vec{p}\sum_h[a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u_s(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} - a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_s^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 2.3.3. $\psi^*(\vec{r},t) = \psi(\vec{r},t)$

Cor. 2.3.4.
$$\begin{cases}a_\varsigma(\vec{p},h) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u_s^+(\vec{p},h)\psi(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u_{s\lambda_\varsigma}^*(\vec{p},h)\psi^{\lambda_\varsigma}(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ a_\varsigma^+(\vec{p},h) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u_s^T(\vec{p},h)\psi(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u_{s\lambda_\varsigma}(\vec{p},h)\psi^{\lambda_\varsigma}(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

2.4 Conserved charge of Majorana equation under real representation

Majorana action: $L=-\frac{1}{2}\int\bar{\psi}(\gamma_s^a\partial_a+m)\psi dr^3$ **, Majorana hamiltonian:** $H=\frac{1}{2}\int\bar{\psi}(\gamma_s\cdot\nabla+m)\psi dr^3$

Cor. 2.4.1. $\bar{\psi}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}\bar{u}_s(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}-a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}\bar{u}_s^*(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 2.4.2. $H=i\int\psi^+\partial_t\psi dr^3=\int\sum\limits_{h}\varsigma E[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h)]d^3\vec{p}$

Proof: $H=\int[\bar{\psi}(\gamma_s\cdot\nabla)+m]\psi]dr^3=i\int\psi^+\partial_t\psi dr^3$
$=\int\sum\limits_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')\bar{u}_s(\vec{p},h)(m+i\varsigma\gamma_s\cdot\vec{p})u_s(\vec{p},h')-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h')\bar{u}_s^*(\vec{p},h)(m-i\varsigma\gamma_s\cdot\vec{p})u_s^*(\vec{p},h')]d^3\vec{p}$
$=\int\sum\limits_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')\bar{u}_s(\vec{p},h)\{2m\varsigma[\sum\limits_{s''}u^*(\vec{p},s'')\bar{u}^*(\vec{p},s'')]+\varsigma E\gamma_s^4\}u_s(\vec{p},h')$
$-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h')\bar{u}_s^*(\vec{p},h)\{2m\varsigma[\sum\limits_{s''}u_\varsigma(\vec{p},s'')\bar{u}_\varsigma(\vec{p},s'')]-\varsigma E\gamma_s^4\}u_s^*(\vec{p},h')]d^3\vec{p}$
$=\int\sum\limits_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_s^+(\vec{p},h)\varsigma Eu_s(\vec{p},h')-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h')u_s^T(\vec{p},h)\varsigma Eu_s^*(\vec{p},h')]d^3\vec{p}$
$=\int\sum\limits_{h}\varsigma E[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h)]d^3\vec{p}$ □

Cor. 2.4.3. $\vec{P}=\int-i\psi^+\nabla\psi dr^3=\int\sum\limits_{h}\varsigma\vec{p}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h)]d^3\vec{p}$

Proof: $\vec{P}=\int-i\psi^+\nabla\psi dr^3$
$=-i\frac{1}{(2\pi)^3}\int\sum\limits_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_s^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u_s^T(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}](i\varsigma\vec{p})$
$[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_{s'}(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}-a_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E}}u_{s'}^*(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'d^3\vec{p}dr^3$
$=-i\int\sum\limits_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_s^+(\vec{p},h)(i\varsigma\vec{p})u_s(\vec{p},h')-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h')u_s^T(\vec{p},h)(i\varsigma\vec{p})u_s^*(\vec{p},h')]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}'d^3\vec{p}$
$=-i\int\sum\limits_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_s^+(\vec{p},h)(i\varsigma\vec{p})u_s(\vec{p},h')-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h')u_s^T(\vec{p},h)(i\varsigma\vec{p})u_s^*(\vec{p},h')]d^3\vec{p}$
$=\int\sum\limits_{h}\varsigma\vec{p}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h)]d^3\vec{p}$ □

Cor. 2.4.4. $Q=\int\psi^+\psi dr^3=\int\sum\limits_{h}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)+a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h)]d^3\vec{p}$

Proof: $Q=\int\psi^+\psi dr^3$
$=\frac{1}{(2\pi)^3}\int\sum\limits_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_s^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u_s^T(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_{s'}(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}+a_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E}}u_{s'}^*(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'d^3\vec{p}dr^3$
$=\int\sum\limits_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_s^+(\vec{p},h)u_s(\vec{p},h')+a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h')u_s^T(\vec{p},h)u_s^*(\vec{p},h')]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}'d^3\vec{p}$
$=\int\sum\limits_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_s^+(\vec{p},h)u_s(\vec{p},h')+a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h')u_s^T(\vec{p},h)(i\varsigma\vec{p})u_s^*(\vec{p},h')]d^3\vec{p}$
$=\int\sum\limits_{h}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)+a_\varsigma(\vec{p},h)a_\varsigma^+(\vec{p},h)]d^3\vec{p}$ □

Cor. 2.4.5. $L=-\frac{1}{2}\int\bar{\psi}(\gamma_s^a\partial_a+m)\psi dr^3=0$

2.5 Quantization of Majorana equation under real representation

By using the above conclusions and properties, the following commutative relations can be obtained:

Cor. 2.5.1. $\begin{cases}\{\psi_{\lambda_\varsigma}(\vec{r},t),\psi_{\mu_\varsigma}(\vec{r}',t)\}=\delta_{\lambda_\varsigma\mu_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ \psi^*(\vec{r},t)=\psi(\vec{r},t)\end{cases}\Leftrightarrow\begin{cases}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\ \{a_\varsigma^+(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=0\end{cases}$

Cor. 2.5.2. $\begin{cases}:H:=:\frac{1}{2}\int i\psi^+\partial_t\psi dr^3:=:\frac{1}{2}\int[\bar{\psi}(\gamma_s\cdot\nabla)+m]\psi]dr^3:\overset{\varsigma=1}{=}\int\sum\limits_{h}E(p)a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\\ :\vec{P}:=:\frac{1}{2}\int-i\psi^+\nabla\psi dr^3:\overset{\varsigma=1}{=}\int\sum\limits_{h}\vec{p}a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\end{cases}$

Cor. 2.5.3. $\begin{cases}:P_u:=:\frac{1}{2}\int-i\psi^+\partial_u\psi dr^3:\overset{\varsigma=1}{=}\int\sum\limits_{h}p_u a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\\ :Q:=:\int\psi^+\psi dr^3:=\int\sum\limits_{h}0a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\overset{\varsigma=1}{=}0\end{cases}$

Cor. 2.5.4. $[P_u,P_v]=0,[Q,P_u]=0$

3 Quantization of Majorana equation under arbitrary representation

3.1 Properties of Majorana equation spin basis under arbitrary representation

Majorana equation under arbitrary representation: $(\gamma^a\partial_a+m)\psi=0, \psi_s=S\psi, \psi^*=S^TS\psi, \gamma^a=S^+(\sigma_{-\varsigma}\sigma_{+\varsigma y}, \varsigma\sigma_{+\varsigma z})S$

Properties between two spin bases under arbitrary representation:

Pro. 3.1.1. $\bar{u}_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h')=\varsigma\delta_{hh'}, \bar{u}_\varsigma(\vec{p},h)(S^+S^*)u^*(\vec{p},h')=0$

Pro. 3.1.2. $\sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)=\frac{\varsigma m-i\gamma^a p_a}{2m}$

Pro. 3.1.3. $\begin{cases}\sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)-[\sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)]^*=\varsigma\\ \sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)+[\sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)]^*=\frac{-i\gamma^a p_a}{m}\end{cases}$

Pro. 3.1.4. $u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')=\frac{E}{m}\delta_{hh'}, u_\varsigma^+(\vec{p},h)(S^+S^*)u^*(-\vec{p},h')=0$

Pro. 3.1.5. $\sum\limits_h u_\varsigma(\vec{p},h)u_\varsigma^+(\vec{p},h)+[\sum\limits_h u_\varsigma(-\vec{p},h)u_\varsigma^+(-\vec{p},h)]^*=\frac{E}{m}$

3.2 Plane wave solutions of Majorana equation under arbitrary representation

Cor. 3.2.1.

$$\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_h[a_\varsigma(\vec{p},h)\sqrt{\tfrac{m}{E}}u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+S^+S^*a_\varsigma^+(\vec{p},h)\sqrt{\tfrac{m}{E}}u^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

Cor. 3.2.2.

$$\nabla\psi(\vec{r},t)=i\varsigma\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\vec{p}\sum_h[a_\varsigma(\vec{p},h)\sqrt{\tfrac{m}{E}}u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}-S^+S^*a_\varsigma^+(\vec{p},h)\sqrt{\tfrac{m}{E}}u^*(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

Cor. 3.2.3. $\psi^*(\vec{r},t)=S^TS\psi(\vec{r},t)$

Cor. 3.2.4.

$$\begin{cases}a_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\tfrac{m}{E}}u_\varsigma^+(\vec{p},h)\psi(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\tfrac{m}{E}}u^{+\lambda_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma}(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ a_\varsigma^+(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\tfrac{m}{E}}u_\varsigma^T(\vec{p},h)\psi^*(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\tfrac{m}{E}}u_\varsigma^{\lambda'_\varsigma}(\vec{p},h)\psi_{\lambda'_\varsigma}^+(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

3.3 Quantization of Majorana equation under arbitrary representation

Cor. 3.3.1. $\begin{cases}\{\psi_{\lambda_\varsigma}(\vec{r},t),\psi_{\lambda'_\varsigma}^+(\vec{r}',t)\}=\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ \psi^*(\vec{r},t)=S^TS\psi(\vec{r},t)\end{cases}\Leftrightarrow\begin{cases}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\ \{a_\varsigma^+(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=0\end{cases}$

Cor. 3.3.2. $L=-\frac{1}{2}\int\bar{\psi}(\gamma_s^a\partial_a+m)\psi dr^3=0$

Cor. 3.3.3. $\begin{cases}:H:=:\frac{1}{2}\int i\psi^+\partial_t\psi dr^3:=:\frac{1}{2}\int[\bar{\psi}(\gamma_s\cdot\nabla)+m]\psi]dr^3:\overset{\varsigma=1}{\equiv}\int\sum\limits_h E(p)a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\\ :\vec{P}:=:\frac{1}{2}\int-i\psi^+\nabla\psi dr^3:\overset{\varsigma=1}{\equiv}\int\sum\limits_h\vec{p}a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\end{cases}$

Cor. 3.3.4. $\begin{cases}:P_u:=:\frac{1}{2}\int-i\psi^+\partial_u\psi dr^3:\overset{\varsigma=1}{\equiv}\int\sum\limits_h p_u a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\\ :Q:=:\int\psi^+\psi dr^3:=\int\sum\limits_h 0a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)d^3\vec{p}\overset{\varsigma=1}{\equiv}0\end{cases}$

Cor. 3.3.5. $[P_u,P_v]=0,[Q,P_u]=0$

Under the representation transformation, the annihilation production operator and its commutation relationship are scalar and invariant. The system energy momentum operator and the conserved charge are also scalars and invariants. The wave function operator and its commutation relationship are representational covariates.

4 Equivalence between Majorana equation and massive neutrino equation

4.1 Equivalent anticommutative relations of Majorana and massive neutrino equation

Anticommutative relation of Majorana equation under Dirac representation:

Cor. 4.1.1. $\begin{cases}\{\psi_{\lambda_\varsigma}(\vec{r},t),\psi_{\lambda'_\varsigma}^+(\vec{r}',t)\}=\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ \psi^*(\vec{r},t)=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi(\vec{r},t)\end{cases}\Leftrightarrow\begin{cases}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\ \{a_\varsigma^+(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=0\end{cases}$

Cor. 4.1.2. $\psi^*(\vec{r},t) = -e^{2i\theta}\sigma_y \otimes \sigma_y \psi(\vec{r},t) \Leftrightarrow \psi(\vec{r},t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}$

Equivalent transformation of canonical anticommutative relation for Majorana equation and massive neutrino equation:

Cor. 4.1.3. $\begin{cases}\{\psi_{\lambda_\varsigma}(\vec{r},t), \psi^+_{\lambda'_\varsigma}(\vec{r}',t)\} = \delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \psi(\vec{r},t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}\end{cases} \Leftrightarrow \begin{cases}\{\nu_{A_\varsigma}(\vec{r},t), \nu^+_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r},t), \nu_{B_\varsigma}(\vec{r}',t)\} = 0 \\ \{\nu^+_{A'_\varsigma}(\vec{r},t), \nu^+_{B'_\varsigma}(\vec{r}',t)\} = 0\end{cases}$

Proof:

$$\begin{cases}\{\psi_{\lambda_\varsigma}(\vec{r},t), \psi^+_{\lambda'_\varsigma}(\vec{r}',t)\} = \delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \psi(\vec{r},t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}\frac{1}{2}\{\nu_{A_\varsigma}(\vec{r},t) - e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), \nu^*_{A'_\varsigma}(\vec{r}',t) - e^{2i\theta}(\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \frac{1}{2}\{\nu_{A_\varsigma}(\vec{r},t) - e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), -\nu^*_{A'_\varsigma}(\vec{r}',t) - e^{2i\theta}(\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = 0 \\ \frac{1}{2}\{-\nu_{A_\varsigma}(\vec{r},t) - e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), \nu^*_{A'_\varsigma}(\vec{r}',t) - e^{2i\theta}(\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = 0 \\ \frac{1}{2}\{-\nu_{A_\varsigma}(\vec{r},t) - e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), -\nu^*_{A'_\varsigma}(\vec{r}',t) - e^{2i\theta}(\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}')\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}\{\nu_{A_\varsigma}(\vec{r},t) - e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), \nu^*_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r},t) - e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), -e^{2i\theta}(\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{-\nu_{A_\varsigma}(\vec{r},t) - e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), -e^{2i\theta}(\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r},t) + e^{-2i\theta}(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), \nu^*_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}')\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}\{\nu_{A_\varsigma}(\vec{r},t), \nu^*_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r},t), (\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = 0 \\ \{(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), (\varepsilon\nu)_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{(\varepsilon\nu)^*_{A_\varsigma}(\vec{r},t), \nu^*_{A'_\varsigma}(\vec{r}',t)\} = 0\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}\{\nu_{A_\varsigma}(\vec{r},t), \nu^+_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r},t), \nu_{B_\varsigma}(\vec{r}',t)\} = 0 \\ \{\nu^+_{A'_\varsigma}(\vec{r},t), \nu^+_{B'_\varsigma}(\vec{r}',t)\} = 0\end{cases}$$
□

Cor. 4.1.4. $\begin{cases}\{\nu_{A_\varsigma}(\vec{r},t), \nu^+_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r},t), \nu_{B_\varsigma}(\vec{r}',t)\} = 0 \\ \{\nu^+_{A'_\varsigma}(\vec{r},t), \nu^+_{B'_\varsigma}(\vec{r}',t)\} = 0\end{cases} \Leftrightarrow \begin{cases}\{a_\varsigma(\vec{p},h), a^+_\varsigma(\vec{p}',h')\} = \delta_{hh'}\delta^3(\vec{r}-\vec{r}') \\ \{a_\varsigma(\vec{p},h), a_\varsigma(\vec{p}',h')\} = 0 \\ \{a^+_\varsigma(\vec{p},h), a^+_\varsigma(\vec{p}',h')\} = 0\end{cases}$

4.2 Majorana and neutrino actions under Dirac representation

Majorana lagrangian: $L = -\frac{1}{2}\int\bar{\psi}(\gamma^a\partial_a + m)\psi dr^3$ **, Majorana hamiltonian:** $H = \frac{1}{2}\int\bar{\psi}(\gamma\cdot\nabla + m)\psi dr^3$

Cor. 4.2.1. $\gamma^a\partial_a = \left[\begin{smallmatrix}\varsigma\partial_\pi & -i\sigma\cdot\nabla \\ i\sigma\cdot\nabla & -\varsigma\partial_\pi\end{smallmatrix}\right], \gamma^4\gamma^a\partial_a = \left[\begin{smallmatrix}\partial_\pi & -i\varsigma\sigma\cdot\nabla \\ -i\varsigma\sigma\cdot\nabla & \partial_\pi\end{smallmatrix}\right], \gamma^a = (\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_z)$

Cor. 4.2.2. $\bar{\psi}(\vec{r},t)\psi(\vec{r},t) = \varsigma\{\nu^+(\vec{r},t)[-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] + [ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\nu(\vec{r},t)\}$

Proof: $\bar{\psi}(\vec{r},t)\psi(\vec{r},t) = \psi^+(\vec{r},t)\gamma^4\psi(\vec{r},t)$

$= \frac{1}{2}\varsigma\left[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y, -\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y\right]\begin{bmatrix}\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ \nu(\vec{r},t) + ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}$

$= \frac{1}{2}\varsigma\{[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y][\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - [\nu^+(\vec{r},t) - ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y][\nu(\vec{r},t) + ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]\}$

$= \varsigma\{\nu^+(\vec{r},t)[-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] + [ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\nu(\vec{r},t)\}$ □

Cor. 4.2.3. $\bar{\psi}(\vec{r},t)\gamma^a\partial_a\psi(\vec{r},t) = i\varsigma[\nu^+(\vec{r},t)(\sigma, -i\varsigma)^a\partial_a\nu(\vec{r},t) - \nu^T(\vec{r},t)(\sigma, i\varsigma)^a\partial_a\nu^*(\vec{r},t)]$

Proof: $\bar{\psi}(\vec{r},t)\gamma^a\partial_a\psi(\vec{r},t) = \psi^+(\vec{r},t)\gamma^4\gamma^a\partial_a\psi(\vec{r},t)$

$= \frac{1}{2}\left[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y, -\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y\right]\left[\begin{smallmatrix}\partial_\pi & -i\varsigma\sigma\cdot\nabla \\ -i\varsigma\sigma\cdot\nabla & \partial_\pi\end{smallmatrix}\right]\begin{bmatrix}\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}$

$= \frac{1}{2}\left[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y, -\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y\right]\partial_\pi\begin{bmatrix}\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}$

$+ \frac{1}{2}\left[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y, -\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y\right](-i\varsigma\sigma\cdot\nabla)\begin{bmatrix}-\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ \nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}$

$= \frac{1}{2}\{[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\partial_\pi[\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]$
$+ [\nu^+(\vec{r},t) - ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\partial_\pi[\nu(\vec{r},t) + ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]\}$
$+ \frac{1}{2}\{[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y](i\varsigma\sigma\cdot\nabla)[\nu(\vec{r},t) + ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]$
$+ [\nu^+(\vec{r},t) - ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y](i\varsigma\sigma\cdot\nabla)[\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]\}$
$= [\nu^+(\vec{r},t)\partial_\pi\nu(\vec{r},t) + \nu^T(\vec{r},t)\partial_\pi\nu^*(\vec{r},t)] + [\nu^+(\vec{r},t)(i\varsigma\sigma\cdot\nabla)\nu(\vec{r},t) - \nu^T(\vec{r},t)\sigma_y(i\varsigma\sigma\cdot\nabla)\sigma_y\nu^*(\vec{r},t)]$
$= i\varsigma[\nu^+(\vec{r},t)(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t) - \nu^T(\vec{r},t)\sigma_y(\sigma,i\varsigma)^a\partial_a\sigma_y\nu^*(\vec{r},t)]$ □

Neutrino lagrangian:

Cor. 4.2.4. $L = -\frac{1}{2}\int\bar{\psi}(\vec{r},t)(\gamma^a\partial_a + m)\psi(\vec{r},t)$
$= -\frac{1}{2}i\varsigma\int\nu^+(\vec{r},t)[(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t) - me^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - \nu^T(\vec{r},t)[(-\sigma^*,i\varsigma)^a\partial_a\nu^*(\vec{r},t) - me^{2i\theta}\sigma_y\nu(\vec{r},t)]$

Proof: $L = -\frac{1}{2}\int\bar{\psi}(\vec{r},t)(\gamma^a\partial_a + m)\psi(\vec{r},t)$
$= -\frac{1}{2}\int i\varsigma[\nu^+(\vec{r},t)(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t) - \nu^T(\vec{r},t)\sigma_y(\sigma,i\varsigma)^a\partial_a\sigma_y\nu^*(\vec{r},t)]$
$+ mi\varsigma\{\nu^+(\vec{r},t)[-e^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] + [e^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\nu(\vec{r},t)\}$
$= -\frac{1}{2}i\varsigma\int\nu^+(\vec{r},t)[(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t) - me^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - \nu^T(\vec{r},t)\sigma_y[(\sigma,i\varsigma)^a\partial_a\sigma_y\nu^*(\vec{r},t) + me^{2i\theta}\sigma_y[\sigma_y\nu^*(\vec{r},t)]^*]$
$= -\frac{1}{2}i\varsigma\int\nu^+(\vec{r},t)[(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t) - me^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - \nu^T(\vec{r},t)[(-\sigma^*,i\varsigma)^a\partial_a\nu^*(\vec{r},t) - me^{2i\theta}\sigma_y\nu(\vec{r},t)]$ □

Neutrino hamiltonian:

Cor. 4.2.5. $H = \frac{1}{2}\int\bar{\psi}(\gamma\cdot\nabla + m)\psi dr^3$
$= i\varsigma\frac{1}{2}\int[\nu^+(\vec{r},t)\sigma\cdot\nabla\nu(\vec{r},t) + \nu^T(\vec{r},t)\sigma^*\cdot\nabla\nu^*(\vec{r},t)] - m[e^{-2i\theta}\nu^+(\vec{r},t)\sigma_y\nu^*(\vec{r},t) - e^{2i\theta}\nu^T(\vec{r},t)\sigma_y\nu(\vec{r},t)]dr^3$

Neutrino charge:

Cor. 4.2.6. $Q = \int\psi^+\psi dr^3 = \int\nu^+(\vec{r},t)\nu(\vec{r},t) + \nu^T(\vec{r},t)\nu^*(\vec{r},t)dr^3 \simeq \int\nu^+(\vec{r},t)\nu(\vec{r},t) + \nu^T(\vec{r},t)\nu^*(\vec{r},t)dr^3$

Energy and momentum of neutrino:

Cor. 4.2.7. $P_u = -i\int\psi^+\partial_u\psi dr^3 = -i\int\nu^+(\vec{r},t)\partial_u\nu(\vec{r},t) + \nu^T(\vec{r},t)\partial_u\nu^*(\vec{r},t)dr^3$

Cor. 4.2.8. $[P_u, P_v] = 0, [Q, P_u] = 0$

5 Plane wave solutions and direct quantization of massive neutrino equation [38]

5.1 Properties of spin basis for massive neutrino equation

Cor. 5.1.1. $(\sigma,-i\varsigma)_a\partial^a\nu(x) - me^{-2i\theta}\sigma_y\nu^*(x) = 0$

Cor. 5.1.2. $\begin{cases}\eta(\vec{p},\frac{1}{2}) := \frac{E+m-\varsigma\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}}\begin{bmatrix}1\\0\end{bmatrix} = u_1(\vec{p},\frac{1}{2}) - u_2(\vec{p},\frac{1}{2})\\ \eta(\vec{p},-\frac{1}{2}) := \frac{E+m-\varsigma\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\1\end{bmatrix} = u_1(\vec{p},-\frac{1}{2}) - u_2(\vec{p},-\frac{1}{2})\end{cases}$

Cor. 5.1.3. $\eta(\vec{p},h) = u_1(\vec{p},h) - u_2(\vec{p},h), \eta^+(\vec{p},h)\eta(-\vec{p},h') = \delta_{hh'}, \eta^T(\vec{p},h)\eta^*(-\vec{p},h') = \delta_{hh'}$

Cor. 5.1.4. $\begin{cases}\sum_h\eta(\vec{p},h)\eta^+(\vec{p},h) = \frac{E-\varsigma\sigma\cdot\vec{p}}{m} = \frac{-\varsigma(\sigma,i\varsigma)^a p_a}{m}\\ \sum_h(-1)^{h-\frac{1}{2}}\eta(\vec{p},h)\eta^+(\vec{p},h) = i\sigma_y\end{cases}$

Cor. 5.1.5. $\begin{cases}\sum_h[\eta(\vec{p},h)\eta^+(\vec{p},h) + \eta(-\vec{p},h)\eta^+(-\vec{p},h)] = \frac{2E}{m}\\ \sum_h[\eta^*(\vec{p},h)\eta^T(\vec{p},h) + \eta^*(-\vec{p},h)\eta^T(-\vec{p},h)] = \frac{2E}{m}\\ \sum_h(-1)^{h-\frac{1}{2}}[\eta(\vec{p},h)\eta^T(\vec{p},-h) + \eta(-\vec{p},-h)\eta^T(-\vec{p},h)] = 0\\ \sum_h(-1)^{h-\frac{1}{2}}[\eta^*(\vec{p},h)\eta^+(\vec{p},-h) + \eta^*(-\vec{p},-h)\eta^+(-\vec{p},h)] = 0\end{cases}$

Cor. 5.1.6. $\eta^+(\vec{p},h)\eta(\vec{p}',h') = -(-1)^{h+h'}\eta^T(-\vec{p},-h)\eta^*(-\vec{p}',-h')$

Cor. 5.1.7. $\begin{cases}\eta^+(\vec{p},h)\eta(\vec{p},h') - (-1)^{h+h'}\eta^T(\vec{p},-h)\eta^*(\vec{p},-h') = \frac{2E}{m}\delta_{hh'}\\ \eta^+(\vec{p},h)\eta(\vec{p},h') + \eta^+(-\vec{p},h)\eta(-\vec{p},h') = \frac{2E}{m}\delta_{hh'}\end{cases}$

Cor. 5.1.8. $\begin{cases}\eta^+(\vec{p},h)\eta(-\vec{p},h') - (-1)^{h+h'}\eta^T(\vec{p},-h)\eta^*(-\vec{p},-h') = 0\\ \eta^+(\vec{p},h)\eta(-\vec{p},h') - \eta^+(-\vec{p},h)\eta(\vec{p},h') = 0\end{cases}$

Cor. 5.1.9. $\begin{cases}\eta^+(\vec{p},h)\eta(-\vec{p},-h') - (-1)^{h'-h}\eta^T(\vec{p},-h)\eta^*(-\vec{p},h') = 0\\ \eta^+(\vec{p},h)\eta(-\vec{p},-h') - \eta^+(-\vec{p},h)\eta(\vec{p},-h') = 0\end{cases}$

5.2 Obtain plane wave solutions of massive neutrino equation from Majorana equation

Cor. 5.2.1. $\nu(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\sqrt{\frac{m}{2E}}\sum\limits_{h}\frac{E+m-\varsigma\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}}[a_\varsigma(\vec{p},h)\xi(h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_\varsigma^+(\vec{p},h)ie^{-2i\theta}\sigma_y\xi(h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 5.2.2. $\nu(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\sqrt{\frac{m}{2E}}\sum\limits_{h}[a_\varsigma(\vec{p},h)\eta(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+(-1)^{h-\frac{1}{2}}e^{-2i\theta}a_\varsigma^+(\vec{p},h)\eta(\vec{p},-h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 5.2.3. $a_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)\psi(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

$\Leftrightarrow a_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{E}{2m}}\int[\eta^+(\vec{p},h)\nu(\vec{r},t)-(-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta^T(\vec{p},-h)\nu^*(\vec{r},t)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

Proof: $a_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)\psi(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}\left[u_1^+(\vec{p},h),u_2^+(\vec{p},h)\right]\frac{1}{\sqrt{2}}\begin{bmatrix}\nu(\vec{r},t)-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\\-\nu(\vec{r},t)-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

$=\frac{1}{(2\pi)^{3/2}}\int\sqrt{\frac{m}{2E}}[\eta^+(\vec{p},h)\nu(\vec{r},t)-ie^{-2i\theta}\eta^+(-\vec{p},h)\sigma_y\nu^*(\vec{r},t)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

$=\frac{1}{(2\pi)^{3/2}}\int\sqrt{\frac{m}{2E}}[\eta^+(\vec{p},h)\nu(\vec{r},t)-(-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta^T(\vec{p},-h)\nu^*(\vec{r},t)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$ □

5.3 Direct verification of above plane wave solutions and quantization conditions

Cor. 5.3.1. $a_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{E}{2m}}\int[\eta^+(\vec{p},h)\nu(\vec{r},t)-(-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta^T(\vec{p},-h)\nu^*(\vec{r},t)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

Proof: $\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{2E}}\int[\eta^+(\vec{p},h)\nu(\vec{r},t)-(-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta^+(-\vec{p},-h)\nu^*(\vec{r},t)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

$=\frac{1}{(2\pi)^3}\sqrt{\frac{m}{2E}}\int d^3\vec{r}d^3\vec{p}'\sqrt{\frac{m}{2E'}}$

$\sum\limits_{h'}\eta^+(\vec{p},h)[a_\varsigma(\vec{p}',h')\eta(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}+(-1)^{h'-\frac{1}{2}}e^{-2i\theta}a_\varsigma^+(\vec{p}',h')\eta(\vec{p}',-h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}$

$-(-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta^T(\vec{p},-h)[a_\varsigma^+(\vec{p}',h')\eta^*(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}+(-1)^{h'-\frac{1}{2}}e^{2i\theta}a_\varsigma(\vec{p}',h')\eta^*(\vec{p}',-h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}$

$=\sqrt{\frac{m}{2E}}\int\sqrt{\frac{m}{2E'}}\sum\limits_{h'}[a_\varsigma(\vec{p}',h')\eta^+(\vec{p},h)\eta(\vec{p}',h')\delta^3(\vec{p}-\vec{p}')+(-1)^{h-\frac{1}{2}}e^{-2i\theta}a_\varsigma^+(\vec{p}',h')\eta^+(\vec{p},h)\eta(\vec{p}',-h')\delta^3(\vec{p}+\vec{p}')e^{2i\varsigma Et}]$

$-(-1)^{h'-\frac{1}{2}}e^{-2i\theta}a_\varsigma^+(\vec{p}',h')\eta^T(\vec{p},-h)\eta^*(\vec{p}',h')\delta^3(\vec{p}+\vec{p}')e^{2i\varsigma Et}-(-1)^{h+h'}a_\varsigma(\vec{p}',h')\eta^T(\vec{p},-h)\eta^*(\vec{p}',-h')\delta^3(\vec{p}-\vec{p}')]d^3\vec{p}'$

$=\frac{m}{2E}\sum\limits_{h'}[a_\varsigma(\vec{p},h')\eta^+(\vec{p},h)\eta(\vec{p},h')+(-1)^{h-\frac{1}{2}}e^{-2i\theta}a^+(-\vec{p},h')\eta^+(\vec{p},h)\eta(-\vec{p},-h')e^{2i\varsigma Et}]$

$-(-1)^{h'-\frac{1}{2}}e^{-2i\theta}a^+(-\vec{p},h')\eta^T(\vec{p},-h)\eta^*(-\vec{p},h')e^{2i\varsigma Et}-(-1)^{h+h'}a_\varsigma(\vec{p},h')\eta^T(\vec{p},-h)\eta^*(\vec{p},-h')]$

$=\frac{m}{2E}\sum\limits_{h'}[a_\varsigma(\vec{p},h')[\eta^+(\vec{p},h)\eta(\vec{p},h')-(-1)^{h+h'}\eta^T(\vec{p},-h)\eta^*(\vec{p},-h')]$

$+(-1)^{h-\frac{1}{2}}e^{2i\varsigma Et}e^{-2i\theta}a^+(-\vec{p},h')[\eta^+(\vec{p},h)\eta(-\vec{p},-h')-(-1)^{h'-h}\eta^T(\vec{p},-h)\eta^*(-\vec{p},h')]$

$=a_\varsigma(\vec{p},h)$ □

Cor. 5.3.2. $a_\varsigma^+(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{E}{2m}}\int[\eta^T(\vec{p},h)\nu^*(\vec{r},t)-(-1)^{h-\frac{1}{2}}e^{2i\theta}\eta^+(\vec{p},-h)\nu(\vec{r},t)]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

By using the above two coefficient expansions, it is easy to directly derive the following canonical commutative relation.

Cor. 5.3.3. $\begin{cases}\{\nu_{A_\varsigma}(\vec{r},t),\nu_{A'_\varsigma}^+(\vec{r}',t)\}=\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\\{\nu_{A_\varsigma}(\vec{r},t),\nu_{A'_\varsigma}(\vec{r}',t)\}=0\\\{\nu_{A'_\varsigma}^+(\vec{r},t),\nu_{B'_\varsigma}^+(\vec{r}',t)\}=0\end{cases}\Rightarrow\begin{cases}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\\{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\\{a_\varsigma^+(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=0\end{cases}$

Now, in turn, we can directly derive the following canonical commutative relation by using the wave function expansion.

Cor. 5.3.4. $\begin{cases}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\\{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\\{a_\varsigma^+(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=0\end{cases}\Rightarrow\begin{cases}\{\nu_{A_\varsigma}(x),\nu_{A'_\varsigma}^+(x')\}=-\varsigma(\sigma,i\varsigma)^a\partial_a\Delta(x-x')\\\{\nu_{A_\varsigma}(\vec{r},t),\nu_{A'_\varsigma}^+(\vec{r}',t)\}=\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}')\end{cases}$

Proof:

$\{\nu_{A_\varsigma}(x),\nu_{A'_\varsigma}^+(x')\}$

$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum\limits_{h,h'}\{[a_\varsigma(\vec{p},h)\eta_{A_\varsigma}(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{-2i\theta}a_\varsigma^+(\vec{p},h)(-1)^{h-\frac{1}{2}}\eta_{A_\varsigma}(\vec{p},-h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}],$

$[a_\varsigma^+(\vec{p}',h')\eta_{A'_\varsigma}^+(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}+e^{2i\theta}a_\varsigma(\vec{p}',h')(-1)^{h'-\frac{1}{2}}\eta_{A'_\varsigma}^+(\vec{p}',-h')e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]\}d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}[\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p}',h')]\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}[e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]$
$+\eta_{A_\varsigma}(\vec{p},-h)\eta_{A'_\varsigma}^+(\vec{p}',-h')(-1)^{h-\frac{1}{2}}\{a_\varsigma^+(\vec{p},h),a_\varsigma(\vec{p}',h)\}(-1)^{h'-\frac{1}{2}}[e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}[\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p}',h')]\delta_{hh'}\delta^3(\vec{p}-\vec{p}')[e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]$
$+\eta_{A_\varsigma}(\vec{p},-h)\eta_{A'_\varsigma}^+(\vec{p}',-h')(-1)^{h-\frac{1}{2}}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')(-1)^{h'-\frac{1}{2}}[e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h}\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p},h)e^{i\varsigma p\cdot(x-x')}+\eta_{A_\varsigma}(\vec{p},-h)\eta_{A'_\varsigma}^+(\vec{p},-h)e^{-i\varsigma p\cdot(x-x')}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h}\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p},h)[e^{i\varsigma p\cdot(x-x')}+e^{-i\varsigma p\cdot(x-x')}]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\frac{-\varsigma(\sigma,i\varsigma)^a p_a}{m}[e^{i\varsigma p\cdot(x-x')}+e^{-i\varsigma p\cdot(x-x')}]d^3\vec{p}$
$=-(\sigma,i\varsigma)^a\partial_a\frac{-i}{(2\pi)^3}\int\frac{1}{2E}[e^{i\varsigma p\cdot(x-x')}-e^{-i\varsigma p\cdot(x-x')}]d^3\vec{p}=-\varsigma(\sigma,i\varsigma)^a\partial_a\Delta(x-x')$ □

Proof:
$\{\nu_{A_\varsigma}(\vec{r},t),\nu_{A'_\varsigma}^+(\vec{r}',t)\}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}\{[a_\varsigma(\vec{p},h)\eta_{A_\varsigma}(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{-2i\theta}a_\varsigma^+(\vec{p},h)(-1)^{h-\frac{1}{2}}\eta_{A_\varsigma}(\vec{p},-h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}],$
$[a_\varsigma^+(\vec{p}',h')\eta_{A'_\varsigma}^+(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}+e^{2i\theta}a_\varsigma(\vec{p}',h')(-1)^{h'-\frac{1}{2}}\eta_{A'_\varsigma}^+(\vec{p}',-h')e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]\}d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}[\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p}',h')]\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}[e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]$
$+\eta_{A_\varsigma}(\vec{p},-h)\eta_{A'_\varsigma}^+(\vec{p}',-h')(-1)^{h-\frac{1}{2}}\{a_\varsigma^+(\vec{p},h),a_\varsigma(\vec{p}',h)\}(-1)^{h'-\frac{1}{2}}[e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}[\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p}',h')]\delta_{hh'}\delta^3(\vec{p}-\vec{p}')[e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]$
$+\eta_{A_\varsigma}(\vec{p},-h)\eta_{A'_\varsigma}^+(\vec{p}',-h')(-1)^{h-\frac{1}{2}}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')(-1)^{h'-\frac{1}{2}}[e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h}\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p},h)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}+\eta_{A_\varsigma}(\vec{p},-h)\eta_{A'_\varsigma}^+(\vec{p},-h)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h}\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p},h)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}+\eta_{A_\varsigma}(-\vec{p},h)\eta_{A'_\varsigma}^+(-\vec{p},h)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h}[\eta_{A_\varsigma}(\vec{p},h)\eta_{A'_\varsigma}^+(\vec{p},h)+\eta_{A_\varsigma}(-\vec{p},h)\eta_{A'_\varsigma}^+(-\vec{p},h)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h}[\eta(\vec{p},h)\eta^+(\vec{p},h)+\eta(-\vec{p},h)\eta^+(-\vec{p},h)]_{A_\varsigma A'_\varsigma}e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$=\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}')$ □

Cor. 5.3.5. $\begin{cases}\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\ \{a_\varsigma^+(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}=0\end{cases}\Rightarrow\begin{cases}\{\nu_{A_\varsigma}(x),\nu_{B_\varsigma}(x')\}=i\varsigma me^{-2i\theta}\varepsilon_{A_\varsigma B_\varsigma}\Delta(x-x')\\ \{\nu_{A'_\varsigma}^+(x),\nu_{B'_\varsigma}^+(x')\}=-i\varsigma me^{2i\theta}\varepsilon_{A'_\varsigma B'_\varsigma}\Delta(x-x')\\ \{\nu_{A_\varsigma}(\vec{r},t),\nu_{B_\varsigma}(\vec{r}',t)\}=0\\ \{\nu_{A'_\varsigma}^+(\vec{r},t),\nu_{B'_\varsigma}^+(\vec{r}',t)\}=0\end{cases}$

Proof:
$\{\nu_{A_\varsigma}(x),\nu_{B_\varsigma}(x')\}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}\{[a_\varsigma(\vec{p},h)\eta_{A_\varsigma}(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+e^{-2i\theta}a_\varsigma^+(\vec{p},h)(-1)^{h-\frac{1}{2}}\eta_{A_\varsigma}(\vec{p},-h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}],$
$[a_\varsigma(\vec{p}',h')\eta_{B_\varsigma}(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}+e^{-2i\theta}a_\varsigma^+(\vec{p}',h')(-1)^{h'-\frac{1}{2}}\eta_{B_\varsigma}(\vec{p}',-h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]\}d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac{1}{2}}e^{-2i\theta}[\eta_{A_\varsigma}(\vec{p},h)\eta_{B_\varsigma}(\vec{p}',-h')]\{a_\varsigma(\vec{p},h),a_\varsigma^+(\vec{p}',h')\}[e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]$
$+(-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta_{A_\varsigma}(\vec{p},-h)\eta_{B_\varsigma}(\vec{p}',h')\{a_\varsigma^+(\vec{p},h),a_\varsigma(\vec{p}',h)\}[e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac{1}{2}}e^{-2i\theta}[\eta_{A_\varsigma}(\vec{p},h)\eta_{B_\varsigma}(\vec{p}',-h')]\delta_{hh'}\delta^3(\vec{p}-\vec{p}')[e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]$
$+(-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta_{A_\varsigma}(\vec{p},-h)\eta_{B_\varsigma}(\vec{p}',h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')[e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}]d^3\vec{p}'d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}e^{-2i\theta}\sum_{h}(-1)^{h-\frac{1}{2}}[\eta_{A_\varsigma}(\vec{p},h)\eta_{B_\varsigma}(\vec{p},-h)]e^{i\varsigma p\cdot(x-x')}+(-1)^{h-\frac{1}{2}}\eta_{A_\varsigma}(\vec{p},-h)\eta_{B_\varsigma}(\vec{p},h)e^{-i\varsigma p\cdot(x-x')}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{m}{E}e^{-2i\theta}\sum_{h}[h\eta_{A_\varsigma}(\vec{p},h)\eta_{B_\varsigma}(\vec{p},-h)][e^{i\varsigma p\cdot(x-x')}-e^{-i\varsigma p\cdot(x-x')}]d^3\vec{p}$
$=i\varsigma m\varepsilon_{A_\varsigma B_\varsigma}e^{-2i\theta}\frac{-i\varsigma}{(2\pi)^3}\int\frac{1}{2E}[e^{i\varsigma p\cdot(x-x')}-e^{-i\varsigma p\cdot(x-x')}]d^3\vec{p}$
$=i\varsigma me^{-2i\theta}\varepsilon_{A_\varsigma B_\varsigma}\Delta(x-x')$ □

Proof:

$$\{\nu_{A_\varsigma}(\vec r,t),\nu_{B_\varsigma}(\vec r',t)\}$$
$$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}\{[a_\varsigma(\vec p,h)\eta_{A_\varsigma}(\vec p,h)e^{i\varsigma(\vec p\cdot\vec r-Et)}+e^{-2i\theta}a_\varsigma^+(\vec p,h)(-1)^{h-\frac12}\eta_{A_\varsigma}(\vec p,-h)e^{-i\varsigma(\vec p\cdot\vec r-Et)}],$$
$$[a_\varsigma(\vec p',h')\eta_{B_\varsigma}(\vec p',h')e^{i\varsigma(\vec p'\cdot\vec r'-E't)}+e^{-2i\theta}a_\varsigma^+(\vec p',h')(-1)^{h'-\frac12}\eta_{B_\varsigma}(\vec p',-h')e^{-i\varsigma(\vec p'\cdot\vec r'-E't)}]\}d^3\vec p'd^3\vec p$$
$$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac12}e^{-2i\theta}[\eta_{A_\varsigma}(\vec p,h)\eta_{B_\varsigma}(\vec p',-h')]\{a_\varsigma(\vec p,h),a_\varsigma^+(\vec p',h')\}[e^{i\varsigma(\vec p\cdot\vec r-Et)}e^{-i\varsigma(\vec p'\cdot\vec r'-E't)}]$$
$$+(-1)^{h-\frac12}e^{-2i\theta}\eta_{A_\varsigma}(\vec p,-h)\eta_{B_\varsigma}(\vec p',h')\{a_\varsigma^+(\vec p,h),a_\varsigma(\vec p',h)\}[e^{-i\varsigma(\vec p\cdot\vec r-Et)}e^{i\varsigma(\vec p'\cdot\vec r'-E't)}]d^3\vec p'd^3\vec p$$
$$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac12}e^{-2i\theta}[\eta_{A_\varsigma}(\vec p,h)\eta_{B_\varsigma}(\vec p',-h')]\delta_{hh'}\delta^3(\vec p-\vec p')[e^{i\varsigma(\vec p\cdot\vec r-Et)}e^{-i\varsigma(\vec p'\cdot\vec r'-E't)}]$$
$$+(-1)^{h-\frac12}e^{-2i\theta}\eta_{A_\varsigma}(\vec p,-h)\eta_{B_\varsigma}(\vec p',h')\delta_{hh'}\delta^3(\vec p-\vec p')[e^{-i\varsigma(\vec p\cdot\vec r-Et)}e^{i\varsigma(\vec p'\cdot\vec r'-E't)}]d^3\vec p'd^3\vec p$$
$$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}e^{-2i\theta}\sum_h(-1)^{h-\frac12}[\eta_{A_\varsigma}(\vec p,h)\eta_{B_\varsigma}(\vec p,-h)]e^{i\varsigma\vec p\cdot(\vec r-\vec r')}+(-1)^{h-\frac12}\eta_{A_\varsigma}(\vec p,-h)\eta_{B_\varsigma}(\vec p,h)e^{-i\varsigma\vec p\cdot(\vec r-\vec r')}d^3\vec p$$
$$=\frac{1}{(2\pi)^3}\int\frac{m}{2E}e^{-2i\theta}e^{i\varsigma\vec p\cdot(\vec r-\vec r')}\sum_h(-1)^{h-\frac12}[\eta_{A_\varsigma}(\vec p,h)\eta_{B_\varsigma}(\vec p,-h)+\eta_{A_\varsigma}(-\vec p,-h)\eta_{B_\varsigma}(-\vec p,h)]d^3\vec p$$
$$=0$$
□

5.4 Summary of anticommutative rules for massive neutrino equation

Cor. 5.4.1.
$$\begin{cases}\{a_\varsigma(\vec p,h),a_\varsigma^+(\vec p',h')\}=\delta_{hh'}\delta^3(\vec p-\vec p')\\\{a_\varsigma(\vec p,h),a_\varsigma(\vec p',h')\}=0\\\{a_\varsigma^+(\vec p,h),a_\varsigma^+(\vec p',h')\}=0\end{cases}\Leftrightarrow\begin{cases}\{\psi_{s\lambda_\varsigma}(x),\psi_{s\lambda'_\varsigma}(x')\}=i[(m-\gamma_s^a\partial_a)\gamma_s^4]_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')\\\{\psi_{s\lambda_\varsigma}(\vec r,t),\psi_{s\lambda'_\varsigma}(\vec r',t)\}=\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec r-\vec r')\\\psi_s^*(\vec r,t)=\psi_s(\vec r,t)\end{cases}$$

Cor. 5.4.2.
$$\begin{cases}\{a_\varsigma(\vec p,h),a_\varsigma^+(\vec p',h')\}=\delta_{hh'}\delta^3(\vec p-\vec p')\\\{a_\varsigma(\vec p,h),a_\varsigma(\vec p',h')\}=0\\\{a_\varsigma^+(\vec p,h),a_\varsigma^+(\vec p',h')\}=0\end{cases}\Leftrightarrow\begin{cases}\{\psi_{\lambda_\varsigma}(x),\psi^+_{\lambda'_\varsigma}(x')\}=i[(m-\gamma^a\partial_a)\gamma^4]_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')\\\{\psi_{\lambda_\varsigma}(\vec r,t),\psi^+_{\lambda'_\varsigma}(\vec r',t)\}=\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta^3(\vec r-\vec r')\\\psi^*(\vec r,t)=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi(\vec r,t)\end{cases}$$

Cor. 5.4.3.
$$\begin{cases}\{a_\varsigma(\vec p,h),a_\varsigma^+(\vec p',h')\}=\delta_{hh'}\delta^3(\vec p-\vec p')\\\{a_\varsigma(\vec p,h),a_\varsigma(\vec p',h')\}=0\\\{a_\varsigma^+(\vec p,h),a_\varsigma^+(\vec p',h')\}=0\end{cases}\Leftrightarrow\begin{cases}\{\nu_{A_\varsigma}(x),\nu^+_{A'_\varsigma}(x')\}=-\varsigma(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x-x')\\\{\nu_{A_\varsigma}(x),\nu_{B_\varsigma}(x')\}=i\varsigma me^{-2i\theta}\varepsilon_{A_\varsigma B_\varsigma}\Delta(x-x')\\\{\nu^+_{A'_\varsigma}(x),\nu^+_{B'_\varsigma}(x')\}=-i\varsigma me^{2i\theta}\varepsilon_{A'_\varsigma B'_\varsigma}\Delta(x-x')\end{cases}$$

Cor. 5.4.4.
$$\begin{cases}\{a_\varsigma(\vec p,h),a_\varsigma^+(\vec p',h')\}=\delta_{hh'}\delta^3(\vec p-\vec p')\\\{a_\varsigma(\vec p,h),a_\varsigma(\vec p',h')\}=0\\\{a_\varsigma^+(\vec p,h),a_\varsigma^+(\vec p',h')\}=0\end{cases}\Leftrightarrow\begin{cases}\{\nu_{A_\varsigma}(\vec r,t),\nu^+_{A'_\varsigma}(\vec r',t)\}=\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec r-\vec r')\\\{\nu_{A_\varsigma}(\vec r,t),\nu_{B_\varsigma}(\vec r',t)\}=0\\\{\nu^+_{A'_\varsigma}(\vec r,t),\nu^+_{B'_\varsigma}(\vec r',t)\}=0\end{cases}$$

5.5 Summary of three equivalent descriptions for massive neutrino equation

5.5.1 Construct plane wave solutions from massive neutrino equation

Cor. 5.5.1.
$$\begin{cases}(\sigma,-i\varsigma)_a\partial^a\nu(x)-me^{-2i\theta}\sigma_y\nu^*(x)=0\\\psi(x)=\frac{1}{\sqrt2}\begin{bmatrix}\nu(x)-ie^{-2i\theta}\sigma_y\nu^*(x)\\-\nu(x)-ie^{-2i\theta}\sigma_y\nu^*(x)\end{bmatrix}\end{cases}\Leftrightarrow\begin{cases}(\gamma^a\partial_a+m)\psi(x)=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_z)\\\psi^*(x)=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi(x)\\\nu(x)=\frac{1}{\sqrt2}[\psi_1(x)+ie^{-2i\theta}\sigma_y\psi_1^*(x)]\end{cases}$$

$$\begin{cases}\nu(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{E+m-\varsigma\vec p\cdot\sigma}{\sqrt{2m(E+m)}}\frac{1}{\sqrt2}(\xi_0e^{i\varsigma p\cdot x}+ie^{-2i\theta}\sigma_y\xi_0^*e^{-i\varsigma p\cdot x})d^3\vec p\\\psi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{E+m+\varsigma\vec p\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\begin{bmatrix}\xi_0e^{i\varsigma p\cdot x}\\-ie^{-2i\theta}\sigma_y\xi_0^*e^{-i\varsigma p\cdot x}\end{bmatrix}d^3\vec p=\frac{1}{(2\pi)^{3/2}}\int\begin{bmatrix}\frac{(E+m)\xi_0e^{i\varsigma p\cdot x}-\varsigma\vec p\cdot\sigma(ie^{-2i\theta}\sigma_y\xi_0^*)e^{-i\varsigma p\cdot x}}{\sqrt{2m(E+m)}}\\\frac{-(E+m)(ie^{-2i\theta}\sigma_y\xi_0^*)e^{-i\varsigma p\cdot x}+\varsigma\vec p\cdot\sigma\xi_0e^{i\varsigma p\cdot x}}{\sqrt{2m(E+m)}}\end{bmatrix}d^3\vec p\end{cases}$$

$\xi_0=a(\vec p,\frac12)\begin{bmatrix}1\\0\end{bmatrix}+a(\vec p,-\frac12)\begin{bmatrix}0\\1\end{bmatrix}\ \xi_0=a(\vec p,-\frac{\varsigma}{2})\lambda(\hat p,-\frac{\varsigma}{2})+a(\vec p,\frac{\varsigma}{2})\lambda(\hat p,\frac{\varsigma}{2})$

6 Plane wave solutions and preliminary quantization of massless neutrino equation

(The following chapters will expand in detail.)

6.1 Plane wave solutions of massless neutrino equation

Cor. 6.1.1. $(\sigma,-i\varsigma)^a\partial_a\nu(\vec r,t)=0$

Cor. 6.1.2. $\nu_{A_\varsigma}(\vec r,t)=\int_{\vec p\neq0}\frac12(1-\varsigma\frac{\sigma\cdot\vec p}{|\vec p|})[\xi(\vec p)e^{i\varsigma(\vec p\cdot\vec r-Et)}+\eta(\vec p)e^{-i\varsigma(\vec p\cdot\vec r-Et)}]d^3\vec p$

$=\int_{\vec p\neq0}\lambda(p,-\varsigma)[a_+(\vec p)e^{i\varsigma(\vec p\cdot\vec r-Et)}+a_-^+(\vec p)e^{-i\varsigma(\vec p\cdot\vec r-Et)}]d^3\vec p,\ \frac{\sigma\cdot\vec p}{|\vec p|}\lambda(p,-\varsigma)=-\varsigma\lambda(p,-\varsigma)$

Cor. 6.1.3. $\nabla\nu(\vec{r},t)=\int\limits_{\vec{p}\neq 0} i\varsigma\vec{p}\lambda(p,-\varsigma)[a_+(\vec{p})e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}-a_-^+(\vec{p})e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 6.1.4. $\nu^+(\vec{r},t)=\int\limits_{\vec{p}\neq 0} \lambda^+(p,-\varsigma)[a_+^+(\vec{p})e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_-(\vec{p})e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 6.1.5. $a_+(\vec{p})=\int\lambda^+(p,-\varsigma)\nu(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r},a_-^+(\vec{p})=\int\lambda^+(p,-\varsigma)\nu(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$

Cor. 6.1.6.
$$\begin{cases}L=-i\varsigma\int\nu^+(\vec{r},t)(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t)d^3\vec{r}=0\\ H=i\int\nu^+(\vec{r},t)\partial_t\nu(\vec{r},t)d^3\vec{r}=i\varsigma\int\nu^+(\vec{r},t)\sigma\cdot\nabla\nu(\vec{r},t)d^3\vec{r}=\varsigma\int E(\vec{p})[a_+^+(\vec{p})a_+(\vec{p})-a_-(\vec{p})a_-^+(\vec{p})]d^3\vec{p}\\ \vec{P}=-i\int\nu^+(\vec{r},t)\nabla\nu(\vec{r},t)d^3\vec{r}=\varsigma\int\vec{p}[a_+^+(\vec{p})a_+(\vec{p})-a_-(\vec{p})a_-^+(\vec{p})]d^3\vec{p}\\ Q=\varsigma\int\nu^+(\vec{r},t)\nu(\vec{r},t)d^3\vec{r}=\varsigma\int[a_+^+(\vec{p})a_+(\vec{p})+a_-(\vec{p})a_-^+(\vec{p})]d^3\vec{p}\end{cases}$$

Proof:
$H=i\varsigma\int\nu^+(\vec{r},t)\sigma\cdot\nabla\nu(\vec{r},t)d^3\vec{r}=i\int\nu^+(\vec{r},t)\partial_t\nu(\vec{r},t)d^3\vec{r}$
$=i\varsigma\int d^3\vec{r}d^3\vec{p}d^3\vec{p}'$
$i\varsigma\lambda^+(\vec{p},-\varsigma)[a_+^+(\vec{p})e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_-(\vec{p})e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]\sigma\cdot\vec{p}'\lambda(\vec{p}',-\varsigma)[a_+(\vec{p}')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}-a_-^+(\vec{p}')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]$
$=\varsigma\int E(\vec{p})[a_+^+(\vec{p})a_+(\vec{p})-a_-(\vec{p})a_-^+(\vec{p})]d^3\vec{p}$ □

Proof:
$\vec{P}=-i\int\nu^+(\vec{r},t)\nabla\nu(\vec{r},t)d^3\vec{r}$
$=-i\int d^3\vec{r}d^3\vec{p}d^3\vec{p}'$
$i\varsigma\vec{p}'\lambda^+(\vec{p},-\varsigma)\lambda(\vec{p}',-\varsigma)[a_+^+(\vec{p})e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_-(\vec{p})e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}][a_+(\vec{p}')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}-a_-^+(\vec{p}')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]$
$=\varsigma\int\vec{p}[a_+^+(\vec{p})a_+(\vec{p})-a_-(\vec{p})a_-^+(\vec{p})]d^3\vec{p}$ □

Proof:
$Q=\varsigma\int\nu^+(\vec{r},t)\nu(\vec{r},t)d^3\vec{r}$
$=\varsigma\int\lambda^+(\vec{p},-\varsigma)\lambda(\vec{p}',-\varsigma)[a_+^+(\vec{p})e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_-(\vec{p})e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}][a_+(\vec{p}')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}+a_-^+(\vec{p}')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{r}d^3\vec{p}d^3\vec{p}'$
$=\varsigma\int[a_+^+(\vec{p})a_+(\vec{p})+a_-(\vec{p})a_-^+(\vec{p})]d^3\vec{p}$ □

6.2 Quantization of massless neutrino equation

Cor. 6.2.1. $\begin{cases}\{\nu_{A_\varsigma}(\vec{r},t),\nu_{A'_\varsigma}^+(\vec{r}',t)\}=\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ \{\nu_{A_\varsigma}(\vec{r},t),\nu_{A'_\varsigma}(\vec{r}',t)\}=0\\ \{\nu_{A_\varsigma}^+(\vec{r},t),\nu_{A'_\varsigma}^+(\vec{r}',t)\}=0\end{cases}\Leftrightarrow\begin{cases}\{a_s(\vec{p}),a_{s'}^+(\vec{p}')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ \{a_s(\vec{p}),a_{s'}(\vec{p}')\}=0\\ \{a_s^+(\vec{p}),a_{s'}^+(\vec{p}')\}=0\end{cases}$

Cor. 6.2.2. $\begin{cases}:P_u:\overset{\varsigma\equiv1}{=}\int p_u[a_+^+(\vec{p})a_+(\vec{p})+a_-^+(\vec{p})a_-(\vec{p})]d^3\vec{p}\\ :Q:\overset{\varsigma\equiv1}{=}\int[a_+^+(\vec{p})a_+(\vec{p})-a_-^+(\vec{p})a_-(\vec{p})]d^3\vec{p}\end{cases}$

Cor. 6.2.3. $S_a=\varepsilon_{abcd}S_{bc}P_d=\varsigma P_a$

6.3 From neutrino Weyl equation come back to Dirac representation

Cor. 6.3.1.
$(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t)-me^{-2i\theta}\sigma_y\nu^*(\vec{r},t)=0\Leftrightarrow(\gamma^a\partial_a+m)\begin{bmatrix}\nu(\vec{r},t)\\-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}=0,\gamma_a:=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$

Cor. 6.3.2. $(\sigma,-i\varsigma)^a\partial_a\nu(\vec{r},t)=0\Leftrightarrow\gamma^a\partial_a\begin{bmatrix}\nu(\vec{r},t)\\-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}=0,(\gamma_a,\gamma_5):=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z]$

Cor. 6.3.3. $\sigma_x=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\sigma_z\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix},\sigma_y=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\sigma_y\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$

Cor. 6.3.4. $\gamma^a\partial_a\begin{bmatrix}\nu(\vec{r},t)\\0\end{bmatrix}=0,(\gamma_a,\gamma_5):=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z],$

Cor. 6.3.5. $\begin{cases}\gamma^a\partial_a\psi_M(\vec{r},t)=0,\psi_M(\vec{r},t)=-e^{-2i\theta}\sigma_y\otimes\sigma_y\psi_M^*(\vec{r},t)=\begin{bmatrix}\nu(\vec{r},t)\\-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)\end{bmatrix}\\ \gamma^a\partial_a\psi_W(\vec{r},t)=0,\psi_W(\vec{r},t)=\varsigma\gamma_5\psi_W(\vec{r},t)=\begin{bmatrix}\nu(\vec{r},t)\\0\end{bmatrix};(\gamma_a,\gamma_5):=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z]\end{cases}$

7 Plane wave solutions and alternative quantization schemes for s-spin equation

7.1 Plane wave solutions of s-spin equation

Thm. 7.1.1. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(\vec{r},t) = 0, S_{ab}(s,\varsigma) = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma}(s)$

Cor. 7.1.1.
$\psi(\vec{r},t) = \int\limits_{\vec{p}\neq 0} \lambda(p,-s\varsigma)[\eta(\vec{p},-s\varsigma)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + d^+(\vec{p},-s\varsigma)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}, \frac{\sigma(s)\cdot\vec{p}}{|\vec{p}|}\lambda(p,-s\varsigma) = -s\varsigma\lambda(p,-s\varsigma)$

7.2 Energy momentum operator of s-spin equation

Def. 7.2.1. $H := \frac{i\varsigma}{s}\int \psi^+(\vec{r},t)\sigma(s)\cdot\nabla\psi(\vec{r},t)d^3\vec{r} \qquad \vec{P} := -i\int\psi^+(\vec{r},t)\nabla\psi(\vec{r},t)d^3\vec{r}$

7.3 Quantum Lorentz invariance of s-spin equation

Cor. 7.3.1. $[\psi_A(\vec{r},t), H] = i\frac{\varsigma}{s}\sigma(s)\cdot\nabla\psi_A(\vec{r},t)$

Proof: $[\psi_A(\vec{r},t), H]$
$= [\psi_A(\vec{r},t), i\int\psi^+_B(\vec{r}',t)\frac{\varsigma}{s}\sigma(s)\cdot\nabla'\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}']$
$= i\int[\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)\frac{\varsigma}{s}\sigma(s)\cdot\nabla'\delta^{BC}\psi_C(\vec{r}',t)]d^3\vec{r}'$
$= i\int\{\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)\}\frac{\varsigma}{s}\sigma(s)\cdot\nabla'\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}'| = i\int[\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)]\frac{\varsigma}{s}\sigma(s)\cdot\nabla'\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}'$
$= i\int\delta_{AB}\delta^3(\vec{r}-\vec{r}')\frac{\varsigma}{s}\sigma(s)\cdot\nabla'\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}'$
$= i\frac{\varsigma}{s}\sigma(s)\cdot\nabla\psi_A(\vec{r},t)$ □

Cor. 7.3.2. $[\psi_A(\vec{r},t), \vec{P}] = -i\nabla\psi_A(\vec{r},t)$

Proof: $[\psi_A(\vec{r},t), \vec{P}]$
$= [\psi_A(\vec{r},t), -i\int\psi^+_B(\vec{r}',t)\nabla\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}']$
$= -i\int[\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)\nabla\delta^{BC}\psi_C(\vec{r}',t)]d^3\vec{r}'$
$= -i\int\{\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)\}\nabla\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}'| = -i\int[\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)]\nabla\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}'$
$= -i\int\delta_{AB}\delta^3(\vec{r}-\vec{r}')\nabla\delta^{BC}\psi_C(\vec{r}',t)d^3\vec{r}'$
$= -i\nabla\psi_A(\vec{r},t)$ □

Cor. 7.3.3. $[\sigma(s), -is\varsigma]^a\partial_a\psi(\vec{r},t) = 0 \Leftrightarrow -i\partial_a\psi(\vec{r},t) = [\psi(\vec{r},t), P_a]; \begin{cases}[\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)]_\pm = \delta_{AB}\delta^3(\vec{r}-\vec{r}')\\ [\psi_A(\vec{r},t), \psi_B(\vec{r}',t)]_\pm = 0\\ [\psi^+_A(\vec{r},t), \psi^+_B(\vec{r}',t)]_\pm = 0\end{cases}$

7.4 Self consistency of s-spin equation and quantum Lorentz invariance

Cor. 7.4.1. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(\vec{r},t) = 0 \overset{s=\frac{1}{2}}{\Leftrightarrow} -i\partial_a\psi(\vec{r},t) = [\psi(\vec{r},t), P_a]; \begin{cases}\{\psi_A(\vec{r},t), \psi^+_B(\vec{r}',t)\} = \delta_{AB}\delta^3(\vec{r}-\vec{r}')\\ \{\psi_A(\vec{r},t), \psi_B(\vec{r}',t)\} = 0\\ \{\psi^+_A(\vec{r},t), \psi^+_B(\vec{r}',t)\} = 0\end{cases}$

This scheme is self consistent only when the spin is $\frac{1}{2}$. So only neutrino can be quantized according to this scheme. Other spin particles cannot be quantized in this way, so they are excluded.

Chapter20 Scalar Field Covariant Quantization Scheme

1 Classical canonical quantization scheme for scalar field [25,26,37,38]

1.1 Classical description of real scalar field

1.1.1 Lagrangian density and Hamiltonian density of real scalar fields

Pro. 1.1.1. *Lagrangian density:* $\mathscr{L} = -\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}m^2\phi^2$

Pro. 1.1.2. *Energy momentum tensor density:* $T^{ab} = i\frac{\partial\mathscr{L}}{\partial(\partial_b\phi)}\partial^a\phi - ig^{ab}\mathscr{L}, T^{a\pi} = (\mathscr{P}, i\mathscr{H})^a, \partial_b T^{ab} = 0$

Pro. 1.1.3. *Hamiltonian density:* $\mathscr{H} = \frac{\partial\mathscr{L}}{\partial\dot{\phi}}\dot{\phi} - \mathscr{L} = \frac{1}{2}[\dot{\phi}^2(\vec{r},t) + \nabla\phi(\vec{r},t)\cdot\nabla\phi(\vec{r},t) + m^2\phi^2(\vec{r},t)]$

Pro. 1.1.4. *Momentum density:* $\mathscr{P} = -\frac{\partial\mathscr{L}}{\partial\dot{\phi}}\nabla\phi = -\dot{\phi}\nabla\phi$

1.1.2 Lagrangian density and equation of motion for real scalar field

Pro. 1.1.5. *Lagrangian density:* $\mathscr{L} = -\frac{1}{2}\partial_a\phi(\vec{r},t)\partial^a\phi(\vec{r},t) - \frac{1}{2}m^2\phi^2(\vec{r},t), \phi(\vec{r},t) = \phi^*(\vec{r},t)$

Pro. 1.1.6. *Equation of motion:* $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0, \phi(\vec{r},t) = \phi^*(\vec{r},t)$

1.1.3 Hamiltonian description of real scalar field

Pro. 1.1.7. $\mathscr{H} = \frac{1}{2}[\pi^2(\vec{r},t) + \partial_i\phi(\vec{r},t)\partial^i\phi(\vec{r},t) + m^2\phi^2(\vec{r},t)], \pi(\vec{r},t) = \dot{\phi}(\vec{r},t), \phi(\vec{r},t) = \phi^*(\vec{r},t)$

Pro. 1.1.8. $\mathscr{L} = \pi(\vec{r},t)\dot{\phi}(\vec{r},t) - \frac{1}{2}[\pi^2(\vec{r},t) + \partial_i\phi(\vec{r},t)\partial^i\phi(\vec{r},t) + m^2\phi^2(\vec{r},t)], \pi(\vec{r},t) = \dot{\phi}(\vec{r},t), \phi(\vec{r},t) = \phi^*(\vec{r},t)$

Pro. 1.1.9. *commutative relation:* $\begin{cases}\{\phi(\vec{r},t), \pi(\vec{r},t)\}_p \\ \dot{\pi}(\vec{r},t) = \nabla^2\phi(\vec{r},t) - m^2\phi(\vec{r},t) = \{\pi(\vec{r},t), H\}_p\end{cases}$

Pro. 1.1.10. $\phi(\vec{k},E)(E^2 - \vec{k}^2 - m^2) = 0 \Leftrightarrow \phi(\vec{k},E) = a(\vec{k},E)\delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k},E)\delta_{E^2,\vec{k}^2+m^2}$

Pro. 1.1.11. *Equation of motion:* $\begin{cases}\dot{\phi}(\vec{r},t) = \pi(\vec{r},t) = \{\phi(\vec{r},t), H\}_p \\ \dot{\pi}(\vec{r},t) = \nabla^2\phi(\vec{r},t) - m^2\phi(\vec{r},t) = \{\pi(\vec{r},t), H\}_p\end{cases}$

1.1.4 Plane wave solutions of real scalar field equation [37]

Real scalar field equation: $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0, \phi(\vec{r},t) = \phi^*(\vec{r},t)$ (20.1)

Thm. 1.1.1. $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0, \phi(\vec{r},t) = \phi^*(\vec{r},t)$

$$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^*(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$$

Proof: $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0, \phi(\vec{r},t) = \phi^*(\vec{r},t)$

$$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k},-\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}, \phi(\vec{r},t) = \phi^*(\vec{r},t)$$

$$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^*(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}, a(-\vec{k},-\omega_k) = a^*(\vec{k},\omega_k)$$ □

Cor. 1.1.1. $\phi_+(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{k}, \phi_-(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}a^*(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{k}$

1.2 Quantum description of real scalar field equation

1.2.1 Cannoical commutative relation of real scalar field equation

Thm. 1.2.1. $\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}+a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$

$$\Leftrightarrow\begin{cases}a(\vec{k},\omega_k)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}[i\dot{\phi}(\vec{r},t)+\omega_k\phi(\vec{r},t)]e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{r}\\ a^+(\vec{k},\omega_k)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}[-i\dot{\phi}(\vec{r},t)+\omega_k\phi(\vec{r},t)]e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{r}\end{cases}$$

Proof: $\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}+a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$

$$\Leftrightarrow\begin{cases}\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}+a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}\\ \dot{\phi}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{-i}{2}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}-a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}\phi(\vec{r},t)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{r}=\frac{1}{2\omega_k}[a(\vec{k},\omega_k)+a^+(-\vec{k},\omega_k)e^{2i\omega_k t}]\\ \frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}\dot{\phi}(\vec{r},t)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{r}=\frac{-i}{2}[a(\vec{k},\omega_k)-a^+(-\vec{k},\omega_k)e^{2i\omega_k t}]\end{cases}$$

$$\Leftrightarrow\begin{cases}a(\vec{k},\omega_k)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}[i\dot{\phi}(\vec{r},t)+\omega_k\phi(\vec{r},t)]e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{r}\\ a^+(\vec{k},\omega_k)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}[-i\dot{\phi}(\vec{r},t)+\omega_k\phi(\vec{r},t)]e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}d^3\vec{r}\end{cases}\qquad\square$$

From the above theorem, it can be proved that the regular commutative relation of the following real scalar particles is obtained.

Cor. 1.2.1. $\begin{cases}[\phi(\vec{r}',t),\phi(\vec{r},t)]=0\\ [\dot{\phi}(\vec{r}',t),\dot{\phi}(\vec{r},t)]=0\\ [\phi(\vec{r}',t),\dot{\phi}(\vec{r},t)]=i\delta^3(\vec{r}'-\vec{r})\end{cases}\Leftrightarrow\begin{cases}[a(\vec{k}',\omega_k'),a(\vec{k},\omega_k)]=0\\ [a^+(\vec{k}',\omega_k'),a^+(\vec{k},\omega_k)]=0\\ [a(\vec{k}',\omega_k'),a^+(\vec{k},\omega_k)]=2\omega_k\delta^3(\vec{k}'-\vec{k})\end{cases}$

Def. 1.2.1. $a(k)\equiv\frac{1}{\sqrt{2\omega_k}}a(\vec{k},\omega_k),a^+(k)\equiv\frac{1}{\sqrt{2\omega_k}}a^+(\vec{k},\omega_k)$

Cor. 1.2.2. $[a(k'),a(k)]=0,[a^+(k'),a^+(k)]=0,[a(k'),a^+(k)]=\delta^3(\vec{k}'-\vec{k})$

1.2.2 Energy operator and momentum operator of real scalar field equation

Cor. 1.2.3. $H=\int\limits_{\vec{k}=-\infty}^{+\infty}\omega_k\hat{N}(k)d^3\vec{k}+E(0),H^+=H$

Proof: $H=\int\limits_{\vec{r}=-\infty}^{+\infty}\frac{1}{2}[\dot{\phi}^2(\vec{r},t)+\nabla\phi(\vec{r},t)\cdot\nabla\phi(\vec{r},t)+m^2\phi^2(\vec{r},t)]d^3\vec{r}$

$=\frac{1}{4}\int\limits_{\vec{k}=-\infty}^{+\infty}[a(\vec{k},\omega_k)a^+(\vec{k},\omega_k)+a^+(\vec{k},\omega_k)a(\vec{k},\omega_k)]d^3\vec{k}$

$=\frac{1}{2}\int\limits_{\vec{k}=-\infty}^{+\infty}\omega_k[a(k)a^+(k)+a^+(k)a(k)]d^3\vec{k}=\int\limits_{\vec{k}=-\infty}^{+\infty}\omega_k[a^+(k)a(k)+\frac{1}{2}\delta^3(0)]d^3\vec{k}$

$=\int\limits_{\vec{k}=-\infty}^{+\infty}\omega_k\hat{N}(k)d^3\vec{k}+E(0)$ $\square$

Cor. 1.2.4. $P=\int\limits_{\vec{k}=-\infty}^{+\infty}\vec{k}\hat{N}(k)d^3\vec{k},P^+=P$

Proof: $P=\int\limits_{\vec{k}=-\infty}^{+\infty}-\dot{\phi}(\vec{r},t)\nabla\phi(\vec{r},t)d^3\vec{r}$

$=\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{4\omega_k}\vec{k}[a(\vec{k},\omega_k)a^+(\vec{k},\omega_k)+a^+(\vec{k},\omega_k)a(\vec{k},\omega_k)]d^3\vec{k}$

$$= \frac{1}{2}\int_{\vec{k}=-\infty}^{+\infty} \vec{k}[a(k)a^+(k) + a^+(k)a(k)]d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k}[a^+(k)a(k) + \frac{1}{2}\delta^3(0)]d^3\vec{k}$$

$$= \int_{\vec{k}=-\infty}^{+\infty} \vec{k}a^+(k)a(k)d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k}\hat{N}(k)d^3\vec{k}$$ □

Cor. 1.2.5. $L(t) = -\frac{1}{4}\int_{\vec{k}=-\infty}^{+\infty} [a(\vec{k},\omega_k)a(-\vec{k},\omega_k)e^{-2i\omega_k t} + a^+(\vec{k},\omega_k)a^+(-\vec{k},\omega_k)e^{2i\omega_k t}]d^3\vec{k}$

1.2.3 Summary of quantum theory for real scalar particles

Cor. 1.2.6.
$$\begin{cases} \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k} \\ \dot{\phi}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{k}=-\infty}^{+\infty} \frac{-i}{2}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k} \\ \nabla\phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{k}=-\infty}^{+\infty} \frac{i}{2\omega_k}\vec{k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k} \end{cases}$$

Cor. 1.2.7. $H = \int_{\vec{k}=-\infty}^{+\infty} \omega_k\hat{N}(k)d^3\vec{k} + E(0), \vec{P} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k}\hat{N}(k)d^3\vec{k}, P^a = (\vec{P}, iH)^a, \dot{P}^a = 0$

Cor. 1.2.8. $[P_a, \phi(\vec{r},t)] = i\partial_a\phi(\vec{r},t) \Leftrightarrow \partial_a\phi(\vec{r},t) = i[\phi(\vec{r},t), P_a]$

Cor. 1.2.9. $\dot{\phi}(\vec{r},t) = -i[\phi(\vec{r},t), H], \dddot{\phi}(\vec{r},t) = -i[\dot{\phi}(\vec{r},t), H]$

Cor. 1.2.10. $\dot{\phi}(\vec{r},t) = -i[\phi(\vec{r},t), H] \Leftrightarrow \omega_k a(\vec{k},\omega_k) = [a(\vec{k},\omega_k), H] \Leftrightarrow \omega_k a^+(\vec{k},\omega_k) = -[a^+(\vec{k},\omega_k), H]$

Proof: $\dot{\phi}(\vec{r},t) = -i[\phi(\vec{r},t), H]$

$$\Leftrightarrow \frac{1}{(2\pi)^{3/2}}\int_{\vec{k}=-\infty}^{+\infty} \frac{-i}{2}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - a^+(\vec{k},\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$$

$$= \frac{-i}{(2\pi)^{3/2}}\int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k}\{[a(\vec{k},\omega_k), H]e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + [a^+(\vec{k},\omega_k), H]e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}\}d^3\vec{k}$$

$$\Leftrightarrow \omega_k a(\vec{k},\omega_k) = [a(\vec{k},\omega_k), H] \Leftrightarrow \omega_k a^+(\vec{k},\omega_k) = -[a^+(\vec{k},\omega_k), H]$$ □

Cor. 1.2.11. $\dot{\phi}(\vec{r},t) = -i[\phi(\vec{r},t), H] \Leftrightarrow a(\vec{k},\omega_k) = [a(\vec{k},\omega_k), H] \Leftrightarrow \omega_k a^+(\vec{k},\omega_k) = -[a^+(\vec{k},\omega_k), H]$

Def. 1.2.2. $a(k,t) \equiv a(k)e^{-i\omega_k t}$

Cor. 1.2.12. $\dot{\phi}(\vec{r},t) = -i[\phi(\vec{r},t), H] \Leftrightarrow \dot{a}(k,t) = -i[a(k,t), H], \dot{a}^+(k,t) = -i[a^+(k,t), H]$

Cor. 1.2.13.
$$\begin{cases} \dot{\phi}(\vec{r},t) = \pi(\vec{r},t) = \{\phi(\vec{r},t), H\}_{\hat{p}} = -i[\phi(\vec{r},t), H] \\ \dot{\pi}(\vec{r},t) = \nabla^2\phi(\vec{r},t) - m^2\phi(\vec{r},t) = \{\pi(\vec{r},t), H\}_{\hat{p}} = -i[\dot{\phi}(\vec{r},t), H] \end{cases}$$

For a boson field, the equation of motion in classical theory is identical to the operator equation in quantum theory in form. But the physical meaning is different. The former can be regarded as the classical limit or quantum average of the latter. he equations of motion in classical theory can be written in Poisson bracket form, but cannot be written in commutator form.(Actually, it is zero, which is inconsistent with the equation of motion.)he operator equation of quantum theory can be written in either Poisson bracket form or the commutator form. That is, Poisson brackets in the form of operators are equivalent to commutators.

1.3 Quantum theory of complex scalar particles

1.3.1 Plane wave solutions of complex scalar field equation [37]

complex scalar field equation: $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0 \Leftrightarrow (\nabla^2 - \partial_t^2 - m^2)\phi(\vec{r},t) = 0$ (20.2)

Thm. 1.3.1. $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0 \Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k},-\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$

Proof: $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0 \Rightarrow \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\phi(\vec{k},E)(-\vec{k}^2+E^2-m^2)e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE = 0$

$\Leftrightarrow \phi(\vec{k},E)(E^2-\vec{k}^2-m^2) = 0 \Leftrightarrow \phi(\vec{k},E) = a(\vec{k},E)\delta(E^2-\vec{k}^2-m^2) + \phi_0(\vec{k},E)\delta_{E^2,\vec{k}^2+m^2}$

$\Rightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}[a(\vec{k},E)\delta(E^2-\vec{k}^2-m^2) + \phi_0(\vec{k},E)\delta_{E^2,\vec{k}^2+m^2}]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}a(\vec{k},E)\delta(E^2-\vec{k}^2-m^2)e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$, Apparent Lorentz covariant.

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\frac{1}{2\omega_k}a(\vec{k},E)[\delta(E-\omega_k)+\delta(E+\omega_k)]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\frac{1}{2\omega_k}a(\vec{k},E)[\delta(E-\omega_k)+\delta(E+\omega_k)]e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(\vec{k},-\omega_k)e^{i(\vec{k}\cdot\vec{r}+\omega_k t)}]d^3\vec{k}$

$\Leftrightarrow \phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{1}{2\omega_k}[a(\vec{k},\omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k},-\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}]d^3\vec{k}$ □

Here, a different approach is used than in ordinary books. Four-dimensional rather than three-dimensional Fourier expansion is used. Clearly show the physical concepts of particles in and out of the shell. Lorentz covariance is also evident in it. And it includes a new algebraic solution on the Dirac function. In the process of proof, we also saw the decomposition of positive and negative energy solutions. And the negative energy solution can be understood in two meanings: one is to understand the negative energy solution as a negative mass particle, and the other is still to understand the negative energy solution as a positive mass particle. However, the negative energy solution should be understand as a reflected wave, the positive energy solution should be understood as an incident wave.

Cor. 1.3.1. $a'(e^{\varepsilon}[\vec{k},E])\delta(E^2-\vec{k}^2-m^2) = e^{\frac{1}{2}\varepsilon^{ab}S_{ab}}a(\vec{k},E)\delta(E^2-\vec{k}^2-m^2)$
$\Rightarrow a'(e^{\varepsilon}[\vec{k},\omega_k]) = e^{\frac{1}{2}\varepsilon^{ab}S_{ab}}a(\vec{k},\omega_k), a'(e^{\varepsilon}[\vec{k},-\omega_k]) = e^{\frac{1}{2}\varepsilon^{ab}S_{ab}}a(\vec{k},-\omega_k)$

Cor. 1.3.2. $a(\vec{k},E)\delta(E^2-\vec{k}^2-m^2) = \frac{1}{2\omega_k}a(\vec{k},E)[\delta(E-\omega_k)+\delta(E+\omega_k)], |\vec{k}| << m$
$\approx \frac{1}{2(m+\frac{\vec{k}^2}{2m})}[a(\vec{k},m+\frac{\vec{k}^2}{2m})\delta(E-m-\frac{\vec{k}^2}{2m}) + a(\vec{k},-m-\frac{\vec{k}^2}{2m})\delta(E+m+\frac{\vec{k}^2}{2m})]$

Cor. 1.3.3. $\phi(\vec{r},t) \approx \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{k}=-K}^{+K}\frac{1}{2(m+\frac{\vec{k}^2}{2m})}[a(\vec{k},m+\frac{\vec{k}^2}{2m})e^{i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}e^{-imt} + a(-\vec{k},-m-\frac{\vec{k}^2}{2m})e^{-i(\vec{k}\cdot\vec{r}-\frac{\vec{k}^2}{2m}t)}e^{imt}]d^3\vec{k}$

From the above, Under the non relativistic limit, the plane wave solution of a complex scalar field is divided into two non relativistic positive and negative particles. They can exist simultaneously. This can be analyzed further. Can we prove that the positive and negative energy solutions are independently conserved?

1.3.2 Basic commutative relation of complex scalar field in coordinate space

Complex scalar field can be regard as the addition of two real scalar field equations. The difference is that there is an internal SO (2) symmetry at this time, so it can carry a conserved charge.

Def. 1.3.1. $\phi(\vec{r},t) = \frac{1}{\sqrt{2}}[\phi_1(\vec{r},t)+i\phi_2(\vec{r},t)], \phi^+(\vec{r},t) = \frac{1}{\sqrt{2}}[\phi_1(\vec{r},t)-i\phi_2(\vec{r},t)]$

Cor. 1.3.4. $\begin{cases}[\phi_1(\vec{r}',t),\phi_1(\vec{r},t)]=0\\ [\dot{\phi}_1(\vec{r}',t),\dot{\phi}_1(\vec{r},t)]=0\\ [\phi_1(\vec{r}',t),\dot{\phi}_1(\vec{r},t)]=i\delta^3(\vec{r}'-\vec{r})\end{cases}, \begin{cases}[\phi_2(\vec{r}',t),\phi_2(\vec{r},t)]=0\\ [\dot{\phi}_2(\vec{r}',t),\dot{\phi}_2(\vec{r},t)]=0\\ [\phi_2(\vec{r}',t),\dot{\phi}_2(\vec{r},t)]=i\delta^3(\vec{r}'-\vec{r})\end{cases}, \begin{cases}[\phi_1(\vec{r}',t),\phi_2(\vec{r},t)]=0\\ [\dot{\phi}_1(\vec{r}',t),\dot{\phi}_2(\vec{r},t)]=0\\ [\phi_1(\vec{r}',t),\dot{\phi}_2(\vec{r},t)]=0\end{cases}$

Cor. 1.3.5. $\begin{cases}[\phi(\vec{r}',t),\phi(\vec{r},t)]=0\\ [\dot{\phi}(\vec{r}',t),\dot{\phi}(\vec{r},t)]=0\\ [\phi(\vec{r}',t),\dot{\phi}(\vec{r},t)]=0\end{cases}, \begin{cases}[\phi^+(\vec{r}',t),\phi^+(\vec{r},t)]=0\\ [\dot{\phi}^+(\vec{r}',t),\dot{\phi}^+(\vec{r},t)]=0\\ [\phi^+(\vec{r}',t),\dot{\phi}^+(\vec{r},t)]=0\end{cases}, \begin{cases}[\phi(\vec{r}',t),\phi^+(\vec{r},t)]=0\\ [\dot{\phi}(\vec{r}',t),\dot{\phi}^+(\vec{r},t)]=0\\ [\phi(\vec{r}',t),\dot{\phi}^+(\vec{r},t)]=i\delta^3(\vec{r}'-\vec{r})\end{cases}$

1.3.3 Basic commutative relation of complex scalar field in momentum space

Def. 1.3.2. $a(k) = \frac{1}{\sqrt{2}}[a_1(k)+ia_2(k)], b(k) = \frac{1}{\sqrt{2}}[a_1(k)-ia_2(k)],$

Cor. 1.3.6. $\begin{cases}[a_1(k'),a_1(k)]=0,[a_1^+(k'),a_1^+(k)]=0,[a_1(k'),a_1^+(k)]=\delta^3(\vec{k}'-\vec{k})\\ [a_2(k'),a_2(k)]=0,[a_2^+(k'),a_2^+(k)]=0,[a_2(k'),a_2^+(k)]=\delta^3(\vec{k}'-\vec{k})\\ [a_1(k'),a_2(k)]=0,[a_1^+(k'),a_2^+(k)]=0,[a_1(k'),a_2^+(k)]=0\end{cases}$

Cor. 1.3.7. $\begin{cases}[a(k'),a(k)]=0,[a^+(k'),a^+(k)]=0,[a(k'),a^+(k)]=\delta^3(\vec{k}'-\vec{k})\\ [b(k'),b(k)]=0,[b^+(k'),b^+(k)]=0,[b(k'),b^+(k)]=\delta^3(\vec{k}'-\vec{k})\\ [a(k'),b(k)]=0,[a^+(k'),b^+(k)]=0,[a(k'),b^+(k)]=0\end{cases}$

1.3.4 Conserved charge of complex scalar field

Cor. 1.3.8. $Q=\int\limits_{\vec{k}=-\infty}^{+\infty}[a^+(k)a(k)-b^+(k)b(k)]d^3\vec{k}=\int\limits_{\vec{k}=-\infty}^{+\infty}[\hat{N}_+(k)-\hat{N}_-(k)]d^3\vec{k}$

Proof: $Q=\int\limits_{\vec{r}=-\infty}^{+\infty}[\phi_1(\vec{r},t)\dot{\phi}_2(\vec{r},t)-\phi_2(\vec{r},t)\dot{\phi}_1(\vec{r},t)]d^3\vec{r}$

$\Leftrightarrow Q=\int\limits_{\vec{k}=-\infty}^{+\infty}\frac{i}{2\omega_k}[a_1(\vec{k},\omega_k)a_2^+(\vec{k},\omega_k)-a_2(\vec{k},\omega_k)a_1^+(\vec{k},\omega_k)]d^3\vec{k}$

$\Leftrightarrow Q=\int\limits_{\vec{k}=-\infty}^{+\infty}i[a_1(k)a_2^+(k)-a_2(k)a_1^+(k)]d^3\vec{k}$

$\Leftrightarrow Q=\int\limits_{\vec{k}=-\infty}^{+\infty}[a^+(k)a(k)-b^+(k)b(k)]d^3\vec{k}=\int\limits_{\vec{k}=-\infty}^{+\infty}[\hat{N}_+(k)-\hat{N}_-(k)]d^3\vec{k}$ □

Cor. 1.3.9. $[Q,\phi(\vec{r},t)]=-\phi(\vec{r},t),[Q,\phi^+(\vec{r},t)]=\phi^+(\vec{r},t),$

1.3.5 Energy momentum operator of complex scalar field

Cor. 1.3.10. $P^a=\int\limits_{\vec{k}=-\infty}^{+\infty}k^a[\hat{N}_+(k)+\hat{N}_-(k)]d^3\vec{k},[P^a,\phi(\vec{r},t)]=i\partial^a\phi(\vec{r},t)$

Cor. 1.3.11. $[Q,P^a]=0,[\hat{N},P^a]=0,[\hat{N},Q]=0$

2 Covariant quantization scheme for scalar field [25, 26, 37, 38]

2.1 Conserved charge of scalar field

Cor. 2.1.1.
$H=\int\frac{1}{2}[\dot{\phi}^+(\vec{r},t)\dot{\phi}(\vec{r},t)+\partial_i\phi^+(\vec{r},t)\partial^i\phi(\vec{r},t)+m^2\phi^+(\vec{r},t)\phi(\vec{r},t)]d^3\vec{r}$
$=\int\frac{1}{2}[\dot{\phi}(\vec{r},t)\dot{\phi}(\vec{r},t)+\partial_i\phi(\vec{r},t)\partial^i\phi(\vec{r},t)+m^2\phi(\vec{r},t)\phi(\vec{r},t)]d^3\vec{r}$
$=\int\frac{1}{2}[\dot{\phi}(\vec{r},t)\dot{\phi}(\vec{r},t)-\phi(\vec{r},t)\partial_i\partial^i\phi(\vec{r},t)+m^2\phi(\vec{r},t)\phi(\vec{r},t)]d^3\vec{r}$
$=\int\frac{1}{2}[\dot{\phi}(\vec{r},t)\dot{\phi}(\vec{r},t)-\phi(\vec{r},t)\partial_t^2\phi(\vec{r},t)]d^3\vec{r}$
$=\int\{\dot{\phi}(\vec{r},t)\dot{\phi}(\vec{r},t)-\partial_t[\dot{\phi}(\vec{r},t)\phi(\vec{r},t)]\}d^3\vec{r}$

Cor. 2.1.2. $P=-\int\dot{\phi}(\vec{r},t)\nabla\phi(\vec{r},t)d^3\vec{r}$

Cor. 2.1.3. $M_{ij}=-\int\dot{\phi}(\vec{r},t)(x_i\partial_j-x_j\partial_i)\phi(\vec{r},t)d^3\vec{r}$

2.2 Scalar field equation and its plane wave solutions

Def. 2.2.1. $(\partial_a\partial^a-m^2)\phi_\sigma(\vec{r},t)=0,\phi_\sigma(\vec{r},t)=\phi_\sigma^+(\vec{r},t)$

Cor. 2.2.1. $\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}[a_\sigma(\vec{p},0)e^{i(\vec{p}\cdot\vec{r}-Et)}+a_\sigma^+(\vec{p},0)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

$\Leftrightarrow\begin{cases}\sqrt{2E}a_\sigma(\vec{p},0)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}[E\phi_\sigma(\vec{r},t)+i\dot{\phi}_\sigma(\vec{r},t)]e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ \sqrt{2E}a_\sigma^+(\vec{p},0)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}[E\phi_\sigma(\vec{r},t)-i\dot{\phi}_\sigma(\vec{r},t)]e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$

Cor. 2.2.2. $\partial_t\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{-iE}{\sqrt{2E}}[a_\sigma(\vec{p},0)e^{i(\vec{p}\cdot\vec{r}-Et)}-a_\sigma^+(\vec{p},0)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 2.2.3. $\partial_i\phi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{ip_i}{\sqrt{2E}}[a_\sigma(\vec{p},0)e^{i(\vec{p}\cdot\vec{r}-Et)}-a_\sigma^+(\vec{p},0)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

2.3 General covariant commutation rules of scalar field in mathematics

Thm. 2.3.1.
$$\begin{cases}[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]_\pm=\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]_\pm=0,[a^+_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]_\pm=0\end{cases}\Rightarrow[\phi_\sigma(x),\phi_{\sigma'}(x')]_\pm=i\delta_{\sigma\sigma'}\Delta(x-x')$$

Proof: $[\phi^{(+)}_\sigma(x),\phi^{+(+)}_{\sigma'}(x')]_\pm=[\phi^{(+)}_\sigma(x),\phi^{(-)}_{\sigma'}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int[\frac{1}{\sqrt{2E}}a_\sigma(\vec{p},0)e^{ipx},\frac{1}{\sqrt{2E'_{p'}}}a^+_{\sigma'}(\vec{p}',0)e^{-ip'x'}]_\pm d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]_\pm e^{ipx}e^{-ip'x'}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')e^{ipx}e^{-ip'x'}d^3\vec{p}d^3\vec{p}'$
$=i\delta_\sigma\delta_{\sigma\sigma'}\frac{-i}{(2\pi)^3}\int\frac{1}{2E}e^{ip(x-x')}d^3\vec{p}$
$=i\delta_\sigma\delta_{\sigma\sigma'}\Delta^{(+)}(x-x')$ □

Proof: $[\phi^{(-)}_\sigma(x),\phi^{+(-)}_{\sigma'}(x')]_\pm=[\phi^{(-)}_\sigma(x),\phi^{(+)}_{\sigma'}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int[\frac{1}{\sqrt{2E}}a^+_\sigma(\vec{p},0)e^{-ipx},\frac{1}{\sqrt{2E'_{p'}}}a_{\sigma'}(\vec{p}',0)e^{ip'x'}]_\pm d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}[a^+_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]_\pm e^{-ipx}e^{ip'x'}d^3\vec{p}d^3\vec{p}'$
$=\pm\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')e^{-ipx}e^{ip'x'}d^3\vec{p}d^3\vec{p}'$
$=-\pm i\delta_\sigma\delta_{\sigma\sigma'}\frac{i}{(2\pi)^3}\int\frac{1}{2E}e^{-ip(x-x')}d^3\vec{p}$
$=-\pm i\delta_\sigma\delta_{\sigma\sigma'}\Delta^{(-)}(x-x')$ □

Proof:
$[\phi_\sigma(x),\phi_{\sigma'}(x')]_\pm=\frac{1}{(2\pi)^3}\int[\frac{1}{\sqrt{2E}}[a_\sigma(\vec{p},0)e^{ipx}+a^+_\sigma(\vec{p},0)e^{-ipx},\frac{1}{\sqrt{2E'_{p'}}}[a_{\sigma'}(\vec{p}',0)e^{ip'x'}+a^+_{\sigma'}(\vec{p}',0)e^{-ip'x'}]_\pm d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\{[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]_\pm e^{ipx}e^{-ip'x'}+[a^+_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]_\pm e^{-ipx}e^{ip'x'}\}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}[\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')e^{ipx}e^{-ip'x'}\pm\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')e^{-ipx}e^{ip'x'}]d^3\vec{p}d^3\vec{p}'$
$=i\delta_\sigma\delta_{\sigma\sigma'}\frac{-i}{(2\pi)^3}\int\frac{1}{2E}[e^{ip(x-x')}\pm e^{-ip(x-x')}]d^3\vec{p}$
$=i\delta_\sigma\delta_{\sigma\sigma'}[\Delta^{(+)}(x-x')-\pm\Delta^{(-)}(x-x')]$
$=i\delta_\sigma\delta_{\sigma\sigma'}[(1\pm1)\Delta^{(+)}(x-x')-\pm\Delta(x-x')]$ □

From the above formula, only when $1\pm1=0$ the micro causality can be satisfied. Only when $\delta_\sigma\geq0$, the probability nonnegativity is satisfied. Therefore, among various covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, $\delta_\sigma=1$.(If equality between scalar fields is required, then if it is not 1, it can be uniformly normalized.) And it satisfies the commutative relation. here are actually two other types. That is $\delta_\sigma=0$. And it satisfies the commutative or anticommutative relation, which is just the classic case.

2.4 Covariant commutation rules of physical scalar field

Thm. 2.4.1.
$$\begin{cases}[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]=0,[a^+_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=0\end{cases}\Rightarrow[\phi_\sigma(x),\phi_{\sigma'}(x')]=i\delta_{\sigma\sigma'}\Delta(x-x')$$

Proof:
$[\phi_\sigma(x),\phi_{\sigma'}(x')]=\frac{1}{(2\pi)^3}\int[\frac{1}{\sqrt{2E}}[a_\sigma(\vec{p},0)e^{ipx}+a^+_\sigma(\vec{p},0)e^{-ipx},\frac{1}{\sqrt{2E'_{p'}}}[a_{\sigma'}(\vec{p}',0)e^{ip'x'}+a^+_{\sigma'}(\vec{p}',0)e^{-ip'x'}]d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\{[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]e^{ipx}e^{-ip'x'}+[a^+_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]e^{-ipx}e^{ip'x'}\}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\{\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')e^{ipx}e^{-ip'x'}-\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')e^{-ipx}e^{ip'x'}\}d^3\vec{p}d^3\vec{p}'$
$=i\delta_{\sigma\sigma'}\frac{-i}{(2\pi)^3}\int\frac{1}{2E}[e^{ip(x-x')}-e^{-ip(x-x')}]d^3\vec{p}$
$=i\delta_{\sigma\sigma'}\Delta(x-x')$ □

2.5 Isochronous commutation rules of scalar field

Cor. 2.5.1. $[\phi_\sigma(x),\phi_{\sigma'}(x')]=i\delta_{\sigma\sigma'}\Delta(x-x')\Rightarrow\begin{cases}[\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]=i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\\ [\phi_\sigma(\vec{r},t),\phi_{\sigma'}(\vec{r}',t)]=0,[\dot{\phi}_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]=0\end{cases}$

Cor. 2.5.2.
$$\begin{cases}[\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]=i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\\ [\phi_\sigma(\vec{r},t),\phi_{\sigma'}(\vec{r}',t)]=0,[\dot{\phi}_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]=0\end{cases}\Rightarrow\begin{cases}[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]=0,[a^+_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=0\end{cases}$$

Proof: $[a_\sigma(\vec{p},0), a^+_{\sigma'}(\vec{p}',0)]$
$= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\{[E\phi_\sigma(\vec{r},t), -i\dot{\phi}(\vec{r}',t)] + [i\dot{\phi}(\vec{r},t), E'_{p'}\phi_{\sigma'}(\vec{r}',t)]\}e^{-i(\vec{p}\cdot\vec{r}-E'_{p'}t)}e^{i(\vec{p}'\cdot\vec{r}'-E'_{p'}t)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\{E\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}') + E'_{p'}\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\}e^{-i(\vec{p}\cdot\vec{r}-E'_{p'}t)}e^{i(\vec{p}'\cdot\vec{r}'-E'_{p'}t)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int \delta_{\sigma\sigma'}\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}(E+E'_{p'})e^{-i(\vec{p}-\vec{p}'\cdot\vec{r}}e^{iE'_{p'}t}e^{-iE'_{p'}t}d^3\vec{r}$
$= \delta_{\sigma\sigma'}\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}(E+E'_{p'})\delta^3(\vec{p}-\vec{p}')e^{iE'_{p'}t}e^{-iE'_{p'}t}$
$= \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')$ □

Proof: $[a_\sigma(\vec{p},0), a_{\sigma'}(\vec{p}',0)]$
$= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\{[E\phi_\sigma(\vec{r},t), -i\dot{\phi}(\vec{r}',t)] + [-i\dot{\phi}(\vec{r},t), E'_{p'}\phi_{\sigma'}(\vec{r}',t)]\}e^{-i(\vec{p}\cdot\vec{r}-E'_{p'}t)}e^{-i(\vec{p}'\cdot\vec{r}'-E'_{p'}t)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int \frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}\{E\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}') - E'_{p'}\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\}e^{-i(\vec{p}\cdot\vec{r}-E'_{p'}t)}e^{-i(\vec{p}'\cdot\vec{r}'-E'_{p'}t)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int \delta_{\sigma\sigma'}\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}(E-E'_{p'})e^{-i(\vec{p}+\vec{p}')\cdot\vec{r}}e^{iE'_{p'}t}e^{-iE'_{p'}t}d^3\vec{r}$
$= \delta_{\sigma\sigma'}\frac{1}{\sqrt{2E}}\frac{1}{\sqrt{2E'_{p'}}}(E-E'_{p'})\delta^3(\vec{p}+\vec{p}')e^{iE'_{p'}t}e^{iE'_{p'}t}$
$= 0$ □

2.6 Summary of scalar field commutation rules

The proof in the above sections exactly forms a logical closed-loop, so it has the following properties.

Cor. 2.6.1. $\begin{cases}[a_\sigma(\vec{p},0), a^+_{\sigma'}(\vec{p}',0)] = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0), a_{\sigma'}(\vec{p}',0)] = 0, [a^+_\sigma(\vec{p},0), a^+_{\sigma'}(\vec{p}',0)] = 0\end{cases} \Leftrightarrow \begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0, [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = 0\end{cases}$

$\Updownarrow \qquad \Updownarrow$

Cor. 2.6.2. $\begin{cases}[\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)] = i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\\ [\phi_\sigma(\vec{r},t), \phi_{\sigma'}(\vec{r}',t)] = 0, [\dot{\phi}_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)] = 0\end{cases} \Leftrightarrow [\phi_\sigma(x), \phi_{\sigma'}(x')] = i\delta_{\sigma\sigma'}\Delta(x-x')$

2.7 Single complex scalar field equation and its plane wave solutions

Def. 2.7.1. $(\partial_a\partial^a - m^2)\phi(\vec{r},t) = 0$

Cor. 2.7.1. $\phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}[b_1(\vec{p},0)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+_2(\vec{p},0)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

$\Leftrightarrow \begin{cases}\sqrt{2E}b_1(\vec{p},0) = \frac{1}{(2\pi)^{3/2}}\int_{p=-\infty}^{+\infty}[E\phi(\vec{r},t) + i\dot{\phi}(\vec{r},t)]e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ \sqrt{2E}b^+_2(\vec{p},0) = \frac{1}{(2\pi)^{3/2}}\int_{p=-\infty}^{+\infty}[E\phi(\vec{r},t) - i\dot{\phi}(\vec{r},t)]e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$

2.8 Commutation rules of single complex scalar field

Cor. 2.8.1. $\begin{cases}[b_\sigma(\vec{p},0), b^+_\sigma(\vec{p}',0)] = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [b_\sigma(\vec{p},0), b_\sigma(\vec{p}',0)] = 0, [b^+_\sigma(\vec{p},0), b^+_\sigma(\vec{p}',0)] = 0\\ b_1(\vec{p},0) = \frac{1}{\sqrt{2}}[a_1(\vec{p},0) + ia_2(\vec{p},0)]\\ b_2(\vec{p},0) = \frac{1}{\sqrt{2}}[a_1(\vec{p},0) - ia_2(\vec{p},0)]\end{cases} \Leftrightarrow \begin{cases}[b_\sigma(\vec{p}), b^+_{\sigma'}(\vec{p}')] = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [b_\sigma(\vec{p}), b_\sigma(\vec{p}')] = 0, [b^+_\sigma(\vec{p}), b^+_\sigma(\vec{p}')] = 0\\ b_1(\vec{p}) = \frac{1}{\sqrt{2}}[a_1(\vec{p}) + ia_2(\vec{p})]\\ b_2(\vec{p}) = \frac{1}{\sqrt{2}}[a_1(\vec{p}) - ia_2(\vec{p})]\end{cases}$

$\Updownarrow \qquad \Updownarrow$

Cor. 2.8.2. $\begin{cases}[a_\sigma(\vec{p},0), a^+_{\sigma'}(\vec{p}',0)] = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0), a_{\sigma'}(\vec{p}',0)] = 0\\ [a^+_\sigma(\vec{p},0), a^+_{\sigma'}(\vec{p}',0)] = 0\end{cases} \Leftrightarrow \begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0\\ [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = 0\end{cases}$

$\Updownarrow \qquad \Updownarrow$

Cor. 2.8.3. $\begin{cases}[\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)] = i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\\ [\phi_\sigma(\vec{r},t), \phi_{\sigma'}(\vec{r}',t)] = 0\\ [\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t)] = 0\end{cases} \Leftrightarrow \begin{cases}[\phi_\sigma(x), \phi_{\sigma'}(x')] = i\delta_{\sigma\sigma'}\Delta(x-x')\\ \phi_1(x) = \frac{1}{\sqrt{2}}[\phi(x) + \phi^+(x)]\\ \phi_2(x) = \frac{1}{i\sqrt{2}}[\phi(x) - \phi^+(x)]\end{cases}$

$\Updownarrow \qquad \Updownarrow$

Cor. 2.8.4. $\begin{cases}[\phi(\vec{r},t),\dot{\phi}^+(\vec{r}',t)]=i\delta^3(\vec{r}-\vec{r}')\\ [\phi(\vec{r},t),\phi(\vec{r}',t)]=0,[\phi^+(\vec{r},t),\phi^+(\vec{r}',t)]=0\\ [\dot{\phi}(\vec{r},t),\dot{\phi}(\vec{r}',t)]=0,[\dot{\phi}^+(\vec{r},t),\dot{\phi}^+(\vec{r}',t)]=0\\ [\phi(\vec{r},t),\phi^+(\vec{r}',t)]=0,[\dot{\phi}(\vec{r},t),\dot{\phi}^+(\vec{r}',t)]=0\end{cases} \Leftrightarrow \begin{cases}[\phi(x),\phi^+(x')]=i\Delta(x-x')\\ [\phi(x),\phi(x')]=0,[\phi^+(x),\phi^+(x')]=0\\ \phi(x)=\frac{1}{\sqrt{2}}[\phi_1(x)+i\phi_2(x)]\\ \phi^+(x)=\frac{1}{\sqrt{2}}[\phi_1(x)-i\phi_2(x)]\end{cases}$

$\Updownarrow \qquad\qquad \Updownarrow$

Cor. 2.8.5. $\begin{cases}[\psi(\vec{r},t),\dot{\psi}^+(\vec{r}',t)]=2i\delta^3(\vec{r}-\vec{r}')\\ [\psi(\vec{r},t),\psi(\vec{r}',t)]=0,[\psi^+(\vec{r},t),\psi^+(\vec{r}',t)]=0\\ [\dot{\psi}(\vec{r},t),\dot{\psi}(\vec{r}',t)]=0,[\dot{\psi}^+(\vec{r},t),\dot{\psi}^+(\vec{r}',t)]=0\\ [\psi(\vec{r},t),\psi^+(\vec{r}',t)]=0,[\dot{\psi}(\vec{r},t),\dot{\psi}^+(\vec{r}',t)]=0\end{cases} \Leftrightarrow \begin{cases}[\psi(x),\psi^+(x')]=2i\Delta(x-x')\\ [\psi(x),\psi(x')]=0,[\psi^+(x),\psi^+(x')]=0\\ \psi(x)=\phi_1(x)+i\phi_2(x)\\ \psi^+(x)=\phi_1(x)-i\phi_2(x)\end{cases}$

2.9 Causal function of massless scalar field

Def. 2.9.1. $\begin{cases}\Delta^{(+)}(x):=\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}e^{ip\cdot x}d^3\vec{p}, i\Delta^{(+)}(\vec{r},0)\leftrightarrow\frac{1}{2|\vec{p}|}\\ \Delta^{(-)}(x):=-\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}e^{-ip\cdot x}d^3\vec{p},\Delta^{(-)}(x)=-\Delta^{(+)}(-x)\\ \Delta(x):=\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}[e^{ip\cdot x}-e^{-ip\cdot x}\}d^3\vec{p},\Delta(x)=\Delta^{(+)}(x)+\Delta^{(-)}(x)\end{cases}$

Pro. 2.9.1. $\begin{cases}\Delta^*(x)=\Delta(x),\Delta(-x)=-\Delta(x),(\nabla^2-\partial_t^2)\Delta(x)=0\\ \partial_t\Delta(x)|_{t=0}=-\delta^3(\vec{r}),\partial_k\partial_t\Delta(x)|_{t=0}=\partial_t\partial_k\Delta(x)|_{t=0}=-\partial_k\delta^3(\vec{r})\\ \partial_k\Delta(x)|_{t=0}=0,\partial_k\partial_l\Delta(x)|_{t=0}=0,\partial_t^2\Delta(x)|_{t=0}=0\end{cases}$

Pro. 2.9.2. $\Delta(x-x'):=\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}\}d^3\vec{p}$

$\begin{cases}\partial_u\Delta(x-x')=-\partial_u'\Delta(x-x')\\ \nabla\Delta(x-x')=-\nabla'\Delta(x-x')\\ \partial_\pi\Delta(x-x')=-\partial_\pi'\Delta(x-x')\end{cases} \begin{cases}(\sqrt{-\nabla^2})^n\Delta(x-x')=(\sqrt{-\nabla'^2})^n\Delta(x-x')\\ \frac{1}{(\sqrt{-\nabla^2})^n}\Delta(x-x')=\frac{1}{(\sqrt{-\nabla'^2})^n}\Delta(x-x')\\ \partial_\pi^{2n}\Delta(x-x')=\partial_\pi'^{2n}\Delta(x-x')\end{cases}$

2.10 Commutation function, causality function and Feynman propagator of scalar field

Def. 2.10.1.
$\begin{cases}\Delta(x)=\Delta^{(+)}(x)+\Delta^{(-)}(x)=\Delta^{ret}(x)-\Delta^{adv}(x)\\ \Delta^{(l)}(x)=i[\Delta^{(-)}(x)-\Delta^{(+)}(x)]\\ \Delta_F(x)=\langle T\varphi(x)\varphi(x')\rangle_0=i\Delta^{(c)}(x-x')\end{cases} \begin{cases}\Delta^{(c)}(x)=\theta(t)\Delta^{(+)}(x)-\theta(-t)\Delta^{(-)}(x)\\ \Delta^{ret}(x)=\theta(t)\Delta(x)=\Delta^{(c)}(x)+\Delta^{(-)}(x)\\ \Delta^{adv}(x)=-\theta(-t)\Delta(x)=\Delta^{(c)}(x)-\Delta^{(+)}(x)\end{cases}$

Cor. 2.10.1.
$\begin{cases}(\partial_a\partial^a-m^2)\Delta(x)=0\\ (\partial_a\partial^a-m^2)\Delta^{(+)}(x)=0\\ (\partial_a\partial^a-m^2)\Delta^{(-)}(x)=0\\ (\partial_a\partial^a-m^2)\Delta^{(l)}(x)=0\end{cases} \begin{cases}(\partial_a\partial^a-m^2)\Delta^{(c)}(x)=\delta^4(x)\\ (\partial_a\partial^a-m^2)\Delta^{ret}(x)=\delta^4(x)\\ (\partial_a\partial^a-m^2)\Delta^{adv}(x)=\delta^4(x)\\ (\partial_a\partial^a-m^2)\Delta_F(x)=i\delta^4(x)\end{cases}$

3 Extraction of various operators from scalar field

3.1 Extraction of energy and momentum operators for scalar field

Cor. 3.1.1.
$H=\frac{1}{2}\int\sum_\sigma E[a^+(\vec{p},0)a(\vec{p},0)+a(\vec{p},0)a^+(\vec{p},0)]d^3\vec{p}=\frac{1}{2}\int\sum_\sigma[\nabla\phi_\sigma(\vec{r},t)]^2+\dot{\phi}_\sigma^2(\vec{r},t)+m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$
$\vec{P}=\frac{1}{2}\int\sum_\sigma\vec{p}[a_\sigma^+(\vec{p},0)a_\sigma(\vec{p},0)+a_\sigma(\vec{p},0)a_\sigma^+(\vec{p},0)]d^3\vec{p}=\int\sum_\sigma-\dot{\phi}_\sigma(\vec{r},t)\nabla\phi_\sigma(\vec{r},t)d^3\vec{r}$

Proof: $H=\frac{1}{2}\int\sum_\sigma E[a_\sigma^+(\vec{p},0)a_\sigma(\vec{p},0)+a_\sigma(\vec{p},0)a_\sigma^+(\vec{p},0)]d^3\vec{p}$
$=\frac{1}{2}\frac{1}{(2\pi)^3}\int\sum_\sigma\frac{1}{2}\{[E\phi_\sigma(\vec{r},t)-i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t)+i\dot{\phi}_\sigma(\vec{r}',t)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$
$+[E\phi_\sigma(\vec{r},t)+i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t)-i\dot{\phi}_\sigma(\vec{r}',t)]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_\sigma\{[E_p^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)+\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]$
$+iE[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)-\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_\sigma\{[m^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)+\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]$
$-\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)\nabla^2[e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]+0\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_\sigma\{[m^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)+\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)]\delta^3(\vec{r}-\vec{r}')-\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)\nabla^2\delta^3(\vec{r}-\vec{r}')\}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{2}\int\sum_\sigma[-\nabla^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)+\dot{\phi}_\sigma^2(\vec{r},t)+m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$

$= \frac{1}{2}\int\sum_{\sigma}[-\partial^i[\partial_i\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)] + [\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}_\sigma^2(\vec{r},t) + m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$

$= \frac{1}{2}\int\sum_{\sigma}[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}_\sigma^2(\vec{r},t) + m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$ □

Proof: $\vec{P} = \frac{1}{2}\int\sum_{\sigma}\vec{p}[a_\sigma^+(\vec{p},0)a_\sigma(\vec{p},0) + a_\sigma(\vec{p},0)a_\sigma^+(\vec{p},0)]d^3\vec{p}$

$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{\vec{p}}{2E}\{[E\phi_\sigma(\vec{r},t) - i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t) + i\dot{\phi}_\sigma(\vec{r}',t)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$

$+ [E\phi_\sigma(\vec{r},t) + i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t) - i\dot{\phi}_\sigma(\vec{r}',t)]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{\vec{p}}{E}\{[E_p^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t) + \dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]$

$+ iE[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_{\sigma}i\vec{p}[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\sum_{\sigma}i\vec{p}[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\sum_{\sigma}[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)]\nabla e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{2}\int\sum_{\sigma}[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)]\nabla\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{2}\int\sum_{\sigma}[-\nabla\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t) + \nabla\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)]d^3\vec{r}$

$= \int\sum_{\sigma}-\dot{\phi}_\sigma(\vec{r},t)\nabla\phi_\sigma(\vec{r},t)d^3\vec{r} + \frac{1}{2}\int\sum_{\sigma}\nabla[\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)]d^3\vec{r}$

$= \int\sum_{\sigma}-\dot{\phi}_\sigma(\vec{r},t)\nabla\phi_\sigma(\vec{r},t)d^3\vec{r}$ □

3.2 Extraction of spin operator and particle number operator for scalar field

Cor. 3.2.1. $\hat{S} = \frac{1}{2}\int\sum_{\sigma}E[a_\sigma^+(\vec{p},0)a_\sigma(\vec{p},0) - a_\sigma(\vec{p},0)a_\sigma^+(\vec{p},0)]d^3\vec{p} = \frac{i}{2}\int\sum_{\sigma}[\phi_\sigma(\vec{r},t),\dot{\phi}_\sigma(\vec{r},t)]d^3\vec{r}$

Proof: $\hat{S} = \frac{1}{2}\int\sum_{\sigma}E[a_\sigma^+(\vec{p},0)a_\sigma(\vec{p},0) - a_\sigma(\vec{p},0)a_\sigma^+(\vec{p},0)]d^3\vec{p}$

$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{1}{2E}\{[E\phi_\sigma(\vec{r},t) - i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t) + i\dot{\phi}_\sigma(\vec{r}',t)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$

$- [E\phi_\sigma(\vec{r},t) + i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t) - i\dot{\phi}_\sigma(\vec{r}',t)]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{1}{E}\{[E_p^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t) + \dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]$

$+ iE[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_{\sigma}i[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{2}\int\sum_{\sigma}i[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)]\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$

$= \frac{i}{2}\int\sum_{\sigma}[\phi_\sigma(\vec{r},t),\dot{\phi}_\sigma(\vec{r},t)]d^3\vec{r}$ □

Cor. 3.2.2. $\hat{N} = \frac{1}{2}\int\sum_{\sigma}E[a_\sigma^+(\vec{p},0)a_\sigma(\vec{p},0) + a_\sigma(\vec{p},0)a_\sigma^+(\vec{p},0)]d^3\vec{p}$

$= \frac{1}{2}\int\sum_{\sigma}\frac{1}{\sqrt{m^2-\nabla^2}}[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}_\sigma^2(\vec{r},t) + m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$

Proof: $\hat{N} = \frac{1}{2}\int\sum_{\sigma}E[a_\sigma^+(\vec{p},0)a_\sigma(\vec{p},0) + a_\sigma(\vec{p},0)a_\sigma^+(\vec{p},0)]d^3\vec{p}$

$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{1}{2E}\{[E\phi_\sigma(\vec{r},t) - i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t) + i\dot{\phi}_\sigma(\vec{r}',t)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$

$+ [E\phi_\sigma(\vec{r},t) + i\dot{\phi}_\sigma(\vec{r},t)][E\phi_\sigma(\vec{r}',t) - i\dot{\phi}_\sigma(\vec{r}',t)]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{1}{E}\{[E_p^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t) + \dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]$

$+ iE[\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t) - \dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{1}{E}\{[m^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t) + \dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)][e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]$

$- \phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)\nabla^2[e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] + 0\}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\sum_{\sigma}\frac{1}{\sqrt{m^2-\nabla^2}}\{[m^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t) + \dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)]\delta^3(\vec{r}-\vec{r}') - \phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r}',t)\nabla^2\delta^3(\vec{r}-\vec{r}')\}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{2}\int\sum_{\sigma}\frac{1}{\sqrt{m^2-\nabla^2}}[-\nabla^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t) + \dot{\phi}_\sigma^2(\vec{r},t) + m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$

$= \frac{1}{2}\int\sum_{\sigma}\frac{1}{\sqrt{m^2-\nabla^2}}[-\partial^i[\partial_i\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)] + [\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}_\sigma^2(\vec{r},t) + m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$

$= \frac{1}{2}\int\sum_{\sigma}\frac{1}{\sqrt{m^2-\nabla^2}}[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}_\sigma^2(\vec{r},t) + m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$ □

3.3 Extraction of spatial angular momentum operators for scalar field

Thm. 3.3.1. $M_{ij} = -\int\sum\limits_{\sigma}\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= -\frac{i}{2}\int\sum\limits_{\sigma}[a^+_\sigma(\vec{p},0)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_\sigma(\vec{p},0) - a_\sigma(\vec{p},0)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a^+_\sigma(\vec{p},0)]d^3\vec{p}$

Proof: M_{ij}
$= -\int\sum\limits_{\sigma}\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= -\int\sum\limits_{\sigma}\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= -\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\sum\limits_{\sigma}$
$\frac{-iE'}{\sqrt{2E'}}\frac{i(r_ip_j-r_jp_i)}{\sqrt{2E}}[a_\sigma(\vec{p}',0)e^{i(\vec{p}'\cdot\vec{r}-E't)} - a^+_\sigma(\vec{p}',0)e^{-i(\vec{p}'\cdot\vec{r}-E't)}][a_\sigma(\vec{p},0)e^{i(\vec{p}\cdot\vec{r}-Et)} - a^+_\sigma(\vec{p},0)e^{-i(\vec{p}\cdot\vec{r}-Et)}]$
$= -\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\sum\limits_{\sigma}$
$\frac{E'(r_ip_j-r_jp_i)}{2\sqrt{E'E}}[a_\sigma(\vec{p}',0)a_\sigma(\vec{p},0)e^{i[(\vec{p}'+\vec{p})\cdot\vec{r}-(E'+E)t]} + a^+_\sigma(\vec{p}',0)a^+_\sigma(\vec{p},0)e^{-i[(\vec{p}'+\vec{p})\cdot\vec{r}-(E'+E)t]}]$
$-$
$\frac{E'(r_ip_j-r_jp_i)}{2\sqrt{E'E}}[a_\sigma(\vec{p}',0)a^+_\sigma(\vec{p},0)e^{i[(\vec{p}'-\vec{p})\cdot\vec{r}-(E'-E)t]} + a^+_\sigma(\vec{p}',0)a_\sigma(\vec{p},0)e^{-i[(\vec{p}'-\vec{p})\cdot\vec{r}-(E'-E)t]}]$
$= -\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\sum\limits_{\sigma}\frac{-iE'}{2\sqrt{E'E}}$
$\{[a_\sigma(\vec{p}',0)a_\sigma(\vec{p},0)e^{-i(E'+E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)e^{i(\vec{p}'+\vec{p})\cdot\vec{r}} - a^+_\sigma(\vec{p}',0)a^+_\sigma(\vec{p},0)e^{i(E'+E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)e^{-i(\vec{p}'+\vec{p})\cdot\vec{r}}]$
$+$
$[a_\sigma(\vec{p}',0)a^+_\sigma(\vec{p},0)e^{-i(E'-E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)e^{i(\vec{p}'-\vec{p})\cdot\vec{r}} - a^+_\sigma(\vec{p}',0)a_\sigma(\vec{p},0)e^{i(E'-E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)e^{-i(\vec{p}'-\vec{p})\cdot\vec{r}}]\}$
$= i\int d^3\vec{p}d^3\vec{p}'\sum\limits_{\sigma}\frac{E'}{2\sqrt{E'E}}$
$\{[a_\sigma(\vec{p}',0)a_\sigma(\vec{p},0)e^{-i(E'+E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)\delta^3(\vec{p}+\vec{p}') - a^+_\sigma(\vec{p}',0)a^+_\sigma(\vec{p},0)e^{i(E'+E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)\delta^3(\vec{p}+\vec{p}')]$
$+$
$[a_\sigma(\vec{p}',0)a^+_\sigma(\vec{p},0)e^{-i(E'-E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)\delta^3(\vec{p}-\vec{p}') - a^+_\sigma(\vec{p}',0)a_\sigma(\vec{p},0)e^{i(E'-E)t}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)\delta^3(\vec{p}-\vec{p}')]\}$
$= \frac{i}{2}\int d^3\vec{p}\sum\limits_{\sigma}$
$\{-a_\sigma(\vec{p},0)\sqrt{E}e^{-iEt}(\tilde{\partial}_i[p_ja_\sigma(-\vec{p},0)\frac{1}{\sqrt{E}}e^{-iEt}] - \tilde{\partial}_j[p_ia_\sigma(-\vec{p},0)\frac{1}{\sqrt{E}}e^{-iEt}])$
$+$
$a^+_\sigma(\vec{p},0)\sqrt{E}e^{iEt}(\tilde{\partial}_i[p_ja^+_\sigma(-\vec{p},0)\frac{1}{\sqrt{E}}e^{iEt}] - \tilde{\partial}_j[p_ia^+_\sigma(-\vec{p},0)\frac{1}{\sqrt{E}}e^{iEt}])$
$-$
$a_\sigma(\vec{p},0)\sqrt{E}e^{-iEt}(\tilde{\partial}_i[p_ja^+_\sigma(\vec{p},0)\frac{1}{\sqrt{E}}e^{iEt}] - \tilde{\partial}_j[p_ia^+_\sigma(\vec{p},0)\frac{1}{\sqrt{E}}e^{iEt}])$
$+$
$a^+_\sigma(\vec{p},0)\sqrt{E}e^{iEt}(\tilde{\partial}_i[p_ja_\sigma(\vec{p},0)\frac{1}{\sqrt{E}}e^{-iEt}] - \tilde{\partial}_j[p_ia_\sigma(\vec{p},0)\frac{1}{\sqrt{E}}e^{-iEt}])\}$
$= \frac{i}{2}\int d^3\vec{p}\sum\limits_{\sigma}$
$\{-a_\sigma(\vec{p},0)\sqrt{E}e^{-iEt}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)[a_\sigma(-\vec{p},0)\frac{1}{\sqrt{E}}e^{-iEt}]) + a^+_\sigma(\vec{p},0)\sqrt{E}e^{iEt}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)[a^+_\sigma(-\vec{p},0)\frac{1}{\sqrt{E}}e^{iEt}])$
$- a_\sigma(\vec{p},0)\sqrt{E}e^{-iEt}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)[a^+_\sigma(\vec{p},0)\frac{1}{\sqrt{E}}e^{iEt}]) + a^+_\sigma(\vec{p},0)\sqrt{E}e^{iEt}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)[a_\sigma(\vec{p},0)\frac{1}{\sqrt{E}}e^{-iEt}])\}$
$= \frac{i}{2}\int d^3\vec{p}\sum\limits_{\sigma}$
$\{-a_\sigma(\vec{p},0)e^{-2iEt}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)a_\sigma(-\vec{p},0) + a^+_\sigma(\vec{p},0)e^{2iEt}(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)a^+_\sigma(-\vec{p},0)$
$- a_\sigma(\vec{p},0)(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)a^+_\sigma(\vec{p},0) + a^+_\sigma(\vec{p},0)(p_j\tilde{\partial}_i - p_i\tilde{\partial}_j)a_\sigma(\vec{p},0)\}$
$= -\frac{i}{2}\int d^3\vec{p}\sum\limits_{\sigma}[a^+_\sigma(\vec{p},0)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_\sigma(\vec{p},0) - a_\sigma(\vec{p},0)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a^+_\sigma(\vec{p},0)]$ □

Cor. 3.3.1. $\partial_t\phi_\sigma(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{-iE}{\sqrt{2E}}[a_\sigma(\vec{p},0)e^{i(\vec{p}\cdot\vec{r}-Et)} - a^+_\sigma(\vec{p},0)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 3.3.2. $(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{i(r_ip_j-r_jp_i)}{\sqrt{2E}}[a_\sigma(\vec{p},0)e^{i(\vec{p}\cdot\vec{r}-Et)} - a^+_\sigma(\vec{p},0)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 3.3.3.
$H = \frac{1}{2}\int\sum\limits_{\sigma}E[a^+(\vec{p},0)a(\vec{p},0) + a(\vec{p},0)a^+(\vec{p},0)]d^3\vec{p} = \frac{1}{2}\int\sum\limits_{\sigma}[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t)]d^3\vec{r}$
$\vec{P} = \frac{1}{2}\int\sum\limits_{\sigma}\vec{p}[a^+_\sigma(\vec{p},0)a_\sigma(\vec{p},0) + a_\sigma(\vec{p},0)a^+_\sigma(\vec{p},0)]d^3\vec{p} = \int\sum\limits_{\sigma}-\dot{\phi}_\sigma(\vec{r},t)\nabla\phi_\sigma(\vec{r},t)d^3\vec{r}$

Thm. 3.3.2. $M_{i\pi} = i\int\sum\limits_{\sigma}\{\frac{1}{2}r_i[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t)] + t\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)\}d^3\vec{r}$

3.4 Commutative and anti commutative formulas

Cor. 3.4.1. $\begin{cases}[A,BC]=[A,B]C+B[A,C],[A,CB]=[A,C]B+C[A,B]\\ [A,BC]=\{A,B\}C-B\{A,C\},[A,CB]=\{A,C\}B-C\{A,B\}\end{cases}$

Cor. 3.4.2. $\begin{cases}[A,\{B,C\}]=\{[A,B],C\}+\{B,[A,C]\}\\ [A,[B,C]]=\{\{A,B\},C\}-\{B,\{A,C\}\}\end{cases}$

3.5 Poincare algebra of scalar field

Cor. 3.5.1.
$H=\frac{1}{2}\int\sum\limits_{\sigma}[\nabla\phi_\sigma(\vec{r},t)]^2+\dot{\phi}_\sigma^2(\vec{r},t)+m^2\phi_\sigma^2(\vec{r},t)]d^3\vec{r}$
$=\frac{1}{2}\int\sum\limits_{\sigma}[\dot{\phi}_\sigma^2(\vec{r},t)-\phi_\sigma(\vec{r},t)\partial_t^2\phi_\sigma(\vec{r},t)]d^3\vec{r}$
$\vec{P}=\int\sum\limits_{\sigma}-\dot{\phi}_\sigma(\vec{r},t)\nabla\phi_\sigma(\vec{r},t)d^3\vec{r}$

Proof: $[P_i(t),P_\pi(t)]$
$=\int\sum\limits_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)\partial_\pi\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\partial_\pi\phi_{\sigma'}(\vec{r}',t)]+\dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\partial_\pi\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma,\sigma'}\dot{\phi}_\sigma(\vec{r},t)[\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\partial_\pi\phi_{\sigma'}(\vec{r}',t)]+\dot{\phi}_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)[\partial_i\phi_\sigma(\vec{r},t),\partial_\pi\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma}\dot{\phi}_\sigma(\vec{r},t)\partial_i\delta^3(\vec{r}-\vec{r}')\dot{\phi}_\sigma(\vec{r}',t)]+\dot{\phi}_\sigma(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)\partial_i\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$=-\int\sum\limits_{\sigma}[\partial_i\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)+\dot{\phi}_\sigma(\vec{r},t)\partial_i\dot{\phi}_\sigma(\vec{r},t)]d^3\vec{r}$
$=-\int\sum\limits_{\sigma}\partial_i[\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r}',t)]d^3\vec{r}=0$ □

Proof:
$=\int\sum\limits_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\phi_{\sigma'}(\vec{r}',t)\partial_t^2\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\phi_{\sigma'}(\vec{r}',t)]\partial_t^2\phi_{\sigma'}(\vec{r}',t)]+\phi_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\partial_t^2\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t),\phi_{\sigma'}(\vec{r}',t)]\partial_i\phi_\sigma(\vec{r},t)\partial_t^2\phi_{\sigma'}(\vec{r}',t)]+\phi_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t),\partial_t^2\phi_{\sigma'}(\vec{r}',t)]\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma,\sigma'}-\delta^3(\vec{r}-\vec{r}')\partial_i\phi_\sigma(\vec{r},t)\partial_t^2\phi_{\sigma'}(\vec{r}',t)]-\phi_{\sigma'}(\vec{r}',t)(m^2-\nabla^2)\delta^3(\vec{r}-\vec{r}')\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma,\sigma'}-\partial_i\phi_\sigma(\vec{r},t)\partial_t^2\phi_{\sigma'}(\vec{r},t)-\phi_{\sigma'}(\vec{r},t)(m^2-\nabla^2)\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}$
$=\int\sum\limits_{\sigma,\sigma'}-\partial_i\phi_\sigma(\vec{r},t)\partial_t^2\phi_{\sigma'}(\vec{r},t)-\phi_{\sigma'}(\vec{r},t)\partial_t^2\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}$
$=-\int\sum\limits_{\sigma,\sigma'}\partial_i[\phi_\sigma(\vec{r},t)\partial_t^2\phi_{\sigma'}(\vec{r},t)]d^3\vec{r}=0$ □

Cor. 3.5.2. $[P_a(t),P_b(t)]=0$

Proof: $[M_{ij}(t),P_\pi(t)]$
$=\int\sum\limits_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)\partial_\pi\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum\limits_{\sigma,\sigma'}d^3\vec{r}d^3\vec{r}'$
$[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\partial_\pi\phi_{\sigma'}(\vec{r}',t)+\dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\partial_\pi\phi_{\sigma'}(\vec{r}',t)]$
$=\int\sum\limits_{\sigma,\sigma'}d^3\vec{r}d^3\vec{r}'$
$\dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\partial_\pi\phi_{\sigma'}(\vec{r}',t)+\dot{\phi}_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\partial_\pi\phi_{\sigma'}(\vec{r}',t)]$
$=-i\int\sum\limits_{\sigma,\sigma'}d^3\vec{r}d^3\vec{r}'$
$\dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\dot{\phi}_{\sigma'}(\vec{r}',t)+\dot{\phi}_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]$
$=-i\int\sum\limits_{\sigma}\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\delta^3(\vec{r}-\vec{r}')\dot{\phi}_\sigma(\vec{r}',t)+\dot{\phi}_\sigma(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\delta^3(\vec{r}-\vec{r}')d^3\vec{r}$
$=i\int\sum\limits_{\sigma}\{\partial_j[r_i\dot{\phi}_\sigma(\vec{r},t)]-\partial_i[r_j\dot{\phi}_\sigma(\vec{r},t)]\}\dot{\phi}_\sigma(\vec{r},t)+\dot{\phi}_\sigma(\vec{r},t)\{\partial_j[r_i\dot{\phi}_\sigma(\vec{r},t)]-\partial_i[r_j\dot{\phi}_\sigma(\vec{r},t)]\}d^3\vec{r}$
$=i\int\sum\limits_{\sigma}\{\partial_j[r_i\dot{\phi}_\sigma(\vec{r},t)]-\partial_i[r_j\dot{\phi}_\sigma(\vec{r},t)]\}\dot{\phi}_\sigma(\vec{r},t)+\dot{\phi}_\sigma(\vec{r},t)[r_i\partial_j-r_j\partial_i]\dot{\phi}_\sigma(\vec{r},t)]d^3\vec{r}$
$=i\int\sum\limits_{\sigma}\{\partial_j[r_i\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)]-\partial_i[r_j\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)]\}d^3\vec{r}=0$ □

Proof: $[M_{ij}(t), P_k(t)]$
$= \int \sum_{\sigma,\sigma'} [\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)\partial'_k\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= \int \sum_{\sigma,\sigma'} d^3\vec{r}d^3\vec{r}'$
$[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]\partial'_j\phi_{\sigma'}(\vec{r}',t) + \dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t), \partial'_k\phi_{\sigma'}(\vec{r}',t)]$
$= \int \sum_{\sigma,\sigma'} d^3\vec{r}d^3\vec{r}'$
$\dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]\partial'_k\phi_{\sigma'}(\vec{r}',t) + \dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t), \partial'_k\phi_{\sigma'}(\vec{r}',t)](r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)$
$= i\int \sum_{\sigma,\sigma'} \dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\delta^3(\vec{r}-\vec{r}')\partial'_k\phi_{\sigma'}(\vec{r}',t) - \dot{\phi}_{\sigma'}(\vec{r}',t)\partial'_k\delta^3(\vec{r}-\vec{r}')(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -i\int \sum_{\sigma} \{\partial_j[r_i\dot{\phi}_\sigma(\vec{r},t)] - \partial_i[r_j\dot{\phi}_\sigma(\vec{r},t)]\}\partial_k\phi_\sigma(\vec{r},t) - \partial_k\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= i\int \sum_{\sigma} r_i\dot{\phi}_\sigma(\vec{r},t)\partial_j\partial_k\phi_\sigma(\vec{r},t) - r_j\dot{\phi}_\sigma(\vec{r},t)\partial_i\partial_k\phi_\sigma(\vec{r},t) - \dot{\phi}_\sigma(\vec{r},t)\partial_k(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= i\int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\partial_k\phi_\sigma(\vec{r},t) - \dot{\phi}_\sigma(\vec{r},t)\partial_k(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= -i\int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r},t)[-i(r_i\partial_j - r_j\partial_i), -i\partial_k]\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= -i\int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r},t)(\delta_{ik}\partial_j - \delta_{jk}\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$= -i[P_i(t)\delta_{jk} - P_j(t)\delta_{ik}]$ □

Proof: $[M_{i\pi}(t), P_\pi(t)]$
$= \int \sum_{\sigma,\sigma'} [\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_\pi - it\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)\partial_\pi\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= \int \sum_{\sigma,\sigma'} d^3\vec{r}d^3\vec{r}'$
$[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_\pi - it\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]\partial_\pi\phi_{\sigma'}(\vec{r}',t) + \dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_\pi - it\partial_i)\phi_\sigma(\vec{r},t), \partial_\pi\phi_{\sigma'}(\vec{r}',t)]$
$= -\int \sum_{\sigma,\sigma'} [\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]\dot{\phi}_{\sigma'}(\vec{r}',t) + \dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\int \sum_{\sigma,\sigma'} \dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]\dot{\phi}_{\sigma'}(\vec{r}',t) + \dot{\phi}_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\int \sum_{\sigma,\sigma'} \dot{\phi}_\sigma(\vec{r},t)t[\partial_i\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]\dot{\phi}_{\sigma'}(\vec{r}',t) + \dot{\phi}_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)t[\partial_i\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$ □

3.6 Strict proof of Poincare algebra of scalar field

Cor. 3.6.1. $\begin{cases}[A,BC] = [A,B]C + B[A,C] \\ [BC,A] = [B,A]C + B[C,A] \\ [\phi_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)] = i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\end{cases}$

Cor. 3.6.2.
$P_i = \int \sum_{\sigma} -\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}$
$P_\pi = \frac{i}{2}\int \sum_{\sigma}[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t)]d^3\vec{r}$

Thm. 3.6.1.
$M_{ij} = -\int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)d^3\vec{r}$
$M_{i\pi} = i\int \sum_{\sigma}\{\frac{1}{2}r_i[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t)] + t\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)\}d^3\vec{r}$
$= i\int \sum_{\sigma}\{\frac{1}{2}[-r_i\nabla^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t) - \partial_i\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t) + r_i\dot{\phi}^2_\sigma(\vec{r},t) + m^2r_i\phi^2_\sigma(\vec{r},t)] + t\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)\}d^3\vec{r}$
$= i\int \sum_{\sigma}\{\frac{1}{2}[-r_i\nabla^2\phi_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t) + r_i\dot{\phi}^2_\sigma(\vec{r},t) + m^2r_i\phi^2_\sigma(\vec{r},t)] + t\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)\}d^3\vec{r}$

3.6.1 Lemma–Mathematical preparation

Lem. 3.6.1. $[\dot{\phi}_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2] = -2i\delta_{\sigma\sigma'}\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot\nabla'\delta^3(\vec{r}-\vec{r}')$

Proof: $[\dot{\phi}_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2]$
$= 2\nabla'\phi_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t), \nabla'\phi_{\sigma'}(\vec{r}',t)]$
$= -2i\delta_{\sigma\sigma'}\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot\nabla'\delta^3(\vec{r}-\vec{r}')$ □

Lem. 3.6.2. $[\dot{\phi}_\sigma(\vec{r},t), \dot{\phi}^2_{\sigma'}(\vec{r}',t)] = 0$

Lem. 3.6.3. $[\dot{\phi}_\sigma(\vec{r},t), m^2\phi^2_{\sigma'}(\vec{r}',t)] = -2im^2\delta_{\sigma\sigma'}\phi_{\sigma'}(\vec{r}',t)\delta^3(\vec{r}-\vec{r}')$

Proof: $[\dot{\phi}_\sigma(\vec{r},t),\phi^2_{\sigma'}(\vec{r}',t)]$
$=2\phi_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t),\phi_{\sigma'}(\vec{r}',t)]$
$=-2i\delta_{\sigma\sigma'}\phi_{\sigma'}(\vec{r}',t)\delta^3(\vec{r}-\vec{r}')$ □

Lem. 3.6.4. $[\dot{\phi}_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2+\dot{\phi}^2_{\sigma'}(\vec{r}',t)+m^2\phi^2_{\sigma'}(\vec{r}',t)]=-2i\delta_{\sigma\sigma'}[\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot\nabla'+m^2\phi_{\sigma'}(\vec{r}',t)]\delta^3(\vec{r}-\vec{r}')$

Lem. 3.6.5. $[\phi_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2+\dot{\phi}^2_{\sigma'}(\vec{r}',t)+m^2\phi^2_{\sigma'}(\vec{r}',t)]=2i\delta_{\sigma\sigma'}\dot{\phi}_{\sigma'}(\vec{r}',t)\delta^3(\vec{r}-\vec{r}')$

Lem. 3.6.6. $[\nabla\phi_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2+\dot{\phi}^2_{\sigma'}(\vec{r}',t)+m^2\phi^2_{\sigma'}(\vec{r}',t)]=2i\delta_{\sigma\sigma'}\dot{\phi}_{\sigma'}(\vec{r}',t)\nabla\delta^3(\vec{r}-\vec{r}')$

3.6.2 Momentum commutation rules of scalar field

Thm. 3.6.2. $[P_a(t),P_b(t)]=0$

Proof: $[P_i(t),P_j(t)]$
$=\int\sum_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)\partial'_j\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum_\sigma[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\partial'_j\phi_{\sigma'}(\vec{r}',t)]+\dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),\partial'_j\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum_\sigma\dot{\phi}_\sigma(\vec{r},t)[\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\partial'_j\phi_{\sigma'}(\vec{r}',t)]+\dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t),\partial'_j\phi_{\sigma'}(\vec{r}',t)]\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$=i\int\sum_\sigma\dot{\phi}_\sigma(\vec{r},t)\partial_i\delta^3(\vec{r}-\vec{r}')\partial'_j\phi_{\sigma'}(\vec{r}',t)-\dot{\phi}_{\sigma'}(\vec{r}',t)\partial'_j\delta^3(\vec{r}-\vec{r}')\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$=-i\int\sum_\sigma\dot{\phi}_\sigma(\vec{r},t)\partial'_i\delta^3(\vec{r}-\vec{r}')\partial'_j\phi_\sigma(\vec{r}',t)-\dot{\phi}_\sigma(\vec{r}',t)\partial_j\delta^3(\vec{r}-\vec{r}')\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$=i\int\sum_\sigma\dot{\phi}_\sigma(\vec{r},t)\partial_i\partial_j\phi_\sigma(\vec{r},t)-\dot{\phi}_\sigma(\vec{r},t)\partial_j\partial_i\phi_\sigma(\vec{r},t)d^3\vec{r}=0$ □

Proof: $[P_i(t),P_\pi(t)]$
$=-\frac{i}{2}\int\sum_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2+\dot{\phi}^2_{\sigma'}(\vec{r}',t)+m^2\phi^2_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=-\frac{i}{2}\int\sum_{\sigma,\sigma'}\{[\dot{\phi}_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2+\dot{\phi}^2_{\sigma'}(\vec{r}',t)+m^2\phi^2_{\sigma'}(\vec{r}',t)]\partial_i\phi_\sigma(\vec{r},t)$
$+\dot{\phi}_\sigma(\vec{r},t)[\partial_i\phi_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2+\dot{\phi}^2_{\sigma'}(\vec{r}',t)+m^2\phi^2_{\sigma'}(\vec{r}',t)]\}d^3\vec{r}d^3\vec{r}'$
$=-\frac{i}{2}\int\sum_{\sigma,\sigma'}\{[\dot{\phi}_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2+m^2\phi^2_{\sigma'}(\vec{r}',t)]\partial_i\phi_\sigma(\vec{r},t)+\dot{\phi}_\sigma(\vec{r},t)[\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}^2_{\sigma'}(\vec{r}',t)]\}d^3\vec{r}d^3\vec{r}'$
$=-i\int\sum_{\sigma,\sigma'}\{[\dot{\phi}_\sigma(\vec{r},t),\nabla'\phi_{\sigma'}(\vec{r}',t)]\cdot\nabla'\phi_{\sigma'}(\vec{r}',t)\partial_i\phi_\sigma(\vec{r},t)+[\dot{\phi}_\sigma(\vec{r},t),\phi_{\sigma'}(\vec{r}',t)]m^2\phi_{\sigma'}(\vec{r}',t)\partial_i\phi_\sigma(\vec{r},t)$
$+\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t)[\partial_i\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\}d^3\vec{r}d^3\vec{r}'$
$=-i\int\sum_{\sigma,\sigma'}\{-i\delta_{\sigma\sigma'}\nabla'\delta^3(\vec{r}-\vec{r}')\cdot\nabla'\phi_{\sigma'}(\vec{r}',t)\partial_i\phi_\sigma(\vec{r},t)-i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')m^2\phi_{\sigma'}(\vec{r}',t)\partial_i\phi_\sigma(\vec{r},t)$
$+\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t)i\delta_{\sigma\sigma'}\partial_i\delta^3(\vec{r}-\vec{r}')\}d^3\vec{r}d^3\vec{r}'$
$=\int\{\nabla^2\phi_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t)-m^2\phi_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t)-\partial_i\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)\}d^3\vec{r}$
$=\int\{\nabla\phi_\sigma(\vec{r},t)\partial_i\cdot\nabla\phi_\sigma(\vec{r},t)-m^2\phi_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t)-\partial_i\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)\}d^3\vec{r}$
$=\frac{1}{2}\int\{-\partial_i[\nabla\phi_\sigma(\vec{r},t)\cdot\nabla\phi_\sigma(\vec{r},t)]-m^2\partial_i\phi^2_\sigma(\vec{r},t)-\partial_i\dot{\phi}^2_\sigma(\vec{r},t)\}d^3\vec{r}$
$=-\frac{1}{2}\int\partial_i\{[\nabla\phi_\sigma(\vec{r},t)]^2+m^2\phi^2_\sigma(\vec{r},t)+\dot{\phi}^2_\sigma(\vec{r},t)\}d^3\vec{r}=0$ □

3.6.3 Angular momentum commutation rules of scalar field

Proof: $[M_{ij}(t),M_{kl}(t)]$
$=\int\sum_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)(r'_k\partial'_l-r'_l\partial'_k)\phi_{\sigma'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$=\int\sum_{\sigma,\sigma'}d^3\vec{r}d^3\vec{r}'\{[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)](r'_k\partial'_l-r'_l\partial'_k)\phi_{\sigma'}(\vec{r}',t)$
$+\dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),(r'_k\partial'_l-r'_l\partial'_k)\phi_{\sigma'}(\vec{r}',t)]\}$
$=\int\sum_{\sigma,\sigma'}d^3\vec{r}d^3\vec{r}'\{\dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)](r'_k\partial'_l-r'_l\partial'_k)\phi_{\sigma'}(\vec{r}',t)$
$+\dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}_\sigma(\vec{r},t),(r'_k\partial'_l-r'_l\partial'_k)\phi_{\sigma'}(\vec{r}',t)](r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t)\}$
$=i\int\sum_{\sigma,\sigma'}d^3\vec{r}d^3\vec{r}'$
$\{\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\delta^3(\vec{r}-\vec{r}')(r'_k\partial'_l-r'_l\partial'_k)\phi_{\sigma'}(\vec{r}',t)-\dot{\phi}_{\sigma'}(\vec{r}',t)(r'_k\partial'_l-r'_l\partial'_k)\delta^3(\vec{r}-\vec{r}')(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t)\}$
$=-i\int\sum_\sigma d^3\vec{r}$
$\{\{\partial_j[r_i\dot{\phi}_\sigma(\vec{r},t)]-\partial_i[r_j\dot{\phi}_\sigma(\vec{r},t)]\}(r_k\partial_l-r_l\partial_k)\phi_\sigma(\vec{r},t)-\{\partial_l[r_k\dot{\phi}_\sigma(\vec{r},t)]-\partial_k[r_l\dot{\phi}_\sigma(\vec{r},t)]\}(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t)\}$
$=-i\int\sum_\sigma\dot{\phi}_\sigma(\vec{r},t)[-i(r_i\partial_j-r_j\partial_i),-i(r_k\partial_l-r_l\partial_k)]\phi_\sigma(\vec{r},t)d^3\vec{r}$
$=-i[\delta_{il}M_{jk}(t)-\delta_{ik}M_{jl}(t)+\delta_{jk}M_{il}(t)-\delta_{jl}M_{ik}(t)]$ □

Proof: $[M_{ij}(t), M_{k\pi}(t)]$
$= i\int\sum_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t), \{\frac{1}{2}r'_k[[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + \dot{\phi}^2_{\sigma'}(\vec{r}',t) + m^2\phi^2_{\sigma'}(\vec{r}',t)] + t\partial'_k\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_{\sigma'}(\vec{r}',t)\}]d^3\vec{r}d^3\vec{r}'$
$= i\int\sum_{\sigma,\sigma'}[\dot{\phi}_\sigma(\vec{r},t), \frac{1}{2}r'_k[[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)] + t\partial'_k\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_{\sigma'}(\vec{r}',t)\}](r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)$
$+ \dot{\phi}_\sigma(\vec{r},t)[(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t), \{\frac{1}{2}r'_k\dot{\phi}^2_{\sigma'}(\vec{r}',t) + t\partial'_k\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_{\sigma'}(\vec{r}',t)\}]d^3\vec{r}d^3\vec{r}'$
$= \int\sum_{\sigma,\sigma'}\delta_{\sigma\sigma'}[r'_k\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot\nabla' + m^2r'_k\phi_{\sigma'}(\vec{r}',t) + t\partial'_k\dot{\phi}_{\sigma'}(\vec{r}',t)]\delta^3(\vec{r}-\vec{r}')(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)$
$- \delta_{\sigma\sigma'}[r'_k\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t) + t\dot{\phi}_\sigma(\vec{r},t)\phi_{\sigma'}(\vec{r}',t)\partial'_k](r_i\partial_j - r_j\partial_i)\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$= \int[-\partial_k\phi_\sigma(\vec{r},t) - r_k\nabla^2\phi_\sigma(\vec{r},t) + m^2r_k\phi_\sigma(\vec{r},t) + t\partial_k\dot{\phi}_\sigma(\vec{r},t)](r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)$
$+ [r_k(r_i\partial_j - r_j\partial_i)\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t) - t(r_i\partial_j - r_j\partial_i)\dot{\phi}_\sigma(\vec{r},t)\partial_k\phi_\sigma(\vec{r},t)]d^3\vec{r}$
$= \int[-\partial_k\phi_\sigma(\vec{r},t) - r_k\nabla^2\phi_\sigma(\vec{r},t) + m^2r_k\phi_\sigma(\vec{r},t)](r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)$
$+ r_k(r_i\partial_j - r_j\partial_i)\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)d^3\vec{r}$
$= \int[-\partial_k\phi_\sigma(\vec{r},t) - r_k\dot{\phi}^2_\sigma(\vec{r},t)](r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)$
$+ r_k(r_i\partial_j - r_j\partial_i)\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)d^3\vec{r}$
$=$
$= \int\sum_\sigma g_{jk}\{\frac{1}{2}r_i[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t)] + t\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)\}$
$- g_{ik}\{\frac{1}{2}r_j[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t)] + t\partial_j\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)\}d^3\vec{r}$ □

$$M_{ab} = L_{ab} + S_{ab}, L_{ab} = x_ap_b - x_bp_a, g_{ab} = \delta_{ab} \tag{20.3}$$

$$\begin{cases}[M_{ab}, M_{cd}] = -i(g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac})\\ [M_{ij}, M_{k\pi}] = -i(g_{jk}M_{i\pi} - g_{ik}M_{j\pi})\\ [M_{ab}, p_c] = -i(g_{bc}p_a - g_{ac}p_b), [p_a, p_b] = 0\end{cases} \tag{20.4}$$

Proof: $[M_{i\pi}(t), M_{j\pi}(t)]$
$= -\int\sum_{\sigma\sigma'}[\{\frac{1}{2}r_i[[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t)] + t\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t)\}$
$, \{\frac{1}{2}r'_j[[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + \dot{\phi}^2_{\sigma'}(\vec{r}',t) + m^2\phi^2_{\sigma'}(\vec{r}',t)] + t\partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_{\sigma'}(\vec{r}',t)\}]d^3\vec{r}d^3\vec{r}'$
$= -\int\sum_{\sigma\sigma'}\{\frac{1}{4}r_ir'_j[[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + \dot{\phi}^2_{\sigma'}(\vec{r}',t) + m^2\phi^2_{\sigma'}(\vec{r}',t)]$
$+ t^2[\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_{\sigma'}(\vec{r}',t)]$
$+ \frac{1}{2}r_it[[\nabla\phi_\sigma(\vec{r},t)]^2 + \dot{\phi}^2_\sigma(\vec{r},t) + m^2\phi^2_\sigma(\vec{r},t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_{\sigma'}(\vec{r}',t)]$
$+ \frac{1}{2}r'_jt[\partial_i\dot{\phi}_\sigma(\vec{r},t)\phi_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + \dot{\phi}^2_{\sigma'}(\vec{r}',t) + m^2\phi^2_{\sigma'}(\vec{r}',t)]\}d^3\vec{r}d^3\vec{r}'$
$= -\int\sum_{\sigma\sigma'}d^3\vec{r}d^3\vec{r}'$
$\{\frac{1}{4}r_ir'_j[[\nabla\phi_\sigma(\vec{r},t)]^2 + m^2\phi^2_\sigma(\vec{r},t), \dot{\phi}^2_{\sigma'}(\vec{r}',t)]] + \frac{1}{4}r_ir'_j[\dot{\phi}^2_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)]$
$+ t^2\partial_i\dot{\phi}_\sigma(\vec{r},t)[\phi_\sigma(\vec{r},t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)]\phi_{\sigma'}(\vec{r}',t) + t^2\partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)[\partial_i\dot{\phi}_\sigma(\vec{r},t), \phi_{\sigma'}(\vec{r}',t)]\phi_\sigma(\vec{r},t)$
$+ \frac{1}{2}r_it[[\nabla\phi_\sigma(\vec{r},t)]^2 + m^2\phi^2_\sigma(\vec{r},t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)]\phi_{\sigma'}(\vec{r}',t) + \frac{1}{2}r_it\partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)[\dot{\phi}^2_\sigma(\vec{r},t), \phi_{\sigma'}(\vec{r}',t)]$
$+ \frac{1}{2}r'_jt\partial_i\dot{\phi}_\sigma(\vec{r},t)[\phi_\sigma(\vec{r},t), \dot{\phi}^2_{\sigma'}(\vec{r}',t)] + \frac{1}{2}r'_jt[\partial_i\dot{\phi}_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)]\phi_\sigma(\vec{r},t)\}$
$= -\int\sum_{\sigma\sigma'}d^3\vec{r}d^3\vec{r}'$
$\{\frac{1}{4}r_ir'_j[[\nabla\phi_\sigma(\vec{r},t)]^2 + m^2\phi^2_\sigma(\vec{r},t), \dot{\phi}^2_{\sigma'}(\vec{r}',t)]] + \frac{1}{4}r_ir'_j[\dot{\phi}^2_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)]$
$+ t^2\partial_i\dot{\phi}_\sigma(\vec{r},t)i\delta_{\sigma\sigma'}\partial'_j\delta^3(\vec{r}-\vec{r}')\phi_{\sigma'}(\vec{r}',t) - t^2\partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)i\delta_{\sigma\sigma'}\partial_i\delta^3(\vec{r}-\vec{r}')\phi_\sigma(\vec{r},t)$
$+ r_it[\nabla\phi_\sigma(\vec{r},t)\cdot\nabla + m^2\phi_\sigma(\vec{r},t)]i\delta_{\sigma\sigma'}\partial'_j\delta^3(\vec{r}-\vec{r}')\phi_{\sigma'}(\vec{r}',t) - r_it\partial'_j\dot{\phi}_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')$
$+ r'_jt\partial_i\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t)i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}') - r'_jt[\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot\nabla' + m^2\phi_{\sigma'}(\vec{r}',t)]i\delta_{\sigma\sigma'}\partial_i\delta^3(\vec{r}-\vec{r}')\phi_\sigma(\vec{r},t)\}$
$= -\int\sum_{\sigma\sigma'}d^3\vec{r}d^3\vec{r}'$
$\{\frac{1}{4}r_ir'_j[[\nabla\phi_\sigma(\vec{r},t)]^2 + m^2\phi^2_\sigma(\vec{r},t), \dot{\phi}^2_{\sigma'}(\vec{r}',t)]] + \frac{1}{4}r_ir'_j[\dot{\phi}^2_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)]$
$- it^2\partial_i\dot{\phi}_\sigma(\vec{r},t)\partial_j\phi_\sigma(\vec{r},t) + it^2\partial_j\dot{\phi}_\sigma(\vec{r},t)\partial_i\phi_\sigma(\vec{r},t)$
$- ir_it[\nabla\phi_\sigma(\vec{r},t)\cdot\nabla + m^2\phi_\sigma(\vec{r},t)]\partial_j\phi_\sigma(\vec{r},t) - ir_it\partial_j\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)$
$+ ir_jt\partial_i\dot{\phi}_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t) + ir_jt[\nabla\phi_\sigma(\vec{r},t)\cdot\nabla + m^2\phi_\sigma(\vec{r},t)]\partial_i\phi_\sigma(\vec{r},t)\}$
$= -\int\sum_{\sigma\sigma'}d^3\vec{r}d^3\vec{r}'\frac{1}{4}r_ir'_j$
$\{[[\nabla\phi_\sigma(\vec{r},t)]^2 + m^2\phi^2_\sigma(\vec{r},t), \dot{\phi}^2_{\sigma'}(\vec{r}',t)]] + [\dot{\phi}^2_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)]$
$= -\int\sum_{\sigma\sigma'}d^3\vec{r}d^3\vec{r}'\frac{1}{4}r_ir'_j$
$\{\dot{\phi}_{\sigma'}(\vec{r}',t)[[\nabla\phi_\sigma(\vec{r},t)]^2 + m^2\phi^2_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)] + [[\nabla\phi_\sigma(\vec{r},t)]^2 + m^2\phi^2_\sigma(\vec{r},t), \dot{\phi}_{\sigma'}(\vec{r}',t)]\dot{\phi}_{\sigma'}(\vec{r}',t)$
$+ [\dot{\phi}^2_\sigma(\vec{r},t), [\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)]$

$= -\int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}'\frac{1}{4}r_i r'_j$
$\{2\dot{\phi}_{\sigma'}(\vec{r}',t)\nabla\phi_\sigma(\vec{r},t)\cdot[\nabla\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)] + 2m^2\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_\sigma(\vec{r},t)[\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]$
$+ 2[\nabla\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\cdot\nabla\phi_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t) + 2m^2[\phi_\sigma(\vec{r},t),\dot{\phi}_{\sigma'}(\vec{r}',t)]\phi_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t)$
$+ [\dot{\phi}^2_\sigma(\vec{r},t),[\nabla'\phi_{\sigma'}(\vec{r}',t)]^2 + m^2\phi^2_{\sigma'}(\vec{r}',t)]$
$= -\int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}'\frac{1}{4}r_i r'_j$
$\{2\dot{\phi}_{\sigma'}(\vec{r}',t)\nabla\phi_\sigma(\vec{r},t)\cdot i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r}-\vec{r}') + 2m^2\dot{\phi}_{\sigma'}(\vec{r}',t)\phi_\sigma(\vec{r},t)i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')$
$+ 2i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r}-\vec{r}')\cdot\nabla\phi_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t) + 2m^2 i\delta_{\sigma\sigma'}\delta^3(\vec{r}-\vec{r}')\phi_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t)$
$- 2\dot{\phi}_\sigma(\vec{r},t)\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot i\delta_{\sigma'\sigma}\nabla'\delta^3(\vec{r}'-\vec{r}) - 2m^2\dot{\phi}_\sigma(\vec{r},t)\phi_{\sigma'}(\vec{r}',t)i\delta_{\sigma'\sigma}\delta^3(\vec{r}'-\vec{r})$
$- 2i\delta_{\sigma'\sigma}\nabla'\delta^3(\vec{r}'-\vec{r})\cdot\nabla'\phi_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t) - 2m^2 i\delta_{\sigma'\sigma}\delta^3(\vec{r}'-\vec{r})\phi_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)$
$= -\int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}'\frac{1}{4}r_i r'_j$
$\{2\dot{\phi}_{\sigma'}(\vec{r}',t)\nabla\phi_\sigma(\vec{r},t)\cdot i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r}-\vec{r}') + 2i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r}-\vec{r}')\cdot\nabla\phi_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t)$
$- 2\dot{\phi}_\sigma(\vec{r},t)\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot i\delta_{\sigma'\sigma}\nabla'\delta^3(\vec{r}'-\vec{r}) - 2i\delta_{\sigma'\sigma}\nabla'\delta^3(\vec{r}'-\vec{r})\cdot\nabla'\phi_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)$
$= -\int d^3\vec{r}d^3\vec{r}'\frac{1}{2}$
$\{r'_j\dot{\phi}_\sigma(\vec{r}',t)r_i\nabla\phi_\sigma(\vec{r},t)\cdot i\nabla\delta^3(\vec{r}-\vec{r}') + i\nabla\delta^3(\vec{r}-\vec{r}')\cdot r_i\nabla\phi_\sigma(\vec{r},t)r'_j\dot{\phi}_\sigma(\vec{r}',t)$
$- r_i\dot{\phi}_\sigma(\vec{r},t)r'_j\nabla'\phi_\sigma(\vec{r}',t)\cdot i\nabla'\delta^3(\vec{r}'-\vec{r}) - i\nabla'\delta^3(\vec{r}'-\vec{r})\cdot r'_j\nabla'\phi_\sigma(\vec{r}',t)r_i\dot{\phi}_\sigma(\vec{r},t)$
$= -\int d^3\vec{r}d^3\vec{r}'\frac{i}{2}$
$\{\nabla[r_j\dot{\phi}_\sigma(\vec{r},t)]r_i\cdot\nabla\phi_\sigma(\vec{r},t) + r_i\nabla\phi_\sigma(\vec{r},t)\cdot\nabla[r_j\dot{\phi}_\sigma(\vec{r},t)]$
$- \nabla[r_i\dot{\phi}_\sigma(\vec{r},t)]r_j\cdot\nabla\phi_\sigma(\vec{r},t) - r_j\nabla\phi_\sigma(\vec{r},t)\cdot\nabla[r_i\dot{\phi}_\sigma(\vec{r},t)]$
$= -\int d^3\vec{r}\frac{i}{2}$
$\{\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t) + (r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t)$
$= -i\int d^3\vec{r}\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r},t)$
$= iM_{ij}(t)$ □

Chapter21 New Scheme for Covariant Quantization of Electromagnetic Field

This chapter mainly derives a new scheme for the covariant quantization of electromagnetic field strength from the traditional scheme for the quantization of electromagnetic field potential. It mainly reflects the export process, but does not reflect its integrity. he main purpose is to verify the correctness of the new covariate program. The following chapters will directly and separately give their complete field strength covariant quantization schemes under two representations.

1 Gauge potential analysis of electromagnetic field equation [22, 24]

1.1 Gauge potential description of electromagnetic field equation with mass

Thm. 1.1.1. $\begin{cases}\nabla\cdot\vec{E}=m^2\phi-\rho,\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\nabla\times\vec{B}=m^2\vec{A}-\vec{J}+\partial_t\vec{E}\\ \nabla\cdot\vec{J}+\partial_t\rho=0\\ \vec{E}=-\partial_t\vec{A}-\nabla\phi,\vec{B}=\nabla\times\vec{A}\end{cases}\Leftrightarrow\begin{cases}(\nabla^2-\partial_t^2-m^2)\phi=\rho\\ (\nabla^2-\partial_t^2-m^2)\vec{A}=\vec{J}\\ \nabla\cdot\vec{A}+\partial_t\phi=0\\ \vec{E}=-\partial_t\vec{A}-\nabla\phi,\vec{B}=\nabla\times\vec{A}\end{cases}$

1.2 General gauge potential description of electromagnetic field equation

Lem. 1.2.1. $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$

Lem. 1.2.2. $\nabla\cdot\vec{B}=0,\vec{A}=\frac{\nabla\times\vec{B}}{-\nabla^2}+\nabla\theta\Leftrightarrow\vec{B}=\nabla\times\vec{A},\nabla\cdot\vec{A}=\nabla^2\theta$

Positive proof:

Proof: $\nabla\cdot\vec{B}=0,\vec{A}=\frac{\nabla\times\vec{B}}{-\nabla^2}+\nabla\theta$
$\Rightarrow\nabla\times\vec{A}=\frac{\nabla\times(\nabla\times\vec{B})}{-\nabla^2}+\nabla\times\nabla\theta,\nabla\cdot\vec{A}=\frac{\nabla\cdot(\nabla\times\vec{B})}{-\nabla^2}+\nabla\cdot\nabla\theta$
$\Rightarrow\nabla\times\vec{A}=\frac{\nabla(\nabla\cdot\vec{B})-\nabla^2\vec{B}}{-\nabla^2},\nabla\cdot\vec{A}=\nabla^2\theta$
$\Rightarrow\vec{B}=\nabla\times\vec{A},\nabla\cdot\vec{A}=\nabla^2\theta$ □

Reverse proof:

Proof: $\vec{B}=\nabla\times\vec{A},\nabla\cdot\vec{A}=\nabla^2\theta$
$\Rightarrow\nabla\times\vec{B}=\nabla\times(\nabla\times\vec{A}),\nabla\cdot\vec{B}=\nabla\cdot(\nabla\times\vec{A})$
$\Rightarrow\nabla\times\vec{B}=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A},\nabla\cdot\vec{B}=0$
$\Rightarrow\nabla\cdot\vec{B}=0,\vec{A}=\frac{\nabla\times\vec{B}}{-\nabla^2}+\nabla\theta$ □

Thm. 1.2.1. $\begin{cases}\nabla\cdot\vec{E}=-\rho,\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\nabla\times\vec{B}=-\vec{J}+\partial_t\vec{E}\\ \vec{A}=\frac{\nabla\times\vec{B}}{-\nabla^2}+\nabla\theta,\phi=\frac{\nabla\cdot\vec{E}}{-\nabla^2}-\partial_t\theta\end{cases}\Leftrightarrow\begin{cases}\nabla^2\phi=\rho-\partial_t\nabla^2\theta,\nabla\cdot\vec{A}=\nabla^2\theta\\ \nabla^2\vec{A}-\partial_t^2\vec{A}=\vec{J}+\nabla(\partial_t\phi+\nabla^2\theta)\\ \vec{E}=-\partial_t\vec{A}-\nabla\phi,\vec{B}=\nabla\times\vec{A}\end{cases}$

Positive proof:

Proof: $\nabla^2\phi=\nabla^2\frac{\nabla\cdot\vec{E}}{-\nabla^2}-\nabla^2\partial_t\theta=-\nabla\cdot\vec{E}-\nabla^2\partial_t\theta=\rho-\partial_t\nabla^2\theta$ □

Proof: $\nabla\cdot\vec{A}=\frac{\nabla\cdot\nabla\times\vec{B}}{-\nabla^2}+\nabla\cdot\nabla\theta=\nabla^2\theta$ □

Proof: $\nabla^2\vec{A}-\partial_t^2\vec{A}=(\nabla^2-\partial_t^2)\frac{\nabla\times\vec{B}}{-\nabla^2}+(\nabla^2-\partial_t^2)\nabla\theta$
$=-\nabla\times\vec{B}+\partial_t\frac{\nabla\times\partial_t\vec{B}}{\nabla^2}+(\nabla^2-\partial_t^2)\nabla\theta$
$=-\nabla\times\vec{B}-\partial_t\frac{\nabla\times\nabla\times\vec{E}}{\nabla^2}+(\nabla^2-\partial_t^2)\nabla\theta$
$=-\nabla\times\vec{B}-\partial_t\frac{\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}}{\nabla^2}+(\nabla^2-\partial_t^2)\nabla\theta$
$=-\nabla\times\vec{B}+\partial_t\vec{E}-\partial_t\frac{\nabla(\nabla\cdot\vec{E})}{\nabla^2}+(\nabla^2-\partial_t^2)\nabla\theta$
$=\vec{J}+\partial_t\nabla(\phi+\partial_t\theta)+(\nabla^2-\partial_t^2)\nabla\theta$
$=\vec{J}+\nabla(\partial_t\phi+\nabla^2\theta)$ □

Proof: $\nabla\times\vec{A}=\frac{\nabla\times\nabla\times\vec{B}}{-\nabla^2}+\nabla\times\nabla\theta=\frac{\nabla(\nabla\cdot\vec{B})-\nabla^2\vec{B}}{-\nabla^2}=\frac{0-\nabla^2\vec{B}}{-\nabla^2}=\vec{B}$ □

Proof: $-\partial_t\vec{A}-\nabla\phi=\partial_t\frac{\nabla\times\vec{B}}{\nabla^2}-\partial_t\nabla\theta+\nabla\frac{\nabla\cdot\vec{E}}{\nabla^2}+\nabla\partial_t\theta$
$=\frac{\nabla\times\partial_t\vec{B}}{\nabla^2}+\nabla\frac{\nabla\cdot\vec{E}}{\nabla^2}=-\frac{\nabla\times\nabla\times\vec{E}}{\nabla^2}+\nabla\frac{\nabla\cdot\vec{E}}{\nabla^2}$
$=-\frac{\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}}{\nabla^2}+\frac{\nabla(\nabla\cdot\vec{E})}{\nabla^2}=\vec{E}$ □

Reverse proof:

Proof: $\nabla\cdot\vec{E}=-\nabla\cdot\partial_t\vec{A}-\nabla\cdot\nabla\phi=-\partial_t(\nabla\cdot\vec{A})-\nabla^2\phi=-\partial_t\nabla^2\theta-\rho+\partial_t\nabla^2\theta=-\rho$ □

Proof: $\nabla\times\vec{E}=-\nabla\times\partial_t\vec{A}-\nabla\times\nabla\phi=-\partial_t\nabla\times\vec{A}-0=-\partial_t\vec{B}$ □

Proof: $\nabla\cdot\vec{B}=\nabla\cdot\nabla\times\vec{A}=0$ □

Proof: $\nabla\times\vec{B}-\partial_t\vec{E}$
$=\nabla\times\nabla\times\vec{A}+\partial_t^2\vec{A}+\partial_t\nabla\phi=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}+\partial_t^2\vec{A}+\partial_t\nabla\phi$
$=\nabla(\nabla^2\theta)-\nabla^2\vec{A}+\partial_t^2\vec{A}+\nabla\partial_t\phi=-\nabla^2\vec{A}+\partial_t^2\vec{A}+\nabla(\partial_t\phi+\nabla^2\theta)$
$=-\vec{J}$ □

Proof: $\frac{\nabla\times\vec{B}}{-\nabla^2}+\nabla\theta=\frac{\nabla\times\nabla\times\vec{A}}{-\nabla^2}+\nabla\theta=\frac{\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}}{-\nabla^2}+\nabla\theta=\vec{A}+\frac{\nabla(\nabla^2\theta)}{-\nabla^2}+\nabla\theta=\vec{A}$ □

Proof: $\frac{\nabla\cdot\vec{E}}{-\nabla^2}-\partial_t\theta=\frac{\nabla\cdot(\partial_t\vec{A}+\nabla\phi)}{\nabla^2}-\partial_t\theta=\frac{\partial_t(\nabla\cdot\vec{A})+\nabla^2\phi)}{\nabla^2}-\partial_t\theta=\phi+\frac{\partial_t\nabla^2\theta}{\nabla^2}-\partial_t\theta=\phi$ □

Proof is completed.

Cor. 1.2.1. $\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi=-i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b\\ \vec{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2}+\nabla\theta\\ \phi=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}-\partial_t\theta\end{cases}\Leftrightarrow\begin{cases}\nabla^2\phi=\rho-\partial_t\nabla^2\theta,\nabla\cdot\vec{A}=\nabla^2\theta\\ \nabla^2\vec{A}-\partial_t^2\vec{A}=\vec{J}+\nabla(\partial_t\phi+\nabla^2\theta)\\ \Psi=-\partial_t\vec{A}-\nabla\phi-i\varsigma\nabla\times\vec{A}\end{cases}$

Cor. 1.2.2. $\vec{A}=\tilde{A}-\nabla\theta,\phi=\tilde{\phi}+\partial_t\theta,\tilde{A}:=\frac{\nabla\times\vec{B}}{-\nabla^2},\tilde{\phi}:=\frac{\nabla\cdot\vec{E}}{-\nabla^2}$

When $\theta=0$, it is the radiation gauge; When $\theta=\frac{\partial_t\phi}{-\nabla^2}$, It is the Lorentz gauge.

1.3 Radiation gauge potential description of electromagnetic field equation($\theta=0$)

Thm. 1.3.1. $\begin{cases}\nabla\cdot\vec{E}=-\rho,\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\nabla\times\vec{B}=-\vec{J}+\partial_t\vec{E}\\ \tilde{A}=\frac{\nabla\times\vec{B}}{-\nabla^2},\tilde{\phi}=\frac{\nabla\cdot\vec{E}}{-\nabla^2}\end{cases}\Leftrightarrow\begin{cases}\nabla^2\tilde{A}-\partial_t^2\tilde{A}=\vec{J}+\partial_t\nabla\tilde{\phi}\\ \nabla^2\tilde{\phi}=\rho,\nabla\cdot\tilde{A}=0\\ \vec{E}=-\partial_t\tilde{A}-\nabla\tilde{\phi},\vec{B}=\nabla\times\tilde{A}\end{cases}$

Cor. 1.3.1. $\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi=-i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b\\ \tilde{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2},\tilde{\phi}=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}\end{cases}\Leftrightarrow\begin{cases}\nabla^2\tilde{A}-\partial_t^2\tilde{A}=\vec{J}+\partial_t\nabla\tilde{\phi}\\ \nabla^2\tilde{\phi}=\rho,\nabla\cdot\tilde{A}=0\\ \sqrt{2}\Psi=-\partial_t\tilde{A}-\nabla\tilde{\phi}-i\varsigma\nabla\times\tilde{A}\end{cases}$

Cor. 1.3.2.

$$[\tilde{A}_i(x),\tilde{A}_j(x')]=i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\Rightarrow\begin{cases}[\tilde{A}_i(x),\partial_{t'}\tilde{A}_j(x')]=-i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\partial_t\Delta(x-x')\\ [\partial_t\tilde{A}_i(x),\tilde{A}_j(x')]=i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\partial_t\Delta(x-x')\\ [\tilde{A}_i(x),(\nabla'\times\tilde{A})_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\Delta(x-x')\\ [(\nabla\times\tilde{A})_i(x),\tilde{A}_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\Delta(x-x')\end{cases}$$

Cor. 1.3.3.

$$[\tilde{A}_i(x),\tilde{A}_j(x')]=i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\Rightarrow\begin{cases}[\partial_t\tilde{A}_i(x),\partial_{t'}\tilde{A}_j(x')]=-i(\delta_{ij}\nabla^2-\partial_i\partial_j)\Delta(x-x')\\ [(\nabla\times\tilde{A})_i(x),(\nabla'\times\tilde{A})_j(x')]=-i(\delta_{ij}\nabla^2-\partial_i\partial_j)\Delta(x-x')\\ [\partial_t\tilde{A}_i(x),(\nabla'\times\tilde{A})_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\\ [(\nabla\times\tilde{A})_i(x),\partial_{t'}\tilde{A}_j(x')]=i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\end{cases}$$

Thm. 1.3.2.

$$\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0,[\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi=-i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b\\ \tilde{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2},\tilde{\phi}=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}\end{cases}\Leftrightarrow\begin{cases}[\tilde{A}_i(x),\tilde{A}_j(x')]=i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\\ [\tilde{A}_i(x),\tilde{\phi}(x')]=0,[\tilde{\phi}(x),\tilde{\phi}(x')]=0\\ \nabla^2\tilde{A}-\partial_t^2\tilde{A}=\vec{J}+\partial_t\nabla\tilde{\phi},\nabla^2\tilde{\phi}=\rho,\nabla\cdot\tilde{A}=0\\ \sqrt{2}\Psi=-\partial_t\tilde{A}-\nabla\tilde{\phi}-i\varsigma\nabla\times\tilde{A}\end{cases}$$

Original detailed proof:(Just one.)

Proof: $[\tilde{A}_i(x), \tilde{A}_j(x')]$
$= \frac{i\varsigma}{\sqrt{2}}\frac{1}{\nabla^2}\frac{i\varsigma}{\sqrt{2}}\frac{1}{\nabla'^2}[\varepsilon_i{}^{kl}\partial_k[\Psi_l(x) - \Psi_l^+(x)], \varepsilon_j{}^{mn}\partial'_m[\Psi_n(x') - \Psi_n^+(x')]]$
$= \frac{-1}{2}\frac{1}{\nabla^2\nabla'^2}\varepsilon_i{}^{kl}\varepsilon_j{}^{mn}\partial_k\partial'_m[\Psi_l(x) - \Psi_l^+(x), \Psi_n(x') - \Psi_n^+(x')]$
$= \frac{1}{2}\frac{1}{\nabla^2\nabla'^2}\varepsilon_i{}^{kl}\varepsilon_j{}^{mn}\partial_k\partial'_m\{[\Psi_l(x), \Psi_n^+(x')] + [\Psi_l^+(x), \Psi_n(x')]\}$
$= \frac{1}{2}\frac{1}{\nabla^2\nabla'^2}\varepsilon_i{}^{kl}\varepsilon_j{}^{mn}\partial_k\partial'_m[i\sigma_{ln}^{ab}\partial_a\partial_b\Delta(x - x') - i\sigma_{nl}^{ab}\partial'_a\partial'_b\Delta(x' - x)]$
$= -\frac{1}{2}\frac{1}{\nabla^4}\varepsilon_i{}^{kl}\varepsilon_j{}^{mn}\partial_k\partial_m[i\sigma_{ln}^{ab}\partial_a\partial_b + i\sigma_{nl}^{ab}\partial_a\partial_b]\Delta(x - x')$
$= \frac{i}{2}\frac{1}{\nabla^4}\varepsilon_i{}^{kl}\varepsilon_j{}^{mn}\partial_k\partial_m[(\nabla^2 - \partial_\pi^2)\delta_{ln} - 2\partial_l\partial_n]\Delta(x - x')$
$= \frac{i}{2}\frac{1}{\nabla^4}\varepsilon_i{}^{kl}\delta_{ln}\varepsilon_j{}^{mn}\partial_k\partial_m(\nabla^2 - \partial_\pi^2)\Delta(x - x')$
$= i\frac{1}{\nabla^4}\varepsilon_i{}^{kl}\delta_{ln}\varepsilon_j{}^{mn}\partial_k\partial_m\nabla^2\Delta(x - x')$
$= i\frac{1}{\nabla^2}\varepsilon_i{}^{kl}\delta_{ln}\varepsilon_j{}^{mn}\partial_k\partial_m\Delta(x - x')$
$= i\frac{1}{\nabla^2}(\delta_{ij}\delta_{km} - \delta_i^k\delta_j^m)\partial_k\partial_m\Delta(x - x')$
$= i\frac{1}{\nabla^2}(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x - x')$
$= i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x - x')$ □

Concise proof:

Proof: $[\tilde{A}_i(x), \tilde{A}_j(x')]$
$= [\frac{(\nabla\times\vec{B})_i}{-\nabla^2}(x), \frac{(\nabla'\times\vec{B})_j}{-\nabla'^2}(x')] = \frac{1}{\nabla^2\nabla'^2}[(\nabla\times\vec{B})_i(x), (\nabla'\times\vec{B})_j(x')]$
$= \frac{1}{\nabla^4}i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x - x') = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x - x')$ □

Proof: $[\tilde{A}_i(x), \tilde{\phi}(x')] = [\frac{(\nabla\times\vec{B})_i}{-\nabla^2}(x), \frac{\nabla'\cdot\vec{E}}{-\nabla'^2}(x')] = \frac{1}{\nabla^2\nabla'^2}[(\nabla\times\vec{B})_i(x), \nabla'\cdot\vec{E}(x')]$
$= \frac{1}{\nabla^4}[\varepsilon_i{}^{jk}\partial_j B_k(x), \nabla'\cdot\vec{E}(x')] = \frac{1}{\nabla^4}\varepsilon_i{}^{jk}\partial_j[B_k(x), \nabla'\cdot\vec{E}(x')] = 0$ □

Proof: $[\tilde{\phi}(x), \tilde{\phi}(x')] = [\frac{\nabla\cdot\vec{E}}{-\nabla^2}(x), \frac{\nabla'\cdot\vec{E}}{-\nabla'^2}(x')] = \frac{1}{\nabla^2\nabla'^2}[\nabla\cdot\vec{E}(x), \nabla'\cdot\vec{E}(x')] = 0$ □

Reverse Proof:

Proof: $[\Psi_i(x), \Psi_j^+(x')]$
$= \frac{1}{2}[-\partial_t\tilde{A}_i(x) - \partial_i\tilde{\phi}(x) - i\varsigma(\nabla\times\tilde{A})_i(x), -\partial_{t'}\tilde{A}_j(x') - \partial'_j\tilde{\phi}(x') + i\varsigma(\nabla'\times\tilde{A})_j(x')]$
$= \frac{1}{2}[-\partial_t\tilde{A}_i(x) - i\varsigma(\nabla\times\tilde{A})_i(x), -\partial_{t'}\tilde{A}_j(x') + i\varsigma(\nabla'\times\tilde{A})_j(x')]$
$= \frac{1}{2}\{[\partial_t\tilde{A}_i(x), \partial_{t'}\tilde{A}_j(x')] + [(\nabla\times\tilde{A})_i(x), (\nabla'\times\tilde{A})_j(x')] - i\varsigma[\partial_t\tilde{A}_i(x), (\nabla'\times\tilde{A})_j(x')] + i\varsigma[(\nabla\times\tilde{A})_i(x), \partial_{t'}\tilde{A}_j(x')]\}$
$= -i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x - x') - \varsigma\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x - x')$
$= i\sigma_{ij}^{ab}\partial_a\partial_b\Delta(x - x')$ □

Proof: $[\Psi_i(x), \Psi_j(x')]$
$= \frac{1}{2}[-\partial_t\tilde{A}_i(x) - \partial_i\tilde{\phi}(x) - i\varsigma(\nabla\times\tilde{A})_i(x), -\partial_{t'}\tilde{A}_j(x') - \partial'_j\tilde{\phi}(x') - i\varsigma(\nabla'\times\tilde{A})_j(x')]$
$= \frac{1}{2}[-\partial_t\tilde{A}_i(x) - i\varsigma(\nabla\times\tilde{A})_i(x), -\partial_{t'}\tilde{A}_j(x') - i\varsigma(\nabla'\times\tilde{A})_j(x')]$
$= \frac{1}{2}\{[\partial_t\tilde{A}_i(x), \partial_{t'}\tilde{A}_j(x')] - [(\nabla\times\tilde{A})_i(x), (\nabla'\times\tilde{A})_j(x')] + i\varsigma[\partial_t\tilde{A}_i(x), (\nabla'\times\tilde{A})_j(x')] + i\varsigma[(\nabla\times\tilde{A})_i(x), \partial_{t'}\tilde{A}_j(x')]\}$
$= 0$ □

Proof: $[\Psi_i^+(x), \Psi_j^+(x')]$
$= \frac{1}{2}[-\partial_t\tilde{A}_i(x) - \partial_i\tilde{\phi}(x) + i\varsigma(\nabla\times\tilde{A})_i(x), -\partial_{t'}\tilde{A}_j(x') - \partial'_j\tilde{\phi}(x') + i\varsigma(\nabla'\times\tilde{A})_j(x')]$
$= \frac{1}{2}[-\partial_t\tilde{A}_i(x) + i\varsigma(\nabla\times\tilde{A})_i(x), -\partial_{t'}\tilde{A}_j(x') + i\varsigma(\nabla'\times\tilde{A})_j(x')]$
$= \frac{1}{2}\{[\partial_t\tilde{A}_i(x), \partial_{t'}\tilde{A}_j(x')] - [(\nabla\times\tilde{A})_i(x), (\nabla'\times\tilde{A})_j(x')] - i\varsigma[\partial_t\tilde{A}_i(x), (\nabla'\times\tilde{A})_j(x')] - i\varsigma[(\nabla\times\tilde{A})_i(x), \partial_{t'}\tilde{A}_j(x')]\}$
$= 0$ □

Cor. 1.3.4.

$$\begin{cases}[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x - x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = 0, [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \sqrt{2}\Psi = -\partial_t\tilde{A} - \nabla\tilde{\phi} - i\varsigma\nabla\times\tilde{A}\end{cases} \Rightarrow \begin{cases}[\tilde{A}_i(x), E_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\partial_t\Delta(x - x') \\ [E_i(x), \tilde{A}_j(x')] = -i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\partial_t\Delta(x - x') \\ [\tilde{A}_i(x), B_j(x')] = -i\varepsilon_{ij}{}^k\partial_k\Delta(x - x') \\ [B_i(x), \tilde{A}_j(x')] = -i\varepsilon_{ij}{}^k\partial_k\Delta(x - x')\end{cases}$$

Proof: $[B_i(x), \tilde{A}_j(x')]$
$= [\varepsilon_i{}^{kl}\partial_k\tilde{A}_l(x), \tilde{A}_j(x')]$
$= i\varepsilon_i{}^{kl}\partial_k(\delta_{lj} - \frac{\partial_l\partial_j}{\nabla^2})\Delta(x - x')$
$= i\varepsilon_i{}^{kl}\partial_k\delta_{lj}\Delta(x - x')$
$= -i\varepsilon_{ij}{}^k\partial_k\Delta(x - x')$ □

Proof: $[E_i(x)-i\varsigma B_i(x),\tilde{A}_j(x')]$
$=[-i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\partial_t-\varsigma\varepsilon_{ij}{}^k\partial_k]\Delta(x-x')$
$=i\frac{\partial_t}{\nabla^2}(\partial_i\partial_j-\delta_{ij}\nabla^2+i\varsigma\varepsilon_{ij}{}^k\partial_k\partial_t)\Delta(x-x')$
$=i\sigma^{ab}_{ij}\partial_a\partial_b\frac{\partial_t}{\nabla^2}\Delta(x-x')$ □

Cor. 1.3.5. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b=\partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma}-\frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2+\partial_t^2)+i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_t$

Cor. 1.3.6.

$$\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0,[\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\\ \tilde{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2},\tilde{\phi}=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}\end{cases}\Rightarrow\begin{cases}[\Psi_i(x),\tilde{A}_j(x')]=\frac{i}{\sqrt{2}}\sigma^{ab}_{ij}\partial_a\partial_b\frac{\partial_t}{\nabla^2}\Delta(x-x')\\ [\tilde{A}_i(x),\Psi_j(x')]=-\frac{i}{\sqrt{2}}\sigma^{ab}_{ji}\partial_a\partial_b\frac{\partial_t}{\nabla^2}\Delta(x-x')\end{cases}$$

Cor. 1.3.7.

$$\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0,[\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi=-i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b\\ \tilde{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2},\tilde{\phi}=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}\end{cases}\Rightarrow\begin{cases}[\tilde{\phi}(x),\tilde{A}(x')]=0,[\tilde{\phi}(x),\tilde{\phi}(x')]=0\\ [\tilde{\phi}(x),\Psi(x')]=0,[\tilde{\phi}(x),\Psi^+(x')]=0\\ [J_a(x),\tilde{A}(x')]=0,[J_a(x),\tilde{\phi}(x')]=0\\ [J_a(x),\Psi(x')]=0,[J_a(x),\Psi^+(x')]=0\\ [J_a(x),J_b(x')]=0\end{cases}$$

It can be seen from the above that the electromagnetic field equation and the radiation gauge potential equation, constraints, and covariant commutation relations are compatible. And scalar potential $\tilde{\phi}(x)$ and source $J_a(x)$ is a c-number relative to the electromagnetic field, not an operator. In this sense we know that the scalar potential, that is, the electrostatic field cannot be quantized because it is not even an operator.

1.4 Analysis of commutative relations for general electromagnetic field strength

Thm. 1.4.1.

$$\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi(x)=-i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b(x)\\ \Psi(x)=\frac{1}{\sqrt{2}}[\vec{E}(x)-i\varsigma\vec{B}(x)]\end{cases}\Leftrightarrow\begin{cases}\nabla\cdot\vec{E}(x)=-\rho(x),\nabla\times\vec{E}(x)=-\partial_t\vec{B}(x)\\ \nabla\cdot\vec{B}(x)=0,\nabla\times\vec{B}(x)=-\vec{J}(x)+\partial_t\vec{E}(x)\end{cases}$$

Thm. 1.4.2.

$$\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0,[\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\\ \Psi(x)=\frac{1}{\sqrt{2}}[\vec{E}(x)-i\varsigma\vec{B}(x)]\end{cases}\Leftrightarrow\begin{cases}[E_i(x),E_j(x')]=-i(\delta_{ij}\nabla^2-\partial_i\partial_j)\Delta(x-x')\\ [B_i(x),B_j(x')]=-i(\delta_{ij}\nabla^2-\partial_i\partial_j)\Delta(x-x')\\ [E_i(x),B_j(x')]=i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\\ [B_i(x),E_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\end{cases}$$

Cor. 1.4.1. $$\begin{cases}[\nabla\cdot\vec{E}(x),\vec{E}(x')]=0\\ [\nabla\cdot\vec{B}(x),\vec{B}(x')]=0\\ [\nabla\cdot\vec{E}(x),\vec{B}(x')]=0\\ [\nabla\cdot\vec{B}(x),\vec{E}(x')]=0\end{cases}\quad\begin{cases}[\nabla\cdot\vec{E}(x),\nabla'\cdot\vec{E}(x')]=0\\ [\nabla\cdot\vec{B}(x),\nabla'\cdot\vec{B}(x')]=0\\ [\nabla\cdot\vec{E}(x),\nabla'\cdot\vec{B}(x')]=0\\ [\nabla\cdot\vec{B}(x),\nabla'\cdot\vec{E}(x')]=0\end{cases}$$

Cor. 1.4.2. $$\begin{cases}[\partial_tE_i(x)-(\nabla\times\vec{B})_i(x),\vec{E}(x')]=0\\ [\partial_tE_i(x)-(\nabla\times\vec{B})_i(x),\vec{B}(x')]=0\\ [\partial_tB_i(x)+(\nabla\times\vec{E})_i(x),\vec{E}(x')]=0\\ [\partial_tB_i(x)+(\nabla\times\vec{E})_i(x),\vec{B}(x')]=0\end{cases}$$

Cor. 1.4.3. $$\begin{cases}[J_i(x),\vec{E}(x')]=0,[J_i(x),\vec{B}(x')]=0\\ [\rho(x),\vec{E}(x')]=0,[\rho(x),\vec{B}(x')]=0\\ [J_a(x),\vec{E}(x')]=0,[J_a(x),\vec{B}(x')]=0\\ [J_a(x),J_b(x')]=0\end{cases}$$

Cor. 1.4.4. $$\begin{cases}[(\nabla\times\vec{E})_i(x),(\nabla'\times\vec{E})_j(x')]=i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\\ [(\nabla\times\vec{B})_i(x),(\nabla'\times\vec{B})_j(x')]=i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\\ [(\nabla\times\vec{E})_i(x),(\nabla'\times\vec{B})_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x')\\ [(\nabla\times\vec{B})_i(x),(\nabla'\times\vec{E})_j(x')]=i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x')\end{cases}$$

Cor. 1.4.5. $$\begin{cases}[\partial_tE_i(x),\partial_{t'}E_j(x')]=i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\\ [\partial_tB_i(x),\partial_{t'}B_j(x')]=i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\\ [\partial_tE_i(x),\partial_{t'}B_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x')\\ [\partial_tB_i(x),\partial_{t'}E_j(x')]=i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x')\end{cases}$$

Cor. 1.4.6. $\begin{cases}[\partial_t E_i(x),(\nabla'\times\vec{B})_j(x')]=i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\\ [(\nabla\times\vec{B})_i(x),\partial_{t'}E_j(x')]=i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\\ [\partial_t B_i(x),(\nabla'\times\vec{E})_j(x')]=-i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\\ [(\nabla\times\vec{E})_i(x),\partial_{t'}B_j(x')]=-i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x')\end{cases}$

Cor. 1.4.7. $\begin{cases}[\partial_t E_i(x),(\nabla'\times\vec{E})_j(x')]=i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\\ [(\nabla\times\vec{E})_i(x),\partial_{t'}E_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\\ [\partial_t B_i(x),(\nabla'\times\vec{B})_j(x')]=i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\\ [(\nabla\times\vec{B})_i(x),\partial_{t'}B_j(x')]=-i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\end{cases}$

It can be seen from the above that the general electromagnetic field equations and constraints are compatible with the covariant commutative relationship. And source $J_a(x)$ is a c-number relative to the electromagnetic field.

1.5 Lorentz λ-gauge potential description of electromagnetic field equation($\theta=\frac{\partial_t\phi}{-\nabla^2}$)

Which have inherent contradictions.

Thm. 1.5.1. $\begin{cases}\nabla\cdot\vec{E}=-\rho,\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\nabla\times\vec{B}=-\vec{J}+\partial_t\vec{E}\\ \vec{A}=\frac{\nabla\times\vec{B}+\nabla\partial_t\phi}{-\nabla^2},\phi=\frac{\nabla\cdot\vec{E}-\partial_t^2\phi}{-\nabla^2}\end{cases}\Leftrightarrow\begin{cases}\nabla^2\vec{A}-\partial_t^2\vec{A}=\vec{J},\nabla^2\phi-\partial_t^2\phi=\rho\\ \nabla\cdot\vec{A}+\partial_t\phi=0\\ \vec{E}=-\partial_t\vec{A}-\nabla\phi,\vec{B}=\nabla\times\vec{A}\end{cases}$

Thm. 1.5.2. ??? $\begin{cases}\langle|\nabla\cdot\vec{E}=-\rho|\rangle,\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\langle|\nabla\times\vec{B}=-\vec{J}+\partial_t\vec{E}|\rangle\\ \vec{A}=\frac{\nabla\times\vec{B}+\nabla\partial_t\phi}{-\nabla^2},\phi=\frac{\nabla\cdot\vec{E}-\partial_t^2\phi}{-\nabla^2}\end{cases}$

$$\Leftrightarrow\begin{cases}\nabla\cdot\vec{E}=-\rho-\partial_t(\nabla\cdot\vec{A}+\partial_t\phi),\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\nabla\times\vec{B}=-\vec{J}+\partial_t\vec{E}+\nabla(\nabla\cdot\vec{A}+\partial_t\phi)\\ \langle|\vec{A}=\frac{\nabla\times\vec{B}+\nabla\partial_t\phi}{-\nabla^2}|\rangle,\phi=\frac{\nabla\cdot\vec{E}-\partial_t^2\phi}{-\nabla^2}\end{cases}\Leftrightarrow\begin{cases}\nabla^2\vec{A}-\partial_t^2\vec{A}=\vec{J},\nabla^2\phi-\partial_t^2\phi=\rho\\ \langle|\nabla\cdot\vec{A}+\partial_t\phi|\rangle=0\\ \vec{E}=-\partial_t\vec{A}-\nabla\phi,\vec{B}=\nabla\times\vec{A}\end{cases}$$

Cor. 1.5.1. $\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi=-i\sigma_{\varsigma ab}^{[\beta_\varsigma]}J^b\\ \vec{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2}-\frac{\nabla\partial_t}{\nabla^2}\phi\\ \phi=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}+\frac{\partial_t^2}{\nabla^2}\phi\end{cases}\Leftrightarrow\begin{cases}\nabla^2\vec{A}-\partial_t^2\vec{A}=\vec{J},\nabla^2\phi-\partial_t^2\phi=\rho\\ \nabla\cdot\vec{A}+\partial_t\phi=0\\ \sqrt{2}\Psi=-\partial_t\vec{A}-\nabla\phi-i\varsigma\nabla\times\vec{A}\end{cases}$

Cor. 1.5.2.

$$\begin{cases}[A_a(x),A_b(x')]=i(\delta_{ab}-\frac{\lambda-1}{\lambda}\frac{\partial_a\partial_b}{\Box+i\varepsilon})\Delta(x-x')\\ \phi=-iA_0,\sqrt{2}\Psi=-\partial_t\vec{A}-\nabla\phi-i\varsigma\nabla\times\vec{A}\end{cases}\Rightarrow\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0,[\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\\ [\Psi_i(x),\phi(x')]=[\Psi^+_i(x),\phi(x')]=\frac{i}{\sqrt{2}}\partial_i\Delta(x-x')\\ [\phi(x),\phi(x')]=-i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')\end{cases}$$

Proof: $[\Psi_i(x),\Psi_j(x')]$

$=\frac{1}{2}\{[-\partial_tA_i(x)-\partial_i\phi(x)-i\varsigma(\nabla\times\vec{A})_i(x),-\partial_{t'}A_j(x')-\partial'_j\phi(x')-i\varsigma(\nabla'\times\vec{A})_j(x')]$

$=\frac{1}{2}[-\partial_tA_i(x)-i\varsigma(\nabla\times\vec{A})_i(x),-\partial_{t'}A_j(x')-i\varsigma(\nabla'\times\vec{A})_j(x')]$

$+[\partial_i\phi(x),\partial'_j\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_tA_i(x),\partial'_j\phi(x')]\}$

$=\frac{1}{2}\{[\partial_tA_i(x),\partial_{t'}A_j(x')]-[(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]+i\varsigma[\partial_tA_i(x),(\nabla'\times\vec{A})_j(x')]+i\varsigma[(\nabla\times\vec{A})_i(x),\partial_{t'}A_j(x')]$

$+[\partial_i\phi(x),\partial'_j\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_tA_i(x),\partial'_j\phi(x')]\}$

$=\frac{1}{2}\{[\partial_tA_i(x),\partial_{t'}A_j(x')]-[(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]+[\partial_i\phi(x),\partial'_j\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_tA_i(x),\partial'_j\phi(x')]\}$

$=\frac{1}{2}\{i\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j\nabla^2}{\Box+i\varepsilon}-i\partial_i\partial_j+i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\partial_i\partial_j-i\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j\nabla^2}{\Box+i\varepsilon}-i\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j\nabla^2}{\Box+i\varepsilon}\}\Delta(x-x')$

$=0$ □

Proof: $[\Psi^+_i(x),\Psi^+_j(x')]$

$=\frac{1}{2}\{[-\partial_tA_i(x)-\partial_i\phi(x)+i\varsigma(\nabla\times\vec{A})_i(x),-\partial_{t'}A_j(x')-\partial'_j\phi(x')+i\varsigma(\nabla'\times\vec{A})_j(x')]$

$=\frac{1}{2}[-\partial_tA_i(x)+i\varsigma(\nabla\times\vec{A})_i(x),-\partial_{t'}A_j(x')+i\varsigma(\nabla'\times\vec{A})_j(x')]$

$+[\partial_i\phi(x),\partial'_j\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_tA_i(x),\partial'_j\phi(x')]\}$

$=\frac{1}{2}\{[\partial_tA_i(x),\partial_{t'}A_j(x')]-[(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]-i\varsigma[\partial_tA_i(x),(\nabla'\times\vec{A})_j(x')]-i\varsigma[(\nabla\times\vec{A})_i(x),\partial_{t'}A_j(x')]$

$+[\partial_i\phi(x),\partial'_j\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_tA_i(x),\partial'_j\phi(x')]\}$

$=\frac{1}{2}\{[\partial_tA_i(x),\partial_{t'}A_j(x')]-[(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]+[\partial_i\phi(x),\partial'_j\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_tA_i(x),\partial'_j\phi(x')]\}$

$=\frac{1}{2}\{i\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j\nabla^2}{\Box+i\varepsilon}-i\partial_i\partial_j+i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\partial_i\partial_j-i\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j\nabla^2}{\Box+i\varepsilon}-i\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j\nabla^2}{\Box+i\varepsilon}\}\Delta(x-x')$

$=0$ □

Proof: $[\Psi_i(x),\Psi_j^+(x')]$
$= \frac{1}{2}\{[-\partial_t A_i(x)-\partial_i\phi(x)-i\varsigma(\nabla\times\vec{A})_i(x), -\partial_{t'}A_j(x')-\partial_j'\phi(x')+i\varsigma(\nabla'\times\vec{A})_j(x')]$
$= \frac{1}{2}[-\partial_t A_i(x)-i\varsigma(\nabla\times\vec{A})_i(x), -\partial_{t'}A_j(x')+i\varsigma(\nabla'\times\vec{A})_j(x')]$
$+ [\partial_i\phi(x),\partial_j'\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_t A_i(x),\partial_j'\phi(x')]\}$
$= \frac{1}{2}\{[\partial_t A_i(x),\partial_{t'}A_j(x')]+[(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]-i\varsigma[\partial_t A_i(x),(\nabla'\times\vec{A})_j(x')]+i\varsigma[(\nabla\times\vec{A})_i(x),\partial_{t'}A_j(x')]$
$+ [\partial_i\phi(x),\partial_j'\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_t A_i(x),\partial_j'\phi(x')]\}$
$= [(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]-i\varsigma[\partial_t A_i(x),(\nabla'\times\vec{A})_j(x')]$
$+ \frac{1}{2}\{[\partial_t A_i(x),\partial_{t'}A_j(x')]-[(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]+i\varsigma[\partial_t A_i(x),(\nabla'\times\vec{A})_j(x')]+i\varsigma[(\nabla\times\vec{A})_i(x),\partial_{t'}A_j(x')]$
$+ [\partial_i\phi(x),\partial_j'\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_t A_i(x),\partial_j'\phi(x')]\}$
$= [(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]-i\varsigma[\partial_t A_i(x),(\nabla'\times\vec{A})_j(x')]$
$+ \frac{1}{2}\{[\partial_t A_i(x),\partial_{t'}A_j(x')]-[(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]+[\partial_i\phi(x),\partial_j'\phi(x')]+[\partial_i\phi(x),\partial_{t'}A_j(x')]+[\partial_t A_i(x),\partial_j'\phi(x')]\}$
$= [(\nabla\times\vec{A})_i(x),(\nabla'\times\vec{A})_j(x')]-i\varsigma[\partial_t A_i(x),(\nabla'\times\vec{A})_j(x')]$
$= -i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\nabla^2\Delta(x-x')-\varsigma\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')$
$= i\sigma_{ij}^{ab}\partial_a\partial_b\Delta(x-x')$ □

Proof: $[\phi(x),\phi(x')] = -[A_0(x),A_0(x')] = -i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$ □

Proof: $[\Psi_i(x),\phi(x')]$
$= \frac{1}{\sqrt{2}}[-\partial_t A_i(x)-\partial_i\phi(x)-i\varsigma(\nabla\times\vec{A})_i(x),\phi(x')]$
$= -\frac{1}{\sqrt{2}}[\partial_t A_i(x)+\partial_i\phi(x),\phi(x')]$
$= -\frac{1}{\sqrt{2}}[i\partial_t\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_t}{\Box+i\varepsilon}\Delta(x-x')-i\partial_i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')]$
$= -\frac{i}{\sqrt{2}}[\frac{\lambda-1}{\lambda}\frac{\partial_i\nabla^2}{\Box+i\varepsilon}-\partial_i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})]\Delta(x-x')$
$= \frac{i}{\sqrt{2}}\partial_i\Delta(x-x')$ □

Proof: $[\Psi_i^+(x),\phi(x')]$
$= \frac{1}{\sqrt{2}}[-\partial_t A_i(x)-\partial_i\phi(x)+i\varsigma(\nabla\times\vec{A})_i(x),\phi(x')]$
$= -\frac{1}{\sqrt{2}}[\partial_t A_i(x)+\partial_i\phi(x),\phi(x')]$
$= -\frac{1}{\sqrt{2}}[i\partial_t\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_t}{\Box+i\varepsilon}\Delta(x-x')-i\partial_i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')]$
$= -\frac{i}{\sqrt{2}}[\frac{\lambda-1}{\lambda}\frac{\partial_i\nabla^2}{\Box+i\varepsilon}-\partial_i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})]\Delta(x-x')$
$= \frac{i}{\sqrt{2}}\partial_i\Delta(x-x')$ □

Cor. 1.5.3.

$$\begin{cases}[A_a(x),A_b(x')]=i(\delta_{ab}-\frac{\lambda-1}{\lambda}\frac{\partial_a\partial_b}{\Box+i\varepsilon})\Delta(x-x')\\ \nabla^2\vec{A}-\partial_t^2\vec{A}=\vec{J},\nabla^2\phi-\partial_t^2\phi=\rho\\ ?\nabla\cdot\vec{A}+\partial_t\phi=0?,\phi=-iA_0\\ \sqrt{2}\Psi=-\partial_t\vec{A}-\nabla\phi-i\varsigma\nabla\times\vec{A}\end{cases} \Rightarrow \begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0,[\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\\ [\Psi_i(x),\phi(x')]=[\Psi_i^+(x),\phi(x')]=\frac{i}{\sqrt{2}}\partial_i\Delta(x-x')\\ [\phi(x),\phi(x')]=-i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi=-i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b,A_0(x)=i\phi(x)\\ \vec{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2}-\frac{\nabla\partial_t}{\nabla^2}\phi,\phi=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}+\frac{\partial_t^2}{\nabla^2}\phi\end{cases}$$

Cor. 1.5.4.

$$\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0,[\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\\ [\Psi_i(x),\phi(x')]=[\Psi_i^+(x),\phi(x')]=\frac{i}{\sqrt{2}}\partial_i\Delta(x-x')\\ [\phi(x),\phi(x')]=-i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')\\ \vec{A}=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2}-\frac{\nabla\partial_t}{\nabla^2}\phi,\phi=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}+\frac{\partial_t^2}{\nabla^2}\phi\end{cases} !\Rightarrow \begin{cases}[A_a(x),A_b(x')]=i(\delta_{ab}-\frac{\lambda-1}{\lambda}\frac{\partial_a\partial_b}{\Box+i\varepsilon})\Delta(x-x')\\ \phi=-iA_0\end{cases}$$

Proof: $[A_i(x),A_j(x')]$
$= [\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla\times[\Psi(x)-\Psi^*(x)])_i}{\nabla^2}-\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla'\times[\Psi(x')-\Psi^*(x')])_j}{\nabla'^2}-\frac{\partial_j'\partial_{t'}}{\nabla'^2}\phi(x')]$
$= [\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla\times[\Psi(x)-\Psi^*(x)])_i}{\nabla^2},\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla'\times[\Psi(x')-\Psi^*(x')])_j}{\nabla'^2}]+[\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\frac{\partial_j'\partial_{t'}}{\nabla'^2}\phi(x')]$
$+ [\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla\times[\Psi(x)-\Psi^*(x)])_i}{\nabla^2},-\frac{\partial_j'\partial_{t'}}{\nabla'^2}\phi(x')]+[-\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla'\times[\Psi(x')-\Psi^*(x')])_j}{\nabla'^2}]$
$= [\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla\times[\Psi(x)-\Psi^*(x)])_i}{\nabla^2},\frac{-i\varsigma}{\sqrt{2}}\frac{(\nabla'\times[\Psi(x')-\Psi^*(x')])_j}{\nabla'^2}]+[\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\frac{\partial_j'\partial_{t'}}{\nabla'^2}\phi(x')]$
$= [\frac{(\nabla\times\vec{B})_i}{-\nabla^2},\frac{(\nabla'\times\vec{B})_j}{-\nabla'^2}]+[\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\frac{\partial_j'\partial_{t'}}{\nabla'^2}\phi(x')]$

$= \frac{1}{\nabla^2\nabla'^2}[(\nabla\times\vec{B})_i,(\nabla'\times\vec{B})_j] + \frac{\partial_i\partial_t}{\nabla^2}\frac{\partial'_j\partial_{t'}}{\nabla'^2}[\phi(x),\phi(x')]$
$= \frac{1}{\nabla^2\nabla'^2}i(\delta_{ij}\nabla^2-\partial_i\partial_j)\nabla^2\Delta(x-x') - i\frac{\partial_i\partial_t}{\nabla^2}\frac{\partial'_j\partial_{t'}}{\nabla'^2}(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= \frac{1}{\nabla^2}i(\delta_{ij}\nabla^2-\partial_i\partial_j)\Delta(x-x') - i\frac{\partial_i\partial_j}{\nabla^2}(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= \frac{1}{\nabla^2}i\delta_{ij}\nabla^2\Delta(x-x') - i\frac{\partial_i\partial_j}{\nabla^2}(2+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= i(\delta_{ij}-\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j}{\Box+i\varepsilon})\Delta(x-x') - 2i\frac{\partial_i\partial_j}{\nabla^2}\Delta(x-x')$ □

Proof: $[A_i(x),\phi(x')]$
$= [\frac{i\varsigma}{\sqrt{2}}\frac{(\nabla\times[\Psi(x)-\Psi^*(x)])_i}{\nabla^2} - \frac{\partial_i\partial_t}{\nabla^2}\phi(x),\phi(x')]$
$= -[\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\phi(x')]$
$= i\frac{\partial_i\partial_t}{\nabla^2}(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= i\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_t}{\Box+i\varepsilon})\Delta(x-x') + i\frac{\partial_i\partial_j}{\nabla^2}\Delta(x-x')$ □

Cor. 1.5.5. $[\tilde{A}_i(x),\phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x') \Leftrightarrow \begin{cases}[\Psi_i(x),\phi(x')] = \frac{i}{\sqrt{2}}\partial_i\Delta(x-x')\\ [\Psi_i^+(x),\phi(x')] = \frac{i}{\sqrt{2}}\partial_i\Delta(x-x')\end{cases}$

It can be seen from the above that the constraint conditions of the electromagnetic field equation are incompatible with the covariant commutative relations. How to reasonably reselect the commutative relations of the additional introduced ϕ to solve this problem. Although traditionally, constraints are not considered as operator equations, rather as a selection of physical states. But this is not natural. Therefore it is necessary to seek a more reasonable potential covariant scheme.

1.6 Equivalent conversion of two descriptions for Lorentz and radiation gauge potential

1.6.1 Equivalence of two gauge potential equations

Thm. 1.6.1.

$$\begin{cases}\nabla^2\tilde{A}-\partial_t^2\tilde{A} = \vec{J}+\partial_t\nabla\tilde{\phi}, \nabla^2\tilde{\phi} = \rho\\ \vec{A} = \tilde{A}-\nabla\frac{\partial_t}{\nabla^2}\phi, \phi = \tilde{\phi}+\partial_t\frac{\partial_t}{\nabla^2}\phi\\ \vec{E} = -\partial_t\tilde{A}-\nabla\tilde{\phi}, \vec{B} = \nabla\times\tilde{A}\end{cases} \Leftrightarrow \begin{cases}\nabla^2\vec{A}-\partial_t^2\vec{A} = \vec{J}, \nabla^2\phi-\partial_t^2\phi = \rho\\ \tilde{A} = \vec{A}+\nabla\frac{\partial_t}{\nabla^2}\phi, \tilde{\phi} = \phi-\partial_t\frac{\partial_t}{\nabla^2}\phi\\ \vec{E} = -\partial_t\vec{A}-\nabla\phi, \vec{B} = \nabla\times\vec{A}\end{cases}$$

Thm. 1.6.2.

$$\begin{cases}\nabla^2\tilde{A}-\partial_t^2\tilde{A} = \vec{J}+\partial_t\nabla\tilde{\phi}, \nabla^2\tilde{\phi} = \rho\\ \nabla\cdot\tilde{A} = 0\\ \vec{A} = \tilde{A}-\nabla\frac{\partial_t}{\nabla^2}\phi, \phi = \tilde{\phi}+\partial_t\frac{\partial_t}{\nabla^2}\phi\\ \vec{E} = -\partial_t\tilde{A}-\nabla\tilde{\phi}, \vec{B} = \nabla\times\tilde{A}\end{cases} \Leftrightarrow \begin{cases}\nabla^2\vec{A}-\partial_t^2\vec{A} = \vec{J}, \nabla^2\phi-\partial_t^2\phi = \rho\\ \nabla\cdot\vec{A}+\partial_t\phi = 0\\ \tilde{A} = \vec{A}+\nabla\frac{\partial_t}{\nabla^2}\phi, \tilde{\phi} = \phi-\partial_t\frac{\partial_t}{\nabla^2}\phi\\ \vec{E} = -\partial_t\vec{A}-\nabla\phi, \vec{B} = \nabla\times\vec{A}\end{cases}$$

Thm. 1.6.3.

$$\begin{cases}\nabla^2\tilde{A}-\partial_t^2\tilde{A} = \vec{J}+\partial_t\nabla\tilde{\phi}, \nabla^2\tilde{\phi} = \rho\\ \langle|\nabla\cdot\tilde{A}|\rangle = 0\\ \vec{A} = \tilde{A}-\nabla\frac{\partial_t}{\nabla^2}\phi, \phi = \tilde{\phi}+\partial_t\frac{\partial_t}{\nabla^2}\phi\\ \vec{E} = -\partial_t\tilde{A}-\nabla\tilde{\phi}, \vec{B} = \nabla\times\tilde{A}\end{cases} \Leftrightarrow \begin{cases}\nabla^2\vec{A}-\partial_t^2\vec{A} = \vec{J}, \nabla^2\phi-\partial_t^2\phi = \rho\\ \langle|\nabla\cdot\vec{A}+\partial_t\phi|\rangle = 0\\ \tilde{A} = \vec{A}+\nabla\frac{\partial_t}{\nabla^2}\phi, \tilde{\phi} = \phi-\partial_t\frac{\partial_t}{\nabla^2}\phi\\ \vec{E} = -\partial_t\vec{A}-\nabla\phi, \vec{B} = \nabla\times\vec{A}\end{cases}$$

1.6.2 Equivalence of commutative relations for two gauge potentials

Thm. 1.6.4.

$$\begin{cases}\begin{cases}[\tilde{A}_i(x),\tilde{A}_j(x')] = i(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\\ [\tilde{A}_i(x),\tilde{\phi}(x')] = [\tilde{\phi}(x),\tilde{\phi}(x')] = 0\\ \phi(x) = \tilde{\phi}(x)+\partial_t\frac{\partial_t}{\nabla^2}\phi(x)\\ A_i(x) = \tilde{A}_i(x)-\frac{\partial_i\partial_t}{\nabla^2}\phi(x), A_0(x) = i\phi(x)\end{cases}\\ \begin{cases}[\phi(x),\phi(x')] = -i(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')\\ [\tilde{A}_i(x),\phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x'), [\tilde{\phi}(x),\phi(x')] = 0\end{cases}\end{cases} \Leftrightarrow \begin{cases}[A_a(x),A_b(x')] = i(\delta_{ab}-\frac{\lambda-1}{\lambda}\frac{\partial_a\partial_b}{\Box+i\varepsilon})\Delta(x-x')\\ \tilde{A}_i(x) = A_i(x)+\frac{\partial_i\partial_t}{\nabla^2}\phi(x)\\ \tilde{\phi}(x) = \phi(x)-\partial_t\frac{\partial_t}{\nabla^2}\phi(x)\\ \phi(x) = -iA_0(x)\end{cases}$$

Proof: $[A_i(x),A_j(x')]$
$= [\tilde{A}_i(x),\tilde{A}_j(x')] + [\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\frac{\partial'_j\partial'_t}{\nabla'^2}\phi(x')] - [\tilde{A}_i(x),\frac{\partial'_j\partial'_t}{\nabla'^2}\phi(x')] - [\frac{\partial_i\partial_t}{\nabla^2}\phi(x),\tilde{A}_j(x')]$
$= [\tilde{A}_i(x),\tilde{A}_j(x')] + \frac{\partial_i\partial_t}{\nabla^2}\frac{\partial'_j\partial'_t}{\nabla'^2}[\phi(x),\phi(x')] - \frac{\partial'_j\partial'_t}{\nabla'^2}[\tilde{A}_i(x),\phi(x')] - \frac{\partial_i\partial_t}{\nabla^2}[\phi(x),\tilde{A}_j(x')]$
$= [\tilde{A}_i(x),\tilde{A}_j(x')] - i\frac{\partial_i\partial_t}{\nabla^2}\frac{\partial'_j\partial'_t}{\nabla'^2}(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x') + i\frac{\partial'_j\partial'_t}{\nabla'^2}\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x') - i\frac{\partial_i\partial_t}{\nabla^2}\frac{\partial'_j\partial'_t}{\nabla'^2}\Delta(x'-x)$
$= [\tilde{A}_i(x),\tilde{A}_j(x')] + i\frac{\partial_i\partial_t}{\nabla^2}\frac{\partial_j\partial_t}{\nabla^2}(1-\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= [\tilde{A}_i(x),\tilde{A}_j(x')] + i\frac{\partial_i\partial_j}{\nabla^2}(1-\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= i(\delta_{ij}-\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_j}{\Box+i\varepsilon})\Delta(x-x')$ □

Proof: $[A_0(x), A_0(x')] = -[\phi(x), \phi(x')] = i(1 + \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$ □

Proof: $[A_i(x), A_0(x')] = i[A_i(x), \phi(x')] = i[\tilde{A}_i(x), \phi(x')] + i[-\frac{\partial_i\partial_t}{\nabla^2}\phi(x), \phi(x')]$
$= \frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x') - \frac{\partial_i\partial_t}{\nabla^2}(1 + \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= -\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_t}{\Box+i\varepsilon}\Delta(x-x') = i(\delta_{i\pi} - \frac{\lambda-1}{\lambda}\frac{\partial_i\partial_\pi}{\Box+i\varepsilon})\Delta(x-x')$ □

Reverse proof:

Proof: $[\phi(x), \phi(x')] = -[A_0(x), A_0(x')] = -i(1 + \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$ □

Proof: $[\tilde{A}_i(x), \phi(x')] = [A_i(x), \phi(x')] + [\frac{\partial_i\partial_t}{\nabla^2}\phi(x), \phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x')$ □

Proof: $[\phi(x), \tilde{A}_i(x')] = [\phi(x), A_i(x')] + [\phi(x), \frac{\partial_i'\partial_{t'}}{\nabla^2}\phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x')$ □

Proof: $[\tilde{A}_i(x), \tilde{A}_j(x')]$
$= [A_i(x), A_j(x')] + [\frac{\partial_i\partial_t}{\nabla^2}\phi(x), \frac{\partial_j'\partial_t'}{\nabla'^2}\phi(x')] + [A_i(x), \frac{\partial_j'\partial_t'}{\nabla'^2}\phi(x')] + [\frac{\partial_i\partial_t}{\nabla^2}\phi(x), A_j(x')]$
$= [A_i(x), A_j(x')] + \frac{\partial_i\partial_t}{\nabla^2}\frac{\partial_j'\partial_t'}{\nabla'^2}[\phi(x), \phi(x')] + \frac{\partial_j'\partial_t'}{\nabla'^2}[A_i(x), \phi(x')] + \frac{\partial_i\partial_t}{\nabla^2}[\phi(x), A_j(x')]$
$= [A_i(x), A_j(x')] - i\frac{\partial_i\partial_t}{\nabla^2}\frac{\partial_j'\partial_t'}{\nabla'^2}(1 + \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x') + i\frac{\partial_j'\partial_{t'}}{\nabla'^2}\frac{\lambda-1}{\lambda}\frac{\partial_i\partial_t}{\Box+i\varepsilon}\Delta(x-x') - i\frac{\partial_i\partial_t}{\nabla^2}\frac{\lambda-1}{\lambda}\frac{\partial_j'\partial_{t'}}{\Box'+i\varepsilon}\Delta(x'-x)$
$= [A_i(x), A_j(x')] - i\frac{\partial_i\partial_j}{\nabla^2}(1 - \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= [A_i(x), A_j(x')] - i\frac{\partial_i\partial_t}{\nabla^2}\frac{\partial_j\partial_t}{\nabla^2}(1 - \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= [A_i(x), A_j(x')] - i\frac{\partial_i\partial_j}{\nabla^2}(1 - \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')$
$= i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')$ □

1.6.3 Equivalence of two gauge potential equations and joint commutative relations

Cor. 1.6.1.

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\nabla^2\tilde{A} - \partial_t^2\tilde{A} = \vec{J} + \partial_t\nabla\tilde{\phi}, \nabla^2\tilde{\phi} = \rho\\ \vec{E} = -\partial_t\tilde{A} - \nabla\tilde{\phi}, \vec{B} = \nabla\times\tilde{A}\end{array}\right.\\ \left\{\begin{array}{l}[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\\ [\tilde{A}_i(x), \tilde{\phi}(x')] = [\tilde{\phi}(x), \tilde{\phi}(x')] = 0\\ \phi(x) = \tilde{\phi}(x) + \partial_t\frac{\partial_t}{\nabla^2}\phi(x)\\ A_i(x) = \tilde{A}_i(x) - \frac{\partial_i\partial_t}{\nabla^2}\phi(x), A_0(x) = i\phi(x)\end{array}\right.\\ \left\{\begin{array}{l}[\phi(x), \phi(x')] = -i(1 + \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')\\ [\tilde{A}_i(x), \phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x'), [\tilde{\phi}(x), \phi(x')] = 0\end{array}\right.\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\left\{\begin{array}{l}\nabla^2\vec{A} - \partial_t^2\vec{A} = \vec{J}, \nabla^2\phi - \partial_t^2\phi = \rho\\ \vec{E} = -\partial_t\vec{A} - \nabla\phi, \vec{B} = \nabla\times\vec{A}\end{array}\right.\\ \left\{\begin{array}{l}[A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\lambda-1}{\lambda}\frac{\partial_a\partial_b}{\Box+i\varepsilon})\Delta(x-x')\\ \tilde{A}_i(x) = A_i(x) + \frac{\partial_i\partial_t}{\nabla^2}\phi(x)\\ \tilde{\phi}(x) = \phi(x) - \partial_t\frac{\partial_t}{\nabla^2}\phi(x)\\ \phi(x) = -iA_0(x)\end{array}\right.\end{array}\right.$$

1.6.4 Ncompatibility between guage conditions and commutative relations

Cor. 1.6.2.

$$\left\{\begin{array}{l}\nabla\cdot\tilde{A}(x) = 0\\ [\tilde{A}_i(x), \phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x')\\ A_i(x) = \tilde{A}_i(x) - \frac{\partial_i\partial_t}{\nabla^2}\phi(x)\\ A_0(x) = i\phi(x)\end{array}\right. \quad incompatible. \Leftrightarrow \left\{\begin{array}{l}\partial^a A_a(x) = 0\\ [A_i(x) + \frac{\partial_i\partial_t}{\nabla^2}\phi(x), \phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x')\\ \tilde{A}_i(x) = A_i(x) + \frac{\partial_i\partial_t}{\nabla^2}\phi(x)\\ \phi(x) = -iA_0(x)\end{array}\right. \quad incompatible.$$

It can be seen from the above that the guage condition is incompatible with a commutative relation. Incompatibility essentially stems come from non physical introduction of ϕ.

1.6.5 Solution to incompatibility between guage conditions and commutative relations

Cor. 1.6.3.

$$\left\{\begin{array}{l}\left\{\begin{array}{l}\nabla^2\tilde{A} - \partial_t^2\tilde{A} = \vec{J} + \partial_t\nabla\tilde{\phi}, \nabla^2\tilde{\phi} = \rho\\ \vec{E} = -\partial_t\tilde{A} - \nabla\tilde{\phi}, \vec{B} = \nabla\times\tilde{A}\\ \langle|\nabla\cdot\tilde{A}|\rangle = 0\end{array}\right.\\ \left\{\begin{array}{l}[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\\ [\tilde{A}_i(x), \tilde{\phi}(x')] = [\tilde{\phi}(x), \tilde{\phi}(x')] = 0\\ \phi(x) = \tilde{\phi}(x) + \partial_t\frac{\partial_t}{\nabla^2}\phi(x)\\ A_i(x) = \tilde{A}_i(x) - \frac{\partial_i\partial_t}{\nabla^2}\phi(x), A_0(x) = i\phi(x)\end{array}\right.\\ \left\{\begin{array}{l}[\phi(x), \phi(x')] = -i(1 + \frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')\\ [\tilde{A}_i(x), \phi(x')] = -i\frac{\partial_i\partial_t}{\nabla^2}\Delta(x-x'), [\tilde{\phi}(x), \phi(x')] = 0\end{array}\right.\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\left\{\begin{array}{l}\nabla^2\vec{A} - \partial_t^2\vec{A} = \vec{J}, \nabla^2\phi - \partial_t^2\phi = \rho\\ \vec{E} = -\partial_t\vec{A} - \nabla\phi, \vec{B} = \nabla\times\vec{A}\\ \langle|\nabla\cdot\vec{A} + \partial_t\phi|\rangle = 0\end{array}\right.\\ \left\{\begin{array}{l}[A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\lambda-1}{\lambda}\frac{\partial_a\partial_b}{\Box+i\varepsilon})\Delta(x-x')\\ \tilde{A}_i(x) = A_i(x) + \frac{\partial_i\partial_t}{\nabla^2}\phi(x)\\ \tilde{\phi}(x) = \phi(x) - \partial_t\frac{\partial_t}{\nabla^2}\phi(x)\\ \phi(x) = -iA_0(x)\end{array}\right.\end{array}\right.$$

If the gauge condition is no longer viewed as an operator equation, but as a choice of physical states, the equation and the commutative relations will be completely compatible. And there will be no contradiction. Where $\lambda = 1$ is the Feynman guage, and $\lambda = \infty$ is the Landau guage.

2 Electromagnetic field equation under radiation guage

2.1 Radiation gauge potential description of electromagnetic field equation without source

Cor. 2.1.1.

$$\begin{cases}\nabla\cdot\vec{E}=0,\nabla\times\vec{E}=-\partial_t\vec{B}\\ \nabla\cdot\vec{B}=0,\nabla\times\vec{B}=\partial_t\vec{E}\\ \tilde{A}=\frac{\nabla\times\vec{B}}{-\nabla^2}=\frac{\partial_t\vec{E}}{-\nabla^2}\end{cases}\Leftrightarrow\begin{cases}\nabla^2\tilde{A}-\partial_t^2\tilde{A}=0,\nabla\cdot\tilde{A}=0\\ \vec{E}=-\partial_t\tilde{A},\vec{B}=\nabla\times\tilde{A}\end{cases}\Leftrightarrow\begin{cases}\partial^aF_{ab}=0,F_{ab}=\partial_a\tilde{A}_b-\partial_b\tilde{A}_a\\ \nabla\cdot\tilde{A}=0,\tilde{A}_0=0\end{cases}$$

Pro. 2.1.1. $\tilde{A}(\vec{r},t)=\frac{\partial_t}{-\nabla^2}\vec{E}(\vec{r},t)\Leftrightarrow\vec{E}(\vec{r},t)=-\partial_t\tilde{A}(\vec{r},t)$

Pro. 2.1.2. $\tilde{A}(\vec{r},t)=\frac{\nabla\times\vec{B}(\vec{r},t)}{-\nabla^2}\Leftrightarrow\vec{B}(\vec{r},t)=\nabla\times\tilde{A}(\vec{r},t)$

2.2 Lorentz transformation properties of radiation gauge potential

Def. 2.2.1. $\begin{cases}\nabla'=\nabla-\gamma_v\vec{v}\partial_t+(\gamma_v-1)\vec{v}/v^2(\vec{v}\cdot\nabla)\\ \partial_{t'}=\gamma_v(\partial_t-\vec{v}\cdot\nabla),\gamma_v\equiv(1-v^2)^{-\frac{1}{2}}\end{cases}$

Cor. 2.2.1. $\vec{E}'=\gamma_v(\vec{E}-\vec{v}\times\vec{B})-(\gamma_v-1)(\vec{v}\cdot\vec{E})\vec{v}/v^2,\vec{B}'=\gamma_v(\vec{B}+\vec{v}\times\vec{E})-(\gamma_v-1)(\vec{v}\cdot\vec{B})\vec{v}/v^2$

Cor. 2.2.2. $\tilde{A}'=\frac{\nabla'\times\vec{B}'}{-\nabla'^2}=-\frac{[\nabla-\gamma_v\vec{v}\partial_t+(\gamma_v-1)\vec{v}/v^2(\vec{v}\cdot\nabla)]\times[\gamma_v(\vec{B}+\vec{v}\times\vec{E})-(\gamma_v-1)(\vec{v}\cdot\vec{B})\vec{v}/v^2]}{[\nabla-\gamma_v\vec{v}\partial_t+(\gamma_v-1)\vec{v}/v^2(\vec{v}\cdot\nabla)]^2}=?$

2.3 Analysis and discussion on potential solution of electromagnetic field equation

Def. 2.3.1. $\partial^aF_{ab}=0,F_{ab}=\partial_aA_b-\partial_bA_a$

If you get a solution A_a, then $A_a+\partial_a\theta$ is also another solution. Due to the arbitrariness of θ, the electromagnetic field equation has infinite sets of potential solutions. But the infinite potential solutions only correspond to the same solution F_{ab}. If the gauge is fixed, it is equivalent to select a solution A_a from an infinite number of potential solutions. In this way, there is no redundant potential solution, and at this time, it can correspond to the field strength solution F_{ab} one by one. Considering the completeness of the solution, for the complete field strength solution F_{ab}, which can be completely obtained by an incomplete potential solution A_a. And this incomplete potential solution A_a can also be completely obtained by a complete field strength solution F_{ab}. At this time, the complete field strength solution F_{ab} and the incomplete potential solution A_a are one-to-one correspondent. At this point, the electromagnetic field spin equation is completely equivalent to the electromagnetic field potential equation with gauge condition.

2.4 Electromagnetic potential and field solutions along z-direction under radiation gauge

Cor. 2.4.1. $\partial^a\partial_a\tilde{A}=0,\nabla\cdot\tilde{A}=0$

$$\Rightarrow\tilde{A}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})=\frac{1}{(2\pi)^{3/2}}\frac{-i}{\sqrt{2|\vec{p}|}}\{\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}-a_2^+(\vec{p})e^{-ip\cdot x}]+\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}-a_1^+(\vec{p})e^{-ip\cdot x}]\}$$

Cor. 2.4.2.

$$\vec{E}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})=\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2|\vec{p}|}}\{|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}+a_2^+(\vec{p})e^{-ip\cdot x}]+|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}+a_1^+(\vec{p})e^{-ip\cdot x}]\}$$

Proof: $\vec{E}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})=-\partial_t\tilde{A}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})$

$$=\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2|\vec{p}|}}\{|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}+a_2^+(\vec{p})e^{-ip\cdot x}]+|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}+a_1^+(\vec{p})e^{-ip\cdot x}]\}\quad\square$$

Cor. 2.4.3.

$$\vec{B}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})=\frac{1}{(2\pi)^{3/2}}\frac{-i}{\sqrt{2|\vec{p}|}}\{-|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}+a_2^+(\vec{p})e^{-ip\cdot x}]+|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}+a_1^+(\vec{p})e^{-ip\cdot x}]\}$$

Proof: $\vec{B}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})=\nabla\times\tilde{A}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})$

$$=\frac{1}{(2\pi)^{3/2}}\frac{-i}{\sqrt{2|\vec{p}|}}\{i\vec{p}\times\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}+a_2^+(\vec{p})e^{-ip\cdot x}]+i\vec{p}\times\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}+a_1^+(\vec{p})e^{-ip\cdot x}]\}$$

$$=\frac{1}{(2\pi)^{3/2}}\frac{-i}{\sqrt{2|\vec{p}|}}\{i|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix}\times\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}+a_2^+(\vec{p})e^{-ip\cdot x}]+i|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix}\times\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}+a_1^+(\vec{p})e^{-ip\cdot x}]\}$$

$$=\frac{1}{(2\pi)^{3/2}}\frac{-i}{\sqrt{2|\vec{p}|}}\{-|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}+a_2^+(\vec{p})e^{-ip\cdot x}]+|\vec{p}|\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}+a_1^+(\vec{p})e^{-ip\cdot x}]\}\quad\square$$

Cor. 2.4.4. $\begin{cases}\frac{1}{\sqrt{2}}[\vec{E}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})-i\vec{B}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})]=\frac{1}{(2\pi)^{3/2}}\sqrt{|\vec{p}|}\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)[a_1(\vec{p})e^{ip\cdot x}+a_2^+(\vec{p})e^{-ip\cdot x}]\\ \frac{1}{\sqrt{2}}[\vec{E}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})+i\vec{B}(|\vec{p}|\begin{bmatrix}0\\0\\1\end{bmatrix})]=\frac{1}{(2\pi)^{3/2}}\sqrt{|\vec{p}|}\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)[a_2(\vec{p})e^{ip\cdot x}+a_1^+(\vec{p})e^{-ip\cdot x}]\end{cases}$

2.5 Electromagnetic potential and field general solutions under radiation gauge

Def. 2.5.1. $\begin{cases} a_1(\vec{p},-1) := a_1(\vec{p}) \\ a_1(\vec{p},1) := a_1^+(\vec{p}) \end{cases}$ $\begin{cases} a_2(\vec{p},-1) := a_2(\vec{p}) \\ a_2(\vec{p},1) := a_2^+(\vec{p}) \end{cases}$ $\begin{cases} a_1(\vec{p},-1) = a_1^+(\vec{p},1) = a_1(\vec{p}) \\ a_2(\vec{p},-1) = a_2^+(\vec{p},1) = a_2(\vec{p}) \end{cases}$

Cor. 2.5.1. $\partial^a\partial_a\tilde{A}(\vec{r},t) = 0, \nabla\cdot\tilde{A}(\vec{r},t) = 0$

$\Rightarrow \tilde{A}(\vec{p}) = \frac{1}{(2\pi)^{3/2}}\frac{-i}{\sqrt{2|\vec{p}|}}\{\lambda_m(\hat{p},-1)[a_1(\vec{p})e^{ip\cdot x} - a_2^+(\vec{p})e^{-ip\cdot x}] + \lambda_m(\hat{p},1)[a_2(\vec{p})e^{ip\cdot x} - a_1^+(\vec{p})e^{-ip\cdot x}]\}$

Cor. 2.5.2.

$$\begin{cases} \Psi(\vec{p},1) = \frac{1}{\sqrt{2}}[\vec{E}(\vec{p}) - i\vec{B}(\vec{p})] = \frac{1}{(2\pi)^{3/2}}\sqrt{|\vec{p}|}\lambda_m(\hat{p},-1)[a_1(\vec{p})e^{ip\cdot x} + a_2^+(\vec{p})e^{-ip\cdot x}] \\ \Psi(\vec{p},-1) = \frac{1}{\sqrt{2}}[\vec{E}(\vec{p}) + i\vec{B}(\vec{p})] = \frac{1}{(2\pi)^{3/2}}\sqrt{|\vec{p}|}\lambda_m(\hat{p},1)[a_2(\vec{p})e^{ip\cdot x} + a_1^+(\vec{p})e^{-ip\cdot x}] \end{cases} \quad \Psi(\vec{p},-1) = \Psi^*(\vec{p},1)$$

Cor. 2.5.3.

$$\begin{cases} \tilde{A}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\frac{-i}{\sqrt{2|\vec{p}|}}\{\lambda_m(\hat{p},-1)[a_1(\vec{p})e^{ip\cdot x} - a_2^+(\vec{p})e^{-ip\cdot x}] + \lambda_m(\hat{p},1)[a_2(\vec{p})e^{ip\cdot x} - a_1^+(\vec{p})e^{-ip\cdot x}]\}d^3\vec{p} \\ \Psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\sqrt{|\vec{p}|}\lambda_m(\vec{p},-\varsigma)[a_1(\vec{p},-\varsigma)e^{i\varsigma p\cdot x} + a_2^+(\vec{p},-\varsigma)e^{-i\varsigma p\cdot x}]d^3\vec{p} \\ \tilde{A}(\vec{r},t) = \frac{1}{2}\frac{\partial_t[\Psi(\vec{r},t)+\Psi^+(\vec{r},t)]}{-\nabla^2}, \Psi(\vec{r},t) = -\partial_t\tilde{A}(\vec{r},t) - i\varsigma\nabla\times\tilde{A}(\vec{r},t) \end{cases}$$

2.6 Detailed analysis of potential solutions under radiation gauge

Cor. 2.6.1.

$$\begin{cases} \tilde{A}(\vec{r},t) \\ = \frac{-1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\sqrt{|\vec{p}|}\{[\lambda_m(\hat{p},-1)a_1(\vec{p}) + \lambda_m(\hat{p},1)a_2(\vec{p})]e^{ip\cdot x} + [\lambda_m(\hat{p},1)a_1^+(\vec{p}) + \lambda_m(\hat{p},-1)a_2^+(\vec{p})]e^{-ip\cdot x}\}d^3\vec{p} \\ \nabla\times\tilde{A}(\vec{r},t) \\ = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\frac{\vec{p}}{\sqrt{|\vec{p}|}}\times\{[\lambda_m(\hat{p},-1)a_1(\vec{p}) + \lambda_m(\hat{p},1)a_2(\vec{p})]e^{ip\cdot x} + [\lambda_m(\hat{p},1)a_1^+(\vec{p}) + \lambda_m(\hat{p},-1)a_2^+(\vec{p})]e^{-ip\cdot x}\}d^3\vec{p} \end{cases}$$

Cor. 2.6.2.

$\tilde{A}(\vec{r},t) = \frac{-i}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\frac{1}{\sqrt{|\vec{p}|}}\{[\lambda_m(\hat{p},-1)a_1(\vec{p}) + \lambda_m(\hat{p},1)a_2(\vec{p})]e^{ip\cdot x} - [\lambda_m(\hat{p},1)a_1^+(\vec{p}) + \lambda_m(\hat{p},-1)a_2^+(\vec{p})]e^{-ip\cdot x}\}d^3\vec{p}$

$\Leftrightarrow \lambda_m(\hat{p},-1)a_1(\vec{p}) + \lambda_m(\hat{p},1)a_2(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\{i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{-ip\cdot x}d^3\vec{r}$

$$\Leftrightarrow \begin{cases} a_1(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda_m^+(\hat{p},-1)\{i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{-ip\cdot x}d^3\vec{r} \\ a_2(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda_m^+(\hat{p},1)\{i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{-ip\cdot x}d^3\vec{r} \end{cases}$$

Cor. 2.6.3. $\begin{cases} a_1^+(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda_m^T(\hat{p},-1)\{-i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{ip\cdot x}d^3\vec{r} \\ a_2^+(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda_m^T(\hat{p},1)\{-i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{ip\cdot x}d^3\vec{r} \end{cases}$

3 Commutation rules for electromagnetic potential under radiation guage [25, 26, 37, 38]

3.1 Commutation rules for electromagnetic potential under radiation gauge

Cor. 3.1.1. $\begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')]_\pm = \delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_\pm = 0, [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')]_\pm = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} [\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm = \frac{i(1-\pm 1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \\ [\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)]_\pm = \frac{(1\pm 1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \\ [\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm = \frac{1\pm 1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}|\vec{p}|[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \end{cases}$$

Proof: $[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm$

$= i\cdot\frac{1}{2(2\pi)^3}$

$\int\limits_{\vec{p}\neq 0}\frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}\{\{[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p}',1)[a_1(\vec{p}), a_1^+(\vec{p}')]_\pm + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p}',-1)[a_2(\vec{p}), a_2^+(\vec{p}')]_\pm\}e^{i(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{i(-Et+E't)}$

$-\{\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p}',-1)[a_1^+(\vec{p}), a_1(\vec{p}')]_\pm + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p}',1)[a_2^+(\vec{p}), a_2(\vec{p}')]_\pm\}e^{-i(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{-i(-Et+E't)}\}d^3\vec{p}d^3\vec{p}'$

$= \frac{i}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\delta^3(\vec{p}-\vec{p}')\{[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$

$-\pm[\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}d^3\vec{p}'$

$= \frac{i}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\{[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$

$$- \pm[\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{i}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\{[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$- \pm[\lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{i}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\{[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$- \pm[\lambda_{mi}(-\hat{p},1)\lambda^+_{mj}(-\hat{p},1)\delta_1 + \lambda_{mi}(-\hat{p},-1)\lambda^+_{mj}(-\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{i(1-\pm 1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \quad \square$$

Proof: $[\tilde{A}_i(\vec{r},t),\tilde{A}_j(\vec{r}',t)]_\pm$

$$= -\frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0} d^3\vec{p}d^3\vec{p}'\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}$$
$$\{-\{[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p}',1)[a_1(\vec{p}),a_1^+(\vec{p}')]_\pm + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p}',-1)[a_2(\vec{p}),a_2^+(\vec{p}')]_\pm\}e^{i(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{i(-Et+E't)}$$
$$-\{\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p}',-1)[a_1^+(\vec{p}),a_1(\vec{p}')]_\pm + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p}',1)[a_2^+(\vec{p}),a_2(\vec{p}')]_\pm\}e^{-i(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{-i(-Et+E't)}\}$$
$$= -\frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\delta^3(\vec{p}-\vec{p}')\{-[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$- \pm[\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}d^3\vec{p}'$$
$$= -\frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\{-[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$- \pm[\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= -\frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\{-[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$- \pm[\lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\{[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$\pm[\lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\{[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$\pm[\lambda_{mi}(-\hat{p},1)\lambda^+_{mj}(-\hat{p},1)\delta_1 + \lambda_{mi}(-\hat{p},-1)\lambda^+_{mj}(-\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{(1\pm 1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \quad \square$$

Proof: $[\dot{\tilde{A}}_i(\vec{r},t),\dot{\tilde{A}}_j(\vec{r}',t)]_\pm$

$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0} d^3\vec{p}d^3\vec{p}'\sqrt{|\vec{p}||\vec{p}'|}$$
$$\{\{[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p}',1)[a_1(\vec{p}),a_1^+(\vec{p}')]_\pm + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p}',-1)[a_2(\vec{p}),a_2^+(\vec{p}')]_\pm\}e^{i(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{i(-Et+E't)}$$
$$\pm\{\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p}',-1)[a_1^+(\vec{p}),a_1(\vec{p}')]_\pm + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p}',1)[a_2^+(\vec{p}),a_2(\vec{p}')]_\pm\}e^{-i(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{-i(-Et+E't)}\}$$
$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}|\vec{p}|\delta^3(\vec{p}-\vec{p}')\{[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$\pm[\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}d^3\vec{p}'$$
$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}|\vec{p}|\{[\lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$\pm[\lambda_{mi}(\hat{p},1)\lambda_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda_{mj}(\hat{p},1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}|\vec{p}|\{[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$\pm[\lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}|\vec{p}|\{[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$\pm[\lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_1 + \lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_2]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}|\vec{p}|\{[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$\pm[\lambda_{mi}(-\hat{p},1)\lambda^+_{mj}(-\hat{p},1)\delta_1 + \lambda_{mi}(-\hat{p},-1)\lambda^+_{mj}(-\hat{p},-1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$$
$$= \frac{1\pm 1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}|\vec{p}|[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)\delta_1 + \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \quad \square$$

3.2 Commutation rules for complex field strength under radiation gauge

Cor. 3.2.1.
$$\begin{cases}[a_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]_\pm = \delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}),a_{\sigma'}(\vec{p}')]_\pm = 0\\ [a^+_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]_\pm = 0\end{cases} \Leftrightarrow \begin{cases}[a_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm = \varsigma^0_1\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma),a_{\sigma'}(\vec{p}',-\varsigma)]_\pm = 0\\ [a^+_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm = 0\end{cases}$$

Cor. 3.2.2.

$$\begin{cases}\Psi(\vec{r},t) = -\partial_t\tilde{A}(\vec{r},t) - i\varsigma\nabla\times\tilde{A}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\lambda_m(\vec{p},-\varsigma)\sqrt{|\vec{p}|}[a_1(\vec{p},-\varsigma)e^{i\varsigma p\cdot x} + a_2^+(\vec{p},-\varsigma)e^{-i\varsigma p\cdot x}]d^3\vec{p} \\ \Psi^*(\vec{r},t) = -\partial_t\tilde{A}(\vec{r},t) + i\varsigma\nabla\times\tilde{A}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\lambda_m^*(\vec{p},-\varsigma)\sqrt{|\vec{p}|}[a_1^+(\vec{p},-\varsigma)e^{-i\varsigma p\cdot x} + a_2(\vec{p},-\varsigma)e^{i\varsigma p\cdot x}]d^3\vec{p}\end{cases}$$

Cor. 3.2.3.

$$\begin{cases}[a_\sigma(\vec{p},-\varsigma), a_{\sigma'}^+(\vec{p}',-\varsigma)]_\pm \\ = \varsigma_1^0\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p},-\varsigma), a_{\sigma'}(\vec{p}',-\varsigma)]_\pm = 0 \\ [a_\sigma^+(\vec{p},-\varsigma), a_{\sigma'}^+(\vec{p}',-\varsigma)]_\pm = 0\end{cases} \Leftrightarrow \begin{cases}[\Psi_i(\vec{r},t), \Psi_j^+(\vec{r}',t)]_\pm = \frac{1}{(2\pi)^3}\varsigma_1^0 \\ \int\limits_{\vec{p}\neq 0}|\vec{p}|[\delta_1\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma) \pm \delta_2\lambda_{mi}(\vec{p},\varsigma)\lambda_{mj}^+(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)]_\pm = 0 \\ [\Psi_i^+(\vec{r},t), \Psi_j^+(\vec{r}',t)]_\pm = 0\end{cases}$$

Proof: $[\Psi_i(\vec{r},t), \Psi_j^+(\vec{r}',t)]_\pm = \frac{1}{(2\pi)^3}\int\limits_{\vec{p},\vec{p}'\neq 0}\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p}',-\varsigma)\sqrt{|\vec{p}||\vec{p}'|}$

$\{[a_1(\vec{p},-\varsigma), a_1^+(\vec{p}',-\varsigma)]_\pm e^{i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{i(-Et+E't)} + [a_2^+(\vec{p},-\varsigma), a_2(\vec{p}',-\varsigma)]_\pm e^{-i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}e^{-i(-Et+E't)}\}d^3\vec{p}d^3\vec{p}'$

$= \frac{1}{(2\pi)^3}\varsigma_1^0\int\limits_{\vec{p},\vec{p}'\neq 0}\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)|\vec{p}|\delta^3(\vec{p}-\vec{p}')[\delta_1 e^{i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')} \pm \delta_2 e^{-i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}]d^3\vec{p}d^3\vec{p}'$

$= \frac{1}{(2\pi)^3}\varsigma_1^0\int\limits_{\vec{p}\neq 0}\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)|\vec{p}|[\delta_1 e^{i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')} \pm \delta_2 e^{-i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}]d^3\vec{p}$

$= \frac{1}{(2\pi)^3}\varsigma_1^0\int\limits_{\vec{p}\neq 0}\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)|\vec{p}|\delta_1 e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} \pm \lambda_{mi}(-\vec{p},-\varsigma)\lambda_{mj}^+(-\vec{p},-\varsigma)|-\vec{p}|\delta_2 e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}$

$= \frac{1}{(2\pi)^3}\varsigma_1^0\int\limits_{\vec{p}\neq 0}|\vec{p}|[\delta_1\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma) \pm \delta_2\lambda_{mi}(\vec{p},\varsigma)\lambda_{mj}^+(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$ □

3.3 Homomorphic commutation rules for $\tilde{A}, \Psi$ under radiation gauge

3.3.1 Homomorphic commutation rules for $\tilde{A}, \Psi$ under radiation gauge

Cor. 3.3.1.

$$\begin{cases}[a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')]_\pm = \kappa\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_\pm = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')]_\pm = 0\end{cases} \Leftrightarrow \begin{cases}[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm = \frac{(1-\pm1)}{2}i\kappa(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}') \\ [\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)]_\pm = \frac{(1\pm1)}{2}\kappa\frac{1}{\sqrt{-\nabla^2}}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}') \\ [\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm = \frac{(1\pm1)}{2}\kappa\sqrt{-\nabla^2}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')\end{cases}$$

Proof: $[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm$

$= \frac{i(1-\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1) + \lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{i(1-\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1) + \lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{i(1-\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa[\sum_{h=1}^{-1}\lambda_{mi}(\hat{p},h)\lambda_{mj}^+(\hat{p},h) - \lambda_{mi}(\hat{p},0)\lambda_{mj}^+(\hat{p},0)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{i(1-\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa[\delta_{ij} - \lambda_{mi}(\hat{p},0)\lambda_{mj}^+(\hat{p},0)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{i(1-\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa(\delta_{ij} - \hat{p}_i\hat{p}_j)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{i(1-\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa(\delta_{ij} - \frac{p_ip_j}{p^2})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{(1-\pm1)}{2}i\kappa(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')$ □

Proof: $[\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)]_\pm$

$= \frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa\frac{1}{|\vec{p}|}[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1) + \lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa\frac{1}{|\vec{p}|}(\delta_{ij} - \frac{p_ip_j}{p^2})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{(1\pm1)}{2}\kappa\frac{1}{\sqrt{-\nabla^2}}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')$ □

Proof: $[\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm$

$= \frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa|\vec{p}|[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1) + \lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\kappa|\vec{p}|(\delta_{ij} - \frac{p_ip_j}{p^2})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{(1\pm1)}{2}\kappa\sqrt{-\nabla^2}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')$ □

3.3.2 Homomorphic commutative relations of complex field strength under radiation gauge

Cor. 3.3.2. $\begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')]_- = \kappa\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_- = 0 \\ [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')]_- = 0\end{cases} \Leftrightarrow \begin{cases}[\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)]_- = i\varsigma\kappa\gamma^k{}_{ij}p_k\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)]_- = 0 \\ [\Psi^+_i(\vec{r},t), \Psi^+_j(\vec{r}',t)]_- = 0\end{cases}$

Proof: $[\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)]_-$
$= \frac{1}{(2\pi)^3}\varsigma\kappa \int\limits_{\vec{p}\neq 0} |\vec{p}|[\lambda_{mi}(\vec{p},-\varsigma)\lambda^+_{mj}(\vec{p},-\varsigma) - \lambda_{mi}(\vec{p},\varsigma)\lambda^+_{mj}(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= -\frac{1}{(2\pi)^3}\varsigma\kappa \int\limits_{\vec{p}\neq 0} [-\varsigma|\vec{p}|\lambda_{mi}(\vec{p},-\varsigma)\lambda^+_{mj}(\vec{p},-\varsigma)0\cdot|\vec{p}|\lambda_{mi}(\vec{p},0)\lambda^+_{mj}(\vec{p},0) + \varsigma|\vec{p}|\lambda_{mi}(\vec{p},\varsigma)\lambda^+_{mj}(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= -\frac{1}{(2\pi)^3}\kappa \int\limits_{\vec{p}\neq 0} \gamma^k i^l p_k \sum\limits_{h=1}^{-1} \lambda_{ml}(\vec{p},h)\lambda^+_{mj}(\vec{p},h)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= -\frac{1}{(2\pi)^3}\kappa \int\limits_{\vec{p}\neq 0} \gamma^k{}_i{}^l p_k\delta_{lj}e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= -\frac{1}{(2\pi)^3}\kappa \int\limits_{\vec{p}\neq 0} \gamma^k{}_{ij}p_k e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= i\varsigma\kappa\gamma^k{}_{ij}p_k\delta^3(\vec{r}-\vec{r}')$ □

3.3.3 Homomorphic anticommutative relations of Ψ under radiation gauge

Cor. 3.3.3. $\begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')]_+ = \kappa\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_+ = 0 \\ [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')]_+ = 0\end{cases} \Leftrightarrow \begin{cases}[\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)]_+ = \kappa\sqrt{-\nabla^2}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)]_+ = 0 \\ [\Psi^+_i(\vec{r},t), \Psi^+_j(\vec{r}',t)]_+ = 0\end{cases}$

Proof: $[\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)]_+ = \frac{1}{(2\pi)^3}\kappa \int\limits_{\vec{p}\neq 0} |\vec{p}|[\lambda_{mi}(\vec{p},-\varsigma)\lambda^+_{mj}(\vec{p},-\varsigma) + \lambda_{mi}(\vec{p},\varsigma)\lambda^+_{mj}(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\kappa \int\limits_{\vec{p}\neq 0} |\vec{p}|[\sum\limits_{h=1}^{-1} \lambda_{mi}(\hat{p},h)\lambda^+_{mj}(\hat{p},h) - \lambda_{mi}(\hat{p},0)\lambda^+_{mj}(\hat{p},0)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\kappa \int\limits_{\vec{p}\neq 0} |\vec{p}|(\delta_{ij} - \hat{p}_i\hat{p}_j)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\kappa \int\limits_{\vec{p}\neq 0} |\vec{p}|(\delta_{ij} - \frac{p_ip_j}{p^2})e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \kappa\sqrt{-\nabla^2}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')$ □

3.4 Heterotypic commutative and anticommutative relations of $\tilde{A}, \Psi$ under radiation gauge

3.4.1 Heterotypic commutative and anticommutative relations of $\tilde{A}$ under radiation gauge

Cor. 3.4.1.
$\begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')]_\pm = \kappa(-1)^{\sigma+1}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_\pm = 0 \\ [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')]_\pm = 0\end{cases} \Leftrightarrow \begin{cases}[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm = -\frac{(1-\pm 1)}{2}\kappa\frac{\gamma^k{}_{ij}\partial_k}{\sqrt{-\nabla^2}}\delta^3(\vec{r}-\vec{r}') \\ [\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)]_\pm = -\frac{(1\pm 1)}{2}i\kappa\frac{\gamma^k{}_{ij}\partial_k}{\nabla^2}\delta^3(\vec{r}-\vec{r}') \\ [\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm = \frac{(1\pm 1)}{2}i\kappa\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\end{cases}$

Proof: $[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)]_\pm$
$= \frac{i(1-\pm 1)}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \kappa[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1) - \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \frac{-i(1-\pm 1)}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \kappa[1\cdot\lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1) + 0\cdot\lambda_{mi}(\hat{p},0)\lambda^+_{mj}(\hat{p},0) - 1\cdot\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \frac{-i(1-\pm 1)}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \kappa\frac{\gamma^k{}_i{}^l p_k}{|\vec{p}|} \sum\limits_{h=1}^{-1} \lambda_{ml}(\hat{p},h)\lambda^+_{mj}(\hat{p},h)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \frac{-i(1-\pm 1)}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \kappa\frac{\gamma^k{}_i{}^l p_k}{|\vec{p}|}\delta_{lj}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= \frac{-i(1-\pm 1)}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \kappa\frac{\gamma^k{}_{ij}p_k}{|\vec{p}|}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= -\frac{(1-\pm 1)}{2}\kappa\frac{\gamma^k{}_{ij}\partial_k}{\sqrt{-\nabla^2}}\delta^3(\vec{r}-\vec{r}')$ □

Proof: $[\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)]_\pm$
$= \frac{(1\pm 1)}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\kappa[\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1) - \lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$= -\frac{(1\pm 1)}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\kappa[1\cdot\lambda_{mi}(\hat{p},1)\lambda^+_{mj}(\hat{p},1) + 0\cdot\lambda_{mi}(\hat{p},0)\lambda^+_{mj}(\hat{p},0) - 1\cdot\lambda_{mi}(\hat{p},-1)\lambda^+_{mj}(\hat{p},-1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}\frac{1}{|\vec{p}|}\kappa\frac{\gamma^k{}_i{}^l p_k}{|\vec{p}|}\sum\limits_{h=1}^{-1}\lambda_{mi}(\hat{p},h)\lambda_{mj}^+(\hat{p},h)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}\frac{1}{|\vec{p}|}\kappa\frac{\gamma^k{}_i{}^l p_k}{|\vec{p}|}\delta_{lj}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}\kappa\frac{\gamma^k{}_{ij}p_k}{\vec{p}^2}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2}i\kappa\frac{\gamma^k{}_{ij}\partial_k}{\nabla^2}\delta^3(\vec{r}-\vec{r}')$ □

Proof: $[\dot{\tilde{A}}_i(\vec{r},t),\dot{\tilde{A}}_j(\vec{r}',t)]_\pm$

$= \frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}|\vec{p}|\kappa[\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1)-\lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)\delta_2]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}|\vec{p}|\kappa[1\cdot\lambda_{mi}(\hat{p},1)\lambda_{mj}^+(\hat{p},1)+0\cdot\lambda_{mi}(\hat{p},0)\lambda_{mj}^+(\hat{p},0)-1\cdot\lambda_{mi}(\hat{p},-1)\lambda_{mj}^+(\hat{p},-1)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}|\vec{p}|\kappa\frac{\gamma^k{}_i{}^l p_k}{|\vec{p}|}\sum\limits_{h=1}^{-1}\lambda_{mi}(\hat{p},h)\lambda_{mj}^+(\hat{p},h)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}\kappa\gamma^k{}_i{}^l p_k\delta_{lj}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{(1\pm1)}{2(2\pi)^3}\int\limits_{\vec{p}\neq0}\kappa\gamma^k{}_{ij}p_k e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{(1\pm1)}{2}i\kappa\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')$ □

3.4.2 Heterotypic commutative relation of complex field strength under radiation gauge

Cor. 3.4.2.

$$\begin{cases}[a_\sigma(\vec{p}),a_{\sigma'}^+(\vec{p}')]_-=\kappa(-1)^{\sigma+1}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}),a_{\sigma'}(\vec{p}')]_-=0\\ [a_\sigma^+(\vec{p}),a_{\sigma'}^+(\vec{p}')]_-=0\end{cases}\Leftrightarrow\begin{cases}[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]_-=\varsigma\kappa\sqrt{-\nabla^2}(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t),\Psi_j(\vec{r}',t)]_-=0\\ [\Psi_i^+(\vec{r},t),\Psi_j^+(\vec{r}',t)]_-=0\end{cases}$$

Proof: $[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]_-=\frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}|\vec{p}|[\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)+\lambda_{mi}(\vec{p},\varsigma)\lambda_{mj}^+(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}|\vec{p}|[\sum\limits_{h=1}^{-1}\lambda_{mi}(\hat{p},h)\lambda_{mj}^+(\hat{p},h)-\lambda_{mi}(\hat{p},0)\lambda_{mj}^+(\hat{p},0)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}|\vec{p}|(\delta_{ij}-\hat{p}_i\hat{p}_j)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}|\vec{p}|(\delta_{ij}-\frac{p_ip_j}{p^2})e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= \varsigma\kappa\sqrt{-\nabla^2}(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')$ □

3.4.3 Heterotypic anticommutative relation of complex field strength under radiation gauge

Cor. 3.4.3. $$\begin{cases}[a_\sigma(\vec{p}),a_{\sigma'}^+(\vec{p}')]_+=\kappa(-1)^{\sigma+1}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}),a_{\sigma'}(\vec{p}')]_+=0\\ [a_\sigma^+(\vec{p}),a_{\sigma'}^+(\vec{p}')]_+=0\end{cases}\Leftrightarrow\begin{cases}[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]_+=i\kappa\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t),\Psi_j(\vec{r}',t)]_+=0\\ [\Psi_i^+(\vec{r},t),\Psi_j^+(\vec{r}',t)]_+=0\end{cases}$$

Proof: $[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]_+$

$= \frac{1}{(2\pi)^3}\kappa\int\limits_{\vec{p}\neq0}|\vec{p}|[\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)-\lambda_{mi}(\vec{p},\varsigma)\lambda_{mj}^+(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}[-\varsigma|\vec{p}|\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)0\cdot|\vec{p}|\lambda_{mi}(\vec{p},0)\lambda_{mj}^+(\vec{p},0)+\varsigma|\vec{p}|\lambda_{mi}(\vec{p},\varsigma)\lambda_{mj}^+(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}\gamma^k i^l p_k\sum\limits_{h=1}^{-1}\lambda_{ml}(\vec{p},h)\lambda_{mj}^+(\vec{p},h)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}\gamma^k{}_i{}^l p_k\delta_{lj}e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= -\frac{1}{(2\pi)^3}\varsigma\kappa\int\limits_{\vec{p}\neq0}\gamma^k{}_{ij}p_k e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$

$= i\kappa\gamma^k{}_{ij}p_k\delta^3(\vec{r}-\vec{r}')$ □

3.5 Analysis on heterotypic anticommutative relations of $\tilde{A},\Psi$ under radiation gauge

According to the electromagnetic field covariant quantization scheme, only $\delta_1\pm\delta_2=0$, the micro causality is satisfied. At the same time only when $\delta_1,\delta_2\geq0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1=\delta_2=1$, it satisfies the commutative

relation. There are actually two other options. Namely when $\delta_1 = \delta_2 = 0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

3.5.1 Homomorphic commutative relations of $\tilde{A}, \Psi$ under radiation gauge

Cor. 3.5.1. $$\begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = \kappa\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0\\ [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = 0\end{cases} \Leftrightarrow \begin{cases}[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)] = i\kappa(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')\\ [\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)] = 0\\ [\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)] = 0\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

Cor. 3.5.2. $$\begin{cases}[a_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)] = \kappa\varsigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma), a_{\sigma'}(\vec{p}',-\varsigma)] = 0\\ [a^+_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)] = 0\end{cases} \Leftrightarrow \begin{cases}[\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)] = i\varsigma\kappa\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)] = 0\\ [\Psi^+_i(\vec{r},t), \Psi^+_j(\vec{r}',t)] = 0\end{cases}$$

When $\kappa = 1$, It satisfies both micro causality and non negative probability, so it is physical. When $\kappa = -1$, it satisfies micro causality but violates non negative probability, so it is non physical.

3.5.2 Isomorphic anticommutative relations of $\tilde{A}, \Psi$ under radiation gauge

Cor. 3.5.3.
$$\begin{cases}\{a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')\} = \kappa\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')\} = 0\\ \{a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')\} = 0\end{cases} \Leftrightarrow \begin{cases}\{\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} = 0\\ \{\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)\} = \kappa\frac{1}{\sqrt{-\nabla^2}}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')\\ \{\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} = \kappa\sqrt{-\nabla^2}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

Cor. 3.5.4.
$$\begin{cases}\{a_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)\} = \kappa\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\sigma(\vec{p},-\varsigma), a_{\sigma'}(\vec{p}',-\varsigma)\} = 0\\ \{a^+_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)\} = 0\end{cases} \Leftrightarrow \begin{cases}\{\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)\} = \kappa\sqrt{-\nabla^2}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')\\ \{\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)\} = 0\\ \{\Psi^+_i(\vec{r},t), \Psi^+_j(\vec{r}',t)\} = 0\end{cases}$$

When $\kappa = 1$, it satisfies non negative probability but violates micro causality, so it is non physical. When $\kappa = -1$, it violates both micro causality and non negative probability, so it is non physical.

3.5.3 Heterotypic commutative relations of $\tilde{A}, \Psi$ under radiation gauge

Cor. 3.5.5.
$$\begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = \kappa(-1)^{\sigma+1}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0\\ [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = 0\end{cases} \Leftrightarrow \begin{cases}[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)] = -\kappa\frac{\gamma^k{}_{ij}\partial_k}{\sqrt{-\nabla^2}}\delta^3(\vec{r}-\vec{r}')\\ [\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)] = 0\\ [\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)] = 0\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

Cor. 3.5.6.
$$\begin{cases}[a_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)]\\ = \kappa\varsigma(-1)^{\sigma+1}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma), a_{\sigma'}(\vec{p}',-\varsigma)] = 0\\ [a^+_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)] = 0\end{cases} \Leftrightarrow \begin{cases}[\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)] = \kappa\varsigma\sqrt{-\nabla^2}(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)] = 0\\ [\Psi^+_i(\vec{r},t), \Psi^+_j(\vec{r}',t)] = 0\end{cases}$$

Regardless of the value of κ, it violates both microscopic causality and non negative probability. So it is non physical.

3.5.4 Heterotypic anticommutative relations of $\tilde{A}, \Psi$ under radiation gauge

Cor. 3.5.7.
$$\begin{cases}\{a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')\} = \kappa(-1)^{\sigma+1}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')\} = 0\\ \{a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')\} = 0\end{cases} \Leftrightarrow \begin{cases}\{\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} = 0\\ \{\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)\} = -i\kappa\frac{\gamma^k{}_{ij}\partial_k}{\nabla^2}\delta^3(\vec{r}-\vec{r}')\\ \{\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} = i\kappa\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

Cor. 3.5.8.
$$\begin{cases}\{a_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)\} = \kappa(-1)^{\sigma+1}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\sigma(\vec{p},-\varsigma), a_{\sigma'}(\vec{p}',-\varsigma)\} = 0\\ \{a^+_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)\} = 0\end{cases} \Leftrightarrow \begin{cases}\{\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)\} = i\kappa\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\\ \{\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)\} = 0\\ \{\Psi^+_i(\vec{r},t), \Psi^+_j(\vec{r}',t)\} = 0\end{cases}$$

Regardless of the value of κ, it satisfies the microscopic causality but violates the non negative probability. So it is non physical.

3.6 Summary of commutative relations for physical $\tilde{A}, \Psi$ under radiation gauge

Cor. 3.6.1. $\begin{cases}[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a^+_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p}')] = 0\end{cases} \Leftrightarrow \begin{cases}[\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}') \\ [\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)] = 0 \\ [\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)] = 0\end{cases}$

$[\Updownarrow] \qquad [\Updownarrow]$

Cor. 3.6.2. $\begin{cases}[a_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)] = \varsigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p},-\varsigma), a_{\sigma'}(\vec{p}',-\varsigma)] = 0 \\ [a^+_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)] = 0\end{cases} \Leftrightarrow \begin{cases}[\Psi_i(\vec{r},t), \Psi^+_j(\vec{r}',t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)] = 0 \\ [\Psi^+_i(\vec{r},t), \Psi^+_j(\vec{r}',t)] = 0\end{cases}$

$[\Updownarrow] \qquad [\Updownarrow]$

Cor. 3.6.3. $\begin{cases}[\Psi_{\alpha_\varsigma}(x), \Psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x') \\ [\Psi_{\alpha_\varsigma}(x), \Psi_{\beta_\varsigma}(x')] = 0 \\ [\Psi^+_{\alpha'_\varsigma}(x), \Psi^+_{\beta'_\varsigma}(x')] = 0\end{cases} \Leftrightarrow \begin{cases}[\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\alpha'_\varsigma}(\vec{r}',t)] = \varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta(\vec{r}-\vec{r}') \\ [\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi_{\beta_\varsigma}(\vec{r}',t)] = 0 \\ [\Psi^+_{\alpha'_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)] = 0\end{cases}$

Electromagnetic field only has the one case. It is to satisfy both micro causality and non negative probability. It is actually physical and correct.

3.7 Classical anticommutative relations under radiation gauge(Zero mode)

Cor. 3.7.1. $\begin{cases}\{\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} = \kappa(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}') \\ \{\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)\} = 0 \\ \{\dot{\tilde{A}}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} = 0\end{cases}$

Cor. 3.7.2. $\lambda_m(\hat{p},-1)a_1(\vec{p}) + \lambda_m(\hat{p},1)a_2(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\{i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{-ip\cdot x}d^3\vec{r}$

$$\Rightarrow\begin{cases}a_1(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda^+_m(\hat{p},-1)\{i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{-ip\cdot x}d^3\vec{r} \\ a^+_1(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda^T_m(\hat{p},-1)\{-i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{ip\cdot x}d^3\vec{r} \\ a_2(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda^+_m(\hat{p},1)\{i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{-ip\cdot x}d^3\vec{r} \\ a^+_2(\vec{p}) = \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int\lambda^T_m(\hat{p},1)\{-i\tilde{A}(\vec{r},t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r},t)\frac{1}{\sqrt{|\vec{p}|}}\}e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Proof: $\{a_1(\vec{p}), a^+_1(\vec{p}')\}$

$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\lambda^{+i}_m(\hat{p},-1)\lambda^j_m(\hat{p}',-1)[-i\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}}\{\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} + i\frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}\{\tilde{A}_j(\vec{r}',t), \dot{\tilde{A}}_i(\vec{r},t)\}]e^{-ip\cdot x}e^{ip'\cdot x'}d^3\vec{r}d^3\vec{r}'$

$= -\frac{i}{2}\frac{1}{(2\pi)^3}\int\lambda^{+i}_m(\hat{p},-1)\lambda^j_m(\hat{p}',-1)\kappa(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')[\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} - \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}]e^{-ip\cdot x}e^{ip'\cdot x'}d^3\vec{r}$

$= -\frac{i}{2}\frac{1}{(2\pi)^3}\int\lambda^{+i}_m(\hat{p}',-1)\lambda^j_m(\hat{p},-1)\kappa(\delta_{ij} - \frac{p_ip_j}{\vec{p}^2})[\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} - \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}]e^{-ip\cdot x}e^{ip'\cdot x}d^3\vec{r}$

$= -\frac{i}{2}\lambda^{+i}_m(\hat{p},-1)\lambda^j_m(\hat{p},-1)\kappa(\delta_{ij} - \frac{p_ip_j}{\vec{p}^2})[\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}|}} - \frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}|}}]\delta^3(\vec{p}-\vec{p}')$

$= 0$ □

Proof: $\{a_1(\vec{p}), a_1(\vec{p}')\}$

$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\lambda^{+i}_m(\hat{p},-1)\lambda^{+j}_m(\hat{p}',-1)[-i\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}}\{\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)\} - i\frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}\{\tilde{A}_j(\vec{r}',t), \dot{\tilde{A}}_i(\vec{r},t)\}]e^{-ip\cdot x}e^{-ip'\cdot x'}d^3\vec{r}d^3\vec{r}'$

$= -\frac{i}{2}\frac{1}{(2\pi)^3}\int\lambda^{+i}_m(\hat{p},-1)\lambda^{+j}_m(\hat{p}',-1)\kappa(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}')[\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} + \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}]e^{-ip\cdot x}e^{-ip'\cdot x'}d^3\vec{r}d^3\vec{r}'$

$= -\frac{i}{2}\frac{1}{(2\pi)^3}\int\lambda^{+i}_m(\hat{p},-1)\lambda^{+j}_m(\hat{p}',-1)\kappa(\delta_{ij} - \frac{p'_ip'_j}{\vec{p}'^2})[\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} + \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}]e^{-ip\cdot x}e^{-ip'\cdot x}d^3\vec{r}$

$= -\frac{i}{2}\lambda^{+i}_m(\hat{p},-1)\lambda^{+j}_m(-\hat{p},-1)\kappa(\delta_{ij} - \frac{p_ip_j}{\vec{p}^2})[\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}|}} + \frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}|}}]\delta^3(\vec{p}+\vec{p}')e^{2iEt}$

$= -i\kappa\lambda^{+i}_m(\hat{p},-1)\lambda^j_m(-\hat{p},1)\delta_{ij}e^{2iEt}\delta^3(\vec{p}+\vec{p}')$

$= -i\kappa\frac{\hat{p}_+}{\hat{p}_-}e^{2iEt}\delta^3(\vec{p}+\vec{p}')$ □

It produces zero mode, so it is non physical.

4 Derive new scheme from traditional radiation gauge quantization scheme [25, 26, 37, 38]

4.1 Obtain isochronous commutative relations of Ψ from traditional $\tilde{A}$ case

Cor. 4.1.1. $\mathscr{L} = -\frac{1}{4}F^{uv}F_{uv} \Rightarrow \pi_i = \frac{\partial \mathscr{L}}{\partial \dot{A}_i} = \partial_t \tilde{A}_i + \partial_i \phi = -E_i, \pi_4 = 0$

From the isochronous commutative relation with canonical variables $(\tilde{A}_i, E_i)$, the isochronous commutative relation with basic variables (Ψ_i, Ψ_i^+) is derived.

Cor. 4.1.2.
$$\begin{cases}[\tilde{A}_i(\vec{r},t), E_j(\vec{r}',t)] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}') \\ [E_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\delta^3(\vec{r}-\vec{r}') \\ [\tilde{A}_i(\vec{r},t), \tilde{A}_j(\vec{r}',t)] = 0 \\ [E_i(\vec{r},t), E_j(\vec{r}',t)] = 0 \end{cases} \Rightarrow \begin{cases}[\Psi_i(\vec{r},t), \Psi_j^+(\vec{r}',t)] = \varsigma\varepsilon_{ij}{}^k\partial_k\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)] = 0 \\ [\Psi_i^+(\vec{r},t), \Psi_j^+(\vec{r}',t)] = 0 \end{cases}$$

Proof: $[\Psi_i(\vec{r},t), \Psi_j^+(\vec{r}',t)]$
$= -\frac{1}{2}i\varsigma\varepsilon_i{}^{kl}\partial_{x_k}[\tilde{A}_l(\vec{r},t), E_j(\vec{r}',t)] + \frac{1}{2}i\varsigma\varepsilon_j{}^{kl}\partial_{x'_k}[E_i(\vec{r},t), \tilde{A}_l(\vec{r}',t)]$
$= \frac{1}{2}\varsigma\varepsilon_{ij}{}^k(\partial_{x_k} - \partial_{x'_k})\delta^3(\vec{r}-\vec{r}')$
$= \varsigma\varepsilon_{ij}{}^k\partial_{(x_k - x'_k)}\delta^3(\vec{r}-\vec{r}')$
$= i\varsigma\gamma_{ij}{}^k\partial_k\delta^3(\vec{r}-\vec{r}')$
$= i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')$ □

Proof: $[\Psi_i^+(\vec{r},t), \Psi_j^+(\vec{r}',t)]$
$= \frac{1}{2}i\varsigma\varepsilon_i{}^{kl}\partial_{x_k}[\tilde{A}_l(\vec{r},t), E_j(\vec{r}',t)] + \frac{1}{2}i\varsigma\varepsilon_j{}^{kl}\partial_{x'_k}[E_i(\vec{r},t), \tilde{A}_l(\vec{r}',t)]$
$= -\frac{1}{2}\varsigma\varepsilon_{ij}{}^k(\partial_{x_k} + \partial_{x'_k})\delta^3(\vec{r}-\vec{r}')$
$= 0$ □

Proof: $[\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)]$
$= -\frac{1}{2}i\varsigma\varepsilon_i{}^{kl}\partial_{x_k}[\tilde{A}_l(\vec{r},t), E_j(\vec{r}',t)] - \frac{1}{2}i\varsigma\varepsilon_j{}^{kl}\partial_{x'_k}[E_i(\vec{r},t), \tilde{A}_l(\vec{r}',t)]$
$= \frac{1}{2}\varsigma\varepsilon_{ij}{}^k(\partial_{x_k} + \partial_{x'_k})\delta^3(\vec{r}-\vec{r}')$
$= 0$ □

4.2 Equivalence between potential and field commutative relations under radiation gauge

Cor. 4.2.1.
$$\begin{cases}[\Psi_{\alpha_\varsigma}(x), \Psi_{\alpha'_\varsigma}^+(x')] = i\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\Psi_{\alpha_\varsigma}(x), \Psi_{\beta_\varsigma}(x')] = 0, [\Psi_{\alpha'_\varsigma}^+(x), \Psi_{\beta'_\varsigma}^+(x')] = 0 \end{cases} \Leftrightarrow [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')$$

Proof: $[\tilde{A}_i(x), \tilde{A}_j(x')]$
$= \frac{1}{2}\frac{\partial_t}{-\nabla^2}\frac{\partial_{t'}}{-\nabla'^2}[\psi_i(x) + \psi_i^+(x), \psi_j(x') + \psi_j^+(x')]$
$= \frac{1}{2}\frac{\partial_t}{-\nabla^2}\frac{\partial_{t'}}{-\nabla'^2}\{[\psi_i(x), \psi_j^+(x')] + [\psi_i^+(x), \psi_j(x')]\}$
$= \frac{1}{2}\frac{\partial_t}{-\nabla^2}\frac{\partial_{t'}}{-\nabla'^2}\{[\psi_i(x), \psi_j^+(x')] - [\psi_j(x'), \psi_i^+(x)]\}$
$= \frac{1}{2}\frac{\partial_t}{-\nabla^2}\frac{\partial_{t'}}{-\nabla'^2}\{i\sigma_{ij}^{ab}\partial_a\partial_b\Delta(x-x') - [i\sigma_{ji}^{ab}\partial'_a\partial'_b\Delta(x'-x)]\}$
$= \frac{1}{2}\frac{\partial_t}{-\nabla^2}\frac{-\partial_t}{-\nabla^2}\{i\sigma_{ij}^{ab}\partial_a\partial_b\Delta(x-x') + [i\sigma_{ji}^{ab}\partial_a\partial_b\Delta(x-x')]\}$
$= -\frac{1}{2}\frac{\partial_t^2}{\nabla^4}\{-i[\frac{1}{2}(\nabla^2 - \partial_\pi^2)\delta_{ij} - \varsigma\varepsilon_{ij}{}^k\partial_k\partial_\pi - \partial_i\partial_j] - i[\frac{1}{2}(\nabla^2 - \partial_\pi^2)\delta_{ji} - \varsigma\varepsilon_{ji}{}^k\partial_k\partial_\pi - \partial_j\partial_i]\}\Delta(x-x')$
$= i\frac{1}{\nabla^2}[\frac{1}{2}(\nabla^2 - \partial_\pi^2)\delta_{ij} - \partial_i\partial_j]\Delta(x-x')$
$= i\frac{1}{\nabla^2}[\nabla^2\delta_{ij} - \partial_i\partial_j]\Delta(x-x')$
$= i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')$ □

Reverse proof:

Proof: $[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')$
$\Rightarrow i[\Psi_i(x), \Psi_j^+(x')]$
$= \frac{i}{2}[-i\partial_\pi\tilde{A}_i(x) + i\varsigma\varepsilon_i{}^{kl}\partial_k\tilde{A}_l(x), -i\partial'_\pi\tilde{A}_j(x') - i\varsigma\varepsilon_j{}^{mn}\partial'_m\tilde{A}_n(x')]$
$= \frac{1}{2}[-\partial_\pi^2(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') - \varsigma\varepsilon_j{}^{mn}(\delta_{in} - \frac{\partial_i\partial_n}{\nabla^2})\partial_\pi\partial_m\Delta(x-x') + \varsigma\varepsilon_i{}^{kl}(\delta_{lj} - \frac{\partial_l\partial_j}{\nabla^2})\partial_k\partial_\pi\Delta(x-x')$
$+ \varepsilon_i{}^{kl}(\delta_{ln} - \frac{\partial_l\partial_n}{\nabla^2})\varepsilon_j{}^{mn}\partial_k\partial_m\Delta(x-x')]$
$= \frac{1}{2}[-\partial_\pi^2(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') - 2\varsigma\varepsilon_{ij}{}^k\partial_k\partial_\pi\Delta(x-x') + (\delta_{ij}\delta^{km} - \delta_i^m\delta_j^k)\partial_k\partial_m\Delta(x-x')]$
$= \frac{1}{2}[(-\partial_\pi^2\delta_{ij} - \partial_i\partial_j)\Delta(x-x') - 2\varsigma\varepsilon_{ij}{}^k\partial_k\partial_\pi\Delta(x-x') + (\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x-x')]$
$= [\frac{1}{2}(\nabla^2 - \partial_\pi^2)\delta_{ij} - \varsigma\varepsilon_{ij}{}^k\partial_k\partial_\pi - \partial_i\partial_j]\Delta(x-x')$
$= -\sigma_{ij}^{ab}\partial_a\partial_b\Delta(x-x')$ □

Proof: $[\tilde{A}_i(x), \tilde{A}_j(x')] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\Delta(x-x')$
$\Rightarrow i[\Psi_i(x), \Psi_j(x')]$
$= \frac{i}{2}[-i\partial_\pi \tilde{A}_i(x) + i\varsigma\varepsilon_i{}^{kl}\partial_k \tilde{A}_l(x), -i\partial'_\pi \tilde{A}_j(x') + i\varsigma\varepsilon_j{}^{mn}\partial'_m \tilde{A}_n(x')]$
$= \frac{1}{2}[-\partial_\pi^2(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') + \varsigma\varepsilon_j{}^{mn}(\delta_{in} - \frac{\partial_i\partial_n}{\nabla^2})\partial_\pi\partial_m\Delta(x-x') + \varsigma\varepsilon_i{}^{kl}(\delta_{lj} - \frac{\partial_l\partial_j}{\nabla^2})\partial_k\partial_\pi\Delta(x-x')$
$- \varepsilon_i{}^{kl}(\delta_{ln} - \frac{\partial_l\partial_n}{\nabla^2})\varepsilon_j{}^{mn}\partial_k\partial_m\Delta(x-x')]$
$= \frac{1}{2}[-\partial_\pi^2(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') - (\delta_{ij}\delta^{km} - \delta_i^m\delta_j^k)\partial_k\partial_m\Delta(x-x')]$
$= \frac{1}{2}[(-\partial_\pi^2\delta_{ij} - \partial_i\partial_j)\Delta(x-x') - (\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x-x')]$
$= -\frac{1}{2}\delta_{ij}(\nabla^2 + \partial_\pi^2)\Delta(x-x')$
$= 0$ □

Proof: $[\tilde{A}_i(x), \tilde{A}_j(x')] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\Delta(x-x')$
$\Rightarrow i[\Psi_i^+(x), \Psi_j^+(x')]$
$= \frac{i}{2}[-i\partial_\pi \tilde{A}_i(x) - i\varsigma\varepsilon_i{}^{kl}\partial_k \tilde{A}_l(x), -i\partial'_\pi \tilde{A}_j(x') - i\varsigma\varepsilon_j{}^{mn}\partial'_m \tilde{A}_n(x')]$
$= \frac{1}{2}[-\partial_\pi^2(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') - \varsigma\varepsilon_j{}^{mn}(\delta_{in} - \frac{\partial_i\partial_n}{\nabla^2})\partial_\pi\partial_m\Delta(x-x') - \varsigma\varepsilon_i{}^{kl}(\delta_{lj} - \frac{\partial_l\partial_j}{\nabla^2})\partial_k\partial_\pi\Delta(x-x')$
$- \varepsilon_i{}^{kl}(\delta_{ln} - \frac{\partial_l\partial_n}{\nabla^2})\varepsilon_j{}^{mn}\partial_k\partial_m\Delta(x-x')]$
$= \frac{1}{2}[-\partial_\pi^2(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') - (\delta_{ij}\delta^{km} - \delta_i^m\delta_j^k)\partial_k\partial_m\Delta(x-x')]$
$= \frac{1}{2}[(-\partial_\pi^2\delta_{ij} - \partial_i\partial_j)\Delta(x-x') - (\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x-x')]$
$= -\frac{1}{2}\delta_{ij}(\nabla^2 + \partial_\pi^2)\Delta(x-x')$
$= 0$ □

Cor. 4.2.2. $[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') \Rightarrow [\tilde{A}_i(\vec{r},t), \dot{\tilde{A}}_j(\vec{r}',t)] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta^3(x-x')$

4.3 Energy momentum operator expressed by Ψ in traditional quantization scheme

Energy momentum operator expressed as a basic variable (Ψ_i, Ψ_i^+).

Cor. 4.3.1.

$$\begin{cases} \mathscr{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) = \frac{1}{2}[\Psi^+(\vec{r},t)\Psi(\vec{r},t) + \Psi^T(\vec{r},t)\Psi^*(\vec{r},t)] = \frac{1}{2}\delta^{ij}\{\Psi_i(\vec{r},t), \Psi_j^+(\vec{r},t)\} = \Psi^+(\vec{r},t)\Psi(\vec{r},t) \\ \vec{\mathscr{P}} = \vec{E}\times\vec{B} = -\frac{1}{2}\varsigma[\Psi^+(\vec{r},t)\gamma\Psi(\vec{r},t) - \Psi^T(\vec{r},t)\gamma\Psi^*(\vec{r},t)] = \frac{\varsigma}{2}\gamma^{ij}\{\Psi_i(\vec{r},t), \Psi_j^+(\vec{r},t)\} \\ \vec{\mathscr{M}} = \vec{r}\times(\vec{E}\times\vec{B}) = \frac{1}{2}i\varsigma[\{\Psi_k(\vec{r},t), x^j\Psi_j^+(\vec{r},t)\} - \{x^i\Psi_i(\vec{r},t), \Psi_k^+(\vec{r},t)\}] \end{cases}$$

Proof: $\vec{\mathscr{M}} = \vec{r}\times(\vec{E}\times\vec{B})$
$= \frac{1}{2}i\varsigma\varepsilon_k{}^{lm}x_l\varepsilon_m{}^{ij}\{\Psi_i(\vec{r},t), \Psi_j^+(\vec{r},t)\}$
$= \frac{1}{2}i\varsigma(\delta_k{}^i\delta^{lj} - \delta_k{}^j\delta^{li})x_l\{\Psi_i(\vec{r},t), \Psi_j^+(\vec{r},t)\}$
$= \frac{1}{2}i\varsigma[\{\Psi_k(\vec{r},t), x^j\Psi_j^+(\vec{r},t)\} - \{x^i\Psi_i(\vec{r},t), \Psi_k^+(\vec{r},t)\}]$ □

Proof: $\vec{\mathscr{M}} = \vec{r}\times(\vec{E}\times\vec{B}) = \vec{E}(\vec{r}\cdot\vec{B}) - (\vec{r}\cdot\vec{E})\vec{B}$
$= -i\varsigma\varepsilon_k{}^{lm}x_l\varepsilon_m{}^{ij}\Psi_i^+(\vec{r},t)\Psi_j(\vec{r},t)$
$= -i\varsigma(\delta_k{}^i\delta^{lj} - \delta_k{}^j\delta^{li})x_l\Psi_i^+(\vec{r},t)\Psi_j(\vec{r},t)$
$= -i\varsigma[\Psi_k^+(\vec{r},t)x^j\Psi_j(\vec{r},t) - x^i\Psi_i^+(\vec{r},t)\Psi_k(\vec{r},t)]$ □

Cor. 4.3.2. $\begin{cases} H = \frac{1}{2}\delta^{ij}\int\{\Psi_i(\vec{r},t), \Psi_j^+(\vec{r},t)\}d^3\vec{r} = \int\Psi^+(\vec{r},t)\Psi(\vec{r},t)d^3\vec{r} \\ \vec{P} = \frac{\varsigma}{2}\gamma^{ij}\int\{\Psi_i(\vec{r},t), \Psi_j^+(\vec{r},t)\}d^3\vec{r}, P_a = \frac{\varsigma}{2}(\gamma, -i\varsigma)_a^{ij}\int\{\Psi_i(\vec{r},t), \Psi_j^+(\vec{r},t)\}d^3\vec{r} \end{cases}$

4.4 Motional equation of Ψ operator in traditional quantization schemes

From the operator motional equation with canonical variables $(\tilde{A}_i, E_i)$, the operator motional equation with basic variables (Ψ_i, Ψ_i^+) is derived.

Cor. 4.4.1. $\begin{cases} \dot{\tilde{A}}(\vec{r},t) = -i[\tilde{A}(\vec{r},t), H] \\ \dot{E}(\vec{r},t) = -i[E(\vec{r},t), H] \end{cases} \Rightarrow \begin{cases} \dot{\Psi}(\vec{r},t) = -i[\Psi(\vec{r},t), H] \\ \dot{\Psi}^+(\vec{r},t) = -i[\Psi^+(\vec{r},t), H] \end{cases}$

4.5 Evolution equation and constraint equation of complex field strength Ψ operator

Cor. 4.5.1.

$$\begin{cases} [\Psi_i(\vec{r},t), \Psi_j^+(\vec{r}',t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)] = 0, [\Psi_i^+(\vec{r},t), \Psi_j^+(\vec{r}',t)] = 0 \end{cases} \Rightarrow \begin{cases} [\Psi(\vec{r},t), H] = i\varsigma\gamma^k\partial_k\Psi(\vec{r},t) \\ [\Psi_i(\vec{r},t), P_j] = -i\partial_j\Psi_i(\vec{r},t) + i\delta_{ij}\nabla\cdot\Psi(\vec{r},t) \end{cases}$$

Proof: $[\Psi_i(\vec{r},t), H]$
$= [\Psi_i(\vec{r},t), \frac{1}{2}\delta^{jl}\int\{\Psi_j(\vec{r}',t), \Psi_l^+(\vec{r}',t)\}d^3\vec{r}']$
$= \frac{1}{2}\delta^{jl}\int[\Psi_i(\vec{r},t), \{\Psi_j(\vec{r}',t), \Psi_l^+(\vec{r}',t)\}]d^3\vec{r}'$
$= \frac{1}{2}\delta^{jl}\int\{\Psi_j(\vec{r}',t), [\Psi_i(\vec{r},t), \Psi_l^+(\vec{r}',t)]\} + \{\Psi_l^+(\vec{r}',t), [\Psi_i(\vec{r},t), \Psi_j(\vec{r}',t)]\}d^3\vec{r}'$

$= \frac{1}{2}\delta^{jl}\int[\{\Psi_j(\vec{r}',t), i\varsigma\gamma^k{}_{il}\partial_k\delta^3(\vec{r}-\vec{r}')\}+0]d^3\vec{r}'$
$= i\varsigma\gamma^k{}_i{}^j\partial_k\Psi_j(\vec{r},t)$
$\succ i\varsigma\gamma^k\partial_k\Psi(\vec{r},t)$ □

Proof: $[\Psi_i(\vec{r},t),\vec{P}]$
$= [\Psi_i(\vec{r},t), \frac{\varsigma}{2}\gamma^{jl}\int\{\Psi_j(\vec{r}',t),\Psi_l^+(\vec{r}',t)\}d^3\vec{r}']$
$= \frac{\varsigma}{2}\gamma^{jl}\int[\Psi_i(\vec{r},t),\{\Psi_j(\vec{r}',t),\Psi_l^+(\vec{r}',t)\}]d^3\vec{r}'$
$= \frac{\varsigma}{2}\gamma^{jl}\int\{\Psi_j(\vec{r}',t),[\Psi_i(\vec{r},t),\Psi_l^+(\vec{r}',t)]\}+\{\Psi_l^+(\vec{r}',t),[\Psi_i(\vec{r},t),\Psi_j(\vec{r}',t)]\}d^3\vec{r}'$
$= \frac{\varsigma}{2}\gamma^{jl}\int[\{\Psi_j(\vec{r}',t), i\varsigma\gamma^k{}_{il}\partial_k\delta^3(\vec{r}-\vec{r}')\}+0]d^3\vec{r}'$
$= i\gamma^{jl}\gamma^k{}_{il}\partial_k\Psi_j(\vec{r},t) \succ -i(\gamma\cdot\nabla)\gamma\psi(\vec{r},t)$
$\prec -i(\delta_n{}^k\delta^j{}_i - \delta_{ni}\delta^{jk})\partial_k\Psi_j(\vec{r},t)$
$= -i\partial_n\Psi_i(\vec{r},t) + i\delta_{ni}\nabla\cdot\Psi(\vec{r},t)$ □

Cor. 4.5.2. $[\psi_i(\vec{r},t),P_j] = -i\partial_j\psi_i(\vec{r},t) + iS_m^+(1)_{ij}\nabla\cdot[S_m(1)\psi(\vec{r},t)]$

Cor. 4.5.3. *From this, the coefficients of the energy momentum operator can be uniquely determined.*

$$\begin{cases}(\gamma,-i\varsigma)^a\partial_a\Psi(\vec{r},t)=0\\ \nabla\cdot\Psi(\vec{r},t)=0\end{cases} \Leftrightarrow \begin{cases}\dot{\Psi}(\vec{r},t) = -i[\Psi(\vec{r},t),H]\\ \partial_i\Psi(\vec{r},t) = i[\Psi(\vec{r},t),P_i]\\ [P_a,\Psi(\vec{r},t)] = i\partial_a\Psi(\vec{r},t)\end{cases}; \begin{cases}[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t),\Psi_j(\vec{r}',t)]=0\\ [\Psi_i^+(\vec{r},t),\Psi_j^+(\vec{r}',t)]=0\end{cases}$$

Cor. 4.5.4.

$$[\partial_a + iS_{ab}(\gamma,\varsigma)\partial^b]\Psi = 0 \Leftrightarrow [P_a,\Psi(\vec{r},t)] = i\partial_a\Psi(\vec{r},t); \begin{cases}[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t),\Psi_j(\vec{r}',t)]=0, [\Psi_i^+(\vec{r},t),\Psi_j^+(\vec{r}',t)]=0\end{cases}$$

Cor. 4.5.5. *Electromagnetic field constraints and commutative relations are self consistent.*

$$\begin{cases}[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t),\Psi_j(\vec{r}',t)]=0, [\Psi_i^+(\vec{r},t),\Psi_j^+(\vec{r}',t)]=0\end{cases} \Rightarrow \begin{cases}\nabla\cdot\Psi(\vec{r},t)=0\\ \nabla\cdot\Psi^+(\vec{r},t)=0\end{cases}$$

Cor. 4.5.6. $[P_a,\Psi(\vec{r},t)] = S_{ab}(\gamma,\varsigma)\partial^b\Psi(\vec{r},t)$

4.6 Quantum scalar product equation of complex field strength Ψ operator (Is it the most basic?)

Def. 4.6.1. $\langle\eta|\dot{\Psi}(\vec{r},t) + i[\Psi(\vec{r},t),H]|\varphi\rangle = 0, \langle\eta|\nabla\cdot\Psi(\vec{r},t)|\varphi\rangle = 0$

Def. 4.6.2. $\langle\eta|\partial_a\Psi(\vec{r},t) - i[\Psi(\vec{r},t),P_a]|\varphi\rangle = 0 \Leftrightarrow \langle\eta|[P_a,\Psi(\vec{r},t)] - i\partial_a\Psi(\vec{r},t)|\varphi\rangle = 0$

It has two solutions, one determined by operator equation $\dot{\Psi}(\vec{r},t) = -i[\Psi(\vec{r},t),H], \nabla\cdot\Psi(\vec{r},t)| = 0$. Another solution is their vacuum states for all physical states: $\langle\eta|\Psi(\vec{r},t)|\varphi\rangle = 0$. So it is a complete Fourier expansion solution.

4.7 Fock representation of complex field strength energy momentum operator

Energy momentum operator:

Cor. 4.7.1.
$$\begin{cases}H = \frac{1}{2}\delta^{ij}\int\{\Psi_i(\vec{r},t),\Psi_j^+(\vec{r},t)\}d^3\vec{r} = \frac{1}{2}\int\limits_{\vec{p}\neq 0}|\vec{p}|[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2(\vec{p},-\varsigma),a_2^+(\vec{p},-\varsigma)\}]d^3\vec{p}\\ \vec{P} = \frac{\varsigma}{2}\gamma^{ij}\int\{\Psi_i(\vec{r},t),\Psi_j^+(\vec{r},t)\}d^3\vec{r} = \frac{1}{2}\int\limits_{\vec{p}\neq 0}\vec{p}[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2(\vec{p},-\varsigma),a_2^+(\vec{p},-\varsigma)\}]d^3\vec{p}\end{cases}$$

Proof:
$H = \frac{1}{2}\delta^{ij}\int\{\Psi_i(\vec{r},t),\Psi_j^+(\vec{r},t)\}d^3\vec{r}$
$= \frac{1}{(2\pi)^3}\frac{1}{2}\delta^{ij}\int\limits_{\vec{p},\vec{p}'\neq 0} d^3\vec{p}d^3\vec{p}'d^3\vec{r}\lambda_{mi}(\vec{p},-\varsigma)\lambda^+_{mj}(\vec{p}',-\varsigma)$
$\{\sqrt{|\vec{p}|}[a_1(\vec{p},-\varsigma)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + a_2^+(\vec{p},-\varsigma)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}], \sqrt{|\vec{p}'|}[a_1^+(\vec{p}',-\varsigma)e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)} + a_2(\vec{p}',-\varsigma)e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]\}$
$= \frac{1}{2}\int\limits_{\vec{p},\vec{p}'\neq 0}\lambda_m^+(\vec{p},-\varsigma)\lambda_m(\vec{p},-\varsigma)\delta^3(\vec{p}-\vec{p}')|\vec{p}|[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2^+(\vec{p},-\varsigma),a_2(\vec{p},-\varsigma)\}]+$
$\lambda_m^+(-\vec{p},-\varsigma)\lambda_m(\vec{p},-\varsigma)\delta^3(\vec{p}+\vec{p}')|\vec{p}|[\{a_1(\vec{p},-\varsigma),d(-\vec{p},-\varsigma)\}e^{-2i\varsigma Et}+\{a_2^+(\vec{p},-\varsigma),c^+(-\vec{p},-\varsigma)\}e^{2i\varsigma Et}]d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{2}\int\limits_{\vec{p}\neq 0}|\vec{p}|[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2^+(\vec{p},-\varsigma),a_2(\vec{p},-\varsigma)\}+0]d^3\vec{p}$
$= \frac{1}{2}\int\limits_{\vec{p}\neq 0}|\vec{p}|[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2(\vec{p},-\varsigma),a_2^+(\vec{p},-\varsigma)\}]d^3\vec{p}$ □

Proof:
$P_k = \frac{\varsigma}{2}\gamma_k{}^{ij}\int\{\Psi_i(\vec{r},t),\Psi_j^+(\vec{r},t)\}d^3\vec{r}$
$= \frac{1}{(2\pi)^3}\frac{\varsigma}{2}\gamma_k{}^{ij}\int\limits_{\vec{p},\vec{p}'\neq 0} d^3\vec{p}d^3\vec{p}'d^3\vec{r}\lambda_{mi}(\vec{p},-\varsigma)\lambda^+_{mj}(\vec{p}',-\varsigma)$

$\{\{\sqrt{|\vec{p}|}[a_1(\vec{p},-\varsigma)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+a_2^+(\vec{p},-\varsigma)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}],\{\sqrt{|\vec{p}'|}[a_1^+(\vec{p}',-\varsigma)e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}+a_2(\vec{p}',-\varsigma)e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]\}$
$=-\frac{\varsigma}{2}\int\limits_{\vec{p},\vec{p}'\neq 0}\lambda_m^+(\vec{p},-\varsigma)\gamma_k\lambda_m(\vec{p},-\varsigma)|\vec{p}|\delta^3(\vec{p}-\vec{p}')[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2^+(\vec{p},-\varsigma),a_2(\vec{p},-\varsigma)\}]+$
$\lambda_m^+(-\vec{p},-\varsigma)\gamma_k\lambda_m(\vec{p},-\varsigma)|\vec{p}|\delta^3(\vec{p}+\vec{p}')[\{a_1(\vec{p},-\varsigma),d(-\vec{p},-\varsigma)\}e^{-2i\varsigma Et}+\{a_2^+(\vec{p},-\varsigma),c^+(-\vec{p},-\varsigma)\}e^{2i\varsigma Et}]d^3\vec{p}d^3\vec{p}'$
$=-\frac{\varsigma}{2}\int\limits_{\vec{p}\neq 0}-\varsigma\vec{p}_k[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2^+(\vec{p},-\varsigma),a_2(\vec{p},-\varsigma)\}]+0d^3\vec{p}$
$=\frac{1}{2}\int\limits_{\vec{p}\neq 0}\vec{p}_k[\{a_1(\vec{p},-\varsigma),a_1^+(\vec{p},-\varsigma)\}+\{a_2(\vec{p},-\varsigma),a_2^+(\vec{p},-\varsigma)\}]d^3\vec{p}$ □

4.8 Fock commutative relation of complex field strength Ψ operator

Cor. 4.8.1.

$$\begin{cases}[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]=i\varsigma{\gamma^k}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')\\ [\Psi_i(\vec{r},t),\Psi_j(\vec{r}',t)]=0\\ [\Psi_i^+(\vec{r},t),\Psi_j^+(\vec{r}',t)]=0\end{cases}\Leftrightarrow\begin{cases}[a_1(\vec{p},-\varsigma),a_1^+(\vec{p}',-\varsigma)]=\varsigma\delta^3(\vec{p}-\vec{p}')\\ [a_2(\vec{p},-\varsigma),a_2^+(\vec{p}',-\varsigma)]=\varsigma\delta^3(\vec{p}-\vec{p}')\\ [a_1(\vec{p},-\varsigma),a_1(\vec{p}',-\varsigma)]=0,[a_2(\vec{p},-\varsigma),a_2(\vec{p}',-\varsigma)]=0\\ [a_1(\vec{p},-\varsigma),a_2(\vec{p}',-\varsigma)]=0,[a_1(\vec{p},-\varsigma),a_2^+(\vec{p}',-\varsigma)]=0\end{cases}$$

Cor. 4.8.2. $[a_1(\vec{p},-\varsigma),a_1^+(\vec{p}',-\varsigma)]=\varsigma\delta^3(\vec{p}-\vec{p}')$

Proof: $[a_1(\vec{p},-\varsigma),a_1^+(\vec{p}',-\varsigma)]$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int[\lambda_m^{+i}(\vec{p},-\varsigma)\Psi_i(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)},\lambda_m^j(\vec{p}',-\varsigma)\Psi_j^+(\vec{r}',t)e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p}',-\varsigma)[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p}',-\varsigma){\gamma^k}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p}',-\varsigma){\gamma^k}_{ij}i\varsigma p_k'e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}d^3\vec{r}$
$=-\frac{1}{|\vec{p}|}\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p},h'){\gamma^k}_{ij}p_k\delta^3(\vec{p}-\vec{p}')$
$=-\lambda_m^+(\vec{p},-\varsigma)\frac{\gamma^k p_k}{|\vec{p}|}\lambda_m(\vec{p},-\varsigma)\delta^3(\vec{p}-\vec{p}')$
$=\varsigma\lambda_m^+(\vec{p},-\varsigma)\lambda_m(\vec{p},-\varsigma)\delta^3(\vec{p}-\vec{p}')$
$=\varsigma\delta^3(\vec{p}-\vec{p}')$ □

Cor. 4.8.3. $[a_2^+(\vec{p},-\varsigma),a_2(\vec{p}',-\varsigma)]=-\varsigma\delta^3(\vec{p}-\vec{p}')$

Proof: $[a_2^+(\vec{p},-\varsigma),a_2(\vec{p}',-\varsigma)]$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int[\lambda_m^{+i}(\vec{p},-\varsigma)\Psi_i(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)},\lambda_m^j(\vec{p}',-\varsigma)\Psi_j^+(\vec{r}',t)e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p}',-\varsigma)[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p}',-\varsigma){\gamma^k}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p}',-\varsigma){\gamma^k}_{ij}(-i\varsigma p_k')e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}d^3\vec{r}$
$=\frac{1}{|\vec{p}|}\lambda_m^{+i}(\vec{p},-\varsigma)\lambda_m^j(\vec{p},h'){\gamma^k}_{ij}p_k\delta^3(\vec{p}-\vec{p}')$
$=\lambda_m^+(\vec{p},-\varsigma)\frac{\gamma^k p_k}{|\vec{p}|}\lambda_m(\vec{p},h')\delta^3(\vec{p}-\vec{p}')$
$=-\varsigma\lambda_m^+(\vec{p},-\varsigma)\lambda_m(\vec{p},h')\delta^3(\vec{p}-\vec{p}')$
$=-\varsigma\delta^3(\vec{p}-\vec{p}')$ □

Cor. 4.8.4. $[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]=i\varsigma{\gamma^k}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')$

Proof: $[\Psi_i(\vec{r},t),\Psi_j^+(\vec{r}',t)]$
$=\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}d^3\vec{p}d^3\vec{p}'\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p}',-\varsigma)\sqrt{|\vec{p}||\vec{p}'|}$
$\{[a_1(\vec{p},-\varsigma),a_1^+(\vec{p}',-\varsigma)]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}+[a_2^+(\vec{p},-\varsigma),a_2(\vec{p}',-\varsigma)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}\}$
$=\frac{1}{(2\pi)^3}\int\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p}',-\varsigma)[\varsigma\delta^3(\vec{p}-\vec{p}')e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}-\varsigma\delta^3(\vec{p}-\vec{p}')e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]\}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)\varsigma|\vec{p}|[e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]\}d^3\vec{p}$
$=-\frac{1}{(2\pi)^3}\int[(-\varsigma|\vec{p}|)\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)+(\varsigma|\vec{p}|)\lambda_{mi}(-\vec{p},-\varsigma)\lambda_{mj}^+(-\vec{p},-\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$
$=-\frac{1}{(2\pi)^3}\int[(-\varsigma|\vec{p}|)\lambda_{mi}(\vec{p},-\varsigma)\lambda_{mj}^+(\vec{p},-\varsigma)+(0|\vec{p}|)\lambda_{mi}(\vec{p},\varsigma)\lambda_{mj}^+(\vec{p},\varsigma)+(\varsigma|\vec{p}|)\lambda_{mi}(\vec{p},\varsigma)\lambda_{mj}^+(\vec{p},\varsigma)]e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{p}$
$=-\frac{1}{(2\pi)^3}\int{\gamma^k}_i{}^l p_k\sum_h\lambda_{ml}(\vec{p},h)\lambda_{mj}^+(\vec{p},h)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int i\varsigma{\gamma^k}_i{}^l i\varsigma p_k\delta_{lj}e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}$
$=i\varsigma{\gamma^k}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}')$ □

4.9 Fock representation of normalized energy momentum operator

Cor. 4.9.1. $: H := \int \sum_\sigma |\vec{p}| a^+_\sigma(\vec{p}) a_\sigma(\vec{p}) d^3\vec{p}, : \vec{P} := \int \sum_\sigma \vec{p} a^+_\sigma(\vec{p}) a_\sigma(\vec{p}) d^3\vec{p}, : P_a := \int \sum_\sigma p_a a^+_\sigma(\vec{p}) a_\sigma(\vec{p}) d^3\vec{p}$

Cor. 4.9.2. $a_\sigma(\vec{p}, \varsigma)|\varphi\rangle = 0, a_\sigma(\vec{p}, 0)|\varphi\rangle = 0, \forall\varphi \in Phys$

5 Field scheme of quantum electrodynamics

5.1 Field representation scheme for electromagnetic interaction

Thm. 5.1.1.

$$\begin{cases}[\Psi_{\alpha_\varsigma}(x), \Psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x') \\ [\Psi_{\alpha_\varsigma}(x), \Psi_{\beta_\varsigma}(x')] = 0, [\Psi^+_{\alpha'_\varsigma}(x), \Psi^+_{\beta'_\varsigma}(x')] = 0 \\ [\partial_a + iS_{ab}(\gamma,\varsigma)\partial^b]\Psi = -i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b \\ \tilde{A} = \frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2}, \tilde{\phi} = -\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}\end{cases} \Leftrightarrow \begin{cases}[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = 0, [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \nabla^2\tilde{A} - \partial_t^2\tilde{A} = \vec{J} + \partial_t\nabla\tilde{\phi}, \nabla^2\tilde{\phi} = \rho, \nabla\cdot\tilde{A} = 0 \\ \sqrt{2}\Psi = -\partial_t\tilde{A} - \nabla\tilde{\phi} - i\varsigma\nabla\times\tilde{A}\end{cases}$$

Cor. 5.1.1. $L = -\int \bar{\psi}(\gamma^a\partial_a + m)\psi dr^3, H = \int \bar{\psi}(\gamma\cdot\nabla + m)\psi dr^3$

Cor. 5.1.2. $L = -\int \bar{\psi}[\gamma^a(\partial_a - ieA_a) + m]\psi dr^3 = -\int \bar{\psi}(\gamma^a\partial_a + m)\psi dr^3 + \int ie\bar{\psi}\gamma^a A_a\psi dr^3$

Cor. 5.1.3. $H = \int \bar{\psi}[\gamma\cdot(\nabla - ie\tilde{A}) + \gamma^4 e\tilde{\phi} + m]\psi dr^3 = \int \bar{\psi}(\gamma\cdot\nabla + m)\psi dr^3 - \int ie\bar{\psi}\gamma^a A_a\psi dr^3$

Cor. 5.1.4. $H_i = -L_i = -\int ie\bar{\psi}\gamma^a A_a\psi dr^3$
$= -\frac{e\varsigma}{\sqrt{2}}\int \bar{\psi}\frac{\gamma\cdot[\nabla\times(\Psi-\Psi^*)]}{\nabla^2}\psi dr^3 - \frac{e}{\sqrt{2}}\int \bar{\psi}\frac{\gamma^4[\nabla\cdot(\Psi+\Psi^*)]}{\nabla^2}\psi dr^3 + \int ie\bar{\psi}\frac{\gamma^a\partial_a\partial_t\phi}{\nabla^2}\psi dr^3$

Thm. 5.1.2. $H_i = -L_i = -\int ie\bar{\psi}\gamma^a A_a\psi dr^3$
$= -ie\int \frac{\bar{\psi}}{\sqrt{-\nabla^2}}(\vec{\gamma}\cdot\partial_t\vec{E})\frac{\psi}{\sqrt{-\nabla^2}}dr^3 + e^2\int \frac{\bar{\psi}\gamma^a\psi}{\sqrt{-\nabla^2}}\frac{\bar{\psi}\gamma_a\psi}{\sqrt{-\nabla^2}}dr^3 + \int \partial_a[ie\bar{\psi}\gamma^a\psi(\frac{\partial_t\phi}{\nabla^2})]dr^3$

Proof: $H_i = -L_i = -\int ie\bar{\psi}\gamma^a A_a\psi dr^3$
$= -\frac{e\varsigma}{\sqrt{2}}\int \bar{\psi}\frac{\vec{\gamma}\cdot[\nabla\times(\Psi-\Psi^*)]}{\nabla^2}\psi dr^3 - \frac{e}{\sqrt{2}}\int \bar{\psi}\frac{\gamma^4[\nabla\cdot(\Psi+\Psi^*)]}{\nabla^2}\psi dr^3 + \int ie\bar{\psi}\frac{\gamma^a\partial_a\partial_t\phi}{\nabla^2}\psi dr^3$
$= \frac{ie}{\sqrt{2}}\int \bar{\psi}\frac{\vec{\gamma}\cdot[\partial_t(\Psi+\Psi^*) - \sqrt{2}\vec{J}]}{\nabla^2}\psi dr^3 + e\int \bar{\psi}\frac{\gamma^4\rho}{\nabla^2}\psi dr^3 + \int ie\bar{\psi}\frac{\gamma^a\partial_a\partial_t\phi}{\nabla^2}\psi dr^3$
$= \frac{ie}{\sqrt{2}}\int \bar{\psi}\frac{\vec{\gamma}\cdot[\partial_t(\Psi+\Psi^*)]}{\nabla^2}\psi dr^3 - ie\int \bar{\psi}\gamma^a\psi(\frac{J_a}{\nabla^2})dr^3 + \int ie\bar{\psi}\gamma^a\psi\partial_a(\frac{\partial_t\phi}{\nabla^2})dr^3$
$= \frac{ie}{\sqrt{2}}\int \bar{\psi}\frac{\vec{\gamma}\cdot[\partial_t(\Psi+\Psi^*)]}{\nabla^2}\psi dr^3 + \int J^a\frac{1}{\nabla^2}J_a dr^3 - \int \partial_a[J^a(\frac{\partial_t\phi}{\nabla^2})]dr^3$
$= \frac{ie}{\sqrt{2}}\int \bar{\psi}\frac{\vec{\gamma}\cdot[\partial_t(\Psi+\Psi^*)]}{\nabla^2}\psi dr^3 - \int \frac{J^a}{\sqrt{-\nabla^2}}\frac{J_a}{\sqrt{-\nabla^2}}dr^3 - \int \partial_a[J^a(\frac{\partial_t\phi}{\nabla^2})]dr^3$
$= \frac{ie}{\sqrt{2}}\int \bar{\psi}\frac{\vec{\gamma}\cdot[\partial_t(\Psi+\Psi^*)]}{\nabla^2}\psi dr^3 + e^2\int \frac{\bar{\psi}\gamma^a\psi}{\sqrt{-\nabla^2}}\frac{\bar{\psi}\gamma_a\psi}{\sqrt{-\nabla^2}}dr^3 + \int \partial_a[ie\bar{\psi}\gamma^a\psi(\frac{\partial_t\phi}{\nabla^2})]dr^3$
$= ie\int \bar{\psi}\frac{\partial_t(\vec{\gamma}\cdot\vec{E})}{\nabla^2}\psi dr^3 + e^2\int \frac{\bar{\psi}\gamma^a\psi}{\sqrt{-\nabla^2}}\frac{\bar{\psi}\gamma_a\psi}{\sqrt{-\nabla^2}}dr^3 + \int \partial_a[ie\bar{\psi}\gamma^a\psi(\frac{\partial_t\phi}{\nabla^2})]dr^3$
$= -ie\int \frac{\bar{\psi}}{\sqrt{-\nabla^2}}(\vec{\gamma}\cdot\partial_t\vec{E})\frac{\bar{\psi}}{\sqrt{-\nabla^2}}dr^3 + e^2\int \frac{\bar{\psi}\gamma^a\psi}{\sqrt{-\nabla^2}}\frac{\bar{\psi}\gamma_a\psi}{\sqrt{-\nabla^2}}dr^3 + \int \partial_a[ie\bar{\psi}\gamma^a\psi(\frac{\partial_t\phi}{\nabla^2})]dr^3$ □

The above theorem unifies covariance and non covariance, radiation gauge and Lorentz gauge, which is very beautiful. In particular, the second term clearly describes the repulsive self interaction energy between electrons, which is greater than zero. At the same time, it is completely possible to perform perturbation expansion based on field quantities without using electromagnetic potential expansion. In this way, the entire deployment process is physical. There are no non physical factors and independent of guages. Can this avoid infinity? No need for renormalization? Further exploration is needed. The third item is a full differential term, which can be removed (?). This item is very delicate and beautiful, and it is the guarantee of Lorentz covariance and the key to unified description.

5.2 S-matrix of field representation for electromagnetic interaction

Cor. 5.2.1. $U(t, t_0) = 1 - i\int_{t_0}^{t} H_i(t_1)U(t_1, t_0)dt_1, S = U(+\infty, -\infty) = Texp\{-i\int_{-\infty}^{+\infty} H_i(t)dt\}$

Cor. 5.2.2. $S = U(+\infty, -\infty) = Texp\{-i\int_{-\infty}^{+\infty} H_i(t)dt\}$
$= Texp\{-i\int_{-\infty}^{+\infty}[-ie\frac{\bar{\psi}}{\sqrt{-\nabla^2}}(\vec{\gamma}\cdot\partial_t\vec{E})\frac{\psi}{\sqrt{-\nabla^2}} + e^2\frac{\bar{\psi}\gamma^a\psi}{\sqrt{-\nabla^2}}\frac{\bar{\psi}\gamma_a\psi}{\sqrt{-\nabla^2}}]dx^4\}$

Chapter22 Covariant Quantization Scheme for Massless Particles

1 New covariant quantization program

1.1 New quantization program

1. Firstly, based on constant invariant tensor analysis, we can reasonably guess the covariant commutation rule.
2. According to the principle of micro causality and the elimination of abnormal particles with negative modes and negative probabilities, a reasonable covariant commutation rule is further determined.
3. According to the obtained covariant commutation rule, the commutation rule of the Fock representation is further obtained.
4. According to the general Fogg representation of energy and momentum in quantum field theory, the energy and momentum operators are inversely deduced, and whether they are true energy and momentum is verified. The spin representation and angular momentum representation are further determined.
5. According to the energy operator, the quantum operator equation in the same form as the classical equation is obtained again, and we will verify whether the quantum Poincare algebra is tenable.
6. Consider the interaction, calculate the scattering matrix, and compare with the experiment.
7. Extend to higher dimensional space and extend to string theory.
8. How to replace potential propagator with field propagator.
9. Think of string theory as potential theory, and what is its corresponding field theory?
10. Can we solve the infinite problem?
11. What is the difference between classical plane wave solutions and quantum plane wave solutions? Is there a significant difference between mode nonexcitation and mode nonexcitation, which may imply a major discovery in physics?

2 Covariant quantization scheme for massless complex scalar field [25, 26, 37, 38]

2.1 complex scalar field And its plane wave solutions

Def. 2.1.1. $\partial_a\partial^a\psi(\vec{r},t)=0$

Cor. 2.1.1. $\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}|\vec{p}|^{-\frac{1}{2}}\lambda(\hat{p},0)[a_1(\vec{p},0)e^{ip\cdot x}+a_2^+(\vec{p},0)e^{-ip\cdot x}]d^3\vec{p}$

$$\Leftrightarrow\begin{cases}|\vec{p}|^{-\frac{1}{2}}a_1(\vec{p},0)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}\lambda^+(\hat{p},0)[\phi(\vec{r},t)+\frac{i}{|\vec{p}|}\dot{\phi}(\vec{r},t)]e^{-ip\cdot x}d^3\vec{r}\\ |\vec{p}|^{-\frac{1}{2}}a_2^+(\vec{p},0)=\frac{1}{(2\pi)^{3/2}}\int\limits_{p=-\infty}^{+\infty}\lambda^+(\hat{p},0)[\phi(\vec{r},t)-\frac{i}{|\vec{p}|}\dot{\phi}(\vec{r},t)]e^{ip\cdot x}d^3\vec{r}\\ \lambda(\hat{p},0):=\frac{1}{\sqrt{2}},\Gamma(0):=\lambda(\hat{p},0)\lambda^+(\hat{p},0)=\frac{1}{2},\lambda^+(\hat{p},0)\lambda(\hat{p},0)=\frac{1}{2}\end{cases}$$

Def. 2.1.2. *Define projection operator:* $\hat{P}(0):=2\lambda(\hat{p},0)\lambda^+(\hat{p},0)=1$

2.2 Properties of covariant constant invariant tensor $\Gamma(0)$ for complex scalar field

Def. 2.2.1. $\lambda(\hat{p},0):=\frac{1}{\sqrt{2}},\Gamma(0):=\lambda(\hat{p},0)\lambda^+(\hat{p},0)=\frac{1}{2},\lambda^+(\hat{p},0)\lambda(\hat{p},0)=\frac{1}{2}$

2.3 General covariant commutation rules in mathematics for complex scalar field

Thm. 2.3.1.

$$\begin{cases}[a_\sigma(\vec{p},0),a_{\sigma'}^+(\vec{p}',0)]_\pm=\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]_\pm=0\\ [a_\sigma^+(\vec{p},0),a_{\sigma'}^+(\vec{p}',0)]_\pm=0\end{cases}\Rightarrow\begin{cases}[\psi(x),\psi^+(x')]_\pm\\ =i2\Gamma(0)[(\delta_1\pm\delta_2)\Delta^{(+)}(x-x')-\pm\delta_2\Delta(x-x')]\\ [\psi(x),\psi(x')]_\pm=0\\ [\psi^+(x),\psi^+(x')]_\pm=0\end{cases}$$

Proof: $[\psi^{(+)}(x),\psi^{(+)+}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|^{-\frac{1}{2}}|\vec{p}'|^{-\frac{1}{2}}\lambda(\hat{p},0)\lambda^+(\hat{p}',0)[a_1(\vec{p},0),a_1^+(\vec{p}',0)]_\pm e^{i\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|^{-1}\delta_1\lambda(\hat{p},0)\lambda^+(\hat{p},0)\delta^3(\vec{p}-\vec{p}')e^{i\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$

$= \frac{1}{(2\pi)^3}\int |\vec{p}|^{-1}\delta_1\lambda(\hat{p},0)\lambda^+(\hat{p},0)e^{i\vec{p}\cdot(x-x')}d^3\vec{p}$
$= i\frac{1}{(2\pi)^3}\int \delta_1\frac{-i}{2|\vec{p}|}2\Gamma(0)e^{ip\cdot(x-x')}d^3\vec{p}$
$= i\delta_1 2\Gamma(0)\Delta^{(+)}(x-x')$ □

Proof: $[\psi^{(-)}(x),\psi^{(-)+}(x')]_\pm$
$= \frac{1}{(2\pi)^3}\int |\vec{p}|^{-\frac{1}{2}}|\vec{p}'|^{-\frac{1}{2}}\lambda(\hat{p},0)\lambda^+(\hat{p},0)[a_2^+(\vec{p},0),a_2(\vec{p}',0)]_\pm e^{-i\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int |\vec{p}|^{-1}\delta_2\lambda(\hat{p},0)\lambda^+(\hat{p},0)\delta^3(\vec{p}-\vec{p}')e^{-i\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int |\vec{p}|^{-1}\delta_2\lambda(\hat{p},0)\lambda^+(\hat{p},0)e^{-i\vec{p}\cdot(x-x')}d^3\vec{p}$
$= -\pm i\frac{1}{(2\pi)^3}\int \delta_2\frac{i}{2|\vec{p}|}2\Gamma(0)e^{-ip\cdot(x-x')}d^3\vec{p}$
$= -\pm i\delta_2 2\Gamma(0)\Delta^{(-)}(x-x')$ □

Proof: $[\psi(x),\psi^+(x')]_\pm$
$= [\psi^{(+)}(x),\psi^{(+)+}(x')]_\pm + [\psi^{(-)}(x),\psi^{(-)+}(x')]_\pm$
$= i\delta_1 2\Gamma(0)\Delta^{(+)}(x-x') - \pm i\delta_2 2\Gamma(0)\Delta^{(-)}(x-x')$
$= i2\Gamma(0)[\delta_1\Delta^{(+)}(x-x') - \pm\delta_2\Delta^{(-)}(x-x')]$
$= i2\Gamma(0)[(\delta_1\pm\delta_2)\Delta^{(+)}(x-x') - \pm\delta_2\Delta(x-x')]$ □

From the above, only $\delta_1\pm\delta_2=0$, the micro causality is satisfied. At the same time only when $\delta_1,\delta_2\geq 0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1=\delta_2=1$, it satisfies the commutative relation. There are actually two other options. Namely when $\delta_1=\delta_2=0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

2.4 Covariant commutation rules for complex scalar field physics

Thm. 2.4.1.
$$\begin{cases}[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]=0\\ [a^+_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=0\end{cases}\Rightarrow\begin{cases}[\psi(x),\psi^+(x')]=i2\Gamma(0)\Delta(x-x')\\ [\psi(x),\psi(x')]=0\\ [\psi^+(x),\psi^+(x')]=0\end{cases}$$

Cor. 2.4.1.
$$\begin{cases}[a_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},0),a_{\sigma'}(\vec{p}',0)]=0\\ [a^+_\sigma(\vec{p},0),a^+_{\sigma'}(\vec{p}',0)]=0\end{cases}\Rightarrow\begin{cases}[\psi^{(+)}(x),\psi^{(+)+}(x')]=i2\Gamma(0)\Delta^{(+)}(x-x')\\ [\psi^{(-)}(x),\psi^{(-)+}(x')]=i2\Gamma(0)\Delta^{(-)}(x-x')\\ [\psi^{(+)}(x),\psi^{(-)+}(x')]=0\end{cases}$$

Cor. 2.4.2.
$$\begin{cases}[\psi(x),\psi^+(x')]=i2\Gamma(0)\Delta(x-x')\\ [\psi(x),\psi(x')]=0\\ [\psi^+(x),\psi^+(x')]=0\end{cases}\Rightarrow\begin{cases}[\psi(\vec{r},t),\psi^+(\vec{r}',t)]=0\\ [\psi(\vec{r},t),\psi(\vec{r}',t)]=0\\ [\psi^+(x),\psi^+(\vec{r}',t)]=0\end{cases}$$

2.5 Commutation function, causality function and Feynman propagator of complex scalar field

Def. 2.5.1.
$$\begin{cases}\Delta^{(+)}(x):=\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}e^{ip\cdot x}d^3\vec{p}, i\Delta^{(+)}(\vec{r},0)\leftrightarrow\frac{1}{2|\vec{p}|}\\ \Delta^{(-)}(x):=-\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}e^{-ip\cdot x}d^3\vec{p},\Delta^{(-)}(x)=-\Delta^{(+)}(-x)\\ \Delta(x):=\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}[e^{ip\cdot x}-e^{-ip\cdot x}\}d^3\vec{p},\Delta(x)=\Delta^{(+)}(x)+\Delta^{(-)}(x)\end{cases}$$

Pro. 2.5.1.
$$\begin{cases}\Delta^*(x)=\Delta(x),\Delta(-x)=-\Delta(x),(\nabla^2-\partial_t^2)\Delta(x)=0\\ \partial_t\Delta(x)|_{t=0}=-\delta^3(\vec{r}),\partial_k\partial_t\Delta(x)|_{t=0}=\partial_t\partial_k\Delta(x)|_{t=0}=-\partial_k\delta^3(\vec{r})\\ \partial_k\Delta(x)|_{t=0}=0,\partial_k\partial_l\Delta(x)|_{t=0}=0,\partial_t^2\Delta(x)|_{t=0}=0\end{cases}$$

Pro. 2.5.2. $\Delta(x-x'):=\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}\}d^3\vec{p}$
$$\begin{cases}\partial_u\Delta(x-x')=-\partial'_u\Delta(x-x')\\ \nabla\Delta(x-x')=-\nabla'\Delta(x-x')\\ \partial_\pi\Delta(x-x')=-\partial'_\pi\Delta(x-x')\end{cases}\quad\begin{cases}(\sqrt{-\nabla^2})^n\Delta(x-x')=(\sqrt{-\nabla'^2})^n\Delta(x-x')\\ \frac{1}{(\sqrt{-\nabla^2})^n}\Delta(x-x')=\frac{1}{(\sqrt{-\nabla'^2})^n}\Delta(x-x')\\ \partial_\pi^{2n}\Delta(x-x')=\partial'^{2n}_\pi\Delta(x-x')\end{cases}$$

Def. 2.5.2.
$$\begin{cases}\Delta(x)=\Delta^{(+)}(x)+\Delta^{(-)}(x)=\Delta^{ret}(x)-\Delta^{adv}(x)\\ \Delta^{(l)}(x)=i[\Delta^{(-)}(x)-\Delta^{(+)}(x)]\\ \Delta_F(x)=\langle T\varphi(x)\varphi(x')\rangle_0=i\Delta^{(c)}(x)=\frac{1}{(2\pi)^4}\int\Delta_F(p)e^{ipx}d^4p\\ \Delta_F(p)=\frac{-i}{p^2-i\varepsilon}\end{cases}$$

$$\begin{cases}\Delta^{(c)}(x)=\theta(t)\Delta^{(+)}(x)-\theta(-t)\Delta^{(-)}(x)\\ \Delta^{ret}(x)=\theta(t)\Delta(x)=\Delta^{(c)}(x)+\Delta^{(-)}(x)\\ \Delta^{adv}(x)=-\theta(-t)\Delta(x)=\Delta^{(c)}(x)-\Delta^{(+)}(x)\end{cases}$$

Cor. 2.5.1.

$$\begin{cases}\partial_a\partial^a\Delta(x)=0\\ \partial_a\partial^a\Delta^{(+)}(x)=0\\ \partial_a\partial^a\Delta^{(-)}(x)=0\\ \partial_a\partial^a\Delta^{(l)}(x)=0\end{cases}\qquad\begin{cases}\partial_a\partial^a\Delta^{(c)}(x)=\delta^4(x)\\ \partial_a\partial^a\Delta^{ret}(x)=\delta^4(x)\\ \partial_a\partial^a\Delta^{adv}(x)=\delta^4(x)\\ \partial_a\partial^a\Delta_F(x)=i\delta^4(x)\end{cases}$$

Cor. 2.5.2. $\Delta(x)\partial_t\delta(t)=-\partial_t\Delta(x)\delta(t)=\delta^4(x)$

Proof: $\int f(t)\Delta(x)\partial_t\delta(t)dt=-\partial_t[f(t)\Delta(x)]|_{t=0}=f(0)\delta^3(\vec{r})$ □

Proof: $\int f(t)\partial_t\Delta(x)\delta(t)dt=f(t)\partial_t\Delta(x)|_{t=0}=-f(0)\delta^3(\vec{r})$ □

Cor. 2.5.3. $\partial_t^2[\theta(t)\Delta(x)]=-\delta^4(x)+\theta(t)\partial_t^2\Delta(x)$

Proof: $\partial_t^2[\theta(t)\Delta(x)]$
$=\partial_t[\partial_t\theta(t)\Delta(x)+\theta(t)\partial_t\Delta(x)]$
$=\partial_t^2\theta(t)\Delta(x)+2\partial_t\theta(t)\partial_t\Delta(x)+\theta(t)\partial_t^2\Delta(x)$
$=\partial_t\delta(t)\Delta(x)+2\delta(t)\partial_t\Delta(x)+\theta(t)\partial_t^2\Delta(x)$
$=\delta(t)\partial_t\Delta(x)+\theta(t)\partial_t^2\Delta(x)$
$=-\delta^4(x)+\theta(t)\partial_t^2\Delta(x)$ □

Cor. 2.5.4. $\Delta(x)\partial_t^n\delta(t)=\sum\limits_{l=0}^{[(n-1)/2]}C_n^{2l+1}\nabla^{2l}\partial_t^{n-2l-1}\delta^4(x)$

Proof: $\int f(t)\Delta(x)\partial_t^n\delta(t)dt$
$=(-1)^n\partial_t^n[f(t)\Delta(x)]_{t=0}=f(0)\delta^3(\vec{r})$
$=(-1)^n\sum\limits_{k=0}^{n}C_n^k\partial_t^{n-k}f(t)\partial_t^k\Delta(x)|_{t=0}=f(0)\delta^3(\vec{r})$
$=(-1)^n\sum\limits_{l=0}^{[(n-1)/2]}C_n^{2l+1}\partial_t^{n-2l-1}f(t)\partial_t^{2l+1}\Delta(x)|_{t=0}$
$=(-1)^{n+1}\sum\limits_{l=0}^{[(n-1)/2]}C_n^{2l+1}\partial_t^{n-2l-1}f(t)|_{t=0}\nabla^{2l}\delta^3(\vec{r})$
$=(-1)^{n+1}\sum\limits_{l=0}^{[(n-1)/2]}C_n^{2l+1}\nabla^{2l}\delta^3(\vec{r})\int\partial_t^{n-2l-1}f(t)\delta(t)dt$
$=\sum\limits_{l=0}^{[(n-1)/2]}C_n^{2l+1}\nabla^{2l}\delta^3(\vec{r})\int f(t)\partial_t^{n-2l-1}\delta(t)dt$
$=\int f(t)\sum\limits_{l=0}^{[(n-1)/2]}C_n^{2l+1}\nabla^{2l}\partial_t^{n-2l-1}\delta^4(x)dt$ □

Cor. 2.5.5. $\Delta(x)\partial_t^2\delta(t)=2\partial_t\delta^4(x)$

Cor. 2.5.6. $\Delta(x)\partial_t^3\delta(t)=3\partial_t^2\delta^4(x)+\nabla^2\delta^4(x)$

2.6 Extraction of energy and momentum operators in complex scalar field

Cor. 2.6.1. $H=\int|\vec{p}|[a_1^+(\vec{p},0)a_1(\vec{p},0)+a_2(\vec{p},0)a_2^+(\vec{p},0)]d^3\vec{p}$
$=i\varsigma\int\psi^+(\vec{r},t)\sigma\cdot\nabla\psi(\vec{r},t)d^3\vec{r}=i\int\psi^+(\vec{r},t)\partial_t\psi(\vec{r},t)d^3\vec{r}$

Proof: $H=\int|\vec{p}|[a_1^+(\vec{p},0)a_1(\vec{p},0)+a_2(\vec{p},0)a_2^+(\vec{p},0)]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int[\lambda(\hat{p},0)\psi^+(\vec{r}',t)e^{ip\cdot x'}\lambda^+(\hat{p},0)\psi(\vec{r},t)e^{-ip\cdot x}+\lambda(\hat{p},0)\psi^+(\vec{r}',t)e^{-ip\cdot x'}\lambda^+(\hat{p},0)\psi(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\lambda^+(\hat{p},0)\lambda(\hat{p},0)\psi^+(\vec{r}',t)\psi(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\psi^+(\vec{r}',t)\psi(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$ □

2.7 Poincare symmetry of complex scalar field

Cor. 2.7.1. $\hat{P}_a(0) = \int \psi^+(\vec{r},t)\hat{P}_a i\dot{\psi}(\vec{r},t)d^3\vec{r}, M_{ab}(n) = \int \psi^+(\vec{r},t)\hat{M}_{ab}i\dot{\psi}(\vec{r},t)d^3\vec{r}$

Lem. 2.7.1. $[\dot{\psi}_{k_\varsigma}(\vec{r},t), \psi^+_{l_\varsigma}(\vec{r}',t)] = -i\delta_{k_\varsigma l_\varsigma}\delta^3(\vec{r}-\vec{r}')$

Thm. 2.7.1. $\begin{cases}[L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0\end{cases}$

Proof: $[L_{ab}, L_{cd}]$
$= -\int d^3\vec{r}d^3\vec{r}'[\psi^+(\vec{r},t)(r_a\partial_b - r_b\partial_a)i\dot{\psi}(\vec{r},t), \psi^+(\vec{r}',t)(r'_c\partial'_d - r'_d\partial'_c)i\dot{\psi}(\vec{r}',t)]$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'[\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t), \psi^+_{k'_\varsigma}(\vec{r}',t)(r'_c\partial'_d - r'_d\partial'_c)\dot{\psi}_{l'_\varsigma}(\vec{r}',t)]$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)[(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t), \psi^+_{k'_\varsigma}(\vec{r}',t)](r'_c\partial'_d - r'_d\partial'_c)\dot{\psi}_{l'_\varsigma}(\vec{r}',t)$
$+ \psi^+_{k'_\varsigma}(\vec{r}',t)[\psi^+_{k_\varsigma}(\vec{r},t), (r'_c\partial'_d - r'_d\partial'_c)\dot{\psi}_{l'_\varsigma}(\vec{r}',t)](r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial_b - r_b\partial_a)(-i)\delta_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\dot{\psi}_{l'_\varsigma}(\vec{r}',t)$
$- \psi^+_{k'_\varsigma}(\vec{r}',t)(r'_c\partial'_d - r'_d\partial'_c)(-i)\delta_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial'_b - r_b\partial'_a)(-i)\delta_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\dot{\psi}_{l'_\varsigma}(\vec{r}',t)$
$- \psi^+_{k'_\varsigma}(\vec{r}',t)(r'_c\partial_d - r'_d\partial_c)(-i)\delta_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial_b - r_b\partial_a)(-i)\delta_{l_\varsigma k'_\varsigma}(r_c\partial_d - r_d\partial_c)\dot{\psi}_{l'_\varsigma}(\vec{r},t)$
$- \psi^+_{k'_\varsigma}(\vec{r},t)(r_c\partial_d - r_d\partial_c)(-i)\delta_{l'_\varsigma k_\varsigma}(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= -\int \psi^+(\vec{r},t)[-i(r_a\partial_b - r_b\partial_a), -i(r_c\partial_d - r_d\partial_c)](-i)\dot{\psi}(\vec{r},t)d^3\vec{r}$
$= \int \psi^+(\vec{r},t)[\hat{L}_{ab}, \hat{L}_{cd}]i\dot{\psi}(\vec{r},t)d^3\vec{r}$
$= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})$ □

Proof: $[L_{ab}, P_c]$
$= -\int d^3\vec{r}d^3\vec{r}'[\psi^+(\vec{r},t)(r_a\partial_b - r_b\partial_a)i\dot{\psi}(\vec{r},t), \psi^+(\vec{r}',t)\partial'_c i\dot{\psi}(\vec{r}',t)]$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'[\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t), \psi^+_{k'_\varsigma}(\vec{r}',t)\partial'_c\dot{\psi}_{l'_\varsigma}(\vec{r}',t)]$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)[(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t), \psi^+_{k'_\varsigma}(\vec{r}',t)]\partial'_c\dot{\psi}_{l'_\varsigma}(\vec{r}',t) + \psi^+_{k'_\varsigma}(\vec{r}',t)[\psi^+_{k_\varsigma}(\vec{r},t), \partial'_c\dot{\psi}_{l'_\varsigma}(\vec{r}',t)](r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial_b - r_b\partial_a)(-i)\delta_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\dot{\psi}_{l'_\varsigma}(\vec{r}',t)$
$- \psi^+_{k'_\varsigma}(\vec{r}',t)\partial'_c(-i)\delta_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial'_b - r_b\partial'_a)(-i)\delta_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\dot{\psi}_{l'_\varsigma}(\vec{r}',t)$
$- \psi^+_{k'_\varsigma}(\vec{r}',t)\partial_c(-i)\delta_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(r_a\partial_b - r_b\partial_a)(-i)\delta_{l_\varsigma k'_\varsigma}\partial_c\dot{\psi}_{l'_\varsigma}(\vec{r},t) - \psi^+_{k'_\varsigma}(\vec{r},t)\partial_c(-i)\delta_{l'_\varsigma k_\varsigma}(r_a\partial_b - r_b\partial_a)\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= -\int \psi^+(\vec{r},t)[-i(r_a\partial_b - r_b\partial_a), -i\partial'_c](-i)\dot{\psi}(\vec{r},t)d^3\vec{r}$
$= \int \psi^+(\vec{r},t)[\hat{L}_{ab}, \hat{P}_c]i\dot{\psi}(\vec{r},t)d^3\vec{r}$
$= -i(g_{bc}P_a - g_{ac}P_b)$ □

Proof: $[P_a, P_b]$
$= -\int[\psi^+(\vec{r},t)\partial_a i\dot{\psi}(\vec{r},t), \psi^+(\vec{r}',t)\partial'_b i\dot{\psi}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int[\psi^+_{k_\varsigma}(\vec{r},t)\partial_a\dot{\psi}_{l_\varsigma}(\vec{r},t), \psi^+_{k'_\varsigma}(\vec{r}',t)\partial'_b\dot{\psi}_{l'_\varsigma}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'\{\psi^+_{k_\varsigma}(\vec{r},t)[\partial_a\dot{\psi}_{l_\varsigma}(\vec{r},t), \psi^+_{k'_\varsigma}(\vec{r}',t)]\partial'_b\dot{\psi}_{l'_\varsigma}(\vec{r}',t) + \psi^+_{k'_\varsigma}(\vec{r}',t)[\psi^+_{k_\varsigma}(\vec{r},t), \partial'_b\dot{\psi}_{l'_\varsigma}(\vec{r}',t)]\partial_a\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(-i)\delta_{l_\varsigma k'_\varsigma}\partial_a\delta^3(\vec{r}-\vec{r}')\partial'_b\dot{\psi}_{l'_\varsigma}(\vec{r}',t) - \psi^+_{k'_\varsigma}(\vec{r}',t)(-i)\delta_{l'_\varsigma k_\varsigma}\partial'_b\delta^3(\vec{r}'-\vec{r})\partial_a\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_{k_\varsigma}(\vec{r},t)(-i)\delta_{l_\varsigma k'_\varsigma}\partial'_a\delta^3(\vec{r}-\vec{r}')\partial'_b\dot{\psi}_{l'_\varsigma}(\vec{r}',t) - \psi^+_{k'_\varsigma}(\vec{r}',t)(-i)\delta_{l'_\varsigma k_\varsigma}\partial_b\delta^3(\vec{r}-\vec{r}')\partial_a\dot{\psi}_{l_\varsigma}(\vec{r},t)\}$
$= \int\{\psi^+_{k_\varsigma}(\vec{r},t)(-i)\delta^{k_\varsigma l'_\varsigma}\partial_a\partial_b\dot{\psi}_{l'_\varsigma}(\vec{r},t) - \psi^+_{k'_\varsigma}(\vec{r},t)(-i)\delta^{k'_\varsigma l_\varsigma}\partial_b\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$

$= \int \psi^+(\vec{r},t)(\partial_a\partial_b - \partial_b\partial_a)(-i)\dot{\psi}(\vec{r},t)d^3\vec{r}$
$= -\int \psi^+(\vec{r},t)(\partial_a\partial_b - \partial_b\partial_a)i\dot{\psi}(\vec{r},t)d^3\vec{r}$
$= \int \psi^+(\vec{r},t)[\hat{P}_a,\hat{P}_b]i\dot{\psi}(\vec{r},t)d^3\vec{r} = 0$ □

3 Covariant quantization scheme for neutrino field

3.1 Neutrino spin operator equation and its plane wave solution

Thm. 3.1.1. $[\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\psi(x) = 0 \Leftrightarrow (\sigma,-i\varsigma)^a\partial_a\psi(x) = 0$

Cor. 3.1.1. $\begin{cases}\psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int \lambda(\hat{p},-\frac{\varsigma}{2})[a_1(\vec{p},-\frac{\varsigma}{2})e^{ip\cdot x} + a_2^+(\vec{p},-\frac{\varsigma}{2})e^{-ip\cdot x}]d^3\vec{p} \\ a_1(\vec{p},-\frac{\varsigma}{2}) = \frac{1}{(2\pi)^{3/2}}\int \lambda^+(\hat{p},-\frac{\varsigma}{2})\psi(\vec{r},t)e^{-ip\cdot x}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int \frac{i}{|\vec{p}|}\lambda^+(\hat{p},-\frac{\varsigma}{2})\dot{\psi}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r} \\ a_2^+(\vec{p},-\frac{\varsigma}{2}) = \frac{1}{(2\pi)^{3/2}}\int \lambda^+(\hat{p},-\frac{\varsigma}{2})\psi(\vec{r},t)e^{ip\cdot x}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int \frac{-i}{|\vec{p}|}\lambda^+(\hat{p},-\frac{\varsigma}{2})\dot{\psi}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$

Def. 3.1.1. *Projection operator:* $\hat{P}_{A_\varsigma A'_\varsigma}(\frac{1}{2},\varsigma) := \lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2}), \hat{P}^2(\frac{1}{2},\varsigma) = \hat{P}(\frac{1}{2},\varsigma), \hat{P}^+(\frac{1}{2},\varsigma) = \hat{P}(\frac{1}{2},\varsigma)$

3.2 Neutrino Lorentz transformation of plane wave solutions for spin operator equation

Cor. 3.2.1. $\Lambda_{\vec{v}} = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(\frac{1}{2})} = \frac{1}{\sqrt{2(1+\gamma_v)}}[1+\gamma_v - \gamma_v v\hat{v}\cdot\sigma] = \frac{1}{\sqrt{2(1+\gamma_v)}}\begin{bmatrix}1+\gamma_v(1-v_z) & -\gamma_v v_x + i\gamma_v v_y \\ -\gamma_v v_x - i\gamma_v v_y & 1+\gamma_v(1+v_z)\end{bmatrix}$

Cor. 3.2.2. $L_{\vec{v}} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L} = 1 - \gamma_v(\vec{v}\cdot L) + \frac{\gamma_v - 1}{v^2}(\vec{v}\cdot L)^2 = \gamma_v(1-\vec{v}\cdot L) - \frac{\gamma_v-1}{v^2}(\vec{v}\cdot R)^2$

Cor. 3.2.3. $\psi'(L_{\vec{v}}x) = \frac{1}{(2\pi)^{3/2}}\int \Lambda_{\vec{v}}\lambda(\hat{p},-\frac{\varsigma}{2})[a_1(\vec{p},-\frac{\varsigma}{2})e^{iL_{\vec{v}}p\cdot L_{\vec{v}}x} + a_2^+(\vec{p},-\frac{\varsigma}{2})e^{-iL_{\vec{v}}p\cdot L_{\vec{v}}x}]d^3\vec{p}$

Cor. 3.2.4. $L_{\vec{v}}p =, \Lambda_{\vec{v}}\lambda(\hat{p},-\frac{\varsigma}{2}) = \lambda(L_{\vec{v}}\hat{p},-\frac{\varsigma}{2})$

Cor. 3.2.5. $\begin{bmatrix}\gamma_{u'}\vec{u'} \\ i\gamma_{u'}\end{bmatrix} = L_{\vec{v}}\begin{bmatrix}\gamma_u\vec{u} \\ i\gamma_u\end{bmatrix}, \begin{bmatrix}\vec{p'} \\ iE'\end{bmatrix} = L_{\vec{v}}\begin{bmatrix}\vec{p} \\ iE\end{bmatrix}$

Cor. 3.2.6. $\lambda(\hat{p},\frac{1}{2}) = \frac{1}{\sqrt{2}\sqrt{1+\hat{p}_z}}\begin{bmatrix}1+\hat{p}_z \\ \hat{p}_x + i\hat{p}_y\end{bmatrix}, \lambda(\hat{p},-\frac{1}{2}) = \frac{1}{\sqrt{2}\sqrt{1+\hat{p}_z}}\begin{bmatrix}-\hat{p}_x + i\hat{p}_y \\ 1+\hat{p}_z\end{bmatrix}$

Proof: $\Lambda_{-\vec{v}}\lambda(\hat{p},\frac{1}{2})$
$= \frac{1}{\sqrt{2(1+\hat{p}_z)}}\Lambda_{-\vec{v}}\begin{bmatrix}1+\hat{p}_z \\ \hat{p}_x + i\hat{p}_y\end{bmatrix} = \frac{1}{\sqrt{2(1+\hat{u}_z)}}\Lambda_{-\vec{v}}\begin{bmatrix}1+\hat{u}_z \\ \hat{u}_x + i\hat{u}_y\end{bmatrix}$
$= \frac{1}{\sqrt{2(1+\hat{u}_z)}}\frac{1}{\sqrt{2(1+\gamma_v)}}\begin{bmatrix}1+\gamma_v(1+v_z) & \gamma_v v_x - i\gamma_v v_y \\ \gamma_v v_x + i\gamma_v v_y & 1+\gamma_v(1-v_z)\end{bmatrix}\begin{bmatrix}1+\hat{u}_z \\ \hat{u}_x + i\hat{u}_y\end{bmatrix}$
$= \frac{1}{2\sqrt{(1+\hat{u}_z)(1+\gamma_v)}}\begin{bmatrix}[1+\gamma_v(1+v_z)](1+\hat{u}_z) + (\gamma_v v_x - i\gamma_v v_y)(\hat{u}_x + i\hat{u}_y) \\ (\gamma_v v_x + i\gamma_v v_y)(1+\hat{u}_z) + [1+\gamma_v(1-v_z)](\hat{u}_x + i\hat{u}_y)\end{bmatrix}$
$= \frac{1}{2\sqrt{(1+\hat{u}_z)(1+\gamma_v)}}\begin{bmatrix}(1+\gamma_v)(1+\hat{u}_z) + \gamma_v[v_z + i(\vec{v}\times\hat{u})_z + \vec{v}\cdot\hat{u}] \\ (1+\gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x+iv_y) + i[(\vec{v}\times\hat{u})_x + i(\vec{v}\times\hat{u})_y]\}\end{bmatrix}$ □

Cor. 3.2.7. $\hat{u}' = [\hat{u} + \gamma_v\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\hat{u})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\hat{u})]$

Cor. 3.2.8. $\hat{p}' = [\hat{p} + \gamma_v\vec{v} + (\gamma_v - 1)(\vec{v}\cdot\hat{p})\vec{v}/v^2]/[\gamma_v(1+\vec{v}\cdot\hat{p})]$

Cor. 3.2.9. $1+\hat{u}'_z = 1 + [\hat{u}_z + \gamma_v\vec{v}_z + (\gamma_v - 1)(\vec{v}\cdot\hat{u})\vec{v}_z/v^2]/[\gamma_v(1+\vec{v}\cdot\hat{u})]$

Cor. 3.2.10. $\hat{u}'_x + i\hat{u}'_y = \{(\hat{u}_x + i\hat{u}_y) + \gamma_v(v_x + iv_y) + (\gamma_v - 1)(\vec{v}\cdot\hat{u})(v_x + iv_y)/v^2\}/[\gamma_v(1+\vec{v}\cdot\hat{u})]$

Cor. 3.2.11.
$\{(1+\gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x+iv_y) + i[(\vec{v}\times\hat{u})_x + i(\vec{v}\times\hat{u})_y]\}\}$
$\{\gamma_v(1+\vec{v}\cdot\hat{u}) + [\hat{u}_z + \gamma_v\vec{v}_z + (\gamma_v - 1)(\vec{v}\cdot\hat{u})\vec{v}_z/v^2]\}$
$= \{(1+\gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x+iv_y) + i[(\vec{v}\times\hat{u})_x + i(\vec{v}\times\hat{u})_y]\}\}$
$\{\gamma_v(1+\vec{v}\cdot\hat{u}) + [\hat{u}_z + \gamma_v\vec{v}_z + (\gamma_v - 1)(\vec{v}\cdot\hat{u})\vec{v}_z/v^2]\}$
$= \{(1+\gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x+iv_y) + i[(\vec{v}\times\hat{u})_x + i(\vec{v}\times\hat{u})_y]\}\}$
$\{1+\hat{u}_z + \gamma_v(1+\vec{v}\cdot\hat{u}) - 1 + \gamma_v\vec{v}_z + (\gamma_v - 1)(\vec{v}\cdot\hat{u})\vec{v}_z/v^2\}$
$= \{(1+\gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v v\{(\hat{v}_x+i\hat{v}_y) + i[(\hat{v}\times\hat{u})_x + i(\hat{v}\times\hat{u})_y]\}\}$
$\{1+\hat{u}_z + \gamma_v(1+\vec{v}\cdot\hat{u}) - 1 + \gamma_v v\hat{v}_z + (\gamma_v - 1)(\hat{v}\cdot\hat{u})\hat{v}_z\}$

Cor. 3.2.12. $\{(1+\gamma_v)(1+\hat{u}_z) + \gamma_v[v_z + i(\vec{v}\times\hat{u})_z + \vec{v}\cdot\hat{u}]\}$
$\{(\hat{u}_x + i\hat{u}_y) + \gamma_v(v_x+iv_y) + (\gamma_v - 1)(\vec{v}\cdot\hat{u})(v_x+iv_y)/v^2\}$
$= \{(1+\gamma_v)(1+\hat{u}_z) + \gamma_v v[\hat{v}_z + i(\hat{v}\times\hat{u})_z + \hat{v}\cdot\hat{u}]\}$
$\{(\hat{u}_x + i\hat{u}_y) + \gamma_v v(\hat{v}_x + i\hat{v}_y) + (\gamma_v - 1)(\hat{v}\cdot\hat{u})(\hat{v}_x + i\hat{v}_y)\}$
$=$

Cor. 3.2.13.
$[(1+\gamma_v)(\hat{u}_x+i\hat{u}_y)-\gamma_v v(\hat{u}_x+i\hat{u}_y)]\{\gamma_v(1+v\hat{u}_z)+[\hat{u}_z+\gamma_v v+(\gamma_v-1)\hat{u}_z)]\}$
$=(1+\gamma_v-\gamma_v v)(\hat{u}_x+i\hat{u}_y)[\gamma_v(1+v\hat{u}_z)+\gamma_v(v+\hat{u}_z)]$
$=(1+\gamma_v-\gamma_v v)(\hat{u}_x+i\hat{u}_y)\gamma_v(1+v)(1+\hat{u}_z)$
$=(1+\gamma_v+\gamma_v v)(1+\hat{u}_z)(\hat{u}_x+i\hat{u}_y)$

Cor. 3.2.14. $[(1+\gamma_v)(1+\hat{u}_z)+\gamma_v v(1+\hat{u}_z)](\hat{u}_x+i\hat{u}_y)$
$=(1+\gamma_v+\gamma_v v)(1+\hat{u}_z)(\hat{u}_x+i\hat{u}_y)$

3.3 Neutrino properties of covariant constant invariant tensor

Cor. 3.3.1.
$\Gamma^a_{A_\varsigma A'_\varsigma}(\frac{1}{2}):=\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}$
$\Gamma^\pi_{A_\varsigma A'_\varsigma}(\frac{1}{2})=(\frac{1}{\sqrt{2}})^1\delta_{A_\varsigma A'_\varsigma}$
$\Gamma^i_{A_\varsigma A'_\varsigma}(\frac{1}{2})=-i\varsigma(\frac{1}{\sqrt{2}})^1 2\sigma^i(\frac{1}{2})_{A_\varsigma A'_\varsigma}$

Lem. 3.3.1. $\Gamma^a_{A_\varsigma A'_\varsigma}p_a=i\sqrt{2}|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2}), \lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})=-\frac{\varsigma}{2}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\hat{p}_a$

Proof: $\Gamma^a_{A_\varsigma A'_\varsigma}p_a$
$=(\frac{1}{\sqrt{2}})^1 i\{-2\varsigma[\sigma(\frac{1}{2})\cdot\vec{p}]_{A_\varsigma}{}^{B_\varsigma}[\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})+\lambda_{B_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]+|\vec{p}|[\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})$
$+\lambda_{A_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]\}$
$=(\frac{1}{\sqrt{2}})^1 i\{[|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})-|\vec{p}|\lambda_{A_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]+|\vec{p}|[\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})+\lambda_{A_\varsigma}(\hat{p},\varsigma)\lambda^+_{A'_\varsigma}(\hat{p},\varsigma)]\}$
$=i\sqrt{2}|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})$ □

Cor. 3.3.2. $|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})=\frac{-\varsigma}{2}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\vec{p}_a$

Cor. 3.3.3. *Projection operator:* $\hat{P}_{A_\varsigma A'_\varsigma}(\frac{1}{2},\varsigma)=-\frac{i}{\sqrt{2}}\Gamma^a_{A_\varsigma A'_\varsigma}\hat{p}_a\to-\frac{1}{\sqrt{2}}\Gamma^a_{A_\varsigma A'_\varsigma}\hat{\partial}_a$

3.4 General covariant commutation rules in mathematics for neutrino field

Thm. 3.4.1.

$$\begin{cases}[a_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm=\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma),a_{\sigma'}(\vec{p}',-\varsigma)]_\pm=0\\ [a^+_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm=0\end{cases}\Rightarrow\begin{cases}[\psi_{A_\varsigma}(x),\psi^+_{A'_\varsigma}(x')]_\pm\\ =-i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a[(\delta_1-\pm\delta_2)\Delta^{(+\varsigma)}(x-x')\pm\delta_2\Delta(x-x')]\\ [\psi_{A_\varsigma}(x),\psi_{B_\varsigma}(x')]_\pm=0\\ [\psi^+_{A'_\varsigma}(x),\psi^+_{B'_\varsigma}(x')]_\pm=0\end{cases}$$

Proof: $[\psi^{(+)}_{A_\varsigma}(x),\psi^{(+)+}_{A'_\varsigma}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})[a_1(\vec{p},-\frac{\varsigma}{2}),a^+_1(\vec{p}',-\frac{\varsigma}{2})]_\pm e^{i(p\cdot x-p'\cdot x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\delta_1\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\delta_1\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})e^{ip\cdot(x-x')}d^3\vec{p}$
$=-i\frac{1}{(2\pi)^3}\int\delta_1\frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}p_a e^{ip\cdot(x-x')}d^3\vec{p}$
$=-\frac{1}{(2\pi)^3}\int\delta_1\frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a e^{ip\cdot(x-x')}d^3\vec{p}$
$=-i\sqrt{2}\delta_1\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}e^{ip\cdot(x-x')}d^3\vec{p}$
$=-i\sqrt{2}\delta_1\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(+)}(x-x')$ □

Proof: $[\psi^{(-)}_{A_\varsigma}(x),\psi^{(-)+}_{A'_\varsigma}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})[a^+_2(\vec{p},-\frac{\varsigma}{2}),a_2(\vec{p}',-\frac{\varsigma}{2})]_\pm e^{-i(p\cdot x-p'\cdot x')}d^3\vec{p}d^3\vec{p}'$
$=\pm\frac{1}{(2\pi)^3}\int\delta_2\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\pm\frac{1}{(2\pi)^3}\int\delta_2\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})e^{-ip\cdot(x-x')}d^3\vec{p}$
$=-\pm i\frac{1}{(2\pi)^3}\int\delta_2\frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}p_a e^{-ip\cdot(x-x')}d^3\vec{p}$
$=\pm\frac{1}{(2\pi)^3}\int\delta_2\frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a e^{-ip\cdot(x-x')}d^3\vec{p}$
$=\pm i\sqrt{2}\delta_2\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}e^{-ip\cdot(x-x')}d^3\vec{p}$
$=-\pm i\sqrt{2}\delta_2\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(-)}(x-x')$ □

Proof: $[\psi_{A_\varsigma}(x),\psi^+_{A'_\varsigma}(x')]_\pm$
$=[\psi^{(+)}_{A_\varsigma}(x),\psi^{(+)+}_{A'_\varsigma}(x')]_\pm+[\psi^{(-)}_{A_\varsigma}(x),\psi^{(-)+}_{A'_\varsigma}(x')]_\pm$
$=-i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a[\delta_1\Delta^{(+\varsigma)}(x-x')\pm\delta_2\Delta^{(-\varsigma)}(x-x')]$
$=-i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a[(\delta_1-\pm\delta_2)\Delta^{(+\varsigma)}(x-x')\pm\delta_2\Delta(x-x')]$ □

From the above, only $\delta_1 \mp \delta_2 = 0$, the micro causality is satisfied. At the same time only when $\delta_1, \delta_2 \geq 0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1 = \delta_2 = 1$, it satisfies the commutative relation. There are actually two other options. Namely when $\delta_1 = \delta_2 = 0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

3.5 Physical covariant anticommutative rules for neutrino field

Thm. 3.5.1.

$$\begin{cases}\{a_\sigma(\vec{p}, -\frac{\varsigma}{2}), a^+_{\sigma'}(\vec{p}', -\frac{\varsigma}{2})\} = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{\varsigma}{2}), a_{\sigma'}(\vec{p}', -\frac{\varsigma}{2})\} = 0 \\ \{a^+_\sigma(\vec{p}, -\frac{\varsigma}{2}), a^+_{\sigma'}(\vec{p}', -\frac{\varsigma}{2})] = 0\end{cases} \Rightarrow \begin{cases}\{\psi_{A_\varsigma}(x), \psi^+_{A'_\varsigma}(x')\} = -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x - x') \\ \{\psi_{A_\varsigma}(x), \psi_{B_\varsigma}(x')\} = 0 \\ \{\psi^+_{A'_\varsigma}(x), \psi^+_{B'_\varsigma}(x')\} = 0\end{cases}$$

Proof: $\{\psi_{A_\varsigma}(x), \psi^+_{A'_\varsigma}(x')\}$

$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}', -\tfrac{\varsigma}{2})\{\{a_1(\vec{p}, -\tfrac{\varsigma}{2}), a^+_1(\vec{p}', -\tfrac{\varsigma}{2})\}e^{i(p\cdot x - p'\cdot x')} + \{a^+_2(\vec{p}, -\tfrac{\varsigma}{2}), a_2(\vec{p}', -\tfrac{\varsigma}{2})\}e^{-ip\cdot(x-x')}\}d^3\vec{p}d^3\vec{p}'$$
$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})[\delta^3(\vec{p} - \vec{p}')e^{ip\cdot(x-x')} + \delta^3(\vec{p} - \vec{p}')e^{-ip\cdot(x-x')}]\}d^3\vec{p}d^3\vec{p}'$$
$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})[e^{ip\cdot(x-x')} + e^{-ip\cdot(x-x')}\}d^3\vec{p}$$
$$= -i\frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}p_a[e^{ip\cdot(x-x')} + e^{-ip\cdot(x-x')}]d^3\vec{p}$$
$$= -\frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec{p}$$
$$= -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\frac{-i}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec{p}$$
$$= -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x - x')$$ □

Thm. 3.5.2.

$$\begin{cases}\{a_\sigma(\vec{p}, -\frac{\varsigma}{2}), a^+_{\sigma'}(\vec{p}', -\frac{\varsigma}{2})\} = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{\varsigma}{2}), a_{\sigma'}(\vec{p}', -\frac{\varsigma}{2})\} = 0 \\ \{a^+_\sigma(\vec{p}, -\frac{\varsigma}{2}), a^+_{\sigma'}(\vec{p}', -\frac{\varsigma}{2})] = 0\end{cases} \Rightarrow \begin{cases}\{\psi^{(\tau)}_{A_\varsigma}(x), \psi^{(\kappa)+}_{A'_\varsigma}(x')\} = -i\sqrt{2}\delta^{\tau\kappa}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(\tau)}(x - x') \\ \{\psi^{(\tau)}_{A_\varsigma}(x), \psi^{(\kappa)}_{B_\varsigma}(x')\} = 0 \\ \{\psi^{(\tau)+}_{A'_\varsigma}(x), \psi^{(\kappa)+}_{B'_\varsigma}(x')\} = 0\end{cases}$$

Proof: $\{\psi^{(+)}_{A_\varsigma}(x), \psi^{(+)+}_{A'_\varsigma}(x')\}$

$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}', -\tfrac{\varsigma}{2})\{a_1(\vec{p}, -\tfrac{\varsigma}{2}), a^+_1(\vec{p}', -\tfrac{\varsigma}{2})\}e^{i(p\cdot x - p'\cdot x')}d^3\vec{p}d^3\vec{p}'$$
$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\delta^3(\vec{p} - \vec{p}')e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$$
$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})e^{ip\cdot(x-x')}d^3\vec{p}$$
$$= -i\frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}p_a e^{ip\cdot(x-x')}d^3\vec{p}$$
$$= -\frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a e^{ip\cdot(x-x')}d^3\vec{p}$$
$$= -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\frac{-i}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}e^{ip\cdot(x-x')}d^3\vec{p}$$
$$= -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(+)}(x - x')$$ □

Proof: $\{\psi^{(-)}_{A_\varsigma}(x), \psi^{(-)+}_{A'_\varsigma}(x')\}$

$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}', -\tfrac{\varsigma}{2})\{a^+_2(\vec{p}, -\tfrac{\varsigma}{2}), a_2(\vec{p}', -\tfrac{\varsigma}{2})\}e^{i(p\cdot x - p'\cdot x')}d^3\vec{p}d^3\vec{p}'$$
$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\delta^3(\vec{p} - \vec{p}')e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$$
$$= \frac{1}{(2\pi)^3}\int \lambda_{A_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}, -\tfrac{\varsigma}{2})e^{-ip\cdot(x-x')}d^3\vec{p}$$
$$= -i\frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}p_a e^{-ip\cdot(x-x')}d^3\vec{p}$$
$$= \frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a e^{-ip\cdot(x-x')}d^3\vec{p}$$
$$= i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\frac{-i}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}e^{-ip\cdot(x-x')}d^3\vec{p}$$
$$= -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(-)}(x - x')$$ □

3.6 Isochronous anticommutation rules of neutrino field

Cor. 3.6.1.

$$\begin{cases}\{\psi_{A_\varsigma}(x), \psi^+_{A'_\varsigma}(x')\} = -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x - x') \\ \{\psi_{A_\varsigma}(x), \psi_{B_\varsigma}(x')\} = 0 \\ \{\psi^+_{A'_\varsigma}(x), \psi^+_{B'_\varsigma}(x')\} = 0\end{cases} \Rightarrow \begin{cases}\{\psi_{A_\varsigma}(\vec{r}, t), \psi^+_{A'_\varsigma}(\vec{r}', t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r} - \vec{r}') \\ \{\dot{\psi}_{A_\varsigma}(\vec{r}, t), \psi^+_{A'_\varsigma}(\vec{r}', t)\} = (\sigma\cdot\nabla)_{A_\varsigma A'_\varsigma}\delta^3(\vec{r} - \vec{r}') \\ \{\psi_{A_\varsigma}(\vec{r}, t), \psi_{B_\varsigma}(\vec{r}', t)\} = 0 \\ \{\psi^+_{A'_\varsigma}(\vec{r}, t), \psi^+_{B'_\varsigma}(\vec{r}', t)\} = 0\end{cases}$$

Proof: $\{\psi_{A_\varsigma}(x), \psi^+_{A'_\varsigma}(x')\} = -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x - x')$

$\Rightarrow \{\psi_{A_\varsigma}(\vec{r}, t), \psi^+_{A'_\varsigma}(\vec{r}', t)\} = -i\sqrt{2}\Gamma^\pi_{A_\varsigma A'_\varsigma}\partial_\pi\Delta(x - x')|_{t=t'}$

$\Leftrightarrow \{\psi_{A_\varsigma}(\vec{r}, t), \psi^+_{A'_\varsigma}(\vec{r}', t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r} - \vec{r}')$ □

Cor. 3.6.2.
$$\begin{cases}\{\psi_{A_\varsigma}(\vec r,t),\psi^+_{A'_\varsigma}(\vec r',t)\}=\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec r-\vec r')\\ \{\psi_{A_\varsigma}(\vec r,t),\psi_{B_\varsigma}(\vec r',t)\}=0\\ \{\psi^+_{A'_\varsigma}(\vec r,t),\psi^+_{B'_\varsigma}(\vec r',t)\}=0\end{cases}\Rightarrow\begin{cases}\{a_\sigma(\vec p,-\frac{\varsigma}{2}),a^+_{\sigma'}(\vec p',-\frac{\varsigma}{2})\}=\delta_{\sigma\sigma'}\delta^3(\vec p-\vec p')\\ \{a_\sigma(\vec p,-\frac{\varsigma}{2}),a_{\sigma'}(\vec p',-\frac{\varsigma}{2})\}=0\\ \{a^+_\sigma(\vec p,-\frac{\varsigma}{2}),a^+_{\sigma'}(\vec p',-\frac{\varsigma}{2})]=0\end{cases}$$

Proof: $\{a_1(\vec p,-\frac{\varsigma}{2}),a^+_1(\vec p',-\frac{\varsigma}{2})\}$
$=\frac{1}{(2\pi)^3}\int\{\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec r,t)e^{-i(\vec p\cdot\vec r-Et)},\lambda^{A'_\varsigma}(\hat p',-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec r',t)e^{i(\vec p'\cdot\vec r'-E't)}\}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\int\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^{A'_\varsigma}(\hat p',-\frac{\varsigma}{2})\{\psi_{A_\varsigma}(\vec r,t),\psi^+_{A'_\varsigma}(\vec r',t)\}e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\int\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^{A'_\varsigma}(\hat p',-\frac{\varsigma}{2})\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec r-\vec r')e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\int\lambda^{+}(\hat p,-\frac{\varsigma}{2})\lambda(\hat p',-\frac{\varsigma}{2})e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r-E't)}d^3\vec r$
$=\lambda^+(\hat p,-\frac{\varsigma}{2})\lambda(\hat p,-\frac{\varsigma}{2})\delta^3(\vec p-\vec p')$
$=\delta^3(\vec p-\vec p')$ □

Proof: $\{a^+_2(\vec p,-\frac{\varsigma}{2}),a_2(\vec p',-\frac{\varsigma}{2})\}$
$=\frac{1}{(2\pi)^3}\int\{\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec r,t)e^{i(\vec p\cdot\vec r-Et)},\lambda^{A'_\varsigma}(\hat p',-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec r',t)e^{-i(\vec p'\cdot\vec r'-E't)}\}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\int\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^{A'_\varsigma}(\hat p',-\frac{\varsigma}{2})\{\psi_{A_\varsigma}(\vec r,t),\psi^+_{A'_\varsigma}(\vec r',t)\}e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\int\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^{A'_\varsigma}(\hat p',-\frac{\varsigma}{2})\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec r-\vec r')e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\int\lambda^{+}(\hat p,-\frac{\varsigma}{2})\lambda(\hat p',-\frac{\varsigma}{2})e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r-E't)}d^3\vec r$
$=\lambda^+(\hat p,-\frac{\varsigma}{2})\lambda(\hat p,-\frac{\varsigma}{2})\delta^3(\vec p-\vec p')$
$=\delta^3(\vec p-\vec p')$ □

3.7 Summary of anticommutation rules for neutrino field

he proof in the above sections exactly forms a logical closed-loop, so it has the following properties:

Cor. 3.7.1.
$$\begin{cases}\{a_\sigma(\vec p,-\frac{\varsigma}{2}),a^+_{\sigma'}(\vec p',-\frac{\varsigma}{2})\}=\delta_{\sigma\sigma'}\delta^3(\vec p-\vec p')\\ \{a_\sigma(\vec p,-\frac{\varsigma}{2}),a_{\sigma'}(\vec p',-\frac{\varsigma}{2})\}=0\\ \{a^+_\sigma(\vec p,-\frac{\varsigma}{2}),a^+_{\sigma'}(\vec p',-\frac{\varsigma}{2})]=0\end{cases}\Leftrightarrow\begin{cases}\{a_\sigma(\vec p),a^+_{\sigma'}(\vec p')\}=\delta_{\sigma\sigma'}\delta^3(\vec p-\vec p')\\ \{a_\sigma(\vec p),a_{\sigma'}(\vec p')\}=0\\ \{a^+_\sigma(\vec p),a^+_{\sigma'}(\vec p')]=0\end{cases}$$

$$\Updownarrow\qquad\qquad\qquad\qquad\Updownarrow$$

Cor. 3.7.2.
$$\begin{cases}\{\psi_{A_\varsigma}(x),\psi^+_{A'_\varsigma}(x')\}=-i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x-x')\\ \{\psi_{A_\varsigma}(x),\psi_{B_\varsigma}(x')\}=0\\ \{\psi^+_{A'_\varsigma}(x),\psi^+_{B'_\varsigma}(x')\}=0\end{cases}\Leftrightarrow\begin{cases}\{\psi_{A_\varsigma}(\vec r,t),\psi^+_{A'_\varsigma}(\vec r',t)\}=\delta_{A_\varsigma A'_\varsigma}\delta^3(\vec r-\vec r')\\ \{\psi_{A_\varsigma}(\vec r,t),\psi_{B_\varsigma}(\vec r',t)\}=0\\ \{\psi^+_{A'_\varsigma}(\vec r,t),\psi^+_{B'_\varsigma}(\vec r',t)\}=0\end{cases}$$

3.8 Commutative function, causal function, and feynman propagator of neutrino field

Cor. 3.8.1.
$$\begin{cases}\Delta_{A_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x)\\ \Delta^{(+)}_{A_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(+)}(x)\\ \Delta^{(-)}_{A_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(-)}(x)\\ \Delta^{(l)}_{A_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(l)}(x)\end{cases}$$

Cor. 3.8.2.
$$\begin{cases}\Delta^{(c)}_{A_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(c)}(x)-i\sqrt{2}\Gamma^\pi_{A_\varsigma A'_\varsigma}\delta(t)\Delta(x)=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{(c)}(x)\\ \Delta^{ret}_{A_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{ret}(x)-i\sqrt{2}\Gamma^\pi_{A_\varsigma A'_\varsigma}\delta(t)\Delta(x)=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{ret}(x)\\ \Delta^{adv}_{A_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{adv}(x)-i\sqrt{2}\Gamma^\pi_{A_\varsigma A'_\varsigma}\delta(t)\Delta(x)=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta^{adv}(x)\\ \Delta_{FA_\varsigma A'_\varsigma}(\frac12;x):=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta_F(x)+\sqrt{2}\Gamma^\pi_{A_\varsigma A'_\varsigma}\delta(t)\Delta(x)=-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta_F(x)\\ =i\Delta^{(c)}_{A_\varsigma A'_\varsigma}(\frac12;x)=\frac{1}{(2\pi)^4}\int\Delta_{FA_\varsigma A'_\varsigma}(\frac12;p)e^{ipx}d^4p\\ \Delta_{FA_\varsigma A'_\varsigma}(\frac12;p)=\frac{-\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}p_a}{p^2-i\varepsilon}=\frac{i\varsigma(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a}{p^2-i\varepsilon}\end{cases}$$

Cor. 3.8.3.
$$\begin{cases}[\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta(\frac12;x)=0\\ [\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta^{(+)}(\frac12;x)=0\\ [\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta^{(-)}(\frac12;x)=0\\ [\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta^{(l)}(\frac12;x)=0\end{cases}\quad\begin{cases}[\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta^{(c)}(\frac12;x)=-\varsigma[\sigma(\frac12),i\frac12\varsigma]_a\delta(t)\Delta(\frac12;x)|_{t=0}\\ [\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta^{ret}(\frac12;x)=-\varsigma[\sigma(\frac12),i\frac12\varsigma]_a\delta(t)\Delta(\frac12;x)|_{t=0}\\ [\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta^{adv}(\frac12;x)=-\varsigma[\sigma(\frac12),i\frac12\varsigma]_a\delta(t)\Delta(\frac12;x)|_{t=0}\\ [\frac12\partial_a+iS_{ab}(\frac12,\varsigma)\partial^b]\Delta_F(\frac12;x)=-i\varsigma[\sigma(\frac12),i\frac12\varsigma]_a\delta(t)\Delta(\frac12;x)|_{t=0}\end{cases}$$

$$\Updownarrow\qquad\qquad\qquad\qquad\Updownarrow$$

Cor. 3.8.4.

$$\begin{cases}[\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta(\frac{1}{2};x)=0\\ [\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta^{(+)}(\frac{1}{2};x)=0\\ [\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta^{(-)}(\frac{1}{2};x)=0\\ [\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta^{(l)}(\frac{1}{2};x)=0\end{cases}\quad \begin{cases}[\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta^{(c)}(\frac{1}{2};x)=-\frac{1}{\sqrt{2}}\Gamma_a\delta^4(x)\\ [\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta^{ret}(\frac{1}{2};x)=-\frac{1}{\sqrt{2}}\Gamma_a\delta^4(x)\\ [\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta^{adv}(\frac{1}{2};x)=-\frac{1}{\sqrt{2}}\Gamma_a\delta^4(x)\\ [\frac{1}{2}\partial_a+iS_{ab}(\frac{1}{2},\varsigma)\partial^b]\Delta_F(\frac{1}{2};x)=-i\frac{1}{\sqrt{2}}\Gamma_a\delta^4(x)\end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

Cor. 3.8.5.

$$\begin{cases}(\sigma,-i\varsigma)^a\partial_a\Delta(\frac{1}{2};x)=0\\ (\sigma,-i\varsigma)^a\partial_a\Delta^{(+)}(\frac{1}{2};x)=0\\ (\sigma,-i\varsigma)^a\partial_a\Delta^{(-)}(\frac{1}{2};x)=0\\ (\sigma,-i\varsigma)^a\partial_a\Delta^{(l)}(\frac{1}{2};x)=0\end{cases}\quad \begin{cases}(\sigma,-i\varsigma)^a\partial_a\Delta^{(c)}(\frac{1}{2};x)=i\varsigma\delta^4(x)\\ (\sigma,-i\varsigma)^a\partial_a\Delta^{ret}(\frac{1}{2};x)=i\varsigma\delta^4(x)\\ (\sigma,-i\varsigma)^a\partial_a\Delta^{adv}(\frac{1}{2};x)=i\varsigma\delta^4(x)\\ (\sigma,-i\varsigma)^a\partial_a\Delta_F(\frac{1}{2};x)=-\varsigma\delta^4(x)\end{cases}$$

3.9 Extraction of energy momentum operator in neutrino field

Cor. 3.9.1. $H=\int|\vec{p}|[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})-a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$
$=i\varsigma\int\psi^+(\vec{r},t)\sigma\cdot\nabla\psi(\vec{r},t)d^3\vec{r}=i\int\psi^+(\vec{r},t)\partial_t\psi(\vec{r},t)d^3\vec{r}$

Proof: $H=\int|\vec{p}|[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})-a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|[\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)e^{ip\cdot x'}\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec{r},t)e^{-ip\cdot x}$
$-\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)e^{-ip\cdot x'}\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int(i\sqrt{2})^{-1}(\Gamma_a)^{A'_\varsigma A_\varsigma}p^a\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\int\psi^+_{A'_\varsigma}(\vec{r},t)(\sigma\cdot\nabla)^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma}(\vec{r},t)d^3\vec{r}$
$=i\varsigma\int\psi^+(\vec{r},t)\sigma\cdot\nabla\psi(\vec{r},t)d^3\vec{r}$
$=i\int\psi^+(\vec{r},t)\partial_t\psi(\vec{r},t)d^3\vec{r}$ □

Cor. 3.9.2. $\vec{P}=\int\vec{p}[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})-a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}=-i\int\psi^+(\vec{r},t)\nabla\psi(\vec{r},t)d^3\vec{r}$

Proof: $\vec{P}=\int\vec{p}[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})-a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\vec{p}[\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)e^{ip\cdot x'}\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec{r},t)e^{-ip\cdot x}$
$-\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)e^{-ip\cdot x'}\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\vec{p}\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\frac{\hat{p}}{|\vec{p}|}(i\sqrt{2})^{-1}(\Gamma_a)^{A'_\varsigma A_\varsigma}p^a\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{2}\int\vec{p}\delta^{A'_\varsigma A_\varsigma}\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=-\frac{1}{(2\pi)^3}\int\vec{p}\delta^{A'_\varsigma A_\varsigma}\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=i\frac{1}{(2\pi)^3}\int\delta^{A'_\varsigma A_\varsigma}\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)\nabla e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=i\int\delta^{A'_\varsigma A_\varsigma}\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)\nabla\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$=-i\int\psi^+(\vec{r},t)\nabla\psi(\vec{r},t)d^3\vec{r}$ □

Cor. 3.9.3. $P^a=\int p^a[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})-a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}=-i\int\psi^+(\vec{r},t)\partial^a\psi(\vec{r},t)d^3\vec{r}$

3.10 Extraction of lepton number operator in neutrino field

Cor. 3.10.1. $Q=\int[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})+a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}=\int\psi^+(\vec{r},t)\psi(\vec{r},t)d^3\vec{r}$

Proof: $Q=\int[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})+a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\lambda^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int(\Gamma^a)^{A'_\varsigma A_\varsigma}\hat{p}_a\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int(\frac{i}{\sqrt{2}})^1\delta^{A'_\varsigma A_\varsigma}\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\int\psi^+(\vec{r}',t)\psi(\vec{r},t)\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$=\int\psi^+(\vec{r},t)\psi(\vec{r},t)d^3\vec{r}$ □

3.11 Extraction of particle number operator in neutrino field

Cor. 3.11.1. $N=\int[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})-a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}=\int\psi^+(\vec{r},t)\frac{i\partial_t}{\sqrt{-\nabla^2}}\psi(\vec{r},t)d^3\vec{r}$

Proof: $N=\int[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})-a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\lambda^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-ip\cdot(\vec{r}-\vec{r}')}-e^{ip\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int(\Gamma^a)^{A'_\varsigma A_\varsigma}\hat{p}_a\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-ip\cdot(\vec{r}-\vec{r}')}-e^{ip\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int[-i\varsigma(\frac{1}{\sqrt{2}})^1(\sigma^i)^{A'_\varsigma A_\varsigma}\hat{p}_i]\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)[e^{-ip\cdot(\vec{r}-\vec{r}')}-e^{ip\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=-\varsigma\frac{1}{(2\pi)^3}\int\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)(\sigma\cdot\hat{p})^{A'_\varsigma A_\varsigma}e^{-ip\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=-i\frac{1}{(2\pi)^3}\int\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)\frac{(\sigma\cdot\nabla)^{A'_\varsigma A_\varsigma}}{\sqrt{-\nabla^2}}e^{-ip\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=-i\varsigma\int\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)\frac{(\sigma\cdot\nabla)^{A'_\varsigma A_\varsigma}}{\sqrt{-\nabla^2}}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\int\psi^+_{A'_\varsigma}(\vec{r},t)\frac{(\sigma\cdot\nabla)^{A'_\varsigma A_\varsigma}}{\sqrt{-\nabla^2}}\psi_{A_\varsigma}(\vec{r},t)d^3\vec{r}$
$=i\varsigma\int\psi^+(\vec{r},t)\frac{\sigma\cdot\nabla}{\sqrt{-\nabla^2}}\psi(\vec{r},t)d^3\vec{r}$
$=\int\psi^+(\vec{r},t)\frac{i\partial_t}{\sqrt{-\nabla^2}}\psi(\vec{r},t)d^3\vec{r}$ □

3.12 Extraction of angular momentum operator in neutrino field

3.12.1 Space orbit angular momentum operator in neutrino field

Thm. 3.12.1. $L_{ij}=-i\int\psi^+(\vec{r},t)(r_i\partial_j-r_j\partial_i)\psi(\vec{r},t)d^3\vec{r}$
$=i\int\{a_1^+(\vec{p},-\frac{\varsigma}{2})(p_j\partial_{p^i}-p_i\partial_{p^j})a_1(\vec{p},-\frac{\varsigma}{2})+a_2(\vec{p},-\frac{\varsigma}{2})(p_j\partial_{p^i}-p_i\partial_{p^j})a_2^+(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$

Proof: $L_{ij}^{(+)}=-i\int\psi^{(+)+}(\vec{r},t)(r_i\partial_j-r_j\partial_i)\psi^{(+)}(\vec{r},t)d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})e^{i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})\delta^3(\vec{p}-\vec{p}')d^3\vec{p}d^3\vec{p}'$
$=i\int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_1^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]d^3\vec{p}$
$=i\int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_1^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\lambda(\hat{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}(p_j\partial_{p^i}-p_i\partial_{p^j})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$=i\int a_1^+(\vec{p},-\frac{\varsigma}{2})(p_j\partial_{p^i}-p_i\partial_{p^j})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}+i\int a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})p_i\frac{-ip_y\delta_{jx}+ip_x\delta_{jy}}{2p(1+\hat{p}_z)}d^3\vec{p}$ □

Proof: $L_{ij}^{(-\varsigma)}=-i\int\psi^{(-\varsigma)+}(\vec{r},t)(r_i\partial_j-r_j\partial_i)\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{-i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{-i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})e^{-i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})\delta^3(\vec{p}-\vec{p}')d^3\vec{p}d^3\vec{p}'$
$=i\int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_2(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]d^3\vec{p}$
$=i\int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_2(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\lambda(\hat{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}(p_j\partial_{p^i}-p_i\partial_{p^j})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$=i\int a_2(\vec{p},-\frac{\varsigma}{2})(p_j\partial_{p^i}-p_i\partial_{p^j})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$ □

Proof: $L_{ij}^{(+-\varsigma)}=-i\int\psi^{(+\varsigma)+}(\vec{r},t)(r_i\partial_j-r_j\partial_i)\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{-i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{-i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})e^{-i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})\delta^3(\vec{p}+\vec{p}')d^3\vec{p}d^3\vec{p}'$
$=i\int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_1^+(-\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]d^3\vec{p}$
$=i\int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_1^+(-\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\lambda(\hat{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}(p_j\partial_{p^i}-p_i\partial_{p^j})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$=0$ □

Proof: $L_{ij}^{(-+\varsigma)}=-i\int\psi^{(-\varsigma)+}(\vec{r},t)(r_i\partial_j-r_j\partial_i)\psi^{(+\varsigma)}(\vec{r},t)d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}][(r_i\partial_j-r_j\partial_i)e^{i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})e^{i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'$
$=-i\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})\delta^3(\vec{p}+\vec{p}')d^3\vec{p}d^3\vec{p}'$
$=i\int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_2(-\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}](p_j\partial_{p^i}-p_i\partial_{p^j})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]d^3\vec{p}$

$= i\int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_2(-\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\lambda(\hat{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}(p_j\partial_{p^i}-p_i\partial_{p^j})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= 0$ □

Proof: $M_{ij} = -i\int\psi^{(+\varsigma)+}(\vec{r},t)(r_i\partial_j - r_j\partial_i)\psi^{(+\varsigma)}(\vec{r},t)d^3\vec{r}$
$= -i\frac{1}{(2\pi)^{3/2}}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$\lambda^+(\hat{p}',-\frac{\varsigma}{2})\lambda(\hat{p},-\frac{\varsigma}{2})[a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma p'\cdot x}+a_2(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma p'\cdot x}](r_i\partial_j - r_j\partial_i)[a_1(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma p\cdot x}+a_2^+(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma p\cdot x}]$
$= L_{ij}^{(+\varsigma)} + L_{ij}^{(-\varsigma)} + L_{ij}^{(+-\varsigma)} + L_{ij}^{(-+\varsigma)}$
$= i\int\{a_1^+(\vec{p},-\frac{\varsigma}{2})(p_j\partial_{p^i}-p_i\partial_{p^j})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})(p_j\partial_{p^i}-p_i\partial_{p^j})a_2^+(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$ □

Cor. 3.12.1. $\int(p_j\partial_{p^i}-p_i\partial_{p^j})[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p} = 0, \int(p_j\partial_{p^i}-p_i\partial_{p^j})[a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p} = 0$

3.12.2 Time orbit angular momentum operator in neutrino field

Thm. 3.12.2. $L_{i\pi} = -i\int\psi^+(\vec{r},t)[r_i\partial_\pi - it\partial_i]\psi(\vec{r},t)d^3\vec{r}$
$= -\int a_1^+(\vec{p},-\frac{\varsigma}{2})\partial_{p^i}\{|\vec{p}|a_1(\vec{p},-\frac{\varsigma}{2})\} + a_2(\vec{p},-\frac{\varsigma}{2})\partial_{p^i}\{|\vec{p}|a_2^+(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$

Proof: $L_{i\pi}^{(+\varsigma)} = -i\int\psi^{(+\varsigma)+}(\vec{r},t)[r_i\partial_\pi - it\partial_i]\psi^{(+\varsigma)}(\vec{r},t)d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}][i\varsigma r_i e^{i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$- i\varsigma t\int p^i a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}][i\varsigma r_i e^{i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$- i\varsigma t\int p^i a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\partial_{p^i}e^{i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}' - i\varsigma t\int p^i a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\partial_{p^i}\delta^3(\vec{p}-\vec{p}')d^3\vec{p}d^3\vec{p}' - i\varsigma t\int p^i a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= -\int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_1^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\partial_{p^i}[|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]d^3\vec{p} - i\varsigma t\int p^i a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= -\int a_1^+(\vec{p},-\frac{\varsigma}{2})(\partial_{p^i}|\vec{p}| - i\varsigma tp^i)a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p} - i\varsigma t\int p^i a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= -\int a_1^+(\vec{p},-\frac{\varsigma}{2})\partial_{p^i}\{|\vec{p}|a_1(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$ □

Proof: $L_{i\pi}^{(-\varsigma)} = -i\int\psi^{(-\varsigma)+}(\vec{r},t)[r_i\partial_\pi - it\partial_i]\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r}$
$= i\int\psi^{(-\varsigma)+}(\vec{r},t)r_i i\partial_t\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r} - it[-i\int\psi^{(-\varsigma)+}(\vec{r},t)\partial_i\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r}]$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][-i\varsigma r_i e^{-i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$+ i\varsigma t\int p^i a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][-i\varsigma r_i e^{-i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$+ i\varsigma t\int p^i a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\partial_{p^i}e^{i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}' + i\varsigma t\int p^i a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\partial_{p^i}\delta^3(\vec{p}-\vec{p}')d^3\vec{p}d^3\vec{p}' + i\varsigma t\int p^i a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= -\int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_2(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\partial_{p^i}[|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]d^3\vec{p} + i\varsigma t\int p^i a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= -\int a_2(\vec{p},-\frac{\varsigma}{2})(\partial_{p^i}|\vec{p}| + i\varsigma tp^i)a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p} + i\varsigma t\int p^i a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= -\int a_2(\vec{p},-\frac{\varsigma}{2})\partial_{p^i}\{|\vec{p}|a_2^+(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$ □

Proof: $L_{i\pi}^{(+-\varsigma)} = -i\int\psi^{(+\varsigma)+}(\vec{r},t)[r_i\partial_\pi - it\partial_i]\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r}$
$= i\int\psi^{(+\varsigma)+}(\vec{r},t)r_i i\partial_t\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r} - it[-i\int\psi^{(+\varsigma)+}(\vec{r},t)\partial_i\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r}]$
$= i\int\psi^{(+\varsigma)+}(\vec{r},t)r_i i\partial_t\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r} + 0$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][-i\varsigma r_i e^{-i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}][-i\varsigma r_i e^{-i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\partial_{p^i}e^{i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\partial_{p^i}\delta^3(\vec{p}+\vec{p}')d^3\vec{p}d^3\vec{p}'$
$= -\int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_1^+(-\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]\partial_{p^i}[|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma|\vec{p}|t}]d^3\vec{p}$
$= 0$ □

Proof: $L_{i\pi}^{(-+\varsigma)} = -i\int\psi^{(-\varsigma)+}(\vec{r},t)[r_i\partial_\pi - it\partial_i]\psi^{(+\varsigma)}(\vec{r},t)d^3\vec{r}$
$= i\int\psi^{(-\varsigma)+}(\vec{r},t)r_i i\partial_t\psi^{(+\varsigma)}(\vec{r},t)d^3\vec{r} - it[-i\int\psi^{(-\varsigma)+}(\vec{r},t)\partial_i\psi^{(+\varsigma)}(\vec{r},t)d^3\vec{r}]$
$= i\int\psi^{(+\varsigma)+}(\vec{r},t)r_i i\partial_t\psi^{(-\varsigma)}(\vec{r},t)d^3\vec{r} + 0$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}][i\varsigma r_i e^{i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}][i\varsigma r_i e^{i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\partial_{p^i}e^{-i\varsigma(\vec{p}+\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}'|t}][|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\partial_{p^i}\delta^3(\vec{p}+\vec{p}')d^3\vec{p}d^3\vec{p}'$
$= -\int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_2(-\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]\partial_{p^i}[|\vec{p}|\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma|\vec{p}|t}]d^3\vec{p}$
$= 0$ □

Proof: $M_{i\pi} = -i\int \psi^+(\vec{r},t)[r_i\partial_\pi - it\partial_i]\psi(\vec{r},t)d^3\vec{r}$
$= i\int \psi^+(\vec{r},t)r_i i\partial_t\psi(\vec{r},t)d^3\vec{r} - it[-i\int \psi^{(+\varsigma)+}(\vec{r},t)\partial_i\psi^{(+\varsigma)}(\vec{r},t)d^3\vec{r}]$
$= \frac{1}{(2\pi)^{3/2}}\int |\vec{p}|\lambda^+(\hat{p}',-\frac{\varsigma}{2})\lambda(\hat{p},-\frac{\varsigma}{2})[a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{-i\varsigma p'\cdot x} + a_2(\vec{p}',-\frac{\varsigma}{2})e^{i\varsigma p'\cdot x}]i\varsigma r_i[a_1(\vec{p},-\frac{\varsigma}{2})e^{i\varsigma p\cdot x} - a_2^+(\vec{p},-\frac{\varsigma}{2})e^{-i\varsigma p\cdot x}]$
$d^3\vec{p}d^3\vec{p}'d^3\vec{r} - i\varsigma t\int p^i a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$
$= L_{i\pi}^{(+\varsigma)} + L_{i\pi}^{(-\varsigma)} + L_{i\pi}^{(+-\varsigma)} + L_{i\pi}^{(-+\varsigma)}$
$= -\int a_1^+(\vec{p},-\frac{\varsigma}{2})\partial_{p^i}\{|\vec{p}|a_1(\vec{p},-\frac{\varsigma}{2})\} + a_2(\vec{p},-\frac{\varsigma}{2})\partial_{p^i}\{|\vec{p}|a_2^+(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$ □

3.12.3 Spin angular momentum operator in neutrino field

Thm. 3.12.3. $S_{ab} = \int \psi^+(\vec{r},t)S_{ab}(\frac{1}{2},\varsigma)\psi(\vec{r},t)d^3\vec{r} = i\sigma_{\varsigma ab}^{\alpha_\varsigma}\int \psi^+(\vec{r},t)\sigma_{\alpha_\varsigma}(\frac{1}{2})\psi(\vec{r},t)d^3\vec{r}$
$= \frac{-i\varsigma}{2}\sigma_{\varsigma ab}^{\alpha_\varsigma}\int \hat{p}_{\alpha_\varsigma}[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$

Thm. 3.12.4. $\hat{s}_{\alpha_\varsigma} = \int \psi^+(\vec{r},t)\sigma_{\alpha_\varsigma}(\frac{1}{2})\psi(\vec{r},t)d^3\vec{r} = -\frac{\varsigma}{2}\int \hat{p}_{\alpha_\varsigma}[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$

Proof: $\hat{s}_{\alpha_\varsigma}^{(+)} = \int \psi^{(+)+}(\vec{r},t)\sigma_{\alpha_\varsigma}\psi^{(+)}(\vec{r},t)d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}d^3\vec{p}'$
$= \int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_1^+(\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]d^3\vec{p}$
$= -\frac{\varsigma}{2}\int a_1^+(\vec{p},-\frac{\varsigma}{2})\hat{p}_{\alpha_\varsigma}a_1(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$ □

Proof: $\hat{s}_{\alpha_\varsigma}^{(-)} = \int \psi^{(-)+}(\vec{r},t)\sigma_{\alpha_\varsigma}\psi^{(-)}(\vec{r},t)d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]\delta^3(\vec{p}-\vec{p}')d^3\vec{p}d^3\vec{p}'$
$= \int[\lambda^+(\hat{p},-\frac{\varsigma}{2})a_2(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]d^3\vec{p}$
$= -\frac{\varsigma}{2}\int a_2(\vec{p},-\frac{\varsigma}{2})\hat{p}_{\alpha_\varsigma}a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$ □

Proof: $\hat{s}_{\alpha_\varsigma}^{(+-)} = \int \psi^{(+)+}(\vec{r},t)\sigma_{\alpha_\varsigma}\psi^{(-)}(\vec{r},t)d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]e^{-i(\vec{p}+\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]\delta^3(\vec{p}+\vec{p}')d^3\vec{p}d^3\vec{p}'$
$= \int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_1^+(-\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})e^{i|\vec{p}|t}]d^3\vec{p}$
$\neq 0$ □

Proof: $\hat{s}_{\alpha_\varsigma}^{(-+)} = \int \psi^{(-)+}(\vec{r},t)\sigma_{\alpha_\varsigma}\psi^{(-)}(\vec{r},t)d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]e^{i(\vec{p}+\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$= \int[\lambda^+(\hat{p}',-\frac{\varsigma}{2})a_2(\vec{p}',-\frac{\varsigma}{2})e^{-i|\vec{p}'|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]\delta^3(\vec{p}+\vec{p}')d^3\vec{p}d^3\vec{p}'$
$= \int[\lambda^+(-\hat{p},-\frac{\varsigma}{2})a_2(-\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]\sigma_{\alpha_\varsigma}(\frac{1}{2})[\lambda(\hat{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})e^{-i|\vec{p}|t}]d^3\vec{p}$
$\neq 0$ □

Proof: $\hat{s}_{\alpha_\varsigma} = \int \psi^+(\vec{r},t)\sigma_{\alpha_\varsigma}(\frac{1}{2})\psi(\vec{r},t)d^3\vec{r}$
$= \frac{1}{(2\pi)^{3/2}}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}$
$\lambda^+(\hat{p}',-\frac{\varsigma}{2})\sigma_{\alpha_\varsigma}(\frac{1}{2})\lambda(\hat{p},-\frac{\varsigma}{2})[a_1^+(\vec{p}',-\frac{\varsigma}{2})e^{-ip'\cdot x} + a_2(\vec{p}',-\frac{\varsigma}{2})e^{ip'\cdot x}][a_1(\vec{p},-\frac{\varsigma}{2})e^{ip\cdot x} + a_2^+(\vec{p},-\frac{\varsigma}{2})e^{-ip\cdot x}]$
$= \hat{s}_{\alpha_\varsigma}^{(+)} + \hat{s}_{\alpha_\varsigma}^{(-)} + \hat{s}_{\alpha_\varsigma}^{(+-)} + \hat{s}_{\alpha_\varsigma}^{(-+)}$
$= -\frac{\varsigma}{2}\int \hat{p}_{\alpha_\varsigma}[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$ □

Proof: $[\hat{s}_{\alpha_\varsigma},\hat{s}_{\beta_\varsigma}]$
$= \int \hat{p}_{\alpha_\varsigma}\hat{p}'_{\beta_\varsigma}[[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})],[a_1^+(\vec{p}',-\frac{\varsigma}{2})a_1(\vec{p}',-\frac{\varsigma}{2}) + a_2(\vec{p}',-\frac{\varsigma}{2})a_2^+(\vec{p}',-\frac{\varsigma}{2})]]d^3\vec{p}d^3\vec{p}'$
$\neq 0$ □

Cor. 3.12.2. $[\hat{s}_{\alpha_\varsigma},\hat{s}_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\hat{s}_{\gamma_\varsigma}$

Proof: $[\hat{s}_{\alpha_\varsigma},\hat{s}_{\beta_\varsigma}] = \int d^3\vec{r}d^3\vec{r}'\sigma_{\alpha_\varsigma}(\frac{1}{2})^{A'_\varsigma A_\varsigma}\sigma_{\beta_\varsigma}(\frac{1}{2})^{B'_\varsigma B_\varsigma}[\psi_{A'_\varsigma}^+(\vec{r},t)\psi_{A_\varsigma}(\vec{r},t),\psi_{B'_\varsigma}^+(\vec{r}',t)\psi_{B_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'\sigma_{\alpha_\varsigma}(\frac{1}{2})^{A'_\varsigma A_\varsigma}(\sigma_{\beta_\varsigma})^{B'_\varsigma B_\varsigma}$
$\{-[\psi_{B'_\varsigma}^+(\vec{r}',t),\psi_{A'_\varsigma}^+(\vec{r},t)\psi_{A_\varsigma}(\vec{r},t)]\psi_{B_\varsigma}(\vec{r}',t) - \psi_{B'_\varsigma}^+(\vec{r}',t)[\psi_{B_\varsigma}(\vec{r}',t),\psi_{A'_\varsigma}^+(\vec{r},t)\psi_{A_\varsigma}(\vec{r},t)]\}$
$= \int d^3\vec{r}d^3\vec{r}'\sigma_{\alpha_\varsigma}(\frac{1}{2})^{A'_\varsigma A_\varsigma}\sigma_{\beta_\varsigma}(\frac{1}{2})^{B'_\varsigma B_\varsigma}$
$\{\psi_{A'_\varsigma}^+(\vec{r},t)\{\psi_{B'_\varsigma}^+(\vec{r}',t),\psi_{A_\varsigma}(\vec{r},t)\}\psi_{B_\varsigma}(\vec{r}',t) - \psi_{B'_\varsigma}^+(\vec{r}',t)\{\psi_{B_\varsigma}(\vec{r}',t),\psi_{A'_\varsigma}^+(\vec{r},t)\}\psi_{A_\varsigma}(\vec{r},t)\}$
$= \int d^3\vec{r}\sigma_{\alpha_\varsigma}(\frac{1}{2})^{A'_\varsigma A_\varsigma}\sigma_{\beta_\varsigma}(\frac{1}{2})^{B'_\varsigma B_\varsigma}\{\psi_{A'_\varsigma}^+(\vec{r},t)\delta_{A_\varsigma B'_\varsigma}\psi_{B_\varsigma}(\vec{r}',t) - \psi_{B'_\varsigma}^+(\vec{r}',t)\delta_{A'_\varsigma B_\varsigma}\psi_{A_\varsigma}(\vec{r},t)\}$
$= \int d^3\vec{r}\{\psi^+(\vec{r},t)\sigma_{\alpha_\varsigma}(\frac{1}{2})\sigma_{\beta_\varsigma}(\frac{1}{2})\psi(\vec{r}',t) - \psi^+(\vec{r}',t)\sigma_{\beta_\varsigma}(\frac{1}{2})\sigma_{\alpha_\varsigma}(\frac{1}{2})\psi(\vec{r},t)\}$
$= \int d^3\vec{r}\psi^+(\vec{r},t)[\sigma_{\alpha_\varsigma}(\frac{1}{2}),\sigma_{\beta_\varsigma}(\frac{1}{2})]\psi(\vec{r}',t)$
$= i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\hat{s}_{\gamma_\varsigma}$ □

Combining the above two points, we have come to some strange conclusions below the free field. The physical meaning is that positive and negative particles must be produced and annihilated in pairs.

Cor. 3.12.3. $\hat{s}_{\alpha_\varsigma} \neq 0$

3.13 Summary of angular momentum operator in neutrino field

Def. 3.13.1. $\tilde{\partial}_a := \partial_{p^a}, \tilde{\partial}_\pi \equiv \frac{1}{i|\vec{p}|}$

Cor. 3.13.1. $L_{ij} = -i\int \psi^+(\vec{r},t)(r_i\partial_j - r_j\partial_i)\psi(\vec{r},t)d^3\vec{r}$
$= -i\int\{a_1^+(\vec{p},-\frac{\varsigma}{2})(p_i\partial_{p^j} - p_j\partial_{p^i})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})(p_i\partial_{p^j} - p_j\partial_{p^i})a_2^+(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$

Cor. 3.13.2. $L_{i\pi} = -i\int \psi^+(\vec{r},t)[r_i\partial_\pi - it\partial_i]\psi(\vec{r},t)d^3\vec{r}$
$= -i\int a_1^+(\vec{p},-\frac{\varsigma}{2})(\frac{p_i}{i|\vec{p}|} - i|\vec{p}|\partial_{p_i})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})(\frac{p_i}{i|\vec{p}|} - i|\vec{p}|\partial_{p_i})a_2^+(\vec{p},-\frac{\varsigma}{2})d^3\vec{p}$

Cor. 3.13.3. $S_{ab} = \int \psi^+(\vec{r},t)S_{ab}(\frac{1}{2},\varsigma)\psi(\vec{r},t)d^3\vec{r} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\int \psi^+(\vec{r},t)\sigma_{\alpha_\varsigma}\psi(\vec{r},t)d^3\vec{r}$
$= -i\int[a_1^+(\vec{p},-\frac{\varsigma}{2})\frac{\varsigma}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\hat{p}_{\alpha_\varsigma}a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})\frac{\varsigma}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\hat{p}_{\alpha_\varsigma}a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$

Cor. 3.13.4. $\hat{M}_{ab} = -i(x_a\partial_b - x_b\partial_a) + \hat{S}_{ab}, \tilde{M}_{ab} = -i(p_a\tilde{\partial}_b - p_b\tilde{\partial}_a) + \frac{-i\varsigma}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\hat{p}_{\alpha_\varsigma}$

The following important theorems are obtained.

Thm. 3.13.1. $M_{ab} = \int \psi^+(\vec{r},t)\hat{M}_{ab}\psi(\vec{r},t)d^3\vec{r} = \int\{a_1^+(\vec{p},-\frac{\varsigma}{2})\tilde{M}_{ab}a_1(\vec{p},-\frac{\varsigma}{2}) + a_2(\vec{p},-\frac{\varsigma}{2})\tilde{M}_{ab}a_2^+(\vec{p},-\frac{\varsigma}{2})\}d^3\vec{p}$

3.14 Normalized energy momentum operator of neutrino field

Cor. 3.14.1. $H_0 = \varsigma\int|\vec{p}|[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2^+(\vec{p},-\frac{\varsigma}{2})a_2(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$
$= \frac{i\varsigma}{2}\int[\psi^+_{A'_\varsigma}(\vec{r},t),(\sigma\cdot\nabla)^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma}(\vec{r},t)]d^3\vec{r} + \frac{\varsigma}{2}\int\{\psi^+_{A'_\varsigma}(\vec{r},t),\delta^{A'_\varsigma A_\varsigma}\sqrt{-\nabla^2}\psi_{A_\varsigma}(\vec{r},t)\}d^3\vec{r}$

Proof: $H_0 = \varsigma\int|\vec{p}|[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2}) + a_2^+(\vec{p},-\frac{\varsigma}{2})a_2(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\varsigma\int|\vec{p}|[\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)e^{i\varsigma p\cdot x'}\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec{r},t)e^{-i\varsigma p\cdot x}$
$+ \lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi_{A_\varsigma}(\vec{r},t)e^{i\varsigma p\cdot x}\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\psi^+_{A'_\varsigma}(\vec{r}',t)e^{-i\varsigma p\cdot x'}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\varsigma\int|\vec{p}|\lambda_m^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_m^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})[\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + \psi_{A_\varsigma}(\vec{r},t)\psi^+_{A'_\varsigma}(\vec{r}',t)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\varsigma\int(i\sqrt{2})^{-1}(\Gamma_a)^{A'_\varsigma A_\varsigma}p^a[\psi^+_{A'_\varsigma}(\vec{r}',t)\psi_{A_\varsigma}(\vec{r},t)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + \psi_{A_\varsigma}(\vec{r},t)\psi^+_{A'_\varsigma}(\vec{r}',t)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{i\varsigma}{2}\int[\psi^+_{A'_\varsigma}(\vec{r},t),(\sigma\cdot\nabla)^{A'_\varsigma A_\varsigma}\psi_{A_\varsigma}(\vec{r},t)]d^3\vec{r} + \frac{\varsigma}{2}\int\{\psi^+_{A'_\varsigma}(\vec{r},t),\delta^{A'_\varsigma A_\varsigma}\sqrt{-\nabla^2}\psi_{A_\varsigma}(\vec{r},t)\}d^3\vec{r}$
$= i\varsigma\int\psi^+(\vec{r},t)\sigma\cdot\nabla\psi(\vec{r},t)d^3\vec{r} + \frac{\varsigma}{2}\int\{\psi^+_{A'_\varsigma}(\vec{r},t),\delta^{A'_\varsigma A_\varsigma}\sqrt{-\nabla^2}\psi_{A_\varsigma}(\vec{r},t)\}d^3\vec{r}$ □

3.15 Quantum equation of neutrino field

Cor. 3.15.1.

$$[\partial_a + iS_{ab}(\tfrac{1}{2},\varsigma)\partial^b]\psi = 0 \Leftrightarrow [P_a,\psi(\vec{r},t)] = i\partial_a\psi(\vec{r},t); \begin{cases}\{\psi_A(\vec{r},t),\psi_B^+(\vec{r}',t)\} = \delta_{AB}\delta^3(\vec{r}-\vec{r}')\\ \{\psi_A(\vec{r},t),\psi_B(\vec{r}',t)\} = 0, \{\psi_A^+(\vec{r},t),\psi_B^+(\vec{r}',t)\} = 0\end{cases}$$

Cor. 3.15.2.

$$\begin{cases}[A,BC] = [A,B]C + B[A,C],[A,CB] = [A,C]B + C[A,B]\\ [A,BC] = \{A,B\}C - B\{A,C\},[A,CB] = \{A,C\}B - C\{A,B\}\end{cases}$$

$$\begin{cases}[BC,A] = [B,A]C + B[C,A]\\ [BC,A] = -\{B,A\}C + B\{C,A\}\end{cases}$$

3.16 Mathematical lemma

Lem. 3.16.1.

$$\begin{cases}[AB,A'B'] = [AB,A']B' + A'[AB,B'],[AB,B'A'] = [AB,B']A' + B'[AB,A']\\ [AB,A'B'] = \{AB,A'\}B' - A'\{AB,B'\},[AB,B'A'] = \{AB,B'\}A' - B'\{AB,A'\}\end{cases}$$

$$\begin{cases}[A'B',AB] = [A',AB]B' + A'[B',AB]\\ [A'B',AB] = -\{A',AB\}B' + A'\{B',AB\}\end{cases}$$

Cor. 3.16.1.

$$\begin{cases}[A,BC] = [A,B]C + B[A,C]\\ [A,BC] = \{A,B\}C - B\{A,C\}\end{cases} \quad \begin{cases}[BC,A] = [B,A]C + B[C,A]\\ [BC,A] = -\{B,A\}C + B\{C,A\}\end{cases}$$

Lem. 3.16.2. $[AB,A'B'] = [AB,A']B' + A'[AB,B'] = [A,A']BB' + A[B,A']B' + A'A[B,B'] + A'[A,B']B$

Lem. 3.16.3. $[AB,A'B'] = [AB,A']B' + A'[AB,B'] = -\{A,A'\}BB' + A\{B,A'\}B' - A'\{A,B'\}B + A'A\{B,B'\}$

Lem. 3.16.4. $[A,A'] = [B,B'] = 0 \Rightarrow [AB,A'B'] = A[B,A']B' + A'[A,B']B$

Lem. 3.16.5. $\{A,A'\} = \{B,B'\} = 0 \Rightarrow [AB,A'B'] = A\{B,A'\}B' - A'\{A,B'\}B$

3.17 Poincare symmetry of neutrino field

Lem. 3.17.1. $\begin{cases} P_a = -i\int \psi^+(\vec{r},t)\partial_a\psi(\vec{r},t)d^3\vec{r} = \int \psi^+(\vec{r},t)\hat{P}_a\psi(\vec{r},t)d^3\vec{r} \\ L_{ab} = -i\int \psi^+(\vec{r},t)(r_a\partial_b - r_b\partial_a)\psi(\vec{r},t)d^3\vec{r} = \int \psi^+(\vec{r},t)\hat{L}_{ab}\psi(\vec{r},t)d^3\vec{r} \\ M_{ab} = \int \psi^+(\vec{r},t)[-i(r_a\partial_b - r_b\partial_a) + \hat{S}_{ab}]\psi(\vec{r},t)d^3\vec{r} = \int \psi^+(\vec{r},t)\hat{M}_{ab}\psi(\vec{r},t)d^3\vec{r} \\ \tilde{M}_{ab} = -i\int \psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)\dot{\psi}(\vec{r},t)d^3\vec{r} \\ \bar{M}_{ab} = -i\int \psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)\psi(\vec{r},t)d^3\vec{r} \end{cases}$

Thm. 3.17.1. $\begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [S_{ab}, S_{cd}] = -i(g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}) \\ [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [S_{ab}, L_{cd}] = 0, [S_{ab}, P_c] = 0, [P_a, P_b] = 0 \end{cases}$

$[\Rightarrow] \begin{cases} [M_{ab}, M_{cd}] = -i(g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac}) \\ [M_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0 \end{cases}$

Proof: $[L_{ab}, L_{cd}]$
$= -\int[\psi^+(\vec{r},t)(r_a\partial_b - r_b\partial_a)\psi(\vec{r},t), \psi^+(\vec{r}',t)(r'_c\partial'_d - r'_d\partial'_c)\psi(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{AB}\delta^{A'B'}\int[\psi^+_A(\vec{r},t)(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)(r'_c\partial'_d - r'_d\partial'_c)\psi_{B'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{AB}\delta^{A'B'}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_A(\vec{r},t)\{(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\}(r'_c\partial'_d - r'_d\partial'_c)\psi_{B'}(\vec{r}',t)$
$-\psi^+_{A'}(\vec{r}',t)\{\psi^+_A(\vec{r},t), (r'_c\partial'_d - r'_d\partial'_c)\psi_{B'}(\vec{r}',t)\}(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t)\}$
$= -\delta^{AB}\delta^{A'B'}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_A(\vec{r},t)\delta_{A'B}(r_a\partial_b - r_b\partial_a)\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\psi_{B'}(\vec{r}',t) - \psi^+_{A'}(\vec{r}',t)\delta_{AB'}(r'_c\partial'_d - r'_d\partial'_c)\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t)\}$
$= -\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+(\vec{r},t)(r_a\partial_b - r_b\partial_a)\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\psi(\vec{r}',t) - \psi^+(\vec{r}',t)(r'_c\partial'_d - r'_d\partial'_c)\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\psi(\vec{r},t)\}$
$= \int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+(\vec{r},t)(r_a\partial'_b - r_b\partial'_a)\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\psi(\vec{r}',t) - \psi^+(\vec{r}',t)(r'_c\partial_d - r'_d\partial_c)\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\psi(\vec{r},t)\}$
$= -\int d^3\vec{r}\{\psi^+(\vec{r},t)(r_a\partial_b - r_b\partial_a)(r_c\partial_d - r_d\partial_c)\psi(\vec{r},t) - \psi^+(\vec{r},t)(r'_c\partial_d - r_d\partial_c)(r_a\partial_b - r_b\partial_a)\psi(\vec{r},t)\}$
$= \int \psi^+(\vec{r},t)[-i(r_a\partial_b - r_b\partial_a), -i(r_c\partial_d - r_d\partial_c)]\psi(\vec{r},t)d^3\vec{r}$
$= \int \psi^+(\vec{r},t)[\hat{L}_{ab}, \hat{L}_{cd}]\psi(\vec{r},t)d^3\vec{r}$
$= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})$ □

Proof: $[L_{ab}, P_c]$
$= -\delta^{AB}\delta^{A'B'}\int[\psi^+(\vec{r},t)(r_a\partial_b - r_b\partial_a)\psi(\vec{r},t), \psi^+(\vec{r}',t)\partial'_c\psi(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{AB}\delta^{A'B'}\int[\psi^+_A(\vec{r},t)(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\partial'_c\psi_{B'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{AB}\delta^{A'B'}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_A(\vec{r},t)[(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)]\partial'_c\psi_{B'}(\vec{r}',t) - \psi^+_{A'}(\vec{r}',t)[\psi^+_A(\vec{r},t), \partial'_c\psi_{B'}(\vec{r}',t)](r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t)\}$
$= -\delta^{AB}\delta^{A'B'}\int d^3\vec{r}d^3\vec{r}'$
$\{\psi^+_A(\vec{r},t)\delta_{A'B}(r_a\partial_b - r_b\partial_a)\delta^3(\vec{r}-\vec{r}')\partial'_c\psi_{B'}(\vec{r}',t) - \psi^+_{A'}(\vec{r}',t)\delta_{AB'}\partial'_c\delta^3(\vec{r}-\vec{r}')\psi_{B'}(\vec{r}',t)](r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t)\}$
$= \int\{\psi^+_A(\vec{r},t)\delta^{AB'}(r_a\partial'_b - r_b\partial'_a)\delta^3(\vec{r}-\vec{r}')\partial'_c\psi_{B'}(\vec{r}',t) - \psi^+_{A'}(\vec{r}',t)\delta^{A'B}\partial_c\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t)\}d^3\vec{r}d^3\vec{r}'$
$= -\int\{\psi^+_A(\vec{r},t)\delta^{AB'}(r_a\partial_b - r_b\partial_a)\partial_c\psi_{B'}(\vec{r},t) - \psi^+_{A'}(\vec{r},t)\delta^{A'B}\partial_c(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r},t)\}d^3\vec{r}$
$= \int \psi^+(\vec{r},t)[\hat{L}_{ab}, \hat{P}_c]\psi(\vec{r},t)d^3\vec{r}$
$= -i(g_{bc}P_a - g_{ac}P_b)$ □

Proof: $[P_a, P_b]$
$= -\int[\psi^+(\vec{r},t)\partial_a\psi(\vec{r},t), \psi^+(\vec{r}',t)\partial'_b\psi(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{AB}\delta^{A'B'}\int[\psi^+_A(\vec{r},t)\partial_a\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\partial'_b\psi_{B'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{AB}\delta^{A'B'}\int\{\psi^+_A(\vec{r},t)\{\partial_a\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\}\partial'_b\psi_{B'}(\vec{r}',t) - \psi^+_{A'}(\vec{r}',t)\{\psi^+_A(\vec{r},t), \partial'_b\psi_{B'}(\vec{r}',t)\}\partial_a\psi_B(\vec{r},t)\}d^3\vec{r}d^3\vec{r}'$
$= -\delta^{AB}\delta^{A'B'}\int\{\psi^+_A(\vec{r},t)\delta_{A'B}\partial_a\delta^3(\vec{r}-\vec{r}')\partial'_b\psi_{B'}(\vec{r}',t) - \psi^+_{A'}(\vec{r}',t)\delta_{AB'}\partial'_b\delta^3(\vec{r}-\vec{r}')\partial_a\psi_B(\vec{r},t)\}d^3\vec{r}d^3\vec{r}'$
$= \delta^{AB}\delta^{A'B'}\int\{\psi^+_A(\vec{r},t)\delta_{A'B}\partial'_a\delta^3(\vec{r}-\vec{r}')\partial'_b\psi_{B'}(\vec{r}',t) - \psi^+_{A'}(\vec{r}',t)\delta_{AB'}\partial_b\delta^3(\vec{r}-\vec{r}')\partial_a\psi_B(\vec{r},t)\}d^3\vec{r}d^3\vec{r}'$
$= -\int\{\psi^+_A(\vec{r},t)\delta^{AB'}\partial_a\partial_b\psi_{B'}(\vec{r},t) - \psi^+_{A'}(\vec{r},t)\delta^{A'B}\partial_b\partial_a\psi_B(\vec{r},t)\}d^3\vec{r}$
$= \int \psi^+(\vec{r},t)[\hat{P}_a, \hat{P}_b]\psi(\vec{r},t)d^3\vec{r} = 0$ □

Proof: $[S_{ab}(t), S_{cd}(t)]$
$= \int[\psi^{+A}(\vec{r},t)S_{abA}{}^B\psi_B(\vec{r},t), \psi^{+C}(\vec{r},t)S_{cdC}{}^D\psi_D(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= \int[\psi^{+A}(\vec{r},t)S_{abA}{}^B\psi_B(\vec{r},t), \psi^{+C}(\vec{r},t)]S_{cdC}{}^D\psi_D(\vec{r}',t) + \psi^{+C}(\vec{r},t)[\psi^{+A}(\vec{r},t)S_{abA}{}^B\psi_B(\vec{r},t), S_{cdC}{}^D\psi_D(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= \int \psi^{+A}(\vec{r},t)\{S_{abA}{}^B\psi_B(\vec{r},t), \psi^{+C}(\vec{r},t)\}S_{cdC}{}^D\psi_D(\vec{r}',t) - \psi^{+C}(\vec{r},t)\{\psi^{+A}(\vec{r},t), S_{cdC}{}^D\psi_D(\vec{r}',t)\}S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= \int \psi^{+A}(\vec{r},t)S_{abA}{}^C\delta^3(\vec{r}-\vec{r}')S_{cdC}{}^D\psi_D(\vec{r}',t) - \psi^{+C}(\vec{r},t)S_{cdC}{}^A\delta^3(\vec{r}-\vec{r}')S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= \int \psi^+(\vec{r},t)[S_{ab}, S_{cd}]\psi(\vec{r},t)d^3\vec{r}$
$= -i(g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac})$ □

Proof: $[S_{ab}(t), L_{cd}]$
$= -i\int[\psi^{+A}(\vec{r},t)S_{abA}{}^B\psi_B(\vec{r},t), \psi^{+C}(\vec{r}',t)(r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -i\int\{\{\psi^{+A}(\vec{r},t)S_{abA}{}^B\psi_B(\vec{r},t), \psi^{+C}(\vec{r}',t)\}(r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}',t)$
$- \psi^{+C}(\vec{r}',t)\{\psi^{+A}(\vec{r},t), (r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}',t)\}S_{abA}{}^B\psi_B(\vec{r},t)\}d^3\vec{r}d^3\vec{r}'$
$= -i\int\psi^{+A}(\vec{r},t)S_{abA}{}^B\delta^C_B\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}',t) - \psi^{+C}(\vec{r}',t)\delta^A_C(r'_c\partial'_d - r'_d\partial'_c)\delta^3(\vec{r}-\vec{r}')S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -i\int\psi^{+A}(\vec{r},t)S_{abA}{}^B\delta^C_B\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}',t) + \psi^{+C}(\vec{r}',t)\delta^A_C(r'_c\partial_d - r'_d\partial_c)\delta^3(\vec{r}-\vec{r}')S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -i\int\psi^{+A}(\vec{r},t)S_{abA}{}^B(r_c\partial_d - r_d\partial_c)\psi_B(\vec{r},t) - \psi^{+A}(\vec{r},t)(r_c\partial_d - r_d\partial_c)S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= \int\psi^+(\vec{r},t)[S_{ab}, \hat{L}_{cd}]\psi(\vec{r},t)d^3\vec{r} = 0$ □

Proof: $[S_{ab}(t), P_c]$
$= -i\int[\psi^{+A}(\vec{r},t)S_{abA}{}^B\psi_B(\vec{r},t), \psi^{+C}(\vec{r}',t)\partial'_c\psi_C(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -i\int\{\psi^{+A}(\vec{r},t)S_{abA}{}^B\psi_B(\vec{r},t), \psi^{+C}(\vec{r}',t)\}\partial'_c\psi_C(\vec{r}',t) - \psi^{+C}(\vec{r}',t)\{\psi^{+A}(\vec{r},t), \partial'_c\psi_C(\vec{r}',t)\}S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -i\int\psi^{+A}(\vec{r},t)S_{abA}{}^B\delta^C_B\delta^3(\vec{r}-\vec{r}')\partial'_c\psi_C(\vec{r}',t) - \psi^{+C}(\vec{r}',t)\delta^A_C\partial'_c\delta^3(\vec{r}-\vec{r}')S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -i\int\psi^{+A}(\vec{r},t)S_{abA}{}^B\delta^C_B\delta^3(\vec{r}-\vec{r}')\partial'_c\psi_C(\vec{r}',t) + \psi^{+C}(\vec{r}',t)\delta^A_C(r'_c\partial_d - r'_d\partial_c)\delta^3(\vec{r}-\vec{r}')S_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -i\int\psi^{+A}(\vec{r},t)S_{abA}{}^B\partial_c\psi_B(\vec{r},t) - \psi^{+A}(\vec{r},t)\partial_cS_{abA}{}^B\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= \int\psi^+(\vec{r},t)[S_{ab}, \hat{P}_c]\psi(\vec{r},t)d^3\vec{r} = 0$ □

Cor. 3.17.1.

$$\begin{cases}\{\psi_{A_\varsigma}(x), \psi^+_{A'_\varsigma}(x')\} = -i\sqrt{2}\Gamma^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x-x') \\ \{\psi_{A_\varsigma}(x), \psi_{B_\varsigma}(x')\} = 0 \\ \{\psi^+_{A'_\varsigma}(x), \psi^+_{B'_\varsigma}(x')\} = 0\end{cases} \Rightarrow \begin{cases}\{\psi_{A_\varsigma}(\vec{r},t), \psi^+_{A'_\varsigma}(\vec{r}',t)\} = \delta_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\dot{\psi}_{A_\varsigma}(\vec{r},t), \psi^+_{A'_\varsigma}(\vec{r}',t)\} = (\sigma\cdot\nabla)_{A_\varsigma A'_\varsigma}\delta^3(\vec{r}-\vec{r}') \\ \{\psi_{A_\varsigma}(\vec{r},t), \psi_{B_\varsigma}(\vec{r}',t)\} = 0 \\ \{\psi^+_{A'_\varsigma}(\vec{r},t), \psi^+_{B'_\varsigma}(\vec{r}',t)\} = 0\end{cases}$$

Lem. 3.17.2. $\{A, A'\} = \{B, B'\} = 0 \Rightarrow [AB, A'B'] = A\{B, A'\}B' - A'\{A, B'\}B$

Proof: $[\tilde{M}_{ab}, \tilde{M}_{cd}]$
$= -\int[\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)\dot{\psi}(\vec{r},t), \psi^+(\vec{r}',t)(r'_c\sigma_d - r'_d\sigma_c)\dot{\psi}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\int[\psi^+_A(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)\{(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}',t)$
$- \psi^+_{A'}(\vec{r}',t)\{\psi^+_A(\vec{r},t), (r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}',t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)\{\dot{\psi}_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}',t)$
$- \psi^+_{A'}(\vec{r}',t)\{\psi^+_A(\vec{r},t), \dot{\psi}_{B'}(\vec{r}',t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)(\sigma\cdot\nabla)_{BA'}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}',t)$
$- \psi^+_{A'}(\vec{r}',t)(\sigma\cdot\nabla')_{B'A}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -\int -\psi^+_A(\vec{r},t)(\sigma\cdot\nabla')_{BA'}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}',t)$
$+ \psi^+_{A'}(\vec{r}',t)(\sigma\cdot\nabla)_{B'A}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)^{AB}(\sigma\cdot\nabla)_{BA'}[(r_c\sigma_d - r_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r},t)]$
$- \psi^+_{A'}(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)^{A'B'}(\sigma\cdot\nabla)_{B'A}[(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r},t)]d^3\vec{r}$
$= -\int\{\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)(\sigma\cdot\nabla)[(r_c\sigma_d - r_d\sigma_c)\dot{\psi}(\vec{r},t)] - \psi^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)(\sigma\cdot\nabla)[(r_a\sigma_b - r_b\sigma_a)\dot{\psi}(\vec{r},t)]\}d^3\vec{r}$
$= -\int\{\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)[\sigma_c, \sigma_d]\dot{\psi}(\vec{r},t)] - \psi^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)[\sigma_a, \sigma_b]\dot{\psi}(\vec{r},t)]\}d^3\vec{r}$
$- \int\{\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)[(r_c\sigma_d - r_d\sigma_c)(\sigma\cdot\nabla)\dot{\psi}(\vec{r},t)] - \psi^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)[(r_a\sigma_b - r_b\sigma_a)(\sigma\cdot\nabla)\dot{\psi}(\vec{r},t)]\}d^3\vec{r}$
$= -\int\{\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)[\sigma_c, \sigma_d]\dot{\psi}(\vec{r},t) - \psi^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)[\sigma_a, \sigma_b]\dot{\psi}(\vec{r},t)\}d^3\vec{r}$
$- \int\{\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)(r_c\sigma_d - r_d\sigma_c)(\sigma\cdot\nabla)\dot{\psi}(\vec{r},t) - \psi^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)(r_a\sigma_b - r_b\sigma_a)(\sigma\cdot\nabla)\dot{\psi}(\vec{r},t)\}d^3\vec{r}$
$= -\int\{\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)[\sigma_c, \sigma_d]\dot{\psi}(\vec{r},t) - \psi^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)[\sigma_a, \sigma_b]\dot{\psi}(\vec{r},t)\}d^3\vec{r}$
$- \int\psi^+(\vec{r},t)[(r_a\sigma_b - r_b\sigma_a), (r_c\sigma_d - r_d\sigma_c)](\sigma\cdot\nabla)\dot{\psi}(\vec{r},t)d^3\vec{r}$
=??? □

Proof: $[\bar{M}_{ab}, \bar{M}_{cd}]$
$= -\int[\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)\psi(\vec{r},t), \psi^+(\vec{r}',t)(r'_c\sigma_d - r'_d\sigma_c)\psi(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\int[\psi^+_A(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}',t)]d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)\{(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}',t)$
$- \psi^+_{A'}(\vec{r}',t)\{\psi^+_A(\vec{r},t), (r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}',t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)\{\psi_B(\vec{r},t), \psi^+_{A'}(\vec{r}',t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}',t)$
$- \psi^+_{A'}(\vec{r}',t)\{\psi^+_A(\vec{r},t), \psi_{B'}(\vec{r}',t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)\delta_{BA'}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}',t)$
$- \psi^+_{A'}(\vec{r}',t)\delta_{B'A}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$
$= -\int\psi^+_A(\vec{r},t)\delta_{BA'}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}',t)$
$- \psi^+_{A'}(\vec{r}',t)\delta_{B'A}\delta^3(\vec{r}-\vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_B(\vec{r},t)d^3\vec{r}d^3\vec{r}'$

$= -\int \psi_A^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)^{AB}\delta_{BA'}[(r_c\sigma_d - r_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r},t)]$
$- \psi_{A'}^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)^{A'B'}\delta_{B'A}[(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r},t)]d^3\vec{r}$
$= -\int\{\psi^+(\vec{r},t)(r_a\sigma_b - r_b\sigma_a)[(r_c\sigma_d - r_d\sigma_c)\psi(\vec{r},t)] - \psi^+(\vec{r},t)(r_c\sigma_d - r_d\sigma_c)[(r_a\sigma_b - r_b\sigma_a)\psi(\vec{r},t)]\}d^3\vec{r}$
$= -\int \psi^+(\vec{r},t)[(r_a\sigma_b - r_b\sigma_a),(r_c\sigma_d - r_d\sigma_c)]\psi(\vec{r},t)d^3\vec{r}$ □

4 Photon spinor field Covariant quantization scheme

4.1 Photon spinor spin operator equation and its plane wave solution

Thm. 4.1.1. $[\partial_a + iS_{ab}(1,\varsigma)\partial^b]\psi(x) = 0$

Cor. 4.1.1. $\begin{cases}\psi(\vec{r},t) := \frac{1}{(2\pi)^{3/2}}\int\sqrt{|\vec{p}|}\lambda(\hat{p},-\varsigma)[a_1(\vec{p},-\varsigma)e^{ip\cdot x} + a_2^+(\vec{p},-\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\ \sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-\varsigma)\psi(\vec{r},t)e^{-ip\cdot x}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int\frac{i}{|\vec{p}|}\lambda^+(\hat{p},-\varsigma)\dot{\psi}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ \sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-\varsigma)\psi(\vec{r},t)e^{ip\cdot x}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int\frac{-i}{|\vec{p}|}\lambda^+(\hat{p},-\varsigma)\dot{\psi}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$

Def. 4.1.1. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(1,\varsigma) := \lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\varsigma), \hat{P}^2(1,\varsigma) = \hat{P}(1,\varsigma), \hat{P}^+(1,\varsigma) = \hat{P}(1,\varsigma)$

4.2 Properties of covariant constant invariant tensor for photon spinor field

Cor. 4.2.1.
$\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}(1) = (\frac{1}{\sqrt{2}})^2\delta_{k_\varsigma k'_\varsigma}$
$\Gamma^{i\pi}_{k_\varsigma k'_\varsigma}(1) = -i\varsigma(\frac{1}{\sqrt{2}})^2\sigma^i(1)_{k_\varsigma k'_\varsigma}$
$\Gamma^{ij}_{k_\varsigma k'_\varsigma}(1) = -(\frac{1}{\sqrt{2}})^2[\sigma^{\{i}(1)\sigma^{j\}}(1) - \delta^{ij}]_{k_\varsigma k'_\varsigma} = -(\frac{1}{\sqrt{2}})^2 2\frac{1}{2!}[\sigma^{\{i}(1)\sigma^{j\}}(1) - \frac{1}{2}\delta^{\{ij\}}]_{k_\varsigma k'_\varsigma}$

Lem. 4.2.1. $\Gamma^{ab}_{k_\varsigma k'_\varsigma}p_ap_b = -2|\vec{p}|^2\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\varsigma)$

Proof: $\Gamma^{ab}_{k_\varsigma k'_\varsigma}p_ap_b$
$= C_2^2\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}(1)p_\pi^2 + C_2^1\Gamma^{i\pi}_{k_\varsigma k'_\varsigma}(1)p_ip_\pi + C_2^0\Gamma^{ij}_{k_\varsigma k'_\varsigma}(1)p_ip_j$
$= (\frac{1}{\sqrt{2}})^2\{-|\vec{p}|^2 + 2|\vec{p}|\varsigma[\sigma^i(1)\cdot\vec{p}] - 2[\sigma^i(1)\cdot\vec{p}]^2 + |\vec{p}|^2\}_{k_\varsigma k'_\varsigma}$
$= (\frac{1}{\sqrt{2}})^2|\vec{p}|^2\{2\varsigma[\sigma^i(1)\cdot\hat{p}] - 2[\sigma^i(1)\cdot\hat{p}]^2\}_{k_\varsigma k'_\varsigma}$
$= (\frac{1}{\sqrt{2}})^2|\vec{p}|^2\{2\varsigma[\sigma^i(1)\cdot\hat{p}] - 2[\sigma^i(1)\cdot\hat{p}]^2\}\sum_{h=1}^{-1}\lambda(\hat{p},h)\lambda^+(\hat{p},h)\}_{k_\varsigma k'_\varsigma}$
$= -2|\vec{p}|^2\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\varsigma)$ □

Cor. 4.2.2. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(1,\varsigma) = -\Gamma^{ab}_{k_\varsigma k'_\varsigma}\hat{p}_a\hat{p}_b \to \Gamma^{ab}_{k_\varsigma k'_\varsigma}\hat{\partial}_a\hat{\partial}_b$

4.3 General covariant commutation rules in mathematics for photon spinor field

Thm. 4.3.1.

$\begin{cases}[a_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm = \delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma), a_{\sigma'}(\vec{p}',-\varsigma)]_\pm = 0\\ [a^+_\sigma(\vec{p},-\varsigma), a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm = 0\end{cases} \Rightarrow \begin{cases}[\Psi_{k_\varsigma}(x), \Psi^+_{k'_\varsigma}(x')]_\pm\\ = i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b[\delta_1\Delta(x-x') - (\delta_1\pm\delta_2)\Delta^{(-)}(x-x')]\\ [\Psi_{k_\varsigma}(x), \Psi_{\beta_\varsigma}(x')]_\pm = 0\\ [\Psi^+_{k'_\varsigma}(x), \Psi^+_{\beta'_\varsigma}(x')]_\pm = 0\end{cases}$

Proof: $[\Psi^{(+)}_{k_\varsigma}(x), \Psi^{(+)+}_{k'_\varsigma}(x')]_\pm$
$= \frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\hat{p}',-\varsigma)\sqrt{|\vec{p}||\vec{p}'|}[a_1(\vec{p},-\varsigma), a_1^+(\vec{p}',-\varsigma)]_\pm e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\varsigma)|\vec{p}|\delta_1\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\varsigma)\delta_1|\vec{p}|e^{ip\cdot(x-x')}d^3\vec{p}$
$= \frac{-\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\Gamma^{ab}_{k_\varsigma k'_\varsigma}p_ap_be^{ip\cdot(x-x')}d^3\vec{p}$
$= i\delta_1\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(+)}(x-x')$ □

Proof: $[\Psi^{(-)}_{k_\varsigma}(x), \Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$
$= \frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-\varsigma)\sqrt{|\vec{p}||\vec{p}'|}[a_2^+(\vec{p},-\varsigma), a_2(\vec{p}',-\varsigma)]_\pm e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-\varsigma)|\vec{p}|\delta_2\delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\varsigma)\delta_2|\vec{p}|e^{-ip\cdot(x-x')}d^3\vec{p}$
$= \pm\frac{-\delta_2}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\Gamma^{ab}_{k_\varsigma k'_\varsigma}p_ap_be^{-ip\cdot(x-x')}d^3\vec{p}$
$= -\pm i\delta_2\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(-)}(x-x')$ □

Proof: $[\Psi_{k_\varsigma}(x), \Psi^+_{k'_\varsigma}(x')]_\pm$
$= [\Psi^{(+)}_{k_\varsigma}(x), \Psi^{(+)+}_{k'_\varsigma}(x')]_\pm + [\Psi^{(-)}_{k_\varsigma}(x), \Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$
$= i\delta_1 \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(+)}(x-x') - \pm i\delta_2\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(-)}(x-x')$
$= i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b[\delta_1\Delta^{(+)}(x-x') - \pm\delta_2\Delta^{(-)}(x-x')]$
$= i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b[(\delta_1 \pm \delta_2)\Delta^{(+)}(x-x') - \pm\delta_2\Delta(x-x')]$
$= i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b[\delta_1\Delta(x-x') - (\delta_1 \pm \delta_2)\Delta^{(-)}(x-x')]$ □

From the above, only $\delta_1 \pm \delta_2 = 0$, the micro causality is satisfied. At the same time only when $\delta_1, \delta_2 \geq 0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1 = \delta_2 = 1$, it satisfies the commutative relation. There are actually two other options. Namely when $\delta_1 = \delta_2 = 0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

4.4 Covariant commutation rules of photon spinor field

From the previous section, we can see that the commutation rules with physical significance are as follows:(In order to confirm each other, a new proof has been made.)

Thm. 4.4.1.
$$\begin{cases}[a_\sigma(\vec p,-\varsigma), a^+_{\sigma'}(\vec p\,',-\varsigma)] = \delta_{\sigma\sigma'}\delta^3(\vec p-\vec p\,')\\ [a_\sigma(\vec p,-\varsigma), a_{\sigma'}(\vec p\,',-\varsigma)] = 0\\ [a^+_\sigma(\vec p,-\varsigma), a^+_{\sigma'}(\vec p\,',-\varsigma)] = 0\end{cases} \Rightarrow \begin{cases}[\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')] = i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x), \psi_{l_\varsigma}(x')] = 0\\ [\psi^+_{k'_\varsigma}(x), \psi^+_{l'_\varsigma}(x')] = 0\end{cases}$$

Proof: $\{\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')\}$
$= \frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p\,'$
$\lambda_{k_\varsigma}(\hat p,-\varsigma)\lambda^+_{k'_\varsigma}(\hat p,-\varsigma)|\vec p|^{1/2}|\vec p\,'|^{1/2}\{[a_1(\vec p,-\varsigma), a^+_1(\vec p\,',-\varsigma)]e^{ip\cdot(x-x')} + [a^+_2(\vec p,-\varsigma), a_2(\vec p\,',-\varsigma)]e^{-ip\cdot(x-x')}\}$
$= \frac{1}{(2\pi)^3}\int |\vec p|\lambda_{k_\varsigma}(\hat p,-\varsigma)\lambda^+_{k'_\varsigma}(\hat p,-\varsigma)[\delta^3(\vec p-\vec p\,')e^{ip\cdot(x-x')} - \delta^3(\vec p-\vec p\,')e^{-ip\cdot(x-x')}]d^3\vec p d^3\vec p\,'$
$= \frac{1}{(2\pi)^3}\int |\vec p|\lambda_{k_\varsigma}(\hat p,-\varsigma)\lambda^+_{k'_\varsigma}(\hat p,-\varsigma)[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec p$
$= -\frac{1}{(2\pi)^3}\int \frac{1}{2|\vec p|}\Gamma^{ab}_{k_\varsigma k'_\varsigma}p_a p_b[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec p$
$= \frac{1}{(2\pi)^3}\int \frac{1}{2|\vec p|}\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec p$
$= i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\frac{-i}{(2\pi)^3}\int \frac{1}{2|\vec p|}[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec p$
$= i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

4.5 Isochronous commutation rules for photon spinor field

Cor. 4.5.1.
$$\begin{cases}[\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')] = i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x), \psi_{l_\varsigma}(x')] = 0\\ [\psi^+_{k'_\varsigma}(x), \psi^+_{l'_\varsigma}(x')] = 0\end{cases} \Rightarrow \begin{cases}[\psi_{k_\varsigma}(\vec r,t), \psi^+_{k'_\varsigma}(\vec r\,',t)] = i\varsigma[\sigma(1)\cdot\nabla]_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r\,')\\ [\psi_{k_\varsigma}(\vec r,t), \psi_{l_\varsigma}(\vec r\,',t)] = 0\\ [\psi^+_{k'_\varsigma}(\vec r,t), \psi^+_{l'_\varsigma}(\vec r\,',t)] = 0\end{cases}$$

Cor. 4.5.2.
$$\begin{cases}[\psi_{k_\varsigma}(\vec r,t), \psi^+_{k'_\varsigma}(\vec r\,',t)] = i\varsigma[\sigma(1)\cdot\nabla]_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r\,')\\ [\psi_{k_\varsigma}(\vec r,t), \psi_{l_\varsigma}(\vec r\,',t)] = 0\\ [\psi^+_{k'_\varsigma}(\vec r,t), \psi^+_{l'_\varsigma}(\vec r\,',t)] = 0\end{cases} \Rightarrow \begin{cases}[a_\sigma(\vec p,-\varsigma), a^+_{\sigma'}(\vec p\,',-\varsigma)] = \delta_{\sigma\sigma'}\delta^3(\vec p-\vec p\,')\\ [a_\sigma(\vec p,-\varsigma), a_{\sigma'}(\vec p\,',-\varsigma)] = 0\\ [a^+_\sigma(\vec p,-\varsigma), a^+_{\sigma'}(\vec p\,',-\varsigma)] = 0\end{cases}$$

Proof: $[a_1(\vec p,-\varsigma), a^+_1(\vec p\,',-\varsigma)]$
$= \frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec p||\vec p\,'|}}\int[\lambda^{+k_\varsigma}(\hat p,-\varsigma)\Psi_{k_\varsigma}(\vec r,t)e^{-i(\vec p\cdot\vec r-Et)}, \lambda^{k'_\varsigma}(\vec p\,',-\varsigma)\Psi^+_{k'_\varsigma}(\vec r\,',t)e^{i(\vec p\,'\cdot\vec r\,'-E't)}]d^3\vec r d^3\vec r\,'$
$= \frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec p||\vec p\,'|}}\int \lambda^{+k_\varsigma}(\hat p,-\varsigma)\lambda^{k'_\varsigma}(\vec p\,',-\varsigma)[\Psi_{k_\varsigma}(\vec r,t), \Psi^+_{k'_\varsigma}(\vec r\,',t)]e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p\,'\cdot\vec r\,'-E't)}d^3\vec r d^3\vec r\,'$
$= i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec p||\vec p\,'|}}\int \lambda^{+k_\varsigma}(\hat p,-\varsigma)\lambda^{k'_\varsigma}(\vec p\,',-\varsigma)[\sigma(1)\cdot\nabla]_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r\,')e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p\,'\cdot\vec r\,'-E't)}d^3\vec r d^3\vec r\,'$
$= i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec p||\vec p\,'|}}\int \lambda^{+k_\varsigma}(\hat p,-\varsigma)\lambda^{k'_\varsigma}(\vec p\,',-\varsigma)[\sigma(1)\cdot\vec p]_{k_\varsigma k'_\varsigma}ie^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p\,'\cdot\vec r-E't)}d^3\vec r$
$= -\varsigma\frac{1}{|\vec p|}\lambda^{+k_\varsigma}(\hat p,-\varsigma)\lambda^{k'_\varsigma}(\hat p,-\varsigma)[\sigma(1)\cdot\vec p]_{k_\varsigma k'_\varsigma}\delta^3(\vec p-\vec p\,')$
$= -\varsigma\lambda^+(\hat p,-\varsigma)\frac{\sigma(1)\cdot\vec p}{|\vec p|}\lambda(\hat p,-\varsigma)\delta^3(\vec p-\vec p\,')$
$= \lambda^+(\hat p,-\varsigma)\lambda(\hat p,-\varsigma)\delta^3(\vec p-\vec p\,')$
$= \delta^3(\vec p-\vec p\,')$ □

Proof: $[a^+_2(\vec p,-\varsigma), a_2(\vec p\,',-\varsigma)]$
$= \frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec p||\vec p\,'|}}\int[\lambda^{+k_\varsigma}(\hat p,-\varsigma)\Psi_{k_\varsigma}(\vec r,t)e^{i(\vec p\cdot\vec r-Et)}, \lambda^{k'_\varsigma}(\vec p\,',-\varsigma)\Psi^+_{k'_\varsigma}(\vec r\,',t)e^{-i(\vec p\,'\cdot\vec r\,'-E't)}]d^3\vec r d^3\vec r\,'$
$= \frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec p||\vec p\,'|}}\int \lambda^{+k_\varsigma}(\hat p,-\varsigma)\lambda^{k'_\varsigma}(\vec p\,',-\varsigma)[\Psi_{k_\varsigma}(\vec r,t), \Psi^+_{k'_\varsigma}(\vec r\,',t)]e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p\,'\cdot\vec r\,'-E't)}d^3\vec r d^3\vec r\,'$

$= i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+k_\varsigma}(\hat{p},-\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-\varsigma)[\sigma(1)\cdot\nabla]_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')e^{i(\vec{p}\cdot\vec{r}-Et)}e^{-i(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$= i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+k_\varsigma}(\hat{p},-\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-\varsigma)[\sigma(1)\cdot\vec{p}]_{k_\varsigma k'_\varsigma}(-i)e^{i(\vec{p}\cdot\vec{r}-Et)}e^{-i(\vec{p}'\cdot\vec{r}-E't)}d^3\vec{r}$
$= \varsigma\frac{1}{|\vec{p}|}\lambda^{+k_\varsigma}(\hat{p},-\varsigma)\lambda^{k'_\varsigma}(\hat{p},-\varsigma)[\sigma(1)\cdot\vec{p}]_{k_\varsigma k'_\varsigma}\delta^3(\vec{p}-\vec{p}')$
$= \varsigma\lambda^{+}(\hat{p},-\varsigma)\frac{\sigma(1)\cdot\vec{p}}{|\vec{p}|}\lambda(\hat{p},-\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= -\lambda^{+}(\hat{p},-\varsigma)\lambda(\hat{p},-\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= -\delta^3(\vec{p}-\vec{p}')$ □

4.6 Summary of commutation rules for photon spinor field

he proof in the above sections exactly forms a logical closed-loop, so it has the following properties:

Cor. 4.6.1. $\begin{cases}[a_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma),a_{\sigma'}(\vec{p}',-\varsigma)]=0\\ [a^+_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]=0\end{cases} \Leftrightarrow \begin{cases}[a_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}),a_{\sigma'}(\vec{p}')]=0\\ [a^+_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]=0\end{cases}$

$\Updownarrow \qquad\qquad \Updownarrow$

Cor. 4.6.2. $\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases} \Leftrightarrow \begin{cases}[\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]=i\varsigma[\sigma(1)\cdot\nabla]_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ [\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)]=0\\ [\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)]=0\end{cases}$

Cor. 4.6.3. $\sigma_{-\varsigma}=S_{em}(\varsigma)(\sigma\otimes I)S^+_{em}(\varsigma),\sigma_{+\varsigma}=S_{em}(\varsigma)(I\otimes\sigma)S^+_{em}(\varsigma),\gamma=S_m(1)\sigma(1)S^-_m(1)$

4.7 Equivalence commutation rules of multiple spinor forms for electromagnetic field

Thm. 4.7.1.

$\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0\\ [\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\end{cases} \Leftrightarrow \begin{cases}[\Psi_{A_\varsigma B_\varsigma}(x),\Psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ =-\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{A_\varsigma B_\varsigma}(x),\Psi_{C_\varsigma D_\varsigma}(x')]=0\\ [\Psi^+_{A'_\varsigma B'_\varsigma}(x),\Psi^+_{C'_\varsigma D'_\varsigma}(x')]=0\end{cases}$

Proof: $[\Psi_{A_\varsigma B_\varsigma}(x),\Psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= [\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\Psi_{\alpha_\varsigma}(x),\frac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\Psi_{\alpha'_\varsigma}(x')]$
$= \frac{1}{2}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}[\Psi_{\alpha_\varsigma}(x),\Psi_{\alpha'_\varsigma}(x')]$
$= \frac{1}{2}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= \frac{i}{2}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^a{}_{C_\varsigma C'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma,i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{C'_\varsigma D'_\varsigma}_{\alpha'_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{8}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\alpha'_\varsigma}(\sigma,i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{8}\delta^{C_\varsigma}_{\{A_\varsigma}\delta^{D_\varsigma}_{B_\varsigma\}}\delta^{C'_\varsigma}_{(A'_\varsigma}\delta^{D'_\varsigma}_{B'_\varsigma)}(\sigma,i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{8}(\sigma,i\varsigma)^a{}_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma\}B'_\varsigma)}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

Proof: $[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]$
$= [\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\Psi_{A_\varsigma B_\varsigma}(x),\frac{-i\varsigma}{\sqrt{2}}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\Psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= \frac{1}{2}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}[\Psi_{A_\varsigma B_\varsigma}(x),\Psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= -\frac{i}{4}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

Thm. 4.7.2.

$\begin{cases}[\Psi_{A_\varsigma B_\varsigma}(x),\Psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ =-\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{A_\varsigma B_\varsigma}(x),\Psi_{C_\varsigma D_\varsigma}(x')]=0\\ [\Psi^+_{A'_\varsigma B'_\varsigma}(x),\Psi^+_{C'_\varsigma D'_\varsigma}(x')]=0\end{cases} \Leftrightarrow \begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases}$

Proof: $[\Psi_{k_\varsigma}(x),\Psi^+_{k'_\varsigma}(x')]$
$= [\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\Psi_{A_\varsigma B_\varsigma}(x),\Gamma^{A'_\varsigma B'_\varsigma}_{k'_\varsigma}(1)\Psi^+_{A'_\varsigma B'_\varsigma}(x')]$

$= \Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\Gamma^{A'_\varsigma B'_\varsigma}_{k'_\varsigma}(1)[\Psi_{A_\varsigma B_\varsigma}(x), \Psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= -\frac{i}{2}\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\Gamma^{A'_\varsigma B'_\varsigma}_{k'_\varsigma}(1)(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x - x')$
$= i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x - x')$ □

Proof: $[\Psi_{A_\varsigma B_\varsigma}(x), \Psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= [\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}(x), \Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)\psi^+_{k'_\varsigma}(x')]$
$= \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)[\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')]$
$= \Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x - x')$
$= -\frac{i}{2}\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)\Gamma^{C_\varsigma D_\varsigma}_{k_\varsigma}(1)\Gamma^{C'_\varsigma D'_\varsigma}_{k'_\varsigma}(1)(\sigma, i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x - x')$
$= -\frac{i}{8}\delta^{\{C_\varsigma}_{A_\varsigma}\delta^{D_\varsigma\}}_{B_\varsigma}\delta^{(C'_\varsigma}_{A'_\varsigma}\delta^{D'_\varsigma)}_{B'_\varsigma}(\sigma, i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x - x')$
$= -\frac{i}{8}(\sigma, i\varsigma)^a{}_{\{A_\varsigma(A'_\varsigma}(\sigma, i\varsigma)^b{}_{B\}_\varsigma B'_\varsigma)}\partial_a\partial_b\Delta(x - x')$
$= -\frac{i}{2}(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x - x')$ □

Lem. 4.7.1. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} = \Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)\Gamma^{ab}_{k_\varsigma k'_\varsigma}, \Gamma^{ab}_{k_\varsigma k'_\varsigma} = \Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}$

Thm. 4.7.3.

$$\begin{cases}[\Psi_{\alpha_\varsigma}(x), \Psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x - x')\\ [\Psi_{\alpha_\varsigma}(x), \Psi_{\beta_\varsigma}(x')] = 0\\ [\Psi^+_{\alpha'_\varsigma}(x), \Psi^+_{\beta'_\varsigma}(x')] = 0\end{cases} \Leftrightarrow \begin{cases}[\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')] = i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x - x')\\ [\psi_{k_\varsigma}(x), \psi_{l_\varsigma}(x')] = 0\\ [\psi^+_{k'_\varsigma}(x), \psi^+_{l'_\varsigma}(x')] = 0\end{cases}$$

Proof: $[\Psi_{k_\varsigma}(x), \Psi^+_{k'_\varsigma}(x')]$
$= [\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Psi_{\alpha_\varsigma}(x), \Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)\Psi^+_{\alpha'_\varsigma}(x')]$
$= \Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)[\Psi_{k_\varsigma}(x), \Psi^+_{k'_\varsigma}(x')]$
$= \Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)i\sigma^{ab}_{\alpha_\varsigma\alpha_\varsigma}\partial_a\partial_b\Delta(x - x')$
$= i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x - x')$ □

Proof: $[\Psi_{\alpha_\varsigma}(x), \Psi^+_{\alpha'_\varsigma}(x')]$
$= [\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Psi_{k_\varsigma}(x), \Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)\Psi^+_{k'_\varsigma}(x')]$
$= \Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)[\Psi_{\alpha_\varsigma}(x), \Psi^+_{\alpha'_\varsigma}(x')]$
$= \Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x - x')$
$= i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x - x')$ □

4.8 Commutative function, causal function and feynman propagator of photon spinor field

Cor. 4.8.1.

$$\begin{cases}\Delta_{k_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x)\\ \Delta^{(+)}_{k_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(+)}(x)\\ \Delta^{(-)}_{k_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(-)}(x)\\ \Delta^{(l)}_{k_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(l)}(x)\end{cases}$$

Cor. 4.8.2.

$$\begin{cases}\Delta^{(c)}_{k_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(c)}(x) + [\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t) + 2i\Gamma^{i\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i]\Delta(x) = \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(c)}(x) + \Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta^4(x)\\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{ret}(x) + [\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t) + 2i\Gamma^{i\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i]\Delta(x) = \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(c)}(x) + \Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta^4(x)\\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{adv}(x) + [\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t) + 2i\Gamma^{i\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i]\Delta(x) = \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(c)}(x) + \Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta^4(x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(1; x) := \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta_F(x) + i[\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t) + 2i\Gamma^{i\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i]\Delta(x) = \Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta^{(c)}(x) + i\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}\delta^4(x)\\ = i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(1; x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(1; p) = \frac{i\Gamma^{ab}_{k_\varsigma k'_\varsigma}p_a p_b}{p^2 - i\varepsilon} + \cdots\end{cases}$$

Cor. 4.8.3.

$$\begin{cases}[\partial_a + iS_{ab}(1, \varsigma)\partial^b]\Delta(1; x) = 0\\ [\partial_a + iS_{ab}(1, \varsigma)\partial^b]\Delta^{(+)}(1; x) = 0\\ [\partial_a + iS_{ab}(1, \varsigma)\partial^b]\Delta^{(-)}(1; x) = 0\\ [\partial_a + iS_{ab}(1, \varsigma)\partial^b]\Delta^{(l)}(1; x) = 0\end{cases} \quad \begin{cases}[\partial^a + iS^{ab}(1, \varsigma)\partial_b]\Delta^{(c)}(1; x) = -\varsigma[\sigma(1), i\varsigma]_a\delta(t)\Delta(1; x)|_{t=0}\\ [\partial^a + iS^{ab}(1, \varsigma)\partial_b]\Delta^{ret}(1; x) = -\varsigma[\sigma(1), i\varsigma]_a\delta(t)\Delta(1; x)|_{t=0}\\ [\partial^a + iS^{ab}(1, \varsigma)\partial_b]\Delta^{adv}(1; x) = -\varsigma[\sigma(1), i\varsigma]_a\delta(t)\Delta(1; x)|_{t=0}\\ [\partial^a + iS^{ab}(1, \varsigma)\partial_b]\Delta_F(1; x) = -i\varsigma[\sigma(1), i\varsigma]_a\delta(t)\Delta(1; x)|_{t=0}\end{cases}$$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 4.8.4.

$$\begin{cases}(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta(1;x)=0\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(+)}(1;x)=0\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(-)}(1;x)=0\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(l)}(1;x)=0\end{cases}\quad\begin{cases}(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(c)}(1;x)=-\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{ret}(1;x)=-\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{adv}(1;x)=-\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta_F(1;x)=-i\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\end{cases}$$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 4.8.5.

$$\begin{cases}(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta(1;x)\bar{N}(1)=0\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(+)}(1;x)\bar{N}(1)=0\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(-)}(1;x)\bar{N}(1)=0\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(l)}(1;x)\bar{N}(1)=0\end{cases}\quad\begin{cases}(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{(c)}(1;x)\bar{N}(1)=-\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\bar{N}(1)\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{ret}(1;x)\bar{N}(1)=-\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\bar{N}(1)\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta^{adv}(1;x)\bar{N}(1)=-\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\bar{N}(1)\\(\sigma\otimes I,-i\varsigma)_a\partial^a N(1)\Delta_F(1;x)\bar{N}(1)=-i\varsigma\delta(t)N(1)\Delta(1;x)|_{t=0}\bar{N}(1)\end{cases}$$

$[\Downarrow]$ $[\Downarrow]$

Cor. 4.8.6.

$$\begin{cases}[\sigma(1),-i\varsigma]_a\partial^a\Delta(1;x)=0\\[\sigma(1),-i\varsigma]_a\partial^a\Delta^{(+)}(1;x)=0\\[\sigma(1),-i\varsigma]_a\partial^a\Delta^{(-)}(1;x)=0\\[\sigma(1),-i\varsigma]_a\partial^a\Delta^{(l)}(1;x)=0\end{cases}\quad\begin{cases}[\sigma(1),-i\varsigma]_a\partial^a\Delta^{(c)}(1;x)=-\varsigma\delta(t)\Delta(1;x)|_{t=0}\\[\sigma(1),-i\varsigma]_a\partial^a\Delta^{ret}(1;x)=-\varsigma\delta(t)\Delta(1;x)|_{t=0}\\[\sigma(1),-i\varsigma]_a\partial^a\Delta^{adv}(1;x)=-\varsigma\delta(t)\Delta(1;x)|_{t=0}\\[\sigma(1),-i\varsigma]_a\partial^a\Delta_F(1;x)=-i\varsigma\delta(t)\Delta(1;x)|_{t=0}\end{cases}$$

4.9 Quantum equation of photon spinor field

Cor. 4.9.1.

$$[\partial_a+iS_{ab}(1,\varsigma)\partial^b]\psi=0\Leftrightarrow[P_a,\psi(\vec{r},t)]=i\partial_a\psi(\vec{r},t);\begin{cases}[\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]=i\varsigma\sigma^i(1)_{k_\varsigma k'_\varsigma}\partial_i\delta^3(\vec{r}-\vec{r}')\\[\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)]=0,[\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)]=0\end{cases}$$

4.10 Poincare symmetry of photon spinor field

Cor. 4.10.1.

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i\sum_{l=0}^{[s]}(-1)^lC^{2l}_{2s}\Gamma^{\overbrace{ij\cdots}^{2s-2l}\overbrace{\pi\cdots\pi}^{2l}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2l}\nabla^{2l}\delta^3(\vec{r}-\vec{r}')\\\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i\sum_{l=0}^{[s]}(-1)^lC^{2l}_{2s}\Gamma^{\overbrace{ij\cdots}^{2s-2l}\overbrace{\pi\cdots\pi}^{2l}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdots}^{2s-2l}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

Cor. 4.10.2.

$\Gamma^{\pi\pi}_{k_\varsigma k'_\varsigma}(1)=(\frac{1}{\sqrt{2}})^2\delta_{k_\varsigma k'_\varsigma}$

$\Gamma^{i\pi}_{k_\varsigma k'_\varsigma}(1)=-i\varsigma(\frac{1}{\sqrt{2}})^2\sigma^i(1)_{k_\varsigma k'_\varsigma}$

$\Gamma^{ij}_{k_\varsigma k'_\varsigma}(1)=-(\frac{1}{\sqrt{2}})^2[\sigma^{\{i}(1)\sigma^{j\}}(1)-\delta^{ij}]_{k_\varsigma k'_\varsigma}=-(\frac{1}{\sqrt{2}})^2 2\frac{1}{2!}[\sigma^{\{i}(1)\sigma^{j\}}(1)-\frac{1}{2}\delta^{\{ij\}}]_{k_\varsigma k'_\varsigma}$

Cor. 4.10.3. $\Gamma^{ab}(1)\partial_a\partial_b\partial_\pi\Delta(x-x')|_{t=t'}=i\{\Gamma^{ij}(1)\partial_i\partial_j\delta^3(\vec{r}-\vec{r}')-\Gamma^{\pi\pi}(1)\nabla^2\delta^3(\vec{r}-\vec{r}')\}=-i[\sigma(1)\cdot\nabla]^2\delta^3(\vec{r}-\vec{r}')$

Cor. 4.10.4.

$$\begin{cases}[\dot{\psi}_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=-\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b|\partial_\pi\Delta(x-x')\\[\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0\\[\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases}\Rightarrow\begin{cases}[\frac{\dot{\psi}_{k_\varsigma}(\vec{r},t)}{\sqrt{(-\nabla^2)}},\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{(-\nabla'^2)}}]=-i[\sigma(1)\cdot\hat{\nabla}]^2\delta^3(\vec{r}-\vec{r}')\\[\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)]=0\\[\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)]=0\end{cases}$$

Cor. 4.10.5.

$\hat{P}_a(n)=\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{P}_a\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=-i\int\frac{\dot{\psi}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{P}_a\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$

$M_{ab}(n)=\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{M}_{ab}\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=-i\int\frac{\dot{\psi}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{M}_{ab}\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$

Thm. 4.10.1. $\begin{cases}[L_{ab},L_{cd}]=-i(g_{ad}L_{bc}-g_{ac}L_{bd}+g_{bc}L_{ad}-g_{bd}L_{ac})\\[L_{ab},P_c]=-i(g_{bc}P_a-g_{ac}P_b),[P_a,P_b]=0\end{cases}$

Proof: $[L_{ab},L_{cd}]$

$=-\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b-r_b\partial_a)\frac{i\dot{\psi}(\vec{r},t)}{\sqrt{-\nabla^2}},\frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}}(r'_c\partial'_d-r'_d\partial'_c)\frac{i\dot{\psi}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]$

$=\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b-r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}},\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}(r'_c\partial'_d-r'_d\partial'_c)\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]$

$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}[(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}](r'_c\partial'_d - r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$$
$$+ \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, (r'_c\partial'_d - r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}](r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}(r'_c\partial'_d - r'_d\partial'_c)\{-i[\sigma(1)\cdot\hat\nabla']^2\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial'_b - r_b\partial'_a)\{-i[\sigma(1)\cdot\hat\nabla']^2\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}(r'_c\partial_d - r'_d\partial_c)\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l_\varsigma k'_\varsigma}(r_c\partial_d - r_d\partial_c)\frac{\dot\psi_{l'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_c\partial_d - r_d\partial_c)\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l'_\varsigma k_\varsigma}(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= -\int\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[-i(r_a\partial_b - r_b\partial_a), -i(r_c\partial_d - r_d\partial_c)]\{-i[\sigma(1)\cdot\hat\nabla]^2\}\frac{\dot\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$$
$$= \int\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[\hat{L}_{ab}, \hat{L}_{cd}]\frac{i\dot\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$$
$$= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})$$ □

Proof: $[L_{ab}, P_c]$

$$= -\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\frac{i\dot\psi(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_c\frac{i\dot\psi(\vec{r}',t)}{\sqrt{-\nabla'^2}}]$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}[(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} + \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}](r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_c\{-i[\sigma(1)\cdot\hat\nabla']^2\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial'_b - r_b\partial'_a)\{-i[\sigma(1)\cdot\hat\nabla']^2\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial_c\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l_\varsigma k'_\varsigma}\partial_c\frac{\dot\psi_{l'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\partial_c\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l'_\varsigma k_\varsigma}(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= -\int\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[-i(r_a\partial_b - r_b\partial_a), -i\partial'_c]\{-i[\sigma(1)\cdot\hat\nabla]^2\}\frac{\dot\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$$
$$= \int\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[\hat{L}_{ab}, \hat{P}_c]\frac{i\dot\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$$
$$= -i(g_{bc}P_a - g_{ac}P_b)$$ □

Proof: $[P_a, P_b]$

$$= -\int[\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}\partial_a\frac{i\dot\psi(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_b\frac{i\dot\psi(\vec{r}',t)}{\sqrt{-\nabla'^2}}]d^3\vec{r}d^3\vec{r}'$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\partial_a\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]d^3\vec{r}d^3\vec{r}'$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}[\partial_a\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} + \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]\partial_a\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l_\varsigma k'_\varsigma}\partial_a\delta^3(\vec{r}-\vec{r}')\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\{-i[\sigma(1)\cdot\hat\nabla']^2\}_{l'_\varsigma k_\varsigma}\partial'_b\delta^3(\vec{r}'-\vec{r})\partial_a\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\{-i[\sigma(1)\cdot\hat\nabla']^2\}_{l_\varsigma k'_\varsigma}\partial'_a\delta^3(\vec{r}-\vec{r}')\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\{-i[\sigma(1)\cdot\hat\nabla]^2\}_{l'_\varsigma k_\varsigma}\partial_b\delta^3(\vec{r}-\vec{r}')\partial_a\frac{\dot\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$$
$$= \int\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\{-i[\sigma(1)\cdot\hat\nabla]^2\}^{k_\varsigma l'_\varsigma}\partial_a\partial_b\frac{\dot\psi_{l'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\{-i[\sigma(1)\cdot\hat\nabla]^2\}^{k'_\varsigma l_\varsigma}\partial_b\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$$

$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}(\partial_a\partial_b - \partial_b\partial_a)\{-i[\sigma(1)\cdot\hat{\nabla}]^2\}\frac{\dot{\psi}(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}(\partial_a\partial_b - \partial_b\partial_a)\frac{-i\dot{\psi}(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[\hat{P}_a,\hat{P}_b]\frac{i\dot{\psi}(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r} = 0$ □

4.11 Poincare symmetry of photon spin

Thm. 4.11.1. $\begin{cases}\nabla\cdot\vec{E} = -\rho, \nabla\times\vec{E} = -\partial_t\vec{B}\\ \nabla\cdot\vec{B} = 0, \nabla\times\vec{B} = -\vec{J} + \partial_t\vec{E}\\ \tilde{A} = \frac{\nabla\times\vec{B}}{-\nabla^2}, \tilde{\phi} = \frac{\nabla\cdot\vec{E}}{-\nabla^2}\end{cases} \Leftrightarrow \begin{cases}\nabla^2\tilde{A} - \partial_t^2\tilde{A} = \vec{J} + \partial_t\nabla\tilde{\phi}\\ \nabla^2\tilde{\phi} = \rho, \nabla\cdot\tilde{A} = 0\\ \vec{E} = -\partial_t\tilde{A} - \nabla\tilde{\phi}, \vec{B} = \nabla\times\tilde{A}\end{cases}$

Cor. 4.11.1. $\begin{cases}[\partial_a + iS_{ab}(\gamma,\varsigma)\partial^b]\Psi = -i\sigma^{[\beta_\varsigma]}_{\varsigma ab}J^b\\ \tilde{A} = \frac{-i}{\sqrt{2}}\frac{\nabla\times(\Psi-\Psi^*)}{\nabla^2}, i\tilde{\phi} = \frac{-i}{\sqrt{2}}\frac{\nabla\cdot(\Psi+\Psi^*)}{\nabla^2}\\ F_{ab} = \frac{i}{2}(\sigma^{\alpha'}_{-ab}\psi_{\alpha'} + \sigma^{\alpha}_{+ab}\psi_\alpha)\end{cases} \Leftrightarrow \begin{cases}\nabla^2\tilde{A} - \partial_t^2\tilde{A} = \vec{J} + \partial_t\nabla\tilde{\phi}\\ \nabla^2\tilde{\phi} = \rho, \nabla\cdot\tilde{A} = 0\\ \sqrt{2}\Psi = -\partial_t\tilde{A} - \nabla\tilde{\phi} - i\varsigma\nabla\times\tilde{A}\end{cases}$

Def. 4.11.1. *Electromagnetic complex vector* $\psi_{\alpha_\varsigma} := \frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab} = i\varsigma(E - i\varsigma B)_{\alpha_\varsigma} = (i\varsigma E + B)_{\alpha_\varsigma}$

Def. 4.11.2. $\psi_\alpha = i(E - iB)_\alpha, \psi^*_\alpha = \psi_{\alpha'} = -i(E + iB)_{\alpha'}$

$The positive branch of SO(4) group generator matrix:$

$$\sigma_+ = R + L = \{\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{bmatrix}, \begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}\} \tag{22.1a}$$

$The negative branch of SO(4) group generator matrix:$

$$\sigma_- = R - L = \{\begin{bmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{bmatrix}, \begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix}\} \tag{22.2a}$$

Thm. 4.11.2. $\Sigma_{ij\pi} = F_{i\pi}A_j - F_{j\pi}A_i = -i(E_iA_j - E_jA_i)$
$= \frac{i}{2}(\sigma^\alpha_{-i\pi}\psi^*_\alpha + \sigma^\alpha_{+i\pi}\psi_\alpha)\frac{-i}{\sqrt{2}}\varepsilon_{jlm}\frac{\partial^l(\Psi-\Psi^*)^m}{\nabla^2} - \frac{i}{2}(\sigma^\alpha_{-j\pi}\psi^*_\alpha + \sigma^\alpha_{+j\pi}\psi_\alpha)\frac{-i}{\sqrt{2}}\varepsilon_{ilm}\frac{\partial^l(\Psi-\Psi^*)^m}{\nabla^2}$
$= \frac{1}{2\sqrt{2}}[(-i\psi^*_i + i\psi_i)\varepsilon_{jlm} - (-i\psi^*_j + i\psi_j)\varepsilon_{ilm}]\frac{\partial^l(\Psi-\Psi^*)^m}{\nabla^2}$
$= \frac{1}{4}[(\psi_i - \psi^*_i)\varepsilon_{jlm} - (\psi_j - \psi^*_j)\varepsilon_{ilm}]\frac{\partial^l(\psi+\psi^*)^m}{\nabla^2}$
$= i(E_i\varepsilon_{jlm} - E_j\varepsilon_{ilm})\frac{\partial^l B^m}{\nabla^2}$

Thm. 4.11.3. $\varepsilon^{kij}\Sigma_{ij\pi} = \varepsilon^{kij}i(E_i\varepsilon_{jlm} - E_j\varepsilon_{ilm})\frac{\partial^l B^m}{\nabla^2} = -2i[\frac{\vec{E}}{\sqrt{-\nabla^2}}\cdot\partial^k\frac{\vec{B}}{\sqrt{-\nabla^2}} - (\frac{\vec{E}}{\sqrt{-\nabla^2}}\cdot\nabla)\frac{B^k}{\sqrt{-\nabla^2}}]$
$\varepsilon_{\alpha_\varsigma\beta_\varsigma\gamma_\varsigma}\varepsilon^{\gamma_\varsigma}{}_{\rho_\varsigma\sigma_\varsigma} = \delta_{\alpha_\varsigma\rho_\varsigma}\delta_{\beta_\varsigma\sigma_\varsigma} - \delta_{\alpha_\varsigma\sigma_\varsigma}\delta_{\beta_\varsigma\rho_\varsigma}$

Thm. 4.11.4. $L_{ij\pi} = x_iF_{k\pi}\partial_jA^k - x_jF_{k\pi}\partial_iA^k = -iE_k(x_i\partial_j - x_j\partial_i)A^k = -iE_k(x_i\partial_j - x_j\partial_i)\varepsilon^{klm}\frac{\partial_l B_m}{-\nabla^2}$

Thm. 4.11.5. $\varepsilon^{kij}L_{ij\pi} = -iE_n(x_i\partial_j - x_j\partial_i)\varepsilon^{kij}\varepsilon^{nlm}\frac{\partial_l B_m}{-\nabla^2}$

Thm. 4.11.6. $\Sigma_{i\pi\pi} = F_{i\pi}A_\pi - F_{\pi\pi}A_i = E_i\phi$

Thm. 4.11.7. $L_{i\pi\pi} = x_iF_{k\pi}\partial_\pi A^k - x_\pi F_{k\pi}\partial_iA^k - \frac{1}{2}x_i\vec{E}^2 + \frac{1}{2}x_i\vec{B}^2 = -iE_k(x_i\partial_\pi - x_\pi\partial_i)A^k - \frac{1}{2}x_i\vec{E}^2 + \frac{1}{2}x_i\vec{B}^2$

4.12 Poincare symmetry of photon angular momentum

Thm. 4.12.1. $\Psi := \frac{1}{\sqrt{2}}(\vec{E} - i\varsigma\vec{B})$
$= -\varsigma\int\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t)d^3\vec{r}$
$i\Sigma_{ij\pi} = [(E_iA_j - E_jA_i) + E_k(x_i\partial_j - x_j\partial_i)A^k]$
$= [(E_iA_j - E_jA_i) + (x_i\partial_j - x_j\partial_i)(\vec{E}\cdot\vec{A})_{\vec{E}}]$
$= [(E_iA_j - E_jA_i) + x_i(\vec{E}\times\vec{B})_j + x_i(\vec{E}\cdot\nabla)\vec{A}_j - x_j(\vec{E}\times\vec{B})_i - x_j(\vec{E}\cdot\nabla)\vec{A}_i]$
$= [x_i(\vec{E}\times\vec{B})_j - x_j(\vec{E}\times\vec{B})_i]$
$= i\varsigma[x_i(\Psi^+\times\Psi)_j - x_j(\Psi^+\times\Psi)_i]$
$= -\varsigma\Psi^+(x_i\gamma_j - x_j\gamma_i)\Psi$

Thm. 4.12.2. $M_{i\pi\pi} = E_i\phi + \vec{E}\cdot(-x_i\partial_t - t\partial_i)\vec{A} - \frac{1}{2}x_i\vec{E}^2 + \frac{1}{2}x_i\vec{B}^2$
$= E_i\phi + x_i\vec{E}\cdot(\vec{E} + \nabla\phi) - t\partial_i(\vec{E}\cdot\vec{A})_{\vec{E}} - \frac{1}{2}x_i\vec{E}^2 + \frac{1}{2}x_i\vec{B}^2$
$= -t\partial_i(\vec{E}\cdot\vec{A})_{\vec{E}} + \frac{1}{2}x_i(\vec{E}^2 + \vec{B}^2)$
$= -t(\vec{E}\times\vec{B})_i + \frac{1}{2}x_i(\vec{E}^2 + \vec{B}^2)$
$= -i\varsigma t(\Psi^+\times\Psi)_i + x_i\Psi^+\Psi$
$= -i\varsigma\pi\Psi^+\gamma_i\Psi + x_i\Psi^+\Psi$
$iM_{i\pi\pi} = -\varsigma\Psi^+[x_i(-i\varsigma) - \pi\gamma_i]\Psi$
$iM_{ab\pi} = -\varsigma\Psi^+(x_a\gamma_b - x_b\gamma_a)\Psi, \gamma_a = (\gamma, -i\varsigma)$

Thm. 4.12.3. $M_{ij} = -\varsigma \int \Psi^+(\vec{r},t)(x_i\gamma_j - x_j\gamma_i)\Psi(\vec{r},t)d^3\vec{r}$

Proof: $[M_{ab}(\vec{r},t), M_{a'b'}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+(\vec{r},t)(r_a\gamma_b - r_b\gamma_a)\Psi(\vec{r},t), \Psi^+(\vec{r}',t)(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})\Psi(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(r_a\gamma_b - r_b\gamma_a)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})^{\beta'_\varsigma\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_a\gamma_b - r_b\gamma_a)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})^{\beta'_\varsigma\beta_\varsigma}[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_a\gamma_b - r_b\gamma_a)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})^{\beta'_\varsigma\beta_\varsigma}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)]\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)[\Psi_{\beta_\varsigma}(\vec{r}',t), \Psi^+_{\alpha'_\varsigma}(\vec{r},t)]\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= \int d^3\vec{r}d^3\vec{r}'(r_a\gamma_b - r_b\gamma_a)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})^{\beta'_\varsigma\beta_\varsigma}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)i\varsigma[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}\delta^3(\vec{r}-\vec{r}')\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)i\varsigma[\gamma\cdot\nabla']_{\beta_\varsigma\alpha'_\varsigma}\delta^3(\vec{r}'-\vec{r})\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{(r_a\gamma_b - r_b\gamma_a)^{\alpha'_\varsigma\alpha_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}[(r_{a'}\gamma_{b'} - r_{b'}\gamma_{a'})^{\beta'_\varsigma\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- [(r_{a'}\gamma_{b'} - r_{b'}\gamma_{a'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\beta'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\beta_\varsigma\alpha'_\varsigma}(r_a\gamma_b - r_b\gamma_a)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\}$ □

Proof: $[M_{ij}(\vec{r},t), M_{i'j'}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t), \Psi^+(\vec{r}',t)(r'_{i'}\gamma_{j'} - r'_{j'}\gamma_{i'})\Psi(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)(r'_{i'}\gamma_{j'} - r'_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{i'}\gamma_{j'} - r'_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{i'}\gamma_{j'} - r'_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)]\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)[\Psi_{\beta_\varsigma}(\vec{r}',t), \Psi^+_{\alpha'_\varsigma}(\vec{r},t)]\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{i'}\gamma_{j'} - r'_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)i\varsigma[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}\delta^3(\vec{r}-\vec{r}')\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)i\varsigma[\gamma\cdot\nabla']_{\beta_\varsigma\alpha'_\varsigma}\delta^3(\vec{r}'-\vec{r})\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}[(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- [(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\beta'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\beta_\varsigma\alpha'_\varsigma}(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)i\varepsilon_{i'j'k'}(\gamma^{k'})_{\alpha_\varsigma}{}^{\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- [(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\beta'_\varsigma}(\vec{r},t)i\varepsilon_{ijk}(\gamma^k)_{\beta_\varsigma}{}^{\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$+ \{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}[(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(\gamma^{k'})_{\alpha_\varsigma\beta'_\varsigma}\partial_{k'}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- (r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}[(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\beta'_\varsigma}(\vec{r},t)(\gamma^k)_{\beta_\varsigma\alpha'_\varsigma}\partial_k\Psi_{\alpha_\varsigma}(\vec{r},t)]\}$
$= i\varsigma\int d^3\vec{r}\{i\varepsilon_{i'j'}{}^{k'}\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_{k'}\Psi(\vec{r},t)] - i\varepsilon_{ij}{}^k\Psi^+(\vec{r},t)[(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})\gamma_k\Psi(\vec{r},t)\}$
$+ \{\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_{k'}(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})\partial^{k'}\Psi(\vec{r},t) - \Psi^+(\vec{r},t)[(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})\gamma_k(r_i\gamma_j - r_j\gamma_i)\partial^k\Psi(\vec{r},t)]\}$ □

Proof: $[M_{ij}(\vec{r},t), M_{i'\pi}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t), \Psi^+(\vec{r}',t)(r'_{i'}\gamma_\pi - \pi\gamma_{i'})\Psi(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)(r'_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)]\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)[\Psi_{\beta_\varsigma}(\vec{r}',t), \Psi^+_{\alpha'_\varsigma}(\vec{r},t)]\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(r'_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)i\varsigma[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}\delta^3(\vec{r}-\vec{r}')\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)i\varsigma[\gamma\cdot\nabla']_{\beta_\varsigma\alpha'_\varsigma}\delta^3(\vec{r}'-\vec{r})\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}[(r_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- [(r_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\beta'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\beta_\varsigma\alpha'_\varsigma}(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(\gamma_{i'}\gamma_\pi)_{\alpha_\varsigma}{}^{\beta_\varsigma}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- [(r_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\beta'_\varsigma}(\vec{r},t)i\varepsilon_{ijk}(\gamma^k)_{\beta_\varsigma}{}^{\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$+ \{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}[(r_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(\gamma^{k'})_{\alpha_\varsigma\beta'_\varsigma}\partial_{k'}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- (r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}[(r_{i'}\gamma_\pi - \pi\gamma_{i'})^{\beta'_\varsigma\beta_\varsigma}\Psi^+_{\beta'_\varsigma}(\vec{r},t)(\gamma^k)_{\beta_\varsigma\alpha'_\varsigma}\partial_k\Psi_{\alpha_\varsigma}(\vec{r},t)]\}$
$= i\varsigma\int d^3\vec{r}\{\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_{i'}\gamma_\pi\Psi(\vec{r},t)] - i\varepsilon_{ijk}\Psi^+(\vec{r},t)(r_{i'}\gamma_\pi - \pi\gamma_{i'})\gamma^k\Psi(\vec{r},t)\}$
$+ \{\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_{k'}(r_{i'}\gamma_\pi - \pi\gamma_{i'})\partial^{k'}\Psi(\vec{r},t)] - \Psi^+(\vec{r},t)(r_{i'}\gamma_\pi - \pi\gamma_{i'})\gamma_k(r_i\gamma_j - r_j\gamma_i)\partial^k\Psi(\vec{r},t)]\}$
$= \int d^3\vec{r}\{\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_{i'}\Psi(\vec{r},t)] - i\varepsilon_{ij}{}^k\Psi^+(\vec{r},t)(r_{i'} + \varsigma t\gamma_{i'})\gamma_k\Psi(\vec{r},t)\}$
$+ \{\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_{k'}(r_{i'} + \varsigma t\gamma_{i'})\partial^{k'}\Psi(\vec{r},t)] - \Psi^+(\vec{r},t)(r_{i'} + \varsigma t\gamma_{i'})\gamma_k(r_i\gamma_j - r_j\gamma_i)\partial^k\Psi(\vec{r},t)]\}$ □

Proof: $-\varsigma\int\Psi^+(\vec{r},t)(\gamma,-i\varsigma)^{a'}\Psi(\vec{r},t)d^3\vec{r}$
$[M_{ij}(\vec{r},t), P_{a'}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t), \Psi^+(\vec{r}',t)(\gamma,-i\varsigma)_{a'}\Psi(\vec{r}',t)]$

$= \int d^3\vec{r}d^3\vec{r}'[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)\Psi_{\beta_\varsigma}(\vec{r}',t)]$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\Psi_{\alpha_\varsigma}(\vec{r},t), \Psi^+_{\beta'_\varsigma}(\vec{r}',t)]\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)[\Psi_{\beta_\varsigma}(\vec{r}',t), \Psi^+_{\alpha'_\varsigma}(\vec{r},t)]\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}$
$\{\Psi^+_{\alpha'_\varsigma}(\vec{r},t)i\varsigma[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}\delta^3(\vec{r}-\vec{r}')\Psi_{\beta_\varsigma}(\vec{r}',t) - \Psi^+_{\beta'_\varsigma}(\vec{r}',t)i\varsigma[\gamma\cdot\nabla']_{\beta_\varsigma\alpha'_\varsigma}\delta^3(\vec{r}'-\vec{r})\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\alpha_\varsigma\beta'_\varsigma}[(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- [(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}\Psi^+_{\beta'_\varsigma}(\vec{r},t)[\gamma\cdot\nabla]_{\beta_\varsigma\alpha'_\varsigma}(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{-[(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}\Psi^+_{\beta'_\varsigma}(\vec{r},t)i\varepsilon_{ijk}(\gamma^k)_{\beta_\varsigma}{}^{\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\}$
$+ \{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}[(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(\gamma^{k'})_{\alpha_\varsigma\beta'_\varsigma}\partial_{k'}\Psi_{\beta_\varsigma}(\vec{r},t)]$
$- (r_i\gamma_j - r_j\gamma_i)^{\alpha'_\varsigma\alpha_\varsigma}[(\gamma,-i\varsigma)^{\beta'_\varsigma\beta_\varsigma}_{a'}\Psi^+_{\beta'_\varsigma}(\vec{r},t)(\gamma^k)_{\beta_\varsigma\alpha'_\varsigma}\partial_k\Psi_{\alpha_\varsigma}(\vec{r},t)]\}$
$= i\varsigma\int d^3\vec{r}\{-i\varepsilon_{ijk}\Psi^+(\vec{r},t)(\gamma,-i\varsigma)_{a'}\gamma^k\Psi(\vec{r},t)$
$+ \Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_{k'}(\gamma,-i\varsigma)_{a'}\partial^{k'}\Psi(\vec{r},t) - \Psi^+(\vec{r},t)(\gamma,-i\varsigma)_{a'}\gamma_k(r_i\gamma_j - r_j\gamma_i)\partial^k\Psi(\vec{r},t)\}$
$= i\varsigma\int d^3\vec{r}\{\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\gamma_k(\gamma,-i\varsigma)_{a'}\partial^k\Psi(\vec{r},t) - \Psi^+(\vec{r},t)(\gamma,-i\varsigma)_{a'}\gamma_k\partial^k[(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t)]\}$ □

Proof:
$= \int d^3\vec{r}\Psi^+(\vec{r},t)\{(r_i\gamma_j - r_j\gamma_i)[\gamma\cdot\nabla][(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'}) - [(r_{i'}\gamma_{j'} - r_{j'}\gamma_{i'})[\gamma\cdot\nabla](r_i\gamma_j - r_j\gamma_i)\}\Psi(\vec{r},t)]$ □

5 New Scheme for covariant quantization of complex electromagnetic field strength

This section is replaced by an electromagnetic representation. Once again a complete description of the photon covariant quantization scheme is given for easy using in later chapters.

5.1 Various equivalent forms of electromagnetic field [22, 24]

Def. 5.1.1. $\Psi_{\alpha_\varsigma} := \frac{-i\varsigma}{\sqrt{2}}\psi_{\alpha_\varsigma} = \frac{-i\varsigma}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab} = \frac{-i\varsigma}{\sqrt{2}}i\varsigma(E - i\varsigma B)_{\alpha_\varsigma}$

Def. 5.1.2. $\Psi := \frac{1}{\sqrt{2}}(\vec{E} - i\varsigma\vec{B}) = \frac{1}{\sqrt{2}}(\vec{E} - i\varsigma\nabla\times\vec{A}), \Psi_i = \frac{1}{\sqrt{2}}(E_i - i\varsigma\varepsilon_i{}^{jk}\partial_jA_k), p\cdot x := \vec{p}\cdot\vec{r} - Et$

Thm. 5.1.1.
$$\begin{cases}\partial^aF_{ab} = 0\\ \partial^a * F_{ab} = 0\end{cases} \Leftrightarrow \begin{cases}\nabla\cdot\vec{E} = 0, \nabla\times\vec{E} = -\partial_t\vec{B}\\ \nabla\cdot\vec{B} = 0, \nabla\times\vec{B} = \partial_t\vec{E}\end{cases} \Leftrightarrow \begin{cases}(\gamma,-i\varsigma)^a\partial_a\Psi = 0\\ \nabla\cdot\Psi = 0\end{cases} \Leftrightarrow \begin{cases}[\partial_a + iS_{ab}(\gamma,\varsigma)\partial^b]\Psi = 0\\ S_{ab}(\gamma,\varsigma) = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}(s)\end{cases}$$

5.2 Spin equation and plane wave solutions of complex electromagnetic field strength

Thm. 5.2.1. $[\partial_a + iS_{ab}(\gamma,\varsigma)\partial^b]\Psi(x) = 0$

Cor. 5.2.1.
$$\begin{cases}\Psi(\vec{r},t) := \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\sqrt{|\vec{p}|}\lambda_m(\hat{p},-\varsigma)[a_1(\vec{p},-\varsigma)e^{i\varsigma p\cdot x} + a_2^+(\vec{p},-\varsigma)e^{-i\varsigma p\cdot x}]d^3\vec{p}\\ \sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda_m^+(\hat{p},-\varsigma)\Psi(\vec{r},t)e^{-i\varsigma p\cdot x}d^3\vec{r}\\ \sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda_m^+(\hat{p},-\varsigma)\Psi(\vec{r},t)e^{i\varsigma p\cdot x}d^3\vec{r}\end{cases}$$

Cor. 5.2.2. $(\gamma,-i\varsigma)^a\partial_a\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}\sum\limits_{h=1}^{-1}\sqrt{|\vec{p}|}\lambda_m(\hat{p},h)[a_1(\vec{p},-\varsigma)e^{i\varsigma p\cdot x} + a_2^+(\vec{p},-\varsigma)e^{-i\varsigma p\cdot x}]d^3\vec{p} = 0$

$$\int\limits_{\vec{p}\neq 0}\sum\limits_{h=1}^{-1}\sqrt{|\vec{p}|}(\gamma,-i\varsigma)^ap_a\lambda_m(\hat{p},h)[a_1(\vec{p},-\varsigma)e^{i\varsigma p\cdot x} - a_2^+(\vec{p},-\varsigma)e^{-i\varsigma p\cdot x}]d^3\vec{p} = 0$$

5.3 Properties of constant invariant tensor $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}$ in electromagnetic field

From constant invariant tensor analysis, it can be seen that:

Cor. 5.3.1.
$\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma} = \frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}$
$\sigma^{k\pi}_{\alpha_\varsigma\alpha'_\varsigma} = \sigma^{\pi k}_{\alpha_\varsigma\alpha'_\varsigma} = -\frac{\varsigma}{2}\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}$
$\sigma^{kl}_{\alpha_\varsigma\alpha'_\varsigma} = \frac{1}{2}(\delta^k_{\alpha_\varsigma}\delta^l_{\alpha'_\varsigma} + \delta^k_{\alpha'_\varsigma}\delta^l_{\alpha_\varsigma} - \delta^{kl}\delta_{\alpha_\varsigma\alpha'_\varsigma})$

Cor. 5.3.2. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b = \partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma} - \frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2 + \partial_t^2) + i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_t$

Proof: $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b = \sigma^{kl}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_l + 2\sigma^{k\pi}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_\pi + \sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\partial_\pi\partial_\pi$
$= \partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma} - \frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2 - \partial_\pi^2) - \varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_\pi$
$= \partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma} - \frac{1}{2}\delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2 + \partial_t^2) + i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_t$ □

Cor. 5.3.3. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} p_a p_b = p_{\alpha_\varsigma} p_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2 - i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma} p_k|\vec{p}|$

Cor. 5.3.4. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x) = (\partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma}\nabla^2 + i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_t)\Delta(x) = -\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} p_a p_b\Delta(x)$

Cor. 5.3.5.
$\begin{cases}\sigma^{ab}_{\{\alpha_\varsigma\alpha'_\varsigma\}}\partial_a\partial_b = 2\partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma}(\nabla^2+\partial_t^2)\\ \sigma^{ab}_{[\alpha_\varsigma\alpha'_\varsigma]}\partial_a\partial_b = 2i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_t = -2\varsigma(\gamma\cdot\nabla)_{\alpha_\varsigma\alpha'_\varsigma}\partial_t\end{cases}$ $\begin{cases}\sigma^{ab}_{\{\alpha_\varsigma\alpha'_\varsigma\}}\partial_a\partial_b\Delta(x) = 2(\partial_{\alpha_\varsigma}\partial_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma}\nabla^2)\Delta(x)\\ \sigma^{ab}_{[\alpha_\varsigma\alpha'_\varsigma]}\partial_a\partial_b\Delta(x) = 2i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_t\Delta(x)\end{cases}$

Cor. 5.3.6.
$\begin{cases}\sigma^{ab}_{\{\alpha_\varsigma\alpha'_\varsigma\}} p_a p_b = 2(p_{\alpha_\varsigma}p_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2)\\ \sigma^{ab}_{[\alpha_\varsigma\alpha'_\varsigma]} p_a p_b = -2i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k|\vec{p}| = 2\varsigma\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k|\vec{p}|\end{cases}$ $\begin{cases}\sigma^{ab}_{\{\alpha_\varsigma\alpha'_\varsigma\}}\hat{p}_a\hat{p}_b = 2(\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma})\\ \sigma^{ab}_{[\alpha_\varsigma\alpha'_\varsigma]}\hat{p}_a\hat{p}_b = -2i\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\hat{p}_k = 2\varsigma\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}\hat{p}_k\end{cases}$

Lem. 5.3.1. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} p_a p_b = -2|\vec{p}|^2\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma)$

Proof: $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} p_a p_b$
$= p_{\alpha_\varsigma}p_{\alpha'_\varsigma} + \varsigma\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k|\vec{p}| - \delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$= p_{\alpha_\varsigma}p_{\alpha'_\varsigma} + \varsigma|\vec{p}|\gamma^k{}_{\alpha_\varsigma}{}^{\beta_\varsigma}p_k\delta_{\beta_\varsigma\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$= \lambda_{m\alpha_\varsigma}(\hat{p},0)\lambda^+_{m\alpha'_\varsigma}(\hat{p},0)|\vec{p}|^2 + \varsigma|\vec{p}|\gamma^k{}_{\alpha_\varsigma}{}^{\beta_\varsigma}p_k\sum_{h=1}^{-1}\lambda_{m\beta_\varsigma}(\hat{p},h)\lambda^+_{m\alpha'_\varsigma}(\hat{p},h) - \delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$= \lambda_{m\alpha_\varsigma}(\hat{p},0)\lambda^+_{m\alpha'_\varsigma}(\hat{p},0)|\vec{p}|^2 + \varsigma|\vec{p}|[\varsigma|\vec{p}|\lambda_{m\beta_\varsigma}(\hat{p},\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat{p},\varsigma) - \varsigma|\vec{p}|\lambda_{m\beta_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma)] - \delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2$
$= |\vec{p}|^2\sum_{h=1}^{-1}\lambda_{m\alpha_\varsigma}(\hat{p},h)\lambda^+_{m\alpha'_\varsigma}(\hat{p},h) - \delta_{\alpha_\varsigma\alpha'_\varsigma}|\vec{p}|^2 - 2|\vec{p}|^2\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma)$
$= -2|\vec{p}|^2\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma)$ □

The above lemma links constant invariant tensor analysis with helicity analysis.

Cor. 5.3.7. $\begin{cases}(\sigma^{ab}\hat{p}_a\hat{p}_b)^n = (-2)^{n-1}\sigma^{ab}\hat{p}_a\hat{p}_b\\ (\frac{\sigma^{ab}\partial_a\partial_b}{\nabla^2})^n = (-2)^{n-1}\frac{\sigma^{ab}\partial_a\partial_b}{\nabla^2}\end{cases}$ $\begin{cases}(\hat{p}^T\hat{p}-1)^n = (-1)^{n-1}(\hat{p}^T\hat{p}-1)\\ (\frac{\nabla^T\nabla}{\nabla^2}-1)^n = (-2)^{n-1}(\frac{\nabla^T\nabla}{\nabla^2}-1)\end{cases}$

Cor. 5.3.8. $\begin{cases}(\varsigma\gamma\cdot\hat{p})^{2n} = -(\hat{p}^T\hat{p}-1)\\ (\frac{-i\varsigma\gamma\cdot\nabla}{\sqrt{-\nabla^2}})^{2n} = -(\frac{\nabla^T\nabla}{\nabla^2}-1)\end{cases}$ $\begin{cases}(\varsigma\gamma\cdot\hat{p})^{2n-1} = (\varsigma\gamma\cdot\hat{p})\\ (\frac{-i\varsigma\gamma\cdot\nabla}{\sqrt{-\nabla^2}})^{2n-1} = (\frac{-i\varsigma\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\end{cases}$

Cor. 5.3.9. $\begin{cases}(\hat{p}^T\hat{p}-1)(\varsigma\gamma\cdot\hat{p}) = (\varsigma\gamma\cdot\hat{p})(\hat{p}^T\hat{p}-1) = -(\varsigma\gamma\cdot\hat{p})\\ (\frac{\nabla^T\nabla}{\nabla^2}-1)(\frac{-i\varsigma\gamma\cdot\nabla}{\sqrt{-\nabla^2}}) = (\frac{-i\varsigma\gamma\cdot\nabla}{\sqrt{-\nabla^2}})(\frac{\nabla^T\nabla}{\nabla^2}-1) = -(\frac{-i\varsigma\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\end{cases}$

5.4 General covariant commutation rules for electromagnetic field in mathematics

Thm. 5.4.1.

$$\begin{cases}[a_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm = \varsigma^{\overset{0}{1}}\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma),a_{\sigma'}(\vec{p}',-\varsigma)]_\pm = 0\\ [a^+_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]_\pm = 0\end{cases} \Rightarrow \begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]_\pm\\ = i\varsigma^{\overset{1}{0}}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b[\delta_1\Delta^{(+\varsigma)}(x-x') - \pm\delta_2\Delta^{(-\varsigma)}(x-x')]\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]_\pm = 0\\ [\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]_\pm = 0\end{cases}$$

Proof: $[\Psi^{(+\varsigma)}_{\alpha_\varsigma}(x),\Psi^{(+\varsigma)+}_{\alpha'_\varsigma}(x')]_\pm$
$= \frac{1}{(2\pi)^3}\int_{\vec{p}\neq 0}\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec{p}',-\varsigma)\sqrt{|\vec{p}||\vec{p}'|}[a_1(\vec{p},-\varsigma),a^+_1(\vec{p}',-\varsigma)]_\pm e^{i\varsigma\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec{p}',-\varsigma)|\vec{p}|\varsigma^{\overset{0}{1}}\delta_1\delta^3(\vec{p}-\vec{p}')e^{i\varsigma\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma)\varsigma^{\overset{0}{1}}\delta_1|\vec{p}|e^{i\varsigma\vec{p}\cdot(x-x')}d^3\vec{p}$
$= \frac{-\varsigma^{\overset{0}{1}}\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_a p_b e^{i\varsigma p\cdot(x-x')}d^3\vec{p}$
$= i\varsigma^{\overset{1}{0}}\delta_1\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(+\varsigma)}(x-x')$ □

Proof: $[\Psi^{(-\varsigma)}_{\alpha_\varsigma}(x),\Psi^{(-\varsigma)+}_{\alpha'_\varsigma}(x')]_\pm$
$= \frac{1}{(2\pi)^3}\int_{\vec{p}\neq 0}\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec{p}',-\varsigma)\sqrt{|\vec{p}||\vec{p}'|}[a^+_2(\vec{p},-\varsigma),a_2(\vec{p}',-\varsigma)]_\pm e^{-i\varsigma\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec{p}',-\varsigma)|\vec{p}|\varsigma^{\overset{0}{1}}\delta_2\delta^3(\vec{p}-\vec{p}')e^{-i\varsigma\vec{p}\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat{p},-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat{p},-\varsigma)\varsigma^{\overset{0}{1}}\delta_2|\vec{p}|e^{-i\varsigma\vec{p}\cdot(x-x')}d^3\vec{p}$
$= \pm\frac{-\varsigma^{\overset{0}{1}}\delta_2}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_a p_b e^{-i\varsigma p\cdot(x-x')}d^3\vec{p}$
$= -\pm i\varsigma^{\overset{1}{0}}\delta_2\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(-\varsigma)}(x-x')$ □

Proof: $[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]_\pm$
$=[\Psi^{(+\varsigma)}_{\alpha_\varsigma}(x),\Psi^{(+\varsigma)+}_{\alpha'_\varsigma}(x')]_\pm+[\Psi^{(-\varsigma)}_{\alpha_\varsigma}(x),\Psi^{(-\varsigma)+}_{\alpha'_\varsigma}(x')]_\pm$
$=i\varsigma^{\frac{1}{0}}\delta_1\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(+\varsigma)}(x-x')-\pm i\varsigma^{\frac{1}{0}}\delta_2\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(-\varsigma)}(x-x')$
$=i\varsigma^{\frac{1}{0}}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b[\delta_1\Delta^{(+\varsigma)}(x-x')-\pm\delta_2\Delta^{(-\varsigma)}(x-x')]$
$=i\varsigma^{\frac{1}{0}}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b[(\delta_1\pm\delta_2)\Delta^{(+\varsigma)}(x-x')-\pm\delta_2\Delta(x-x')]$ □

From the above, only $\delta_1\pm\delta_2=0$, the micro causality is satisfied. At the same time only when $\delta_1,\delta_2\geq 0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1=\delta_2=1$, it satisfies the commutative relation. There are actually two other options. Namely when $\delta_1=\delta_2=0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

5.5 Physical covariant commutation rules for electromagnetic field

From the previous section, we can see that the commutation rules with physical significance are as follows:(In order to confirm each other, a new proof has been made.)

Thm. 5.5.1.
$$\begin{cases}[a_\sigma(\vec p,-\varsigma),a^+_{\sigma'}(\vec p',-\varsigma)]=\varsigma\delta_{\sigma\sigma'}\delta^3(\vec p-\vec p')\\ [a_\sigma(\vec p,-\varsigma),a_{\sigma'}(\vec p',-\varsigma)]=0\\ [a^+_\sigma(\vec p,-\varsigma),a^+_{\sigma'}(\vec p',-\varsigma)]=0\end{cases}\Rightarrow\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0\\ [\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\end{cases}$$

Proof: $[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]$
$=\frac{1}{(2\pi)^3}\int\limits_{\vec p\neq 0}d^3\vec p d^3\vec p'$
$\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec p',-\varsigma)\sqrt{|\vec p||\vec p'|}\{[a_1(\vec p,-\varsigma),a^+_1(\vec p',-\varsigma)]e^{i\varsigma\vec p\cdot(x-x')}+[a^+_2(\vec p,-\varsigma),a_2(\vec p',-\varsigma)]e^{-i\varsigma\vec p\cdot(x-x')}\}$
$=\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec p',-\varsigma)|\vec p|[\varsigma\delta^3(\vec p-\vec p')e^{i\varsigma\vec p\cdot(x-x')}-\varsigma\delta^3(\vec p-\vec p')e^{-i\varsigma\vec p\cdot(x-x')}]\}d^3\vec p d^3\vec p'$
$=\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat p,-\varsigma)\varsigma|\vec p|[e^{i\varsigma\vec p\cdot(x-x')}-e^{-i\varsigma\vec p\cdot(x-x')}\}d^3\vec p$
$=\frac{-\varsigma}{(2\pi)^3}\int\frac{1}{2|\vec p|}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b[e^{i\varsigma p\cdot(x-x')}-e^{-i\varsigma p\cdot(x-x')}]d^3\vec p$
$=i\varsigma\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta[\varsigma(x-x')]$
$=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

Thm. 5.5.2.
$$\begin{cases}[a_\sigma(\vec p,-\varsigma),a^+_{\sigma'}(\vec p',-\varsigma)]=\varsigma\delta_{\sigma\sigma'}\delta^3(\vec p-\vec p')\\ [a_\sigma(\vec p,-\varsigma),a_{\sigma'}(\vec p',-\varsigma)]=0\\ [a^+_\sigma(\vec p,-\varsigma),a^+_{\sigma'}(\vec p',-\varsigma)]=0\end{cases}\Rightarrow\begin{cases}[\Psi^{(\tau)}_{\alpha_\varsigma}(x),\Psi^{(\kappa)+}_{\alpha'_\varsigma}(x')]=i\delta^{\tau\kappa}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(\tau)}(x-x')\\ [\Psi^{(\tau)}_{\alpha_\varsigma}(x),\Psi^{(\kappa)}_{\beta_\varsigma}(x')]=0\\ [\Psi^{(\tau)+}_{\alpha'_\varsigma}(x),\Psi^{(\kappa)+}_{\beta'_\varsigma}(x')]=0\end{cases}$$

Proof: $[\Psi^{(+\varsigma)}_{\alpha_\varsigma}(x),\Psi^{(+\varsigma)+}_{\alpha'_\varsigma}(x')]$
$=\frac{1}{(2\pi)^3}\int\limits_{\vec p\neq 0}\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec p',-\varsigma)\sqrt{|\vec p||\vec p'|}[a_1(\vec p,-\varsigma),a^+_1(\vec p',-\varsigma)]e^{i\varsigma\vec p\cdot(x-x')}d^3\vec p d^3\vec p'$
$=\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec p',-\varsigma)|\vec p|\varsigma\delta^3(\vec p-\vec p')e^{i\varsigma\vec p\cdot(x-x')}d^3\vec p d^3\vec p'$
$=\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat p,-\varsigma)\varsigma|\vec p|e^{i\varsigma\vec p\cdot(x-x')}d^3\vec p$
$=\frac{-\varsigma}{(2\pi)^3}\int\frac{1}{2|\vec p|}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_be^{i\varsigma p\cdot(x-x')}d^3\vec p$
$=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(+\varsigma)}(x-x')$ □

Proof: $[\Psi^{(-\varsigma)}_{\alpha_\varsigma}(x),\Psi^{(-\varsigma)+}_{\alpha'_\varsigma}(x')]$
$=\frac{1}{(2\pi)^3}\int\limits_{\vec p\neq 0}\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec p',-\varsigma)\sqrt{|\vec p||\vec p'|}[a^+_2(\vec p,-\varsigma),a_2(\vec p',-\varsigma)]e^{-i\varsigma\vec p\cdot(x-x')}d^3\vec p d^3\vec p'$
$=-\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\vec p',-\varsigma)|\vec p|\varsigma\delta^3(\vec p-\vec p')e^{-i\varsigma\vec p\cdot(x-x')}d^3\vec p d^3\vec p'$
$=-\frac{1}{(2\pi)^3}\int\lambda_{m\alpha_\varsigma}(\hat p,-\varsigma)\lambda^+_{m\alpha'_\varsigma}(\hat p,-\varsigma)\varsigma|\vec p|e^{-i\varsigma\vec p\cdot(x-x')}d^3\vec p$
$=\frac{\varsigma}{(2\pi)^3}\int\frac{1}{2|\vec p|}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_be^{-i\varsigma p\cdot(x-x')}d^3\vec p$
$=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(-\varsigma)}(x-x')$ □

5.6 Isochronous commutation rules for electromagnetic field

Cor. 5.6.1.
$$\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0\\ [\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\end{cases}\Rightarrow\begin{cases}[\Psi_{\alpha_\varsigma}(\vec r,t),\Psi^+_{\alpha'_\varsigma}(\vec r',t)]=\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta(\vec r-\vec r')\\ [\Psi_{\alpha_\varsigma}(\vec r,t),\Psi_{\beta_\varsigma}(\vec r',t)]=0\\ [\Psi^+_{\alpha'_\varsigma}(\vec r,t),\Psi^+_{\beta'_\varsigma}(\vec r',t)]=0\end{cases}$$

Proof: $[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$\Rightarrow[\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)]=2i\sigma^{k\pi}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\partial_\pi\Delta(x-x')|_{t=t'}$
$\Leftrightarrow[\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)]=\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta(\vec{r}-\vec{r}')$ □

Cor. 5.6.2. $\begin{cases}[\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)]=\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta(\vec{r}-\vec{r}')\\ [\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi_{\beta_\varsigma}(\vec{r}',t)]=0\\ [\Psi^+_{\alpha'_\varsigma}(\vec{r},t),\Psi^+_{\beta'_\varsigma}(\vec{r}',t)]=0\end{cases}\Rightarrow\begin{cases}[a_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]=\varsigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma),a_{\sigma'}(\vec{p}',-\varsigma)]=0\\ [a^+_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]=0\end{cases}$

Proof: $[a_1(\vec{p},-\varsigma),a^+_1(\vec{p}',-\varsigma)]$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int[\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)},\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)[\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)]e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta^3(\vec{r}-\vec{r}')e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}i\varsigma p_ke^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}d^3\vec{r}$
$=-\frac{1}{|\vec{p}|}\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\hat{p},h')\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k\delta^3(\vec{p}-\vec{p}')$
$=-\lambda^+_m(\hat{p},-\varsigma)\frac{\gamma^kp_k}{|\vec{p}|}\lambda_m(\hat{p},-\varsigma)\delta^3(\vec{p}-\vec{p}')$
$=\varsigma\lambda^+_m(\hat{p},-\varsigma)\lambda_m(\hat{p},-\varsigma)\delta^3(\vec{p}-\vec{p}')$
$=\varsigma\delta^3(\vec{p}-\vec{p}')$ □

Proof: $[a^+_2(\vec{p},-\varsigma),a_2(\vec{p}',-\varsigma)]$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int[\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)},\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)[\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)]e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta^3(\vec{r}-\vec{r}')e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$=i\varsigma\frac{1}{(2\pi)^3}\frac{1}{\sqrt{|\vec{p}||\vec{p}'|}}\int\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\vec{p}',-\varsigma)\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}(-i\varsigma p_k)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}d^3\vec{r}$
$=\frac{1}{|\vec{p}|}\lambda^{+\alpha_\varsigma}_m(\hat{p},-\varsigma)\lambda^{\alpha'_\varsigma}_m(\hat{p},h')\gamma^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k\delta^3(\vec{p}-\vec{p}')$
$=\lambda^+_m(\hat{p},-\varsigma)\frac{\gamma^kp_k}{|\vec{p}|}\lambda_m(\hat{p},h')\delta^3(\vec{p}-\vec{p}')$
$=-\varsigma\lambda^+_m(\hat{p},-\varsigma)\lambda_m(\hat{p},h')\delta^3(\vec{p}-\vec{p}')$
$=-\varsigma\delta^3(\vec{p}-\vec{p}')$ □

5.7 Summary of commutation rules for electromagnetic field

he proof in the above sections exactly forms a logical closed-loop, so it has the following properties:

Cor. 5.7.1. $\begin{cases}[a_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]=\varsigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\varsigma),a_{\sigma'}(\vec{p}',-\varsigma)]=0\\ [a^+_\sigma(\vec{p},-\varsigma),a^+_{\sigma'}(\vec{p}',-\varsigma)]=0\end{cases}\Leftrightarrow\begin{cases}[a_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}),a_{\sigma'}(\vec{p}')]=0\\ [a^+_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]=0\end{cases}$

$\Updownarrow$ $\Updownarrow$

Cor. 5.7.2. $\begin{cases}[\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi_{\alpha_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]=0\\ [\Psi^+_{\alpha'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]=0\end{cases}\Leftrightarrow\begin{cases}[\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)]=\varsigma\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta(\vec{r}-\vec{r}')\\ [\Psi_{\alpha_\varsigma}(\vec{r},t),\Psi_{\beta_\varsigma}(\vec{r}',t)]=0\\ [\Psi^+_{\alpha'_\varsigma}(\vec{r},t),\Psi^+_{\beta'_\varsigma}(\vec{r}',t)]=0\end{cases}$

5.8 Commutative function, causal function and feynman propagator of electromagnetic field (It seems like there's a minus sign missing from Bogoliubov.)

Def. 5.8.1.
$\begin{cases}[\varphi(x),\varphi(x')]=i\Delta(x-x'),\varphi^+(x)=\varphi(x)\\ \Delta(x)=\Delta^{(+)}(x)+\Delta^{(-)}(x)=\Delta^{ret}(x)-\Delta^{adv}(x)\\ \Delta^{(l)}(x)=i[\Delta^{(-)}(x)-\Delta^{(+)}(x)]\\ N_m(1)=\begin{bmatrix}I_3\\0\end{bmatrix},\bar{N}_m(1)=[I_3,0]\end{cases}\quad\begin{cases}\Delta^{(c)}(x)=\theta(t)\Delta^{(+)}(x)-\theta(-t)\Delta^{(-)}(x)\\ \Delta^{ret}(x)=\theta(t)\Delta(x)=\Delta^{(c)}(x)+\Delta^{(-)}(x)\\ \Delta^{adv}(x)=-\theta(-t)\Delta(x)=\Delta^{(c)}(x)-\Delta^{(+)}(x)\\ \Delta_F(x)=\langle T\varphi(x)\varphi(x')\rangle_0=i\Delta^{(c)}(x-x')\end{cases}$

Def. 5.8.2. $\tilde{\Delta}^{()}(x):=\begin{bmatrix}\Delta^{()}(x)\\0\end{bmatrix},\Delta^{(c)}(x-x'):=i\langle T\varphi(x)\varphi(x')\rangle_0,\vec{\partial}:=(\partial_x,\partial_y,\partial_z)$

Cor. 5.8.1.

$$\begin{cases}\Delta_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x)\\ \Delta^{(+)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(+)}(x)\\ \Delta^{(-)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(-)}(x)\\ \Delta^{(l)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(l)}(x)\end{cases}
\quad
\begin{cases}\Delta^{(c)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(c)}(x)+\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4(x)\\ \Delta^{ret}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{ret}(x)+\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4(x)\\ \Delta^{adv}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{adv}(x)+\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4(x)\\ \Delta_{F\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta_F(x)+i\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4(x)=i\Delta^{(c)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x)\\ \Delta_{F\alpha_\varsigma\alpha'_\varsigma}(\gamma;x)=\frac{i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b}{p^2-i\varepsilon}+\cdots\end{cases}$$

Cor. 5.8.2.

$$\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta(\gamma;x)=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{(+)}(\gamma;x)=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{(-)}(\gamma;x)=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{(l)}(\gamma;x)=0\end{cases}
\quad
\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{(c)}(\gamma;x)=-\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta(\gamma;x)|_{t=0}\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{ret}(\gamma;x)=-\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta(\gamma;x)|_{t=0}\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{adv}(\gamma;x)=-\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta(\gamma;x)|_{t=0}\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta_F(\gamma;x)=-i\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta(\gamma;x)|_{t=0}\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

Cor. 5.8.3.

$$\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta(\gamma;x)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(+)}(\gamma;x)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(-)}(\gamma;x)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(l)}(\gamma;x)=0\end{cases}
\quad
\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(c)}(\gamma;x)=-\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{ret}(\gamma;x)=-\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{adv}(\gamma;x)=-\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta_F(\gamma;x)=-i\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

Cor. 5.8.4.

$$\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta(\gamma;x)\bar N_m(1)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(+)}(\gamma;x)\bar N_m(1)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(-)}(\gamma;x)\bar N_m(1)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(l)}(\gamma;x)\bar N_m(1)=0\end{cases}
\quad
\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{(c)}(\gamma;x)\bar N_m(1)=-\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\bar N_m(1)\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{ret}(\gamma;x)\bar N_m(1)=-\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\bar N_m(1)\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{adv}(\gamma;x)\bar N_m(1)=-\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\bar N_m(1)\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta_F(\gamma;x)\bar N_m(1)=-i\varsigma\delta(t)N_m(1)\Delta(\gamma;x)|_{t=0}\bar N_m(1)\end{cases}$$

$$[\Downarrow] \qquad\qquad [\Downarrow]$$

Cor. 5.8.5.

$$\begin{cases}(\gamma,-i\varsigma)_a\partial^a\Delta(\gamma;x)=0\\ (\gamma,-i\varsigma)_a\partial^a\Delta^{(+)}(\gamma;x)=0\\ (\gamma,-i\varsigma)_a\partial^a\Delta^{(-)}(\gamma;x)=0\\ (\gamma,-i\varsigma)_a\partial^a\Delta^{(l)}(\gamma;x)=0\end{cases}
\quad
\begin{cases}(\gamma,-i\varsigma)_a\partial^a\Delta^{(c)}(\gamma;x)=-\varsigma\delta(t)\Delta(\gamma;x)|_{t=0}\\ (\gamma,-i\varsigma)_a\partial^a\Delta^{ret}(\gamma;x)=-\varsigma\delta(t)\Delta(\gamma;x)|_{t=0}\\ (\gamma,-i\varsigma)_a\partial^a\Delta^{adv}(\gamma;x)=-\varsigma\delta(t)\Delta(\gamma;x)|_{t=0}\\ (\gamma,-i\varsigma)_a\partial^a\Delta_F(\gamma;x)=-i\varsigma\delta(t)\Delta(\gamma;x)|_{t=0}\end{cases}$$

5.9 Extraction of energy momentum operator in electromagnetic field

Cor. 5.9.1. $H=\int\limits_{\vec p\neq 0}|\vec p|[a_1^+(\vec p,-\varsigma)a_1(\vec p,-\varsigma)+a_2(\vec p,-\varsigma)a_2^+(\vec p,-\varsigma)]d^3\vec p=\int\Psi^+(\vec r,t)\Psi(\vec r,t)d^3\vec r$

Proof: $H=\int\limits_{\vec p\neq 0}|\vec p|[a_1^+(\vec p,-\varsigma)a_1(\vec p,-\varsigma)+a_2(\vec p,-\varsigma)a_2^+(\vec p,-\varsigma)]d^3\vec p$

$=\frac{1}{(2\pi)^3}\int\limits_{\vec p\neq 0}[\lambda_m^{\alpha'_\varsigma}(\hat p,-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec r',t)e^{i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat p,-\varsigma)\Psi_{\alpha_\varsigma}(\vec r,t)e^{-i\varsigma p\cdot x}$

$+\lambda_m^{\alpha'_\varsigma}(\hat p,-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec r',t)e^{-i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat p,-\varsigma)\Psi_{\alpha_\varsigma}(\vec r,t)e^{i\varsigma p\cdot x}]d^3\vec pd^3\vec rd^3\vec r'$

$=\frac{1}{(2\pi)^3}\int\limits_{\vec p\neq 0}\lambda_m^{\alpha'_\varsigma}(\hat p,-\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat p,-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec r',t)\Psi_{\alpha_\varsigma}(\vec r,t)[e^{-i\varsigma\vec p\cdot(\vec r-\vec r')}+e^{i\varsigma\vec p\cdot(\vec r-\vec r')}]d^3\vec pd^3\vec rd^3\vec r'$

$=\frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec p\neq 0}\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}p^ap^b\Psi^+_{\alpha'_\varsigma}(\vec r',t)\Psi_{\alpha_\varsigma}(\vec r,t)[e^{-i\varsigma\vec p\cdot(\vec r-\vec r')}+e^{i\varsigma\vec p\cdot(\vec r-\vec r')}]d^3\vec pd^3\vec rd^3\vec r'$

$=\frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec p\neq 0}(\hat p^{\alpha'_\varsigma}\hat p^{\alpha_\varsigma}+\varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat p^k-\delta^{\alpha'_\varsigma\alpha_\varsigma})\Psi^+_{\alpha'_\varsigma}(\vec r',t)\Psi_{\alpha_\varsigma}(\vec r,t)[e^{-i\varsigma\vec p\cdot(\vec r-\vec r')}+e^{i\varsigma\vec p\cdot(\vec r-\vec r')}]d^3\vec pd^3\vec rd^3\vec r'$

$=\frac{1}{(2\pi)^3}\int\limits_{\vec p\neq 0}(\delta^{\alpha'_\varsigma\alpha_\varsigma}-\hat p^{\alpha'_\varsigma}\hat p^{\alpha_\varsigma})e^{i\vec p\cdot(\vec r-\vec r')}\Psi^+_{\alpha'_\varsigma}(\vec r',t)\Psi_{\alpha_\varsigma}(\vec r,t)d^3\vec pd^3\vec rd^3\vec r'$

$=\int[\delta^{\alpha'_\varsigma\alpha_\varsigma}-\frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2}]\delta^3(\vec r-\vec r')\Psi^+_{\alpha'_\varsigma}(\vec r',t)\Psi_{\alpha_\varsigma}(\vec r,t)d^3\vec rd^3\vec r'$

$=\int\Psi^+_{\alpha'_\varsigma}(\vec r,t)[\delta^{\alpha'_\varsigma\alpha_\varsigma}-\frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2}]\Psi_{\alpha_\varsigma}(\vec r,t)d^3\vec r$(The equation of motion have been used: $\nabla\cdot\Psi(\vec r,t)=0$)

$=\int\Psi^+(\vec r,t)\Psi(\vec r,t)d^3\vec r$ □

Cor. 5.9.2. $\vec P=\int\limits_{\vec p\neq 0}\vec p[a_1^+(\vec p,-\varsigma)a_1(\vec p,-\varsigma)+a_2(\vec p,-\varsigma)a_2^+(\vec p,-\varsigma)]d^3\vec p=-\varsigma\int\Psi^+(\vec r,t)\gamma\Psi(\vec r,t)d^3\vec r$

Proof: $\vec{P} = \int\limits_{\vec{p}\neq 0} \vec{p}[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) + a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0} \hat{p}[\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma p\cdot x}$
$+ \lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{-i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{i\varsigma p\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0} \hat{p}\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec{p}\neq 0} \hat{p}\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}p^a p^b\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec{p}\neq 0} \hat{p}(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} + \varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k - \delta^{\alpha'_\varsigma\alpha_\varsigma})\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= -\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0} \hat{p}\varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\varsigma\nabla(\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}\frac{1}{|\vec{p}|^2}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \int \Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\varsigma\frac{\nabla(\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}}{-\nabla^2}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$= \varsigma\int \Psi^+(\vec{r},t)\frac{\nabla(\gamma\cdot\nabla)}{-\nabla^2}\Psi(\vec{r},t)d^3\vec{r}$
$= -\varsigma\int \Psi^+(\vec{r},t)\gamma\Psi(\vec{r},t)d^3\vec{r}$ □

Cor. 5.9.3. $P^a = -\varsigma\int \Psi^+(\vec{r},t)(\gamma,-i\varsigma)^a\Psi(\vec{r},t)d^3\vec{r}$

5.10 Extraction of similar charge operators in electromagnetic field

Cor. 5.10.1. $Q = \varsigma\int\limits_{\vec{p}\neq 0}[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)]d^3\vec{p} = i\varsigma\int \Psi^+(\vec{r},t)\frac{\gamma\cdot\nabla}{-\nabla^2}\Psi(\vec{r},t)d^3\vec{r}$

Proof: $Q = \varsigma\int\limits_{\vec{p}\neq 0}[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)]d^3\vec{p}$
$= \varsigma\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}[\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma p\cdot x}$
$- \lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{-i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{i\varsigma p\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \varsigma\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \varsigma\frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}p^a p^b\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \varsigma\frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} + \varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k - \delta^{\alpha'_\varsigma\alpha_\varsigma})\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \varsigma\frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \varsigma\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k\Psi_{\alpha_\varsigma}(\vec{r},t)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= -i\varsigma\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\frac{(\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}}{-\nabla^2}\delta^3(\vec{r}-\vec{r}')d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= i\varsigma\int \Psi^+(\vec{r},t)\frac{\gamma\cdot\nabla}{-\nabla^2}\Psi(\vec{r},t)d^3\vec{r} = \int \Psi^+(\vec{r},t)\frac{i\partial_t}{-\nabla^2}\Psi(\vec{r},t)d^3\vec{r}$ □

5.11 Extraction of particle number operator in electromagnetic field

Cor. 5.11.1. $N = \int\limits_{\vec{p}\neq 0}[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) + a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)]d^3\vec{p} = \int \Psi^+(\vec{r},t)\frac{1}{\sqrt{-\nabla^2}}\Psi(\vec{r},t)d^3\vec{r}$

Proof: $N = \int\limits_{\vec{p}\neq 0}[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) + a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}[\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma p\cdot x}$
$+ \lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{-i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{i\varsigma p\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}p^a p^b\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{-1}{2}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} + \varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k - \delta^{\alpha'_\varsigma\alpha_\varsigma})\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\frac{1}{|\vec{p}|}(\delta^{\alpha'_\varsigma\alpha_\varsigma} - \hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \int \frac{1}{\sqrt{-\nabla^2}}[\delta^{\alpha'_\varsigma\alpha_\varsigma} - \frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2}]\delta^3(\vec{r}-\vec{r}')\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)d^3\vec{r}d^3\vec{r}'$

$= \int \Psi^+_{\alpha'_\varsigma}(\vec{r},t)\frac{1}{\sqrt{-\nabla^2}}[\delta^{\alpha'_\varsigma\alpha_\varsigma} - \frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2}]\Psi_{\alpha_\varsigma}(\vec{r},t)d^3\vec{r}(\nabla\cdot\Psi(\vec{r},t)=0)$

$= \int \Psi^+(\vec{r},t)\frac{1}{\sqrt{-\nabla^2}}\Psi(\vec{r},t)d^3\vec{r}$ □

5.12 Energy momentum normalization operator of electromagnetic field

Cor. 5.12.1. $H_0 = \int\limits_{\vec{p}\neq 0} |\vec{p}|[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) + a_2^+(\vec{p},-\varsigma)a_2(\vec{p},-\varsigma)]d^3\vec{p}$

$= \int \Psi^+(\vec{r},t)\Psi(\vec{r},t)d^3\vec{r} - \frac{i}{2}\int[\Psi^+(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\Psi(\vec{r},t) + \Psi^T(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\Psi^*(\vec{r},t)]d^3\vec{r}$

Proof: $H_0 = \int\limits_{\vec{p}\neq 0} |\vec{p}|[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) + a_2^+(\vec{p},-\varsigma)a_2(\vec{p},-\varsigma)]d^3\vec{p}$

$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}[\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma p\cdot x}$

$+ \lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{i\varsigma p\cdot x}\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{-i\varsigma p\cdot x'}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)[\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + \Psi_{\alpha_\varsigma}(\vec{r},t)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{-1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} + \varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k - \delta^{\alpha'_\varsigma\alpha_\varsigma})[\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + \Psi_{\alpha_\varsigma}(\vec{r},t)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \int \Psi^+(\vec{r},t)\Psi(\vec{r},t)d^3\vec{r}$

$+ \frac{-1}{2(2\pi)^3}\int\limits_{\vec{p}\neq 0}\varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k[\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + \Psi_{\alpha_\varsigma}(\vec{r},t)\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \int \Psi^+(\vec{r},t)\Psi(\vec{r},t)d^3\vec{r} + \frac{-1}{2}\int i[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t) - i[(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t)\Psi^+_{\alpha'_\varsigma}(\vec{r},t)d^3\vec{r}$

$= \int \Psi^+(\vec{r},t)\Psi(\vec{r},t)d^3\vec{r} - \frac{i}{2}\int[\Psi^+_{\alpha'_\varsigma}(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})^{\alpha'_\varsigma\alpha_\varsigma}\Psi_{\alpha_\varsigma}(\vec{r},t) + [\Psi_{\alpha_\varsigma}(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})^{\alpha_\varsigma\alpha'_\varsigma}\Psi^+_{\alpha'_\varsigma}(\vec{r},t)d^3\vec{r}$

$= \int \Psi^+(\vec{r},t)\Psi(\vec{r},t)d^3\vec{r} - \frac{i}{2}\int[\Psi^+(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\Psi(\vec{r},t) + \Psi^T(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\Psi^*(\vec{r},t)]d^3\vec{r}$

$= H - H_g$ □

Cor. 5.12.2. $H = H_0 + H_g$

$H_g = \frac{i}{2}\int[\Psi^+(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\Psi(\vec{r},t) + \Psi^T(\vec{r},t)(\frac{\gamma\cdot\nabla}{\sqrt{-\nabla^2}})\Psi^*(\vec{r},t)]d^3\vec{r} = \int\limits_{\vec{p}\neq 0}|\vec{p}|[a_2(\vec{p},-\varsigma),a_2^+(\vec{p},-\varsigma)]d^3\vec{p}$

5.13 Extraction of angular momentum operator in electromagnetic field

5.13.1 Extraction of space orbit angular momentum operator in electromagnetic field

Lem. 5.13.1. $\lambda_m^+(\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma) =?0, \lambda_m^+(-\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma) =?0$

Cor. 5.13.1. $M_{ij}(1,\varsigma) = -\varsigma\int\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t)d^3\vec{r}$

$= -i\varsigma\int\limits_{\vec{p}\neq 0} a_1^+(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_2^+(\vec{p},-\varsigma)d^3\vec{p}$

$+ i\int\limits_{\vec{p}\neq 0}|\vec{p}|[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)]\lambda_m^+(\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)d^3\vec{p}$

Proof: $M_{ij}(1,\varsigma) = -\varsigma\int\Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t)d^3\vec{r}$

$= \frac{-\varsigma}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}'|}a_1^+(\vec{p}',-\varsigma)e^{-i\varsigma\vec{p}'\cdot\vec{r}}e^{i\varsigma|\vec{p}'|t}\lambda_m^+(\hat{p}',-\varsigma)](r_i\gamma_j - r_j\gamma_i)[\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma)e^{i\varsigma\vec{p}\cdot\vec{r}}e^{-i\varsigma|\vec{p}|t}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$

$- \frac{\varsigma}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}'|}a_2(\vec{p}',-\varsigma)e^{i\varsigma\vec{p}'\cdot\vec{r}}e^{-i\varsigma|\vec{p}'|t}\lambda_m^+(\hat{p}',-\varsigma)](r_i\gamma_j - r_j\gamma_i)[\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma)e^{-i\varsigma\vec{p}\cdot\vec{r}}e^{i\varsigma|\vec{p}|t}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$

$= \frac{-\varsigma}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}'|}a_1^+(\vec{p}',-\varsigma)e^{i\varsigma|\vec{p}'|t}\lambda_m^+(\hat{p}',-\varsigma)][-i\varsigma(\tilde{\partial}_i\gamma_j - \tilde{\partial}_j\gamma_i)e^{i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}][\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$

$- \frac{\varsigma}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}'|}a_2(\vec{p}',-\varsigma)e^{-i\varsigma|\vec{p}'|t}\lambda_m^+(\hat{p}',-\varsigma)][i\varsigma(\tilde{\partial}_i\gamma_j - \tilde{\partial}_j\gamma_i)e^{-i\varsigma(\vec{p}-\vec{p}')\cdot\vec{r}}][\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}]d^3\vec{p}d^3\vec{p}'d^3\vec{r}$

$= i\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}'|}a_1^+(\vec{p}',-\varsigma)e^{i\varsigma|\vec{p}'|t}\lambda_m^+(\hat{p}',-\varsigma)][(\tilde{\partial}_i\gamma_j - \tilde{\partial}_j\gamma_i)\delta^3(\vec{p}-\vec{p}')][\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}]d^3\vec{p}d^3\vec{p}'$

$- i\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}'|}a_2(\vec{p}',-\varsigma)e^{-i\varsigma|\vec{p}'|t}\lambda_m^+(\hat{p}',-\varsigma)][(\tilde{\partial}_i\gamma_j - \tilde{\partial}_j\gamma_i)\delta^3(\vec{p}-\vec{p}')][\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}]d^3\vec{p}d^3\vec{p}'$

$= -i\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}|}a_1^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}\lambda_m^+(\hat{p},-\varsigma)]\{(\tilde{\partial}_i\gamma_j - \tilde{\partial}_j\gamma_i)[\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}]\}d^3\vec{p}$

$+ i\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}|}a_2(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}\lambda_m^+(\hat{p},-\varsigma)]\{(\tilde{\partial}_i\gamma_j - \tilde{\partial}_j\gamma_i)[\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}]\}d^3\vec{p}$

$= -i\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}|}a_1^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}\lambda_m^+(\hat{p},-\varsigma)]\{(\gamma_j\tilde{\partial}_i - \gamma_i\tilde{\partial}_j)[\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}]\}d^3\vec{p}$

$+ i\int\limits_{\vec{p}\neq 0}[\sqrt{|\vec{p}|}a_2(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}\lambda_m^+(\hat{p},-\varsigma)]\{(\gamma_j\tilde{\partial}_i - \gamma_i\tilde{\partial}_j)[\lambda_m(\hat{p},-\varsigma)\sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}]\}d^3\vec{p}$

$= i\varsigma \int\limits_{\vec{p}\neq 0} [\sqrt{|\vec{p}|}a_1^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}]\{(\hat{p}_j\tilde{\partial}_i - \hat{p}_i\tilde{\partial}_j)[\sqrt{|\vec{p}|}a_1(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}]\}d^3\vec{p}$
$+ i \int\limits_{\vec{p}\neq 0} [|\vec{p}|a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)]\{(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)\}d^3\vec{p}$
$- i\varsigma \int\limits_{\vec{p}\neq 0} [\sqrt{|\vec{p}|}a_2(\vec{p},-\varsigma)e^{-i\varsigma|\vec{p}|t}]\{(\hat{p}_j\tilde{\partial}_i - \hat{p}_i\tilde{\partial}_j)[\sqrt{|\vec{p}|}a_2^+(\vec{p},-\varsigma)e^{i\varsigma|\vec{p}|t}]\}d^3\vec{p}$
$- i \int\limits_{\vec{p}\neq 0} [|\vec{p}|a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)]\{(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)\}d^3\vec{p}$
$= -i\varsigma \int\limits_{\vec{p}\neq 0} a_1^+(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_1(\vec{p},-\varsigma)d^3\vec{p} + i \int\limits_{\vec{p}\neq 0} |\vec{p}|a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)d^3\vec{p}$
$+ i\varsigma \int\limits_{\vec{p}\neq 0} a_2(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_2^+(\vec{p},-\varsigma)d^3\vec{p} - i \int\limits_{\vec{p}\neq 0} |\vec{p}|a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)d^3\vec{p}$
$= -i\varsigma \int\limits_{\vec{p}\neq 0} a_1^+(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_2^+(\vec{p},-\varsigma)d^3\vec{p}$
$+ i \int\limits_{\vec{p}\neq 0} |\vec{p}|[a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)]\lambda_m^+(\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j - \gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)d^3\vec{p}$ □

Attempt to reverse inference:

Proof: $M_{ij}(1,\varsigma) = \varsigma \int\limits_{\vec{p}\neq 0} \{a_1^+(\vec{p},-\varsigma)\tilde{M}_{ij}(1,\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)\tilde{M}_{ij}(1,\varsigma)a_2^+(\vec{p},-\varsigma)\}d^3\vec{p}$
$= -i\varsigma \int\limits_{\vec{p}\neq 0} \{a_1^+(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)a_2^+(\vec{p},-\varsigma)\}d^3\vec{p}$
$= -i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{\sqrt{|\vec{p}|}}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)e^{i\varsigma p\cdot x'}(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)[\frac{1}{\sqrt{|\vec{p}|}}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)e^{-i\varsigma p\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= -i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{\sqrt{|\vec{p}|}}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)e^{i\varsigma p\cdot x'}e^{-i\varsigma p\cdot x}(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)[\frac{1}{\sqrt{|\vec{p}|}}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$- i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{\sqrt{|\vec{p}|}}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)e^{i\varsigma p\cdot x'}(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)[\frac{1}{\sqrt{|\vec{p}|}}e^{-i\varsigma p\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= -i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$- i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}[-i\varsigma(p_ir_j - p_jr_i)]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{i\varsigma}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)(\sigma)^{ab}_{\alpha'_\varsigma\alpha_\varsigma}\hat{p}_a\hat{p}_b[-i\varsigma(p_ir_j - p_jr_i)]e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{i\varsigma}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)(\sigma)^{ab}_{\alpha'_\varsigma\alpha_\varsigma}\hat{p}_a\hat{p}_b[-i\varsigma(p_ir_j - p_jr_i)][e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{i\varsigma}{2(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} + \varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k - \delta^{\alpha'_\varsigma\alpha_\varsigma})[-i\varsigma(p_ir_j - p_jr_i)][e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= -i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{1}{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k[-i(r_ip_j - r_jp_i)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)(r_i\partial_j - r_j\partial_i)\frac{(\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}}{-\nabla^2}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \varsigma \int \Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)(r_i\partial_j - r_j\partial_i)\frac{(\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}}{-\nabla^2}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$= \varsigma \int \Psi^+(\vec{r},t)\frac{\gamma\cdot\nabla}{-\nabla^2}(\partial_jr_i - \partial_ir_j)\Psi(\vec{r},t)d^3\vec{r}$
$= -\varsigma \int \Psi^+(\vec{r},t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r},t)d^3\vec{r}$ □

5.13.2 Extraction of time orbit angular momentum operator in electromagnetic field

Cor. 5.13.2. $L_{i\pi}(1,\varsigma) = \varsigma \int\limits_{\vec{p}\neq 0} \{a_1^+(\vec{p},-\varsigma)\tilde{M}_{i\pi}(1,\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)\tilde{M}_{i\pi}(1,\varsigma)a_2^+(\vec{p},-\varsigma)\}d^3\vec{p}$
$= \int \Psi^+(\vec{r},t)(ir_i + i\varsigma t\gamma_i)\Psi(\vec{r},t)d^3\vec{r}$

Proof: $L_{i\pi}(1,\varsigma) = \varsigma \int\limits_{\vec{p}\neq 0} \{a_1^+(\vec{p},-\varsigma)\tilde{M}_{i\pi}(1,\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)\tilde{M}_{i\pi}(1,\varsigma)a_2^+(\vec{p},-\varsigma)\}d^3\vec{p}$
$= -i\varsigma \int\limits_{\vec{p}\neq 0} \{a_1^+(\vec{p},-\varsigma)(p_i\tilde{\partial}_\pi - p_\pi\tilde{\partial}_i)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)(p_i\tilde{\partial}_\pi - p_\pi\tilde{\partial}_i)a_2^+(\vec{p},-\varsigma)\}d^3\vec{p}$
$= -\varsigma \int\limits_{\vec{p}\neq 0} \hat{p}_i\{a_1^+(\vec{p},-\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)a_2^+(\vec{p},-\varsigma)\}d^3\vec{p} - \varsigma \int\limits_{\vec{p}\neq 0} |\vec{p}|\{a_1^+(\vec{p},-\varsigma)\tilde{\partial}_ia_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)\tilde{\partial}_ia_2^+(\vec{p},-\varsigma)\}d^3\vec{p}$
$= -\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \sqrt{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)e^{i\varsigma p\cdot x'}\tilde{\partial}_i[\frac{1}{\sqrt{|\vec{p}|}}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)(e^{-i\varsigma p\cdot x})]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$- \varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \sqrt{|\vec{p}|}\Psi^+_{\alpha'_\varsigma}(\vec{r}',t)\Psi_{\alpha_\varsigma}(\vec{r},t)\lambda_m^{\alpha'_\varsigma}(\hat{p},-\varsigma)e^{-i\varsigma p\cdot x'}\tilde{\partial}_i[\frac{1}{\sqrt{|\vec{p}|}}\lambda_m^{+\alpha_\varsigma}(\hat{p},-\varsigma)(-e^{i\varsigma p\cdot x})]d^3\vec{p}d^3\vec{r}d^3\vec{r}' - \hat{s}_i(1,\varsigma)$

$= -\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)e^{i\varsigma p\cdot x'}\tilde{\partial}_i[\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)(e^{-i\varsigma p\cdot x})]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$- \varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)e^{-i\varsigma p\cdot x'}\tilde{\partial}_i[\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)(-e^{i\varsigma p\cdot x})]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= -\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)e^{i\varsigma p\cdot x'}\tilde{\partial}_i(e^{-i\varsigma p\cdot x})d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$- \varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)e^{-i\varsigma p\cdot x'}\tilde{\partial}_i(-e^{i\varsigma p\cdot x})d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= -\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]\tilde{\partial}_i(-i\varsigma p\cdot x)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= i \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}](r_i - t\hat{p}_i)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= -i/2 \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)(\sigma)^{ab}_{\alpha'_\varsigma\alpha_\varsigma}\hat{p}_a\hat{p}_b[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}](r_i - t\hat{p}_i)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= -i/2 \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} + \varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k - \delta^{\alpha'_\varsigma\alpha_\varsigma})[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}](r_i - t\hat{p}_i)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= -i \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)r_i\Psi_{\alpha_\varsigma}(\vec{r}, t)(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} - \delta^{\alpha'_\varsigma\alpha_\varsigma})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$+ i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)t\Psi_{\alpha_\varsigma}(\vec{r}, t)(\gamma\cdot\hat{p})^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}_i e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= -i \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)r_i\Psi_{\alpha_\varsigma}(\vec{r}, t)(\frac{-\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{-\nabla^2} - \delta^{\alpha'_\varsigma\alpha_\varsigma})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$+ i\varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)t\Psi_{\alpha_\varsigma}(\vec{r}, t)\frac{(-i\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}(-i\partial_i)}{-\nabla^2}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= -i\int \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)r_i\Psi_{\alpha_\varsigma}(\vec{r}, t)(\frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2} - \delta^{\alpha'_\varsigma\alpha_\varsigma})\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$

$+ i\varsigma\int \Psi^+_{\alpha'_\varsigma}(\vec{r}', t)t\Psi_{\alpha_\varsigma}(\vec{r}, t)\frac{(\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}(\partial_i)}{\nabla^2}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$

$= -i\int \Psi^+_{\alpha'_\varsigma}(\vec{r}, t)(\frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2} - \delta^{\alpha'_\varsigma\alpha_\varsigma})r_i\Psi_{\alpha_\varsigma}(\vec{r}, t)d^3\vec{r} + i\varsigma\int \Psi^+_{\alpha'_\varsigma}(\vec{r}, t)\frac{(\gamma\cdot\nabla)^{\alpha'_\varsigma\alpha_\varsigma}(\partial_i)}{\nabla^2}t\Psi_{\alpha_\varsigma}(\vec{r}, t)d^3\vec{r}$

$= i\int \Psi^+(\vec{r}, t)r_i\Psi(\vec{r}, t)d^3\vec{r} + i\varsigma\int \Psi^+(\vec{r}, t)t\gamma_i\Psi(\vec{r}, t)d^3\vec{r}$

$= \int \Psi^+(\vec{r}, t)(ir_i + i\varsigma t\gamma_i)\Psi(\vec{r}, t)d^3\vec{r}$ □

5.13.3 Extraction of electromagnetic field orbit angular momentum operator

Cor. 5.13.3. $M_{ab}(1,\varsigma) = \varsigma \int\limits_{\vec{p}\neq 0} \{a_1^+(\vec{p}, -\varsigma)\tilde{M}_{ab}(1,\varsigma)a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma)\tilde{M}_{ab}(1,\varsigma)a_2^+(\vec{p}, -\varsigma)\}d^3\vec{p}$

$= \int \Psi^+(\vec{r}, t)[r_a\gamma_b(1,\varsigma) - r_b\gamma_a(1,\varsigma)]\Psi(\vec{r}, t)d^3\vec{r}$

5.13.4 Extraction of spin angular momentum operator in electromagnetic field

Cor. 5.13.4. $\hat{s}(1,\varsigma) = \varsigma \int\limits_{\vec{p}\neq 0} \hat{p}\{a_1^+(\vec{p}, -\varsigma)a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma)a_2^+(\vec{p}, -\varsigma)\}d^3\vec{p} = \int \Psi^+(\vec{r}, t)\frac{-i\nabla}{-\nabla^2}\Psi(\vec{r}, t)d^3\vec{r}$

Proof: $\hat{s}(1,\varsigma) = \varsigma \int\limits_{\vec{p}\neq 0} \hat{p}\{a_1^+(\vec{p}, -\varsigma)a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma)a_2^+(\vec{p}, -\varsigma)\}d^3\vec{p}$

$= \varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{\hat{p}}{|\vec{p}|}[\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)e^{i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)\Psi_{\alpha_\varsigma}(\vec{r}, t)e^{-i\varsigma p\cdot x}$

$- \lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)e^{-i\varsigma p\cdot x'}\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)\Psi_{\alpha_\varsigma}(\vec{r}, t)e^{i\varsigma p\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \varsigma \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{\hat{p}}{|\vec{p}|}\lambda_m^{\alpha'_\varsigma}(\hat{p}, -\varsigma)\lambda_m^{+\alpha_\varsigma}(\hat{p}, -\varsigma)\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \varsigma \frac{1}{(2\pi)^3}\frac{-1}{2} \int\limits_{\vec{p}\neq 0} \frac{\hat{p}}{|\vec{p}|}\sigma_{ab}^{\alpha'_\varsigma\alpha_\varsigma}p^a p^b\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \varsigma \frac{1}{(2\pi)^3}\frac{-1}{2} \int\limits_{\vec{p}\neq 0} \frac{\hat{p}}{|\vec{p}|}(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} + \varsigma\gamma_k{}^{\alpha'_\varsigma\alpha_\varsigma}\hat{p}^k - \delta^{\alpha'_\varsigma\alpha_\varsigma})\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \varsigma \frac{1}{(2\pi)^3}\frac{-1}{2} \int\limits_{\vec{p}\neq 0} \frac{\hat{p}}{|\vec{p}|}(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} - \delta^{\alpha'_\varsigma\alpha_\varsigma})\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)[e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{\hat{p}}{|\vec{p}|}(\hat{p}^{\alpha'_\varsigma}\hat{p}^{\alpha_\varsigma} - \delta^{\alpha'_\varsigma\alpha_\varsigma})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \frac{-i\nabla}{-\nabla^2}(\frac{-\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{-\nabla^2} - \delta^{\alpha'_\varsigma\alpha_\varsigma})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= i\int \frac{\nabla}{\nabla^2}(\frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2} - \delta^{\alpha'_\varsigma\alpha_\varsigma})\delta^3(\vec{r}-\vec{r}')\Psi^+_{\alpha'_\varsigma}(\vec{r}', t)\Psi_{\alpha_\varsigma}(\vec{r}, t)d^3\vec{r}d^3\vec{r}'$

$= -i\int \Psi^+_{\alpha'_\varsigma}(\vec{r}, t)\frac{\nabla}{\nabla^2}(\frac{\partial^{\alpha'_\varsigma}\partial^{\alpha_\varsigma}}{\nabla^2} - \delta^{\alpha'_\varsigma\alpha_\varsigma})\Psi_{\alpha_\varsigma}(\vec{r}, t)d^3\vec{r}$

$= \int \Psi^+(\vec{r}, t)\frac{-i\nabla}{-\nabla^2}\Psi(\vec{r}, t)d^3\vec{r}$ □

5.14 Summary of angular momentum operator in electromagnetic field

Cor. 5.14.1. $L_{ab}(1,\varsigma) = \varsigma \int\limits_{\vec{p}\neq 0} \{a_1^+(\vec{p},-\varsigma)\tilde{M}_{ab}(1,\varsigma)a_1(\vec{p},-\varsigma) - a_2(\vec{p},-\varsigma)\tilde{M}_{ab}(1,\varsigma)a_2^+(\vec{p},-\varsigma)\}d^3\vec{p}$
$= \int \Psi^+(\vec{r},t)[r_a\gamma_b(1,\varsigma) - r_b\gamma_a(1,\varsigma) - \frac{\sigma^{\alpha_\varsigma}_{\varsigma ab}\partial_{\alpha_\varsigma}}{-\nabla^2}]\Psi(\vec{r},t)d^3\vec{r}$

Proof: $[\vec{r}\times(\vec{E}\times\vec{B}), \vec{r}'\times(\vec{E}'\times\vec{B}')]$

$$
\begin{aligned}
&= [E_i(\vec{r}\cdot\vec{B}) - (\vec{r}\cdot\vec{E})B_i, E_j'(\vec{r}'\cdot\vec{B}') - (\vec{r}'\cdot\vec{E}')B_j']\\
&= [E_i(\vec{r}\cdot\vec{B}), E_j'(\vec{r}'\cdot\vec{B}')] - [E_i(\vec{r}\cdot\vec{B}), (\vec{r}'\cdot\vec{E}')B_j'] - [(\vec{r}\cdot\vec{E})B_i, E_j'(\vec{r}'\cdot\vec{B}')] + [(\vec{r}\cdot\vec{E})B_i, (\vec{r}'\cdot\vec{E}')B_j']\\
&= E_i[(\vec{r}\cdot\vec{B}), E_j'](\vec{r}'\cdot\vec{B}') + E_j'[E_i, (\vec{r}'\cdot\vec{B}')](\vec{r}\cdot\vec{B})\\
&- E_i[(\vec{r}\cdot\vec{B}), (\vec{r}'\cdot\vec{E}')]B_j' - (\vec{r}'\cdot\vec{E}')[E_i, B_j'](\vec{r}\cdot\vec{B})\\
&- (\vec{r}\cdot\vec{E})[B_i, E_j'](\vec{r}'\cdot\vec{B}') - E_j'[(\vec{r}\cdot\vec{E}), (\vec{r}'\cdot\vec{B}')]B_i\\
&+ (\vec{r}\cdot\vec{E})[B_i, (\vec{r}'\cdot\vec{E}')]B_j' + (\vec{r}'\cdot\vec{E}')[(\vec{r}\cdot\vec{E}), B_j']B_i\\
&= r^k r'^l\{E_i[B_k, E_j']B_l' + E_j'[E_i, B_l']B_k\\
&- E_i[B_k, E_l']B_j' - E_l'[E_i, B_j']B_k\\
&- E_k[B_i, E_j']B_l' - E_j'[E_k, B_l']B_i\\
&+ E_k[B_i, E_l']B_j' + E_l'[E_k, B_j']B_i\}\\
&= r^k r'^l\\
&\{E_iB_l'[i\varepsilon_{kj}{}^m\partial_m\delta(x-x')] + E_j'B_k[-i\varepsilon_{il}{}^m\partial_m\delta(x-x')]\\
&- E_iB_j'[i\varepsilon_{kl}{}^m\partial_m\delta(x-x')] - E_l'B_k[-i\varepsilon_{ij}{}^m\partial_m\delta(x-x')]\\
&- E_kB_l'[i\varepsilon_{ij}{}^m\partial_m\delta(x-x')] - E_j'B_i[-i\varepsilon_{kl}{}^m\partial_m\delta(x-x')]\\
&+ E_kB_j'[i\varepsilon_{il}{}^m\partial_m\delta(x-x')] + E_l'B_i[-i\varepsilon_{kj}{}^m\partial_m\delta(x-x')]\}\\
&= -r^k r'^l\\
&\{\partial_mE_iB_l'[i\varepsilon_{kj}{}^m\delta(x-x')] + \partial_m'E_j'B_k[i\varepsilon_{il}{}^m\delta(x-x')] - \partial_mE_iB_j'[i\varepsilon_{kl}{}^m\delta(x-x')] - \partial_m'E_l'B_k[i\varepsilon_{ij}{}^m\delta(x-x')]\\
&- \partial_mE_kB_l'[i\varepsilon_{ij}{}^m\delta(x-x')] - \partial_m'E_j'B_i[i\varepsilon_{kl}{}^m\delta(x-x')] + \partial_mE_kB_j'[i\varepsilon_{il}{}^m\delta(x-x')] + \partial_m'E_l'B_i[i\varepsilon_{kj}{}^m\delta(x-x')]\}\\
&+ r^kE_l'B_k[i\varepsilon_{ij}{}^l\delta(x-x')] + r'^lE_kB_l'[i\varepsilon_{ij}{}^k\delta(x-x')] - r'^lE_kB_j'[i\varepsilon_{il}{}^k\delta(x-x')] - r^kE_l'B_i[i\varepsilon_{kj}{}^l\delta(x-x')]\\
&= -ir^kr'^l\delta(x-x')\\
&\{\partial_mE_iB_l'\varepsilon_{kj}{}^m + \partial_m'E_j'B_k\varepsilon_{il}{}^m - \partial_mE_iB_j'\varepsilon_{kl}{}^m - \partial_m'E_l'B_k\varepsilon_{ij}{}^m\\
&- \partial_mE_kB_l'\varepsilon_{ij}{}^m - \partial_m'E_j'B_i\varepsilon_{kl}{}^m + \partial_mE_kB_j'\varepsilon_{il}{}^m + \partial_m'E_l'B_i\varepsilon_{kj}{}^m\}\\
&+ r^kE_l'B_k[i\varepsilon_{ij}{}^l\delta(x-x')] + r'^lE_kB_l'[i\varepsilon_{ij}{}^k\delta(x-x')] - r'^lE_kB_j'[i\varepsilon_{il}{}^k\delta(x-x')] - r^kE_l'B_i[i\varepsilon_{kj}{}^l\delta(x-x')]\\
&= -i\delta(x-x')\\
&\{\varepsilon_{kj}{}^mr^k\partial_mE_ir^lB_l + \varepsilon_{il}{}^mr^l\partial_mE_jr^kB_k - \varepsilon_{kl}{}^mr^kr^l\partial_mE_iB_j - \varepsilon_{ij}{}^mr^l\partial_mE_lr^kB_k\\
&- \varepsilon_{ij}{}^mr^k\partial_mE_kr^lB_l - \varepsilon_{kl}{}^mr^kr^l\partial_mE_jB_i + \varepsilon_{il}{}^mr^kr^l\partial_mE_kB_j + \varepsilon_{kj}{}^mr^kr^l\partial_mE_lB_i\\
&- \varepsilon_{ij}{}^lE_lr^kB_k - \varepsilon_{ij}{}^kE_kr^lB_l + \varepsilon_{il}{}^kr^lE_kB_j + \varepsilon_{kj}{}^lr^kE_lB_i\}\\
&= -i\delta(x-x')\\
&\{\varepsilon_{kj}{}^mr^k\partial_mE_i(\vec{r}\cdot\vec{B}) + \varepsilon_{il}{}^mr^l\partial_mE_j(\vec{r}\cdot\vec{B}) - \varepsilon_{kl}{}^mr^kr^l\partial_mE_iB_j - \varepsilon_{ij}{}^mr^l\partial_mE_l(\vec{r}\cdot\vec{B})\\
&- \varepsilon_{ij}{}^mr^k\partial_mE_k(\vec{r}\cdot\vec{B}) - \varepsilon_{kl}{}^mr^kr^l\partial_mE_jB_i + \varepsilon_{il}{}^mr^kr^l\partial_mE_kB_j + \varepsilon_{kj}{}^mr^kr^l\partial_mE_lB_i\\
&- \varepsilon_{ij}{}^lE_l(\vec{r}\cdot\vec{B}) - \varepsilon_{ij}{}^kE_k(\vec{r}\cdot\vec{B}) + \varepsilon_{il}{}^kr^lE_kB_j + \varepsilon_{kj}{}^lr^kE_lB_i\}\\
&= -i\delta(x-x')\\
&\{-\varepsilon_{jk}{}^lr^k\partial_lE_i(\vec{r}\cdot\vec{B}) + \varepsilon_{ik}{}^lr^k\partial_lE_j(\vec{r}\cdot\vec{B}) - 2\varepsilon_{ij}{}^kr^l\partial_kE_l(\vec{r}\cdot\vec{B}) - 2\varepsilon_{ij}{}^kE_k(\vec{r}\cdot\vec{B})\\
&+ \varepsilon_{ik}{}^lr^kr^m\partial_lE_mB_j - \varepsilon_{jk}{}^lr^kr^m\partial_lE_mB_i + \varepsilon_{ik}{}^lr^kE_lB_j - \varepsilon_{jk}{}^lr^kE_lB_i\}
\end{aligned}
$$
□

Thm. 5.14.1.

$$
\begin{cases}
[\Psi_{\alpha_\varsigma}(x), \Psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\
[\Psi_{\alpha_\varsigma}(x), \Psi_{\beta_\varsigma}(x')] = 0, [\Psi^+_{\alpha'_\varsigma}(x), \Psi^+_{\beta'_\varsigma}(x')] = 0\\
\Psi(x) = \frac{1}{\sqrt{2}}[\vec{E}(x) - i\varsigma\vec{B}(x)]
\end{cases}
\Leftrightarrow
\begin{cases}
[E_i(x), E_j(x')] = -i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x-x')\\
[B_i(x), B_j(x')] = -i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x-x')\\
[E_i(x), B_j(x')] = i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')\\
[B_i(x), E_j(x')] = -i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x')
\end{cases}
$$

5.15 Commutative and anticommutative formulas

Cor. 5.15.1. $\begin{cases}[A,BC] = [A,B]C + B[A,C]\\ [BC,A] = [B,A]C + B[C,A]\end{cases}$

$$
\begin{cases}
[AB, A'B'] = [AB, A']B' + A'[AB, B'] = [A, A']BB' + A[B, A']B' + A'[A, B']B + A'A[B, B']\\
[AB, A'B'] = A[B, A'B'] + [A, A'B']B = AA'[B, B'] + A[B, A']B' + A'[A, B']B + [A, A']B'B
\end{cases}
$$

Cor. 5.15.2. $\begin{cases}[A,BC] = [A,B]C + B[A,C], [A,CB] = [A,C]B + C[A,B]\\ [A,BC] = \{A,B\}C - B\{A,C\}, [A,CB] = \{A,C\}B - C\{A,B\}\end{cases}$

Cor. 5.15.3. $\begin{cases}[A,\{B,C\}]=\{[A,B],C\}+\{B,[A,C]\}\\ [A,[B,C]]=\{\{A,B\},C\}-\{B,\{A,C\}\}\end{cases}$

6 New scheme for covariant quantization of free uncoupled Yang-Mills field

6.1 Various equivalent forms of free uncoupled YM field equation [22, 24]

Def. 6.1.1. $\Psi^{\rho}_{\alpha_\varsigma}:=\frac{-i\varsigma}{\sqrt{2}}\psi^{\rho}_{\alpha_\varsigma}=\frac{-i\varsigma}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F^{\rho}_{ab}=\frac{-i\varsigma}{\sqrt{2}}i\varsigma(E^{\rho}-i\varsigma B^{\rho})_{\alpha_\varsigma}$

Def. 6.1.2. $\Psi^{\rho}:=\frac{1}{\sqrt{2}}(\vec{E}^{\rho}-i\varsigma\vec{B}^{\rho})=\frac{1}{\sqrt{2}}(\vec{E}^{\rho}-i\varsigma\nabla\times\vec{A}^{\rho}),\Psi^{\rho}_i=\frac{1}{\sqrt{2}}(E^{\rho}_i-i\varsigma\varepsilon_i{}^{jk}\partial_jA^{\rho}_k),p\cdot x:=\vec{p}\cdot\vec{r}-Et$

Thm. 6.1.1.

$$\begin{cases}\partial^aF^{\rho}_{ab}=0\\ \partial^a*F^{\rho}_{ab}=0\end{cases}\Leftrightarrow\begin{cases}\nabla\cdot\vec{E}^{\rho}=0,\nabla\times\vec{E}^{\rho}=-\partial_t\vec{B}^{\rho}\\ \nabla\cdot\vec{B}^{\rho}=0,\nabla\times\vec{B}^{\rho}=\partial_t\vec{E}^{\rho}\end{cases}\Leftrightarrow\begin{cases}(\gamma,-i\varsigma)^a\partial_a\Psi^{\rho}=0\\ \nabla\cdot\Psi^{\rho}=0\end{cases}\Leftrightarrow\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi^{\rho}=0\\ S_{ab}(\gamma,\varsigma)=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}(s)\end{cases}$$

6.2 Spin equation and plane wave solution of free uncoupled YM complex field strength

Thm. 6.2.1. $[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi^{\rho}(x)=0$

Cor. 6.2.1.

$$\begin{cases}\Psi^{\rho}(\vec{r},t):=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}\sqrt{|\vec{p}|}\lambda_m(\hat{p},-\varsigma)[a^{\rho}_1(\vec{p},-\varsigma)e^{i\varsigma p\cdot x}+a^{\rho+}_2(\vec{p},-\varsigma)e^{-i\varsigma p\cdot x}]d^3\vec{p}\\ \sqrt{|\vec{p}|}a^{\rho}_1(\vec{p},-\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+_m(\hat{p},-\varsigma)\Psi^{\rho}(\vec{r},t)e^{-i\varsigma p\cdot x}d^3\vec{r}\\ \sqrt{|\vec{p}|}a^{\rho+}_2(\vec{p},-\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+_m(\hat{p},-\varsigma)\Psi^{\rho}(\vec{r},t)e^{i\varsigma p\cdot x}d^3\vec{r}\end{cases}$$

6.3 Commutation rules of free uncoupled YM field equation

Cor. 6.3.1.

$$\begin{cases}[a^{\rho}_{\sigma}(\vec{p},-\varsigma),a^{\rho'+}_{\sigma'}(\vec{p}',-\varsigma)]=\varsigma\delta^{\rho\rho'}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a^{\rho}_{\sigma}(\vec{p},-\varsigma),a^{\tau}_{\sigma'}(\vec{p}',-\varsigma)]=0\\ [a^{\rho'+}_{\sigma}(\vec{p},-\varsigma),a^{\tau'+}_{\sigma'}(\vec{p}',-\varsigma)]=0\end{cases}\Leftrightarrow\begin{cases}[a^{\rho}_{\sigma}(\vec{p}),a^{\rho'+}_{\sigma'}(\vec{p}')]=\delta^{\rho\rho'}\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a^{\rho}_{\sigma}(\vec{p}),a^{\tau}_{\sigma'}(\vec{p}')]=0\\ [a^{\rho'+}_{\sigma}(\vec{p}),{}^{\tau'+}_{\sigma'}(\vec{p}')]=0\end{cases}$$

$$\Updownarrow\qquad\qquad\Updownarrow$$

Cor. 6.3.2.

$$\begin{cases}[\Psi^{\rho}_{\alpha_\varsigma}(x),\Psi^{\rho'+}_{\alpha'_\varsigma}(x')]=i\delta^{\rho\rho'}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\Psi^{\rho}_{\alpha_\varsigma}(x),\Psi^{\tau}_{\beta_\varsigma}(x')]=0\\ [\Psi^{\rho'+}_{\alpha'_\varsigma}(x),\Psi^{\tau'+}_{\beta'_\varsigma}(x')]=0\end{cases}\Leftrightarrow\begin{cases}[\Psi^{\rho}_{\alpha_\varsigma}(\vec{r},t),\Psi^{\rho'+}_{\alpha'_\varsigma}(\vec{r}',t)]=\varsigma\delta^{\rho\rho'}\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\partial_k\delta(\vec{r}-\vec{r}')\\ [\Psi^{\rho}_{\alpha_\varsigma}(\vec{r},t),\Psi^{\tau}_{\beta_\varsigma}(\vec{r}',t)]=0\\ [\Psi^{\rho'+}_{\alpha'_\varsigma}(\vec{r},t),\Psi^{\tau'+}_{\beta'_\varsigma}(\vec{r}',t)]=0\end{cases}$$

6.4 Free uncoupled YM commutative function, causal function and feynman propagator

Cor. 6.4.1.

$$\begin{cases}\Delta^{\rho\rho'}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x)\\ \Delta^{\rho\rho'(+)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(+)}(x)\\ \Delta^{\rho\rho'(-)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(-)}(x)\\ \Delta^{\rho\rho'(l)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(l)}(x)\end{cases}\begin{cases}\Delta^{\rho\rho'(c)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}[\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{(c)}(x)+\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4]\\ \Delta^{\rho\rho' ret}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}[\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{ret}(x)+\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4(x)]\\ \Delta^{\rho\rho' adv}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}[\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta^{adv}(x)+\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4(x)]\\ \Delta^{\rho\rho'}_{F\alpha_\varsigma\alpha'_\varsigma}(\gamma;x):=\delta^{\rho\rho'}[\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta_F(x)+i\sigma^{\pi\pi}_{\alpha_\varsigma\alpha'_\varsigma}\delta^4(x)]=i\Delta^{\rho\rho'(c)}_{\alpha_\varsigma\alpha'_\varsigma}(\gamma;x)\\ \Delta_{F\alpha_\varsigma\alpha'_\varsigma}(\gamma;x)=\frac{i\delta^{\rho\rho'}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b}{p^2-i\varepsilon}+\cdots\end{cases}$$

Cor. 6.4.2.

$$\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho'}(\gamma;x)=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho'(+)}(\gamma;x)=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho'(-)}(\gamma;x)=0\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho'(l)}(\gamma;x)=0\end{cases}\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho'(c)}(\gamma;x)=-\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho' ret}(\gamma;x)=-\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho' adv}(\gamma;x)=-\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\ [\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Delta^{\rho\rho'}_F(\gamma;x)=-i\varsigma(\gamma,i\varsigma)_a\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\end{cases}$$

$$[\Updownarrow]\qquad\qquad[\Updownarrow]$$

Cor. 6.4.3.

$$\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'}(\gamma;x)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(+)}(\gamma;x)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(-)}(\gamma;x)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(l)}(\gamma;x)=0\end{cases}\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(c)}(\gamma;x)=-\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho' ret}(\gamma;x)=-\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho' adv}(\gamma;x)=-\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta_F(\gamma;x)=-i\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\end{cases}$$

$$[\Updownarrow]\qquad\qquad[\Updownarrow]$$

Cor. 6.4.4.

$$\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'}(\gamma;x)\bar{N}_m(1)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(+)}(\gamma;x)\bar{N}_m(1)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(-)}(\gamma;x)\bar{N}_m(1)=0\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(l)}(\gamma;x)\bar{N}_m(1)=0\end{cases}\begin{cases}(\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho'(c)}(\gamma;x)\bar{N}_m(1)=-\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\bar{N}_m(1)\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho' ret}(\gamma;x)\bar{N}_m(1)=-\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\bar{N}_m(1)\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta^{\rho\rho' adv}(\gamma;x)\bar{N}_m(1)=-\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\bar{N}_m(1)\\ (\sigma_{-\varsigma},-i\varsigma)_a\partial^aN_m(1)\Delta_F(\gamma;x)\bar{N}_m(1)=-i\varsigma\delta(t)N_m(1)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\bar{N}_m(1)\end{cases}$$

$[\Downarrow]$ $[\Downarrow]$

Cor. 6.4.5.

$$\begin{cases}(\gamma,-i\varsigma)_a\partial^a\Delta^{\rho\rho'}(\gamma;x)=0\\(\gamma,-i\varsigma)_a\partial^a\Delta^{\rho\rho'(+)}(\gamma;x)=0\\(\gamma,-i\varsigma)_a\partial^a\Delta^{\rho\rho'(-)}(\gamma;x)=0\\(\gamma,-i\varsigma)_a\partial^a\Delta^{\rho\rho'(l)}(\gamma;x)=0\end{cases}\quad\begin{cases}(\gamma,-i\varsigma)_a\partial^a\Delta^{\rho\rho'(c)}(\gamma;x)=-\varsigma\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\(\gamma,-i\varsigma)_a\partial^a\Delta^{\rho\rho'ret}(\gamma;x)=-\varsigma\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\(\gamma,-i\varsigma)_a\partial^a\Delta^{\rho\rho'adv}(\gamma;x)=-\varsigma\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\\(\gamma,-i\varsigma)_a\partial^a\Delta_F(\gamma;x)=-i\varsigma\delta(t)\Delta^{\rho\rho'}(\gamma;x)|_{t=0}\end{cases}$$

6.5 Equivalent commutative relations of free uncoupled $\tilde{\phi}^\rho,\Psi^\rho$ under radiation guage

Lem. 6.5.1.

$$\begin{cases}\nabla^2\tilde{A}^\rho-\partial_t^2\tilde{A}^\rho=\vec{J}^\rho+\partial_t\nabla\tilde{\phi}^\rho,\nabla^2\tilde{\phi}^\rho=\rho^\rho\\\sqrt{2}\Psi^\rho=-\partial_t\tilde{A}^\rho-\nabla\tilde{\phi}^\rho-i\varsigma\nabla\times\tilde{A}^\rho,\nabla\cdot\tilde{A}^\rho=0\end{cases}\quad[\Leftrightarrow]\begin{cases}[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi^\rho=-i\sigma_{\varsigma ab}^{[\beta_\varsigma]}J^{b\rho}\\\tilde{A}^\rho=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi^\rho-\Psi^{+\rho})}{\nabla^2},\tilde{\phi}^\rho=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi^\rho+\Psi^{+\rho})}{\nabla^2}\end{cases}$$

Lem. 6.5.2.

$$\begin{cases}[\tilde{A}_i^\rho(x),\tilde{A}_j^\tau(x')]=i\delta^{\rho\tau}(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\\[\tilde{A}_i^\rho(x),\tilde{\phi}^\tau(x')]=0,[\tilde{\phi}^\rho(x),\tilde{\phi}^\tau(x')]=0\\\sqrt{2}\Psi^\rho=-\partial_t\tilde{A}^\rho-\nabla\tilde{\phi}^\rho-i\varsigma\nabla\times\tilde{A}^\rho,\nabla\cdot\tilde{A}^\rho=0\end{cases}\Rightarrow\begin{cases}[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\alpha'_\varsigma}^{\rho'+}(x')]=i\delta^{\rho\rho'}\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab}\partial_a\partial_b\Delta(x-x')\\[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\beta_\varsigma}^\tau(x')]=0,[\Psi_{\alpha'_\varsigma}^{\rho'+}(x),\Psi_{\beta'_\varsigma}^{\tau'+}(x')]=0\\\tilde{A}^\rho=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi^\rho-\Psi^{+\rho})}{\nabla^2},\tilde{\phi}^\rho=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi^\rho+\Psi^{+\rho})}{\nabla^2}\end{cases}$$

Lem. 6.5.3.

$$\begin{cases}[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\alpha'_\varsigma}^{\rho'+}(x')]=i\delta^{\rho\rho'}\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab}\partial_a\partial_b\Delta(x-x')\\[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\beta_\varsigma}^\tau(x')]=0,[\Psi_{\alpha'_\varsigma}^{\rho'+}(x),\Psi_{\beta'_\varsigma}^{\tau'+}(x')]=0\\\tilde{A}^\rho=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi^\rho-\Psi^{+\rho})}{\nabla^2},\tilde{\phi}^\rho=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi^\rho+\Psi^{+\rho})}{\nabla^2}\end{cases}\Rightarrow\begin{cases}[\tilde{A}_i^\rho(x),\tilde{A}_j^\tau(x')]=i\delta^{\rho\tau}(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\\[\tilde{A}_i^\rho(x),\tilde{\phi}^\tau(x')]=0,[\tilde{\phi}^\rho(x),\tilde{\phi}^\tau(x')]=0\end{cases}$$

Thm. 6.5.1.

$$\begin{cases}[\tilde{A}_i^\rho(x),\tilde{A}_j^\tau(x')]=i\delta^{\rho\tau}(\delta_{ij}-\frac{\partial_i\partial_j}{\nabla^2})\Delta(x-x')\\[\tilde{A}_i^\rho(x),\tilde{\phi}^\tau(x')]=0,[\tilde{\phi}^\rho(x),\tilde{\phi}^\tau(x')]=0\\\nabla^2\tilde{A}^\rho-\partial_t^2\tilde{A}^\rho=\vec{J}^\rho+\partial_t\nabla\tilde{\phi}^\rho,\nabla^2\tilde{\phi}^\rho=\rho^\rho\\\sqrt{2}\Psi^\rho=-\partial_t\tilde{A}^\rho-\nabla\tilde{\phi}^\rho-i\varsigma\nabla\times\tilde{A}^\rho,\nabla\cdot\tilde{A}^\rho=0\end{cases}[\Leftrightarrow]\begin{cases}[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\alpha'_\varsigma}^{\rho'+}(x')]=i\delta^{\rho\rho'}\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab}\partial_a\partial_b\Delta(x-x')\\[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\beta_\varsigma}^\tau(x')]=0,[\Psi_{\alpha'_\varsigma}^{\rho'+}(x),\Psi_{\beta'_\varsigma}^{\tau'+}(x')]=0\\[\partial_a+iS_{ab}(\gamma,\varsigma)\partial^b]\Psi^\rho=-i\sigma_{\varsigma ab}^{[\beta_\varsigma]}J^{b\rho}\\\tilde{A}^\rho=\frac{-i\varsigma}{\sqrt{2}}\frac{\nabla\times(\Psi^\rho-\Psi^{+\rho})}{\nabla^2},\tilde{\phi}^\rho=-\frac{1}{\sqrt{2}}\frac{\nabla\cdot(\Psi^\rho+\Psi^{+\rho})}{\nabla^2}\end{cases}$$

6.6 Covariant commutation rules for free uncoupled YM field under radiation $\lambda-$guage

Cor. 6.6.1.

$$\begin{cases}[A_a^\rho(x),A_b^\tau(x')]=i\delta^{\rho\tau}(\delta_{ab}-\frac{\lambda-1}{\lambda}\frac{\partial_a\partial_b}{\Box+i\varepsilon})\Delta(x-x')\\\phi=-iA_0,\sqrt{2}\Psi^\rho=-\partial_t\vec{A}^\rho-\nabla\phi^\rho-i\varsigma\nabla\times\vec{A}^\rho\end{cases}\Rightarrow\begin{cases}[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\alpha'_\varsigma}^{+\rho'}(x')]=i\delta^{\rho\rho'}\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab}\partial_a\partial_b\Delta(x-x')\\[\Psi_{\alpha_\varsigma}^\rho(x),\Psi_{\beta_\varsigma}^\tau(x')]=0,[\Psi_{\alpha'_\varsigma}^{+\rho'}(x),\Psi_{\beta'_\varsigma}^{+\tau'}(x')]=0\\[\Psi_i^\rho(x),\phi^\tau(x')]=[\Psi_i^{+\rho}(x),\phi^\tau(x')]=\frac{i}{\sqrt{2}}\delta^{\rho\tau}\partial_i\Delta(x-x')\\[\phi^\rho(x),\phi^\tau(x')]=-i\delta^{\rho\tau}(1+\frac{\lambda-1}{\lambda}\frac{\nabla^2}{\Box+i\varepsilon})\Delta(x-x')\end{cases}$$

7 Gravitino field covariant quantization scheme

7.1 Gravitino spin operator equation and its plane wave solution

Thm. 7.1.1. $[\frac{3}{2}\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\psi(x)=0$

Cor. 7.1.1.
$$\begin{cases}\psi(\vec{r},t):=\frac{1}{(2\pi)^{3/2}}\int|\vec{p}|\lambda(\hat{p},-\frac{3}{2}\varsigma)[a_1(\vec{p},-\frac{3}{2}\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-\frac{3}{2}\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\|\vec{p}|a_1(\vec{p},-\frac{3}{2}\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-\frac{3}{2}\varsigma)\psi(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\|\vec{p}|a_2^+(\vec{p},-\frac{3}{2}\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-\frac{3}{2}\varsigma)\psi(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Def. 7.1.1. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(\frac{3}{2},\varsigma):=\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda_{k'_\varsigma}^+(\hat{p},-\frac{3}{2}\varsigma),\hat{P}^2(\frac{3}{2},\varsigma)=\hat{P}(\frac{3}{2},\varsigma),\hat{P}^+(\frac{3}{2},\varsigma)=\hat{P}(\frac{3}{2},\varsigma)$

Cor. 7.1.2. $H_2=\int|\vec{p}|[a_1^+(\vec{p},-\frac{3}{2}\varsigma)a_1(\vec{p},-\frac{3}{2}\varsigma)-a_2(\vec{p},-\frac{3}{2}\varsigma)a_2^+(\vec{p},-\frac{3}{2}\varsigma)]d^3\vec{p}=\int\psi_{k'_\varsigma}^+(\vec{r},t)\frac{i\partial_t}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$

Proof: $H_2=\int|\vec{p}|[a_1^+(\vec{p},-\frac{3}{2}\varsigma)a_1(\vec{p},-\frac{3}{2}\varsigma)-a_2(\vec{p},-\frac{3}{2}\varsigma)a_2^+(\vec{p},-\frac{3}{2}\varsigma)]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|}[\lambda^{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\psi_{k'_\varsigma}^+(\vec{r}',t)e^{ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{-ip\cdot x}$
$-\lambda^{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\psi_{k'_\varsigma}^+(\vec{r}',t)e^{-ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|}\lambda^{+k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\psi_{k'_\varsigma}^+(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int(-2\sqrt{2}i)^{-1}\frac{1}{|\vec{p}|^4}\Gamma_{k_\varsigma k'_\varsigma}^{abc}p_ap_bp_c\psi_{k'_\varsigma}^+(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int(-2\sqrt{2}i)^{-1}\frac{1}{|\vec{p}|^4}(\frac{1}{\sqrt{2}})^3\frac{i}{6}\{[3|\vec{p}|^3-2\varsigma|\vec{p}|^2[\sigma(\frac{3}{2})\cdot\vec{p}]-12|\vec{p}|[\sigma(\frac{3}{2})\cdot\vec{p}]^2+8\varsigma[\sigma(\frac{3}{2})\cdot\vec{p}]^3\}\psi_{k'_\varsigma}^+(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)$
$[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\frac{-1}{48}\frac{1}{|\vec{p}|^4}\{[3|\vec{p}|^3-2\varsigma|\vec{p}|^2[\sigma(\frac{3}{2})\cdot\vec{p}]-12|\vec{p}|[\sigma(\frac{3}{2})\cdot\vec{p}]^2+8\varsigma[\sigma(\frac{3}{2})\cdot\vec{p}]^3\}\psi_{k'_\varsigma}^+(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)$

$$[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$$
$$=\frac{1}{(2\pi)^3}\int\frac{-1}{48}\frac{1}{|\vec{p}|^4}\{[-2\varsigma|\vec{p}|^2[\sigma(\tfrac{3}{2})\cdot\vec{p}]+8\varsigma[\sigma(\tfrac{3}{2})\cdot\vec{p}]^3\}\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}-e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$$
$$=\frac{1}{(2\pi)^3}\int\frac{i\varsigma}{24}\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{[\frac{1}{|\vec{p}|^2}[\sigma(\tfrac{3}{2})\cdot\nabla]+\frac{1}{|\vec{p}|^4}4[\sigma(\tfrac{3}{2})\cdot\nabla]^3\}[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$$
$$=\frac{-i\varsigma}{12}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}-4\frac{[\sigma(\frac{3}{2})\cdot\nabla]^3}{\nabla^4}\}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$$
$$=\frac{i\varsigma}{12}\int\psi^+_{k'_\varsigma}(\vec{r},t)\{\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}-4\frac{[\sigma(\frac{3}{2})\cdot\nabla]^3}{\nabla^4}\}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$$
$$=\frac{-i\varsigma}{3/2}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$$
$$=\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{i\partial_t}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$$
□

7.2 Gravitino properties of covariant constant invariant tensor

Cor. 7.2.1.

$$\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}(\tfrac{3}{2})=(\tfrac{1}{\sqrt{2}})^3\delta_{k_\varsigma k'_\varsigma}$$
$$\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}(\tfrac{3}{2})=-i\varsigma(\tfrac{1}{\sqrt{2}})^3\tfrac{2}{3}\sigma^i(\tfrac{3}{2})_{k_\varsigma k'_\varsigma}$$
$$\Gamma^{ij\pi}_{k_\varsigma k'_\varsigma}(\tfrac{3}{2})=-(\tfrac{1}{\sqrt{2}})^3\tfrac{1}{3}[\sigma^{\{i}(\tfrac{3}{2})\sigma^{j\}}(\tfrac{3}{2})-\tfrac{3}{2}\delta^{ij}]_{k_\varsigma k'_\varsigma}=-(\tfrac{1}{\sqrt{2}})^3\tfrac{2}{3}\tfrac{1}{2!}[\sigma^{\{i}(\tfrac{3}{2})\sigma^{j\}}(\tfrac{3}{2})-\tfrac{3}{4}\delta^{\{ij\}}]_{k_\varsigma k'_\varsigma}$$
$$\Gamma^{ijk}_{k_\varsigma k'_\varsigma}(\tfrac{3}{2})=(\tfrac{1}{\sqrt{2}})^3\tfrac{2i\varsigma}{3}\{\sigma^{\{j}(\tfrac{3}{2})[\sigma^i(\tfrac{3}{2})]\sigma^{k\}}(\tfrac{3}{2})-[\tfrac{1}{2}\sigma^i(\tfrac{3}{2})\delta^{jk}+\tfrac{3}{2}\delta^{i\{j}\sigma^{k\}}(\tfrac{3}{2})]\}_{k_\varsigma k'_\varsigma}$$
$$=(\tfrac{1}{\sqrt{2}})^3\tfrac{4i\varsigma}{3}\tfrac{1}{3!}[\sigma^{\{i}(\tfrac{3}{2})\sigma^j(\tfrac{3}{2})\sigma^{k\}}(\tfrac{3}{2})-\tfrac{7}{4}\delta^{\{ij}\sigma^{k\}}(\tfrac{3}{2})]_{k_\varsigma k'_\varsigma}$$

Cor. 7.2.2. $\Gamma^{abc}(\frac{3}{2})\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}=\frac{i}{4\sqrt{2}}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}\delta^3(\vec{r}-\vec{r}')$

Proof: $\Gamma^{abc}(\frac{3}{2})\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}$

$$=i\sum_{l=0}^{1}(-1)^lC_3^{2l+1}\Gamma^{\overbrace{ij\cdots}^{2-2l}\overbrace{\pi\cdots\pi}^{2l+1}}(\tfrac{3}{2})\overbrace{\partial_i\partial_j\cdots}^{2-2l}\nabla^{2l}\delta^3(\vec{r}-\vec{r}')$$
$$=i[C_3^1\Gamma^{ij\pi}(\tfrac{3}{2})\partial_i\partial_j-C_3^3\Gamma^{\pi\pi\pi}(\tfrac{3}{2})\nabla^2]\delta^3(\vec{r}-\vec{r}')$$
$$=i\{-(\tfrac{1}{\sqrt{2}})^3[\sigma^{\{i}(\tfrac{3}{2})\sigma^{j\}}(\tfrac{3}{2})-\tfrac{3}{2}\delta^{ij}]\partial_i\partial_j-(\tfrac{1}{\sqrt{2}})^3\nabla^2\}\delta^3(\vec{r}-\vec{r}')$$
$$=-i(\tfrac{1}{\sqrt{2}})^3\{[\sigma^{\{i}(\tfrac{3}{2})\sigma^{j\}}(\tfrac{3}{2})-\tfrac{3}{2}\delta^{ij}]\partial_i\partial_j+\nabla^2\}\delta^3(\vec{r}-\vec{r}')$$
$$=i\frac{1}{4\sqrt{2}}\{\nabla^2-4[\sigma(\tfrac{3}{2})\cdot\nabla]^2\}\delta^3(\vec{r}-\vec{r}')$$
$$=\frac{i}{4\sqrt{2}}\{\nabla^2-9[\tfrac{2}{3}\sigma(\tfrac{3}{2})\cdot\nabla]^2\}\delta^3(\vec{r}-\vec{r}')$$
□

Cor. 7.2.3. $\Gamma^{abc}(\frac{3}{2})\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}\partial_\pi\Delta(x-x')|_{t=t'}=\frac{\varsigma}{4\sqrt{2}}\{\nabla^2-9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\nabla]^2\}[\frac{2}{3}\sigma(\frac{3}{2})\cdot\nabla]\delta^3(\vec{r}-\vec{r}')$

Proof: $\Gamma^{abc}(\frac{3}{2})\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}\partial_\pi\Delta(x-x')|_{t=t'}$

$$=i\sum_{l=0}^{1}(-1)^lC_3^{2l}\Gamma^{\overbrace{ij\cdots}^{3-2l}\overbrace{\pi\cdots\pi}^{2l}}(\tfrac{3}{2})\overbrace{\partial_i\partial_j\cdots}^{3-2l}\nabla^{2l}\delta^3(\vec{r}-\vec{r}')$$
$$=i[C_3^0\Gamma^{ijk}(\tfrac{3}{2})\partial_i\partial_j\partial_k-C_3^2\Gamma^{i\pi\pi}(\tfrac{3}{2})\partial_i\nabla^2]\delta^3(\vec{r}-\vec{r}')$$
$$=i[(\tfrac{1}{\sqrt{2}})^3\tfrac{4i\varsigma}{3}\tfrac{1}{3!}[\sigma^{\{i}(\tfrac{3}{2})\sigma^j(\tfrac{3}{2})\sigma^{k\}}(\tfrac{3}{2})-\tfrac{7}{4}\delta^{\{ij}\sigma^{k\}}(\tfrac{3}{2})]\partial_i\partial_j\partial_k+2i\varsigma(\tfrac{1}{\sqrt{2}})^3\sigma^i(\tfrac{3}{2})\partial_i\nabla^2]\delta^3(\vec{r}-\vec{r}')$$
$$=i(\tfrac{1}{\sqrt{2}})^32i\varsigma\{\tfrac{2}{3}\{[\sigma(\tfrac{3}{2})\cdot\nabla]^3-\tfrac{7}{4}[\sigma(\tfrac{3}{2})\cdot\nabla]\nabla^2\}+[\sigma(\tfrac{3}{2})\cdot\nabla]\nabla^2\}\delta^3(\vec{r}-\vec{r}')$$
$$=-i(\tfrac{1}{\sqrt{2}})^3\tfrac{i\varsigma}{3}\{\nabla^2-4[\sigma(\tfrac{3}{2})\cdot\nabla]^2\}[\sigma(\tfrac{3}{2})\cdot\nabla]\delta^3(\vec{r}-\vec{r}')$$
$$=\frac{\varsigma}{4\sqrt{2}}\{\nabla^2-9[\tfrac{2}{3}\sigma(\tfrac{3}{2})\cdot\nabla]^2\}[\tfrac{2}{3}\sigma(\tfrac{3}{2})\cdot\nabla]\delta^3(\vec{r}-\vec{r}')$$
□

Lem. 7.2.1. $\Gamma^{abc}_{k_\varsigma k'_\varsigma}p_ap_bp_c=-2\sqrt{2}i|\vec{p}|^3\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)$

Proof: $\Gamma^{abc}_{k_\varsigma k'_\varsigma}p_ap_bp_c$

$$\succ=C_3^3\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}(1)p_\pi^3+C_3^2\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}(1)p_ip_\pi^2+C_3^1\Gamma^{ij\pi}_{k_\varsigma k'_\varsigma}(1)p_ip_jp_\pi+C_3^0\Gamma^{ijk}_{k_\varsigma k'_\varsigma}(1)p_ip_jp_k$$
$$=(\tfrac{1}{\sqrt{2}})^3[-i|\vec{p}|^3+2i\varsigma|\vec{p}|^2\sigma(\tfrac{3}{2})\cdot\vec{p}-2i|\vec{p}|[\sigma(\tfrac{3}{2})\cdot\vec{p}]^2+i\tfrac{3}{2}|\vec{p}|^3+\tfrac{4i\varsigma}{3}\{[\sigma(\tfrac{3}{2})\cdot\vec{p}]^3-\tfrac{7}{4}|\vec{p}|^2[\sigma(\tfrac{3}{2})\cdot\vec{p}]\}$$
$$=(\tfrac{1}{\sqrt{2}})^3\tfrac{i}{6}[3|\vec{p}|^3-2\varsigma|\vec{p}|^2[\sigma(\tfrac{3}{2})\cdot\vec{p}]-12|\vec{p}|[\sigma(\tfrac{3}{2})\cdot\vec{p}]^2+8\varsigma[\sigma(\tfrac{3}{2})\cdot\vec{p}]^3$$
$$=(\tfrac{1}{\sqrt{2}})^3\tfrac{i}{6}|\vec{p}|^3[3-2\varsigma[\sigma(\tfrac{3}{2})\cdot\hat{p}]-12[\sigma(\tfrac{3}{2})\cdot\hat{p}]^2+8\varsigma[\sigma(\tfrac{3}{2})\cdot\hat{p}]^3$$
$$=\{(\tfrac{1}{\sqrt{2}})^3\tfrac{i}{6}|\vec{p}|^3[3-2\varsigma[\sigma(\tfrac{3}{2})\cdot\hat{p}]-12[\sigma(\tfrac{3}{2})\cdot\hat{p}]^2+8\varsigma[\sigma(\tfrac{3}{2})\cdot\hat{p}]^3\}\sum_{h=3/2}^{-3/2}\lambda(\hat{p},h)\lambda^+(\hat{p},h)$$
$$=\prec-2\sqrt{2}i|\vec{p}|^3\lambda_{k_\varsigma}(\hat{p},-\tfrac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\tfrac{3}{2}\varsigma)$$
□

Cor. 7.2.4. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(\frac{3}{2},\varsigma)=\frac{i}{2\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\hat{p}_a\hat{p}_b\hat{p}_c\rightarrow-\frac{1}{2\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c$

7.3 General covariant commutation rules for gravitino field in mathematics

Thm. 7.3.1.

$$\begin{cases}[a_\sigma(\vec{p},-\frac{3}{2}\varsigma),a^+_{\sigma'}(\vec{p}',-\frac{3}{2}\varsigma)]_\pm\\=\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-\frac{3}{2}\varsigma),a_{\sigma'}(\vec{p}',-\frac{3}{2}\varsigma)]_\pm=0\\ [a^+_\sigma(\vec{p},-\frac{3}{2}\varsigma),a^+_{\sigma'}(\vec{p}',-\frac{3}{2}\varsigma)]_\pm=0\end{cases}\Rightarrow\begin{cases}[\Psi_{k_\varsigma}(x),\Psi^+_{k'_\varsigma}(x')]_\pm\\=-\frac{i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c[(\delta_1-\pm\delta_2)\Delta^{(+)}(x-x')\pm\delta_2\Delta(x-x')]\\ [\Psi_{k_\varsigma}(x),\Psi_{l_\varsigma}(x')]_\pm=0\\ [\Psi^+_{k'_\varsigma}(x),\Psi^+_{l'_\varsigma}(x')]_\pm=0\end{cases}$$

Proof: $[\Psi^{(+)}_{k_\varsigma}(x),\Psi^{(+)+}_{k'_\varsigma}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-\frac{3}{2}\varsigma)|\vec{p}||\vec{p}'|[a_1(\vec{p},-\frac{3}{2}\varsigma),a^+_1(\vec{p}',-\frac{3}{2}\varsigma)]_\pm e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-\frac{3}{2}\varsigma)|\vec{p}|^2\delta_1\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\delta_1|\vec{p}|^2e^{ip\cdot(x-x')}d^3\vec{p}$
$=\frac{\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}p_ap_bp_ce^{ip\cdot(x-x')}d^3\vec{p}$
$=\frac{\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_ce^{ip\cdot(x-x')}d^3\vec{p}$
$=-\frac{i}{\sqrt{2}}\delta_1\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{(+)}(x-x')$ □

Proof: $[\Psi^{(-)}_{k_\varsigma}(x),\Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-\frac{3}{2}\varsigma)|\vec{p}||\vec{p}'|[a_2(\vec{p},-\frac{3}{2}\varsigma),a^+_2(\vec{p}',-\frac{3}{2}\varsigma)]_\pm e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\pm\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-\frac{3}{2}\varsigma)|\vec{p}|^2\delta_2\delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\pm\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\delta_2|\vec{p}|^2e^{-ip\cdot(x-x')}d^3\vec{p}$
$=\pm\frac{\delta_2}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}p_ap_bp_ce^{-ip\cdot(x-x')}d^3\vec{p}$
$=\pm\frac{\delta_2}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_ce^{-ip\cdot(x-x')}d^3\vec{p}$
$=-\pm\frac{i}{\sqrt{2}}\delta_2\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{(-)}(x-x')$ □

Proof: $[\Psi_{k_\varsigma}(x),\Psi^+_{k'_\varsigma}(x')]_\pm$
$=[\Psi^{(+)}_{k_\varsigma}(x),\Psi^{(+)+}_{k'_\varsigma}(x')]_\pm+[\Psi^{(-)}_{k_\varsigma}(x),\Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$
$=-\frac{i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c[\delta_1\Delta^{(+)}(x-x')\pm\delta_2\Delta^{(-)}(x-x')]$
$=-\frac{i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c[(\delta_1-\pm\delta_2)\Delta^{(+)}(x-x')\pm\delta_2\Delta(x-x')]$ □

From the above, only $\delta_1\mp\delta_2=0$, the micro causality is satisfied. At the same time only when $\delta_1,\delta_2\geq0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1=\delta_2=1$, it satisfies the commutative relation. There are actually two other options. Namely when $\delta_1=\delta_2=0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

7.4 Physical covariant anticommutative rules for gravitino field

Thm. 7.4.1. $\begin{cases}\{a_\sigma(\vec{p},-\frac{3}{2}\varsigma),a^+_{\sigma'}(\vec{p}',-\frac{3}{2}\varsigma)\}=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\sigma(\vec{p},-\frac{3}{2}\varsigma),a_{\sigma'}(\vec{p}',-\frac{3}{2}\varsigma)\}=0\\ \{a^+_\sigma(\vec{p},-\frac{3}{2}\varsigma),a^+_{\sigma'}(\vec{p}',-\frac{3}{2}\varsigma)]=0\end{cases}\Rightarrow\begin{cases}\{\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')\}=\frac{-i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta(x-x')\\ \{\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')\}=0\\ \{\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')\}=0\end{cases}$

Proof: $\{\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')\}$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)|\vec{p}||\vec{p}'|$
$\{\{a_1(\vec{p},-\frac{3}{2}\varsigma),a^+_1(\vec{p}',-\frac{3}{2}\varsigma)\}e^{ip\cdot(x-x')}+\{a^+_2(\vec{p},-\frac{3}{2}\varsigma),a_2(\vec{p}',-\frac{3}{2}\varsigma)\}e^{-ip\cdot(x-x')}\}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|^2\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)[\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')}+\delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}]d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int|\vec{p}|^2\lambda_{k_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-\frac{3}{2}\varsigma)[e^{ip\cdot(x-x')}+e^{-ip\cdot(x-x')}]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}p_ap_bp_c[e^{ip\cdot(x-x')}+e^{-ip\cdot(x-x')}]d^3\vec{p}$
$=i\frac{1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]d^3\vec{p}$
$=\frac{-i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]d^3\vec{p}$
$=\frac{-i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta(x-x')$ □

7.5 Isochronous anticommutation rules for gravitino field

Cor. 7.5.1.

$$\begin{cases}\{\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')\}=\frac{-i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta[(x-x')]\\ \{\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')\}=0\\ \{\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')\}=0\end{cases}\Rightarrow\begin{cases}\{\psi_{k_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)\}\\ =\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\\ \{\psi_{k_\varsigma}(\vec r,t),\psi_{l_\varsigma}(\vec r',t)\}=0,\{\psi^+_{k'_\varsigma}(\vec r,t),\psi^+_{l'_\varsigma}(\vec r',t)\}=0\end{cases}$$

Pro. 7.5.1. $$\begin{cases}\Delta^*(x)=\Delta(x),\Delta(-x)=-\Delta(x),(\nabla^2-\partial_t^2)\Delta(x)=0\\ \partial_t\Delta(x)|_{t=0}=-\delta^3(\vec r),\partial_k\partial_t\Delta(x)|_{t=0}=\partial_t\partial_k\Delta(x)|_{t=0}=-\partial_k\delta^3(\vec r)\\ \partial_k\Delta(x)|_{t=0}=0,\partial_k\partial_l\Delta(x)|_{t=0}=0,\partial_t^2\Delta(x)|_{t=0}=0\end{cases}$$

Proof: $\{\psi_{k_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)\}=\frac{-i}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta[(x-x')]|_{t=t'}$
$=C_3^1\frac{-i}{\sqrt{2}}\Gamma^{ij\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_j\partial_\pi\Delta[(x-x')]|_{t=t'}+\frac{-i}{\sqrt{2}}\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_\pi\partial_\pi\partial_\pi\Delta[(x-x')]|_{t=t'}$
$=\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')$ □

Cor. 7.5.2.

$$\begin{cases}\{\psi_{k_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)\}\\ =\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\\ \{\psi_{k_\varsigma}(\vec r,t),\psi_{l_\varsigma}(\vec r',t)\}=0,\{\psi^+_{k'_\varsigma}(\vec r,t),\psi^+_{l'_\varsigma}(\vec r',t)\}=0\end{cases}\Rightarrow\begin{cases}\{a_\sigma(\vec p,-\frac{3}{2}\varsigma),a^+_{\sigma'}(\vec p',-\frac{3}{2}\varsigma)\}=\delta_{\sigma\sigma'}\delta^3(\vec p-\vec p')\\ \{a_\sigma(\vec p,-\frac{3}{2}\varsigma),a_{\sigma'}(\vec p',-\frac{3}{2}\varsigma)\}=0\\ \{a^+_\sigma(\vec p,-\frac{3}{2}\varsigma),a^+_{\sigma'}(\vec p',-\frac{3}{2}\varsigma)]=0\end{cases}$$

Proof: $\{a_1(\vec p,-\frac{3}{2}\varsigma),a_1^+(\vec p',-\frac{3}{2}\varsigma)\}$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\{\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\Psi_{k_\varsigma}(\vec r,t)e^{-i(\vec p\cdot\vec r-Et)},\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\Psi^+_{k'_\varsigma}(\vec r',t)e^{i(\vec p'\cdot\vec r'-E't)}\}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)[\Psi_{k_\varsigma}(\vec r,t),\Psi^+_{k'_\varsigma}(\vec r',t)]e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\frac{-1}{8}\{\vec p^2-4[\sigma(\frac{3}{2})\cdot\vec p]^2\}_{k_\varsigma k'_\varsigma}e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r-E't)}d^3\vec r$
$=\frac{1}{|\vec p||\vec p'|}\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\frac{-1}{8}\{\vec p^2-4[\sigma(\frac{3}{2})\cdot\vec p]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec p-\vec p')$
$=\lambda^+(\hat p,-\frac{3}{2}\varsigma)\frac{-1}{8}\{\vec p^2-4[\sigma(\frac{3}{2})\cdot\vec p]^2\}\lambda(\hat p,-\frac{3}{2}\varsigma)\delta^3(\vec p-\vec p')$
$=\lambda^+(\hat p,-\frac{3}{2}\varsigma)\lambda(\hat p,-\frac{3}{2}\varsigma)\delta^3(\vec p-\vec p')$
$=\delta^3(\vec p-\vec p')$ □

Proof: $\{a_2^+(\vec p,-\frac{3}{2}\varsigma),a_2(\vec p',-\frac{3}{2}\varsigma)\}$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\{\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\Psi_{k_\varsigma}(\vec r,t)e^{i(\vec p\cdot\vec r-Et)},\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\Psi^+_{k'_\varsigma}(\vec r',t)e^{-i(\vec p'\cdot\vec r'-E't)}\}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)[\Psi_{k_\varsigma}(\vec r,t),\Psi^+_{k'_\varsigma}(\vec r',t)]e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r'-E't)}d^3\vec r d^3\vec r'$
$=\frac{1}{(2\pi)^3}\frac{1}{|\vec p||\vec p'|}\int\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\frac{-1}{8}\{\vec p^2-4[\sigma(\frac{3}{2})\cdot\vec p]^2\}_{k_\varsigma k'_\varsigma}e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r-E't)}d^3\vec r$
$=\frac{1}{|\vec p||\vec p'|}\lambda^{+k_\varsigma}(\hat p,-\frac{3}{2}\varsigma)\lambda^{k'_\varsigma}(\vec p',-\frac{3}{2}\varsigma)\frac{-1}{8}\{\vec p^2-4[\sigma(\frac{3}{2})\cdot\vec p]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec p-\vec p')$
$=\lambda^+(\hat p,-\frac{3}{2}\varsigma)\frac{-1}{8}\{\vec p^2-4[\sigma(\frac{3}{2})\cdot\vec p]^2\}\lambda(\hat p,-\frac{3}{2}\varsigma)\delta^3(\vec p-\vec p')$
$=\lambda^+(\hat p,-\frac{3}{2}\varsigma)\lambda(\hat p,-\frac{3}{2}\varsigma)\delta^3(\vec p-\vec p')$
$=\delta^3(\vec p-\vec p')$ □

7.6 Commutative function, causal function and feynman propagator of gravitino field

Cor. 7.6.1.

$$\begin{cases}\Delta_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta(x)\\ \Delta^{(+)}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{(+)}(x)\\ \Delta^{(-)}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{(-)}(x)\\ \Delta^{(l)}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{(l)}(x)\end{cases}$$

Cor. 7.6.2.

$$\begin{cases}\Delta^{(c)}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{(c)}(x)+\frac{i}{\sqrt{2}}[\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i-3\Gamma^{ij\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j]\Delta(x)\\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{ijk}_{k_\varsigma k'_\varsigma}\partial_i\partial_j\partial_k\Delta^{ret}(x)+\frac{i}{\sqrt{2}}[\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i-3\Gamma^{ij\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j]\Delta(x)\\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{adv}(x)+\frac{i}{\sqrt{2}}[\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i-3\Gamma^{ij\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j]\Delta(x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta_F(x)+\frac{-1}{\sqrt{2}}[\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i-3\Gamma^{ij\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j]\Delta(x)\\ =i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x)\end{cases}$$

Cor. 7.6.3.

$$\begin{cases}\Delta^{(c)}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{(c)}(x)+\frac{i}{\sqrt{2}}[2\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_t\delta^4(x)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\delta^4(x)]\\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{ijk}_{k_\varsigma k'_\varsigma}\partial_i\partial_j\partial_k\Delta^{ret}(x)+\frac{i}{\sqrt{2}}[2\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_t\delta^4(x)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\delta^4(x)]\\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta^{adv}(x)+\frac{i}{\sqrt{2}}[2\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_t\delta^4(x)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\delta^4(x)]\\ \Delta_{Fk_\varsigma k'_\varsigma}(\frac{3}{2};x):=\frac{-1}{\sqrt{2}}\Gamma^{abc}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\Delta_F(x)+\frac{-1}{\sqrt{2}}[2\Gamma^{\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_t\delta^4(x)+3i\Gamma^{i\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\delta^4(x)]\\ =i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(\frac{3}{2};x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(\frac{3}{2};p)=\frac{1}{\sqrt{2}}\frac{\Gamma^{abc}_{k_\varsigma k'_\varsigma}p_ap_bp_c}{p^2-i\varepsilon}+\cdots\end{cases}$$

Cor. 7.6.4.

$$\begin{cases}[s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta(\frac{3}{2};x)=0\\ [s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta^{(+)}(\frac{3}{2};x)=0\\ [s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta^{(-)}(\frac{3}{2};x)=0\\ [s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta^{(l)}(\frac{3}{2};x)=0\end{cases}\quad\begin{cases}[s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta^{(c)}(\frac{3}{2};x)=-\varsigma[\sigma(\frac{3}{2}),i\frac{3}{2}\varsigma]_a\delta(t)\Delta(\frac{3}{2};x)|_{t=0}\\ [s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta^{ret}(\frac{3}{2};x)=-\varsigma[\sigma(\frac{3}{2}),i\frac{3}{2}\varsigma]_a\delta(t)\Delta(\frac{3}{2};x)|_{t=0}\\ [s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta^{adv}(\frac{3}{2};x)=-\varsigma[\sigma(\frac{3}{2}),i\frac{3}{2}\varsigma]_a\delta(t)\Delta(\frac{3}{2};x)|_{t=0}\\ [s\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\Delta_F(\frac{3}{2};x)=-i\varsigma[\sigma(\frac{3}{2}),i\frac{3}{2}\varsigma]_a\delta(t)\Delta(\frac{3}{2};x)|_{t=0}\end{cases}$$

7.7 Gravitino quantum equation

Cor. 7.7.1.

$$\begin{cases}\Delta^{(c)}_{k_\varsigma k'_\varsigma}(s;x):=\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta^{(c)}(x)+\frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1}i^nC^n_{2s}\Gamma^{\overbrace{ab\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_a\partial_b\cdots}^{n}\Delta(x)\\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(s;x):=\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta^{ret}(x)+\frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1}i^nC^n_{2s}\Gamma^{\overbrace{ab\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_a\partial_b\cdots}^{n}\Delta(x)\\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(s;x):=\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta^{adv}(x)+\frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1}i^nC^n_{2s}\Gamma^{\overbrace{ab\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_a\partial_b\cdots}^{n}\Delta(x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(s;x):=\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta_F(x)+\frac{i^{2s-1}}{2^{s-1}}\sum_{n=0}^{2s-1}i^nC^n_{2s}\Gamma^{\overbrace{ab\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_a\partial_b\cdots}^{n}\Delta(x)\\ =i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(s;x)\end{cases}$$

Cor. 7.7.2. $[\frac{3}{2}\partial_a+iS_{ab}(\frac{3}{2},\varsigma)\partial^b]\psi(x)=0\Rightarrow\begin{cases}\dot{\psi}(\vec{r},t)=-i[\psi(\vec{r},t),H]\\ \nabla\psi(\vec{r},t)=i[\psi(\vec{r},t),\vec{P}]\\ \partial_a\psi(\vec{r},t)=i[\psi(\vec{r},t),P_a]\end{cases}$

Thm. 7.7.1.

$$\begin{cases}\{\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)\}=\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ \{\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)\}=0,\{\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)\}=0\\ H=\frac{-i\varsigma}{3/2}\int\psi^+(\vec{r},t)\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}\psi(\vec{r},t)d^3\vec{r},\vec{P}=\int\psi^+(\vec{r},t)\frac{-i\nabla}{-\nabla^2}\psi(\vec{r},t)d^3\vec{r}\end{cases}$$

$$\Rightarrow$$

$$\begin{cases}[\psi(\vec{r},t),H]=\frac{-i\varsigma}{3/2}\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}\psi(\vec{r},t)\\ [\psi(\vec{r},t),\vec{P}]=\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}\frac{-i\nabla}{-\nabla^2}\psi(\vec{r},t)\end{cases}$$

Proof: $[\psi(\vec{r},t),H]$

$=\frac{-i\varsigma}{3/2}\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)\frac{\sigma(\frac{3}{2})\cdot\nabla'}{\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)]$

$=\frac{-i\varsigma}{3/2}\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]\frac{\sigma(\frac{3}{2})\cdot\nabla'}{\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)$

$=\frac{-i\varsigma}{3/2}\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{j_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\frac{\sigma(\frac{3}{2})\cdot\nabla'}{\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)$

$=\frac{-i\varsigma}{3/2}\frac{1}{8}\delta^{k'_\varsigma k_\varsigma}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}\psi_{k_\varsigma}(\vec{r},t)$

$=\frac{-i\varsigma}{3/2}\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}\psi(\vec{r},t)$ □

Proof: $[\psi(\vec{r},t),P]$

$=\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)\frac{-i\nabla}{-\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)]$

$=\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]\frac{-i\nabla}{-\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)$

$=\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{j_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\frac{-i\nabla}{-\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)$

$=\frac{1}{8}\delta^{k'_\varsigma k_\varsigma}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}_{k_\varsigma k'_\varsigma}\frac{-i\nabla}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)$

$=\frac{1}{8}\{\nabla^2-4[\sigma(\frac{3}{2})\cdot\nabla]^2\}\frac{-i\nabla}{-\nabla^2}\psi(\vec{r},t)$ □

Cor. 7.7.3.
$$\begin{cases}\dot\psi(\vec r,t)=-i[\psi(\vec r,t),H]\\ \nabla\psi(\vec r,t)=i[\psi(\vec r,t),\vec P]\end{cases}\Leftrightarrow\begin{cases}\dot\psi(\vec r,t)=\frac{-\varsigma}{12}\{\sigma(\frac32)\cdot\nabla-\frac{4}{\nabla^2}[\sigma(\frac32)\cdot\nabla]^3\}\psi(\vec r,t)\\ \nabla\psi(\vec r,t)=-\frac18\{1-\frac{4}{\nabla^2}[\sigma(\frac32)\cdot\nabla]^2\}\nabla\psi(\vec r,t)\end{cases}\Leftrightarrow\begin{cases}\partial^a\partial_a\psi(\vec r,t)=0\\ [\sigma(\frac32),-\frac32 i\varsigma]^a\partial_a\psi(\vec r,t)=0\end{cases}$$

Cor. 7.7.4.
$$\begin{cases}(\frac32)^2\nabla\psi=\frac32\sigma(\frac32)\cdot\nabla\sigma(\frac32)\psi-\frac12\sigma(\frac32)[\sigma(\frac32)\cdot\nabla]\psi\\ [\sigma(\frac32),-\frac32 i\varsigma]^a\partial_a\psi(\vec r,t)=0\end{cases}\Rightarrow\begin{cases}\partial^a\partial_a\psi(\vec r,t)=0\\ [\sigma(2),-\frac32 i\varsigma]^a\partial_a\psi(\vec r,t)=0\end{cases}$$

Cor. 7.7.5. $$\frac62\begin{bmatrix}3\partial_z&\sqrt3\partial_-&0&0\\ \sqrt3\partial_+&\partial_z&2\partial_-&0\\ 0&2\partial_+&-\partial_z&\sqrt3\partial_-\\ 0&0&\sqrt3\partial_+&-3\partial_z\end{bmatrix}\frac12\begin{bmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{bmatrix}=\frac64\begin{bmatrix}9\partial_z&\sqrt3\partial_-&0&0\\ 3\sqrt3\partial_+&\partial_z&-2\partial_-&0\\ 0&2\partial_+&\partial_z&-3\sqrt3\partial_-\\ 0&0&-\sqrt3\partial_+&9\partial_z\end{bmatrix}$$

Cor. 7.7.6. $$-\frac22\begin{bmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{bmatrix}\frac12\begin{bmatrix}3\partial_z&\sqrt3\partial_-&0&0\\ \sqrt3\partial_+&\partial_z&2\partial_-&0\\ 0&2\partial_+&-\partial_z&\sqrt3\partial_-\\ 0&0&\sqrt3\partial_+&-3\partial_z\end{bmatrix}=-\frac24\begin{bmatrix}9\partial_z&3\sqrt3\partial_-&0&0\\ \sqrt3\partial_+&\partial_z&2\partial_-&0\\ 0&-2\partial_+&\partial_z&-\sqrt3\partial_-\\ 0&0&-3\sqrt3\partial_+&9\partial_z\end{bmatrix}$$

$$\sigma(\tfrac32)=(\frac12\begin{bmatrix}0&\sqrt3&0&0\\ \sqrt3&0&2&0\\ 0&2&0&\sqrt3\\ 0&0&\sqrt3&0\end{bmatrix},\frac i2\begin{bmatrix}0&-\sqrt3&0&0\\ \sqrt3&0&-2&0\\ 0&2&0&-\sqrt3\\ 0&0&\sqrt3&0\end{bmatrix},\frac12\begin{bmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{bmatrix})\tag{22.3}$$

7.8 Gravitino poincare symmetry

Lem. 7.8.1. $\nabla^2(r_i\partial_j-r_j\partial_i)=(r_i\partial_j-r_j\partial_i)\nabla^2$

Lem. 7.8.2. $[\sigma(s)\cdot\nabla](r_i\partial_j-r_j\partial_i)=(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]+[\sigma_i(s)\partial_j-\sigma_j(s)\partial_i]$

Lem. 7.8.3. $[\sigma(s)\cdot\nabla]^2(r_i\partial_j-r_j\partial_i)=(r_i\partial_j-r_j\partial_i)$

Proof: $[\sigma(s)\cdot\nabla]^2(r_i\partial_j-r_j\partial_i)$
$=[\sigma(s)\cdot\nabla]\{(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]+[\sigma_i(s)\partial_j-\sigma_j(s)\partial_i]\}$
$=(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]^2+[\sigma_i(s)\partial_j-\sigma_j(s)\partial_i][\sigma(s)\cdot\nabla]+[\sigma(s)\cdot\nabla][\sigma_i(s)\partial_j-\sigma_j(s)\partial_i]$ □

Cor. 7.8.1. $\frac{-i}{\sqrt2}\Gamma^{abc}(\frac32)\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}=\frac18\{\{\nabla^2-9[\frac23\sigma(\frac32)\cdot\nabla]^2\}\}\delta^3(\vec r-\vec r')$

Cor. 7.8.2. $\frac{-i}{\sqrt2}\Gamma^{abc}(\frac32)\partial_a\partial_b\partial_c\partial_\pi\Delta(x-x')|_{t=t'}=\frac18\{\nabla^2-9[\frac23\sigma(\frac32)\cdot\nabla]^2\}[-i\varsigma\frac23\sigma(\frac32)\cdot\nabla]\delta^3(\vec r-\vec r')$

7.9 Poincare symmetry of gravitino field

Lem. 7.9.1.
$$\begin{cases}P_a=-i\int\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}\partial_a\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}}d^3\vec r=\int\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}\hat P_a\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}}d^3\vec r\\ L_{ab}=-i\int\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}(r_a\partial_b-r_b\partial_a)\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}}d^3\vec r=\int\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}\hat L_{ab}\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}}d^3\vec r\\ M_{ab}=\int\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}[-i(r_a\partial_b-r_b\partial_a)+\hat S_{ab}]\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}}d^3\vec r=\int\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}\hat M_{ab}\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}}d^3\vec r\end{cases}$$

Cor. 7.9.1.
$$\begin{cases}\{\frac{\psi_{k_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}\}=\frac18\{9[\frac23\sigma(\frac32)\cdot\hat\nabla]^2-1\}_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r'),\hat\nabla:=\frac{-i\nabla}{\sqrt{-\nabla^2}}\\ \{\frac{\psi_{k_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}},\frac{\psi_{l_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}\}=0,\{\frac{\psi^+_{k'_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}},\frac{\psi^+_{l'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}\}=0\end{cases}$$

Thm. 7.9.1.
$$\begin{cases}[L_{ab},L_{cd}]=-i(g_{ad}L_{bc}-g_{ac}L_{bd}+g_{bc}L_{ad}-g_{bd}L_{ac})\\ [L_{ab},P_c]=-i(g_{bc}P_a-g_{ac}P_b),[P_a,P_b]=0\end{cases}$$

Proof: $[L_{ab},L_{cd}]$
$=-\int d^3\vec r d^3\vec r'[\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}(r_a\partial_b-r_b\partial_a)\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}},\frac{\psi^+(\vec r',t)}{\sqrt{-\nabla'^2}}(r'_c\partial'_d-r'_d\partial'_c)\frac{\psi(\vec r',t)}{\sqrt{-\nabla'^2}}]$
$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}}(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}(r'_c\partial'_d-r'_d\partial'_c)\frac{\psi_{l'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}]$
$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$
$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}}\{(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}\}(r'_c\partial'_d-r'_d\partial'_c)\frac{\psi_{l'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}$
$-\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}},(r'_c\partial'_d-r'_d\partial'_c)\frac{\psi_{l'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}\}(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}}\}$
$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$
$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}}(r_a\partial_b-r_b\partial_a)\frac18\{9[\frac23\sigma(\frac32)\cdot\hat\nabla]^2-1\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')(r'_c\partial'_d-r'_d\partial'_c)\frac{\psi_{l'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}$
$-\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{\sqrt{-\nabla'^2}}(r'_c\partial'_d-r'_d\partial'_c)\frac18\{9[\frac23\sigma(\frac32)\cdot\hat\nabla']^2-1\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r'-\vec r)(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{\sqrt{-\nabla^2}}\}$
$=\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial'_b - r_b\partial'_a)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}']^2-1\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$
$-\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}(r'_c\partial_d - r'_d\partial_c)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$
$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l_\varsigma k'_\varsigma}(r_c\partial_d - r_d\partial_c)\frac{\psi_{l'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}$
$-\frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_c\partial_d - r_d\partial_c)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l'_\varsigma k_\varsigma}(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[-i(r_a\partial_b - r_b\partial_a), -i(r_c\partial_d - r_d\partial_c)]\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[\hat{L}_{ab},\hat{L}_{cd}]\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})$ □

Proof: $[L_{ab}, P_c]$
$= -\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_c\frac{\psi(\vec{r}',t)}{\sqrt{-\nabla'^2}}]$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_c\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\{(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\}\partial'_c\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \partial'_c\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\}(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$
$-\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_c\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}']^2-1\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial'_b - r_b\partial'_a)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}']^2-1\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$
$-\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial_c\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$
$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}(r_a\partial_b - r_b\partial_a)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l_\varsigma k'_\varsigma}\partial_c\frac{\psi_{l'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\partial_c\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l'_\varsigma k_\varsigma}(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[-i(r_a\partial_b - r_b\partial_a), -i\partial'_c]\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[\hat{L}_{ab},\hat{P}_c]\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= -i(g_{bc}P_a - g_{ac}P_b)$ □

Proof: $[P_a, P_b]$
$= -\int[\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}\partial_a\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_b\frac{\psi(\vec{r}',t)}{\sqrt{-\nabla'^2}}]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\partial'_b\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]d^3\vec{r}d^3\vec{r}'$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\{\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\}\partial'_b\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \partial'_b\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\}\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l_\varsigma k'_\varsigma}\partial_a\delta^3(\vec{r}-\vec{r}')\partial'_b\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}']^2-1\}_{l'_\varsigma k_\varsigma}\partial'_b\delta^3(\vec{r}'-\vec{r})\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$
$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}']^2-1\}_{l_\varsigma k'_\varsigma}\partial'_a\delta^3(\vec{r}-\vec{r}')\partial'_b\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}_{l'_\varsigma k_\varsigma}\partial_b\delta^3(\vec{r}-\vec{r}')\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}$
$= -\int\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}^{k_\varsigma l'_\varsigma}\partial_a\partial_b\frac{\psi_{l'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}^{k'_\varsigma l_\varsigma}\partial_b\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$
$= -\int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}(\partial_a\partial_b - \partial_b\partial_a)\frac{1}{8}\{9[\frac{2}{3}\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2-1\}\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= -\int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}(\partial_a\partial_b - \partial_b\partial_a)\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$
$= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}[\hat{P}_a,\hat{P}_b]\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r} = 0$ □

Lem. 7.9.2. $[\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2\sigma(\frac{3}{2})\psi = \{2[\sigma(\frac{3}{2})\cdot\hat{\nabla}]\hat{\nabla} + \frac{1}{4}\sigma(\frac{3}{2})\}\psi, S_{ab}(\frac{3}{2}) = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma}(\frac{3}{2})$

Lem. 7.9.3. $[\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2 S_{ab}(\frac{3}{2})\psi = \{2[\sigma(\frac{3}{2})\cdot\hat{\nabla}]i\sigma^{\alpha_\varsigma}_{\varsigma ab}\hat{\nabla}_{\alpha_\varsigma} + \frac{1}{4}S_{ab}(\frac{3}{2})\}\psi$

Lem. 7.9.4. $\int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}\sigma_{\alpha_\varsigma}(\frac{3}{2})[\sigma(\frac{3}{2})\cdot\hat{\nabla}]\hat{\nabla}_{\beta_\varsigma}\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}? = \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}\sigma_{\alpha_\varsigma}(\frac{3}{2})\sigma_{\beta_\varsigma}(\frac{3}{2})\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$

Proof: $[S_{ab}(t), S_{cd}(t)]$
$= \int[\frac{\psi^{+k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{abk_\varsigma}{}^{l_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}S_{cdm_\varsigma}{}^{n_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}]d^3\vec{r}d^3\vec{r}'$
$= \int\{\frac{\psi^{+k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\{S_{abk_\varsigma}{}^{l_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\}S_{cdm_\varsigma}{}^{n_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$
$- \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\{\frac{\psi^{+k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}, S_{cdm_\varsigma}{}^{n_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}\}S_{abk_\varsigma}{}^{l_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}d^3\vec{r}'$
$= \int\{\frac{\psi^{+k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{abk_\varsigma}{}^{l_\varsigma}(\frac{3}{2},\varsigma)S_{cdm_\varsigma}{}^{n_\varsigma}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4\frac{[\sigma(\frac{3}{2})\cdot\nabla']^2}{\nabla'^2}\}_{l_\varsigma}{}^{m_\varsigma}\delta^3(\vec{r}-\vec{r}')\frac{\psi_{n_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}$
$- \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{\sqrt{-\nabla'^2}}S_{cdm_\varsigma}{}^{n_\varsigma}(\frac{3}{2},\varsigma)S_{abk_\varsigma}{}^{l_\varsigma}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4\frac{[\sigma(\frac{3}{2})\cdot\nabla]^2}{\nabla^2}\}_{n_\varsigma}{}^{k_\varsigma}\delta^3(\vec{r}-\vec{r}')\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}d^3\vec{r}'$
$= \int\{\frac{\psi^{+k_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{abk_\varsigma}{}^{l_\varsigma}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4\frac{[\sigma(\frac{3}{2})\cdot\nabla]^2}{\nabla^2}\}_{l_\varsigma}{}^{m_\varsigma}S_{cdm_\varsigma}{}^{n_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}$
$- \frac{\psi^{+m_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{cdm_\varsigma}{}^{n_\varsigma}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4\frac{[\sigma(\frac{3}{2})\cdot\nabla]^2}{\nabla^2}\}_{n_\varsigma}{}^{k_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(\frac{3}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$
$= \int\{\frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{ab}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4\frac{[\sigma(\frac{3}{2})\cdot\nabla]^2}{\nabla^2}\}S_{cd}(\frac{3}{2},\varsigma)\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{cd}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4\frac{[\sigma(\frac{3}{2})\cdot\nabla]^2}{\nabla^2}\}S_{ab}(\frac{3}{2},\varsigma)\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$
$= \int\{\frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{ab}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4[\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2\}S_{cd}(\frac{3}{2},\varsigma)\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{cd}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4[\sigma(\frac{3}{2})\cdot\hat{\nabla}]^2\}S_{ab}(\frac{3}{2},\varsigma)\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$
$= \int\{\frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{ab}(\frac{3}{2},\varsigma)[\sigma(\frac{3}{2})\cdot\hat{\nabla}]i\sigma^{\alpha_\varsigma}_{\varsigma cd}\hat{\nabla}_{\alpha_\varsigma}\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{cd}(\frac{3}{2},\varsigma)[\sigma(\frac{3}{2})\cdot\hat{\nabla}]i\sigma^{\alpha_\varsigma}_{\varsigma ab}\hat{\nabla}_{\alpha_\varsigma}\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$
$? = \int\{\frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{ab}(\frac{3}{2},\varsigma)i\sigma^{\alpha_\varsigma}_{\varsigma cd}\sigma_{\alpha_\varsigma}(\frac{3}{2})\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}S_{cd}(\frac{3}{2},\varsigma)i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma}(\frac{3}{2})\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}\}d^3\vec{r}$
$= \int\frac{\psi^{+}(\vec{r},t)}{\sqrt{-\nabla^2}}[S_{ab}(\frac{3}{2},\varsigma), S_{cd}(\frac{3}{2},\varsigma)]\frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}}d^3\vec{r}$ □

Proof: $[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi = \{s[s^n-(s-1)^n](\varsigma\hat{\partial}_t)^{n-1}\hat{\nabla} + (s-1)^n(\varsigma\hat{\partial}_t)^n\sigma(s)\}\psi$
$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi = \{s[s^n-(s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1}\hat{\nabla} + (s-1)^n\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\}\psi$ □

8 Free uncoupled graviton field covariant quantization scheme

8.1 Graviton spin operator equation and its plane wave solution

Thm. 8.1.1. $[2\partial_a + iS_{ab}(2,\varsigma)\partial^b]\psi(x) = 0$

Cor. 8.1.1.
$$\begin{cases}\psi(\vec{r},t) := \frac{1}{(2\pi)^{3/2}}\int|\vec{p}|^{3/2}\lambda(\hat{p},-2\varsigma)[a_1(\vec{p},-2\varsigma)e^{ip\cdot x} + a_2^+(\vec{p},-2\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\ \vec{p}|^{3/2}a_1(\vec{p},-2\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-2\varsigma)\psi(\vec{r},t)e^{-ip\cdot x}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int\frac{i}{|\vec{p}|}\lambda^+(\hat{p},-2\varsigma)\dot{\psi}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ |\vec{p}|^{3/2}a_2^+(\vec{p},-2\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-2\varsigma)\psi(\vec{r},t)e^{ip\cdot x}d^3\vec{r} = \frac{1}{(2\pi)^{3/2}}\int\frac{-i}{|\vec{p}|}\lambda^+(\hat{p},-2\varsigma)\dot{\psi}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Def. 8.1.1. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(2,\varsigma) := \lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma), \hat{P}^2(2,\varsigma) = \hat{P}(2,\varsigma), \hat{P}^+(2,\varsigma) = \hat{P}(2,\varsigma)$

Cor. 8.1.2. $H_2 = \int|\vec{p}|[a_1^+(\vec{p},-2\varsigma)a_1(\vec{p},-2\varsigma) + a_2(\vec{p},-2\varsigma)a_2^+(\vec{p},-2\varsigma)]d^3\vec{p} = \int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{1}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$

Proof: $H_2 = \int|\vec{p}|[a_1^+(\vec{p},-2\varsigma)a_1(\vec{p},-2\varsigma) + a_2(\vec{p},-2\varsigma)a_2^+(\vec{p},-2\varsigma)]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|^2}[\lambda^{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{-ip\cdot x}$
$+ \lambda^{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{-ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|^2}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int\frac{1}{4|\vec{p}|^6}4|\vec{p}|^4\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_ap_bp_cp_d\frac{1}{|\vec{p}|^6}[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{4}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{(\frac{1}{\sqrt{2}})^4\frac{1}{3}|\vec{p}|^4\{0+4\varsigma[\sigma(2)\cdot\hat{p}] - 2[\sigma(2)\cdot\hat{p}]^2 - 4\varsigma[\sigma(2)\cdot\hat{p}]^3 + 2[\sigma(2)\cdot\hat{p}]^4\}\frac{1}{|\vec{p}|^6}[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{24}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{\{-[\sigma(2)\cdot\hat{p}]^2 + [\sigma(2)\cdot\hat{p}]^4\}\frac{1}{|\vec{p}|^2}[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{12}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{-[\sigma(2)\cdot\hat{p}]^2 + [\sigma(2)\cdot\hat{p}]^4\}\frac{1}{|\vec{p}|^2}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{12}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{\frac{1}{|\vec{p}|^4}[\sigma(2)\cdot i\vec{p}]^2 + \frac{1}{|\vec{p}|^6}[\sigma(2)\cdot i\vec{p}]^4\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{12}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^4} - \frac{[\sigma(2)\cdot\nabla]^4}{\nabla^6}\}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{12}\int\psi^+_{k'_\varsigma}(\vec{r},t)\{\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^4} - \frac{[\sigma(2)\cdot\nabla]^4}{\nabla^6}\}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$= \int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{1}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$ □

Cor. 8.1.3. $P_2 = \int\vec{p}[a_1^+(\vec{p},-2\varsigma)a_1(\vec{p},-2\varsigma) + a_2(\vec{p},-2\varsigma)a_2^+(\vec{p},-2\varsigma)]d^3\vec{p} = \frac{-\varsigma}{2}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{\sigma(2)}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$

Proof: $P_2 = \int\vec{p}[a_1^+(\vec{p},-2\varsigma)a_1(\vec{p},-2\varsigma) + a_2(\vec{p},-2\varsigma)a_2^+(\vec{p},-2\varsigma)]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{\hat{p}}{|\vec{p}|^2}[\lambda^{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{-ip\cdot x}$
$+ \lambda^{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{-ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int \frac{\hat{p}}{|\vec{p}|^2}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\int \frac{\hat{p}}{4|\vec{p}|^6}4|\vec{p}|^4\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{4}\frac{1}{(2\pi)^3}\int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_ap_bp_cp_d\frac{\hat{p}}{|\vec{p}|^6}[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{4}\frac{1}{(2\pi)^3}\int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{(\frac{1}{\sqrt{2}})^4\frac{1}{3}|\vec{p}|^4\{0+4\varsigma[\sigma(2)\cdot\hat{p}]-2[\sigma(2)\cdot\hat{p}]^2-4\varsigma[\sigma(2)\cdot\hat{p}]^3+2[\sigma(2)\cdot\hat{p}]^4\}\frac{\hat{p}}{|\vec{p}|^6}[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{12}\frac{1}{(2\pi)^3}\int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{\varsigma[\sigma(2)\cdot\hat{p}]-\varsigma[\sigma(2)\cdot\hat{p}]^3\}\frac{\hat{p}}{|\vec{p}|^2}[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{\varsigma}{6}\frac{1}{(2\pi)^3}\int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{[\sigma(2)\cdot\hat{p}]-[\sigma(2)\cdot\hat{p}]^3\}\frac{\hat{p}}{|\vec{p}|^2}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{\varsigma}{6}\frac{1}{(2\pi)^3}\int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{-[\sigma(2)\cdot i\vec{p}]\frac{i\vec{p}}{|\vec{p}|^4}-[\sigma(2)\cdot i\vec{p}]^3\frac{i\vec{p}}{|\vec{p}|^6}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$= \frac{\varsigma}{6}\int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\{-[\sigma(2)\cdot\nabla]\frac{\nabla}{\nabla^4}+[\sigma(2)\cdot\nabla]^3\frac{\nabla}{\nabla^6}\}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$= \frac{\varsigma}{6}\int \psi^+_{k'_\varsigma}(\vec{r},t)\{-[\sigma(2)\cdot\nabla]\frac{\nabla}{\nabla^4}+[\sigma(2)\cdot\nabla]^3\frac{\nabla}{\nabla^6}\}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$= \frac{\varsigma}{2}\int \psi^+_{k'_\varsigma}(\vec{r},t)[\sigma(2)\cdot\nabla]\frac{\nabla}{\nabla^4}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$= \frac{-\varsigma}{2}\int \psi^+_{k'_\varsigma}(\vec{r},t)\frac{\sigma(2)}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$ □

Cor. 8.1.4. $P_a=\int p_a[a_1^+(\vec{p},-2\varsigma)a_1(\vec{p},-2\varsigma)+a_2(\vec{p},-2\varsigma)a_2^+(\vec{p},-2\varsigma)]d^3\vec{p}=\frac{-\varsigma}{2}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{[\sigma(2),-i2\varsigma]_a}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$

Cor. 8.1.5. $P_2=\int \vec{p}[a_1^+(\vec{p},-2\varsigma)a_1(\vec{p},-2\varsigma)+a_2(\vec{p},-2\varsigma)a_2^+(\vec{p},-2\varsigma)]d^3\vec{p}=\frac{\varsigma}{2}\int\psi^+_{k'_\varsigma}(\vec{r},t)[\sigma(2)\cdot\nabla]\frac{\nabla}{\nabla^4}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$

8.2 Graviton properties of covariant constant invariant tensor

Cor. 8.2.1.
$\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}(2)=(\frac{1}{\sqrt{2}})^4\delta_{k_\varsigma k'_\varsigma}$
$\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}(2)=-i\varsigma(\frac{1}{\sqrt{2}})^4\frac{1}{2}\sigma^i(2)_{k_\varsigma k'_\varsigma}$
$\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}(2)=-(\frac{1}{\sqrt{2}})^4\frac{1}{6}[\sigma^{\{i}(2)\sigma^{j\}}(2)-2\delta^{ij}]_{k_\varsigma k'_\varsigma}=-(\frac{1}{\sqrt{2}})^4\frac{1}{3}\frac{1}{2!}[\sigma^{\{i}(2)\sigma^{j\}}(2)-\delta^{\{ij\}}]_{k_\varsigma k'_\varsigma}$
$\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}(2)=(\frac{1}{\sqrt{2}})^4\frac{i\varsigma}{6}\{\sigma^{\{j}(2)[\sigma^i(2)]\sigma^{k\}}(2)-[\sigma^i(2)\delta^{jk}+2\delta^{i\{j}\sigma^{k\}}(2)]\}_{k_\varsigma k'_\varsigma}$
$=(\frac{1}{\sqrt{2}})^4\frac{i\varsigma}{3}\frac{1}{3!}\{\sigma^{\{i}(2)\sigma^j(2)\sigma^{k\}}(2)-\frac{5}{2}\sigma^{\{i}(2)\delta^{jk\}}\}_{k_\varsigma k'_\varsigma}$
$\Gamma^{ijkl}_{k_\varsigma k'_\varsigma}(2)=(\frac{1}{\sqrt{2}})^4\frac{2}{3}\frac{1}{4!}[\sigma^{\{i}(2)\sigma^j(2)\sigma^k(2)\sigma^{l\}}(2)-4\sigma^{\{i}(2)\sigma^j(2)\delta^{kl\}}+\frac{3}{2}\delta^{\{ij}\delta^{kl\}}]_{k_\varsigma k'_\varsigma}$

Lem. 8.2.1. $\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_ap_bp_cp_d=4|\vec{p}|^4\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)$

Proof: $\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_ap_bp_cp_d$
$=C_4^4\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}(1)p_\pi^4+C_4^3\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}(1)p_ip_\pi^3+C_4^2\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}(1)p_ip_jp_\pi^2+C_4^1\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}(1)p_ip_jp_kp_\pi+C_4^0\Gamma^{ijkl}_{k_\varsigma k'_\varsigma}(1)p_ip_jp_kp_l$
$=(\frac{1}{\sqrt{2}})^4\frac{1}{3}\{3|\vec{p}|^4-6\varsigma|\vec{p}|^3[\sigma(2)\cdot\vec{p}]+6|\vec{p}|^2\{[\sigma(2)\cdot\vec{p}]^2-|\vec{p}|^2\}-4\varsigma|\vec{p}|\{[\sigma(2)\cdot\vec{p}]^3-\frac{5}{2}|\vec{p}|^2[\sigma(2)\cdot\vec{p}]\}$
$+2[[\sigma(2)\cdot\vec{p}]^4-4|\vec{p}|^2[\sigma(2)\cdot\vec{p}]^2+\frac{3}{2}|\vec{p}|^4]\}_{k_\varsigma k'_\varsigma}$
$=\{(\frac{1}{\sqrt{2}})^4\frac{1}{3}|\vec{p}|^4\{0+4\varsigma[\sigma(2)\cdot\hat{p}]-2[\sigma(2)\cdot\hat{p}]^2-4\varsigma[\sigma(2)\cdot\hat{p}]^3+2[\sigma(2)\cdot\hat{p}]^4\}\sum_{h=2}^{-2}\lambda(\hat{p},h)\lambda^+(\hat{p},h)\}_{k_\varsigma k'_\varsigma}$
$=4|\vec{p}|^4\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)$ □

Cor. 8.2.2. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(2,\varsigma)=\frac{1}{4}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\hat{p}_a\hat{p}_b\hat{p}_c\hat{p}_d\to\frac{1}{4}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\hat{\partial}_d$

8.3 General covariant commutation rules for graviton field in mathematics

Thm. 8.3.1.

$$\begin{cases}[a_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]_\pm\\=\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-2\varsigma),a_{\sigma'}(\vec{p}',-2\varsigma)]_\pm=0\\ [a^+_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]_\pm=0\end{cases}\Rightarrow\begin{cases}[\Psi_{k_\varsigma}(x),\Psi^+_{k'_\varsigma}(x')]_\pm\\=\frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d[\delta_1\Delta(x-x')-(\delta_1\pm\delta_2)\Delta^{(-)}(x-x')]\\ [\Psi_{k_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]_\pm=0\\ [\Psi^+_{k'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]_\pm=0\end{cases}$$

Proof: $[\Psi^{(+)}_{k_\varsigma}(x),\Psi^{(+)+}_{k'_\varsigma}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-2\varsigma)|\vec{p}|^{3/2}|\vec{p}'|^{3/2}[a_1(\vec{p},-2\varsigma),a_1^+(\vec{p}',-2\varsigma)]_\pm e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-2\varsigma)|\vec{p}|^3\delta_1\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)\delta_1|\vec{p}|^3e^{ip\cdot(x-x')}d^3\vec{p}$
$=\frac{\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_ap_bp_cp_de^{ip\cdot(x-x')}d^3\vec{p}$
$=\frac{\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_de^{ip\cdot(x-x')}d^3\vec{p}$
$=\frac{i}{2}\delta_1\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{(+)}(x-x')$ □

Proof: $[\Psi^{(-)}_{k_\varsigma}(x),\Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$
$= \frac{1}{(2\pi)^3}\int \lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-2\varsigma)|\vec{p}|^{3/2}|\vec{p}'|^{3/2}[a^+_2(\vec{p},-2\varsigma),a_2(\vec{p}',-2\varsigma)]_\pm e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int \lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-2\varsigma)|\vec{p}|^3\delta_2\delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$= \pm\frac{1}{(2\pi)^3}\int \lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)\delta_2|\vec{p}|^3 e^{-ip\cdot(x-x')}d^3\vec{p}$
$= \pm\frac{\delta_2}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_a p_b p_c p_d e^{-ip\cdot(x-x')}d^3\vec{p}$
$= \pm\frac{\delta_2}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d e^{-ip\cdot(x-x')}d^3\vec{p}$
$= -\pm\frac{i}{2}\delta_1\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{(-)}(x-x')$ □

Proof: $[\Psi_{k_\varsigma}(x),\Psi^+_{k'_\varsigma}(x')]_\pm$
$= [\Psi^{(+)}_{k_\varsigma}(x),\Psi^{(+)+}_{k'_\varsigma}(x')]_\pm + [\Psi^{(-)}_{k_\varsigma}(x),\Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$
$= \frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d[\delta_1\Delta^{(+)}(x-x') - \pm\delta_2\Delta^{(-)}(x-x')]$
$= \frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d[(\delta_1\pm\delta_2)\Delta^{(+)}(x-x') - \pm\delta_2\Delta(x-x')]$
$= \frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d[\delta_1\Delta(x-x') - (\delta_1\pm\delta_2)\Delta^{(-)}(x-x')]$ □

From the above, only $\delta_1\pm\delta_2=0$, the micro causality is satisfied. At the same time only when $\delta_1,\delta_2\geq 0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1=\delta_2=1$, it satisfies the commutative relation. There are actually two other options. Namely when $\delta_1=\delta_2=0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

8.4 Physical Covariant commutation rules for graviton field

From the previous section, we can see that the commutation rules with physical significance are as follows:(In order to confirm each other, a new proof has been made.)

Thm. 8.4.1.
$$\begin{cases}[a_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-2\varsigma),a_{\sigma'}(\vec{p}',-2\varsigma)]=0\\ [a^+_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]=0\end{cases} \Rightarrow \begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=\frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases}$$

Proof: $\{\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')\}$
$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)|\vec{p}|^{3/2}|\vec{p}'|^{3/2}$
$\{[a_1(\vec{p},-2\varsigma),a^+_1(\vec{p}',-2\varsigma)]e^{ip\cdot(x-x')} + [a^+_2(\vec{p},-2\varsigma),a_2(\vec{p}',-2\varsigma)]e^{-ip\cdot(x-x')}\}$
$= \frac{1}{(2\pi)^3}\int |\vec{p}|^3\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)[\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')} - \delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}]d^3\vec{p}d^3\vec{p}'$
$= \frac{1}{(2\pi)^3}\int |\vec{p}|^3\lambda_{k_\varsigma}(\hat{p},-2\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-2\varsigma)[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_a p_b p_c p_d[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec{p}$
$= \frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\frac{-i}{(2\pi)^3}\int \frac{1}{2|\vec{p}|}[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]d^3\vec{p}$
$= \frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')$ □

8.5 Isochronous commutation rules for graviton field

Cor. 8.5.1.
$$\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=\frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases} \Rightarrow \begin{cases}[\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]\\ =\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ [\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)]=0,[\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)]=0\end{cases}$$

Proof: $[\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)] = \frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')|_{t=t'}$
$= C^1_4\frac{i}{2}\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_j\partial_k\partial_\pi\Delta(x-x')|_{t=t'} + C^3_4\frac{i}{2}\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_\pi\partial_\pi\partial_\pi\Delta(x-x')|_{t=t'}$
$= \frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2 - [\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')$ □

Cor. 8.5.2.
$$\begin{cases}[\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]\\ =\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ [\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)]=0,[\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)]=0\end{cases} \Rightarrow \begin{cases}[a_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]=\varsigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-2\varsigma),a_{\sigma'}(\vec{p}',-2\varsigma)]=0\\ [a^+_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]=0\end{cases}$$

Proof: $[a_1(\vec{p},-2\varsigma),a^+_1(\vec{p}',-2\varsigma)]$
$= \frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\int[\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\Psi_{k_\varsigma}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)},\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\Psi^+_{k'_\varsigma}(\vec{r}',t)e^{i(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\int \lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)[\Psi_{k_\varsigma}(\vec{r},t),\Psi^+_{k'_\varsigma}(\vec{r}',t)]e^{-i(\vec{p}\cdot\vec{r}-Et)}e^{i(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}$
$\int \lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')e^{-i(\vec{p}\cdot\vec{r}-Et)}e^{i(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\int \lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\frac{1}{6}\{[\sigma(2)\cdot\vec{p}]\vec{p}^2-[\sigma(2)\cdot\vec{p}]^3]\}_{k_\varsigma k'_\varsigma}e^{-i(\vec{p}\cdot\vec{r}-Et)}e^{i(\vec{p}'\cdot\vec{r}-E't)}d^3\vec{r}$
$= \varsigma\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\frac{1}{6}\{[\sigma(2)\cdot\vec{p}]\vec{p}^2-[\sigma(2)\cdot\vec{p}]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{p}-\vec{p}')$
$= \varsigma\lambda^{+}(\hat{p},-2\varsigma)\frac{1}{6}\{[\sigma(2)\cdot\hat{p}]-[\sigma(2)\cdot\hat{p}]^3]\}\lambda(\hat{p},-2\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= \lambda^{+}(\hat{p},-2\varsigma)\lambda(\hat{p},-2\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= \delta^3(\vec{p}-\vec{p}')$ □

Proof: $[a_2^+(\vec{p},-2\varsigma),a_2(\vec{p}',-2\varsigma)]$
$= \frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\int [\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\Psi_{k_\varsigma}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)},\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\Psi^+_{k'_\varsigma}(\vec{r}',t)e^{-i(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\int \lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)[\Psi_{k_\varsigma}(\vec{r},t),\Psi^+_{k'_\varsigma}(\vec{r}',t)]e^{i(\vec{p}\cdot\vec{r}-Et)}e^{-i(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}$
$\int \lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')e^{i(\vec{p}\cdot\vec{r}-Et)}e^{-i(\vec{p}'\cdot\vec{r}'-E't)}d^3\vec{r}d^3\vec{r}'$
$= \varsigma\frac{1}{(2\pi)^3}\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\int \lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\frac{-1}{6}\{[\sigma(2)\cdot\vec{p}]\vec{p}^2-[\sigma(2)\cdot\vec{p}]^3]\}_{k_\varsigma k'_\varsigma}e^{i(\vec{p}\cdot\vec{r}-Et)}e^{-i(\vec{p}'\cdot\vec{r}-E't)}d^3\vec{r}$
$= \varsigma\frac{1}{|\vec{p}|^{3/2}|\vec{p}'|^{3/2}}\lambda^{+k_\varsigma}(\hat{p},-2\varsigma)\lambda^{k'_\varsigma}(\vec{p}',-2\varsigma)\frac{-1}{6}\{[\sigma(2)\cdot\vec{p}]\vec{p}^2-[\sigma(2)\cdot\vec{p}]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{p}-\vec{p}')$
$= \varsigma\lambda^{+}(\hat{p},-2\varsigma)\frac{-1}{6}\{[\sigma(2)\cdot\hat{p}]-[\sigma(2)\cdot\hat{p}]^3]\}\lambda(\hat{p},-2\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= -\lambda^{+}(\hat{p},-2\varsigma)\lambda(\hat{p},-2\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= -\delta^3(\vec{p}-\vec{p}')$ □

8.6 Summary of commutation rules for graviton field

he proof in the above sections exactly forms a logical closed-loop, so it has the following properties:

Cor. 8.6.1.

$$\begin{cases}[a_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-2\varsigma),a_{\sigma'}(\vec{p}',-2\varsigma)]=0\\ [a^+_\sigma(\vec{p},-2\varsigma),a^+_{\sigma'}(\vec{p}',-2\varsigma)]=0\end{cases}\Leftrightarrow\begin{cases}[a_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]=\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p}),a_{\sigma'}(\vec{p}')]=0\\ [a^+_\sigma(\vec{p}),a^+_{\sigma'}(\vec{p}')]=0\end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

Cor. 8.6.2.

$$\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=\frac{i}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]\\ =\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ [\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)]=0,[\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)]=0\end{cases}$$

8.7 Commutative function, causal function and feynman propagator of graviton field

Cor. 8.7.1.

$$\begin{cases}\Delta_{k_\varsigma k'_\varsigma}(2;x):=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x)\\ \Delta^{(+)}_{k_\varsigma k'_\varsigma}(2;x):=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{(+)}(x)\\ \Delta^{(-)}_{k_\varsigma k'_\varsigma}(2;x):=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{(-)}(x)\\ \Delta^{(l)}_{k_\varsigma k'_\varsigma}(2;x):=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{(l)}(x)\end{cases}$$

Cor. 8.7.2.

$$\begin{cases}\Delta^{(c)}_{k_\varsigma k'_\varsigma}(2;x)\\ :=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{(c)}(x)-\frac{1}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta'''(t)+4i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)\partial_i-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i\partial_j-4i\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j\partial_k]\Delta(x)\\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(2;x)\\ :=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{ret}(x)-\frac{1}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta'''(t)+4i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)\partial_i-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i\partial_j-4i\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j\partial_k]\Delta(x)\\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(2;x)\\ :=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{adv}(x)-\frac{1}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta'''(t)+4i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)\partial_i-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i\partial_j-4i\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j\partial_k]\Delta(x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(2;x)=i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(2;x)\\ :=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta_F(x)-\frac{i}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta'''(t)+4i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\delta''(t)\partial_i-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\delta'(t)\partial_i\partial_j-4i\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}\delta(t)\partial_i\partial_j\partial_k]\Delta(x)\end{cases}$$

Cor. 8.7.3.

$$\begin{cases}\Delta^{(c)}_{k_\varsigma k'_\varsigma}(2;x)=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{(c)}(x)-\frac{1}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}(3\partial_t^2+\nabla^2)+8i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_t-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_j]\delta^4(x)\\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(2;x)=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{ret}(x)-\frac{1}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}(3\partial_t^2+\nabla^2)+8i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_t-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_j]\delta^4(x)\\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(2;x)=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta^{adv}(x)-\frac{1}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}(3\partial_t^2+\nabla^2)+8i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_t-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_j]\delta^4(x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(2;x)=i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(2;x)=\frac{1}{2}\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta_F(x)-\frac{i}{2}[\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}(3\partial_t^2+\nabla^2)+8i\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_t-6\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}\partial_i\partial_j]\delta^4(x)\\ \Delta_{Fk_\varsigma k'_\varsigma}(2;p)=\frac{-i}{2}\frac{\Gamma^{abcd}_{k_\varsigma k'_\varsigma}p_ap_bp_cp_d}{p^2-i\varepsilon}+\cdots\end{cases}$$

Cor. 8.7.4.

$$\begin{cases}[s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta(2;x)=0\\ [s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta^{(+)}(2;x)=0\\ [s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta^{(-)}(2;x)=0\\ [s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta^{(l)}(2;x)=0\end{cases}\quad\begin{cases}[s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta^{(c)}(2;x)=-\varsigma[\sigma(2),i2\varsigma]_a\delta(t)\Delta(2;x)|_{t=0}\\ [s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta^{ret}(2;x)=-\varsigma[\sigma(2),i2\varsigma]_a\delta(t)\Delta(2;x)|_{t=0}\\ [s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta^{adv}(2;x)=-\varsigma[\sigma(2),i2\varsigma]_a\delta(t)\Delta(2;x)|_{t=0}\\ [s\partial_a+iS_{ab}(2,\varsigma)\partial^b]\Delta_F(2;x)=-i\varsigma[\sigma(2),i2\varsigma]_a\delta(t)\Delta(2;x)|_{t=0}\end{cases}$$

8.8 Quantum equation of graviton field

Thm. 8.8.1.
$H=\frac{1}{2}\int\{\psi^+_{k'_\varsigma}(\vec r,t),\Gamma(\nabla)\psi_{k_\varsigma}(\vec r,t)\}d^3\vec r$

Thm. 8.8.2. $[\psi_{j_\varsigma}(\vec r,t),\int d^3\vec r'\psi^+_{k'_\varsigma}(\vec r',t)\Gamma(\nabla')^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)]=\frac{1}{2}[\psi_{j_\varsigma}(\vec r,t),\int d^3\vec r'\{\psi^+_{k'_\varsigma}(\vec r',t),\Gamma(\nabla')^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)\}]$

Proof: $\int d^3\vec r'[\psi_{j_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)\Gamma(\nabla')^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)]$
$=\int d^3\vec r'[\psi_{j_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]\Gamma(\nabla')^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)]$
$=\int d^3\vec r'\Gamma(\nabla')^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)[\psi_{j_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]$
$=\int d^3\vec r'[\psi_{j_\varsigma}(\vec r,t),\Gamma(\nabla')^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)\psi^+_{k'_\varsigma}(\vec r',t)]$ □

8.9 Commutative and anticommutative formulas

Cor. 8.9.1. $\begin{cases}[A,BC]=[A,B]C+B[A,C],[A,CB]=[A,C]B+C[A,B]\\ [A,BC]=\{A,B\}C-B\{A,C\},[A,CB]=\{A,C\}B-C\{A,B\}\end{cases}$

Cor. 8.9.2. $\begin{cases}[A,\{B,C\}]=\{[A,B],C\}+\{B,[A,C]\}\\ [A,[B,C]]=\{\{A,B\},C\}-\{B,\{A,C\}\}\end{cases}$

Thm. 8.9.1.

$$\begin{cases}[\psi_{k_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]=\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\\ [\psi_{k_\varsigma}(\vec r,t),\psi_{l_\varsigma}(\vec r',t)]=0,[\psi^+_{k'_\varsigma}(\vec r,t),\psi^+_{l'_\varsigma}(\vec r',t)]=0\\ H=\int\psi^+_{k'_\varsigma}(\vec r,t)\frac{[\frac{1}{2}\sigma(2)\cdot\nabla]^2}{-\nabla^4}\psi_{k_\varsigma}(\vec r,t)d^3\vec r|=\int\psi^+_{k'_\varsigma}(\vec r,t)\frac{1}{-\nabla^2}\psi_{k_\varsigma}(\vec r,t)d^3\vec r,\vec P=\frac{\varsigma}{2}\int\psi^+_{k'_\varsigma}(\vec r,t)[\sigma(2)\cdot\nabla]\frac{\nabla}{\nabla^4}\psi_{k_\varsigma}(\vec r,t)d^3\vec r\end{cases}$$

$$\Rightarrow\begin{cases}[\psi(\vec r,t),H]=\frac{i}{6}\varsigma\{-[\sigma(2)\cdot\nabla]+\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3\}\psi(\vec r,t)\\ [\psi(\vec r,t),\vec P]=\frac{i}{12}\{\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^2}-\frac{[\sigma(2)\cdot\nabla]^4}{\nabla^4}]\}\nabla\psi(\vec r,t)\end{cases}$$

Proof: $[\psi(\vec r,t),H]$
$=\int d^3\vec r[\psi_{j_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)\{\frac{[\frac{1}{2}\sigma(2)\cdot\nabla']^2}{-\nabla'^4}\}^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)]$
$=\int d^3\vec r[\psi_{j_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]\{\frac{[\frac{1}{2}\sigma(2)\cdot\nabla']^2}{-\nabla'^4}\}^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)$
$=\int d^3\vec r\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3\}_{j_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\{\frac{[\frac{1}{2}\sigma(2)\cdot\nabla']^2}{-\nabla'^4}\}^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r',t)$
$=\frac{i}{6}\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3\}_{j_\varsigma k'_\varsigma}\{\frac{[\frac{1}{2}\sigma(2)\cdot\nabla]^2}{-\nabla^4}\}^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec r,t)$
$=\frac{i}{24}\varsigma\{-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3+\frac{1}{\nabla^4}[\sigma(2)\cdot\nabla]^5\}\psi(\vec r,t)$
$=\frac{i}{6}\varsigma\{-[\sigma(2)\cdot\nabla]+\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3\}\psi(\vec r,t)$ □

Proof: $[\psi(\vec r,t),H]$
$=\delta^{k'_\varsigma k_\varsigma}\int d^3\vec r[\psi_{j_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)\frac{1}{-\nabla'^2}\psi_{k_\varsigma}(\vec r',t)]$
$=\delta^{k'_\varsigma k_\varsigma}\int d^3\vec r[\psi_{j_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]\frac{1}{-\nabla'^2}\psi_{k_\varsigma}(\vec r',t)$
$=\delta^{k'_\varsigma k_\varsigma}\int d^3\vec r\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3\}_{j_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\frac{1}{-\nabla'^2}\psi_{k_\varsigma}(\vec r',t)$
$=\frac{i}{6}\varsigma\delta^{k'_\varsigma k_\varsigma}\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3\}_{j_\varsigma k'_\varsigma}\frac{1}{-\nabla^2}\psi_{k_\varsigma}(\vec r,t)$
$=\frac{i}{6}\varsigma\{-[\sigma(2)\cdot\nabla]+\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3\}\psi(\vec r,t)$ □

Proof: $[\psi(\vec{r},t),\vec{P}]$
$= \frac{\varsigma}{2}\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}'[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)[\sigma(2)\cdot\nabla']\frac{\nabla'}{\nabla'^4}\psi_{k_\varsigma}(\vec{r}',t)]$
$= \frac{\varsigma}{2}\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}'[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)][\sigma(2)\cdot\nabla']\frac{\nabla'}{\nabla'^4}\psi_{k_\varsigma}(\vec{r}',t)$
$= \frac{\varsigma}{2}\delta^{k'_\varsigma k_\varsigma}\int d^3\vec{r}'\frac{-1}{6}i\varsigma\{[\sigma(2)\cdot\nabla']\nabla'^2-[\sigma(2)\cdot\nabla']^3]\}_{j_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')[\sigma(2)\cdot\nabla']\frac{\nabla'}{\nabla'^4}\psi_{k_\varsigma}(\vec{r},t)$
$= \frac{i}{12}\{[\sigma(2)\cdot\nabla]^2\nabla^2-[\sigma(2)\cdot\nabla]^4]\}\frac{\nabla}{\nabla^4}\psi(\vec{r},t)$
$= \frac{i}{12}\{\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^2}-\frac{[\sigma(2)\cdot\nabla]^4}{\nabla^4}]\}\nabla\psi(\vec{r},t)$ □

Cor. 8.9.3.

$$\begin{cases}\sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^1\sigma(s)\}=[\sigma^2(s)-1][\sigma(s)\cdot\hat{p}]\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^2\sigma(s)\}=[\sigma^2(s)-3][\sigma(s)\cdot\hat{p}]^2+\sigma^2(s)\\ \sigma(s)\cdot\{[\sigma(s)\cdot\hat{p}]^3\sigma(s)\}\\ =[\sigma^2(s)-6][\sigma(s)\cdot\hat{p}]^3+[3\sigma^2(s)-1]\sigma(s)\cdot\hat{p}\end{cases}\Rightarrow\begin{cases}\sigma(2)\cdot\{[\sigma(2)\cdot\hat{p}]^1\sigma(2)\}=5[\sigma(2)\cdot\hat{p}]\\ \sigma(2)\cdot\{[\sigma(2)\cdot\hat{p}]^2\sigma(2)\}=3[\sigma(2)\cdot\hat{p}]^2+6\\ \sigma(2)\cdot\{[\sigma(2)\cdot\hat{p}]^3\sigma(2)\}=17[\sigma(2)\cdot\hat{p}]\end{cases}$$

Cor. 8.9.4.

$$\begin{cases}\nabla\psi(\vec{r},t)=i[\psi(\vec{r},t),P]\\ \dot{\psi}(\vec{r},t)=-i[\psi(\vec{r},t),H]\end{cases}\Leftrightarrow\begin{cases}\nabla\psi(\vec{r},t)=-\frac{1}{12}\{\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^2}-\frac{[\sigma(2)\cdot\nabla]^4}{\nabla^4}\}\nabla\psi(\vec{r},t)\\ \dot{\psi}(\vec{r},t)=-\frac{1}{6}\varsigma\{[\sigma(2)\cdot\nabla]-\frac{[\sigma(2)\cdot\nabla]^3}{\nabla^2}\}\psi(\vec{r},t)\end{cases}\Leftrightarrow\begin{cases}\partial^a\partial_a\psi(\vec{r},t)=0\\ [\sigma(2),-2i\varsigma]^a\partial_a\psi(\vec{r},t)=0\end{cases}$$

Cor. 8.9.5. $\begin{cases}\nabla\psi(\vec{r},t)=-\frac{1}{12}\{\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^2}-\frac{[\sigma(2)\cdot\nabla]^4}{\nabla^4}\}\nabla\psi(\vec{r},t)\\ \psi(\vec{r},t)=\int\lambda(\hat{p},-2\varsigma)[a_1(\vec{p})e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+a_2^+(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]d^3\vec{p}\end{cases}$

$$\Leftrightarrow\begin{cases}\nabla\psi(\vec{r},t)=-\frac{1}{12}\{[\sigma(2)\cdot\nabla]-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3]\}\sigma(2)\psi(\vec{r},t)\\ \psi(\vec{r},t)=\int\lambda(\hat{p},-2\varsigma)[a_1(\vec{p})e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+a_2^+(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]d^3\vec{p}\end{cases}$$

Cor. 8.9.6. $[2\partial_a+iS_{ab}(2,\varsigma)\partial^b]\psi(x)=0\Rightarrow\begin{cases}\partial^a\partial_a\psi(\vec{r},t)=0\\ [\sigma(2),-2i\varsigma]^a\partial_a\psi(\vec{r},t)=0\end{cases}$

Cor. 8.9.7. $[2\partial_a+iS_{ab}(2,\varsigma)\partial^b]\psi(x)=0\Rightarrow\partial_a\psi(\vec{r},t)=i[\psi(\vec{r},t),P_a]$

8.10 Second quantum equation of graviton field

Thm. 8.10.1.

$$\begin{cases}[\psi_{k_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]=\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla^2-[\sigma(2)\cdot\nabla]^3]\}_{k_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\\ [\psi_{k_\varsigma}(\vec{r},t),\psi_{l_\varsigma}(\vec{r}',t)]=0,[\psi^+_{k'_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t)]=0\\ H=\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{1}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r},\vec{P}=\frac{-\varsigma}{2}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{\sigma(2)}{-\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}\end{cases}$$

$$\Rightarrow\begin{cases}[\psi(\vec{r},t),H]=\frac{i}{6}\varsigma\{-[\sigma(2)\cdot\nabla]+\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3\}\psi(\vec{r},t)\\ [\psi(\vec{r},t),\vec{P}]=\frac{i}{12}\{[\sigma(2)\cdot\nabla]-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3]\}\sigma(2)\psi(\vec{r},t)\end{cases}$$

Proof: $[\psi(\vec{r},t),P]$
$= \frac{\varsigma}{2}\sigma(2)^{k'_\varsigma k_\varsigma}\int d^3\vec{r}'[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)\frac{1}{\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)]$
$= \frac{\varsigma}{2}\sigma(2)^{k'_\varsigma k_\varsigma}\int d^3\vec{r}'[\psi_{j_\varsigma}(\vec{r},t),\psi^+_{k'_\varsigma}(\vec{r}',t)]\frac{1}{\nabla'^2}\psi_{k_\varsigma}(\vec{r}',t)$
$= \frac{\varsigma}{2}\sigma(2)^{k'_\varsigma k_\varsigma}\int d^3\vec{r}'\frac{-1}{6}i\varsigma\{[\sigma(2)\cdot\nabla']\nabla'^2-[\sigma(2)\cdot\nabla']^3]\}_{j_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\frac{1}{\nabla'^2}\psi_{k_\varsigma}(\vec{r},t)$
$= \frac{\varsigma}{2}\sigma(2)^{k'_\varsigma k_\varsigma}\frac{1}{6}i\varsigma\{[\sigma(2)\cdot\nabla]\nabla'^2-[\sigma(2)\cdot\nabla]^3]\}_{j_\varsigma k'_\varsigma}\frac{1}{\nabla^2}\psi_{k_\varsigma}(\vec{r},t)$
$= \frac{i}{12}\{[\sigma(2)\cdot\nabla]-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3]\}\sigma(2)\psi(\vec{r},t)$
$? = -i\nabla\psi(\vec{r},t)$ □

Pro. 8.10.1. $i\sigma(s)\times\nabla=\sigma(s)\cdot\nabla\sigma(s)-\sigma(s)[\sigma(s)\cdot\nabla],\sigma(s)\cdot\nabla\sigma(s)=i\sigma(s)\times\nabla+\sigma(s)[\sigma(s)\cdot\nabla]$

Cor. 8.10.1. $\{[\sigma(2)\cdot\nabla]-\frac{[\sigma(2)\cdot\nabla]^3}{\nabla^2}\}\sigma(2)$
$=i\sigma(s)\times\nabla+\sigma(s)[\sigma(s)\cdot\nabla]-\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^2}\{i\sigma(s)\times\nabla+\sigma(s)[\sigma(s)\cdot\nabla]\}$
$=i\sigma(s)\times\nabla+\sigma(s)[\sigma(s)\cdot\nabla]-\frac{[\sigma(2)\cdot\nabla]}{\nabla^2}i\{i\sigma(s)\times\nabla+\sigma(s)[\sigma(s)\cdot\nabla]\}\times\nabla+\frac{[\sigma(2)\cdot\nabla]}{\nabla^2}\{i\sigma(s)\times\nabla+\sigma(s)[\sigma(s)\cdot\nabla]\}[\sigma(s)\cdot\nabla]$

Pro. 8.10.2.
$\sigma(2)\cdot\hat{\nabla}\equiv-\frac{1}{12}\sigma_\alpha(2)\{[\sigma(2)\cdot\hat{\nabla}]-[\sigma(2)\cdot\hat{\nabla}]^3]\}\sigma^\alpha(2),[\sigma(2)\cdot\hat{\nabla}]^5\equiv-4[\sigma(2)\cdot\hat{\nabla}]+5[\sigma(2)\cdot\hat{\nabla}]^3,\hat{\nabla}:=\frac{-i\nabla}{\sqrt{-\nabla^2}},\hat{\nabla}^2=1$

Cor. 8.10.2.

$$\begin{cases}\dot{\psi}(\vec{r},t)=\frac{1}{6}\varsigma\{-[\sigma(2)\cdot\nabla]+\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3\}\psi(\vec{r},t)\\ \nabla\psi(\vec{r},t)=-\frac{1}{12}\{[\sigma(2)\cdot\nabla]-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3]\}\sigma(2)\psi(\vec{r},t)\end{cases}\Rightarrow\partial_t^2\psi(\vec{r},t)=\nabla^2\psi(\vec{r},t)$$

Proof:

$$\begin{cases}\dot\psi(\vec r,t)=\frac{1}{6}\varsigma\{-[\sigma(2)\cdot\nabla]+\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3\}\psi(\vec r,t)\\ \nabla\psi(\vec r,t)=-\frac{1}{12}\{[\sigma(2)\cdot\nabla]-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3]\}\sigma(2)\psi(\vec r,t)\end{cases}$$

$\Rightarrow \partial_t^2\psi(\vec r,t)=\frac{1}{36}\{[\sigma(2)\cdot\nabla]^2-2\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^4+\frac{1}{\nabla^4}[\sigma(2)\cdot\nabla]^6\}\psi(\vec r,t)$
$=\frac{1}{36}\{[\sigma(2)\cdot\nabla]^2-2\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^4-4[\sigma(2)\cdot\nabla]^2+5\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^4\}\psi(\vec r,t)$
$=-\frac{1}{12}\{[\sigma(2)\cdot\nabla]^2-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^4\}\psi(\vec r,t)$
$=\nabla^2\psi(\vec r,t)$ □

Cor. 8.10.3.

$$\begin{cases}\dot\psi(\vec r,t)=\frac{1}{6}\varsigma\{-[\sigma(2)\cdot\nabla]+\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3\}\psi(\vec r,t)\\ \nabla\psi(\vec r,t)=-\frac{1}{12}\{[\sigma(2)\cdot\nabla]-\frac{1}{\nabla^2}[\sigma(2)\cdot\nabla]^3]\}\sigma(2)\psi(\vec r,t)\end{cases} \quad !\Rightarrow [\sigma(2),-2i\varsigma]^a\partial_a\psi(\vec r,t)=0$$

8.11 Poincare symmetry of graviton field

Cor. 8.11.1.

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i\sum_{l=0}^{[s]}(-1)^lC_{2s}^{2l}\Gamma^{\overbrace{ij\cdots}^{2s-2l}\overbrace{\pi\cdots\pi}^{2l}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2l}\nabla^{2l}\delta^3(\vec r-\vec r')\\ \Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i\sum_{l=0}^{[s]}(-1)^lC_{2s}^{2l}\Gamma^{\overbrace{ij\cdots}^{2s-2l}\overbrace{\pi\cdots\pi}^{2l}}(s)\overbrace{\hat\partial_i\hat\partial_j\cdots}^{2s-2l}\delta^3(\vec r-\vec r')\end{cases}$$

Cor. 8.11.2.
$\Gamma^{\pi\pi\pi\pi}_{k_\varsigma k'_\varsigma}(2)=(\frac{1}{\sqrt2})^4\delta_{k_\varsigma k'_\varsigma}$
$\Gamma^{i\pi\pi\pi}_{k_\varsigma k'_\varsigma}(2)=-i\varsigma(\frac{1}{\sqrt2})^4\frac12\sigma^i(2)_{k_\varsigma k'_\varsigma}$
$\Gamma^{ij\pi\pi}_{k_\varsigma k'_\varsigma}(2)=-(\frac{1}{\sqrt2})^4\frac16[\sigma^{\{i}(2)\sigma^{j\}}(2)-2\delta^{ij}]_{k_\varsigma k'_\varsigma}=-(\frac{1}{\sqrt2})^4\frac13\frac{1}{2!}[\sigma^{\{i}(2)\sigma^{j\}}(2)-\delta^{\{ij\}}]_{k_\varsigma k'_\varsigma}$
$\Gamma^{ijk\pi}_{k_\varsigma k'_\varsigma}(2)=(\frac{1}{\sqrt2})^4\frac{i\varsigma}{6}\{\sigma^{\{j}(2)[\sigma^i(2)]\sigma^{k\}}(2)-[\sigma^i(2)\delta^{jk}+2\delta^{i\{j}\sigma^{k\}}(2)]\}_{k_\varsigma k'_\varsigma}$
$=(\frac{1}{\sqrt2})^4\frac{i\varsigma}{3}\frac{1}{3!}\{\sigma^{\{i}(2)\sigma^j(2)\sigma^{k\}}(2)-\frac52\sigma^{\{i}(2)\delta^{jk\}}\}_{k_\varsigma k'_\varsigma}$
$\Gamma^{ijkl}_{k_\varsigma k'_\varsigma}(2)=(\frac{1}{\sqrt2})^4\frac23\frac{1}{4!}[\sigma^{\{i}(2)\sigma^j(2)\sigma^k(2)\sigma^{l\}}(2)-4\sigma^{\{i}(2)\sigma^j(2)\delta^{kl\}}+\frac32\delta^{\{ij}\delta^{kl\}}]_{k_\varsigma k'_\varsigma}$

Cor. 8.11.3. $\Gamma^{abcd}(2)\partial_a\partial_b\partial_c\partial_d\partial_\pi\Delta(x-x')|_{t=t'}$
$=i\sum_{l=0}^{2}(-1)^lC_4^{2l}\Gamma^{\overbrace{ij\cdots}^{4-2l}\overbrace{\pi\cdots\pi}^{2l}}(2)\overbrace{\partial_i\partial_j\cdots}^{4-2l}\nabla^{2l}\delta^3(\vec r-\vec r')$
$=i\{\Gamma^{ijkl}(2)\partial_i\partial_j\partial_k\partial_l\delta^3(\vec r-\vec r')-6\Gamma^{ij\pi\pi}(2)\partial_i\partial_j\nabla^2\delta^3(\vec r-\vec r')+\Gamma^{\pi\pi\pi\pi}(2)\nabla^4\delta^3(\vec r-\vec r')\}$
$=i\{(\frac{1}{\sqrt2})^4\frac23\frac{1}{4!}[\sigma^{\{i}(2)\sigma^j(2)\sigma^k(2)\sigma^{l\}}(2)-4\sigma^{\{i}(2)\sigma^j(2)\delta^{kl\}}+\frac32\delta^{\{ij}\delta^{kl\}}]\partial_i\partial_j\partial_k\partial_l\delta^3(\vec r-\vec r')+6(\frac{1}{\sqrt2})^4\frac13\frac{1}{2!}[\sigma^{\{i}(2)\sigma^{j\}}(2)-\delta^{\{ij\}}]\partial_i\partial_j\nabla^2\delta^3(\vec r-\vec r')+(\frac{1}{\sqrt2})^4\nabla^4\delta^3(\vec r-\vec r')\}$
$=i\{\frac16\{[\sigma(2)\cdot\nabla]^4-4[\sigma(2)\cdot\nabla]^2\nabla^2+\frac32\nabla^4\}\delta^3(\vec r-\vec r')+\frac12\{[\sigma(2)\cdot\nabla]^2\nabla^2-\nabla^4\}\delta^3(\vec r-\vec r')+\frac14\nabla^4\delta^3(\vec r-\vec r')\}$
$=\frac{i}{6}\{[\sigma(2)\cdot\nabla]^4-[\sigma(2)\cdot\nabla]^2\nabla^2\}\delta^3(\vec r-\vec r')$

Cor. 8.11.4.

$$\begin{cases}[\dot\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=-\frac12\Gamma^{abcd}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\partial_c\partial_d\partial_\pi\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases}\Rightarrow\begin{cases}[\frac{\dot\psi_{k_\varsigma}(\vec r,t)}{-\nabla^2},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}]\\ =\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}\delta^3(\vec r-\vec r')\\ [\psi_{k_\varsigma}(\vec r,t),\psi_{l_\varsigma}(\vec r',t)]=0,[\psi^+_{k'_\varsigma}(\vec r,t),\psi^+_{l'_\varsigma}(\vec r',t)]=0\end{cases}$$

Cor. 8.11.5. $\hat P_a(2)=\int\frac{\psi^+(\vec r,t)}{-\nabla^2}\hat P_a\frac{i\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r, M_{ab}(2)=\int\frac{\psi^+(\vec r,t)}{-\nabla^2}\hat M_{ab}\frac{i\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r$

Thm. 8.11.1. $\begin{cases}[L_{ab},L_{cd}]=-i(g_{ad}L_{bc}-g_{ac}L_{bd}+g_{bc}L_{ad}-g_{bd}L_{ac})\\ [L_{ab},P_c]=-i(g_{bc}P_a-g_{ac}P_b),[P_a,P_b]=0\end{cases}$

Proof: $[L_{ab},L_{cd}]$
$=-\int d^3\vec rd^3\vec r'[\frac{\psi^+(\vec r,t)}{-\nabla^2}(r_a\partial_b-r_b\partial_a)\frac{i\dot\psi(\vec r,t)}{-\nabla^2},\frac{\psi^+(\vec r',t)}{-\nabla'^2}(r'_c\partial'_d-r'_d\partial'_c)\frac{i\dot\psi(\vec r',t)}{-\nabla'^2}]$
$=\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec rd^3\vec r'[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}(r'_c\partial'_d-r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}]$
$=\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec rd^3\vec r'$
$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}[(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}](r'_c\partial'_d-r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}$
$+\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2},(r'_c\partial'_d-r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}](r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$
$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec rd^3\vec r'$
$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial_b-r_b\partial_a)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')(r'_c\partial'_d-r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}$

$$- \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}(r'_c\partial'_d - r'_d\partial'_c)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla']^2-[\sigma(2)\cdot\hat\nabla']^4\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r'-\vec r)(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial'_b - r_b\partial'_a)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla']^2-[\sigma(2)\cdot\hat\nabla']^4\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')(r'_c\partial'_d - r'_d\partial'_c)\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}(r'_c\partial_d - r'_d\partial_c)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r-\vec r')(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial_b - r_b\partial_a)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l_\varsigma k'_\varsigma}(r_c\partial_d - r_d\partial_c)\frac{\dot\psi_{l'_\varsigma}(\vec r,t)}{-\nabla^2}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec r,t)}{-\nabla^2}(r_c\partial_d - r_d\partial_c)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l'_\varsigma k_\varsigma}(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= -\int \frac{\psi^+(\vec r,t)}{-\nabla^2}[-i(r_a\partial_b - r_b\partial_a), -i(r_c\partial_d - r_d\partial_c)]\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}\frac{\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r$$
$$= \int \frac{\psi^+(\vec r,t)}{-\nabla^2}[\hat L_{ab},\hat L_{cd}]\frac{i\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r$$
$$= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \qquad \square$$

Proof: $[L_{ab}, P_c]$
$$= -\int d^3\vec r d^3\vec r'[\frac{\psi^+(\vec r,t)}{-\nabla^2}(r_a\partial_b - r_b\partial_a)\frac{i\dot\psi(\vec r,t)}{-\nabla^2}, \frac{\psi^+(\vec r',t)}{-\nabla'^2}\partial'_c\frac{i\dot\psi(\vec r',t)}{-\nabla'^2}]$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}, \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}]$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}[(r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}, \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}]\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2} + \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}, \partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}](r_a\partial_b - r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial_b - r_b\partial_a)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}\partial'_c\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla']^2-[\sigma(2)\cdot\hat\nabla']^4\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r'-\vec r)(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial'_b - r_b\partial'_a)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla']^2-[\sigma(2)\cdot\hat\nabla']^4\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\partial'_c\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}$$
$$- \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}\partial_c\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r-\vec r')(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}(r_a\partial_b - r_b\partial_a)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l_\varsigma k'_\varsigma}\partial_c\frac{\dot\psi_{l'_\varsigma}(\vec r,t)}{-\nabla^2} - \frac{\psi^+_{k'_\varsigma}(\vec r,t)}{-\nabla^2}\partial_c\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l'_\varsigma k_\varsigma}(r_a\partial_b-r_b\partial_a)\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= -\int \frac{\psi^+(\vec r,t)}{-\nabla^2}[-i(r_a\partial_b - r_b\partial_a), -i\partial'_c]\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}\frac{\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r$$
$$= \int \frac{\psi^+(\vec r,t)}{-\nabla^2}[\hat L_{ab},\hat P_c]\frac{i\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r$$
$$= -i(g_{bc}P_a - g_{ac}P_b) \qquad \square$$

Proof: $[P_a, P_b]$
$$= -\int[\frac{\psi^+(\vec r,t)}{-\nabla^2}\partial_a\frac{i\dot\psi(\vec r,t)}{-\nabla^2}, \frac{\psi^+(\vec r',t)}{-\nabla'^2}\partial'_b\frac{i\dot\psi(\vec r',t)}{-\nabla'^2}]d^3\vec r d^3\vec r'$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}\partial_a\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}, \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}]d^3\vec r d^3\vec r'$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}[\partial_a\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}, \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}]\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2} + \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}, \partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2}]\partial_a\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l_\varsigma k'_\varsigma}\partial_a\delta^3(\vec r-\vec r')\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2} - \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla']^2-[\sigma(2)\cdot\hat\nabla']^4\}_{l'_\varsigma k_\varsigma}\partial'_b\delta^3(\vec r'-\vec r)\partial_a\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla']^2-[\sigma(2)\cdot\hat\nabla']^4\}_{l_\varsigma k'_\varsigma}\partial'_a\delta^3(\vec r-\vec r')\partial'_b\frac{\dot\psi_{l'_\varsigma}(\vec r',t)}{-\nabla'^2} - \frac{\psi^+_{k'_\varsigma}(\vec r',t)}{-\nabla'^2}\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}_{l'_\varsigma k_\varsigma}\partial_b\delta^3(\vec r-\vec r')\partial_a\frac{\dot\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}$$
$$= \int\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{-\nabla^2}\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}^{k_\varsigma l'_\varsigma}\partial_a\partial_b\frac{\dot\psi_{l'_\varsigma}(\vec r,t)}{-\nabla^2} - \frac{\psi^+_{k'_\varsigma}(\vec r,t)}{-\nabla^2}\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}^{k'_\varsigma l_\varsigma}\partial_b\partial_a\frac{\psi_{l_\varsigma}(\vec r,t)}{-\nabla^2}\}d^3\vec r$$
$$= \int \frac{\psi^+(\vec r,t)}{-\nabla^2}(\partial_a\partial_b - \partial_b\partial_a)\frac{i}{12}\{[\sigma(2)\cdot\hat\nabla]^2-[\sigma(2)\cdot\hat\nabla]^4\}\frac{\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r$$
$$= \int \frac{\psi^+(\vec r,t)}{-\nabla^2}(\partial_a\partial_b - \partial_b\partial_a)\frac{-i\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r$$
$$= \int \frac{\psi^+(\vec r,t)}{-\nabla^2}[\hat P_a,\hat P_b]\frac{i\dot\psi(\vec r,t)}{-\nabla^2}d^3\vec r = 0 \qquad \square$$

Chapter23 Covariant Quantization Scheme for s-spin Equation

Self comment: In this chapter, I have finally established a corresponding quantum field theory for all massless spin particles in a unified manner. Without knowing the Hamiltonian, various spin particles can be quantized by using a unified new program. Unified quantization commutative rules and energy momentum operator forms have been given. And partial quantum Poincare algebras have been given. However, the angular momentum operator has only achieved partial success and has not been thoroughly resolved. Efforts are still needed.

1 Fourier transform properties of spin wave functions

(No need to satisfy the spin equation.)

1.1 First order correspondent properties between coordinate and momentum space

Pro. 1.1.1. $\begin{cases}\int\psi^+(\vec{r},t)\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)\psi(\vec{p},t)d^3\vec{p}=\int[a_1^+(\vec{p})a_1(\vec{p})+a_2(\vec{p})a_2^+(\vec{p})]d^3\vec{p}\\ \int\psi^+(\vec{r},t)\sigma(s)\psi(\vec{r},t)d^3\vec{r}[=\int\psi^+(\vec{p},t)\sigma(s)\psi(\vec{p},t)d^3\vec{p}]\end{cases}$

Pro. 1.1.2. $\begin{cases}\int\psi^+(\vec{r},t)\hat{\nabla}\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)\hat{p}\psi(\vec{p},t)d^3\vec{p}\\ \int\psi^+(\vec{r},t)\vec{r}\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)(i\tilde{\nabla})\psi(\vec{p},t)d^3\vec{p}\\ \int\psi^+(\vec{r},t)[\sigma(s)\cdot\hat{\nabla}]\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)[\sigma(s)\cdot\hat{p}]\psi(\vec{p},t)d^3\vec{p}\end{cases}$

Pro. 1.1.3. $\begin{cases}\int\psi^+(\vec{r},t)[\sigma(s)\cdot\hat{\nabla}]\hat{\nabla}\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)[\sigma(s)\cdot\hat{p}]\hat{p}\psi(\vec{p},t)d^3\vec{p}\\ \int\psi^+(\vec{r},t)\sigma(s)[\sigma(s)\cdot\hat{\nabla}]\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)\sigma(s)[\sigma(s)\cdot\hat{p}]\psi(\vec{p},t)d^3\vec{p}\\ \int\psi^+(\vec{r},t)[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)[\sigma(s)\cdot\hat{p}]\sigma(s)\psi(\vec{p},t)d^3\vec{p}\end{cases}$

Pro. 1.1.4. $\begin{cases}\int\psi^+(\vec{r},t)r_i\sigma_j(s)\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)\sigma_j(s)(i\tilde{\partial}_i)\psi(\vec{p},t)d^3\vec{p}\\ \int\psi^+(\vec{r},t)\sigma_i(s)\partial_j\psi(\vec{r},t)d^3\vec{r}=i\int\psi^+(\vec{p},t)\sigma_i(s)p_j\psi(\vec{p},t)d^3\vec{p}\end{cases}$

Pro. 1.1.5. $\begin{cases}\int\psi^+(\vec{r},t)[r_i\sigma_j(s)-r_j\sigma_i(s)]\psi(\vec{r},t)d^3\vec{r}=-i\int\psi^+(\vec{p},t)[\sigma_i(s)\tilde{\partial}_j-\sigma_j(s)\tilde{\partial}_i]\psi(\vec{p},t)d^3\vec{p}\\ \int\psi^+(\vec{r},t)[\sigma_i(s)\partial_j-\sigma_j(s)\partial_i]\psi(\vec{r},t)d^3\vec{r}=i\int\psi^+(\vec{p},t)[\sigma_i(s)p_j-\sigma_j(s)p_i]\psi(\vec{p},t)d^3\vec{p}\end{cases}$

1.2 Second order correspondent properties between coordinate and momentum space

Pro. 1.2.1. $\begin{cases}\int\psi^+(\vec{r},t)r_i\partial_j\psi(\vec{r},t)d^3\vec{r}=\int d^3\vec{p}\psi^+(\vec{p},t)(-\delta_{ij}-p_j\tilde{\partial}_i)\psi(\vec{p},t)\\ \int\psi^+(\vec{r},t)(\delta_{ij}+r_i\partial_j)\psi(\vec{r},t)d^3\vec{r}=\int d^3\vec{p}\psi^+(\vec{p},t)(-p_j\tilde{\partial}_i)\psi(\vec{p},t)\\ \int\psi^+(\vec{r},t)(r_i\partial_j-r_j\partial_i)\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{p},t)(p_i\tilde{\partial}_j-p_j\tilde{\partial}_i)\psi(\vec{p},t)d^3\vec{p}\end{cases}$

Proof: $\int\psi^+(\vec{r},t)r_i\partial_j\psi(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p},t)r_i\partial_je^{i\vec{p}\cdot\vec{r}}$
$=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p},t)ip_j(-i\tilde{\partial}_i)e^{i\vec{p}\cdot\vec{r}}$
$=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p},t)p_j\tilde{\partial}_ie^{i\vec{p}\cdot\vec{r}}$
$=-\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)\tilde{\partial}_i[\psi(\vec{p},t)p_j]e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}$
$=-\int d^3\vec{p}d^3\vec{p}'\psi^+(\vec{p}',t)\tilde{\partial}_i[\psi(\vec{p},t)p_j]\delta^3(\vec{p}-\vec{p}')$
$=-\int d^3\vec{p}\psi^+(\vec{p},t)\tilde{\partial}_i[\psi(\vec{p},t)p_j]$
$=\int\psi^+(\vec{p},t)(-\delta_{ij}-p_j\tilde{\partial}_i)\psi(\vec{p},t)d^3\vec{p}$ □

Pro. 1.2.2.
$\begin{cases}\int\psi^+(\vec{r},t)r_i\partial_j[\sigma(s)\cdot\nabla]\psi(\vec{r},t)d^3\vec{r}=-i\{\int\psi^+(\vec{p},t)\delta_{ij}[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)d^3\vec{p}+\int\psi^+(\vec{p},t)p_j\tilde{\partial}_i\{[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)\}d^3\vec{p}\}\\ =-i\{\int\psi^+(\vec{p},t)\delta_{ij}[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)d^3\vec{p}+\int\psi^+(\vec{p},t)[\sigma(s)\cdot\vec{p}]p_j\tilde{\partial}_i\psi(\vec{p},t)d^3\vec{p}+\int\psi^+(\vec{p},t)p_j\sigma_i(s)\psi(\vec{p},t)\}d^3\vec{p}\}\end{cases}$

Proof: $\int\psi^+(\vec{r},t)r_i\partial_j[\sigma(s)\cdot\nabla]\psi(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p},t)r_i\partial_j[\sigma(s)\cdot\nabla]e^{i\vec{p}\cdot\vec{r}}$
$=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p},t)ip_j[i\sigma(s)\cdot\vec{p}](-i\tilde{\partial}_i)e^{i\vec{p}\cdot\vec{r}}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p},t)p_j[i\sigma(s)\cdot\vec{p}]\tilde{\partial}_i e^{i\vec{p}\cdot\vec{r}}$
$= -\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}',t)\tilde{\partial}_i\{\psi(\vec{p},t)p_j[i\sigma(s)\cdot\vec{p}]\}e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}$
$= -\int d^3\vec{p}d^3\vec{p}'\psi^+(\vec{p}',t)\tilde{\partial}_i\{\psi(\vec{p},t)p_j[i\sigma(s)\cdot\vec{p}]\}\delta^3(\vec{p}-\vec{p}')$
$= -\int d^3\vec{p}\psi^+(\vec{p},t)\tilde{\partial}_i\{\psi(\vec{p},t)p_j[i\sigma(s)\cdot\vec{p}]\}$
$= \int \psi^+(\vec{p},t)(-\delta_{ij}-p_j\tilde{\partial}_i)\{[i\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)\}d^3\vec{p}$
$= -i\{\int \psi^+(\vec{p},t)\delta_{ij}[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)d^3\vec{p}+\int \psi^+(\vec{p},t)[\sigma(s)\cdot\vec{p}]p_j\tilde{\partial}_i\psi(\vec{p},t)d^3\vec{p}+\int \psi^+(\vec{p},t)p_j\sigma_i(s)\psi(\vec{p},t)\}d^3\vec{p}\}$ □

Pro. 1.2.3.
$$\begin{cases}\int \psi^+(\vec{r},t)(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]\psi(\vec{r},t)d^3\vec{r}=i\int\psi^+(\vec{p},t)(p_i\tilde{\partial}_j-p_j\tilde{\partial}_i)\{[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)\}d^3\vec{p}\\ =i\{\int\psi^+(\vec{p},t)[\sigma(s)\cdot\vec{p}](p_i\tilde{\partial}_j-p_j\tilde{\partial}_i)\psi(\vec{p},t)d^3\vec{p}+\int\psi^+(\vec{p},t)[p_i\sigma_j(s)-p_j\sigma_i(s)]\psi(\vec{p},t)\}d^3\vec{p}\}\end{cases}$$

Pro. 1.2.4.
$\int \psi^+(\vec{r},t)[\sigma(s)\cdot\nabla]\{r_i\partial_j\psi(\vec{r},t)\}d^3\vec{r}=-i\{\int\psi^+(\vec{p},t)\delta_{ij}[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)d^3\vec{p}+\int\psi^+(\vec{p},t)[\sigma(s)\cdot\vec{p}]p_j\tilde{\partial}_i\psi(\vec{p},t)d^3\vec{p}\}$

Proof:
$\int \psi^+(\vec{r},t)[\sigma(s)\cdot\nabla]\{r_i\partial_j\psi(\vec{r},t)\}d^3\vec{r}$
$= \int \psi^+(\vec{r},t)\sigma_i(s)\partial_j\psi(\vec{r},t)d^3\vec{r}+\int\psi^+(\vec{r},t)r_i\partial_j[\sigma(s)\cdot\nabla]\psi(\vec{r},t)d^3\vec{r}$
$= i\int\psi^+(\vec{p},t)\sigma_i(s)p_j\psi(\vec{p},t)d^3\vec{p}-i\{\int\psi^+(\vec{p},t)\delta_{ij}[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)d^3\vec{p}+\int\psi^+(\vec{p},t)p_j\tilde{\partial}_i\{[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)\}d^3\vec{p}\}$
$= -i\{\int\psi^+(\vec{p},t)\delta_{ij}[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t)d^3\vec{p}+\int\psi^+(\vec{p},t)[\sigma(s)\cdot\vec{p}]p_j\tilde{\partial}_i\psi(\vec{p},t)d^3\vec{p}\}$ □

Pro. 1.2.5. $\int\psi^+(\vec{r},t)[\sigma(s)\cdot\nabla][(r_i\partial_j-r_j\partial_i)\psi(\vec{r},t)]d^3\vec{r}=i\int\psi^+(\vec{p},t)[\sigma(s)\cdot\vec{p}](p_i\tilde{\partial}_j-p_j\tilde{\partial}_i)\psi(\vec{p},t)d^3\vec{p}$

1.3 Higher order correspondent properties between coordinate and momentum space

2 Spin equation in coordinate space

2.1 s-spin equation and its plane wave solution

Thm. 2.1.1. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(x)=0$

Cor. 2.1.1.
$$\begin{cases}\psi(\vec{r},t):=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\ |\vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\frac{i}{|\vec{p}|}\lambda^+(\hat{p},-s\varsigma)\dot{\psi}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ |\vec{p}|^{(s-\frac{1}{2})}a_2^+(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{ip\cdot x}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\frac{-i}{|\vec{p}|}\lambda^+(\hat{p},-s\varsigma)\dot{\psi}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Def. 2.1.1. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(s,\varsigma):=\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma),\hat{P}^2(s,\varsigma)=\hat{P}(s,\varsigma),\hat{P}^+(s,\varsigma)=\hat{P}(s,\varsigma)$

Def. 2.1.2. $A(\vec{r},t):=\frac{\partial_t}{\nabla^2}\psi(\vec{r},t)\Leftrightarrow\psi(\vec{r},t)=\partial_tA(\vec{r},t)$

2.2 Plane wave solutions of spin equation in momentum space

Cor. 2.2.1. $\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}\psi(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p},\psi(\vec{p},t)=|\vec{p}|^{(s-\frac{1}{2})}[a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t)}+a_2^+(-\vec{p},-s\varsigma)\lambda(-\hat{p},-s\varsigma)e^{i|\vec{p}|t)}]$

Proof: $\psi(\vec{r},t):=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma)e^{-ip\cdot x}]d^3\vec{p}$

$\Leftrightarrow\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+a_2^+(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]d^3\vec{p}$

$\Leftrightarrow\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{-i|\vec{p}|t)}e^{i\vec{p}\cdot\vec{r}}+a_2^+(\vec{p},-s\varsigma)e^{i|\vec{p}|t)}e^{-i\vec{p}\cdot\vec{r}}]d^3\vec{p}$

$\Leftrightarrow\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}[a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t)}+a_2^+(-\vec{p},-s\varsigma)\lambda(-\hat{p},-s\varsigma)e^{i|\vec{p}|t)}]e^{i\vec{p}\cdot\vec{r}}d^3\vec{p}$

$\Leftrightarrow\psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}\psi(\vec{p},t)e^{i\vec{p}\cdot\vec{r}}d^3\vec{p},\psi(\vec{p},t)=|\vec{p}|^{(s-\frac{1}{2})}[a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t)}+a_2^+(-\vec{p},-s\varsigma)\lambda(-\hat{p},-s\varsigma)e^{i|\vec{p}|t)}]$
$\Leftrightarrow\psi(\vec{p},t)=\frac{1}{(2\pi)^{3/2}}\int\psi(\vec{r},t)e^{-i\vec{p}\cdot\vec{r}}d^3\vec{p}$ □

2.3 Several important lemmas

Lem. 2.3.1. $s\nabla\psi(\vec{r},t)=[i\sigma(s)\times\nabla+\varsigma\sigma(s)\partial_t]\psi(\vec{r},t)\Rightarrow[\frac{1}{s}\sigma(s)\cdot\nabla]\psi(\vec{r},t)=\varsigma\partial_t\psi(\vec{r},t)$

Proof: $s\nabla\psi(\vec{r},t)=[i\sigma(s)\times\nabla+\varsigma\sigma(s)\partial_t]\psi(\vec{r},t)$
$\Rightarrow s\sigma(s)\cdot\nabla\psi(\vec{r},t)=\sigma(s)\cdot[i\sigma(s)\times\nabla+\varsigma\sigma(s)\partial_t]\psi(\vec{r},t)$
$\Leftrightarrow s[\sigma(s)\cdot\nabla]\psi(\vec{r},t)=-[\sigma(s)\cdot\nabla]\psi(\vec{r},t)+\varsigma\sigma^2(s)\partial_t\psi(\vec{r},t)$
$\Leftrightarrow(s+1)[\sigma(s)\cdot\nabla]\psi(\vec{r},t)=\varsigma s(s+1)\partial_t\psi(\vec{r},t)$
$\Leftrightarrow[\frac{1}{s}\sigma(s)\cdot\nabla]\psi(\vec{r},t)=\varsigma\partial_t\psi(\vec{r},t)$ □

Lem. 2.3.2. $s\nabla\psi(\vec{r},t) = [\sigma(s)\cdot\nabla - \varsigma(s-1)\partial_t]\sigma(s)\psi(\vec{r},t) \overset{s\neq 1}{\Rightarrow} [\frac{1}{s}\sigma(s)\cdot\nabla]\psi(\vec{r},t) = \varsigma\partial_t\psi(\vec{r},t)$

Proof: $s\nabla\psi(\vec{r},t) = [\sigma(s)\cdot\nabla - \varsigma(s-1)\partial_t]\sigma(s)\psi(\vec{r},t)$
$\Rightarrow s\sigma(s)\cdot\nabla\psi(\vec{r},t) = \sigma(s)\cdot[\sigma(s)\cdot\nabla - \varsigma(s-1)\partial_t]\sigma(s)\psi(\vec{r},t)$
$\Leftrightarrow s[\sigma(s)\cdot\nabla]\psi(\vec{r},t) = \sigma(s)\cdot[\sigma(s)\cdot\nabla]\sigma(s)\psi(\vec{r},t) - \varsigma(s-1)\sigma^2(s)\partial_t\psi(\vec{r},t)$
$\Leftrightarrow s[\sigma(s)\cdot\nabla]\psi(\vec{r},t) = [\sigma^2(s)-1][\sigma(s)\cdot\nabla]\psi(\vec{r},t) - \varsigma(s-1)\sigma^2(s)\partial_t\psi(\vec{r},t)$
$\Leftrightarrow (s+1)[\sigma(s)\cdot\nabla]\psi(\vec{r},t) = \sigma^2(s)[\sigma(s)\cdot\nabla]\psi(\vec{r},t) - \varsigma(s-1)\sigma^2(s)\partial_t\psi(\vec{r},t)$
$\Leftrightarrow (s-1)[\sigma(s)\cdot\nabla]\psi(\vec{r},t) = \varsigma(s-1)s\partial_t\psi(\vec{r},t)$
$\overset{s\neq 1}{\Leftarrow\Rightarrow} [\frac{1}{s}\sigma(s)\cdot\nabla]\psi(\vec{r},t) = \varsigma\partial_t\psi(\vec{r},t)$ □

Cor. 2.3.1.
$$\begin{cases} s\nabla\psi(\vec{r},t) = \{[\sigma(s)\cdot\nabla,\sigma(s)] + \varsigma\sigma(s)\partial_t\}\psi(\vec{r},t) \overset{s\neq 1}{\Leftrightarrow} s\nabla\psi(\vec{r},t) = [\sigma(s)\cdot\nabla - \varsigma(s-1)\partial_t]\sigma(s)\psi(\vec{r},t) \\ s\nabla\psi(\vec{r},t) = \{[\sigma(s)\cdot\nabla,\sigma(s)] + \varsigma\sigma(s)\partial_t\}\psi(\vec{r},t) \overset{s=1}{\Rightarrow} s\nabla\psi(\vec{r},t) = [\sigma(s)\cdot\nabla]\sigma(s)\psi(\vec{r},t) \end{cases}$$

Cor. 2.3.2. $\nabla\psi(\vec{r},t) = \{[\sigma(s)\cdot\nabla,\sigma(s)] + \varsigma\sigma(s)\partial_t\}\psi(\vec{r},t) \overset{s=1}{\Rightarrow} \begin{cases} \nabla\psi(\vec{r},t) = [\sigma(s)\cdot\nabla]\sigma(s)\psi(\vec{r},t) \\ [\sigma(s)\cdot\nabla]\psi(\vec{r},t) = \varsigma\partial_t\psi(\vec{r},t) \end{cases}$

Cor. 2.3.3. $[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = [\sigma(s)\cdot\hat{\nabla}]^{n-1}\hat{\nabla}\psi(\vec{r},t), s=1$

Lem. 2.3.3.
$$\begin{cases} s^2\nabla\psi(\vec{r},t) = \{is\sigma(s)\times\nabla + \sigma(s)[\sigma(s)\cdot\nabla]\}\psi(\vec{r},t) \Rightarrow \nabla^2\psi(\vec{r},t) = [\frac{1}{s}\sigma(s)\cdot\nabla]^2\psi(\vec{r},t) \\ \Updownarrow \\ s^2\nabla\psi(\vec{r},t) = \{s\sigma(s)\cdot\nabla\sigma(s) - (s-1)\sigma(s)[\sigma(s)\cdot\nabla]\}\psi(\vec{r},t) \Rightarrow \nabla^2\psi(\vec{r},t) = [\frac{1}{s}\sigma(s)\cdot\nabla]^2\psi(\vec{r},t) \end{cases}$$

2.4 Several equivalent forms of s-spin equation(Proof is omitted.)

Thm. 2.4.1.
$$\begin{array}{ccc} [s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(x) = 0 & [\Leftrightarrow] & \sigma(s)\cdot\nabla\psi = s\varsigma\partial_t\psi, O(s)\cdot\nabla\psi(x) = 0 \\ [\Updownarrow] & & [\Updownarrow] \\ s\nabla\psi(x) = [i\sigma(s)\times\nabla + \varsigma\sigma(s)\partial_t]\psi(x) & [\Leftrightarrow] & \begin{cases} \sigma(s)\cdot\nabla\psi(x) = s\varsigma\partial_t\psi(x) \\ s^2\nabla\psi(x) = is\sigma(s)\times\nabla + \sigma(s)[\sigma(s)\cdot\nabla]\psi(x) \end{cases} \\ [\Updownarrow] & & [\Updownarrow] \\ s\nabla\psi(x) = \{[\sigma(s)\cdot\nabla,\sigma(s)] + \varsigma\sigma(s)\partial_t\}\psi(x) & [\Leftrightarrow] & \begin{cases} \sigma(s)\cdot\nabla\psi(x) = s\varsigma\partial_t\psi(x) \\ s^2\nabla\psi(x) = \{s\sigma(s)\cdot\nabla\sigma(s) - (s-1)\sigma(s)[\sigma(s)\cdot\nabla]\}\psi(x) \end{cases} \end{array}$$

2.5 Several equivalent forms of constraint equations

Cor. 2.5.1. $O(1) = \frac{1}{\sqrt{2}}\{\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}, i\begin{bmatrix} -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2} & 0 \end{bmatrix}\}$

Cor. 2.5.2. $O(1)S_m^+(1) = \{\begin{bmatrix} i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i \end{bmatrix}\}$

Cor. 2.5.3. $S_m(1) = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}, S_m^+(1) = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}, S_m(1)S_m^+(1) = S_m^+(1)S_m(1) = I_3$

Cor. 2.5.4. $[\gamma\cdot\nabla]\gamma\Psi = \nabla\Psi[\Leftrightarrow]O(1)S_m^+(1)\cdot\nabla\Psi = 0[\Leftrightarrow]\nabla\cdot\Psi = 0$

Cor. 2.5.5. $O_x(s) = -\sqrt{s(s-\frac{1}{2})}[\bar{N}_{1_\varsigma}(s-\frac{1}{2})\bar{N}_{1_\varsigma}(s) - \bar{N}_{2_\varsigma}(s-\frac{1}{2})\bar{N}_{2_\varsigma}(s)]$
$$= \frac{1}{2}\begin{bmatrix} -\sqrt{2s\cdot(2s-1)} & 0 & \sqrt{2\cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{(2s-1)\cdot(2s-2)} & 0 & \sqrt{3\cdot 2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\sqrt{2\cdot 1} & 0 & \sqrt{2s\cdot(2s-1)} \end{bmatrix}$$

Cor. 2.5.6. $O_y(s) = -i\sqrt{s(s-\frac{1}{2})}[\bar{N}_{1_\varsigma}(s-\frac{1}{2})\bar{N}_{1_\varsigma}(s) + \bar{N}_{2_\varsigma}(s-\frac{1}{2})\bar{N}_{2_\varsigma}(s)]$
$$= \frac{i}{2}\begin{bmatrix} -\sqrt{2s\cdot(2s-1)} & 0 & -\sqrt{2\cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{(2s-1)\cdot(2s-2)} & 0 & -\sqrt{3\cdot 2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\sqrt{2\cdot 1} & 0 & -\sqrt{2s\cdot(2s-1)} \end{bmatrix}$$

Cor. 2.5.7. $O_z(s) = \sqrt{s(s-\frac{1}{2})}[\bar{N}_{1_\varsigma}(s-\frac{1}{2})\bar{N}_{2_\varsigma}(s) + \bar{N}_{2_\varsigma}(s-\frac{1}{2})\bar{N}_{1_\varsigma}(s)]$
$$= \begin{bmatrix} 0 & \sqrt{1\cdot(2s-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2\cdot(2s-2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{(2s-1)\cdot 1} & 0 \end{bmatrix}, \bar{N}_{1_\varsigma}(s-\frac{1}{2})\bar{N}_{2_\varsigma}(s) = \bar{N}_{2_\varsigma}(s-\frac{1}{2})\bar{N}_{1_\varsigma}(s)$$

$$\sigma(s)=(\frac{1}{2}\begin{bmatrix}0&A_1&0&0&0\\A_1&0&A_2&0&0\\0&A_2&0&\cdots&0\\0&0&\cdots&0&A_{2s}\\0&0&0&A_{2s}&0\end{bmatrix},\frac{i}{2}\begin{bmatrix}0&-A_1&0&0&0\\A_1&0&-A_2&0&0\\0&A_2&0&\cdots&0\\0&0&\cdots&0&-A_{2s}\\0&0&0&A_{2s}&0\end{bmatrix},\begin{bmatrix}s&0&0&0&0\\0&s-1&0&0&0\\0&0&\cdots&0&0\\0&0&0&-(s-1)&0\\0&0&0&0&-s\end{bmatrix}) \tag{23.1a}$$

$$A_n=\sqrt{n}\cdot\sqrt{2s+1-n},n=1,2,\cdots,2s;\sigma(s)\prec\sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(s)\simeq\sigma_{\alpha'_\varsigma}{}^{k'_\varsigma}{}_{l'_\varsigma}(s) \tag{23.1b}$$

$$\sigma(s)\times\sigma(s)=i\sigma(s),\sigma^2(s)=s(s+1),\sigma^+(s)=\sigma(s),s=\frac{1}{2},1,\frac{3}{2},2,\cdots \tag{23.1c}$$

2.6 Important corollaries of constraint equations

Cor. 2.6.1. $O(s)\cdot\nabla\psi=0\Leftrightarrow$

$$\frac{1}{2}\begin{bmatrix}-\sqrt{2s\cdot(2s-1)}(\partial_x+i\partial_y)&2\sqrt{1\cdot(2s-1)}\partial_z&\sqrt{2\cdot 1}(\partial_x-i\partial_y)&0&0&0\\0&-\sqrt{(2s-1)\cdot(2s-2)}(\partial_x+i\partial_y)&2\sqrt{2\cdot(2s-2)}\partial_z&\sqrt{3\cdot 2}(\partial_x-i\partial_y)&0&0\\0&0&\cdots&0&\cdots&0\\0&0&0&-\sqrt{2\cdot 1}(\partial_x+i\partial_y)&2\sqrt{(2s-1)\cdot 1}\partial_z&\sqrt{2s\cdot(2s-1)}(\partial_x-i\partial_y)\end{bmatrix}\psi=0$$

Cor. 2.6.2. $\{s^2\hat{\nabla}+s[\sigma(s),\sigma(s)\cdot\hat{\nabla}]-\sigma(s)[\sigma(s)\cdot\hat{\nabla}]\}\psi=0\Leftrightarrow O(s)\cdot\nabla\psi=0$

Proof: $\{s^2\partial_y+s[\sigma_y(s),\sigma(s)\cdot\hat{\nabla}]-\sigma_y(s)[\sigma(s)\cdot\hat{\nabla}]\}\psi=0$
$\Leftrightarrow\{s^2\partial_y+is[\sigma_x(s)\partial_z-\sigma_z(s)\partial_x]-[\sigma_y^2(s)\partial_y+\sigma_y(s)\sigma_x(s)\partial_x+\sigma_y(s)\sigma_z(s)\partial_z]\}\psi=0$
$\Leftrightarrow\{[s^2-\sigma_y^2(s)]\partial_y+[is\sigma_x(s)-\sigma_y(s)\sigma_z(s)]\partial_z-[is\sigma_z(s)+\sigma_y(s)\sigma_x(s)]\partial_x\}\psi=0$

$$\Leftrightarrow\{[s^2-\frac{1}{4}\begin{bmatrix}A_1^2&0&-A_1A_2&0&0&0&0\\0&A_2^2+A_1^2&0&-A_2A_3&0&0&0\\-A_1A_2&0&A_3^2+A_2^2&0&\cdots&0&0\\0&-A_2A_3&0&\cdots&0&-A_{2s-1}A_{2s}&0\\0&0&\cdots&0&A_{2s-1}^2+A_{2s-2}^2&0&-A_{2s-1}A_{2s}\\0&0&0&-A_{2s-2}A_{2s-1}&0&A_{2s}^2+A_{2s-1}^2&0\\0&0&0&0&-A_{2s-1}A_{2s}&0&A_{2s}^2\end{bmatrix}]\partial_y$$

$$+[is\frac{1}{2}\begin{bmatrix}0&A_1&0&0&0\\A_1&0&A_2&0&0\\0&A_2&0&\cdots&0\\0&0&\cdots&0&A_{2s}\\0&0&0&A_{2s}&0\end{bmatrix}-\frac{i}{2}\begin{bmatrix}0&-(s-1)A_1&0&0&0\\sA_1&0&-(s-2)A_2&0&0\\0&(s-1)A_2&0&\cdots&0\\0&0&\cdots&0&sA_{2s}\\0&0&0&-(s-1)A_{2s}&0\end{bmatrix}]\partial_z$$

$$-[is\begin{bmatrix}s&0&0&0&0\\0&s-1&0&0&0\\0&0&\cdots&0&0\\0&0&0&-(s-1)&0\\0&0&0&0&-s\end{bmatrix}+\frac{i}{4}\begin{bmatrix}-A_1^2&0&-A_1A_2&0&0&0&0\\0&A_1^2-A_2^2&0&-A_2A_3&0&0&0\\A_1A_2&0&A_2^2-A_3^2&0&\cdots&0&0\\0&A_2A_3&0&\cdots&0&-A_{2s-2}A_{2s-1}&0\\0&0&\cdots&0&A_{2s-2}^2-A_{2s-1}^2&0&-A_{2s-1}A_{2s}\\0&0&0&A_{2s-2}A_{2s-1}&0&A_{2s-1}^2-A_{2s}^2&0\\0&0&0&0&A_{2s-1}A_{2s}&0&A_{2s}^2\end{bmatrix}]\partial_x\}\psi=0$$

$$\Leftrightarrow\{[s^2-\frac{1}{4}\begin{bmatrix}A_1^2&0&-A_1A_2&0&0&0&0\\0&A_2^2+A_1^2&0&-A_2A_3&0&0&0\\-A_1A_2&0&A_3^2+A_2^2&0&\cdots&0&0\\0&-A_2A_3&0&\cdots&0&-A_{2s-1}A_{2s}&0\\0&0&\cdots&0&A_{2s-1}^2+A_{2s-2}^2&0&-A_{2s-1}A_{2s}\\0&0&0&-A_{2s-2}A_{2s-1}&0&A_{2s}^2+A_{2s-1}^2&0\\0&0&0&0&-A_{2s-1}A_{2s}&0&A_{2s}^2\end{bmatrix}]\partial_y$$

$$-[is\begin{bmatrix}s&0&0&0&0\\0&s-1&0&0&0\\0&0&\cdots&0&0\\0&0&0&-(s-1)&0\\0&0&0&0&-s\end{bmatrix}+\frac{i}{4}\begin{bmatrix}-A_1^2&0&-A_1A_2&0&0&0&0\\0&A_1^2-A_2^2&0&-A_2A_3&0&0&0\\A_1A_2&0&A_2^2-A_3^2&0&\cdots&0&0\\0&A_2A_3&0&\cdots&0&-A_{2s-2}A_{2s-1}&0\\0&0&\cdots&0&A_{2s-2}^2-A_{2s-1}^2&0&-A_{2s-1}A_{2s}\\0&0&0&A_{2s-2}A_{2s-1}&0&A_{2s-1}^2-A_{2s}^2&0\\0&0&0&0&A_{2s-1}A_{2s}&0&A_{2s}^2\end{bmatrix}]\partial_x$$

$$-\frac{i}{2}\begin{bmatrix}0&-(2s-1)A_1&0&0&0\\0A_1&0&-(2s-2)A_2&0&0\\0&-1A_2&0&\cdots&0\\0&0&\cdots&0&0A_{2s}\\0&0&0&-(2s-1)A_{2s}&0\end{bmatrix}\partial_z\}\psi=0$$ □

Proof: $\{s^2\partial_x+s[\sigma_x(s),\sigma(s)\cdot\hat{\nabla}]-\sigma_x(s)[\sigma(s)\cdot\hat{\nabla}]\}\psi=0$
$\Leftrightarrow\{s^2\partial_x+is[\sigma_z(s)\partial_y-\sigma_y(s)\partial_z]-[\sigma_x^2(s)\partial_x+\sigma_x(s)\sigma_y(s)\partial_y+\sigma_x(s)\sigma_z(s)\partial_z]\}\psi=0$
$\Leftrightarrow\{[s^2-\sigma_x^2(s)]\partial_x+[is\sigma_z(s)-\sigma_x(s)\sigma_y(s)]\partial_y-[is\sigma_y(s)+\sigma_x(s)\sigma_z(s)]\partial_z\}\psi=0$

$$\Leftrightarrow\{[s^2-\frac{1}{4}\begin{bmatrix}A_1^2&0&A_1A_2&0&0&0&0\\0&A_2^2+A_1^2&0&A_2A_3&0&0&0\\A_1A_2&0&A_3^2+A_2^2&0&\cdots&0&0\\0&A_2A_3&0&\cdots&0&A_{2s-2}A_{2s-1}&0\\0&0&\cdots&0&A_{2s-1}^2+A_{2s-2}^2&0&A_{2s-1}A_{2s}\\0&0&0&A_{2s-2}A_{2s-1}&0&A_{2s}^2+A_{2s-1}^2&0\\0&0&0&0&A_{2s-1}A_{2s}&0&A_{2s}^2\end{bmatrix}]\partial_x$$

$$+[is\begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} - \frac{i}{4}\begin{bmatrix} A_1^2 & 0 & -A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2-A_1^2 & 0 & -A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & A_3^2-A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-2}A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2-A_{2s-2}^2 & 0 & -A_{2s-1}A_{2s} \\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s}^2-A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & -A_{2s}^2 \end{bmatrix}]\partial_y$$

$$-[is\frac{i}{2}\begin{bmatrix} 0 & -A_1 & 0 & 0 & 0 \\ A_1 & 0 & -A_2 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & (s-1)A_1 & 0 & 0 & 0 \\ sA_1 & 0 & (s-2)A_2 & 0 & 0 \\ 0 & (s-1)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -sA_{2s} \\ 0 & 0 & 0 & -(s-1)A_{2s} & 0 \end{bmatrix}]\partial_z\}\psi = 0$$

$$\Leftrightarrow \{[s^2 - \frac{1}{4}\begin{bmatrix} A_1^2 & 0 & A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2+A_1^2 & 0 & A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & A_3^2+A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & A_{2s-2}A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2+A_{2s-2}^2 & 0 & A_{2s-1}A_{2s} \\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s}^2+A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & A_{2s}^2 \end{bmatrix}]\partial_x$$

$$+[is\begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} - \frac{i}{4}\begin{bmatrix} A_1^2 & 0 & -A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2-A_1^2 & 0 & -A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & A_3^2-A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-2}A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2-A_{2s-2}^2 & 0 & -A_{2s-1}A_{2s} \\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s}^2-A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & -A_{2s}^2 \end{bmatrix}]\partial_y$$

$$-\frac{1}{2}\begin{bmatrix} 0 & (2s-1)A_1 & 0 & 0 & 0 \\ 0A_1 & 0 & (2s-2)A_2 & 0 & 0 \\ 0 & -1A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0A_{2s} \\ 0 & 0 & 0 & -(2s-1)A_{2s} & 0 \end{bmatrix}\partial_z\}\psi = 0 \qquad \square$$

Cor. 2.6.3. $\begin{bmatrix} -(2s-1)A_1(\partial_x+i\partial_y) & 1(2s-1)2 & 1A_2(\partial_x-i\partial_y) & 0 & 0 \\ 0 & -(2s-2)A_2(\partial_x+i\partial_y) & 2(2s-2)2 & \cdots & 0 \\ 0 & 0 & \cdots & 1(2s-1)2, & (2s-1)A_{2s}(\partial_x-i\partial_y) \end{bmatrix}\psi = 0$

Proof: $\{s^2\partial_z + s[\sigma_z(s), \sigma(s)\cdot\hat{\nabla}] - \sigma_z(s)[\sigma(s)\cdot\hat{\nabla}]\}\psi = 0, A_n = \sqrt{n}\cdot\sqrt{2s+1-n}, n = 1, 2, \cdots, 2s;$
$\Leftrightarrow \{s^2\partial_z + is[\sigma_y(s)\partial_x - \sigma_x(s)\partial_y] - [\sigma_z^2(s)\partial_z + \sigma_z(s)\sigma_x(s)\partial_x + \sigma_z(s)\sigma_y(s)\partial_y]\}\psi = 0$
$\Leftrightarrow \{[s^2 - \sigma_z^2(s)]\partial_z + [is\sigma_y(s) - \sigma_z(s)\sigma_x(s)]\partial_x - [is\sigma_x(s) + \sigma_z(s)\sigma_y(s)]\partial_y\}\psi = 0$
$\Leftrightarrow \{[s^2 - \sigma_z^2(s)]\partial_z$

$$+[-\frac{1}{2}\begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0 \\ sA_1 & 0 & -sA_2 & 0 & 0 \\ 0 & sA_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -sA_{2s} \\ 0 & 0 & 0 & sA_{2s} & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & (s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix}]\partial_x$$

$$-[\frac{i}{2}\begin{bmatrix} 0 & sA_1 & 0 & 0 & 0 \\ sA_1 & 0 & sA_2 & 0 & 0 \\ 0 & sA_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & sA_{2s} \\ 0 & 0 & 0 & sA_{2s} & 0 \end{bmatrix} + \frac{i}{2}\begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & -(s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix}]\partial_y\}\psi = 0$$

$$\Leftrightarrow \{[s^2-\begin{bmatrix} s^2 & 0 & 0 & 0 & 0 \\ 0 & (s-1)^2 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (s-1)^2 & 0 \\ 0 & 0 & 0 & 0 & s^2 \end{bmatrix}]\partial_z-\frac{1}{2}\begin{bmatrix} 0 & -0A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & -1A_2 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(2s-1)A_{2s} \\ 0 & 0 & 0 & 0A_{2s} & 0 \end{bmatrix}\partial_x-\frac{i}{2}\begin{bmatrix} 0 & 0A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & 1A_2 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (2s-1)A_{2s} \\ 0 & 0 & 0 & 0A_{2s} & 0 \end{bmatrix}\partial_y\}$$
0

$$\Leftrightarrow \{\begin{bmatrix} 0(2s) & 0 & 0 & 0 & 0 \\ 0 & 1(2s-1) & 0 & 0 & 0 \\ 0 & 0 & 2(2s-2) & 0 & 0 \\ 0 & 0 & 0 & 1(2s-1) & 0 \\ 0 & 0 & 0 & 0 & 0(2s) \end{bmatrix}\partial_z-\frac{1}{2}\begin{bmatrix} 0 & -0A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & -1A_2 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(2s-1)A_{2s} \\ 0 & 0 & 0 & 0A_{2s} & 0 \end{bmatrix}\partial_x-\frac{i}{2}\begin{bmatrix} 0 & 0A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & 1A_2 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (2s-1)A_{2s} \\ 0 & 0 & 0 & 0A_{2s} & 0 \end{bmatrix}$$
0

$$\Leftrightarrow \begin{bmatrix} 0(2s)2 & 0A_1(\partial_x-i\partial_y) & 0 & 0 & 0 \\ -(2s-1)A_1(\partial_x+i\partial_y) & 1(2s-1)2 & 1A_2(\partial_x-i\partial_y) & 0 & 0 \\ 0 & -(2s-2)A_2(\partial_x+i\partial_y) & 2(2s-2)2 & \cdots & 0 \\ 0 & 0 & \cdots & 1(2s-1)2 & (2s-1)A_{2s}(\partial_x-i\partial_y) \\ 0 & 0 & 0 & -0A_{2s}(\partial_x+i\partial_y) & 0(2s)2 \end{bmatrix}\psi = 0$$

$$\Leftrightarrow \begin{bmatrix} -(2s-1)A_1(\partial_x+i\partial_y) & 1(2s-1)2 & 1A_2(\partial_x-i\partial_y) & 0 & 0 \\ 0 & -(2s-2)A_2(\partial_x+i\partial_y) & 2(2s-2)2 & \cdots & 0 \\ 0 & 0 & \cdots & 1(2s-1)2, & (2s-1)A_{2s}(\partial_x-i\partial_y) \end{bmatrix}\psi = 0 \qquad \square$$

Cor. 2.6.4.

$$\sigma_z(s)\sigma_x(s) = \frac{1}{2}\begin{bmatrix} 0 & sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & (s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix}$$

$$\sigma_z(s)\sigma_y(s) = \frac{i}{2}\begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & -(s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix}$$

2.7 Important corollaries of spin equation

Thm. 2.7.1. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(\vec{r},t) = 0 [\Leftrightarrow] \begin{cases} [\frac{1}{s}\sigma(s)\cdot\nabla]\psi(\vec{r},t) = \varsigma\partial_t\psi(\vec{r},t) \\ [\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t) = [s\hat{\nabla} + \varsigma(s-1)\hat{\partial}_t\sigma(s)]\psi(\vec{r},t) \\ \hat{\nabla} := \frac{-i\nabla}{\sqrt{-\nabla^2}}, \hat{\nabla}^2 = 1, \hat{\partial}_t := \frac{-i\partial_t}{\sqrt{-\nabla^2}} \simeq -1 \end{cases}$

$$[\Leftrightarrow] \begin{cases} [\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = \{s[s^n - (s-1)^n](\varsigma\hat{\partial}_t)^{n-1}\hat{\nabla} + (s-1)^n(\varsigma\hat{\partial}_t)^n\sigma(s)\}\psi(\vec{r},t) \\ [\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = \{s[s^n - (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1}\hat{\nabla} + (s-1)^n\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\}\psi(\vec{r},t) \\ [\frac{1}{s}\sigma(s)\cdot\nabla]^n\psi(\vec{r},t) = \varsigma^n\partial_t^n\psi(\vec{r},t), \nabla^{2n}\psi(\vec{r},t) = [\frac{1}{s}\sigma(s)\cdot\nabla]^{2n}\psi(\vec{r},t) = \partial_t^{2n}\psi(\vec{r},t), n \geq 1 \end{cases}$$

Proof:

$[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t) = [e_1\hat{\nabla} + d_1\sigma(s)]\psi(\vec{r},t), e_1 = s, d_1 = \varsigma(s-1)\hat{\partial}_t$

$\cdot\cdot$

$[\sigma(s)\cdot\hat{\nabla}]^{n-1}\sigma(s)\psi(\vec{r},t) = [e_{n-1}\hat{\nabla} + d_{n-1}\sigma(s)]\psi(\vec{r},t)$

$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = [e_n\hat{\nabla} + d_n\sigma(s)]\psi(\vec{r},t)$

$\cdot\cdot$

$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t)$

$= [\sigma(s)\cdot\hat{\nabla}][e_{n-1}\hat{p} + d_{n-1}\sigma(s)]\psi(\vec{r},t) = [(-s\varsigma e_{n-1} + d_1^{n-1}e_1)\hat{\nabla} + d_{n-1}d_1\sigma(s)]\psi(\vec{r},t)$

$$\begin{cases} e_n = e_{n-1}s\varsigma\hat{\partial}_t + e_1 d_1^{n-1} \\ d_n = d_{n-1}d_1 \\ e_1 = s, d_1 = \varsigma(s-1)\hat{\partial}_t \end{cases} \Leftrightarrow \begin{cases} e_n = s[s^n - (s-1)^n](\varsigma\hat{\partial}_t)^{n-1} \\ d_n = d_1^n = (s-1)^n(\varsigma\hat{\partial}_t)^n \end{cases}$$

$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = \{s[s^n - (s-1)^n](\varsigma\hat{\partial}_t)^{n-1}\hat{\nabla} + (s-1)^n(\varsigma\hat{\partial}_t)^n\sigma(s)\}\psi(\vec{r},t)$

$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = \{s[s^n - (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1}\hat{\nabla} + (s-1)^n\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\}\psi(\vec{r},t), n \geq 1$ □

Cor. 2.7.1. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(\vec{r},t) = 0$

$$[\Rightarrow] \begin{cases} \sigma^\alpha(s)[\sigma(s)\cdot\hat{\nabla}]^n\sigma_\alpha(s)\psi = s[s^{n+1} + (s-1)^n](\varsigma\hat{\partial}_t)^n(\vec{r},t) \\ \sigma^\alpha(s)[\sigma(s)\cdot\hat{\nabla}]^n\sigma_\alpha(s)\psi(\vec{r},t) = s[s^{n+1} + (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\psi(\vec{r},t) \end{cases}$$

Thm. 2.7.2. $[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\}\psi(\vec{r},t)$

$$[\Leftrightarrow] \begin{cases} [\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = \{s[s^n - (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1}\hat{\nabla} + (s-1)^n\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\}\psi(\vec{r},t) \\ [\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{2n}\psi(\vec{r},t) = \psi(\vec{r},t), n \geq 1 \end{cases}$$

Proof:

$[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t) = [e_1\hat{\nabla} + d_1\sigma(s)]\psi(\vec{r},t), e_1 = s, d_1 = (s-1)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]$

$\cdot\cdot$

$[\sigma(s)\cdot\hat{\nabla}]^{n-1}\sigma(s)\psi(\vec{r},t) = [e_{n-1}\hat{\nabla} + d_{n-1}\sigma(s)]\psi(\vec{r},t)$

$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = [e_n\hat{\nabla} + d_n\sigma(s)]\psi(\vec{r},t)$

$\cdot\cdot$

$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t)$

$= [\sigma(s)\cdot\hat{\nabla}][e_{n-1}\hat{p} + d_{n-1}\sigma(s)]\psi(\vec{r},t) = [(e_{n-1}s[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}] + e_1 d_1^{n-1})\hat{\nabla} + d_{n-1}d_1\sigma(s)]\psi(\vec{r},t)$

$$\begin{cases} e_n = e_{n-1}s[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}] + e_1 d_1^{n-1} \\ d_n = d_{n-1}d_1 \\ e_1 = s, d_1 = (s-1)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}] \end{cases} \Leftrightarrow \begin{cases} e_n = s[s^n - (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1} \\ d_n = d_1^n = (s-1)^n[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n \end{cases}$$

$[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi(\vec{r},t) = \{s[s^n - (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1}\hat{\nabla} + (s-1)^n\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\}\psi(\vec{r},t), n \geq 1$ □

Cor. 2.7.2. $[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\}\psi(\vec{r},t)$

$[\Rightarrow]\sigma^\alpha(s)[\sigma(s)\cdot\hat{\nabla}]^n\sigma_\alpha(s)\psi(\vec{r},t) = s[s^{n+1} + (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\psi(\vec{r},t), n \geq 1$

Cor. 2.7.3. $[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\}\psi(\vec{r},t)$

$$[\Rightarrow] \begin{cases} \sigma(s)\times\{[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\}\psi(\vec{r},t) = i\{-s^2[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\sigma(s) + (s^2+s-1)\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\}\psi(\vec{r},t) \\ \sigma(s)\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\}\}\psi(\vec{r},t) = i[s^2+s-1][\sigma(s)\cdot\hat{\nabla}]\psi(\vec{r},t) \end{cases}$$

Cor. 2.7.4. $[\sigma(s)\cdot\hat{\nabla}]\sigma(s)\psi(\vec{r},t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\}\psi(\vec{r},t)$

$$[\Rightarrow] \begin{cases} \sigma(s)\times\{[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\}\psi(\vec{r},t) \\ = i\{-s^2[s^n - (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1} + [s^{n+2} - (s+1)(s-1)^{n+1}]\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\}\psi(\vec{r},t) \\ \sigma(s)\cdot\{\sigma(s)\times\{[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\}\}\psi(\vec{r},t) = is[s^{n+1} + (s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\psi(\vec{r},t), n \geq 1 \end{cases}$$

3 Spin equation in momentum space

3.1 Various equivalent forms of s-spin equation in momentum space

Thm. 3.1.1.

$$[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(x) = 0 \quad [\Leftrightarrow] \quad \sigma(s)\cdot\nabla\psi = s\varsigma\partial_t\psi, O(s)\cdot\nabla\psi(x) = 0$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

$$s\nabla\psi(x) = [i\sigma(s)\times\nabla + \varsigma\sigma(s)\partial_t]\psi(x) \quad [\Leftrightarrow] \quad \begin{cases}\sigma(s)\cdot\nabla\psi(x) = s\varsigma\partial_t\psi(x)\\ s^2\nabla\psi(x) = is\sigma(s)\times\nabla + \sigma(s)[\sigma(s)\cdot\nabla]\psi(x)\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

$$s\nabla\psi(x) = \{[\sigma(s)\cdot\nabla, \sigma(s)] + \varsigma\sigma(s)\partial_t\}\psi(x)[\Leftrightarrow]\begin{cases}\sigma(s)\cdot\nabla\psi(x) = s\varsigma\partial_t\psi(x)\\ s^2\nabla\psi(x) = \{s[\sigma(s)\cdot\nabla]\sigma(s) - (s-1)\sigma(s)[\sigma(s)\cdot\nabla]\}\psi(x)\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

Thm. 3.1.2.

$$[s(\vec{p},-\partial_t)_a + iS_{ab}(s,\varsigma)(\vec{p},-\partial_t)^b]\psi(\vec{p},t) = 0 \quad [\Leftrightarrow] \quad \tfrac{1}{s}\sigma(s)\cdot\vec{p}\psi(\vec{p},t) = -i\varsigma\partial_t\psi, O(s)\cdot\vec{p}\psi(\vec{p},t) = 0$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

$$[s\vec{p} - i\sigma(s)\times\vec{p}]\psi(\vec{p},t) = -\sigma(s)i\varsigma\partial_t\psi(\vec{p},t) \quad [\Leftrightarrow] \quad \begin{cases}\frac{1}{s}\sigma(s)\cdot\vec{p}\psi(\vec{p},t) = -i\varsigma\partial_t\psi(\vec{p},t)\\ \{s^2\vec{p} - is\sigma(s)\times\vec{p} - \sigma(s)[\sigma(s)\cdot\vec{p}]\}\psi(\vec{p},t) = 0\end{cases}$$

$$[\Updownarrow] \qquad\qquad [\Updownarrow]$$

$$\{s\vec{p} - [\sigma(s)\cdot\vec{p}, \sigma(s)]\}\psi(\vec{p},t) = -i\varsigma\sigma(s)\partial_t\psi(\vec{p},t)[\Leftrightarrow]\begin{cases}\frac{1}{s}\sigma(s)\cdot\vec{p}\psi(\vec{p},t) = -i\varsigma\partial_t\psi(\vec{p},t)\\ \{s^2\vec{p} + (s-1)\sigma(s)[\sigma(s)\cdot\vec{p}] - s[\sigma(s)\cdot\vec{p}]\sigma(s)\}\psi(\vec{p},t) = 0\end{cases}$$

Cor. 3.1.1. $\begin{cases}\sigma(s)\cdot\nabla\psi(x) = s\varsigma\partial_t\psi(x)\\ \nabla\psi(x) = [\sigma(s)\cdot\nabla]\sigma(s)\psi(x)\end{cases} [\Leftrightarrow] \begin{cases}[\sigma(s)\cdot\vec{p}]\psi(\vec{p},t) = -i\varsigma\partial_t\psi(\vec{p},t)\\ \vec{p}\psi(\vec{p},t) = [\sigma(s)\cdot\vec{p}]\sigma(s)\psi(\vec{p},t)\end{cases} ; s = 1$

3.2 Plane wave solutions of s-spin equation in momentum space

Cor. 3.2.1. $\psi(\vec{p},t) = |\vec{p}|^{(s-\frac{1}{2})}[a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t)} + a_2^+(-\vec{p},-s\varsigma)\lambda(-\hat{p},-s\varsigma)e^{i|\vec{p}|t)}]$

3.3 Properties of plane wave solutions in momentum space

3.3.1 Important property 1

Cor. 3.3.1. $[\frac{1}{s}\sigma(s)\cdot\hat{p}]\sigma(s)\psi(\vec{p},t) = \{\hat{p} + (1-\frac{1}{s})\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\}\psi(\vec{p},t)$

$[\Rightarrow][\frac{1}{s}\sigma(s)\cdot\hat{p}]^2\sigma(s)\psi(\vec{p},t) = \{s[1-(1-\frac{1}{s})^2][\frac{1}{s}\sigma(s)\cdot\hat{p}]\hat{p} + (1-\frac{1}{s})^2\sigma(s)\}\psi(\vec{p},t)$

Cor. 3.3.2. $[\frac{1}{s}\sigma(s)\cdot\hat{p}]\sigma(s)\psi(\vec{p},t) = \{\hat{p} + (1-\frac{1}{s})\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\}\psi(\vec{p},t)$

$$[\Leftrightarrow]\begin{cases}[\frac{1}{s}\sigma(s)\cdot\hat{p}]^n\sigma(s)\psi(\vec{p},t) = \{s[1-(1-\frac{1}{s})^n][\frac{1}{s}\sigma(s)\cdot\hat{p}]^{n-1}\hat{p} + (1-\frac{1}{s})^n\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]^n\}\psi(\vec{p},t)\\ [\frac{1}{s}\sigma(s)\cdot\hat{p}]^{2n}\psi(\vec{p},t) = \psi(\vec{p},t)\end{cases}$$

$$[\Leftrightarrow]\begin{cases}[\frac{1}{s}\sigma(s)\cdot\hat{p}]^{2k+1}\sigma(s)\psi(\vec{p},t) = \{s[1-(1-\frac{1}{s})^{2k+1}]\hat{p} + (1-\frac{1}{s})^{2k+1}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\}\psi(\vec{p},t)\\ [\frac{1}{s}\sigma(s)\cdot\hat{p}]^{2k}\sigma(s)\psi(\vec{p},t) = \{s[1-(1-\frac{1}{s})^{2k}][\frac{1}{s}\sigma(s)\cdot\hat{p}]\hat{p} + (1-\frac{1}{s})^{2k}\sigma(s)\}\psi(\vec{p},t)\\ [\frac{1}{s}\sigma(s)\cdot\hat{p}]^{2k}\psi(\vec{p},t) = \psi(\vec{p},t)\end{cases}$$

Cor. 3.3.3.

$$\begin{cases}[\sigma(1)\cdot\hat{p}]^{2k+1}\sigma(1)\psi(\vec{p},t) = [\sigma(1)\cdot\hat{p}]\sigma(1)\psi(\vec{p},t) = \hat{p}\psi(\vec{p},t)\\ [\sigma(1)\cdot\hat{p}]^{2k+2}\sigma(1)\psi(\vec{p},t) = [\sigma(1)\cdot\hat{p}]^2\sigma(1)\psi(\vec{p},t) = [\sigma(1)\cdot\hat{p}]\hat{p}\psi(\vec{p},t)\end{cases}$$

3.3.2 Important property 2

Thm. 3.3.1.

$$\begin{cases}\psi^+(\vec{p},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\psi(\vec{p},t) = \psi^+(\vec{p},t)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\frac{1}{s}\sigma(s)\psi(\vec{p},t) = \psi^+(\vec{p},t)\hat{p}\psi(\vec{p},t), s\geq 1\\ \psi^+(\vec{p},t)\{\sigma(\frac{1}{2})[\sigma(\frac{1}{2})\cdot\hat{p}] + [\sigma(\frac{1}{2})\cdot\hat{p}]\sigma(\frac{1}{2})\}\psi(\vec{p},t) = \frac{1}{2}\psi^+(\vec{p},t)\hat{p}\psi(\vec{p},t)\end{cases}$$

Proof: $\psi^+(\vec{p},t)\{s^2\hat{p} + (s-1)\sigma(s)[\sigma(s)\cdot\hat{p}] - s[\sigma(s)\cdot\hat{p}]\sigma(s)\}\psi(\vec{p},t) = 0$

$$\Leftrightarrow\begin{cases}\psi^+(\vec{p},t)\{s^2\hat{p} + (s-1)\sigma(s)[\sigma(s)\cdot\hat{p}] - s[\sigma(s)\cdot\hat{p}]\sigma(s)\}\psi(\vec{p},t) = 0\\ \psi^+(\vec{p},t)\{s^2\hat{p} + (s-1)[\sigma(s)\cdot\hat{p}]\sigma(s) - s\sigma(s)[\sigma(s)\cdot\hat{p}]\}\psi(\vec{p},t) = 0\end{cases}$$

$$\Leftrightarrow\begin{cases}\psi^+(\vec{p},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\psi(\vec{p},t) = \psi^+(\vec{p},t)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\frac{1}{s}\sigma(s)\psi(\vec{p},t) = \psi^+(\vec{p},t)\hat{p}\psi(\vec{p},t), s\geq 1\\ \psi^+(\vec{p},t)\{\sigma(\frac{1}{2})[\sigma(\frac{1}{2})\cdot\hat{p}] + [\sigma(\frac{1}{2})\cdot\hat{p}]\sigma(\frac{1}{2})\}\psi(\vec{p},t) = \frac{1}{2}\psi^+(\vec{p},t)\hat{p}\psi(\vec{p},t)\end{cases}$$

□

Thm. 3.3.2. $[\frac{1}{s}\sigma(s)\cdot\vec{p}]^2\psi(\vec{p},t) = \vec{p}^2\psi(\vec{p},t) = -\partial_t^2\psi(\vec{p},t)$

Cor. 3.3.4.

$$\begin{cases}\int\psi^+(\vec{r},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\frac{1}{s}\sigma(s)\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)\hat{\nabla}\psi(\vec{r},t)d^3\vec{r}, s\geq 1\\ \int\psi^+(\vec{r},t)\{\sigma(\frac{1}{2})[\sigma(\frac{1}{2})\cdot\hat{\nabla}] + [\sigma(\frac{1}{2})\cdot\hat{\nabla}]\sigma(\frac{1}{2})\}\psi(\vec{r},t)d^3\vec{r} = \int\frac{1}{2}\psi^+(\vec{r},t)\hat{\nabla}\psi(\vec{r},t)d^3\vec{r}\end{cases}$$

3.3.3 Important property 3

Lem. 3.3.1. $\psi^+(\vec{p},t)\sigma(s)\psi(\vec{p},t) = (-s\varsigma)|\vec{p}|^{(2s-1)}[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) - a_2(-\vec{p},-s\varsigma)a_2^+(-\vec{p},-s\varsigma)]\hat{p}, s \geq 1$

Proof: $\psi^+(\vec{p},t)\sigma(s)\psi(\vec{p},t), s \geq 1$
$= |\vec{p}|^{(s-\frac{1}{2})}[a_1^+(\vec{p},-s\varsigma)\lambda^+(\hat{p},-s\varsigma)e^{i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)\lambda^+(-\hat{p},-s\varsigma)e^{-i|\vec{p}|t}]\sigma(s)|\vec{p}|^{(s-\frac{1}{2})}[a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t}+a_2^+(-\vec{p},-s\varsigma)\lambda(-$
$= |\vec{p}|^{(2s-1)}[a_1^+(\vec{p},-s\varsigma)\lambda^+(\hat{p},-s\varsigma)e^{i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)\lambda^+(-\hat{p},-s\varsigma)e^{-i|\vec{p}|t}]\sigma(s)[a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t}+a_2^+(-\vec{p},-s\varsigma)\lambda(-\hat{p},-s\varsigma)$
$= (-s\varsigma)|\vec{p}|^{(2s-1)}[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) - a_2(-\vec{p},-s\varsigma)a_2^+(-\vec{p},-s\varsigma)]\hat{p}, s \geq 1$ □

Lem. 3.3.2. $\psi^+(\vec{p},t)\sigma(s)[\sigma(s)\cdot\hat{p}]\psi(\vec{p},t) = s^2|\vec{p}|^{(2s-1)}[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + a_2(-\vec{p},-s\varsigma)a_2^+(-\vec{p},-s\varsigma)]\hat{p}, s \geq 1$

Proof: $\psi^+(\vec{p},t)\sigma(s)[\sigma(s)\cdot\hat{p}]\psi(\vec{p},t), s \geq 1$
$= |\vec{p}|^{(s-\frac{1}{2})}[a_1^+(\vec{p},-s\varsigma)\lambda^+(\hat{p},-s\varsigma)e^{i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)\lambda^+(-\hat{p},-s\varsigma)e^{-i|\vec{p}|t}]\sigma(s)[\sigma(s)\cdot\hat{p}]|\vec{p}|^{(s-\frac{1}{2})}[a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t}+$
$a_2^+(-\vec{p},-s\varsigma)\lambda(-\hat{p},-s\varsigma)e^{i|\vec{p}|t})]$
$= |\vec{p}|^{(2s-1)}[a_1^+(\vec{p},-s\varsigma)\lambda^+(\hat{p},-s\varsigma)e^{i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)\lambda^+(-\hat{p},-s\varsigma)e^{-i|\vec{p}|t}]\sigma(s)[-s\varsigma a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-i|\vec{p}|t}+s\varsigma a_2^+(-\vec{p},-s\varsigma)\lambda(-\hat{p}$
$= s^2|\vec{p}|^{(2s-1)}[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + a_2(-\vec{p},-s\varsigma)a_2^+(-\vec{p},-s\varsigma)]\hat{p}, s \geq 1$ □

Thm. 3.3.3.
$$\begin{cases}\psi^+(\vec{p},t)\sigma(s)\psi(\vec{p},t) = \psi^+(\vec{p},t)[\sigma(s)\cdot\hat{p}]\hat{p}\psi(\vec{p},t), \psi^+(\vec{p},t)\sigma(s)\times\hat{p}\psi(\vec{p},t) = 0, s \geq 1\\ \psi^+(\vec{p},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\psi(\vec{p},t) = \psi^+(\vec{p},t)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\frac{1}{s}\sigma(s)\psi(\vec{p},t) = \psi^+(\vec{p},t)\hat{p}\psi(\vec{p},t), s \geq 1\\ \psi^+(\vec{p},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]^k\psi(\vec{p},t) = \psi^+(\vec{p},t)[\frac{1}{s}\sigma(s)\cdot\hat{p}]^k\frac{1}{s}\sigma(s)\psi(\vec{p},t), s \geq 1\end{cases}$$

Cor. 3.3.5.
$$\begin{cases}\int\psi^+(\vec{r},t)\sigma(s)\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)[\sigma(s)\cdot\hat{\nabla}]\hat{\nabla}\psi(\vec{r},t)d^3\vec{r}, \int\psi^+(\vec{r},t)[\sigma(s)\times\hat{\nabla}]\psi(\vec{r},t)d^3\vec{r} = 0, s \geq 1\\ \int\psi^+(\vec{r},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\frac{1}{s}\sigma(s)\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)\hat{\nabla}\psi(\vec{r},t)d^3\vec{r}, s \geq 1\\ \int\psi^+(\vec{r},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^k\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^k\frac{1}{s}\sigma(s)\psi(\vec{r},t)d^3\vec{r}, s \geq 1\\ \int\psi^+(\vec{r},t)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^i\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^j\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^j\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^i\psi(\vec{r},t)d^3\vec{r}, s \geq 1\end{cases}$$

Cor. 3.3.6.
$$\begin{cases}\psi^+(\vec{p},t)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{p}]\psi(\vec{p},t) = \psi^+(\vec{p},t)\frac{1}{s}\sigma(s)\psi(\vec{p},t), s \geq 1\\ \int\psi^+(\vec{r},t)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)\frac{1}{s}\sigma(s)\psi(\vec{r},t)d^3\vec{r}, s \geq 1\\ \int\psi^+(\vec{r},t)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^i\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^j\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{i+j}\psi(\vec{r},t)d^3\vec{r}, s \geq 1\end{cases}$$

4 Various properties of spin and angular momentum operators
(satisfying spin equation)

4.1 General properties of spin wave function

Def. 4.1.1. $\Gamma(n;m,l) := (\sqrt{-\nabla^2})^n\partial_\pi^m\overbrace{\partial_i\partial_j\cdots}^{l}, n \in Z; m,l \in N$

Cor. 4.1.1. $\int\psi^+(\vec{r},t)\Gamma(n;m,l)\dot{\psi}(\vec{r},t)d^3\vec{r} = -\int\dot{\psi}^+(\vec{r},t)\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r}$

Cor. 4.1.2.
$$\begin{cases}\int\psi^+(\vec{r},t)[\sigma(s)\cdot\hat{\nabla}]\hat{\nabla}\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)\sigma(s)\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r}, s \geq 1\\ \int\psi^+(\vec{r},t)\sigma(s)\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r} = s\varsigma\int\psi^+(\vec{r},t)\frac{\nabla}{\nabla^2}\Gamma(n;m,l)\dot{\psi}(\vec{r},t)d^3\vec{r}, s \geq 1\end{cases}$$

Cor. 4.1.3.
$$\begin{cases}\int\psi^+(\vec{r},t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)\hat{\nabla}\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r}, s \geq 1\\ \int\psi^+(\vec{r},t)\sigma(s)\Gamma(n;m,l)\dot{\psi}(\vec{r},t)d^3\vec{r} = s\varsigma\int\psi^+(\vec{r},t)\nabla\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r}, s \geq 1\end{cases}$$

Pro. 4.1.1.
$$\begin{cases}\int\psi^+(\vec{r},t)[\sigma_i(s)\partial_j - \sigma_j(s)\partial_i]\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r} = 0, s \geq 1\\ \int\psi^+(\vec{r},t)[\sigma_i(s)\partial_j - \sigma_j(s)\partial_i]\Gamma(n;m,l)\dot{\psi}(\vec{r},t)d^3\vec{r} = 0, s \geq 1\end{cases}$$

Cor. 4.1.4.
$$\begin{cases}\int\psi^+(\vec{r},t)[\sigma(s)\cdot\hat{\nabla}]^j\sigma(s)[\sigma(s)\cdot\hat{\nabla}]^k\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r} = \int\psi^+(\vec{r},t)[\sigma(s)\cdot\hat{\nabla}]^k\sigma(s))[\sigma(s)\cdot\hat{\nabla}]^j\Gamma(n;m,l)\psi(\vec{r},t)d^3\vec{r}, s \geq 1\\ \int\psi^+[\sigma(s)\cdot\hat{\nabla}]^j\sigma(s)[\sigma(s)\cdot\hat{\nabla}]^k\Gamma(n;m,l)\psi d^3\vec{r} = s\varsigma\int\psi^+[\sigma(s)\cdot\hat{\nabla}]^{j+k}\frac{\nabla}{\nabla^2}\Gamma(n;m,l)\dot{\psi}d^3\vec{r}, s \geq 1\\ \int\psi^+[\sigma(s)\cdot\hat{\nabla}]^j\sigma(s)[\sigma(s)\cdot\hat{\nabla}]^k\Gamma(n;m,l)\dot{\psi}d^3\vec{r} = s\varsigma\int\psi^+[\sigma(s)\cdot\hat{\nabla}]^{j+k}\nabla\Gamma(n;m,l)\psi d^3\vec{r}, s \geq 1\end{cases}$$

4.2 Properties 1 of angular momentum operator

Lem. 4.2.1. $\nabla^2(r_i\partial_j - r_j\partial_i) = (r_i\partial_j - r_j\partial_i)\nabla^2$

Lem. 4.2.2. $[\sigma(s)\cdot\nabla](r_i\partial_j - r_j\partial_i) = (r_i\partial_j - r_j\partial_i)[\sigma(s)\cdot\nabla] + [\sigma_i(s)\partial_j - \sigma_j(s)\partial_i]$

Lem. 4.2.3. $[\sigma(s)\cdot\nabla]^2(r_i\partial_j - r_j\partial_i)$
$= (r_i\partial_j - r_j\partial_i)[\sigma(s)\cdot\nabla]^2 + [\sigma_i(s)\partial_j - \sigma_j(s)\partial_i][\sigma(s)\cdot\nabla] + [\sigma(s)\cdot\nabla][\sigma_i(s)\partial_j - \sigma_j(s)\partial_i]$

Proof: $[\sigma(s)\cdot\nabla]^2(r_i\partial_j-r_j\partial_i)$
$=[\sigma(s)\cdot\nabla]\{(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]+[\sigma_i(s)\partial_j-\sigma_j(s)\partial_i]\}$
$=(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]^2+[\sigma_i(s)\partial_j-\sigma_j(s)\partial_i][\sigma(s)\cdot\nabla]+[\sigma(s)\cdot\nabla][\sigma_i(s)\partial_j-\sigma_j(s)\partial_i]$ □

Pro. 4.2.1.
$$\begin{cases} s\geq 1, n\in Z, l,m\in N,\\ \int\psi^+(\vec r,t)[\sigma(s)\cdot\nabla](r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]\Gamma(n;m,l)\psi(\vec r,t)d^3\vec r\\ \int\psi^+(\vec r,t)[\sigma(s)\cdot\nabla](r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\dot\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]\Gamma(n;m,l)\dot\psi(\vec r,t)d^3\vec r\end{cases}$$

Pro. 4.2.2.
$$\begin{cases} s\geq 1, n\in Z, l,m\in N,\\ \int\psi^+(\vec r,t)[\sigma(s)\cdot\nabla]^2(r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]^2\Gamma(n;m,l)\psi(\vec r,t)d^3\vec r\\ \int\psi^+(\vec r,t)[\sigma(s)\cdot\nabla]^2(r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\dot\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]^2\Gamma(n;m,l)\dot\psi(\vec r,t)d^3\vec r\end{cases}$$

Cor. 4.2.1.
$$\begin{cases} s\geq 1, n\in Z, l,m\in N,\\ \int\psi^+(\vec r,t)[\frac{1}{s}\sigma(s)\cdot\hat\nabla]^2(r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\psi(\vec r,t)d^3\vec r\\ \int\psi^+(\vec r,t)[\frac{1}{s}\sigma(s)\cdot\hat\nabla]^2(r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\dot\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)\Gamma(n;m,l)\dot\psi(\vec r,t)d^3\vec r\end{cases}$$

4.3 Properties 2 of angular momentum operator

Cor. 4.3.1.
$$\begin{cases}\int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r=i\int\psi^+(\vec p,t)(p_i\tilde\partial_j-p_j\tilde\partial_i)\{[\sigma(s)\cdot\vec p]\psi(\vec p,t)\}d^3\vec p, s\geq\frac12\\ \int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)[\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r=i\int\psi^+(\vec p,t)[\sigma(s)\cdot\vec p](p_i\tilde\partial_j-p_j\tilde\partial_i)\psi(\vec p,t)d^3\vec p, s\geq 1\end{cases}$$

Cor. 4.3.2.
$$\begin{cases}\int\psi^+(\vec r,t)(r_i\hat\partial_j-r_j\hat\partial_i)[\sigma(s)\cdot\hat\nabla]\psi(\vec r,t)d^3\vec r=i\int\psi^+(\vec p,t)(\hat p_i\tilde\partial_j-\hat p_j\tilde\partial_i)\{[\sigma(s)\cdot\hat p]\psi(\vec p,t)\}d^3\vec p, s\geq\frac12\\ \int\psi^+(\vec r,t)(r_i\hat\partial_j-r_j\hat\partial_i)[\sigma(s)\cdot\hat\nabla]\psi(\vec r,t)d^3\vec r=i\int\psi^+(\vec p,t)[\sigma(s)\cdot\hat p](\hat p_i\tilde\partial_j-\hat p_j\tilde\partial_i)\psi(\vec p,t)d^3\vec p, s\geq 1\end{cases}$$

Cor. 4.3.3.
$$\begin{cases}\int\psi^+(\vec r,t)[r_i\sigma_j(s)-r_j\sigma_i(s)]\psi(\vec r,t)d^3\vec r=-i\int\psi^+(\vec p,t)[\sigma_i(s)\tilde\partial_j-\sigma_j(s)\tilde\partial_i]\psi(\vec p,t)d^3\vec p\\ \int\psi^+(\vec r,t)(r_i\partial_j-r_j\partial_i)\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec p,t)(p_i\tilde\partial_j-p_j\tilde\partial_i)\psi(\vec p,t)d^3\vec p\end{cases}$$

Cor. 4.3.4. $-i\int\psi^+(\vec r,t)[r_i\partial_j-r_j\partial_i-\sigma_{\varsigma ij}^{k}\sigma_k(s)]\psi(\vec r,t)d^3\vec r$
$=-i\int\psi^+(\vec r,t)[r_i\partial_j-r_j\partial_i+i\varepsilon_{ij}{}^{k}\sigma_k(s)]\psi(\vec r,t)d^3\vec r$
$=-i\int\psi^+(\vec r,t)\{r_i\partial_j-r_j\partial_i+[\sigma_i(s),\sigma_j(s)]\}\psi(\vec r,t)d^3\vec r$
$=-i\int\psi^+(\vec p,t)\{p_i\tilde\partial_j-p_j\tilde\partial_i+[\sigma_i(s),\sigma_j(s)]\}\psi(\vec p,t)d^3\vec p$

4.4 Properties of angular momentum operator???

Cor. 4.4.1. $\int\psi^+(\vec r,t)r_i\sigma_j(s)[\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r=-\int\psi^+(\vec p,t)\{\sigma_j(s)\sigma_i(s)+\sigma_j(s)[\sigma(s)\cdot\vec p]\tilde\partial_i\}\psi(\vec p,t)d^3\vec p$

Proof: $\int\psi^+(\vec r,t)r_i\sigma_j(s)[\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r$
$=\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)e^{-i\vec p'\cdot\vec r}r_i\sigma_j(s)[\sigma(s)\cdot\nabla][\psi(\vec p,t)e^{i\vec p\cdot\vec r}]$
$=\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)e^{-i\vec p'\cdot\vec r}\sigma_j(s)[\sigma(s)\cdot i\vec p]\psi(\vec p,t)(-i\tilde\partial_i)e^{i\vec p\cdot\vec r}$
$=-\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)\tilde\partial_i\{\sigma_j(s)[\sigma(s)\cdot\vec p]\psi(\vec p,t)\}e^{i(\vec p-\vec p')\cdot\vec r}$
$=-\int d^3\vec p d^3\vec p'\psi^+(\vec p',t)\tilde\partial_i\{\sigma_j(s)[\sigma(s)\cdot\vec p]\psi(\vec p,t)\}\delta^3(\vec p-\vec p')$
$=-\int\psi^+(\vec p,t)\sigma_j(s)\tilde\partial_i\{[\sigma(s)\cdot\vec p]\psi(\vec p,t)\}d^3\vec p$
$=-\int\psi^+(\vec p,t)\{\sigma_j(s)\sigma_i(s)+\sigma_j(s)[\sigma(s)\cdot\vec p]\tilde\partial_i\}\psi(\vec p,t)d^3\vec p$ □

Cor. 4.4.2. $\int\psi^+(\vec r,t)[\sigma(s)\cdot\nabla]\{r_i\sigma_j(s)\psi(\vec r,t)\}d^3\vec r=-\int d^3\vec p\psi^+(\vec p,t)[\sigma(s)\cdot\vec p]\sigma_j(s)\tilde\partial_i\psi(\vec p,t)$

Proof: $\int\psi^+(\vec r,t)[\sigma(s)\cdot\nabla]\{r_i\sigma_j(s)\psi(\vec r,t)\}d^3\vec r$
$=\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)e^{-i\vec p'\cdot\vec r}[\sigma(s)\cdot\nabla][r_i\sigma_j(s)\psi(\vec p,t)e^{i\vec p\cdot\vec r}]$
$=\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)e^{-i\vec p'\cdot\vec r}[\sigma(s)\cdot\nabla][r_ie^{i\vec p\cdot\vec r}]\sigma_j(s)\psi(\vec p,t)$
$=\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)e^{-i\vec p'\cdot\vec r}\{[\sigma(s)\cdot\nabla]r_i[e^{i\vec p\cdot\vec r}]+r_i[\sigma(s)\cdot\nabla]e^{i\vec p\cdot\vec r}\}\sigma_j(s)\psi(\vec p,t)$
$=\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)e^{-i\vec p'\cdot\vec r}\{[\sigma_i(s)\sigma_j(s)e^{i\vec p\cdot\vec r}]+[\sigma(s)\cdot\vec p][\sigma_j(s)\tilde\partial_ie^{i\vec p\cdot\vec r}]\}\psi(\vec p,t)$
$=\frac{1}{(2\pi)^3}\int d^3\vec p d^3\vec p' d^3\vec r\psi^+(\vec p',t)\{\sigma_i(s)\sigma_j(s)\psi(\vec p,t)-\tilde\partial_i[\sigma(s)\cdot\vec p\sigma_j(s)\psi(\vec p,t)]\}e^{i(\vec p-\vec p')\cdot\vec r}$
$=\int d^3\vec p\psi^+(\vec p,t)\{[\sigma_i(s)\sigma_j(s)]\psi(\vec p,t)-\tilde\partial_i[\sigma(s)\cdot\vec p\sigma_j(s)\psi(\vec p,t)]\}$
$=-\int\psi^+(\vec p,t)[\sigma(s)\cdot\vec p]\sigma_j(s)\tilde\partial_i\psi(\vec p,t)d^3\vec p$ □

Cor. 4.4.3.
$$\begin{cases}\int\psi^+(\vec r,t)[r_i\sigma_j(s)-r_j\sigma_i(s)][\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r=\int\psi^+(\vec p,t)[\sigma_i(s)\tilde\partial_j-\sigma_j(s)\tilde\partial_i]\{[\sigma(s)\cdot\vec p]\psi(\vec p,t)\}d^3\vec p\\ \int\psi^+(\vec r,t)[\sigma(s)\cdot\nabla]\{[r_i\sigma_j(s)-r_j\sigma_i(s)]\psi(\vec r,t)\}d^3\vec r=\int\psi^+(\vec p,t)[\sigma(s)\cdot\vec p][\sigma_i(s)\tilde\partial_j-\sigma_j(s)\tilde\partial_i]\psi(\vec p,t)d^3\vec p\end{cases}$$

Lem. 4.4.1. $\begin{cases}\Psi(\vec{p},t)=|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)\lambda_m(\hat{p},-\varsigma)e^{-i|\vec{p}|t)}+a_2^+(-\vec{p},-\varsigma)\lambda_m(-\hat{p},-\varsigma)e^{i|\vec{p}|t)}]\\(\gamma\cdot\vec{p})\Psi(\vec{p},t)=-\varsigma|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)\lambda_m(\hat{p},-\varsigma)e^{-i|\vec{p}|t)}-a_2^+(-\vec{p},-\varsigma)\lambda_m(-\hat{p},-\varsigma)e^{i|\vec{p}|t)}]\end{cases}$

Cor. 4.4.4. $\lambda_m^+(-\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)=0,\lambda_m^+(\hat{p},-\varsigma)(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\lambda_m(-\hat{p},-\varsigma)=0$

Thm. 4.4.1. $\int d^3\vec{r}\{\Psi^+(\vec{r},t)(r_i\gamma_j-r_j\gamma_i)(\gamma\cdot\nabla)\Psi(\vec{r},t)-\Psi^+(\vec{r},t)(\gamma\cdot\nabla)[(r_i\gamma_j-r_j\gamma_i)\Psi(\vec{r},t)]\}=0$

Proof: $\int d^3\vec{r}\{\Psi^+(\vec{r},t)(r_i\gamma_j-r_j\gamma_i)(\gamma\cdot\nabla)\Psi(\vec{r},t)-\Psi^+(\vec{r},t)(\gamma\cdot\nabla)[(r_i\gamma_j-r_j\gamma_i)\Psi(\vec{r},t)]\}$
$=\int d^3\vec{r}\{\Psi^+(\vec{p},t)(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)[(\gamma\cdot\vec{p})\Psi(\vec{p},t)]-\Psi^+(\vec{p},t)(\gamma\cdot\vec{p})(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\Psi(\vec{p},t)\}$
$=\int d^3\vec{r}\{\Psi^+(\vec{p},t)(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)[(\gamma\cdot\vec{p})\Psi(\vec{p},t)]-[(\gamma\cdot\vec{p})\Psi(\vec{p},t)]^+(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\Psi(\vec{p},t)\}$
$=-\varsigma\int d^3\vec{r}\{|\vec{p}|^{\frac{1}{2}}[a_1^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)e^{i|\vec{p}|t)}+a_2(-\vec{p},-\varsigma)\lambda_m^+(-\hat{p},-\varsigma)e^{-i|\vec{p}|t)}]$
$(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\{|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)\lambda_m(\hat{p},-\varsigma)e^{-i|\vec{p}|t)}-a_2^+(-\vec{p},-\varsigma)\lambda_m(-\hat{p},-\varsigma)e^{i|\vec{p}|t)}]\}$
$-\{|\vec{p}|^{\frac{1}{2}}[a_1^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)e^{i|\vec{p}|t)}-a_2(-\vec{p},-\varsigma)\lambda_m^+(-\hat{p},-\varsigma)e^{-i|\vec{p}|t)}]\}$
$(\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\{|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)\lambda_m(\hat{p},-\varsigma)e^{-i|\vec{p}|t)}+a_2^+(-\vec{p},-\varsigma)\lambda_m(-\hat{p},-\varsigma)e^{i|\vec{p}|t)}]\}\}$
$=2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_1^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)e^{i|\vec{p}|t)}](\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\{|\vec{p}|^{\frac{1}{2}}[a_2^+(-\vec{p},-\varsigma)\lambda_m(-\hat{p},-\varsigma)e^{i|\vec{p}|t)}]\}$
$-2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_2(-\vec{p},-\varsigma)\lambda_m^+(-\hat{p},-\varsigma)e^{-i|\vec{p}|t)}](\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\{|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)\lambda_m(\hat{p},-\varsigma)e^{-i|\vec{p}|t)}]\}$
$=2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_1^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)e^{i|\vec{p}|t)}](\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\{|\vec{p}|^{\frac{1}{2}}[a_2^+(-\vec{p},-\varsigma)\lambda_m(-\hat{p},-\varsigma)e^{i|\vec{p}|t)}]\}$
$-2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_2(-\vec{p},-\varsigma)\lambda_m^+(-\hat{p},-\varsigma)e^{-i|\vec{p}|t)}](\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\{|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)\lambda_m(\hat{p},-\varsigma)e^{-i|\vec{p}|t)}]\}$
$=2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_1^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)e^{i|\vec{p}|t)}](\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\lambda_m(-\hat{p},-\varsigma)\{|\vec{p}|^{\frac{1}{2}}[a_2^+(-\vec{p},-\varsigma)e^{i|\vec{p}|t)}]\}$
$-2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_2(-\vec{p},-\varsigma)\lambda_m^+(-\hat{p},-\varsigma)e^{-i|\vec{p}|t)}](\gamma_i\tilde{\partial}_j-\gamma_j\tilde{\partial}_i)\lambda_m(\hat{p},-\varsigma)\{|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)e^{-i|\vec{p}|t)}]\}$
$+2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_1^+(\vec{p},-\varsigma)\lambda_m^+(\hat{p},-\varsigma)e^{i|\vec{p}|t)}][\gamma_i\lambda_m(-\hat{p},-\varsigma)\tilde{\partial}_j-\gamma_j\lambda_m(-\hat{p},-\varsigma)\tilde{\partial}_i]\{|\vec{p}|^{\frac{1}{2}}[a_2^+(-\vec{p},-\varsigma)e^{i|\vec{p}|t)}]\}$
$-2\varsigma\int d^3\vec{r}|\vec{p}|^{\frac{1}{2}}[a_2(-\vec{p},-\varsigma)\lambda_m^+(-\hat{p},-\varsigma)e^{-i|\vec{p}|t)}][\gamma_i\lambda_m(\hat{p},-\varsigma)\tilde{\partial}_j-\gamma_j\lambda_m(\hat{p},-\varsigma)\tilde{\partial}_i]\{|\vec{p}|^{\frac{1}{2}}[a_1(\vec{p},-\varsigma)e^{-i|\vec{p}|t)}]\}$
$=0-0+0-0=0$ □

Cor. 4.4.5. $\int d^3\vec{r}\{\psi^+(\vec{r},t)[r_i\sigma_j(1)-r_j\sigma_i(1)][\sigma(1)\cdot\nabla]\psi(\vec{r},t)-\psi^+(\vec{r},t)[\sigma(1)\cdot\nabla]\{[r_i\sigma_j(1)-r_j\sigma_i(1)]\psi(\vec{r},t)\}\}=0$

4.5 Properties 3 of angular momentum operator

Cor. 4.5.1. $\psi^+(\vec{p},t)\sigma_i(s)[\sigma(s)\cdot\vec{p}]\tilde{\partial}_j\psi(\vec{p},t)??=\psi^+(\vec{p},t)p_i\tilde{\partial}_j\psi(\vec{p},t)$

Lem. 4.5.1. $\begin{cases}P_a=-i\int\psi^+(\vec{r},t)\partial_a\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{r},t)\hat{P}_a\psi(\vec{r},t)d^3\vec{r}\\L_{ab}=-i\int\psi^+(\vec{r},t)(r_a\partial_b-r_b\partial_a)\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{r},t)\hat{L}_{ab}\psi(\vec{r},t)d^3\vec{r}\\M_{ab}=\int\psi^+(\vec{r},t)[-i(r_a\partial_b-r_b\partial_a)+i\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)]\psi(\vec{r},t)d^3\vec{r}=\int\psi^+(\vec{r},t)\hat{M}_{ab}\psi(\vec{r},t)d^3\vec{r}\\\tilde{M}_{ab}=-i\int\psi^+(\vec{r},t)(r_a\sigma_b-r_b\sigma_a)\dot{\psi}(\vec{r},t)d^3\vec{r}\\\bar{M}_{ab}=-i\int\psi^+(\vec{r},t)(r_a\sigma_b-r_b\sigma_a)\psi(\vec{r},t)d^3\vec{r}\end{cases}$

Thm. 4.5.1. $S_{ab}=\int\psi^+(\vec{r},t)S_{ab}(\frac{1}{2},\varsigma)\psi(\vec{r},t)d^3\vec{r}=i\sigma_{\varsigma ab}^{\alpha_\varsigma}\int\psi^+(\vec{r},t)\sigma_{\alpha_\varsigma}(\frac{1}{2})\psi(\vec{r},t)d^3\vec{r}$
$=\frac{-i\varsigma}{2}\sigma_{\varsigma ab}^{\alpha_\varsigma}\int\hat{p}_{\alpha_\varsigma}[a_1^+(\vec{p},-\frac{\varsigma}{2})a_1(\vec{p},-\frac{\varsigma}{2})+a_2(\vec{p},-\frac{\varsigma}{2})a_2^+(\vec{p},-\frac{\varsigma}{2})]d^3\vec{p}$

4.6 Properties 1 of spin operator

Cor. 4.6.1. $\begin{cases}\int\psi^+(\vec{r},t)\sigma(s)\psi(\vec{r},t)d^3\vec{r}=s\varsigma\int\psi^+(\vec{r},t)\frac{\nabla}{\nabla^2}\dot{\psi}(\vec{r},t)d^3\vec{r},s\geq1\\\int\psi^+(\vec{r},t)\sigma(s)\dot{\psi}(\vec{r},t)d^3\vec{r}=s\varsigma\int\psi^+(\vec{r},t)\nabla\psi(\vec{r},t)d^3\vec{r},s\geq1\end{cases}$

Cor. 4.6.2. $\begin{cases}\int\psi^+(\vec{r},t)S_{ab}(s,\varsigma)\psi(\vec{r},t)d^3\vec{r}=is\varsigma\sigma_{\varsigma ab}^{\alpha_\varsigma}\int\psi^+(\vec{r},t)\frac{\nabla_{\alpha_\varsigma}}{\nabla^2}\dot{\psi}(\vec{r},t)d^3\vec{r},s\geq1\\\int\psi^+(\vec{r},t)S_{ab}(s,\varsigma)\dot{\psi}(\vec{r},t)d^3\vec{r}=is\varsigma\sigma_{\varsigma ab}^{\alpha_\varsigma}\int\psi^+(\vec{r},t)\nabla_{\alpha_\varsigma}\psi(\vec{r},t)d^3\vec{r},s\geq1\end{cases}$

Cor. 4.6.3. $\begin{cases}\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\sigma_{\alpha_\varsigma}(n+\frac{1}{2})\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=-(n+\frac{1}{2})\varsigma\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\alpha_\varsigma}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}},n+\frac{1}{2}\geq1\\\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\sigma_{\alpha_\varsigma}(n)\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=n\varsigma\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\nabla_{\alpha_\varsigma}\frac{i\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n},n\geq1\end{cases}$

Cor. 4.6.4. $\begin{cases}\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{ab}(n+\frac{1}{2},\varsigma)\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=-(n+\frac{1}{2})i\varsigma\sigma_{\varsigma ab}^{\alpha_\varsigma}\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\alpha_\varsigma}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}},n+\frac{1}{2}\geq1\\\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{ab}(n,\varsigma)\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=ni\varsigma\sigma_{\varsigma ab}^{\alpha_\varsigma}\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\nabla_{\alpha_\varsigma}\frac{i\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n},n\geq1\end{cases}$

4.7 Properties 2 of spin operator

Cor. 4.7.1. $\int\psi^+(\vec r,t)\sigma_i(s)[\sigma(s)\cdot\nabla]\sigma_j(s)\psi(\vec r,t)d^3\vec r? = \int\psi^+(\vec r,t)\sigma_i(s)\sigma_j(s)[\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r$

Cor. 4.7.2. $\int\psi^+(\vec r,t)\sigma_i(s)\sigma_j(s)\psi(\vec r,t)d^3\vec r = ???$

Cor. 4.7.3. $[\sigma(s)\cdot\nabla]\sigma(s)\psi = \{s\nabla + (s-1)\sigma(s)[\frac{1}{s}\sigma(s)\cdot\nabla]\}\psi$

Proof: $\int\psi^+(\vec r,t)\sigma_i(s)[\sigma(s)\cdot\nabla]\sigma_j(s)\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)\sigma_i(s)\{s\partial_j + (s-1)\sigma_j(s)[\frac{1}{s}\sigma(s)\cdot\nabla]\}\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)s\sigma_i(s)\partial_j\psi(\vec r,t) + \psi^+(\vec r,t)(s-1)\sigma_i(s)\sigma_j(s)[\frac{1}{s}\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)s[\sigma(s)\cdot\nabla]\hat\partial_i\hat\partial_j\psi(\vec r,t) + \psi^+(\vec r,t)(s-1)[s^2\hat\partial_i\hat\partial_j + \frac{s}{2}(\delta_{ij} - \hat\partial_i\hat\partial_j + i\varsigma\varepsilon_{ij}{}^k\hat\partial_k\hat\partial_t)][\frac{1}{s}\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)[s^2\hat\partial_i\hat\partial_j + \frac{s-1}{2}(\delta_{ij} - \hat\partial_i\hat\partial_j + i\varsigma\varepsilon_{ij}{}^k\hat\partial_k\hat\partial_t)][\sigma(s)\cdot\nabla]\psi(\vec r,t)d^3\vec r$ □

Cor. 4.7.4. $[\sigma(s)\cdot\nabla]^2\sigma(s)\psi = \{(2s-1)[\sigma(s)\cdot\nabla]\nabla + (1-\frac{1}{s})^2\sigma(s)[\sigma(s)\cdot\nabla]^2\}\psi$

Proof: $\int\psi^+(\vec r,t)\sigma_i(s)[\sigma(s)\cdot\nabla]^2\sigma_j(s)\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)\sigma_i(s)\{(2s-1)[\sigma(s)\cdot\nabla]\partial_j + (1-\frac{1}{s})^2\sigma_j(s)[\sigma(s)\cdot\nabla]^2\}\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)\{(2s-1)\sigma_i(s)[\sigma(s)\cdot\nabla]\partial_j + (s-1)^2\sigma_i(s)\sigma_j(s)\nabla^2\}\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)\{(2s-1)s^2\partial_i\partial_j + (s-1)^2\sigma_i(s)\sigma_j(s)\nabla^2\}\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)\{(2s-1)s^2\partial_i\partial_j + (s-1)^2[s^2\partial_i\partial_j + \frac{s}{2}(\delta_{ij}\nabla^2 - \partial_i\partial_j + i\varsigma\varepsilon_{ij}{}^k\partial_k\partial_t)]\psi(\vec r,t)d^3\vec r$
$= \int\psi^+(\vec r,t)\{s^4\partial_i\partial_j + \frac{s}{2}(s-1)^2(\delta_{ij}\nabla^2 - \partial_i\partial_j + i\varsigma\varepsilon_{ij}{}^k\partial_k\partial_t)\psi(\vec r,t)d^3\vec r$ □

Proof: $\int\psi^+(\vec r,t)\sigma_{[i}(s)[\sigma(s)\cdot\nabla]^2\sigma_{j]}(s)\psi(\vec r,t)d^3\vec r = \int\psi^+(\vec r,t)s(s-1)^2 i\varsigma\varepsilon_{ij}{}^k\partial_k\partial_t\psi(\vec r,t)d^3\vec r$ □

Proof: $\int\psi^+(\vec r,t)\sigma_{[i}(s)[\sigma(s)\cdot\hat\nabla]^2\sigma_{j]}(s)\psi(\vec r,t)d^3\vec r = (s-1)^2 i\varepsilon_{ij}{}^k\int\psi^+(\vec r,t)\sigma_k(s)\psi(\vec r,t)d^3\vec r$ □

Proof: $\int\psi^+(\vec r,t)\sigma_i(s)\sigma_j(s)\psi(\vec r,t)d^3\vec r = \int\psi^+(\vec r,t)[s^2\hat\partial_i\hat\partial_j + \frac{s}{2}(\delta_{ij}\hat\nabla^2 - \hat\partial_i\hat\partial_j + i\varsigma\varepsilon_{ij}{}^k\hat\partial_k\hat\partial_t)]\psi(\vec r,t)d^3\vec r, s\neq 1$ □

Proof: $\int\psi^+(\vec r,t)\sigma_{[i}(s)\sigma_{j]}(s)\psi(\vec r,t)d^3\vec r = i\varsigma s\int\psi^+(\vec r,t)\varepsilon_{ij}{}^k\hat\partial_k\hat\partial_t\psi(\vec r,t)d^3\vec r = i\varepsilon_{ij}{}^k\int\psi^+(\vec r,t)\sigma_k(s)\psi(\vec r,t)d^3\vec r, s\neq 1$ □

Cor. 4.7.5. $\psi(\vec p,t) = |\vec p|^{(s-\frac{1}{2})}[a_1(\vec p,-s\varsigma)\lambda(\hat p,-s\varsigma)e^{-i|\vec p|t)} + a_2^+(-\vec p,-s\varsigma)\lambda(-\hat p,-s\varsigma)e^{i|\vec p|t)}]$

4.8 Properties 3 of spin operator

Proof: $\int\frac{\psi^+(\vec r,t)}{\sqrt{-\nabla^2}}\sigma_{[i}(\frac{3}{2},\varsigma)\frac{1}{8}\{-1+4[\sigma(\frac{3}{2})\cdot\hat\nabla]^2\}\sigma_{j]}(\frac{3}{2},\varsigma)\frac{\psi(\vec r,t)}{\sqrt{-\nabla^2}}d^3\vec r$ □

Ass. 4.8.1.

$$\begin{cases}P_a(n+\frac{1}{2}) = \int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\hat P_a\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r\\ M_{ab}(n+\frac{1}{2}) = \int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\hat M_{ab}\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r\end{cases}\quad \begin{cases}P_a(n) = \int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\hat P_a\frac{i\dot\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r\\ M_{ab}(n) = \int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\hat M_{ab}\frac{i\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r\end{cases}$$

Ass. 4.8.2.

$$\begin{cases}\hat s(n+\frac{1}{2}) = \int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\sigma(n+\frac{1}{2})\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r = \varsigma(n+\frac{1}{2})\int\frac{\dot\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^{n+1}}d^3\vec r\\ \hat s(n) = \int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\sigma_\alpha(n)\frac{i\dot\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r = i\varsigma n\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\nabla\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r\end{cases}$$

5 Mathematical analysis of constant invariant tensor $\Gamma^{abc\cdots}_{k_\varsigma k'_\varsigma}(s)$

5.1 Properties of covariant constant invariant tensor $\Gamma^{abc\cdots}_{k_\varsigma k'_\varsigma}(s)$ for s-spin field

Pro. 5.1.1. $\Gamma^{\overbrace{\pi\pi\pi\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) = (\frac{-i\varsigma}{\sqrt2})^{2s}\overbrace{(i\varsigma)_{A_\varsigma A'_\varsigma}(i\varsigma)_{B_\varsigma B'_\varsigma}(i\varsigma)_{C_\varsigma C'_\varsigma}\cdots}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{k'_\varsigma}(s)$
$= (\frac{1}{\sqrt2})^{2s}\delta_{k_\varsigma k'_\varsigma}$

Pro. 5.1.2. $\Gamma^{\overbrace{i\pi\pi\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) = (\frac{-i\varsigma}{\sqrt2})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(i\varsigma)_{B_\varsigma B'_\varsigma}(i\varsigma)_{C_\varsigma C'_\varsigma}\cdots}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{k'_\varsigma}(s)$
$= -i\varsigma(\frac{1}{\sqrt2})^{2s}\frac{1}{s}\sigma^i(s)_{k_\varsigma k'_\varsigma}$

Pro. 5.1.3. $\Gamma^{\overbrace{ij\pi\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) = (\frac{-i\varsigma}{\sqrt2})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}(i\varsigma)_{C_\varsigma C'_\varsigma}\cdots}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}^{2s}}_{k'_\varsigma}(s)$
$= -(\frac{1}{\sqrt2})^{2s}\frac{1}{s(s-\frac{1}{2})}\frac{1}{2!}[\sigma^{\{i}(s)\sigma^{j\}}(s) - \frac{s}{2}\delta^{\{ij\}}]_{k_\varsigma k'_\varsigma}$

Pro. 5.1.4. $\Gamma^{\overbrace{ijk\pi\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) = (\frac{-i\varsigma}{\sqrt2})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}(\sigma)^k_{C_\varsigma C'_\varsigma}(i\varsigma)_{D_\varsigma D'_\varsigma}\cdots}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma\cdots}^{2s}}_{k'_\varsigma}(s)$
$= (\frac{1}{\sqrt2})^{2s}\frac{i\varsigma}{s(s-\frac{1}{2})(s-1)}\frac{1}{3!}[\sigma^{\{i}(s)\sigma^j(s)\sigma^{k\}}(s) + \frac{1-3s}{2}\delta^{\{ij}\sigma^{k\}}(s)]_{k_\varsigma k'_\varsigma}$

Pro. 5.1.5. $\Gamma^{\overbrace{ijkl\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}(\sigma)^k_{C_\varsigma C'_\varsigma}(\sigma)^l_{D_\varsigma D'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)$
$=(\frac{1}{\sqrt{2}})^{2s}\frac{1}{s(s-\frac{1}{2})(s-1)(s-\frac{3}{2})}\frac{1}{4!}[\sigma^{\{i}(s)\sigma^j(s)\sigma^k(s)\sigma^{l\}}(s)+(2-3s)\sigma^{\{i}(s)\sigma^j(s)\delta^{kl\}}+\frac{3}{4}s(s-1)\delta^{\{ij}\delta^{kl\}}]_{k_\varsigma k'_\varsigma}$

5.2 Important relations

Lem. 5.2.1.

$$\begin{cases}(\sigma\cdot\nabla)^{A'_\varsigma A_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(s)\psi_{k_\varsigma}(\vec{r},t)=i\varsigma\partial_\pi\delta^{A'_\varsigma A_\varsigma}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(s)\psi_{k_\varsigma}(\vec{r},t),[\sigma(s)\cdot\nabla]^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec{r},t)=is\varsigma\partial_\pi\delta^{k'_\varsigma k_\varsigma}\psi_{k_\varsigma}(\vec{r},t)\\(\sigma\cdot\nabla)_{A_\varsigma A'_\varsigma}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\psi^{k'_\varsigma}(\vec{r},t)=-i\varsigma\partial_\pi\delta^{A_\varsigma A'_\varsigma}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\psi^{k'_\varsigma}(\vec{r},t),[\sigma(s)\cdot\nabla]_{k_\varsigma k'_\varsigma}\psi^{k'_\varsigma}(\vec{r},t)=-is\varsigma\partial_\pi\delta_{k_\varsigma k'_\varsigma}\psi^{k'_\varsigma}(\vec{r},t)\end{cases}$$

Thm. 5.2.1. $\Gamma^{\overbrace{ij\cdot\cdot}^{n}}_{k_\varsigma k'_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{n}\psi(\vec{r},t)=2^{-s}\delta_{k_\varsigma k'_\varsigma}(\partial_\pi)^n\psi^{k'_\varsigma}(\vec{r},t)$

Proof: $\Gamma^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{n}\psi^{k'_\varsigma}(\vec{r},t)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P_\varsigma P'_\varsigma}(i\varsigma)_{Q_\varsigma Q'_\varsigma}\cdot\cdot}^{2s-n}\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{n}\psi^{k'_\varsigma}(\vec{r},t)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma\cdot\nabla)_{A_\varsigma A'_\varsigma}(\sigma\cdot\nabla)_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P_\varsigma P'_\varsigma}(i\varsigma)_{Q_\varsigma Q'_\varsigma}\cdot\cdot}^{2s-n}\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\psi^{k'_\varsigma}(\vec{r},t)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(-i\varsigma\partial_\pi)_{A_\varsigma A'_\varsigma}(-i\varsigma\partial_\pi)_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P_\varsigma P'_\varsigma}(i\varsigma)_{Q_\varsigma Q'_\varsigma}\cdot\cdot}^{2s-n}\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\psi^{k'_\varsigma}(\vec{r},t)$

$=2^{-s}\delta_{k_\varsigma k'_\varsigma}(-\partial_\pi)^n\psi(\vec{r},t)$ □

Cor. 5.2.1.

$$\begin{cases}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)=\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\overbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}^{2s}\\\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(s)\lambda_{k_\varsigma}(\hat{p},-s\varsigma)=\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}_{2s}\end{cases}$$

Thm. 5.2.2. $\Gamma^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{n}\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)=\frac{i^n}{2^s}\lambda_{k_\varsigma}(\hat{p},-s\varsigma),\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdot\cdot}_{n}\underbrace{\pi\cdot\cdot\pi}_{2s-n}}(s)\overbrace{\hat{p}^i\hat{p}^j\cdot\cdot}^{n}\lambda_{k_\varsigma}(\hat{p},s\varsigma)=\frac{i^n}{2^s}\lambda^{k'_\varsigma}(\hat{p},s\varsigma)$

Proof: $\Gamma^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{n}\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)^i_{A_\varsigma A'_\varsigma}(\sigma)^j_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P_\varsigma P'_\varsigma}(i\varsigma)_{Q_\varsigma Q'_\varsigma}\cdot\cdot}^{2s-n}\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{n}\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)$

$\overbrace{(\sigma\cdot\hat{p})_{A_\varsigma A'_\varsigma}(\sigma\cdot\hat{p})_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P_\varsigma P'_\varsigma}(i\varsigma)_{Q_\varsigma Q'_\varsigma}\cdot\cdot}^{2s-n}\overbrace{\lambda^{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot\lambda^{P'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{Q'_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}^{2s}$

$=\frac{i^n}{2^s}\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\overbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot\lambda_{P_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{Q_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}^{2s}$

$=\frac{i^n}{2^s}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)$ □

Thm. 5.2.3. $\Gamma^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_i\hat{p}_j\cdot\cdot}^{n}\lambda^{k'_\varsigma}(\hat{p},s\varsigma)=\frac{(-i)^n}{2^s}\lambda_{k_\varsigma}(\hat{p},s\varsigma),\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdot\cdot}_{n}\underbrace{\pi\cdot\cdot\pi}_{2s-n}}(s)\overbrace{\hat{p}^i\hat{p}^j\cdot\cdot}^{n}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)=\frac{(-i)^n}{2^s}\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)$

Thm. 5.2.4. $\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdot\cdot}_{n}\underbrace{\pi\cdot\cdot\pi}_{2s-n}}(s)\overbrace{\hat{p}^i\hat{p}^j\cdot\cdot}^{n}\lambda_{k_\varsigma}(\hat{p},s\varsigma)=\frac{i^n}{2^s}\lambda^{k'_\varsigma}(\hat{p},s\varsigma)$

Thm. 5.2.5. $\begin{cases} \Gamma^{\overbrace{ij\cdot\cdot}^{n}}_{k'_\varsigma k_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{n}\psi^{k_\varsigma}(\vec{r},t)=2^{-s}\delta_{k'_\varsigma k_\varsigma}\partial^n_\pi\psi(\vec{r},t) \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k'_\varsigma k_\varsigma}(s):=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,i\varsigma)^a_{A'_\varsigma A_\varsigma}(\sigma,i\varsigma)^b_{B'_\varsigma B_\varsigma}(\sigma,i\varsigma)^c_{C'_\varsigma C_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s) \end{cases}$

Proof: $\Gamma^{\overbrace{ij\cdot\cdot}^{n}\overbrace{\pi\cdot\cdot\pi}^{2s-n}}_{k'_\varsigma k_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{n}\psi^{k_\varsigma}(\vec{r},t)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma)^i_{A'_\varsigma A_\varsigma}(\sigma)^j_{B'_\varsigma B_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P'_\varsigma P_\varsigma}(i\varsigma)_{Q'_\varsigma Q_\varsigma}\cdot\cdot}^{2s-n}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{n}\psi^{k_\varsigma}(\vec{r},t)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma\cdot\nabla)_{A'_\varsigma A_\varsigma}(\sigma\cdot\nabla)_{B'_\varsigma B_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P'_\varsigma P_\varsigma}(i\varsigma)_{Q'_\varsigma Q_\varsigma}\cdot\cdot}^{2s-n}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\psi^{k_\varsigma}(\vec{r},t)$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(i\varsigma\partial_\pi)_{A'_\varsigma A_\varsigma}(i\varsigma\partial_\pi)_{B'_\varsigma B_\varsigma}\cdot\cdot}^{n}\overbrace{(i\varsigma)_{P'_\varsigma P_\varsigma}(i\varsigma)_{Q'_\varsigma Q_\varsigma}\cdot\cdot}^{2s-n}\Gamma^{\overbrace{A'_\varsigma B'_\varsigma\cdot\cdot P'_\varsigma Q'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\Gamma^{\overbrace{A_\varsigma B_\varsigma\cdot\cdot P_\varsigma Q_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\psi^{k_\varsigma}(\vec{r},t)$

$=2^{-s}\delta_{k'_\varsigma k_\varsigma}(\partial_\pi)^n\psi^{k_\varsigma}(\vec{r},t)$ □

5.3 Important theorem of covariant constant invariant tensor $\Gamma^{abc\cdot\cdot}_{k_\varsigma k'_\varsigma}(s)$ for s-spin field

Lem. 5.3.1. $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_a\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_b\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_c\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)\lambda^+_{k'_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)$

Proof: $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_a\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_b\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_c\cdot\cdot}^{2s}$

$=(\frac{-i\varsigma}{\sqrt{2}})^{2s}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)\overbrace{\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_a\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_b\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_c\cdot\cdot}^{2s}$

$=\begin{cases}(i\sqrt{2})^{2s}\Gamma^{\overbrace{1_\varsigma 1_\varsigma 1_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{1'_\varsigma 1'_\varsigma 1'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},s)\lambda^+_{k'_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},s),\varsigma=-1\\ (i\sqrt{2})^{2s}\Gamma^{\overbrace{2_\varsigma 2_\varsigma 2_\varsigma\cdot\cdot}^{2s}}_{k_\varsigma}(s)\Gamma^{\overbrace{2'_\varsigma 2'_\varsigma 2'_\varsigma\cdot\cdot}^{2s}}_{k'_\varsigma}(s)=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s)\lambda^+_{k'_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s),\varsigma=1\end{cases}$

$=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)\lambda^+_{k'_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)$ □

Thm. 5.3.1. $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma)$

The above $s=\frac{1}{2},1,\frac{3}{2},2$ have been proved, and $s>2$ is still a conjecture. In the following the constant invariant tensor analysis method is used to uniformly prove it.

Proof: $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_a\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_b\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_c\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)\lambda^+_{k'_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)$

$\Leftrightarrow\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)[exp\{-i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|_a{}^{\tilde{a}}exp\{-i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|_b{}^{\tilde{b}}exp\{-i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|_c{}^{\tilde{c}}\cdot\cdot]\overbrace{\hat{p}_{\tilde{a}}\hat{p}_{\tilde{b}}\hat{p}_{\tilde{c}}\cdot\cdot}^{2s}$

$=(i\sqrt{2})^{2s}exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|_{k_\varsigma}{}^{\tilde{k}_\varsigma}\lambda_{\tilde{k}_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{\tilde{k}'_\varsigma}(\hat{p},-s\varsigma)exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|^{\tilde{k}'_\varsigma}{}_{k'_\varsigma}$

$\Leftrightarrow[exp\{i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|^{\tilde{a}}{}_a exp\{i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|^{\tilde{b}}{}_b exp\{i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|^{\tilde{c}}{}_c\cdot\cdot]$

$exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|_{\tilde{k}_\varsigma}{}^{k_\varsigma}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|^{k'_\varsigma}{}_{\tilde{k}'_\varsigma}\overbrace{\hat{p}_{\tilde{a}}\hat{p}_{\tilde{b}}\hat{p}_{\tilde{c}}\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{\tilde{k}_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{\tilde{k}'_\varsigma}(\hat{p},-s\varsigma)$

$\Leftrightarrow\Gamma^{\overbrace{\tilde{a}\tilde{b}\tilde{c}\cdot\cdot}^{2s}}_{\tilde{k}_\varsigma\tilde{k}'_\varsigma}(s)\overbrace{\hat{p}_{\tilde{a}}\hat{p}_{\tilde{b}}\hat{p}_{\tilde{c}}\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{\tilde{k}_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{\tilde{k}'_\varsigma}(\hat{p},-s\varsigma)$

$\Leftrightarrow\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}=(i\sqrt{2})^{2s}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma)$ □

Cor. 5.3.1. *Projection operator:* $\hat{P}_{k_\varsigma k'_\varsigma}(s,\varsigma) = (i\sqrt{2})^{-2s}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\,\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}, \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\,\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} = (i\sqrt{2})^{2s}\hat{P}_{k_\varsigma k'_\varsigma}(s,\varsigma)$

Cor. 5.3.2. $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\,\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} \succ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\,\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} = (i\sqrt{2})^{2s}\lambda(\hat{p},-s\varsigma)\lambda^{+}(\hat{p},-s\varsigma), s \geq 0$

Cor. 5.3.3. $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\,\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s}\lambda(\hat{p},-s\varsigma) = (i\sqrt{2})^{2s}\lambda(\hat{p},-s\varsigma)$

Cor. 5.3.4. $\begin{cases} \lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\lambda_{k_\varsigma}(\hat{p},-s\varsigma) = 1, \lambda^{+k_\varsigma}(-\hat{p},-s\varsigma)\lambda_{k_\varsigma}(\hat{p},-s\varsigma) = 0 \\ \lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^{+}_{k'_\varsigma}(\hat{p},-s\varsigma) = (i\sqrt{2})^{-2s}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\,\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdot\cdot}^{2s} \end{cases}$

5.4 Re-summarize of operators $\hat{p}_a, \hat{\partial}_a$ and $\Gamma^{abc\cdot\cdot}_{+}(s), \Gamma^{abc\cdot\cdot}_{-}(s)$

Def. 5.4.1. $\hat{p}_a := \frac{p_a}{|\vec{p}|} = (\hat{p}, i); \hat{p} = \frac{\vec{p}}{|\vec{p}|}, \hat{p}_\pi = \frac{p_\pi}{|\vec{p}|} = i; \hat{p}^2 = 1, \hat{p}_\pi^2 = i^2$

Def. 5.4.2. $\hat{\partial}_a := \frac{\partial_a}{i\sqrt{-\nabla^2}} = \frac{-i\partial_a}{\sqrt{-\nabla^2}} = \frac{(-i\nabla, -\partial_t)}{\sqrt{-\nabla^2}}; \hat{\nabla} = \frac{\nabla}{i\sqrt{-\nabla^2}} = \frac{-i\nabla}{\sqrt{-\nabla^2}}; \hat{\nabla}^2 = 1, \hat{\nabla}_\pi^2 = i^2$

Cor. 5.4.1. $p_a \simeq -i\partial_a, |\vec{p}| \simeq \sqrt{-\nabla^2}, \hat{p}_a \simeq \hat{\partial}_a, p_a = |\vec{p}|\hat{p}_a, \partial_a = (i\sqrt{-\nabla^2})\hat{\partial}_a$

Def. 5.4.3. $odd := -, even := +$

Def. 5.4.4. $\begin{cases} \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s) = 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s), 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s), l = 0,\cdot\cdot, 2s \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{+}(s) := 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s), 0\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s), l = 0,\cdot\cdot, 2s \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s) := 0\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s), 1\cdot\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s), l = 0,\cdot\cdot, 2s \end{cases}$

Cor. 5.4.2. $\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s) = \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{+}(s) + \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s)$

5.5 Properties of operators $\Gamma^{abc\cdot\cdot}_{\pm}(s)\partial_a\partial_b\partial_c\cdot\cdot$ and $\Gamma^{abc\cdot\cdot}_{\pm}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot$ under shelly conditions

Cor. 5.5.1. $\partial^a\partial_a\psi = 0 \Rightarrow \begin{cases} \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{+}(s)\,\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\psi = \sum\limits_{l=0}^{[s]} C_{2s}^{2n}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s)\,\overbrace{\partial_i\partial_j\cdot\cdot}^{2s-2l}(\sqrt{-\nabla^2})^{2l}\psi \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s)\,\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\psi = \sum\limits_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2n+1}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\,\overbrace{\partial_i\partial_j\cdot\cdot}^{2s-2n-1}(\sqrt{-\nabla^2})^{2l}\partial_\pi\psi \end{cases}$

Cor. 5.5.2. $\partial^a\partial_a\psi = 0 \Rightarrow \begin{cases} \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{+}(s)\,\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s}\psi = \sum\limits_{n=0}^{[s]}(-1)^l C_{2s}^{2l}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l}\overbrace{\pi\cdot\cdot\pi}^{2l}}(s)\,\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{2s-2l}\psi \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s)\,\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s}\psi = \sum\limits_{n=0}^{[s-\frac{1}{2}]}(-1)^l C_{2s}^{2l+1}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\,\overbrace{\hat{\partial}_i\hat{\partial}_j\cdot\cdot}^{2s-2l-1}\hat{\partial}_\pi\psi \end{cases}$

Cor. 5.5.3. $\partial^a\partial_a\psi = 0 \Rightarrow \begin{cases} \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{+}(s)\,\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s}\psi = \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{+}(s)\,\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}\psi \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s)\,\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s}\psi = -i\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s)\,\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdot\cdot}^{2s,\hat{\partial}_\pi\to i}\hat{\partial}_\pi\psi \end{cases}$

5.6 Properties of operators $\Gamma^{abc\cdot\cdot}(s)\partial_a\partial_b\partial_c\cdot\cdot\Delta(x-x')|_{t=t'}$

Pro. 5.6.1. $\begin{cases} \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s)\,\overbrace{p_a p_b p_c\cdot\cdot}^{2s} := \sum\limits_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\,\overbrace{p_i p_j\cdot\cdot}^{2s-2l-1}p_\pi^{2l+1} \\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{-}(s)\,\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s} := \sum\limits_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1}\Gamma^{\overbrace{ij\cdot\cdot}^{2s-2l-1}\overbrace{\pi\cdot\cdot\pi}^{2l+1}}(s)\,\overbrace{\partial_i\partial_j\cdot\cdot}^{2s-2l-1}\partial_\pi^{2l+1} \end{cases}$

Cor. 5.6.1.

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x')|_{t=t'}=i\sum\limits_{l=0}^{[s-\frac{1}{2}]}(-1)^l C_{2s}^{2l+1}\Gamma^{\overbrace{ij\cdots}^{2s-2l-1}\overbrace{\pi\cdots\pi}^{2l+1}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2l-1}\nabla^{2l}\delta^3(\vec{r}-\vec{r}')\\ \Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s}\Delta(x-x')|_{t=t'}=\frac{1}{\sqrt{-\nabla^2}}\sum\limits_{l=0}^{[s-\frac{1}{2}]}(-1)^l C_{2s}^{2l+1}\Gamma^{\overbrace{ij\cdots}^{2s-2l-1}\overbrace{\pi\cdots\pi}^{2l+1}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdots}^{2s-2l-1}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x')|_{t=t'}=(i\sqrt{-\nabla^2})^{2s-1}\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\\ \Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s}\Delta(x-x')|_{t=t'}=\frac{1}{i\sqrt{-\nabla^2}}\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

5.7 Properties of operators $\Gamma^{abc\cdots}(s)\partial_a\partial_b\partial_c\cdots|\partial_\pi\Delta(x-x')|_{t=t'}$

Cor. 5.7.1.

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i\sum\limits_{l=0}^{[s]}(-1)^l C_{2s}^{2l}\Gamma^{\overbrace{ij\cdots}^{2s-2l}\overbrace{\pi\cdots\pi}^{2l}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2l}\nabla^{2l}\delta^3(\vec{r}-\vec{r}')\\ \Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i\sum\limits_{l=0}^{[s]}(-1)^l C_{2s}^{2l}\Gamma^{\overbrace{ij\cdots}^{2s-2l}\overbrace{\pi\cdots\pi}^{2l}}(s)\overbrace{\hat{\partial}_i\hat{\partial}_j\cdots}^{2s-2l}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i(i\sqrt{-\nabla^2})^{2s}\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\\ \Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s}\partial_\pi\Delta(x-x')|_{t=t'}=i\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

5.8 Several important theorems

Thm. 5.8.1.

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s}\hat{\partial}_\pi\Delta(x-x')|_{t=t'}=i\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\\ \Gamma^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s}\Delta(x-x')|_{t=t'}=\frac{1}{i\sqrt{-\nabla^2}}\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

Thm. 5.8.2.

$$\begin{cases}\Gamma^{\overbrace{abc\cdots}^{2n}}(n)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n}\hat{\partial}_\pi\Delta(x-x')|_{t=t'}=i\Gamma_+^{\overbrace{abc\cdots}^{2n}}(n)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\\ \Gamma^{\overbrace{abc\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n+1}\Delta(x-x')|_{t=t'}=\frac{1}{i\sqrt{-\nabla^2}}\Gamma_-^{\overbrace{abc\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n+1,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

Ass. 5.8.1.

$$\Gamma_+^{\overbrace{ab\cdots}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\cdots}^{2s}\lambda(\hat{p},-s\varsigma)=\Gamma_-^{\overbrace{ab\cdots}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\cdots}^{2s}\lambda(\hat{p},-s\varsigma)=\frac{1}{2}\Gamma^{\overbrace{ab\cdots}^{2s}}(s)\overbrace{\hat{p}_a\hat{p}_b\cdots}^{2s}\lambda(\hat{p},-s\varsigma)=\frac{(i\sqrt{2})^{2s}}{2}\lambda(\hat{p},-s\varsigma)$$

his conjecture has been verified to be correct for the low spin case, but it needs to be strictly proved for the general case.

Cor. 5.8.1.

$$\begin{cases}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdots}^{2n}}(n)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n}\psi(\vec{r},t;n)=(-2)^n\psi(\vec{r},t;n)\\ \Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n+1}\hat{\partial}_\pi\psi(\vec{r},t;n+\frac{1}{2})=-(-2)^n\sqrt{2}\psi(\vec{r},t;n+\frac{1}{2})\end{cases}$$

Proof: $\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdots}^{2n}}(n)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n}\psi(\vec{r},t;n)$

$$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}|\vec{p}|^{(n-\frac{1}{2})}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdots}^{2n}}(n)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdots}^{2n}\lambda(\hat{p},-n\varsigma)[a_1(\vec{p},-n\varsigma)e^{ip\cdot x}+(-1)^{2n}a_2^+(\vec{p},-n\varsigma)e^{-ip\cdot x}]d^3\vec{p}$$

$$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}|\vec{p}|^{(n-\frac{1}{2})}(i\sqrt{2})^{2n}\lambda(\hat{p},-s\varsigma)\lambda(\hat{p},-n\varsigma)[a_1(\vec{p},-n\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-n\varsigma)e^{-ip\cdot x}]d^3\vec{p}$$

$$=(-2)^n\psi(\vec{r},t;n)$$

□

Proof: $\Gamma^{\overbrace{abc\cdots}^{2n+1}}_{k_\varsigma k'_\varsigma}(n+\frac{1}{2})\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2n+1}\hat{\partial}_\pi\psi(\vec{r},t;n+\frac{1}{2})$
$= \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0} d^3\vec{p}$
$|\vec{p}|^n\Gamma^{\overbrace{abc\cdots}^{2n}}_{k_\varsigma k'_\varsigma}(n+\frac{1}{2})\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdots}^{2n+1}\hat{p}_\pi\lambda(\hat{p},-(n+\frac{1}{2})\varsigma)[a_1(\vec{p},-(n+\frac{1}{2})\varsigma)e^{ip\cdot x}-(-1)^{2n+1}a_2^+(\vec{p},-(n+\frac{1}{2})\varsigma)e^{-ip\cdot x}]$
$= \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}|\vec{p}|^n i(i\sqrt{2})^{2n+1}\lambda(\hat{p},-(n+\frac{1}{2})\varsigma)\lambda(\hat{p},-(n+\frac{1}{2})\varsigma)[a_1(\vec{p},-(n+\frac{1}{2})\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-(n+\frac{1}{2})\varsigma)e^{-ip\cdot x}]d^3\vec{p}$
$= i(i\sqrt{2})^{2n+1}\psi(\vec{r},t;n+\frac{1}{2})$ □

6 Commutative rule of s-spin field

6.1 Commutative and anticommutative formulas

Cor. 6.1.1. $\begin{cases}[A,BC]=[A,B]C+B[A,C],[A,CB]=[A,C]B+C[A,B]\\ [A,BC]=\{A,B\}C-B\{A,C\},[A,CB]=\{A,C\}B-C\{A,B\}\end{cases}$

Cor. 6.1.2. $\begin{cases}[A,\{B,C\}]=\{[A,B],C\}+\{B,[A,C]\}\\ [A,[B,C]]=\{\{A,B\},C\}-\{B,\{A,C\}\}\end{cases}$

6.2 General covariant commutation rules in mathematics for s-spin field

Thm. 6.2.1. $\begin{cases}[a_\sigma(\vec{p},-s\varsigma),a^+_{\sigma'}(\vec{p}',-s\varsigma)]_\pm=\delta_\sigma\delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}')\\ [a_\sigma(\vec{p},-s\varsigma),a_{\sigma'}(\vec{p}',-s\varsigma)]_\pm=0,[a^+_\sigma(\vec{p},-s\varsigma),a^+_{\sigma'}(\vec{p}',-s\varsigma)]_\pm=0\end{cases}$

$\Rightarrow\begin{cases}[\Psi_{k_\varsigma}(x),\Psi^+_{k'_\varsigma}(x')]_\pm\\ =i(-\sqrt{2})^{-2(s-1)}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\{[\delta_1\pm(-1)^{2s}\delta_2]\Delta^{(+)}(x-x')\pm(-1)^{2s+1}\delta_2\Delta(x-x')\}\\ [\Psi_{k_\varsigma}(x),\Psi_{\beta_\varsigma}(x')]_\pm=0,[\Psi^+_{k'_\varsigma}(x),\Psi^+_{\beta'_\varsigma}(x')]_\pm=0\end{cases}$

Proof: $[\Psi^{(+)}_{k_\varsigma}(x),\Psi^{(+)+}_{k'_\varsigma}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-s\varsigma)|\vec{p}|^{(2s-1)/2}|\vec{p}'|^{(2s-1)/2}[a_1(\vec{p},-s\varsigma),a^+_1(\vec{p}',-s\varsigma)]_\pm e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-s\varsigma)|\vec{p}|^{2s-1}\delta_1\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma)\delta_1|\vec{p}|^{2s-1}e^{ip\cdot(x-x')}d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int(i\sqrt{2})^{-2s}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdots}^{2s}\delta_1|\vec{p}|^{2s-1}e^{ip\cdot(x-x')}d^3\vec{p}$
$=-(i\sqrt{2})^{-2(s-1)}\frac{\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{p_ap_bp_c\cdots}^{2s}e^{ip\cdot(x-x')}d^3\vec{p}$
$=i^{-2s}(\sqrt{2})^{-2(s-1)}\frac{\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}i^{-2s}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}e^{ip\cdot(x-x')}d^3\vec{p}$
$=(-\sqrt{2})^{-2(s-1)}\frac{\delta_1}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}e^{ip\cdot(x-x')}d^3\vec{p}$
$=i(-\sqrt{2})^{-2(s-1)}\delta_1\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta^{(+)}(x-x')$ □

Proof: $[\Psi^{(-)}_{k_\varsigma}(x),\Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$
$=\frac{1}{(2\pi)^3}\int\limits_{\vec{p}\neq 0}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-s\varsigma)|\vec{p}|^{(2s-1)/2}|\vec{p}'|^{(2s-1)/2}[a^+_2(\vec{p},-s\varsigma),a_2(\vec{p}',-s\varsigma)]_\pm e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\pm\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\vec{p}',-s\varsigma)|\vec{p}|^{2s-1}\delta_2\delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}d^3\vec{p}d^3\vec{p}'$
$=\pm\frac{1}{(2\pi)^3}\int\lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma)\delta_2|\vec{p}|^{2s-1}e^{-ip\cdot(x-x')}d^3\vec{p}$
$=\pm\frac{1}{(2\pi)^3}\int(i\sqrt{2})^{-2s}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\cdots}^{2s}\delta_2|\vec{p}|^{2s-1}e^{-ip\cdot(x-x')}d^3\vec{p}$
$=-\pm(i\sqrt{2})^{-2(s-1)}\frac{\delta_2}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{p_ap_bp_c\cdots}^{2s}e^{-ip\cdot(x-x')}d^3\vec{p}$
$=\pm i^{-2s}(\sqrt{2})^{-2(s-1)}\frac{\delta_2}{(2\pi)^3}\int\frac{1}{2|\vec{p}|}(-i)^{-2s}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}e^{-ip\cdot(x-x')}d^3\vec{p}$

$= \pm(\sqrt{2})^{-2(s-1)} \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} e^{-ip\cdot(x-x')} d^3\vec{p}$

$= - \pm i(\sqrt{2})^{-2(s-1)} \delta_2 \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \Delta^{(-)}(x-x')$ □

Proof: $[\Psi_{k_\varsigma}(x), \Psi^+_{k'_\varsigma}(x')]_\pm$

$= [\Psi^{(+)}_{k_\varsigma}(x), \Psi^{(+)+}_{k'_\varsigma}(x')]_\pm + [\Psi^{(-)}_{k_\varsigma}(x), \Psi^{(-)+}_{k'_\varsigma}(x')]_\pm$

$= i(-\sqrt{2})^{-2(s-1)} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} [\delta_1 \Delta^{(+)}(x-x') \pm (-1)^{2s+1} \delta_2 \Delta^{(-)}(x-x')]$

$= i(-\sqrt{2})^{-2(s-1)} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \{[\delta_1 \pm (-1)^{2s}\delta_2]\Delta^{(+)}(x-x') \pm (-1)^{2s+1}\delta_2\Delta(x-x')\}$ □

From the above, only $\delta_1 \pm (-1)^{-2s}\delta_2 = 0$, the micro causality is satisfied. At the same time only when $\delta_1, \delta_2 \geq 0$, the probability is just nonnegative. Therefore, among the eight covariant commutative or anticommutative schemes in mathematics, there is only one physically reasonable scheme: That is, when $\delta_1 = \delta_2 = 1$,(if not 1,it can be normalized.) It satisfies the commutative relation for bosons and satisfies the anticommutative relation for fermions. There are actually two other options. Namely when $\delta_1 = \delta_2 = 0$, it satisfies the commutative or anticommutative relation, which is just the classic case.

6.3 Covariant commutation rules for s-spin field physics

Def. 6.3.1. $\Delta_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \Delta(x)$

Thm. 6.3.1.

$$\begin{cases} [a_\sigma(\vec{p},-s\varsigma), a^+_{\sigma'}(\vec{p}',-s\varsigma)]_{-^{2s+1}} = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p},-s\varsigma), a_{\sigma'}(\vec{p}',-s\varsigma)]_{-^{2s+1}} = 0 \\ [a^+_\sigma(\vec{p},-s\varsigma), a^+_{\sigma'}(\vec{p}',-s\varsigma)]_{-^{2s+1}} = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')]_{-^{2s+1}}, s \geq 0 \\ = i\frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \Delta(x-x'), \Gamma(0) := 1 \\ [\psi_{k_\varsigma}(x), \psi_{l_\varsigma}(x')]_{-^{2s+1}} = 0, [\psi^+_{k'_\varsigma}(x), \psi^+_{l'_\varsigma}(x')]_{-^{2s+1}} = 0 \end{cases}$$

Proof: $\{\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')\}$

$= \frac{1}{(2\pi)^3} \int\limits_{\vec{p}\neq 0} \lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma)|\vec{p}|^{(2s-1)/2}|\vec{p}'|^{(2s-1)/2}$

$\{[a_1(\vec{p},-s\varsigma), a^+_1(\vec{p}',-s\varsigma)]_{-^{2s+1}} e^{ip\cdot(x-x')} + [a^+_2(\vec{p},-s\varsigma), a_2(\vec{p}',-s\varsigma)]_{-^{2s+1}} e^{-ip\cdot(x-x')}\} d^3\vec{p} d^3\vec{p}'$

$= \frac{1}{(2\pi)^3} \int |\vec{p}|^{2s-1} \lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma)[\delta^3(\vec{p}-\vec{p}')e^{ip\cdot(x-x')} + (-1)^{2s+1}\delta^3(\vec{p}-\vec{p}')e^{-ip\cdot(x-x')}] d^3\vec{p} d^3\vec{p}'$

$= \frac{1}{(2\pi)^3} \int |\vec{p}|^{2s-1} \lambda_{k_\varsigma}(\hat{p},-s\varsigma)\lambda^+_{k'_\varsigma}(\hat{p},-s\varsigma)[e^{ip\cdot(x-x')} + (-1)^{2s+1}e^{-ip\cdot(x-x')}] d^3\vec{p}$

$= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{-1}{(i\sqrt{2})^{2(s-1)}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{p_a p_b p_c\cdots}^{2s} [e^{ip\cdot(x-x')} + (-1)^{2s+1}e^{-ip\cdot(x-x')}] d^3\vec{p}$

$= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{(-i)^{2(s-1)}}{(i\sqrt{2})^{2(s-1)}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} [e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}] d^3\vec{p}$

$= \frac{i}{(-\sqrt{2})^{2(s-1)}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}] d^3\vec{p}$

$= \frac{i}{(-\sqrt{2})^{2(s-1)}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \Delta(x-x')$

$= i\Delta_{k_\varsigma k'_\varsigma}(s; x-x')$ □

6.4 Isochronous commutation rules for s-spin field

Cor. 6.4.1.

$\Delta_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \Delta(x), \Delta_{k_\varsigma k'_\varsigma}(s;x)|_{t=0} = \frac{(-1)^{2s}}{2^{s-1}} (i\sqrt{-\nabla^2})^{2s-1} \Gamma^{\overbrace{abc\cdots}^{2s}}_{-}(s) \overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i} \delta^3(\vec{r})$

Cor. 6.4.2.

$$\begin{cases} [\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')]_{-^{2s+1}}, s \geq 0 \\ = i\frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a\partial_b\partial_c\cdots}^{2s} \Delta(x-x') \\ [\psi_{k_\varsigma}(x), \psi_{l_\varsigma}(x')]_{-^{2s+1}} = 0 \\ [\psi^+_{k'_\varsigma}(x), \psi^+_{l'_\varsigma}(x')]_{-^{2s+1}} = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\varsigma}(\vec{r},t), \psi^+_{k'_\varsigma}(\vec{r}',t)]_{-^{2s+1}}, s > 0 \\ = i\frac{(-1)^{2s}}{2^{s-1}} (i\sqrt{-\nabla^2})^{2s-1} \Gamma^{\overbrace{abc\cdots}^{2s}}_{-}(s) \overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots}^{2s,\hat{\partial}_\pi\to i} \delta^3(\vec{r}-\vec{r}') \\ [\psi_{k_\varsigma}(\vec{r},t), \psi_{l_\varsigma}(\vec{r}',t)]_{-^{2s+1}} = 0 \\ [\psi^+_{k'_\varsigma}(\vec{r},t), \psi^+_{l'_\varsigma}(\vec{r}',t)]_{-^{2s+1}} = 0 \end{cases}$$

Cor. 6.4.3.

$$\begin{cases}[\dot{\psi}_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]_{-^{2s+1}},s\geq 0\\ =i\frac{(-1)^{2s}}{2^{s-1}}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}i\partial_\pi\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]_{-^{2s+1}}=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]_{-^{2s+1}}=0\end{cases}\Rightarrow\begin{cases}[\dot{\psi}_{k_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]_{-^{2s+1}},s>0\\ =i\frac{(-1)^{2s+1}}{2^{s-1}}(i\sqrt{-\nabla^2})^{2s}\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s,\hat\partial_\pi\to i}\delta^3(\vec r-\vec r')\\ [\psi_{k_\varsigma}(\vec r,t),\psi_{l_\varsigma}(\vec r',t)]_{-^{2s+1}}=0\\ [\psi^+_{k'_\varsigma}(\vec r,t),\psi^+_{l'_\varsigma}(\vec r',t)]_{-^{2s+1}}=0\end{cases}$$

Cor. 6.4.4.

$$\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]_{-^{2s+1}},s\geq 0\\ =i\frac{(-1)^{2s}}{2^{s-1}}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]_{-^{2s+1}}=0\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]_{-^{2s+1}}=0\end{cases}\Rightarrow\begin{cases}[\psi_{k_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]_{-^{2s+1}}=\frac{(-1)^{2s+1}}{2^{s-1}},s>0\\ \sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2n-1}\nabla^{2n}\delta^3(\vec r-\vec r')\\ [\psi_{k_\varsigma}(\vec r,t),\psi_{l_\varsigma}(\vec r',t)]_{-^{2s+1}}=0\\ [\psi^+_{k'_\varsigma}(\vec r,t),\psi^+_{l'_\varsigma}(\vec r',t)]_{-^{2s+1}}=0\end{cases}$$

Cor. 6.4.5.

$$\begin{cases}[\psi_{k_\varsigma}(\vec r,t),\psi^+_{k'_\varsigma}(\vec r',t)]_{-^{2s+1}}=\frac{(-1)^{2s+1}}{2^{s-1}},s>0\\ \sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2n-1}\nabla^{2n}\delta^3(\vec r-\vec r')\\ [\psi_{k_\varsigma}(\vec r,t),\psi_{l_\varsigma}(\vec r',t)]_{-^{2s+1}}=0\\ [\psi^+_{k'_\varsigma}(\vec r,t),\psi^+_{l'_\varsigma}(\vec r',t)]_{-^{2s+1}}=0\end{cases}\Rightarrow\begin{cases}[a_\sigma(\vec p,-s\varsigma),a^+_{\sigma'}(\vec p',-s\varsigma)]_{-^{2s+1}}\\ =\delta_{\sigma\sigma'}\delta^3(\vec p-\vec p')\\ [a_\sigma(\vec p,-s\varsigma),a_{\sigma'}(\vec p',-s\varsigma)]_{-^{2s+1}}=0\\ [a^+_\sigma(\vec p,-s\varsigma),a^+_{\sigma'}(\vec p',-s\varsigma)]_{-^{2s+1}}=0\end{cases}$$

Proof: $[a_1(\vec p,-s\varsigma),a_1^+(\vec p',-s\varsigma)]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int[\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\Psi_{k_\varsigma}(\vec r,t)e^{-i(\vec p\cdot\vec r-Et)},\lambda^{k'_\varsigma}(\vec p',-s\varsigma)\Psi^+_{k'_\varsigma}(\vec r',t)e^{i(\vec p'\cdot\vec r'-E't)}]_{-^{2s+1}}d^3\vec rd^3\vec r'$$
$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\lambda^{k'_\varsigma}(\vec p',-s\varsigma)[\Psi_{k_\varsigma}(\vec r,t),\Psi^+_{k'_\varsigma}(\vec r',t)]_{-^{2s+1}}e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r'-E't)}d^3\vec rd^3\vec r'$$
$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int d^3\vec rd^3\vec r'\frac{(-1)^{2s+1}}{2^{s-1}}$$
$$\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\lambda^{k'_\varsigma}(\vec p',-s\varsigma)\sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2n-1}\nabla^{2n}\delta^3(\vec r-\vec r')e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r'-E't)}$$
$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int d^3\vec r\frac{(-1)^{2s+1}}{2^{s-1}}$$
$$i^{2s-1}\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\lambda^{k'_\varsigma}(\vec p',-s\varsigma)\sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{p_ip_j\cdots}^{2s-2n-1}\vec p^{2n}e^{-i(\vec p\cdot\vec r-Et)}e^{i(\vec p'\cdot\vec r-E't)}$$
$$=\frac{(-i)^{2s-1}}{2^{s-1}}\lambda^+(\hat p,-s\varsigma)\sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{\hat p_i\hat p_j\cdots}^{2s-2n-1}\lambda(\vec p',-s\varsigma)\delta^3(\vec p-\vec p')$$
$$=\frac{(-i)^{2s}}{2^{s-1}}\lambda^+(\hat p,-s\varsigma)\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat p_a\hat p_b\hat p_c\cdots}^{2s}\lambda(\vec p',-s\varsigma)\delta^3(\vec p-\vec p')$$
$$=\frac{(-i)^{2s}}{2^{s-1}}\frac12(i\sqrt2)^{2s}\lambda^+(\hat p,-s\varsigma)\lambda(\hat p,-s\varsigma)\delta^3(\vec p-\vec p')$$
$$=\lambda^+(\hat p,-s\varsigma)\lambda(\hat p,-s\varsigma)\delta^3(\vec p-\vec p')$$
$$=\delta^3(\vec p-\vec p')$$
□

Proof: $[a_2^+(\vec p,-s\varsigma),a_2(\vec p',-s\varsigma)]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int[\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\Psi_{k_\varsigma}(\vec r,t)e^{i(\vec p\cdot\vec r-Et)},\lambda^{k'_\varsigma}(\vec p',-s\varsigma)\Psi^+_{k'_\varsigma}(\vec r',t)e^{-i(\vec p'\cdot\vec r'-E't)}]_{-^{2s+1}}d^3\vec rd^3\vec r'$$
$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\lambda^{k'_\varsigma}(\vec p',-s\varsigma)[\Psi_{k_\varsigma}(\vec r,t),\Psi^+_{k'_\varsigma}(\vec r',t)]_{-^{2s+1}}e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r'-E't)}d^3\vec rd^3\vec r'$$
$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int d^3\vec rd^3\vec r'\frac{(-1)^{2s+1}}{2^{s-1}}$$
$$\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\lambda^{k'_\varsigma}(\vec p',-s\varsigma)\sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{\partial_i\partial_j\cdots}^{2s-2n-1}\nabla^{2n}\delta^3(\vec r-\vec r')e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r'-E't)}$$
$$=\frac{1}{(2\pi)^3}\frac{1}{(|\vec p||\vec p'|)^{(2s-1)/2}}\int d^3\vec r\frac{(-1)^{2s+1}}{2^{s-1}}i$$
$$(-i)^{2s-1}\lambda^{+k_\varsigma}(\hat p,-s\varsigma)\lambda^{k'_\varsigma}(\vec p',-s\varsigma)\sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma_{k_\varsigma k'_\varsigma}^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{p_ip_j\cdots}^{2s-2n-1}\vec p^{2n}e^{i(\vec p\cdot\vec r-Et)}e^{-i(\vec p'\cdot\vec r-E't)}$$
$$=\frac{(i)^{2s-1}}{2^{s-1}}\lambda^+(\hat p,-s\varsigma)\sum_{n=0}^{[s-\frac12]}(-1)^nC_{2s}^{2n+1}\Gamma^{\overbrace{ij\cdots}^{2s-2n-1}\overbrace{\pi\cdots\pi}^{2n+1}}(s)\overbrace{\hat p_i\hat p_j\cdots}^{2s-2n-1}\lambda(\vec p',-s\varsigma)\delta^3(\vec p-\vec p')$$
$$=-\frac{(i)^{2s}}{2^{s-1}}\varsigma\lambda^+(\hat p,-s\varsigma)\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat p_a\hat p_b\hat p_c\cdots}^{2s}\lambda(\vec p',-s\varsigma)\delta^3(\vec p-\vec p')$$

$= -\frac{(i)^{2s}}{2^{s-1}}\frac{1}{2}(i\sqrt{2})^{2s}\lambda^{+}(\hat{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= (-1)^{2s+1}\lambda^{+}(\hat{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)\delta^3(\vec{p}-\vec{p}')$
$= (-1)^{2s+1}\delta^3(\vec{p}-\vec{p}')$ □

6.5 Commutation function, causality function and Feynman propagator of s-spin field

Lem. 6.5.1. $[\theta(t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]\Delta^{(+)}(x)-[\theta(-t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]\Delta^{(-)}(x)$
$= \frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1} i^n C_{2s}^n\Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}\Delta(x)$

Proof: $[\theta(t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]$
$= \frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)[\theta(t),\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]$
$= \frac{(-1)^{2s-1}}{2^{s-1}}\sum_{n=0}^{2s-1} C_{2s}^n\Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_\pi^{2s-n}\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}$
$= \frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1} i^n C_{2s}^n\Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-n}\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}$ □

Proof: $[\theta(-t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]$
$= \frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)[\theta(-t),\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]$
$= \frac{(-1)^{2s-1}}{2^{s-1}}\sum_{n=0}^{2s-1} C_{2s}^n\Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_\pi^{2s-n}\theta(-t)]\overbrace{\partial_i\partial_j\cdots}^{n}$
$= \frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1} i^n C_{2s}^n\Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-n}\theta(-t)]\overbrace{\partial_i\partial_j\cdots}^{n}$ □

Proof: $[\theta(t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]\Delta^{(+)}(x)-[\theta(-t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]\Delta^{(-)}(x)$
$= [\theta(t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]\Delta(x)-[\theta(t)+\theta(-t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]\Delta^{(-)}(x)$
$= [\theta(t),\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]\Delta(x)$
$= \frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1} i^n C_{2s}^n\Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-n}\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}\Delta(x)$
$= \frac{i^{2s-2}}{2^{s-1}}\sum_{n=0}^{2s-1} i^n C_{2s}^n\Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s)[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}\Delta(x)$ □

Lem. 6.5.2. $[sp_a+iS_{ab}(s,\varsigma)p^b]\lambda(\hat{p},-s\varsigma)=0$

Proof: $[sp_a+iS_{ab}(s,\varsigma)p^b]\lambda(\hat{p},-s\varsigma)$
$= |\vec{p}|\{s[exp\{i\frac{(R\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\begin{bmatrix}0\\0\\1\\i\end{bmatrix}]_a+iS_{ab}(s,\varsigma)[exp\{i\frac{(R\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\begin{bmatrix}0\\0\\1\\i\end{bmatrix}]^b\}$
$exp\{i\frac{[\sigma(2)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)$
$= exp\{i\frac{(R\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|_a{}^c exp\{i\frac{[\sigma(2)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|\vec{p}|[s\begin{bmatrix}0\\0\\1\\i\end{bmatrix}_c+iS_{cd}(s,\varsigma)\begin{bmatrix}0\\0\\1\\i\end{bmatrix}^d]\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},-s\varsigma)$
$= exp\{i\frac{(R\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|_a{}^c exp\{i\frac{[\sigma(2)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}|\vec{p}|\cdot 0$
$= 0$ □

Lem. 6.5.3. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]_{j_\varsigma}{}^{k_\varsigma}\Delta_{k_\varsigma k'_\varsigma}(s;x)=0,[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\Delta(s;x)=0$

Proof: $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]_{j_\varsigma}{}^{k_\varsigma}\Delta_{k_\varsigma k'_\varsigma}(s;x)$
$= (\frac{-1}{\sqrt{2}})^{2(s-1)}[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]_{j_\varsigma}{}^{k_\varsigma}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x)$

$= (\frac{-1}{\sqrt{2}})^{2(s-1)} i^{2s+1} \frac{-i}{(2\pi)^3} \int [s p_a + i S_{ab}(s,\varsigma) p^b]_{j_\varsigma}{}^{k_\varsigma} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{p_a p_b p_c \cdots}^{2s} \frac{1}{2|\vec{p}|} [e^{ip\cdot x} - (-1)^{2s+1} e^{-ip\cdot x}] d^3\vec{p}$
$= (-\frac{1}{2})^{2s-1} \frac{-1}{(2\pi)^3} \int [s p_a + i S_{ab}(s,\varsigma) p^b]_{j_\varsigma}{}^{k_\varsigma} \lambda_{k_\varsigma}(\hat{p}, -s\varsigma) \lambda^+_{k'_\varsigma}(\hat{p}, -s\varsigma) \overbrace{p_a p_b p_c \cdots}^{2s} \frac{1}{2|\vec{p}|^{2s+1}} [e^{ip\cdot x} - (-1)^{2s+1} e^{-ip\cdot x}] d^3\vec{p}$
$= (-\frac{1}{2})^{2s-1} \frac{-1}{(2\pi)^3} \int 0 \cdot \lambda^+_{k'_\varsigma}(\hat{p}, -s\varsigma) \overbrace{p_a p_b p_c \cdots}^{2s} \frac{1}{2|\vec{p}|^{2s+1}} [e^{ip\cdot x} - (-1)^{2s+1} e^{-ip\cdot x}] d^3\vec{p}$
$= 0$ □

Def. 6.5.1.

$$\begin{cases} \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) = \Delta^{ret}(x) - \Delta^{adv}(x) \\ \Delta^{(l)}(x) = i[\Delta^{(-)}(x) - \Delta^{(+)}(x)] \\ \Delta_F(x) = \langle T\varphi(x)\varphi(x')\rangle_0 = i\Delta^{(c)}(x-x') \end{cases} \quad \begin{cases} \Delta^{(c)}(x) = \theta(t)\Delta^{(+)}(x) - \theta(-t)\Delta^{(-)}(x) \\ \Delta^{ret}(x) = \theta(t)\Delta(x) = \Delta^{(c)}(x) + \Delta^{(-)}(x) \\ \Delta^{adv}(x) = -\theta(-t)\Delta(x) = \Delta^{(c)}(x) - \Delta^{(+)}(x) \end{cases}$$

Cor. 6.5.1.

$$\begin{cases} \Delta_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x) \\ \Delta^{(+)}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(+)}(x) \\ \Delta^{(-)}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(-)}(x) \\ \Delta^{(l)}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(l)}(x) \end{cases}$$

Cor. 6.5.2.

$$\begin{cases} \Delta^{(c)}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(c)}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C^n_{2s} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) [\partial_t^{2s-1-n}\delta(t)] \overbrace{\partial_i \partial_j \cdots}^{n} \Delta(x) \\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{ret}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C^n_{2s} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) [\partial_t^{2s-1-n}\delta(t)] \overbrace{\partial_i \partial_j \cdots}^{n} \Delta(x) \\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{adv}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C^n_{2s} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) [\partial_t^{2s-1-n}\delta(t)] \overbrace{\partial_i \partial_j \cdots}^{n} \Delta(x) \\ \Delta_{F k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta_F(x) + \frac{i^{2s-1}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C^n_{2s} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) [\partial_t^{2s-1-n}\delta(t)] \overbrace{\partial_i \partial_j \cdots}^{n} \Delta(x) \\ = i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(s;x) \end{cases}$$

Cor. 6.5.3.

$$\begin{cases} \Delta^{(c)}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(c)}(x) \\ + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} i^n C^n_{2s} C^{2l+1}_{2s-1-n} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_i \partial_j \cdots}^{n} \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\ \Delta^{ret}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{ret}(x) \\ + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} i^n C^n_{2s} C^{2l+1}_{2s-1-n} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_i \partial_j \cdots}^{n} \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\ \Delta^{adv}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{adv}(x) \\ + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} i^n C^n_{2s} C^{2l+1}_{2s-1-n} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_i \partial_j \cdots}^{n} \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\ \Delta_{F k_\varsigma k'_\varsigma}(s;x) = i\Delta^{(c)}_{k_\varsigma k'_\varsigma}(s;x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta_F(x) \\ + \frac{i^{2s-1}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} i^n C^n_{2s} C^{2l+1}_{2s-1-n} \Gamma^{\overbrace{ij\cdots}^{n}\overbrace{\pi\cdots}^{2s-n}}_{k_\varsigma k'_\varsigma}(s) \overbrace{\partial_i \partial_j \cdots}^{n} \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\ \Delta_{F k_\varsigma k'_\varsigma}(s;p) = \frac{(-i)^{2s+1}}{2^{s-1}} \frac{\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s) \overbrace{p_a p_b p_c \cdots}^{2s}}{p^2 - i\varepsilon} + \cdots \end{cases}$$

Lem. 6.5.4. $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\theta(t) = -\varsigma[\sigma(s), is\varsigma]_a \delta(t)$

Proof: $[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\theta(t)$
$= [-is\delta_{a4} + S_{a4}(s,\varsigma)]\delta(t) = [-is\delta_{a4} - \varsigma\sigma_a(s)]\delta(t) = -\varsigma[\sigma(s), is\varsigma]_a\delta(t)$ □

Lem. 6.5.5. $\frac{1}{\sqrt{-\nabla^2}}\delta^3(\vec{r}) = 2\Delta^{(+)}(x)|_{t=0} = 2\Delta^{(-)}(x)|_{t=0}$

Lem. 6.5.6. $[sD_a + iS_{ab}(s,\varsigma)D^b]\psi(s,\varsigma) = -\sqrt{2}\varsigma s\bar{Z}_a(s,\varsigma)\tilde{J}(s) \Leftrightarrow (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a\tilde{\psi}(s,\varsigma) = i\tilde{J}(s,\varsigma)$

Cor. 6.5.4.

$$\begin{cases}[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta(s;x) = 0\\ [s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta^{(+)}(s;x) = 0\\ [s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta^{(-)}(s;x) = 0\\ [s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta^{(l)}(s;x) = 0\end{cases} \quad \begin{cases}[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta^{(c)}(s;x) = -\varsigma[\sigma(s), is\varsigma]_a\delta(t)\Delta(s;x)|_{t=0}\\ = -\sqrt{2}\varsigma s[\frac{-i\varsigma}{\sqrt{2}}\bar{N}(s)(\sigma, i\varsigma)_a]i\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\\ [s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta^{ret}(s;x) = -\varsigma[\sigma(s), is\varsigma]_a\delta(t)\Delta(s;x)|_{t=0}\\ [s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta^{adv}(s;x) = -\varsigma[\sigma(s), is\varsigma]_a\delta(t)\Delta(s;x)|_{t=0}\\ [s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\Delta_F(s;x) = -i\varsigma[\sigma(s), is\varsigma]_a\delta(t)\Delta(s;x)|_{t=0}\end{cases}$$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 6.5.5.

$$\begin{cases}(\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta(s;x) = 0\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(+)}(s;x) = 0\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(-)}(s;x) = 0\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(l)}(s;x) = 0\end{cases} \quad \begin{cases}(\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(c)}(s;x) = -\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{ret}(s;x) = -\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{adv}(s;x) = -\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta_F(s;x) = -i\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\end{cases}$$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 6.5.6.

$$\begin{cases}(\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta(s;x) = 0\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(+)}(s;x) = 0\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(-)}(s;x) = 0\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(l)}(s;x) = 0\end{cases} \quad \begin{cases}(\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(c)}(s;x) = -\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{ret}(s;x) = -\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{adv}(s;x) = -\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta_F(s;x) = -i\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\end{cases}$$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 6.5.7.

$$\begin{cases}(\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta(s;x)\bar{\Gamma}(s) = 0\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(+)}(s;x)\bar{\Gamma}(s) = 0\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(-)}(s;x)\bar{\Gamma}(s) = 0\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(l)}(s;x)\bar{\Gamma}(s) = 0\end{cases} \quad \begin{cases}(\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{(c)}(s;x)\bar{\Gamma}(s) = -\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\bar{\Gamma}(s)\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{ret}(s;x)\bar{\Gamma}(s) = -\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\bar{\Gamma}(s)\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta^{adv}(s;x)\bar{\Gamma}(s) = -\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\bar{\Gamma}(s)\\ (\sigma\otimes I_{2^{2s-1}}, -i\varsigma)_a\partial^a\Gamma(s)\Delta_F(s;x)\bar{\Gamma}(s) = -i\varsigma\delta(t)\Gamma(s)\Delta(s;x)|_{t=0}\bar{\Gamma}(s)\end{cases}$$

$[\Updownarrow]$ $[\Updownarrow]$

Cor. 6.5.8.

$$\begin{cases}(\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta(s;x)\bar{N}(s) = 0\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(+)}(s;x)\bar{N}(s) = 0\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(-)}(s;x)\bar{N}(s) = 0\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(l)}(s;x)\bar{N}(s) = 0\end{cases} \quad \begin{cases}(\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{(c)}(s;x)\bar{N}(s) = -\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\bar{N}(s)\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{ret}(s;x)\bar{N}(s) = -\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\bar{N}(s)\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta^{adv}(s;x)\bar{N}(s) = -\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\bar{N}(s)\\ (\sigma\otimes I_{2s}, -i\varsigma)_a\partial^a N(s)\Delta_F(s;x)\bar{N}(s) = -i\varsigma\delta(t)N(s)\Delta(s;x)|_{t=0}\bar{N}(s)\end{cases}$$

$[\Downarrow]$ $[\Downarrow]$

Cor. 6.5.9.

$$\begin{cases}[\sigma(s), -is\varsigma]_a\partial^a\Delta(s;x) = 0\\ [\sigma(s), -is\varsigma]_a\partial^a\Delta^{(+)}(s;x) = 0\\ [\sigma(s), -is\varsigma]_a\partial^a\Delta^{(-)}(s;x) = 0\\ [\sigma(s), -is\varsigma]_a\partial^a\Delta^{(l)}(s;x) = 0\end{cases} \quad \begin{cases}[\sigma(s), -is\varsigma]_a\partial^a\Delta^{(c)}(s;x) = -s\varsigma\delta(t)\Delta(s;x)|_{t=0}\\ [\sigma(s), -is\varsigma]_a\partial^a\Delta^{ret}(s;x) = -s\varsigma\delta(t)\Delta(s;x)|_{t=0}\\ [\sigma(s), -is\varsigma]_a\partial^a\Delta^{adv}(s;x) = -s\varsigma\delta(t)\Delta(s;x)|_{t=0}\\ [\sigma(s), -is\varsigma]_a\partial^a\Delta_F(s;x) = -is\varsigma\delta(t)\Delta(s;x)|_{t=0}\end{cases}$$

6.6 Extraction of energy momentum operator for s-spin field

Cor. 6.6.1.

$$\begin{cases}\psi(\vec{r},t) := \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq 0}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p}, -s\varsigma)[a_1(\vec{p}, -s\varsigma)e^{ip\cdot x} + a_2^+(\vec{p}, -s\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\ \vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p}, -s\varsigma)\psi(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ |\vec{p}|^{(s-\frac{1}{2})}a_2^+(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p}, -s\varsigma)\psi(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Lem. 6.6.1.
$\Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}_{k_\varsigma k'_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{2s-n}\psi(\vec{r},t)=(\frac{1}{\sqrt{2}})^{2s}\delta_{k_\varsigma k'_\varsigma}\partial_\pi^{2s-n}\psi(\vec{r},t),\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdot\cdot}_{2s-n}\underbrace{\pi\cdot\cdot\pi}_{n}}(s)\overbrace{\partial^i\partial^j\cdot\cdot}^{2s-n}\psi(\vec{r},t)=(\frac{1}{\sqrt{2}})^{2s}\delta^{k'_\varsigma k_\varsigma}\partial_\pi^{2s-n}\psi(\vec{r},t)$

Lem. 6.6.2. $\Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}_{k_\varsigma k'_\varsigma}(s)\underbrace{\partial_i\partial_j\cdot\cdot}_{2s-n}\partial_\pi^n\psi(\vec{r},t)=(\frac{1}{\sqrt{2}})^{2s}\delta_{k_\varsigma k'_\varsigma}\partial_\pi^{2s}\psi(\vec{r},t)$

Pro. 6.6.1. $\begin{cases}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{p_ap_bp_c\cdot\cdot}^{2s}=\sum\limits_{n=0}^{2s}C_{2s}^n\Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}(s)\overbrace{p_ip_j\cdot\cdot}^{2s-n}p_\pi^n\\ \Gamma^{\overbrace{abc\cdot\cdot}^{2s}}(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}=\sum\limits_{n=0}^{2s}C_{2s}^n\Gamma^{\overbrace{ij\cdot\cdot}^{2s-n}\overbrace{\pi\cdot\cdot\pi}^{n}}(s)\overbrace{\partial_i\partial_j\cdot\cdot}^{2s-n}\partial_\pi^n\end{cases}$

Thm. 6.6.1.
$H(s)=\int|\vec{p}|[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a_2^+(\vec{p},-s\varsigma)]d^3\vec{p}=\int\psi^+(\vec{r},t)\frac{(i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}}\psi(\vec{r},t)d^3\vec{r}$

Proof: $H(s)=\int|\vec{p}|[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a_2^+(\vec{p},-s\varsigma)]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|^{2s-2}}[\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{-ip\cdot x}$
$+(-1)^{2s}\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{-ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|^{2s-2}}\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)\psi_{k_\varsigma}(\vec{r},t)\psi^+_{k'_\varsigma}(\vec{r}',t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+(-1)^{2s}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=(i\sqrt{2})^{-2s}\frac{1}{(2\pi)^3}\int\frac{1}{|\vec{p}|^{2s-2}}\psi^+_{k'_\varsigma}(\vec{r}',t)\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)\underbrace{\hat{p}^a\hat{p}^b\hat{p}^c\cdot\cdot}_{2s}\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+(-1)^{2s}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=(-i\sqrt{2})^{-2s}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\frac{1}{|\vec{p}|^{4s-2}}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)(\underbrace{p^ap^bp^c\cdot\cdot}_{2s}+\underbrace{p^{+a}p^{+b}p^{+c}\cdot\cdot}_{2s})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=(-i\sqrt{2})^{-2s}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\frac{1}{|\vec{p}|^{4s-2}}\sum\limits_{n=0}^{2s}C_{2s}^n\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdot\cdot}_{2s-n}\underbrace{\pi\cdot\cdot\pi}_{n}}(s)(\overbrace{p^ip^j\cdot\cdot}^{2s-n}p_\pi^n+\overbrace{p^ip^j\cdot\cdot}^{2s-n}p_\pi^{+n})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=(\sqrt{2})^{-2s}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\frac{1}{(-\nabla^2)^{2s-1}}\sum\limits_{n=0}^{2s}C_{2s}^n(\sqrt{-\nabla^2})^n\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdot\cdot}_{2s-n}\underbrace{\pi\cdot\cdot\pi}_{n}}(s)\overbrace{\partial^i\partial^j\cdot\cdot}^{2s-n}[1+(-1)^n]\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$=(-\sqrt{2})^{-2s}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{1}{(-\nabla^2)^{2s-1}}\sum\limits_{n=0}^{2s}C_{2s}^n(\sqrt{-\nabla^2})^n\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdot\cdot}_{2s-n}\underbrace{\pi\cdot\cdot\pi}_{n}}(s)\overbrace{\partial^i\partial^j\cdot\cdot}^{2s-n}[1+(-1)^n]\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$=(\frac{-1}{\sqrt{2}})^{2s}(\sqrt{2})^{-2s}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{1}{(-\nabla^2)^{2s-1}}\sum\limits_{n=0}^{2s}C_{2s}^n(\sqrt{-\nabla^2})^n\delta^{k'_\varsigma k_\varsigma}\partial_\pi^{2s-n}[1+(-1)^n]\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$=(\frac{-1}{\sqrt{2}})^{2s}(\sqrt{2})^{-2s}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{1}{(-\nabla^2)^{2s-1}}\sum\limits_{n=0}^{2s}C_{2s}^n\partial_\pi^n\delta^{k'_\varsigma k_\varsigma}\partial_\pi^{2s-n}[1+(-1)^n]\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(-2)^{2s}}\int\psi^{+k_\varsigma}(\vec{r},t)\frac{1}{(-\nabla^2)^{2s-1}}\sum\limits_{n=0}^{2s}C_{2s}^n(-i\partial_t)^n(-i\partial_t)^{2s-n}[1+(-1)^n]\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(-2)^{2s}}\int\psi^+_{k'_\varsigma}(\vec{r},t)\frac{(-i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}}\sum\limits_{n=0}^{2s}C_{2s}^n[1+(-1)^n]\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(-2)^{2s}}\int\psi^{+k_\varsigma}(\vec{r},t)\frac{(-i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}}\sum\limits_{n=0}^{2s}C_{2s}^n[1+(-1)^n]\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}$
$=\int\psi^+(\vec{r},t)\frac{(i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}}\psi(\vec{r},t)d^3\vec{r}$ □

Thm. 6.6.2.
$P(s)=\int\vec{p}[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a_2^+(\vec{p},-s\varsigma)]d^3\vec{p}=\int\psi^+(\vec{r},t)\frac{-i\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi(\vec{r},t)d^3\vec{r}$

Proof: $P(s)=\int\vec{p}[a_1^+(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a_2^+(\vec{p},-s\varsigma)]d^3\vec{p}$
$=\frac{1}{(2\pi)^3}\int\frac{\hat{p}}{|\vec{p}|^{2s-2}}[\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{-ip\cdot x}$
$+(-1)^{2s}\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)\psi^+_{k'_\varsigma}(\vec{r}',t)e^{-ip\cdot x'}\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\psi_{k_\varsigma}(\vec{r},t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=\frac{1}{(2\pi)^3}\int\frac{\hat{p}}{|\vec{p}|^{2s-2}}\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\lambda^{k'_\varsigma}(\hat{p},-s\varsigma)\psi_{k_\varsigma}(\vec{r},t)\psi^+_{k'_\varsigma}(\vec{r}',t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+(-1)^{2s}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=(i\sqrt{2})^{-2s}\frac{1}{(2\pi)^3}\int\frac{\hat{p}}{|\vec{p}|^{2s-2}}\psi^+_{k'_\varsigma}(\vec{r}',t)(\Gamma)^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)\underbrace{\hat{p}^a\hat{p}^b\hat{p}^c\cdot\cdot}_{2s}\psi_{k_\varsigma}(\vec{r},t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}+(-1)^{2s}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}'$
$=(-i\sqrt{2})^{-2s}\frac{1}{(2\pi)^3}\int\psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t)\frac{\hat{p}}{|\vec{p}|^{4s-2}}\Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{abc\cdot\cdot}_{2s}}(s)(\underbrace{p^ap^bp^c\cdot\cdot}_{2s}-\underbrace{p^{+a}p^{+b}p^{+c}\cdot\cdot}_{2s})e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}'$

$= (-i\sqrt{2})^{-2s} \frac{1}{(2\pi)^3} \int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t) \frac{\hat{p}}{|\vec{p}|^{4s-2}} \sum_{n=0}^{2s} C^n_{2s} \Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdots}_{2s-n}\underbrace{\pi\cdots\pi}_{n}}(s)(\overbrace{p^i p^j \cdots}^{2s-n} p^n_\pi - \overbrace{p^i p^j \cdots}^{2s-n} p^{+n}_\pi) e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} d^3\vec{r} d^3\vec{r}'$

$= (\sqrt{2})^{-2s} \int \psi^+_{k'_\varsigma}(\vec{r}',t)\psi_{k_\varsigma}(\vec{r},t) \frac{\hat{\nabla}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C^n_{2s} (\sqrt{-\nabla^2})^n \Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdots}_{2s-n}\underbrace{\pi\cdots\pi}_{n}}(s) \overbrace{\partial^i \partial^j \cdots}^{2s-n} [(-1)^n - 1]\delta^3(\vec{r}-\vec{r}') d^3\vec{r} d^3\vec{r}'$

$= (-\sqrt{2})^{-2s} \int \psi^+_{k'_\varsigma}(\vec{r},t) \frac{\hat{\nabla}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C^n_{2s} (\sqrt{-\nabla^2})^n \Gamma^{k'_\varsigma k_\varsigma}_{\underbrace{ij\cdots}_{2s-n}\underbrace{\pi\cdots\pi}_{n}}(s) \overbrace{\partial^i \partial^j \cdots}^{2s-n} [(-1)^n - 1]\psi_{k_\varsigma}(\vec{r},t) d^3\vec{r}$

$= (\frac{1}{\sqrt{2}})^{2s} (-\sqrt{2})^{-2s} \int \psi^+_{k'_\varsigma}(\vec{r},t) \frac{\hat{\nabla}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C^n_{2s} (\sqrt{-\nabla^2})^n \delta^{k'_\varsigma k_\varsigma} \partial^{2s-n}_\pi [(-1)^n - 1]\psi_{k_\varsigma}(\vec{r},t) d^3\vec{r}$

$= \frac{1}{(-2)^{2s}} \int \psi^{+k_\varsigma}(\vec{r},t) \frac{\hat{\nabla}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C^n_{2s} \sqrt{-\nabla^2} (-i\partial_t)^{n-1} (-i\partial_t)^{2s-n} [(-1)^n - 1]\psi_{k_\varsigma}(\vec{r},t) d^3\vec{r}$

$= \frac{1}{(-2)^{2s}} \int \psi^+_{k'_\varsigma}(\vec{r},t) \frac{-i\nabla(-i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C^n_{2s} [(-1)^n - 1]\psi_{k_\varsigma}(\vec{r},t) d^3\vec{r}$

$= \frac{1}{(-2)^{2s}} \int \psi^{+k_\varsigma}(\vec{r},t) \frac{-i\nabla(-i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C^n_{2s} [(-1)^n - 1]\psi_{k_\varsigma}(\vec{r},t) d^3\vec{r}$

$= \int \psi^+(\vec{r},t) \frac{-i\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r},t) d^3\vec{r}$ □

Thm. 6.6.3.
$P_u(s) = \int p_u [a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + (-1)^{2s} a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)] d^3\vec{p} = \int \psi^+(\vec{r},t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r},t) d^3\vec{r}$

6.7 Various physical operators of s-spin field equation

Cor. 6.7.1. $\begin{cases} \psi(\vec{r},t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}\neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x} + a^+_2(\vec{p},-s\varsigma)e^{-ip\cdot x}] d^3\vec{p} \\ \vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p},-s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{-ip\cdot x} d^3\vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a^+_2(\vec{p},-s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{ip\cdot x} d^3\vec{r} \end{cases}$

Thm. 6.7.1.
$P_u(s) = \int \psi^+(\vec{r},t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r},t) d^3\vec{r} = \int p_u [a^+(\vec{p},-s\varsigma)a(\vec{p},-s\varsigma) + (-1)^{2s} b(\vec{p},-s\varsigma)b^+(\vec{p},-s\varsigma)] d^3\vec{p}$

Proof: $P_u(s) = \int \psi^+(\vec{r},t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r},t) d^3\vec{r}$
$= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^+(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma) \frac{p_u}{|\vec{p}|^{2s-1}}$
$[a^+_1(\vec{p}',-s\varsigma)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}',-s\varsigma)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s} a^+_2(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$
$= \int \vec{p}|^{2s-1} \lambda^+(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma) \frac{p_u}{|\vec{p}|^{2s-1}} \{[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + (-1)^{2s} a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]\delta^3(\vec{p}'-\vec{p})$
$+ [(-1)^{2s} a^+_1(-\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)e^{-2i|\vec{p}|t} + a_2(-\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\} d^3\vec{p}' d^3\vec{p}$
$= \int p_u [a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + (-1)^{2s} a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)] d^3\vec{p}$ □

Thm. 6.7.2. $Q(s) = \int \psi^+(\vec{r},t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r},t) d^3\vec{r} = \int [a^+(\vec{p},-s\varsigma)a(\vec{p},-s\varsigma) + (-1)^{2s-1} b(\vec{p},-s\varsigma)b^+(\vec{p},-s\varsigma)] d^3\vec{p}$

Proof: $Q(s) = \int \psi^+(\vec{r},t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r},t) d^3\vec{r}$
$= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^+(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma) \frac{1}{|\vec{p}|^{2s-1}}$
$[a^+_1(\vec{p}',-s\varsigma)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}',-s\varsigma)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s-1} a^+_2(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$
$= \int \vec{p}|^{2s-1} \lambda^+(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma) \frac{1}{|\vec{p}|^{2s-1}} \{[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + (-1)^{2s-1} a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]\delta^3(\vec{p}'-\vec{p})$
$+ [(-1)^{2s-1} a^+_1(-\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)e^{-2i|\vec{p}|t} + a_2(-\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\} d^3\vec{p}' d^3\vec{p}$
$= \int [a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + (-1)^{2s-1} a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)] d^3\vec{p}$ □

Thm. 6.7.3. $N(s) = \int \psi^+(\vec{r},t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi(\vec{r},t) d^3\vec{r} = \int [a^+(\vec{p},-s\varsigma)a(\vec{p},-s\varsigma) + (-1)^{2s} b(\vec{p},-s\varsigma)b^+(\vec{p},-s\varsigma)] d^3\vec{p}$

Proof: $N(s) = \int \psi^+(\vec{r},t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi(\vec{r},t) d^3\vec{r}$
$= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^+(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma) \frac{1}{|\vec{p}|^{2s-1}}$
$[a^+_1(\vec{p}',-s\varsigma)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}',-s\varsigma)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s} a^+_2(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$
$= \int \vec{p}|^{2s-1} \lambda^+(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma) \frac{1}{|\vec{p}|^{2s-1}} \{[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + (-1)^{2s} a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]\delta^3(\vec{p}'-\vec{p})$
$+ [(-1)^{2s} a^+_1(-\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)e^{-2i|\vec{p}|t} + a_2(-\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\} d^3\vec{p}' d^3\vec{p}$
$= \int [a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma) + (-1)^{2s} a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)] d^3\vec{p}$ □

Thm. 6.7.4. $\vec{S}(s)=\int\psi^{+}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi(\vec{r},t)d^3\vec{r}=\int\hat{p}[a^{+}(\vec{p},-s\varsigma)a(\vec{p},-s\varsigma)+(-1)^{2s-1}b(\vec{p},-s\varsigma)b^{+}(\vec{p},-s\varsigma)]d^3\vec{p}$

Proof: $\vec{S}(s)=\int\psi^{+}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int d^3\vec{p}'d^3\vec{p}d^3\vec{r}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\lambda^{+}(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma)\frac{\hat{p}}{|\vec{p}|^{2s-1}}$
$[a_1^{+}(\vec{p}',-s\varsigma)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}+a_2(\vec{p}',-s\varsigma)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+(-1)^{2s}a_2^{+}(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$
$=\int\vec{p}|^{2s-1}\lambda^{+}(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma)\frac{\hat{p}}{|\vec{p}|^{2s-1}}\{[a_1^{+}(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a_2^{+}(\vec{p},-s\varsigma)]\delta^3(\vec{p}'-\vec{p})$
$+[(-1)^{2s}a_1^{+}(-\vec{p},-s\varsigma)a_2^{+}(\vec{p},-s\varsigma)e^{-2i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\}d^3\vec{p}'d^3\vec{p}$
$=\int\hat{p}[a_1^{+}(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a_2^{+}(\vec{p},-s\varsigma)]d^3\vec{p}$ □

Thm. 6.7.5. $\vec{M}(s)=\int\psi^{+}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi(\vec{r},t)d^3\vec{r}=\int\hat{p}[a^{+}(\vec{p},-s\varsigma)a(\vec{p},-s\varsigma)+(-1)^{2s}b(\vec{p},-s\varsigma)b^{+}(\vec{p},-s\varsigma)]d^3\vec{p}$

Proof: $\vec{M}(s)=\int\psi^{+}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi(\vec{r},t)d^3\vec{r}$
$=\frac{1}{(2\pi)^3}\int d^3\vec{p}'d^3\vec{p}d^3\vec{r}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\lambda^{+}(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma)\frac{\hat{p}}{|\vec{p}|^{2s-1}}$
$[a_1^{+}(\vec{p}',-s\varsigma)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}+a_2(\vec{p}',-s\varsigma)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+(-1)^{2s-1}a_2^{+}(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$
$=\int\vec{p}|^{2s-1}\lambda^{+}(\hat{p}',-s\varsigma)\lambda(\hat{p},-s\varsigma)\frac{\hat{p}}{|\vec{p}|^{2s-1}}\{[a_1^{+}(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s-1}a_2(\vec{p},-s\varsigma)a_2^{+}(\vec{p},-s\varsigma)]\delta^3(\vec{p}'-\vec{p})$
$+[(-1)^{2s-1}a_1^{+}(-\vec{p},-s\varsigma)a_2^{+}(\vec{p},-s\varsigma)e^{-2i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\}d^3\vec{p}'d^3\vec{p}$
$=\int\hat{p}[a_1^{+}(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s-1}a_2(\vec{p},-s\varsigma)a_2^{+}(\vec{p},-s\varsigma)]d^3\vec{p}$ □

6.8 Summary of energy momentum operator for s-spin field

Thm. 6.8.1. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\frac{\psi(x)}{(\sqrt{-\nabla^2})^{[s]}}=0$

Thm. 6.8.2. $P_a(s)=\int\psi^{+}(\vec{r},t)\frac{-i\partial_a(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-l}}\psi(\vec{r},t)d^3\vec{r}$

Thm. 6.8.3. $P_a(n)=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\partial_a\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r},P_a(n+\frac{1}{2})=-i\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\partial_a\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$

Thm. 6.8.4. $\begin{cases}M_{ab}(n)=i\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[-i(x_a\partial_b-x_b\partial_a)+S_{ab}(n,\varsigma)]\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\\M_{ab}(n+\frac{1}{2})=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[-i(x_a\partial_b-x_b\partial_a)+S_{ab}(n+\frac{1}{2},\varsigma)]\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\end{cases}$

6.9 Reasonable hamiltonian type energy momentum operator for s-spin field

Thm. 6.9.1.
$\begin{cases}\hat{H}(\frac{1}{2})=\frac{i\varsigma}{1/2}\int\psi^{+}(\vec{r},t)\sigma(\frac{1}{2})\cdot\nabla\psi(\vec{r},t)d^3\vec{r}\\\hat{H}(1)=\int\psi^{+}(\vec{r},t)\frac{[\sigma(1)\cdot\nabla]^2}{\nabla^2}\psi(\vec{r},t)d^3\vec{r}\\\hat{H}(\frac{3}{2})=\frac{-i\varsigma}{3/2}\int\psi^{+}(\vec{r},t)\frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2}\psi(\vec{r},t)d^3\vec{r}\\\hat{H}(2)=(\frac{-i\varsigma}{2})^2\int\psi^{+}(\vec{r},t)\frac{[\sigma(2)\cdot\nabla]^2}{\nabla^4}\psi(\vec{r},t)d^3\vec{r}\end{cases}$ $\begin{cases}\hat{P}(\frac{1}{2})=-\int\psi^{+}(\vec{r},t)i\nabla\psi(\vec{r},t)d^3\vec{r}\\\hat{P}(1)=i\varsigma\int\psi^{+}(\vec{r},t)\frac{[\sigma(1)\cdot\nabla]i\nabla}{\nabla^2}\psi(\vec{r},t)d^3\vec{r}\\\hat{P}(\frac{3}{2})=\int\psi^{+}(\vec{r},t)\frac{i\nabla}{\nabla^2}\psi(\vec{r},t)d^3\vec{r}\\\hat{P}(2)=(\frac{-i\varsigma}{2})\int\psi^{+}(\vec{r},t)\frac{[\sigma(2)\cdot\nabla]i\nabla}{\nabla^4}\psi(\vec{r},t)d^3\vec{r}\end{cases}$

Thm. 6.9.2.
$\begin{cases}\hat{H}(n+\frac{1}{2})=\int\psi^{+}(\vec{r},t)\frac{\frac{i\varsigma}{n+1/2}\sigma(n+\frac{1}{2})\cdot\nabla}{(\sqrt{-\nabla^2})^{2n}}\psi(\vec{r},t)d^3\vec{r}\\\hat{P}(n+\frac{1}{2})=\int\psi^{+}(\vec{r},t)\frac{-i\nabla}{(\sqrt{-\nabla^2})^{2n}}\psi(\vec{r},t)d^3\vec{r}\end{cases}$ $\begin{cases}\hat{H}(n)=\int\psi^{+}(\vec{r},t)\frac{[\frac{i\varsigma}{n}\sigma(n)\cdot\nabla]^2}{(\sqrt{-\nabla^2})^{2n}}\psi(\vec{r},t)d^3\vec{r}\\\hat{P}(n)=\int\psi^{+}(\vec{r},t)\frac{-i\nabla[\frac{i\varsigma}{n}\sigma(n)\cdot\nabla]}{(\sqrt{-\nabla^2})^{2n}}\psi(\vec{r},t)d^3\vec{r}\end{cases}$

6.10 Derived energy momentum operator and angular momentum operator

Def. 6.10.1. $\begin{cases}\hat{M}_{ab}(s,\varsigma)=x_a\hat{P}_b-x_b\hat{P}_a+i\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s)\delta(s-\frac{1}{2}),\hat{P}_a=-i\partial_a\\\Gamma_{ab}(s,\varsigma)=x_a\Gamma_b(s,\varsigma)-x_b\Gamma_a(s,\varsigma),\Gamma_a(s,\varsigma):=-\varsigma[\frac{1}{s}\sigma(s),-i\varsigma]_a\end{cases}$

Cor. 6.10.1.
$\begin{cases}P_a(n+\frac{1}{2})=\int\psi^{+}(\vec{r},t)\frac{-i\partial_a}{(\sqrt{-\nabla^2})^{2n}}\psi(\vec{r},t)d^3\vec{r}\\P_a(n+\frac{1}{2})=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{P}_a\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\\M_{ab}(n+\frac{1}{2})=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{M}_{ab}\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\end{cases}$ $\begin{cases}P_a(n)=\int\psi^{+}(\vec{r},t)\frac{-i\partial_a[\frac{i\varsigma}{n}\sigma(n)\cdot\nabla]}{(\sqrt{-\nabla^2})^{2n}}\psi(\vec{r},t)d^3\vec{r}\\\hat{P}_a(n)=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{P}_a\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\\M_{ab}(n)=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{M}_{ab}\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\end{cases}$

Thm. 6.10.1.
$\begin{cases}H(1)=\int\psi_{k'_\varsigma}^{+}(\vec{r},t)\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}\\H(2)=-\int\psi_{k'_\varsigma}^{+}(\vec{r},t)\frac{1}{\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}\end{cases}$ $\begin{cases}P(1)=-\varsigma\int\psi_{k'_\varsigma}^{+}(\vec{r},t)\sigma(1)\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}\\P(2)=(\frac{\varsigma}{2})\int\psi_{k'_\varsigma}^{+}(\vec{r},t)\frac{\sigma(2)}{\nabla^2}\psi_{k_\varsigma}(\vec{r},t)d^3\vec{r}\end{cases}$

Cor. 6.10.2.

$$\begin{cases}P_a(n-\frac{1}{2})=\int\psi^+(\vec r,t)\frac{-i\partial_a}{(\sqrt{-\nabla^2})^{2(n-1)}}\psi(\vec r,t)d^3\vec r\\ P_a(n-\frac{1}{2})=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}\hat P_a\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}d^3\vec r\\ M_{ab}(n-\frac{1}{2})=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}\hat M_{ab}\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}d^3\vec r\end{cases}\quad\begin{cases}P_a(n)=\int\psi^+(\vec r,t)\frac{-\varsigma[\frac{1}{n}\sigma(n),-i\varsigma]_a}{(\sqrt{-\nabla^2})^{2(n-1)}}\psi(\vec r,t)d^3\vec r\\ P_a(n)=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}\Gamma_a\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}d^3\vec r\\ M_{ab}(n)=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}\Gamma_{ab}\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^{n-1}}d^3\vec r\end{cases}$$

6.11 Extraction of quantum equation for s-spin field

Thm. 6.11.1. $[\psi(\vec r,t),H(s)]=\frac{(-1)^{2s}}{4^{s-1}}\sqrt{-\nabla^2}[\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}][\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}]\psi(\vec r,t)$

Proof: $[\psi(\vec r,t),H(s)]=[\psi(\vec r,t),\frac{i^{-2s}}{2^{s-1}}\int\frac{\psi^+(\vec r',t)}{(\sqrt{-\nabla^2})^{s-1}}\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat\partial'_a\hat\partial'_b\hat\partial'_c\cdots}^{2s}\frac{\psi(\vec r',t)}{(\sqrt{-\nabla^2})^{s-1}}d^3\vec r']$

$=\frac{i^{-2s}}{2^{s-1}}\int[\psi(\vec r,t),\frac{\psi^+(\vec r',t)}{(\sqrt{-\nabla^2})^{s-1}}]_{-2s+1}\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat\partial'_a\hat\partial'_b\hat\partial'_c\cdots}^{2s}\frac{\psi(\vec r',t)}{(\sqrt{-\nabla^2})^{s-1}}d^3\vec r'$

$=\frac{i^{-2s}}{2^{s-1}}\int i\frac{(-1)^{2s}}{2^{s-1}}(i\sqrt{-\nabla^2})^{2s-1}[\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}]\delta^3(\vec r-\vec r')[\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)\overbrace{\hat\partial'_a\hat\partial'_b\hat\partial'_c\cdots}^{2s}]\frac{\psi(\vec r',t)}{(\sqrt{-\nabla^2})^{2(s-1)}}d^3\vec r'$

$=\frac{(-1)^{2s}}{4^{s-1}}\sqrt{-\nabla^2}[\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}][\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}]\psi(\vec r,t)$ □

Thm. 6.11.2. $[\psi(\vec r,t),\vec P(s)]=\frac{(-1)^{2s}}{4^{s-1}}(-i\nabla)[\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}][\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}]\psi(\vec r,t)$

Thm. 6.11.3.

???$[\psi(\vec r,t),P(s)]=\frac{(-1)^{2s}}{4^{s-1}}\sqrt{-\nabla^2}\{[\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\hat\nabla,i\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}]\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}\}[\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)^{\hat\partial_\pi\to i}\overbrace{\hat\partial_a\hat\partial_b\hat\partial_c\cdots}^{2s}]\psi(\vec r,t)$
$=\frac{(-1)^{2s}}{4^{s-1}}\sqrt{-\nabla^2}\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq0}d^3\vec p|\vec p|^{(s-\frac{1}{2})}$

$\{\{[\Gamma_-^{\overbrace{ab\cdots}^{2s}}(s)\hat\nabla,i\Gamma_+^{\overbrace{ab\cdots}^{2s}}(s)]\overbrace{(\varsigma\hat p,i)_a(\varsigma\hat p,i)_b\cdots}^{2s}\}[\Gamma_-^{\overbrace{ab\cdots}^{2s}}(s)\overbrace{(\varsigma\hat p,i)_a(\varsigma\hat p,i)_b\cdots}^{2s}]\lambda(\hat p,-s\varsigma)a_1(\vec p,-s\varsigma)e^{ip\cdot x}$
$+[\Gamma_-^{\overbrace{ab\cdots}^{2s}}(s)\hat\nabla,i\Gamma_+^{\overbrace{ab\cdots}^{2s}}(s)]\overbrace{(-\varsigma\hat p,i)_a(-\varsigma\hat p,i)_b\cdots}^{2s}\}[\Gamma_-^{\overbrace{ab\cdots}^{2s}}(s)\overbrace{(-\varsigma\hat p,i)_a(-\varsigma\hat p,i)_b\cdots}^{2s}]\lambda(\hat p,-s\varsigma)a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}\}$
$!=\frac{(-1)^{2s}}{4^{s-1}}\sqrt{-\nabla^2}\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq0}|\{[\Gamma_-^{\overbrace{abc\cdots}^{2s}}(s)\hat\nabla,i\Gamma_+^{\overbrace{abc\cdots}^{2s}}(s)]\overbrace{\hat p_a\hat p_b\hat p_c\cdots}^{2s}\}\frac{2^{s-1}}{i^{-2s}}\lambda(\hat p,-s\varsigma)$
$\vec p|^{(s-\frac{1}{2})}[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^3\vec p$
$=\frac{(-1)^{2s}}{4^{s-1}}\sqrt{-\nabla^2}\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq0}(\hat\nabla,i)(\frac{2^{s-1}}{i^{-2s}})^2\lambda(\hat p,-s\varsigma)|\vec p|^{(s-\frac{1}{2})}[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^3\vec p$
$=\sqrt{-\nabla^2}(\hat\nabla,i)\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq0}|\vec p|^{(s-\frac{1}{2})}\lambda(\hat p,-s\varsigma)[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^3\vec p$
$=\sqrt{-\nabla^2}(\hat\nabla,i)\psi$
$=(-i\nabla,i\sqrt{-\nabla^2})\psi$

6.12 Commutative and anticommutative formulas

Cor. 6.12.1. $\begin{cases}[A,BC]=[A,B]C+B[A,C],[A,CB]=[A,C]B+C[A,B]\\ [A,BC]=\{A,B\}C-B\{A,C\},[A,CB]=\{A,C\}B-C\{A,B\}\end{cases}$

Cor. 6.12.2. $\begin{cases}[A,\{B,C\}]=\{[A,B],C\}+\{B,[A,C]\}\\ [A,[B,C]]=\{\{A,B\},C\}-\{B,\{A,C\}\}\end{cases}$

6.13 Misrepresentation of energy momentum and angular momentum operator with s-spin

Cor. 6.13.1.

$$\begin{cases}\hat M_{ab}(s,\varsigma)=-i(x_a\partial_b-x_b\partial_a)+i\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma}(s)\\ \gamma_{ab}(s,\varsigma)=x_a\gamma_b(s,\varsigma)-x_b\gamma_a(s,\varsigma)+\frac{\sigma^{\alpha_\varsigma}_{\varsigma ab}\partial_{\alpha_\varsigma}}{(\sqrt{-\nabla^2})^{2s}},\gamma_a(s,\varsigma):=-\varsigma[\frac{1}{s}\sigma(s),-i\varsigma]_a\\ \tilde M_{ab}(s,\varsigma)=-i(p_a\tilde\partial_b-p_b\tilde\partial_a)-is\varsigma\sigma^{\alpha_\varsigma}_{\varsigma ab}\hat p_{\alpha_\varsigma},\tilde\partial_\pi\equiv\frac{1}{i|\vec p|}\\ \tilde M_{ab}(s,\varsigma)?=?-i(p_a\tilde\partial_b-p_b\tilde\partial_a)-is\varsigma\sigma^{\alpha_\varsigma}_{\varsigma ab}\hat p_{\alpha_\varsigma},\tilde\partial_\pi\equiv\frac{1}{i|\vec p|}\end{cases}$$

Cor. 6.13.2.
$$\begin{cases}P_a(s,\varsigma)=\int\limits_{\vec{p}\neq 0}\{a_1^+(\vec{p},-s\varsigma)p_a a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)p_a a_2^+(\vec{p},-s\varsigma)\}d^3\vec{p}\\ M_{ab}(s,\varsigma)=\int\limits_{\vec{p}\neq 0}\{a_1^+(\vec{p},-s\varsigma)\tilde{M}_{ab}(s,\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s+1}a_2(\vec{p},-s\varsigma)\tilde{M}_{ab}(s,\varsigma)a_2^+(\vec{p},-s\varsigma)\}d^3\vec{p}\end{cases}$$

6.14 Quantum equation of s-spin field

Cor. 6.14.1. $[2\partial_a+iS_{ab}(2,\varsigma)\partial^b]\psi(x)=0\Rightarrow\begin{cases}\dot{\psi}(\vec{r},t)=-i[\psi(\vec{r},t),H]\\ \nabla\psi(\vec{r},t)=i[\psi(\vec{r},t),\vec{P}]\\ \partial_a\psi(\vec{r},t)=i[\psi(\vec{r},t),P_a]\end{cases}$

Cor. 6.14.2. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(x)=0[\textit{Wave field}]\Leftrightarrow\begin{cases}\partial^a\partial_a\psi(\vec{r},t)=0\\ [\sigma(s),-is\varsigma]^a\partial_a\psi(\vec{r},t)=0\end{cases}\Leftrightarrow\partial_a\psi(\vec{r},t)=i[\psi(\vec{r},t),P_a]$

Cor. 6.14.3. $\begin{cases}s^2\vec{p}\lambda(\hat{p},-s\varsigma)=s\sigma(s)\cdot\vec{p}\sigma(s)\lambda(\hat{p},-s\varsigma)-(s-1)\sigma(s)[\sigma(s)\cdot\vec{p}]\lambda(\hat{p},-s\varsigma)\\ [\sigma(s)\cdot\vec{p}+s\varsigma p]\sigma(s)\lambda(\hat{p},-s\varsigma)=(s\vec{p}+\varsigma p\sigma(s))]\lambda(\hat{p},-s\varsigma)\\ [\sigma(s)\cdot\vec{p}+s\varsigma p]\lambda(\hat{p},-s\varsigma)=0\end{cases}$

Cor. 6.14.4. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(x)=0\Rightarrow$
$$\begin{cases}[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi=\{(\varsigma\hat{\partial}_t)^{n-1}s[s^n-(s-1)^n]\hat{\nabla}+(\varsigma\hat{\partial}_t)^n(s-1)^n\sigma(s)\}\psi\\ [\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi=\{[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^{n-1}s[s^n-(s-1)^n]\hat{\nabla}+(s-1)^n\sigma(s)[\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\}\psi\\ \sigma(s)\cdot[\sigma(s)\cdot\hat{\nabla}]^n\sigma(s)\psi=[s^{n+2}+s(s-1)^n][\frac{1}{s}\sigma(s)\cdot\hat{\nabla}]^n\psi\end{cases}$$

6.15 Poincare commutative algebra of s-spin field

Def. 6.15.1. $\begin{cases}\hat{M}_{ab}(s,\varsigma)=x_a\hat{P}_b-x_b\hat{P}_a+i\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\alpha_\varsigma}(s),\hat{P}_a=-i\partial_a\\ \Gamma_{ab}(s,\varsigma)=x_a\Gamma_b(s,\varsigma)-x_b\Gamma_a(s,\varsigma),\Gamma_a(s,\varsigma):=-\varsigma[\frac{1}{s}\sigma(s),-i\varsigma]_a\end{cases}$

Ass. 6.15.1.
$$\begin{cases}P_a(n+\frac{1}{2})=\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{P}_a\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\\ M_{ab}(n+\frac{1}{2})=\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{M}_{ab}\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\end{cases}\quad\begin{cases}P_a(n)=\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{P}_a\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\\ M_{ab}(n)=\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{M}_{ab}\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}\end{cases}$$

Proof: $[P_a(x),P_b(x')]=[\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\hat{P}_a\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r},\int\frac{\psi^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n}\hat{P}_b\frac{\psi(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n}d^3\vec{r}']$
$=-\int\frac{1}{\nabla^{2n}\nabla'^{2n}}[\psi^+(\vec{r},t)\partial_a\psi(\vec{r},t),\psi^+(\vec{r}',t')\partial'_b\psi(\vec{r}',t')]d^3\vec{r}d^3\vec{r}'$
$=-\int\frac{\delta^{k_\varsigma k'_\varsigma}\delta^{l_\varsigma l'_\varsigma}}{\nabla^{2n}\nabla'^{2n}}[\psi^+_{k'_\varsigma}(\vec{r},t)\partial_a\psi_{k_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t')\partial'_b\psi_{l_\varsigma}(\vec{r}',t')]d^3\vec{r}d^3\vec{r}'$
$=-\int d^3\vec{r}d^3\vec{r}'\frac{\delta^{k_\varsigma k'_\varsigma}\delta^{l_\varsigma l'_\varsigma}}{\nabla^{2n}\nabla'^{2n}}$
$\{[\psi^+_{k'_\varsigma}(\vec{r},t)\partial_a\psi_{k_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t')]\partial'_b\psi_{l_\varsigma}(\vec{r}',t')+\psi^+_{l'_\varsigma}(\vec{r}',t')[\psi^+_{k'_\varsigma}(\vec{r},t)\partial_a\psi_{k_\varsigma}(\vec{r},t),\partial'_b\psi_{l_\varsigma}(\vec{r}',t')]\}$
$=-\int d^3\vec{r}d^3\vec{r}'\frac{\delta^{k_\varsigma k'_\varsigma}\delta^{l_\varsigma l'_\varsigma}}{\nabla^{2n}\nabla'^{2n}}$
$\{\psi^+_{k'_\varsigma}(\vec{r},t)\{\partial_a\psi_{k_\varsigma}(\vec{r},t),\psi^+_{l'_\varsigma}(\vec{r}',t')\}\partial'_b\psi_{l_\varsigma}(\vec{r}',t')-\psi^+_{l'_\varsigma}(\vec{r}',t')\{\psi^+_{k'_\varsigma}(\vec{r},t),\partial'_b\psi_{l_\varsigma}(\vec{r}',t')\}\partial_a\psi_{k_\varsigma}(\vec{r},t)\}$
$=-\int d^3\vec{r}d^3\vec{r}'\frac{\delta^{k_\varsigma k'_\varsigma}\delta^{l_\varsigma l'_\varsigma}}{\nabla^{2n}\nabla'^{2n}}\frac{-i}{2^{2n}}$
$\{\psi^+_{k'_\varsigma}(\vec{r},t)\partial_a\Gamma^{\overbrace{cd\cdots}^{2n+1}}_{k_\varsigma l'_\varsigma}(n+\frac{1}{2})\overbrace{\partial_c\partial_d\cdots}^{2n+1}\Delta(x-x')\partial'_b\psi_{l_\varsigma}(\vec{r}',t')-\psi^+_{l'_\varsigma}(\vec{r}',t')\partial'_b\Gamma^{\overbrace{cd\cdots}^{2n+1}}_{l_\varsigma k'_\varsigma}(n+\frac{1}{2})\overbrace{\partial'_c\partial'_d\cdots}^{2n+1}\Delta(x'-x)\partial_a\psi_{k_\varsigma}(\vec{r},t)\}$
$=\int d^3\vec{r}d^3\vec{r}'\frac{i\delta^{k_\varsigma k'_\varsigma}\delta^{l_\varsigma l'_\varsigma}}{2^{2n}\nabla^{4n}}$
$\{\psi^+_{k'_\varsigma}(\vec{r},t)\partial'_b\psi_{l_\varsigma}(\vec{r}',t')\Gamma^{\overbrace{cd\cdots}^{2n+1}}_{k_\varsigma l'_\varsigma}(n+\frac{1}{2})\partial_a\overbrace{\partial_c\partial_d\cdots}^{2n+1}\Delta(x-x')+\psi^+_{l'_\varsigma}(\vec{r}',t')\partial_a\psi_{k_\varsigma}(\vec{r},t)\Gamma^{\overbrace{cd\cdots}^{2n+1}}_{l_\varsigma k'_\varsigma}(n+\frac{1}{2})\partial_b\overbrace{\partial_c\partial_d\cdots}^{2n+1}\Delta(x-x')\}$
$?=0$ □

7 poincare symmetry of s-spin particles

7.1 Poincare symmetry of bosons

Lem. 7.1.1. $\begin{cases}[\frac{\dot{\psi}_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]=\frac{i}{(-2)^{n-1}}\Gamma_+^{\overbrace{ef\cdots}^{2n}}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\\ [\frac{\psi_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]=\frac{i}{(-2)^{n-1}}\frac{-1}{i\sqrt{-\nabla^2}}\Gamma_-^{\overbrace{ef\cdots}^{2n}}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}'),n>0\end{cases}$

Thm. 7.1.1. $\begin{cases}[L_{ab},L_{cd}]=-i(g_{ad}L_{bc}-g_{ac}L_{bd}+g_{bc}L_{ad}-g_{bd}L_{ac})\\ [L_{ab},P_c]=-i(g_{bc}P_a-g_{ac}P_b),[P_a,P_b]=0\end{cases}$

Proof: $[L_{ab}, L_{cd}]$

$= -\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}(r'_c\partial'_d - r'_d\partial'_c)\frac{i\dot{\psi}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]$

$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}(r'_c\partial'_d - r'_d\partial'_c)\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]$

$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}](r'_c\partial'_d - r'_d\partial'_c)\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$+ \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, (r'_c\partial'_d - r'_d\partial'_c)\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}](r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}(r'_c\partial'_d - r'_d\partial'_c)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdots}^{2n,\hat{\partial}'_\pi\to i}\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial'_b - r_b\partial'_a)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdots}^{2n,\hat{\partial}'_\pi\to i}\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')(r'_c\partial'_d - r'_d\partial'_c)\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}(r'_c\partial_d - r'_d\partial_c)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l_\varsigma k'_\varsigma}(r_c\partial_d - r_d\partial_c)\frac{\dot{\psi}_{l'_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$

$- \frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_c\partial_d - r_d\partial_c)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l'_\varsigma k_\varsigma}(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= -\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[-i(r_a\partial_b - r_b\partial_a), -i(r_c\partial_d - r_d\partial_c)]\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$

$= \int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[\hat{L}_{ab}, \hat{L}_{cd}]\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$

$= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})$ □

Proof: $[L_{ab}, P_c]$

$= -\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\partial'_c\frac{i\dot{\psi}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]$

$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\partial'_c\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]$

$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]\partial'_c\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} + \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}[\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \partial'_c\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}](r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\partial'_c\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdots}^{2n,\hat{\partial}'_\pi\to i}\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial'_b - r_b\partial'_a)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdots}^{2n,\hat{\partial}'_\pi\to i}\}_{l_\varsigma k'_\varsigma}\delta^3(\vec{r}-\vec{r}')\partial'_c\frac{\dot{\psi}_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$- \frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\partial_c\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l'_\varsigma k_\varsigma}\delta^3(\vec{r}-\vec{r}')(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}$

$\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l_\varsigma k'_\varsigma}\partial_c\frac{\dot{\psi}_{l'_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} - \frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\partial_c\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma^{ef\cdots}_+}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l'_\varsigma k_\varsigma}$

$(r_a\partial_b - r_b\partial_a)\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= -\int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[-i(r_a\partial_b - r_b\partial_a), -i\partial_c']\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$
$= \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[\hat{L}_{ab}, \hat{P}_c]\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$
$= -i(g_{bc}P_a - g_{ac}P_b)$ □

Proof: $[P_a, P_b]$
$= -\int[\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\partial_a\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\partial_b'\frac{i\dot{\psi}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]d^3\vec{r}d^3\vec{r}'$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int[\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\partial_a\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\partial_b'\frac{\dot{\psi}_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]d^3\vec{r}d^3\vec{r}'$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int d^3\vec{r}d^3\vec{r}'\{\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[\partial_a\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]\partial_b'\frac{\dot{\psi}_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} + \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}[\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \partial_b'\frac{\dot{\psi}_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]\partial_a\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l_\varsigma k_\varsigma'}\partial_a\delta^3(\vec{r}-\vec{r}')\partial_b'\frac{\dot{\psi}_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} - \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e'\hat{\partial}_f'\cdots}^{2n,\hat{\partial}_\pi'\to i}\}_{l_\varsigma' k_\varsigma}$
$\partial_b'\delta^3(\vec{r}'-\vec{r})\partial_a\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e'\hat{\partial}_f'\cdots}^{2n,\hat{\partial}_\pi'\to i}\}_{l_\varsigma k_\varsigma'}\partial_a'\delta^3(\vec{r}-\vec{r}')\partial_b'\frac{\dot{\psi}_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} - \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}_{l_\varsigma' k_\varsigma}$
$\partial_b\delta^3(\vec{r}-\vec{r}')\partial_a\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$

$= \int\frac{i}{(-2)^{n-1}}\{\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}^{k_\varsigma l_\varsigma'}\partial_a\partial_b\frac{\dot{\psi}_{l_\varsigma'}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} - \frac{\psi_{k_\varsigma'}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}^{k_\varsigma' l_\varsigma}\partial_b\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}d^3\vec{r}$
$= \int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(\partial_a\partial_b - \partial_b\partial_a)\frac{i}{(-2)^{n-1}}\{\overbrace{\Gamma_+^{ef\cdots}}^{2n}(n)\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n,\hat{\partial}_\pi\to i}\}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$
$= \int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(\partial_a\partial_b - \partial_b\partial_a)\frac{-i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$
$= \int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[\hat{P}_a, \hat{P}_b]\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r} = 0$ □

7.2 Poincare symmetry of fermions

Lem. 7.2.1. $\{\frac{\psi_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\} = \frac{i}{(-2)^{n-1}\sqrt{2}}\overbrace{\Gamma_-^{ef\cdots}}^{2n+1}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')$

Thm. 7.2.1. $\begin{cases}[L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})\\ [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0\end{cases}$

Proof: $[L_{ab}, L_{cd}]$
$= -\int d^3\vec{r}d^3\vec{r}'[\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}(r_c'\partial_d' - r_d'\partial_c')\frac{\psi(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int d^3\vec{r}d^3\vec{r}'[\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}(r_c'\partial_d' - r_d'\partial_c')\frac{\psi_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int d^3\vec{r}d^3\vec{r}'$
$\{\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\{(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\}(r_c'\partial_d' - r_d'\partial_c')\frac{\psi_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$
$-\frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\{\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, (r_c'\partial_d' - r_d'\partial_c')\frac{\psi_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\}(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$
$= -\delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int d^3\vec{r}d^3\vec{r}'$

$\{\frac{\psi_{k_\varsigma}^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b - r_b\partial_a)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\overbrace{\Gamma_-^{ef\cdots}}^{2n+1}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}_{l_\varsigma k_\varsigma'}\delta^3(\vec{r}-\vec{r}')(r_c'\partial_d' - r_d'\partial_c')\frac{\psi_{l_\varsigma'}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$-\frac{\psi_{k_\varsigma'}^+(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}(r_c'\partial_d' - r_d'\partial_c')\frac{i}{(-2)^{n-1}\sqrt{2}}\{\overbrace{\Gamma_-^{ef\cdots}}^{2n+1}(n+\frac{1}{2})\overbrace{\hat{\partial}_e'\hat{\partial}_f'\cdots}^{2n+1,\hat{\partial}_\pi'\to i}\}_{l_\varsigma' k_\varsigma}\delta^3(\vec{r}'-\vec{r})(r_a\partial_b - r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$
$= \delta^{k_\varsigma l_\varsigma}\delta^{k_\varsigma' l_\varsigma'}\int d^3\vec{r}d^3\vec{r}'$

$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_a\partial'_b-r_b\partial'_a)\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial'_e\hat\partial'_f\cdots}^{2n+1,\hat\partial'_\pi\to i}\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')(r'_c\partial'_d-r'_d\partial'_c)\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}(r'_c\partial_d-r'_d\partial_c)\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r-\vec r')(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\}$$
$$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b-r_b\partial_a)\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}_{l_\varsigma k'_\varsigma}(r_c\partial_d-r_d\partial_c)\frac{\psi_{l'_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_c\partial_d-r_d\partial_c)\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}_{l'_\varsigma k_\varsigma}(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\}$$
$$=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}[-i(r_a\partial_b-r_b\partial_a),-i(r_c\partial_d-r_d\partial_c)]\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r$$
$$=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}[\hat L_{ab},\hat L_{cd}]\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r$$
$$=-i(g_{ad}L_{bc}-g_{ac}L_{bd}+g_{bc}L_{ad}-g_{bd}L_{ac})$$ □

Proof: $[L_{ab},P_c]$
$$=-\int d^3\vec r d^3\vec r'[\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b-r_b\partial_a)\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\partial'_c\frac{\psi(\vec r',t)}{(\sqrt{-\nabla'^2})^n}]$$
$$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\partial'_c\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}]$$
$$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\{(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\}\partial'_c\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}-\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n},\partial'_c\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\}(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\}$$
$$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b-r_b\partial_a)\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\partial'_c\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\partial'_c\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial'_e\hat\partial'_f\cdots}^{2n+1,\hat\partial'_\pi\to i}\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r'-\vec r)(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\}$$
$$=\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r d^3\vec r'$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_a\partial'_b-r_b\partial'_a)\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial'_e\hat\partial'_f\cdots}^{2n+1,\hat\partial'_\pi\to i}\}_{l_\varsigma k'_\varsigma}\delta^3(\vec r-\vec r')\partial'_c\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\partial_c\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}_{l'_\varsigma k_\varsigma}\delta^3(\vec r-\vec r')(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\}$$
$$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec r$$
$$\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}(r_a\partial_b-r_b\partial_a)\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}_{l_\varsigma k'_\varsigma}\partial_c\frac{\psi_{l'_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\partial_c\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}_{l'_\varsigma k_\varsigma}(r_a\partial_b-r_b\partial_a)\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\}$$
$$=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}[-i(r_a\partial_b-r_b\partial_a),-i\partial'_c]\frac{i}{(-2)^{n-1}\sqrt2}\{\overbrace{\Gamma^{ef\cdots}_-}^{2n+1}(n+\tfrac12)\overbrace{\hat\partial_e\hat\partial_f\cdots}^{2n+1,\hat\partial_\pi\to i}\}\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r$$
$$=\int\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}[\hat L_{ab},\hat P_c]\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n}d^3\vec r$$
$$=-i(g_{bc}P_a-g_{ac}P_b)$$ □

Proof: $[P_a,P_b]$
$$=-\int[\frac{\psi^+(\vec r,t)}{(\sqrt{-\nabla^2})^n}\partial_a\frac{\psi(\vec r,t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\partial'_b\frac{\psi(\vec r',t)}{(\sqrt{-\nabla'^2})^n}]d^3\vec r d^3\vec r'$$
$$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int[\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\partial_a\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\partial'_b\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}]d^3\vec r d^3\vec r'$$
$$=-\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}$$
$$\int d^3\vec r d^3\vec r'\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\{\partial_a\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\}\partial'_b\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}-\frac{\psi^+_{k'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\{\frac{\psi^+_{k_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n},\partial'_b\frac{\psi_{l'_\varsigma}(\vec r',t)}{(\sqrt{-\nabla'^2})^n}\}\partial_a\frac{\psi_{l_\varsigma}(\vec r,t)}{(\sqrt{-\nabla^2})^n}\}$$

$$= -\delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}_{l_\varsigma k'_\varsigma}\partial_a\delta^3(\vec{r}-\vec{r}')\partial'_b\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdots}^{2n+1,\hat{\partial}'_\pi\to i}\}_{l'_\varsigma k_\varsigma}\partial'_b\delta^3(\vec{r}'-\vec{r})\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$$
$$= \delta^{k_\varsigma l_\varsigma}\delta^{k'_\varsigma l'_\varsigma}\int d^3\vec{r}d^3\vec{r}'\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdots}^{2n+1,\hat{\partial}'_\pi\to i}\}_{l_\varsigma k'_\varsigma}\partial'_a\delta^3(\vec{r}-\vec{r}')\partial'_b\frac{\psi_{l'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}_{l'_\varsigma k_\varsigma}\partial_b\delta^3(\vec{r}-\vec{r}')\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}$$
$$= -\int\{\frac{\psi^+_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}^{k_\varsigma l'_\varsigma}\partial_a\partial_b\frac{\psi_{l'_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$$
$$-\frac{\psi^+_{k'_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}^{k'_\varsigma l_\varsigma}\partial_b\partial_a\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\}d^3\vec{r}$$
$$= -\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(\partial_a\partial_b-\partial_b\partial_a)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$$
$$= -\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}(\partial_a\partial_b-\partial_b\partial_a)\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$$
$$= \int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}[\hat{P}_a,\hat{P}_b]\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=0$$ □

7.3 Poincare symmetry of fermion spin

Lem. 7.3.1.
$$\begin{cases}\{\frac{\psi_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\}=\frac{i}{(-2)^{n-1}\sqrt{2}}\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\\ \{\frac{\dot{\psi}_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\}=\frac{i}{(-2)^{n-1}\sqrt{2}}(-i\sqrt{-\nabla^2})\Gamma_+^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\\ \{\frac{\dot{\psi}_{k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}},\frac{\psi^+_{k'_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}}\}=\frac{i}{(-2)^{n-1}\sqrt{2}}\frac{1}{i\sqrt{-\nabla^2}}\Gamma_+^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\delta^3(\vec{r}-\vec{r}')\end{cases}$$

Cor. 7.3.1. $\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\sigma_{\alpha_\varsigma}(n+\frac{1}{2})\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=-\varsigma(n+\frac{1}{2})\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\alpha_\varsigma}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}},n+\frac{1}{2}\ge 1$

Cor. 7.3.2. $\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\sigma_{\alpha_\varsigma}(n)\frac{i\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}=n\varsigma\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\nabla_{\alpha_\varsigma}\frac{i\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n},n\ge 1$

Proof: $[S_{ab}(t),S_{cd}(t)]$
$$=\int[\frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{abk_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}S_{cdm_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}]d^3\vec{r}d^3\vec{r}'$$
$$=\int\frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}\{S_{abk_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n},\frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\}S_{cdm_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$$
$$-\frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\{\frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n},S_{cdm_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}\}S_{abk_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}d^3\vec{r}'$$
$$=\int\frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{abk_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2},\varsigma)S_{cdm_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2},\varsigma)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdots}^{2n+1,\hat{\partial}'_\pi\to i}\}_{l_\varsigma}{}^{m_\varsigma}\delta^3(\vec{r}-\vec{r}')\frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$$
$$-\frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}S_{cdm_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2},\varsigma)S_{abk_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2},\varsigma)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}_{n_\varsigma}{}^{k_\varsigma}\delta^3(\vec{r}-\vec{r}')\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}d^3\vec{r}'$$
$$=\int\frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{abk_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2},\varsigma)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}_{l_\varsigma}{}^{m_\varsigma}S_{cdm_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{n_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$$
$$-\frac{\psi^{+m_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{cdm_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2},\varsigma)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}_{n_\varsigma}{}^{k_\varsigma}S_{abk_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2},\varsigma)\frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$$
$$=\int\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{ab}(n+\frac{1}{2},\varsigma)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}S_{cd}(n+\frac{1}{2},\varsigma)\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$$
$$-\frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n}S_{cd}(n+\frac{1}{2},\varsigma)\frac{i}{(-2)^{n-1}\sqrt{2}}\{\Gamma_-^{\overbrace{ef\cdots}^{2n+1}}(n+\frac{1}{2})\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}S_{ab}(n+\frac{1}{2},\varsigma)\frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}d^3\vec{r}$$

$= \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} S_{ab}(n+\frac{1}{2},\varsigma) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} S_{cd}(n+\frac{1}{2},\varsigma) \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$

$- \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} S_{cd}(n+\frac{1}{2},\varsigma) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} S_{ab}(n+\frac{1}{2},\varsigma) \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$

$??? = \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} [S_{ab}(n+\frac{1}{2},\varsigma), S_{cd}(n+\frac{1}{2},\varsigma)] \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$

$= \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} [S_{ab}(n+\frac{1}{2},\varsigma), S_{cd}(n+\frac{1}{2},\varsigma)] \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$

$= -i[g_{ad}S_{bc}(n+\frac{1}{2},\varsigma) - g_{ac}S_{bd}(n+\frac{1}{2},\varsigma) + g_{bc}S_{ad}(n+\frac{1}{2},\varsigma) - g_{bd}S_{ac}(n+\frac{1}{2},\varsigma)]$ □

Proof: $[\sigma_{\alpha_\varsigma}(t), \sigma_{\beta_\varsigma}(t)]$

$= \int [\frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2}) \frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} \sigma_{\beta_\varsigma m_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2}) \frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}] d^3\vec{r} d^3\vec{r}'$

$= \int \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \{ \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2}) \frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} \} \sigma_{\beta_\varsigma m_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2}) \frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$- \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} \{ \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \sigma_{\beta_\varsigma m_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2}) \frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} \} \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2}) \frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} d^3\vec{r}'$

$= \int \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2}) \sigma_{\beta_\varsigma m_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdot\cdot}^{2n+1,\hat{\partial}'_\pi\to i} \}_{l_\varsigma}{}^{m_\varsigma} \delta^3(\vec{r}-\vec{r}') \frac{\psi_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n}$

$- \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^n} \sigma_{\beta_\varsigma m_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2}) \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \}_{n_\varsigma}{}^{k_\varsigma} \delta^3(\vec{r}-\vec{r}') \frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} d^3\vec{r}'$

$= \int \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \}_{l_\varsigma}{}^{m_\varsigma} \sigma_{\beta_\varsigma m_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2}) \frac{\psi_{n_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$

$- \frac{\psi^{+m_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\beta_\varsigma m_\varsigma}{}^{n_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \}_{n_\varsigma}{}^{k_\varsigma} \sigma_{\alpha_\varsigma k_\varsigma}{}^{l_\varsigma}(n+\frac{1}{2}) \frac{\psi_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$

$= \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} \sigma_{\beta_\varsigma}(n+\frac{1}{2}) \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$

$- \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\beta_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} \sigma_{\alpha_\varsigma}(n+\frac{1}{2}) \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$

$= \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} \sigma_{\beta_\varsigma}(n+\frac{1}{2}) \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}$

$- \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\beta_\varsigma}(n+\frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} \sigma_{\alpha_\varsigma}(n+\frac{1}{2}) \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$

$??? = \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} [\sigma_{\alpha_\varsigma}(n+\frac{1}{2}), \sigma_{\beta_\varsigma}(n+\frac{1}{2})] \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \overbrace{\Gamma_-^{ef\cdot\cdot}}^{2n+1}(n+\frac{1}{2}) \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$

$= \int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} [\sigma_{\alpha_\varsigma}(n+\frac{1}{2}), \sigma_{\beta_\varsigma}(n+\frac{1}{2})] \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(t)$ □

Proof: $[\sigma_{\alpha_\varsigma}(t), \sigma_{\beta_\varsigma}(t)](n+\frac{1}{2})^{-2}$

$= \int [\frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma} \frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}, \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}} \nabla'_{\beta_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma} \frac{\dot{\psi}_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}}] d^3\vec{r} d^3\vec{r}'$

$= \int \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \{ \nabla_{\alpha_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma} \frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}, \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}} \} \nabla'_{\beta_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma} \frac{\dot{\psi}_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}}$

$- \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}} \{ \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}, \nabla'_{\beta_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma} \frac{\dot{\psi}_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}} \} \nabla_{\alpha_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma} \frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3\vec{r} d^3\vec{r}'$

$= \int \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla'_{\alpha_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma} \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla'^2}} \overbrace{\Gamma_+^{ef\cdot\cdot}}^{2n+1} \overbrace{\hat{\partial}'_e\hat{\partial}'_f\cdot\cdot}^{2n+1,\hat{\partial}'_\pi\to i} \}_{l_\varsigma}{}^{m_\varsigma} \delta^3(\vec{r}-\vec{r}') \nabla'_{\beta_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma} \frac{\dot{\psi}_{n_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}}$

$- \frac{\psi^{+m_\varsigma}(\vec{r}',t)}{(\sqrt{-\nabla'^2})^{n+1}} \nabla_{\beta_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma} \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \overbrace{\Gamma_+^{ef\cdot\cdot}}^{2n+1} \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \}_{n_\varsigma}{}^{k_\varsigma} \delta^3(\vec{r}-\vec{r}') \nabla_{\alpha_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma} \frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3\vec{r} d^3\vec{r}'$

$= \int \frac{\psi^{+k_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma} \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \overbrace{\Gamma_+^{ef\cdot\cdot}}^{2n+1} \overbrace{\hat{\partial}_e\hat{\partial}_f\cdot\cdot}^{2n+1,\hat{\partial}_\pi\to i} \}_{l_\varsigma}{}^{m_\varsigma} \nabla_{\beta_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma} \frac{\dot{\psi}_{n_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}$

$-\frac{\psi^{+m_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\beta_\varsigma}\delta_{m_\varsigma}{}^{n_\varsigma}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\frac{1}{i\sqrt{-\nabla^2}}\Gamma_{+}^{\overbrace{ef\cdots}^{2n+1}}\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}_{n_\varsigma}{}^{k_\varsigma}\nabla_{\alpha_\varsigma}\delta_{k_\varsigma}{}^{l_\varsigma}\frac{\dot{\psi}_{l_\varsigma}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}d^3\vec{r}$

$=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\alpha_\varsigma}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\frac{1}{i\sqrt{-\nabla^2}}\Gamma_{+}^{\overbrace{ef\cdots}^{2n+1}}\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}\nabla_{\beta_\varsigma}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}$

$-\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\beta_\varsigma}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\frac{1}{i\sqrt{-\nabla^2}}\Gamma_{+}^{\overbrace{ef\cdots}^{2n+1}}\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}\nabla_{\alpha_\varsigma}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}d^3\vec{r}$

$=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\alpha_\varsigma}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\frac{1}{i\sqrt{-\nabla^2}}\Gamma_{+}^{\overbrace{ef\cdots}^{2n+1}}\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}\nabla_{\beta_\varsigma}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}$

$-\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}\nabla_{\beta_\varsigma}\frac{i}{(-2)^{n-1}\sqrt{2}}\{\frac{1}{i\sqrt{-\nabla^2}}\Gamma_{+}^{\overbrace{ef\cdots}^{2n+1}}\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}\nabla_{\alpha_\varsigma}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}d^3\vec{r}$

$=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}[\nabla_{\alpha_\varsigma},\nabla_{\beta_\varsigma}]\frac{i}{(-2)^{n-1}\sqrt{2}}\{\frac{1}{i\sqrt{-\nabla^2}}\Gamma_{+}^{\overbrace{ef\cdots}^{2n+1}}\overbrace{\hat{\partial}_e\hat{\partial}_f\cdots}^{2n+1,\hat{\partial}_\pi\to i}\}\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}d^3\vec{r}$

$=\int\frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}[\nabla_{\alpha_\varsigma},\nabla_{\beta_\varsigma}]\frac{\dot{\psi}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}d^3\vec{r}=0?=i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}\sigma_{\gamma_\varsigma}(t)$ □

Cor. 7.3.3. $\lambda^{+}(\hat{p},-s\varsigma)\sigma_i(s)[\sigma(s)\cdot\hat{p}]^n\sigma_j(s)\lambda(\hat{p},-s\varsigma)$
$=\lambda^{+}(\hat{p},-s\varsigma)\sigma_i(s)\{(-\varsigma)^{n-1}s[s^n-(s-1)^n]\hat{p}_j+(-\varsigma)^n(s-1)^n\sigma_j(s)\}\lambda(\hat{p},-s\varsigma)$
$=(-\varsigma)^n s^2[s^n-(s-1)^n]\hat{p}_i\hat{p}_j+(-\varsigma)^n(s-1)^n\lambda^{+}(\hat{p},-s\varsigma)\sigma_i(s)\sigma_j(s)\lambda(\hat{p},-s\varsigma)$
$=(-\varsigma)^n s^2[s^n-(s-1)^n]\hat{p}_i\hat{p}_j+(-\varsigma)^n(s-1)^n[s^2\hat{p}_i\hat{p}_j+\frac{s}{2}(\delta_{ij}-\hat{p}_i\hat{p}_j-i\varsigma\varepsilon_{ij}{}^{k}\hat{p}_k)]$
$=(-\varsigma)^n s^2 s^n\hat{p}_i\hat{p}_j+(-\varsigma)^n(s-1)^n[\frac{s}{2}(\delta_{ij}-\hat{p}_i\hat{p}_j-i\varsigma\varepsilon_{ij}{}^{k}\hat{p}_k)]$

8 Covariate quantization of fully symmetric Penrose equation

Self comment: Since Penrose fully symmetric equation is completely equivalent to the spin equation, the covariant quantization of Penrose fully symmetric equation has also been successfully completed in principle. It only needs to be equivalently converted from the spin equation. But starting directly from Penrose fully symmetric equation can provide a completely new solution.It has implications for the covariant quantization of massive particles. As detailed conclusions have been obtained by the spin equation method, only the essence of the Penrose fully symmetric equation solution is given below. I no longer seek perfection. And it is a supplement to the spin equation method.

8.1 Penrose fully symmetric equation [1,2] and its plane wave solutions

Thm. 8.1.1.

$[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(x)=0\Leftrightarrow(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)=0,\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)=\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s)\psi_{k_\varsigma}(x)$

Cor. 8.1.1.

$$\begin{cases}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s)\lambda_{k_\varsigma}(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^{+}(\vec{p},-s\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\ \vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s)\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)e^{-ip\cdot x}d^3\vec{r}\\ |\vec{p}|^{(s-\frac{1}{2})}a_2^{+}(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^{+k_\varsigma}(\hat{p},-s\varsigma)\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s)\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Cor. 8.1.2.

$$\begin{cases}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}^{2s}}(s)\overbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}^{2s}\\ \Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s)\lambda_{k_\varsigma}(\hat{p},-s\varsigma)=\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}_{2s}\end{cases}$$

Cor. 8.1.3.

$$\begin{cases}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}_{2s}[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^{+}(\vec{p},-s\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\ \vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}^{2s}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)e^{-ip\cdot x}d^3\vec{r}\\ |\vec{p}|^{(s-\frac{1}{2})}a_2^{+}(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}^{2s}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

8.2 Spin bases of and its properties of Penrose fully symmetric equation

Def. 8.2.1. $\lambda_{A_\varsigma B_\varsigma\cdot\cdot}(\hat{p},-s\varsigma):=\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}_{2s}$

Cor. 8.2.1. $\begin{cases}\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})=1,\lambda^{+A_\varsigma}(-\hat{p},-\frac{\varsigma}{2})\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})=1\\ \lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+}_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})=-\frac{\varsigma}{2}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\hat{p}_a\end{cases}$

Cor. 8.2.2. $\begin{cases}\lambda^{+A_\varsigma B_\varsigma\cdot\cdot}(\hat{p},-s\varsigma)\lambda_{A_\varsigma B_\varsigma\cdot\cdot}(\hat{p},-s\varsigma)=1,\lambda^{+A_\varsigma B_\varsigma\cdot\cdot}(-\hat{p},-s\varsigma)\lambda_{A_\varsigma B_\varsigma\cdot\cdot}(\hat{p},-s\varsigma)=0\\ \lambda_{A_\varsigma B_\varsigma\cdot\cdot}(\hat{p},-s\varsigma)\lambda^{+}_{A'_\varsigma B'_\varsigma\cdot\cdot}(\hat{p},-s\varsigma)=(-\frac{\varsigma}{2})^{2s}\frac{1}{[(2s)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot\})}^{2s}\overbrace{\hat{p}_a\hat{p}_b\cdot\cdot}^{2s}\end{cases}$

8.3 Various physical operators of Penrose fully symmetric equation

Thm. 8.3.1. $P_u(s)$

$=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)d^3\vec{r}\int p_u[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]d^3\vec{p}$

Proof: $P_u(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)d^3\vec{r}$

$=\frac{1}{(2\pi)^3}\int d^3\vec{p}'d^3\vec{p}d^3\vec{r}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\overbrace{\lambda^{+A_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\cdot\cdot}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}_{2s}\frac{p_u}{|\vec{p}|^{2s-1}}$

$[a^+_1(\vec{p}',-s\varsigma)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}+a_2(\vec{p}',-s\varsigma)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+(-1)^{2s}a^+_2(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$

$=\int\vec{p}|^{2s-1}\overbrace{\lambda^{+A_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\cdot\cdot}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}_{2s}\frac{p_u}{|\vec{p}|^{2s-1}}$

$\{[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]\delta^3(\vec{p}'-\vec{p})$
$+[(-1)^{2s}a^+_1(-\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)e^{-2i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\}d^3\vec{p}'d^3\vec{p}$

$=\int\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}_{2s}$

$p_u[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]d^3\vec{p}$
$=\int p_u[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]d^3\vec{p}$ □

Thm. 8.3.2. $Q(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)d^3\vec{r}$

$=\int[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s-1}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]d^3\vec{p}$

Proof: $Q(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)d^3\vec{r}$

$=\frac{1}{(2\pi)^3}\int d^3\vec{p}'d^3\vec{p}d^3\vec{r}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\overbrace{\lambda^{+A_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\cdot\cdot}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}_{2s}\frac{1}{|\vec{p}|^{2s-1}}$

$[a^+_1(\vec{p}',-s\varsigma)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}+a_2(\vec{p}',-s\varsigma)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+(-1)^{2s-1}a^+_2(\vec{p},-s\varsigma)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$

$=\int\vec{p}|^{2s-1}\overbrace{\lambda^{+A_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\cdot\cdot}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot}_{2s}\frac{1}{|\vec{p}|^{2s-1}}$

$\{[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s-1}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]\delta^3(\vec{p}'-\vec{p})$
$+[(-1)^{2s-1}a^+_1(-\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)e^{-2i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\}d^3\vec{p}'d^3\vec{p}$
$=\int[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s-1}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]d^3\vec{p}$ □

Thm. 8.3.3. $N(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)d^3\vec{r}$

$=\int[a^+_1(\vec{p},-s\varsigma)a_1(\vec{p},-s\varsigma)+(-1)^{2s}a_2(\vec{p},-s\varsigma)a^+_2(\vec{p},-s\varsigma)]d^3\vec{p}$

Proof: $N(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$$=\frac{1}{(2\pi)^3}\int d^3\vec p'd^3\vec p d^3\vec r|\vec p'|^{s-\frac12}|\vec p|^{s-\frac12}\overbrace{\lambda^{+A_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\cdots}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdots}_{2s}\frac{1}{|\vec p|^{2s-1}}$$

$$[a_1^+(\vec p',-s\varsigma)e^{-i(\vec p'\cdot\vec r-|\vec p'|t)}+a_2(\vec p',-s\varsigma)e^{i(\vec p'\cdot\vec r-|\vec p'|t)}][a_1(\vec p,-s\varsigma)e^{i(\vec p\cdot\vec r-|\vec p|t)}+(-1)^{2s}a_2^+(\vec p,-s\varsigma)e^{-i(\vec p\cdot\vec r-|\vec p|t)}]$$

$$=\int\vec p|^{2s-1}\overbrace{\lambda^{+A_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\cdots}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdots}_{2s}\frac{1}{|\vec p|^{2s-1}}$$

$$\{[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]\delta^3(\vec p'-\vec p)$$
$$+[(-1)^{2s}a_1^+(-\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)e^{-2i|\vec p|t}+a_2(-\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)e^{2i|\vec p|t}]\delta^3(\vec p'+\vec p)\}d^3\vec p'd^3\vec p$$
$$=\int[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]d^3\vec p$$ □

Thm. 8.3.4. $\vec S(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$=\int\hat p[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]d^3\vec p$

Proof: $\vec S(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$$=\frac{1}{(2\pi)^3}\int d^3\vec p'd^3\vec p d^3\vec r|\vec p'|^{s-\frac12}|\vec p|^{s-\frac12}\overbrace{\lambda^{+A_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\cdots}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdots}_{2s}\frac{\hat p}{|\vec p|^{2s-1}}$$

$$[a_1^+(\vec p',-s\varsigma)e^{-i(\vec p'\cdot\vec r-|\vec p'|t)}+a_2(\vec p',-s\varsigma)e^{i(\vec p'\cdot\vec r-|\vec p'|t)}][a_1(\vec p,-s\varsigma)e^{i(\vec p\cdot\vec r-|\vec p|t)}+(-1)^{2s}a_2^+(\vec p,-s\varsigma)e^{-i(\vec p\cdot\vec r-|\vec p|t)}]$$

$$=\int\vec p|^{2s-1}\overbrace{\lambda^{+A_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\cdots}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdots}_{2s}\frac{\hat p}{|\vec p|^{2s-1}}$$

$$\{[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]\delta^3(\vec p'-\vec p)$$
$$+[(-1)^{2s}a_1^+(-\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)e^{-2i|\vec p|t}+a_2(-\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)e^{2i|\vec p|t}]\delta^3(\vec p'+\vec p)\}d^3\vec p'd^3\vec p$$
$$=\int\hat p[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]d^3\vec p$$ □

Thm. 8.3.5. $\vec M(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$=\int\hat p[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s-1}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]d^3\vec p$

Proof: $\vec M(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$$=\frac{1}{(2\pi)^3}\int d^3\vec p'd^3\vec p d^3\vec r|\vec p'|^{s-\frac12}|\vec p|^{s-\frac12}\overbrace{\lambda^{+A_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\cdots}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdots}_{2s}\frac{\hat p}{|\vec p|^{2s-1}}$$

$$[a_1^+(\vec p',-s\varsigma)e^{-i(\vec p'\cdot\vec r-|\vec p'|t)}+a_2(\vec p',-s\varsigma)e^{i(\vec p'\cdot\vec r-|\vec p'|t)}][a_1(\vec p,-s\varsigma)e^{i(\vec p\cdot\vec r-|\vec p|t)}+(-1)^{2s-1}a_2^+(\vec p,-s\varsigma)e^{-i(\vec p\cdot\vec r-|\vec p|t)}]$$

$$=\int\vec p|^{2s-1}\overbrace{\lambda^{+A_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p',-\tfrac{\varsigma}{2})\cdots}^{2s}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdots}_{2s}\frac{\hat p}{|\vec p|^{2s-1}}$$

$$\{[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s-1}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]\delta^3(\vec p'-\vec p)$$
$$+[(-1)^{2s-1}a_1^+(-\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)e^{-2i|\vec p|t}+a_2(-\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)e^{2i|\vec p|t}]\delta^3(\vec p'+\vec p)\}d^3\vec p'd^3\vec p$$
$$=\int\hat p[a_1^+(\vec p,-s\varsigma)a_1(\vec p,-s\varsigma)+(-1)^{2s-1}a_2(\vec p,-s\varsigma)a_2^+(\vec p,-s\varsigma)]d^3\vec p$$ □

8.4 Covariant commutation rules for Penrose fully symmetric equation

Thm. 8.4.1.

$$\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]_{-^{2s+1}}=i\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x'),\Gamma(0):=1\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]_{-^{2s+1}}=0,[\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]_{-^{2s+1}}=0,s\geq 0\end{cases}\Leftrightarrow$$

$$\begin{cases}[\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot\})}^{2s}\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\Delta(x-x')\\ [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi_{\underbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=0,[\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=0,s\geq 0\end{cases}$$

Lem. 8.4.1.

$$\begin{cases}\lambda_{A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat p,-\frac{\varsigma}{2})=-\frac{\varsigma}{2}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\hat p_a\\ \lambda_{A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat p,-\frac{\varsigma}{2})=\lambda_{A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat p,-\frac{\varsigma}{2})\end{cases}$$

$\Leftrightarrow\Leftrightarrow$

$$\begin{cases}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}p_b=(\sigma,i\varsigma)^a_{B_\varsigma A'_\varsigma}p_a(\sigma,i\varsigma)^b_{A_\varsigma B'_\varsigma}p_b,p^ap_a=0\\ (\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}p_b=(\sigma,i\varsigma)^a_{A_\varsigma B'_\varsigma}p_a(\sigma,i\varsigma)^b_{B_\varsigma A'_\varsigma}p_b,p^ap_a=0\end{cases}$$

Lem. 8.4.2. $\begin{cases}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b=(\sigma,i\varsigma)^a_{B_\varsigma A'_\varsigma}\partial_a(\sigma,i\varsigma)^b_{A_\varsigma B'_\varsigma}\partial_b,\partial^a\partial_a=0\\ (\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b=(\sigma,i\varsigma)^a_{A_\varsigma B'_\varsigma}\partial_a(\sigma,i\varsigma)^b_{B_\varsigma A'_\varsigma}\partial_b,\partial^a\partial_a=0\end{cases}$

Direct verification can prove the above two lemmas.

Cor. 8.4.1. $\frac{1}{[(2s)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot\})}^{2s}\overbrace{p_ap_bp_c\cdot\cdot}^{2s}=\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\overbrace{p_ap_bp_c\cdot\cdot}^{2s}$

Cor. 8.4.2. $\frac{1}{[(2s)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot\})}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta(x-x')=\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta(x-x')$

Cor. 8.4.3.

$$\begin{cases}[\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\Delta(x-x')\\ [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi_{\underbrace{E_\varsigma F_\varsigma G_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=0,[\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=0,s\geq 0\end{cases}$$

Cor. 8.4.4. $\psi_{\alpha_\varsigma\beta_\varsigma}=\frac{i\varsigma}{\sqrt2}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\frac{i\varsigma}{\sqrt2}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\psi_{A_\varsigma B_\varsigma C_\kappa D_\kappa}=-\frac12\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\psi_{A_\varsigma B_\varsigma C_\kappa D_\kappa},[\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}]^*=\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}$
$\Psi_{\alpha_\varsigma}:=\frac{-i\varsigma}{\sqrt2}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}$

Cor. 8.4.5. $[\psi_{\alpha_\varsigma\beta_\varsigma},\psi^+_{\alpha'_\varsigma\beta'_\varsigma}]=\frac{i}{2}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')$

Proof: $[\psi_{\alpha_\varsigma\beta_\varsigma},\psi^+_{\alpha'_\varsigma\beta'_\varsigma}]$
$=\frac14\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\beta'_\varsigma}[\psi_{A_\varsigma B_\varsigma C_\kappa D_\kappa},\psi^+_{A'_\varsigma B'_\varsigma C'_\kappa D'_\kappa}]$
$=\frac{i}{32}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\beta'_\varsigma}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^d_{D_\varsigma D'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')$
$=\frac{i}{2}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')$ □

Cor. 8.4.6. $[\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdot\cdot}_{2n}},\psi^+_{\underbrace{\alpha'_\varsigma\beta'_\varsigma\cdot\cdot}_{2n}}]=i\frac{(-1)^n}{2^{n-1}}\underbrace{\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\cdot\cdot}_{n}\underbrace{\partial_a\partial_b\partial_c\partial_d\cdot\cdot}_{n}\Delta(x-x')$

Cor. 8.4.7. $\psi_{\alpha_\varsigma\beta_\varsigma}=\frac{i\varsigma}{\sqrt2}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma},[\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}]^*=\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}$

Cor. 8.4.8. $[\psi_{\alpha_\varsigma},\psi^+_{\alpha'_\varsigma}]=\frac{i}{2}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$

Proof: $[\psi_{\alpha_\varsigma},\psi^+_{\alpha'_\varsigma}]$
$=-\frac12\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}[\psi_{A_\varsigma B_\varsigma},\psi^+_{A'_\varsigma B'_\varsigma}]$
$=\frac{i}{4}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$=-i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

8.5 Covariant quantization of photon Penrose fully symmetric equation

Thm. 8.5.1.
$[\partial_a+iS_{ab}(1,\varsigma)\partial^b]\psi(x)=0\Leftrightarrow(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a\psi_{A_\varsigma B_\varsigma}(x)=0,\psi_{A_\varsigma B_\varsigma}(x)=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}\psi_{k_\varsigma}(x)$

Cor. 8.5.1.

$$\begin{cases}\psi_{A_\varsigma B_\varsigma}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq 0}|\vec p|^{\frac12}\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}\lambda_{k_\varsigma}(\hat p,-\varsigma)[a_1(\vec p,-\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-\varsigma)e^{-ip\cdot x}]d^3\vec p\\ \vec p|^{\frac12}a_1(\vec p,-\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^{+k_\varsigma}(\hat p,-\varsigma)\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}\psi_{A_\varsigma B_\varsigma}(x)e^{-ip\cdot x}d^3\vec r\\ |\vec p|^{\frac12}a_2^+(\vec p,-\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^{+k_\varsigma}(\hat p,-\varsigma)\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}\psi_{A_\varsigma B_\varsigma}(x)e^{ip\cdot x}d^3\vec r\end{cases}$$

Thm. 8.5.2.

$$\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0,[\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\end{cases}\Leftrightarrow\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ =-\frac{i}{8}(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma\}B'_\varsigma)}\partial_a\partial_b\Delta(x-x')\\ [\psi_{A_\varsigma B_\varsigma}(x),\psi_{C_\varsigma D_\varsigma}(x')]=0,[\psi^+_{A'_\varsigma B'_\varsigma}(x),\psi^+_{C'_\varsigma D'_\varsigma}(x')]=0\end{cases}$$

Thm. 8.5.3. $H(1)=\int\psi^+(\vec r,t)\frac{[\sigma(1)\cdot\nabla]^2}{\nabla^2}\psi(\vec r,t)d^3\vec r=\int\psi^+_{A'_\varsigma B'_\varsigma}(\vec r,t)\Gamma^{A'_\varsigma B'_\varsigma}_{k'_\varsigma}\frac{[\sigma(1)\cdot\nabla]^2|^{k'_\varsigma k_\varsigma}}{\nabla^2}\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}\psi_{A_\varsigma B_\varsigma}(\vec r,t)d^3\vec r$

8.6 Covariant commutation rules for general photon Penrose equation

Ass. 8.6.1. $[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]=-\frac{i}{2}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')+ik\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}\Delta(x-x')$
$[\Leftrightarrow][\Psi_{\alpha_\varsigma}(x),\Psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x'),[\phi(x),\phi^+(x')]=i\Delta(x-x'),[\Psi_{\alpha_\varsigma}(x),\phi^+(x')]=0$

Self comment: The above are all equivalent conversions from known conclusions of spin equation. The following will directly start from the Penrose fully symmetric equation and provide a new solution.

8.7 Direct plane wave solutions of Penrose fully symmetric equation

Thm. 8.7.1. $(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot\}}_{2s}}(x)$

$$\Leftrightarrow\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq 0}|\vec p|^{(s-\frac12)}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdot\cdot}_{2s}[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^3\vec p$$

Proof: $(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot\}}_{2s}}(x)$

$$\Rightarrow\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq 0}\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})[a_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(\vec p)e^{ip\cdot x}+b^+_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(\vec p)e^{-ip\cdot x}]d^3\vec p$$

$$\Rightarrow\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-1}}(\vec p)=\frac{1}{(2\pi)^{3/2}}\int\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)e^{-ip\cdot x}d^3\vec r=\frac{1}{(2s)!}\lambda_{\{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot\}}_{2s-1}}(\vec p,-s\varsigma)$$

$$\Rightarrow\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(\vec p)=\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{A_\varsigma C_\varsigma\cdot\cdot}_{2s}}(\vec p)$$

$$\Rightarrow\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})[a_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})+a'_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{B_\varsigma}(\hat p,\tfrac{\varsigma}{2})]$$

$$=\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})[a_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})+a'_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{A_\varsigma}(\hat p,\tfrac{\varsigma}{2})]$$

$$\Rightarrow\lambda^{+A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p,\tfrac{\varsigma}{2})\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})[a_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})+a'_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{B_\varsigma}(\hat p,\tfrac{\varsigma}{2})]$$

$$=\lambda^{+A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p,\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})[a_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})+a'_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)\lambda_{A_\varsigma}(\hat p,\tfrac{\varsigma}{2})]$$

$$\Rightarrow a'_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s-2}}(\vec p)=0,\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(\vec p)=\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{C_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)=\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{B_\varsigma D_\varsigma\cdot\cdot}_{2s-2}}(\vec p)$$

$\Rightarrow\cdot\cdot$

$$\Rightarrow\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})a_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(\vec p)=a(\vec p,-s\varsigma)\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdot\cdot}_{2s}$$

For the same reason: $\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})b^+_{\underbrace{B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(\vec p)=b^+(\vec p,-s\varsigma)\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdot\cdot}_{2s}$

$$\Rightarrow\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq 0}|\vec p|^{(s-\frac12)}\underbrace{\lambda_{A_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat p,-\tfrac{\varsigma}{2})\cdot\cdot}_{2s}[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^3\vec p$$

$|\vec p|^{(s-\frac12)}a_1(\vec p,-s\varsigma)=a(\vec p,-s\varsigma),|\vec p|^{(s-\frac12)}a_2^+(\vec p,-s\varsigma)=b^+(\vec p,-s\varsigma)$
$\Rightarrow(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot\}}_{2s}}(x)$ □

Cor. 8.7.1.

$$\begin{cases} \vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p},-s\varsigma) = \frac{1}{(2\pi)^{3/2}}\int \overbrace{\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}^{2s}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)e^{-ip\cdot x}d^3\vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})}a_2^+(\vec{p},-s\varsigma) = \frac{1}{(2\pi)^{3/2}}\int \overbrace{\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}^{2s}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x)e^{ip\cdot x}d^3\vec{r} \end{cases}$$

8.8 Re-proving covariant commutative relations from Penrose fully symmetric equation

Thm. 8.8.1.

$$\begin{cases} [a_\sigma(\vec{p},-s\varsigma),a^+_{\sigma'}(\vec{p}',-s\varsigma)]_{-^{2s+1}} = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p},-s\varsigma),a_{\sigma'}(\vec{p}',-s\varsigma)]_{-^{2s+1}} = 0, [a^+_\sigma(\vec{p},-s\varsigma),a^+_{\sigma'}(\vec{p}',-s\varsigma)]_{-^{2s+1}} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x') \\ [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi_{\underbrace{E_\varsigma F_\varsigma G_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = 0, [\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{E'_\varsigma F'_\varsigma G'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = 0, s\geq 0 \end{cases}$$

Proof: $[\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'(|\vec{p}||\vec{p}'|)^{(s-\frac{1}{2})}$

$\underbrace{\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}_{2s}\underbrace{\lambda^+_{A'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^+_{C'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\cdots}_{2s}$

$[[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma)e^{-ip\cdot x}],[a_1^+(\vec{p}',-s\varsigma)e^{-ip'\cdot x'}+a_2(\vec{p}',-s\varsigma)e^{ip'\cdot x'}]]_{-^{2s+1}}$

$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'(|\vec{p}||\vec{p}'|)^{(s-\frac{1}{2})}$

$\underbrace{[\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})][\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})][\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{C'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})]\cdots}_{2s}$

$\{[a_1(\vec{p},-s\varsigma),a_1^+(\vec{p}',-s\varsigma)]_{-^{2s+1}}e^{i(p\cdot x-p'\cdot x')}+[a_2^+(\vec{p},-s\varsigma),a_2(\vec{p}',-s\varsigma)]_{-^{2s+1}}e^{-i(p\cdot x-p'\cdot x')}\}$

$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'(|\vec{p}||\vec{p}'|)^{(s-\frac{1}{2})}$

$\underbrace{[\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})][\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})][\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{C'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})]\cdots}_{2s}$

$\delta^3(\vec{p}-\vec{p}')[e^{i(p\cdot x-p'\cdot x')}+(-1)^{2s+1}e^{-i(p\cdot x-p'\cdot x')}]$

$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\underbrace{[|\vec{p}|\lambda_{A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{A'_\varsigma}(\hat{p},-\frac{\varsigma}{2})][|\vec{p}|\lambda_{B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat{p},-\frac{\varsigma}{2})][|\vec{p}|\lambda_{C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^+_{C'_\varsigma}(\hat{p},-\frac{\varsigma}{2})]\cdots}_{2s}$

$|\vec{p}|^{-1}[e^{ip\cdot(x-x')}+(-1)^{2s+1}e^{-ip\cdot(x-x')}]$

$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}(-\frac{\varsigma}{2})^{2s}\underbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}p_b(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}p_c\cdots}_{2s}|\vec{p}|^{-1}[e^{ip\cdot(x-x')}+(-1)^{2s+1}e^{-ip\cdot(x-x')}]$

$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}2i(\frac{i\varsigma}{2})^{2s}\underbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\partial_c\cdots}_{2s}\int\frac{-i}{|2\vec{p}|}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]d^3\vec{p}$

$$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x')$$

□

Proof: $[a_1(\vec{p},-s\varsigma),a_1^+(\vec{p}',-s\varsigma)]$

$= \frac{1}{(2\pi)^3}\int|\vec{p}|^{-(2s-1)}\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p}',-\frac{\varsigma}{2})\cdots}^{2s}$

$$[\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]e^{-i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$=\frac{1}{(2\pi)^3}\int|\vec{p}|^{-(2s-1)}\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\cdots}^{2s}$$
$$i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x')e^{-i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$=\frac{1}{(2\pi)^3}\int|\vec{p}|^{-(2s-1)}\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\cdots}^{2s}$$
$$i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}\cdots}^{2s}\{\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec{p}_0|}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec{p}_0\}e^{-i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$=[\frac{1}{(2\pi)^3}]^2\int d^3\vec{p}_0d^3\vec{r}d^3\vec{r}'|\vec{p}|^{-(2s-1)}\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\cdots}^{2s}$$
$$(-\tfrac{\varsigma}{2})^{2s}\underbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_{0a}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}p_{0b}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}p_{0c}\cdots}_{2s}\frac{1}{|\vec{p}_0|}[e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}+(-1)^{2s+1}e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}]$$

$$=[\frac{1}{(2\pi)^3}]^2\int|\vec{p}|^{-(2s-1)}\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\cdots}^{2s}$$
$$\underbrace{\lambda_{A_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\cdots}_{2s}\underbrace{\lambda^+_{A'_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda^+_{C'_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\cdots}_{2s}$$
$$\frac{1}{|\vec{p}_0|}[e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}+(-1)^{2s+1}e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}]d^3\vec{p}_0d^3\vec{r}d^3\vec{r}'$$

$$=\int|\vec{p}|^{-(2s-1)}\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p}',-\tfrac{\varsigma}{2})\cdots}^{2s}$$
$$\underbrace{\lambda_{A_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\cdots}_{2s}\underbrace{\lambda^+_{A'_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\lambda^+_{C'_\varsigma}(\hat{p}_0,-\tfrac{\varsigma}{2})\cdots}_{2s}$$
$$\frac{1}{|\vec{p}_0|}[\delta^3(\vec{p}_0-\vec{p})\delta^3(\vec{p}_0-\vec{p}')+(-1)^{2s+1}e^{2iE_0(t-t')}\delta^3(\vec{p}_0+\vec{p})\delta^3(\vec{p}_0+\vec{p}')]d^3\vec{p}_0$$

$$=\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}$$
$$\underbrace{\lambda_{A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}_{2s}\underbrace{\lambda^+_{A'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^+_{B'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^+_{C'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}_{2s}\delta^3(\vec{p}-\vec{p}')$$

$$+(-1)^{2s+1}e^{2iE_0(t-t')}\overbrace{\lambda^{+A_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{+C_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}\overbrace{\lambda^{A'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{B'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\lambda^{C'_\varsigma}(\hat{p},-\tfrac{\varsigma}{2})\cdots}^{2s}$$
$$\underbrace{\lambda_{A_\varsigma}(-\hat{p},-\tfrac{\varsigma}{2})\lambda_{B_\varsigma}(-\hat{p},-\tfrac{\varsigma}{2})\lambda_{C_\varsigma}(-\hat{p},-\tfrac{\varsigma}{2})\cdots}_{2s}\underbrace{\lambda^+_{A'_\varsigma}(-\hat{p},-\tfrac{\varsigma}{2})\lambda^+_{B'_\varsigma}(-\hat{p},-\tfrac{\varsigma}{2})\lambda^+_{C'_\varsigma}(-\hat{p},-\tfrac{\varsigma}{2})\cdots}_{2s}\delta^3(\vec{p}-\vec{p}')$$
$$=\delta^3(\vec{p}-\vec{p}')+0=\delta^3(\vec{p}-\vec{p}')$$

□

Self comment: The above proof method is no longer based on the isochronous commutation rule, but directly based on the covariant commutation rule. It seems more difficult, but it's actually simpler. Because there is no need to find complex isochronal commutation rules (see the next section). Even if it is calculated out, it is still difficult to use. The covariant commutation rule itself is known and very regular and can also be decomposed into the product of spin bases. The entire proof process basically depends on the properties of the spin base and hasn't complex calculations. This proof method can be extended to all other similar cases and thereby simplify all similar proofs. The other commutative brackets can also be calculated out by using the same method and will not be listed.

8.9 Isochronous quantization rules for Penrose fully symmetric equation

Thm. 8.9.1. $[\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x-x')$

$$\Rightarrow[\psi_{\underbrace{A_\varsigma B_\varsigma\cdots E_\varsigma F_\varsigma\cdots Z_\varsigma}_{2s}}(\vec{r},t),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots E'_\varsigma F'_\varsigma\cdots Z'_\varsigma}_{2s}}(\vec{r}',t)]_{-^{2s+1}}$$
$$=-\frac{(i\varsigma)^{2s+1}}{2^{2s-1}}\sum_{k=0}^{[s-\frac{1}{2}]}\frac{(2s)!}{(2s-2k-1)!(2k)!1!}\overbrace{(\sigma\cdot\nabla)_{A_\varsigma A'_\varsigma}(\sigma\cdot\nabla)_{B_\varsigma B'_\varsigma}\cdots}^{2s-2k-1}\overbrace{\delta_{E_\varsigma E'_\varsigma}\delta_{F_\varsigma F'_\varsigma}\cdots}^{2k}\nabla^{2k}\delta_{Z_\varsigma Z'_\varsigma}\delta^3(\vec{r}-\vec{r}')$$

8.10 Commutative function, causal function and Feynman propagator of Penrose equation

Lem. 8.10.1.

$$[\theta(t),\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}]=-\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-1}C^n_{2s}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdots}^{n}(i\varsigma)^{2s-n}[\partial_\pi^{2s-n}\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}$$

Proof: $[\theta(t),\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}]$

$$=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}[\theta(t),\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}]$$

$$=-\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-1}C^n_{2s}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdots}^{n}(i\varsigma)^{2s-n}[\partial_\pi^{2s-n}\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}$$

$$=-\tfrac{i^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-1}\varsigma^nC^n_{2s}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdots}^{n}[\partial_t^{2s-n}\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}$$ □

Cor. 8.10.1.

$$\begin{cases}\Delta_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x):=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x)\\ \Delta^{(+)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x):=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta^{(+)}(x)\\ \Delta^{(-)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x):=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta^{(-)}(x)\\ \Delta^{(l)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x):=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta^{(l)}(x)\end{cases}$$

Cor. 8.10.2.

$$\begin{cases}\Delta^{(c)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x)\\ :=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta^{(c)}(x)-\tfrac{i^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-1}\varsigma^nC^n_{2s}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdots}^{n}[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}\Delta(x)\\ \Delta^{(F)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x)=i\Delta^{(c)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x)\\ :=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta_F(x)-\tfrac{i^{2s+1}}{2^{2s-1}}\sum_{n=0}^{2s-1}\varsigma^nC^n_{2s}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdots}^{n}[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}\Delta(x)\end{cases}$$

Cor. 8.10.3.

$$\begin{cases}\Delta^{(ret)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x)\\ :=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta^{(ret)}(x)-\tfrac{i^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-1}\varsigma^nC^n_{2s}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdots}^{n}[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}\Delta(x)\\ \Delta^{(adv)}_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(s;x)\\ :=\tfrac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta^{(adv)}(x)-\tfrac{i^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-1}\varsigma^nC^n_{2s}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdots}^{n}[\partial_t^{2s-1-n}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{n}\Delta(x)\end{cases}$$

Cor. 8.10.4.

$$\begin{cases}\Delta^{(c)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x) := \frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta^{(c)}(x)\\ -\frac{i^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-2}\sum_{l=0}^{[(2s-2-n)/2]}\varsigma^n C^n_{2s}C^{2l+1}_{2s-1-n}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{\partial_i\partial_j\cdot\cdot}^{n}\nabla^{2l}\partial_t^{2s-2-n-2l}\delta^4(x)\\ \Delta^{(F)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x) = i\Delta^{(c)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x) := \frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta_F(x)\\ -\frac{i^{2s+1}}{2^{2s-1}}\sum_{n=0}^{2s-2}\sum_{l=0}^{[(2s-2-n)/2]}\varsigma^n C^n_{2s}C^{2l+1}_{2s-1-n}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{\partial_i\partial_j\cdot\cdot}^{n}\nabla^{2l}\partial_t^{2s-2-n-2l}\delta^4(x)\\ \Delta^{(F)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;p) = -i\frac{(-\varsigma)^{2s}}{2^{2s-1}}\frac{\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{p_ap_b\cdot\cdot}^{2s}}{p^2-i\varepsilon}+\cdots\end{cases}$$

Cor. 8.10.5.

$$\begin{cases}\Delta^{(ret)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x) := \frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta^{(ret)}(x)\\ -\frac{i^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-2}\sum_{l=0}^{[(2s-2-n)/2]}\varsigma^n C^n_{2s}C^{2l+1}_{2s-1-n}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{\partial_i\partial_j\cdot\cdot}^{n}\nabla^{2l}\partial_t^{2s-2-n-2l}\delta^4(x)\\ \Delta^{(adv)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x) := \frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta^{(adv)}(x)\\ -\frac{i^{2s}}{2^{2s-1}}\sum_{n=0}^{2s-2}\sum_{l=0}^{[(2s-2-n)/2]}\varsigma^n C^n_{2s}C^{2l+1}_{2s-1-n}\overbrace{\sigma^i_{A_\varsigma A'_\varsigma}\sigma^j_{B_\varsigma B'_\varsigma}\cdot\cdot}^{n}\overbrace{\partial_i\partial_j\cdot\cdot}^{n}\nabla^{2l}\partial_t^{2s-2-n-2l}\delta^4(x)\end{cases}$$

Lem. 8.10.2. $\Delta_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot E_\varsigma F_\varsigma\cdot\cdot Z_\varsigma}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot E'_\varsigma F'_\varsigma\cdot\cdot Z'_\varsigma}_{2s}}(s;x)|_{t=0}$

$$= i\frac{(i\varsigma)^{2s+1}}{2^{2s-1}}\sum_{k=0}^{[s-\frac{1}{2}]}\frac{(2s)!}{(2s-2k-1)!(2k)!1!}\overbrace{(\sigma\cdot\nabla)_{A_\varsigma A'_\varsigma}(\sigma\cdot\nabla)_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s-2k-1}\overbrace{\delta_{E_\varsigma E'_\varsigma}\delta_{F_\varsigma F'_\varsigma}\cdot\cdot}^{2k}\nabla^{2k}\delta_{Z_\varsigma Z'_\varsigma}\delta^3(\vec{r})$$

Cor. 8.10.6.

$$\begin{cases}(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=0\\ (\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta^{(+)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=0\\ (\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta^{(-)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=0\\ (\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta^{(l)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=0\end{cases}$$

$$\begin{cases}(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta^{(c)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=-\varsigma\delta(t)\Delta_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)|_{t=0}\\ (\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta^{(ret)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=-\varsigma\delta(t)\Delta_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)|_{t=0}\\ (\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta^{(adv)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=-\varsigma\delta(t)\Delta_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)|_{t=0}\\ (\sigma\otimes I_{2^{2s-1}},-i\varsigma)_a\partial^a\Delta^{(F)}_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)=-i\varsigma\delta(t)\Delta_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(s;x)|_{t=0}\end{cases}$$

Cor. 8.10.7.

$$\begin{cases}(\sigma,-i\varsigma)_a\partial^a\Delta_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=0\\ (\sigma,-i\varsigma)_a\partial^a\Delta_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=0\\ (\sigma,-i\varsigma)_a\partial^a\Delta_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=0\\ (\sigma,-i\varsigma)_a\partial^a\Delta_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=0\end{cases}\qquad\begin{cases}(\sigma,-i\varsigma)_a\partial^a\Delta^{(c)}_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=i\varsigma\delta_{A_\varsigma A'_\varsigma}\delta^4(x)\\ (\sigma,-i\varsigma)_a\partial^a\Delta^{(ret)}_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=i\varsigma\delta_{A_\varsigma A'_\varsigma}\delta^4(x)\\ (\sigma,-i\varsigma)_a\partial^a\Delta^{(adv)}_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=i\varsigma\delta_{A_\varsigma A'_\varsigma}\delta^4(x)\\ (\sigma,-i\varsigma)_a\partial^a\Delta^{(F)}_{A_\varsigma A'_\varsigma}(\frac{1}{2};x)=-\varsigma\delta_{A_\varsigma A'_\varsigma}\delta^4(x)\end{cases}$$

8.11 Commutative and anticommutative formulas

Cor. 8.11.1. $\begin{cases} [A,BC]=[A,B]C+B[A,C],[A,CB]=[A,C]B+C[A,B] \\ [A,BC]=\{A,B\}C-B\{A,C\},[A,CB]=\{A,C\}B-C\{A,B\} \end{cases}$

Cor. 8.11.2. $\begin{cases} [A,\{B,C\}]=\{[A,B],C\}+\{B,[A,C]\} \\ [A,[B,C]]=\{\{A,B\},C\}-\{B,\{A,C\}\} \end{cases}$

Chapter24 Field Covariation Scheme for Complex Particles with Mass

Self comment: This chapter and the next chapter describe complex particles with mass. Positive and negative particles are different. It is essentially complex functions in mathematics, And they are different from Majorana particles. Positive and negative particles are same for Majorana particle. It is essentially a real function in mathematics. We will discuss it in detail in the following chapters. The massive particle scheme adopts the opposite steps to the massless particle scheme. It first proves the general spin particle case, and then respectively studies the special cases of $s = \frac{1}{2}, 1, \frac{3}{2}, 2$. The reason for doing this is as follows: Firstly, the new covariant quantization scheme has been relatively clear in general after previous research. The second is to prove the general case first, and the later special cases do not need to be proved again. It saves a lot of trouble and makes the content more compact. And I can also focuse more on physics. In order to prove the general case, it is necessary to first study the properties of the spin basis for Dirac equation. Therefore, the first half of this chapter mainly studies the spin basis for Dirac equation, and the second half is the proof for the general spin particle case. However, the complete covariant quantization scheme for Dirac equation will be studied in the latter chapters. In this chapter, the corresponding quantum field theories for all massive spin complex particles are established in a unified manner. Like massless particles, it is not necessary to know the Hamiltonian to quantize various massive spin particles according to a unified new program. It provides a unified quantization commutation rule and energy momentum operator form. As well as a partial quantum Poincare algebra is given. Like massless particles, the angular momentum operator has only achieved partial success and has not been thoroughly resolved. Efforts are still needed.

1 Foundation preparation

1.1 Introduction of Dirac basis

1.1.1 Four-dimensional Fourier solution of plane wave for Dirac electron equation [4]

Thm. 1.1.1. $(\gamma^a \partial_a + m)\psi(\vec{r}, t) = 0$

$$\Leftrightarrow \begin{cases} \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{p}=-\infty}^{+\infty} \frac{1}{2E_{\vec{p}}}[a(\vec{p}, E_{\vec{p}})e^{i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)} + a(-\vec{p}, -E_{\vec{p}})e^{-i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)}]d^3\vec{p} \\ (i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, (-i\gamma^a p_a + m)a(-\vec{p}, -E_{\vec{p}}) = 0 \end{cases}$$

Proof: $(\gamma^a \partial_a + m)\psi(\vec{r}, t) = 0$

$$\Rightarrow \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{p}=-\infty}^{+\infty} \int\limits_{E=-\infty}^{+\infty} (i\gamma^a p_a + m)\psi(\vec{p}, E)e^{ip\cdot x}d^3\vec{p}dE = 0$$

$$\Leftrightarrow (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0 \Leftrightarrow (i\gamma^a p_a - m)(i\gamma^a p_a + m)\psi(\vec{p}, E) = 0, (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0$$

$$\Leftrightarrow (E^2 - \vec{p}^2 - m^2)\psi(\vec{p}, E) = 0, (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0$$

$$\Leftrightarrow \psi(\vec{p}, E) = a(\vec{p}, E)\delta(E^2 - \vec{p}^2 - m^2) + \psi_0(\vec{p}, E)\delta_{E^2, \vec{p}^2 + m^2}, (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0$$

$$\Rightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{p}=-\infty}^{+\infty} \int\limits_{E=-\infty}^{+\infty} [a(\vec{p}, E)\delta(E^2 - \vec{p}^2 - m^2) + \psi_0(\vec{p}, E)\delta_{E^2, \vec{p}^2 + m^2}]e^{ip\cdot x}d^3\vec{p}dE$$

$$\Leftrightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{p}=-\infty}^{+\infty} \int\limits_{E=-\infty}^{+\infty} a(\vec{p}, E)\delta(E^2 - \vec{p}^2 - m^2)e^{ip\cdot x}d^3\vec{p}dE$$

$$\Leftrightarrow \begin{cases} \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{p}=-\infty}^{+\infty} \frac{1}{2E_{\vec{p}}}[a(\vec{p}, E_{\vec{p}})e^{i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)} + a(\vec{p}, -E_{\vec{p}})e^{i(\vec{p}\cdot\vec{r} + E_{\vec{p}}t)}]d^3\vec{p} \\ (i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, (i\gamma \cdot \vec{p} + \gamma^4 E_{\vec{p}} + m)a(\vec{p}, -E_{\vec{p}}) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{p}=-\infty}^{+\infty} \frac{1}{2E_{\vec{p}}}[a(\vec{p}, E_{\vec{p}})e^{i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)} + a(-\vec{p}, -E_{\vec{p}})e^{-i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)}]d^3\vec{p} \\ (i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, (-i\gamma^a p_a + m)a(-\vec{p}, -E_{\vec{p}}) = 0 \end{cases} \qquad \square$$

Thm. 1.1.2. $(i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), (i\gamma^a p_a + m) = \begin{bmatrix} m & -\varsigma E + \sigma\cdot\vec{p} \\ -\varsigma E - \sigma\cdot\vec{p} & m \end{bmatrix}$

$$\Leftrightarrow \begin{cases} a(\vec{p},E_{\vec{p}}) = \begin{bmatrix} m\varphi(\vec{p}) \\ (\varsigma E_{\vec{p}}+\sigma\cdot\vec{p})\varphi(\vec{p}) \end{bmatrix} = (-i\gamma^a p_a+m)\begin{bmatrix}\varphi(\vec{p})\\0\end{bmatrix} = (-i\gamma^a p_a+m)\begin{bmatrix}0\\ \frac{\varsigma E_{\vec{p}}+\sigma\cdot\vec{p}}{m}\varphi(\vec{p})\end{bmatrix} \\ a(\vec{p},E_{\vec{p}}) = \begin{bmatrix} (\varsigma E_{\vec{p}}-\sigma\cdot\vec{p})\eta(\vec{p}) \\ m\eta(\vec{p}) \end{bmatrix} = (-i\gamma^a p_a+m)\begin{bmatrix}0\\ \eta(\vec{p})\end{bmatrix} = (-i\gamma^a p_a+m)\begin{bmatrix}\frac{\varsigma E_{\vec{p}}-\sigma\cdot\vec{p}}{m}\eta(\vec{p})\\0\end{bmatrix} \end{cases}$$

Cor. 1.1.1. $(i\gamma^a p_a-m)a(-\vec{p},-E_{\vec{p}})=0, \gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x), (i\gamma^a p_a-m)=\begin{bmatrix}-m & -\varsigma E+\sigma\cdot\vec{p}\\ -\varsigma E-\sigma\cdot\vec{p} & -m\end{bmatrix}$

$$\Leftrightarrow \begin{cases} a(-\vec{p},-E_{\vec{p}}) = \begin{bmatrix} -m\varphi(-\vec{p}) \\ (\varsigma E_{\vec{p}}+\sigma\cdot\vec{p})\varphi(-\vec{p}) \end{bmatrix} = (-i\gamma^a p_a-m)\begin{bmatrix}\varphi(-\vec{p})\\0\end{bmatrix} = (-i\gamma^a p_a-m)\begin{bmatrix}0\\ \frac{-\varsigma E_{\vec{p}}-\sigma\cdot\vec{p}}{m}\varphi(-\vec{p})\end{bmatrix} \\ a(-\vec{p},-E_{\vec{p}}) = \begin{bmatrix} (\varsigma E_{\vec{p}}-\sigma\cdot\vec{p})\eta(-\vec{p}) \\ -m\eta(-\vec{p}) \end{bmatrix} = (-i\gamma^a p_a-m)\begin{bmatrix}0\\ \eta(-\vec{p})\end{bmatrix} = (-i\gamma^a p_a-m)\begin{bmatrix}\frac{-\varsigma E_{\vec{p}}+\sigma\cdot\vec{p}}{m}\eta(-\vec{p})\\0\end{bmatrix} \end{cases}$$

Self comment: From the above, it can be seen that the plane wave solutions of Dirac equation have multiple equivalent expressions.There are many intuitive choices for spin bases, essentially unlimited choices. But they are also essentially representation equivalent and lack a unitary transformation. No matter which base is chosen, physics is equivalent and consistent. But if you choose well, it's convenient to calculate. However, it should be noted that for massless particles, the above expressions are not necessarily equivalent.

1.1.2 Non normalized Dirac basis(suitable for arbitrary mass cases)

Cor. 1.1.2. $(\gamma^a\partial_a+m)\psi(\vec{r},t)=0, \gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$

$$\Leftrightarrow \psi(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{2E}[a(\vec{p},E)e^{i(\vec{p}\cdot\vec{r}-Et)}+a(-\vec{p},-E)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$a(\vec{p},E)=(-i\gamma^a p_a+m)\begin{bmatrix}\varphi(\vec{p})\\0\end{bmatrix}, a(-\vec{p},-E)=(-i\gamma^a p_a-m)\begin{bmatrix}\eta(\vec{p})\\0\end{bmatrix}$$

γ^a **: This chapter adopts the above provisions, unless otherwise specified.**

Def. 1.1.1. $X(\vec{p},\frac{\kappa}{2}):=(-i\gamma^a p_a+m)\begin{bmatrix}\lambda(\hat{p},\frac{\kappa}{2})\\0\end{bmatrix}, Y(\vec{p},\frac{\kappa}{2}):=(-i\gamma^a p_a-m)\begin{bmatrix}\lambda(\hat{p},\frac{\kappa}{2})\\0\end{bmatrix}$

Cor. 1.1.3. $X(\vec{p},\frac{\kappa}{2})=\lambda(\hat{p},\frac{\kappa}{2})\otimes\begin{bmatrix}m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}, Y(\vec{p},\frac{\kappa}{2})=\lambda(\hat{p},\frac{\kappa}{2})\otimes\begin{bmatrix}-m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}$

1.1.3 Normalized massless Dirac basis

Def. 1.1.2. $X(\vec{p},\frac{\kappa}{2})=Y(\vec{p},\frac{\kappa}{2}):=-i\gamma^a p_a\begin{bmatrix}\lambda(\hat{p},\frac{\kappa}{2})\\0\end{bmatrix}$

Cor. 1.1.4. $X(\vec{p},\frac{\varsigma}{2})=Y(\vec{p},\frac{\varsigma}{2})=2\varsigma|\vec{p}|\lambda(\hat{p},\frac{\kappa}{2})\otimes\begin{bmatrix}0\\1\end{bmatrix}, X(\vec{p},-\frac{\varsigma}{2})=Y(\vec{p},-\frac{\varsigma}{2})=0$

Cor. 1.1.5. $\bar{X}(\vec{p},\frac{\varsigma}{2})=\bar{Y}(\vec{p},\frac{\varsigma}{2})=\lambda(\hat{p},\frac{\kappa}{2})\otimes\begin{bmatrix}0\\1\end{bmatrix}, \bar{X}(\vec{p},-\frac{\varsigma}{2})=\bar{Y}(\vec{p},-\frac{\varsigma}{2})=0$

Self comment: The massless case is obtained by directly using $m\to 0$. But it is not comprehensive. This is just one set of solutions, and there is another set of solutions. In fact the massless case requires reanalysis.

1.1.4 Definition of Dirac charge basis

Def. 1.1.3. $\mu(\vec{p},\frac{\kappa}{2}):=\begin{bmatrix}\sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2m}}\\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2m}}\end{bmatrix}, \nu(\vec{p},\frac{\kappa}{2}):=\begin{bmatrix}-\sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2m}}\\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2m}}\end{bmatrix}$

Cor. 1.1.6. $\mu(\vec{p},\frac{\kappa}{2}):=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{\frac{m}{E+\kappa\varsigma|\vec{p}|}}\\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{m}}\end{bmatrix}, \nu(\vec{p},\frac{\kappa}{2}):=\frac{1}{\sqrt{2}}\begin{bmatrix}-\sqrt{\frac{m}{E+\kappa\varsigma|\vec{p}|}}\\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{m}}\end{bmatrix}$

Def. 1.1.4. $\tilde{\mu}(\vec{p},\frac{\kappa}{2}):=\begin{bmatrix}\sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2E}}\\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2E}}\end{bmatrix}, \tilde{\nu}(\vec{p},\frac{\kappa}{2}):=\begin{bmatrix}-\sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2E}}\\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2E}}\end{bmatrix}$

Cor. 1.1.7. $\tilde{\mu}(\vec{p},\frac{\kappa}{2})=\sqrt{\frac{m}{E}}\mu(\vec{p},\frac{\kappa}{2}), \tilde{\nu}(\vec{p},\frac{\kappa}{2})=\sqrt{\frac{m}{E}}\nu(\vec{p},\frac{\kappa}{2})$

Self comment: Why is it called a charge base related to the latter concrete analysis? Maybe that's not the right way to call it.

1.1.5 Normalized Dirac basis

Cor. 1.1.8. $u(\vec{p},\frac{\kappa}{2})=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix},v(\vec{p},\frac{\kappa}{2})=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}-m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}$

Cor. 1.1.9. $u(\vec{p},\frac{\kappa}{2})=\lambda(\hat{p},\frac{\kappa}{2})\otimes\mu(\vec{p},\frac{\kappa}{2}),v(\vec{p},\frac{\kappa}{2})=\lambda(\hat{p},\frac{\kappa}{2})\otimes\nu(\vec{p},\frac{\kappa}{2})$

Cor. 1.1.10. $u(\vec{p},h)=-\varsigma\gamma_5v(\vec{p},h),v(\vec{p},h)=-\varsigma\gamma_5u(\vec{p},h),h=-\frac{1}{2},\frac{1}{2}$

Thm. 1.1.3. $(i\gamma^ap_a+m)u(\vec{p},h)=0,(i\gamma^ap_a-m)v(\vec{p},h)=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x,\varsigma I\otimes\sigma_z)$

Def. 1.1.5. $\tilde{u}(\vec{p},\frac{\kappa}{2}):=\sqrt{\frac{m}{E}}u(\vec{p},\frac{\kappa}{2}),\tilde{v}(\vec{p},\frac{\kappa}{2}):=\sqrt{\frac{m}{E}}v(\vec{p},\frac{\kappa}{2})$

Cor. 1.1.11. $\tilde{u}(\vec{p},\frac{\kappa}{2})=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2E(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix},\tilde{v}(\vec{p},\frac{\kappa}{2})=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2E(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}-m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}$

Cor. 1.1.12. $\tilde{u}(\vec{p},\frac{\kappa}{2};m=0)=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2E(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix},\tilde{v}(\vec{p},\frac{\kappa}{2};m=0)=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2E(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}-m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}$

Self comment: Why do we define two normalized spin bases? Because it is corresponding to two different normalization methods. There are two main reasons why we choose such spin bases. The first is that it can be decomposed into a direct product of two bases, which can simplify many calculations. The second is that one of the bases is helicity, which can make full use of previous helicity analysis results and greatly simplify the calculation.

1.1.6 Definition of new charge operator

Def. 1.1.6. $\hat{Q}(\vec{p}):=\frac{i\gamma^ap_a}{m},\hat{q}(\vec{p},\kappa):=\frac{-\varsigma E\sigma_x+i\kappa|\vec{p}|\sigma_y}{m}$

Lem. 1.1.1. $i\gamma^ap_a=\begin{bmatrix}0&-\varsigma E+\sigma\cdot\vec{p}\\ -\varsigma E-\sigma\cdot\vec{p}&0\end{bmatrix}=-\varsigma EI\otimes\sigma_x+i\sigma\cdot\vec{p}\otimes\sigma_y,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$

Thm. 1.1.4. $\hat{Q}(\vec{p})u(\vec{p},\frac{\kappa}{2})=-u(\vec{p},\frac{\kappa}{2}),\hat{Q}(\vec{p})v(\vec{p},\frac{\kappa}{2})=v(\vec{p},\frac{\kappa}{2}),\hat{q}(\vec{p},\kappa)\mu(\vec{p},\frac{\kappa}{2})=-\mu(\vec{p},\frac{\kappa}{2}),\hat{q}(\vec{p},\kappa)\nu(\vec{p},\frac{\kappa}{2})=\nu(\vec{p},\frac{\kappa}{2})$

Proof: $\hat{Q}(\vec{p})u(\vec{p},\frac{\kappa}{2})=\frac{i\gamma^ap_a}{m}u(\vec{p},\frac{\kappa}{2})$
$=(-\varsigma\frac{E}{m}I\otimes\sigma_x+i\frac{1}{m}\sigma\cdot\vec{p}\otimes\sigma_y)\lambda(\hat{p},\frac{\kappa}{2})\otimes\mu(\vec{p},\frac{\kappa}{2})=(I\otimes\frac{-\varsigma E\sigma_x+i\kappa|\vec{p}|\sigma_y}{m})(\lambda(\hat{p},\frac{\kappa}{2}))\otimes\mu(\vec{p},\frac{\kappa}{2})$
$=-\lambda(\hat{p},\frac{\kappa}{2})\otimes\mu(\vec{p},\frac{\kappa}{2})=-u(\vec{p},\frac{\kappa}{2})$ □

Proof: $\hat{Q}(\vec{p})v(\vec{p},\frac{\kappa}{2})=\frac{i\gamma^ap_a}{m}v(\vec{p},\frac{\kappa}{2})$
$=(-\varsigma\frac{E}{m}I\otimes\sigma_x+i\frac{1}{m}\sigma\cdot\vec{p}\otimes\sigma_y)\lambda(\hat{p},\frac{\kappa}{2})\otimes\nu(\vec{p},\frac{\kappa}{2})=(I\otimes\frac{-\varsigma E\sigma_x+i\kappa|\vec{p}|\sigma_y}{m})(\lambda(\hat{p},\frac{\kappa}{2}))\otimes\nu(\vec{p},\frac{\kappa}{2})$
$=\lambda(\hat{p},\frac{\kappa}{2})\otimes\nu(\vec{p},\frac{\kappa}{2})=v(\vec{p},\frac{\kappa}{2})$ □

1.1.7 Dirac basis is a common eigenstate of three operators: spin, helicity and charge

Pro. 1.1.1.
$$\begin{cases}\sigma^2(\frac{1}{2})\otimes Iu(\vec{p},\frac{\kappa}{2})=\frac{1}{2}(\frac{1}{2}+1)u(\vec{p},\frac{\kappa}{2})\\ \sigma(\frac{1}{2})\cdot\hat{p}\otimes Iu(\vec{p},\frac{\kappa}{2})=\frac{\kappa}{2}u(\vec{p},\frac{\kappa}{2})\\ \hat{Q}(\vec{p})u(\vec{p},\frac{\kappa}{2})=-u(\vec{p},\frac{\kappa}{2})\\ \textit{Description electron: }(s,h;Q)=(\frac{1}{2};\frac{\kappa}{2},-1)\end{cases}\quad\begin{cases}\sigma^2(\frac{1}{2})\otimes Iv(\vec{p},\frac{\kappa}{2})=\frac{1}{2}(\frac{1}{2}+1)v(\vec{p},\frac{\kappa}{2})\\ \sigma(\frac{1}{2})\cdot\hat{p}\otimes Iv(\vec{p},\frac{\kappa}{2})=\frac{\kappa}{2}v(\vec{p},\frac{\kappa}{2})\\ \hat{Q}(\vec{p})v(\vec{p},\frac{\kappa}{2})=v(\vec{p},\frac{\kappa}{2})\\ \textit{Description positron: }(s,h;Q)=(\frac{1}{2};\frac{\kappa}{2},1)\end{cases}$$

1.2 Introduction of 4D vector spin basis

1.2.1 4D vector spin basis

Cor. 1.2.1. $\lambda_m(\hat{p},-1)=\lambda_m^*(\hat{p},1),\lambda_m(\hat{p},0)=-\lambda_m^*(\hat{p},0),\lambda_m(\hat{p},1)=\lambda_m^*(\hat{p},-1)$

Def. 1.2.1. $\varepsilon_a(\vec{p},\kappa):=[i\lambda_m(\vec{p},\kappa),0]_a,\varepsilon_a(\vec{p},0):=\frac{1}{m}[iE\lambda_m(\vec{p},0),i|\vec{p}|]_a,\bar{\varepsilon}_a(\vec{p},h):=\varepsilon^+_{a'}(\vec{p},h)\eta_a^{a'}$

Cor. 1.2.2.
$$\begin{cases}\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)=\frac{1}{\sqrt{2}}\begin{bmatrix}i\\-1\\0\end{bmatrix}\\ \lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},0)=\begin{bmatrix}0\\0\\-i\end{bmatrix}\\ \lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\-1\\0\end{bmatrix}\end{cases}$$

Cor. 1.2.3. $\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},1):=\frac{1}{\sqrt{2}}[-1,-i,0,0]_a,\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},0):=\frac{1}{m}[0,0,E,i|\vec{p}|]_a,\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},-1):=\frac{1}{\sqrt{2}}[1,-i,0,0]_a$

Cor. 1.2.4. $\eta_{aa'}\varepsilon^{+a'}(\vec{p},\kappa)=-\varepsilon_a(\vec{p},-\kappa),\eta_{aa'}\varepsilon^{+a'}(\vec{p},0)=\varepsilon_a(\vec{p},0),\eta_{aa'}\varepsilon^{+a'}(\vec{p},h)=(-1)^h\varepsilon_a(\vec{p},-h)$

Thm. 1.2.1. $\varepsilon^+(\vec{p},h)\varepsilon(\vec{p},h')=(\frac{E^2+p^2}{m^2})^{1-|h|}\delta_{hh'},\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h)=\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2},\sum\limits_{h=1}^{-1}h\varepsilon(\vec{p},h)\varepsilon^+(\vec{p},h)=R\cdot\hat{p}$

Thm. 1.2.2. $\bar{\varepsilon}(\vec{p},h)\varepsilon(\vec{p},h')=\delta_{hh'},\sum_{h=1}^{-1}\varepsilon_a(\vec{p},h)\bar{\varepsilon}_b(\vec{p},h)=\delta_{ab}+\frac{p_ap_b}{m^2},\sum_{h=1}^{-1}h\varepsilon(\vec{p},h)\bar{\varepsilon}(\vec{p},h)=R\cdot\hat{p}$

Cor. 1.2.5. $(R\cdot\hat{p})\varepsilon(\vec{p},h)=h\varepsilon(\vec{p},h),(R\cdot\hat{p})\frac{p_{[a]}}{m}=0;R^2\varepsilon(\vec{p},h)=1(1+1)\varepsilon(\vec{p},h)$

Cor. 1.2.6. $(L\cdot\hat{p})\varepsilon(\vec{p},\kappa)=0,(L\cdot\hat{p})\varepsilon(\vec{p},0)=-\frac{p_{[a]}}{m},(L\cdot\hat{p})\frac{p_{[a]}}{m}=-\varepsilon(\vec{p},0)$

Cor. 1.2.7. $\begin{cases}(\sigma_+\cdot\hat{p})\varepsilon(\vec{p},\kappa)=\kappa\varepsilon(\vec{p},\kappa),(\sigma_+\cdot\hat{p})\varepsilon(\vec{p},0)=-\frac{p_{[a]}}{m},(\sigma_+\cdot\hat{p})\frac{p_{[a]}}{m}=-\varepsilon(\vec{p},0)\\(\sigma_-\cdot\hat{p})\varepsilon(\vec{p},\kappa)=\kappa\varepsilon(\vec{p},\kappa),(\sigma_-\cdot\hat{p})\varepsilon(\vec{p},0)=\frac{p_{[a]}}{m},(\sigma_-\cdot\hat{p})\frac{p_{[a]}}{m}=\varepsilon(\vec{p},0)\end{cases}$

Self comment: Why do we choose such spin basis. It is related to the latter concrete analysis. In fact, I extracted this result from the latter concrete analysis. And then I put it here to conduct the necessary advance research. This allows latter chapters to focus more on physics itself.

1.2.2 Relations I between complex vector spin basis and 4D vector spin basis

Cor. 1.2.8. $\begin{cases}[R\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},\kappa)=-i\kappa\varepsilon(\vec{p},h),[R\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},0)=0,[R\cdot\lambda_m(\vec{p},0)]\frac{p_{[a]}}{m}=0\\ [L\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},\kappa)=0,[L\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},0)=\frac{ip_{[a]}}{m},[L\cdot\lambda_m(\vec{p},0)]\frac{p_{[a]}}{m}=i\varepsilon(\vec{p},0)\end{cases}$

Cor. 1.2.9. $\begin{cases}[R\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},1)=[\vec{0},0],[R\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},-1)=-[\lambda_m(\vec{p},0),0]\\ [R\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},0)=-\frac{E}{m}[\lambda_m(\vec{p},1),0],[R\cdot\lambda_m(\vec{p},1)]\frac{p_{[a]}}{m}=-\frac{|\vec{p}|}{m}[\lambda_m(\vec{p},1),0]\\ [L\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},1)=[\vec{0},0],[L\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},-1)=[\vec{0},1]\\ [L\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},0)=-\frac{|\vec{p}|}{m}[\lambda_m(\vec{p},1),0],[L\cdot\lambda_m(\vec{p},1)]\frac{p_{[a]}}{m}=-\frac{E}{m}[\lambda_m(\vec{p},1),0]\end{cases}$

Cor. 1.2.10. $\begin{cases}[R\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},-1)=[\vec{0},0],[R\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},1)=[\lambda_m(\vec{p},0),0]\\ [R\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},0)=\frac{E}{m}[\lambda_m(\vec{p},-1),0],[R\cdot\lambda_m(\vec{p},-1)]\frac{p_{[a]}}{m}=\frac{|\vec{p}|}{m}[\lambda_m(\vec{p},-1),0]\\ [L\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},-1)=[\vec{0},0],[L\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},1)=[\vec{0},1]\\ [L\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},0)=-\frac{|\vec{p}|}{m}[\lambda_m(\vec{p},-1),0],[L\cdot\lambda_m(\vec{p},-1)]\frac{p_{[a]}}{m}=-\frac{E}{m}[\lambda_m(\vec{p},-1),0]\end{cases}$

1.2.3 Relations II between complex vector spin basis and 4D vector spin basis

Cor. 1.2.11. $\begin{cases}[\sigma_+\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},\kappa)=-i\kappa\varepsilon(\vec{p},h),[\sigma_+\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},0)=\frac{ip_{[a]}}{m},[\sigma_+\cdot\lambda_m(\vec{p},0)]\frac{p_{[a]}}{m}=i\varepsilon(\vec{p},0)\\ [\sigma_-\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},\kappa)=-i\kappa\varepsilon(\vec{p},h),[\sigma_-\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},0)=-\frac{ip_{[a]}}{m},[\sigma_-\cdot\lambda_m(\vec{p},0)]\frac{p_{[a]}}{m}=-i\varepsilon(\vec{p},0)\end{cases}$

Cor. 1.2.12. $\begin{cases}[\sigma_+\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},1)=[\vec{0},0],[\sigma_+\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},-1)=-[\lambda_m(\vec{p},0),-1]\\ [\sigma_+\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},0)=-\frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p},1),0],[\sigma_+\cdot\lambda_m(\vec{p},1)]\frac{p_{[a]}}{m}=-\frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p},1),0]\\ [\sigma_-\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},1)=[\vec{0},0],[\sigma_-\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},-1)=-[\lambda_m(\vec{p},0),1]\\ [\sigma_-\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},0)=-\frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p},1),0],[\sigma_-\cdot\lambda_m(\vec{p},1)]\frac{p_{[a]}}{m}=\frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p},1),0]\end{cases}$

Cor. 1.2.13. $\begin{cases}[\sigma_+\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},-1)=[\vec{0},0],[\sigma_+\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},1)=[\lambda_m(\vec{p},0),1]\\ [\sigma_+\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},0)=\frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p},-1),0],[\sigma_+\cdot\lambda_m(\vec{p},-1)]\frac{p_{[a]}}{m}=-\frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p},-1),0]\\ [\sigma_-\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},-1)=[\vec{0},0],[\sigma_-\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},1)=[\lambda_m(\vec{p},0),-1]\\ [\sigma_-\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},0)=\frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p},-1),0],[\sigma_-\cdot\lambda_m(\vec{p},-1)]\frac{p_{[a]}}{m}=\frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p},-1),0]\end{cases}$

1.2.4 Relations III between complex vector spin basis and 4D vector spin basis

Cor. 1.2.14. $\begin{cases}[\sigma_+\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},\kappa)=-i\kappa\varepsilon(\vec{p},h),[\sigma_+\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},0)=\frac{ip_{[a]}}{m},[\sigma_+\cdot\lambda_m(\vec{p},0)]\frac{p_{[a]}}{m}=i\varepsilon(\vec{p},0)\\ [\sigma_-\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},\kappa)=-i\kappa\varepsilon(\vec{p},h),[\sigma_-\cdot\lambda_m(\vec{p},0)]\varepsilon(\vec{p},0)=-\frac{ip_{[a]}}{m},[\sigma_-\cdot\lambda_m(\vec{p},0)]\frac{p_{[a]}}{m}=-i\varepsilon(\vec{p},0)\end{cases}$

Cor. 1.2.15. $\begin{cases}[\sigma_+\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},1)=[\vec{0},0],[\sigma_+\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},-1)=i\frac{E+|\vec{p}|}{m}[\varepsilon(\vec{p},0)-\frac{p_{[a]}}{m}]\\ [\sigma_+\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},0)=i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p},1),[\sigma_+\cdot\lambda_m(\vec{p},1)]\frac{p_{[a]}}{m}=i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p},1)\\ [\sigma_-\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},1)=[\vec{0},0],[\sigma_-\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},-1)=i\frac{E-|\vec{p}|}{m}[\varepsilon(\vec{p},0)+\frac{p_{[a]}}{m}]\\ [\sigma_-\cdot\lambda_m(\vec{p},1)]\varepsilon(\vec{p},0)=i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p},1),[\sigma_-\cdot\lambda_m(\vec{p},1)]\frac{p_{[a]}}{m}=-i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p},1)\end{cases}$

Cor. 1.2.16. $\begin{cases}[\sigma_+\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},-1)=[\vec{0},0],[\sigma_+\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},1)=-i\frac{E-|\vec{p}|}{m}[\varepsilon(\vec{p},0)+\frac{p_{[a]}}{m}]\\ [\sigma_+\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},0)=-i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p},-1),[\sigma_+\cdot\lambda_m(\vec{p},-1)]\frac{p_{[a]}}{m}=i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p},-1)\\ [\sigma_-\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},-1)=[\vec{0},0],[\sigma_-\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},1)=-i\frac{E+|\vec{p}|}{m}[\varepsilon(\vec{p},0)-\frac{p_{[a]}}{m}]\\ [\sigma_-\cdot\lambda_m(\vec{p},-1)]\varepsilon(\vec{p},0)=-i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p},-1),[\sigma_-\cdot\lambda_m(\vec{p},-1)]\frac{p_{[a]}}{m}=-i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p},-1)\end{cases}$

1.2.5 Relations IV between complex vector spin basis and 4D vector spin basis

Cor. 1.2.17. $\begin{cases}[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},\kappa)]\lambda_m^\alpha(\vec{p},0)=-i\kappa\varepsilon^a(\vec{p},h),[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},0)=\frac{ip^a}{m},[\sigma_{+\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p},0)=i\varepsilon^a(\vec{p},0)\\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},\kappa)]\lambda_m^\alpha(\vec{p},0)=-i\kappa\varepsilon^a(\vec{p},h),[\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},0)=-\frac{ip^a}{m},[\sigma_{-\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p},0)=-i\varepsilon^a(\vec{p},0)\end{cases}$

Cor. 1.2.18. $\begin{cases}[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},1)]\lambda_m^\alpha(\vec{p},1)=[\vec{0},0],[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},-1)]\lambda_m^\alpha(\vec{p},1)=i\frac{E+|\vec{p}|}{m}[\varepsilon^a(\vec{p},0)-\frac{p^a}{m}]\\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},1)=i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p},1),[\sigma_{+\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p},1)=i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p},1)\\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},1)]\lambda_m^\alpha(\vec{p},1)=[\vec{0},0],[\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},-1)]\lambda_m^\alpha(\vec{p},1)=i\frac{E-|\vec{p}|}{m}[\varepsilon^a(\vec{p},0)+\frac{p^a}{m}]\\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},1)=i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p},1),[\sigma_{-\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p},1)=-i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p},1)\end{cases}$

Cor. 1.2.19. $\begin{cases}[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},-1)]\lambda_m^\alpha(\vec{p},-1)=[\vec{0},0],[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},1)]\lambda_m^\alpha(\vec{p},-1)=-i\frac{E-|\vec{p}|}{m}[\varepsilon^a(\vec{p},0)+\frac{p^a}{m}]\\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},-1)=-i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p},-1),[\sigma_{+\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p},-1)=i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p},-1)\\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},-1)]\lambda_m^\alpha(\vec{p},-1)=[\vec{0},0],[\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},1)]\lambda_m^\alpha(\vec{p},-1)=-i\frac{E+|\vec{p}|}{m}[\varepsilon^a(\vec{p},0)-\frac{p^a}{m}]\\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},-1)=-i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p},-1),[\sigma_{-\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p},-1)=-i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p},-1)\end{cases}$

1.2.6 Relations V between complex vector spin basis and 4D vector spin basis

Cor. 1.2.20.

$$\begin{cases}[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},1)]\lambda_m^\alpha(\vec{p},0)=-i\varepsilon^a(\vec{p},1),[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},0)=\frac{ip^a}{m},[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},-1)]\lambda_m^\alpha(\vec{p},0)=i\varepsilon^a(\vec{p},-1)\\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},1)]\lambda_m^\alpha(\vec{p},1)=[\vec{0},0],[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},1)=i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p},1)\\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},-1)]\lambda_m^\alpha(\vec{p},1)=i\frac{E+|\vec{p}|}{m}[\varepsilon^a(\vec{p},0)-\frac{p^a}{m}]\\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},1)]\lambda_m^\alpha(\vec{p},-1)=-i\frac{E-|\vec{p}|}{m}[\varepsilon^a(\vec{p},0)+\frac{p^a}{m}],[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},0)]\lambda_m^\alpha(\vec{p},-1)=-i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p},-1)\\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},-1)]\lambda_m^\alpha(\vec{p},-1)=[\vec{0},0]\end{cases}$$

Cor. 1.2.21.

$$\sum_{h,h'=1}^{-1}[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},h)]\lambda_m^\alpha(\vec{p},h')\{[\sigma_{+\alpha'}^{a'b'}\varepsilon_{b'}(\vec{p},h)]\lambda_m^{\alpha'}(\vec{p},h')\}^+$$
$$=\sum_{h=1}^{-1}[\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p},h)]\delta^{\alpha\alpha'}\{[\sigma_{+\alpha'}^{a'b'}\varepsilon_{b'}(\vec{p},h)]\}^+$$
$$=-\delta^{\alpha\alpha'}\sigma_{+\alpha}^{ab}\sigma_{+\alpha'}^{a'b'}\sum_{h=1}^{-1}\varepsilon_b(\vec{p},h)\varepsilon_{b'}^+(\vec{p},h)$$
$$=-(-\delta^{aa'}\delta^{bb'}+\delta^{ab'}\delta^{ba'}+\varepsilon^{aba'b'})\sum_{h=1}^{-1}\varepsilon_b(\vec{p},h)\varepsilon_{b'}^+(\vec{p},h)$$
$$=3\delta^{aa'}-\sum_{h=1}^{-1}\varepsilon^{a'}(\vec{p},h)\varepsilon^{+a}(\vec{p},h)=3\delta^{aa'}-\sum_{h=1}^{-1}\varepsilon^{a}(\vec{p},h)\varepsilon^{+a'}(\vec{p},h)$$

Cor. 1.2.22.

$$\sum_{h,h'=1}^{-1}[\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},h)]\lambda_m^\alpha(\vec{p},h')\{[\sigma_{-\alpha'}^{a'b'}\varepsilon_{b'}(\vec{p},h)]\lambda_m^{\alpha'}(\vec{p},h')\}^+$$
$$=\sum_{h=1}^{-1}[\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p},h)]\delta^{\alpha\alpha'}\{[\sigma_{-\alpha'}^{a'b'}\varepsilon_{b'}(\vec{p},h)]\}^+$$
$$=-\delta^{\alpha\alpha'}\sigma_{-\alpha}^{ab}\sigma_{-\alpha'}^{a'b'}\sum_{h=1}^{-1}\varepsilon_b(\vec{p},h)\varepsilon_{b'}^+(\vec{p},h)$$
$$=-(-\delta^{aa'}\delta^{bb'}+\delta^{ab'}\delta^{ba'}-\varepsilon^{aba'b'})\sum_{h=1}^{-1}\varepsilon_b(\vec{p},h)\varepsilon_{b'}^+(\vec{p},h)$$
$$=3\delta^{aa'}-\sum_{h=1}^{-1}\varepsilon^{a'}(\vec{p},h)\varepsilon^{+a}(\vec{p},h)=3\delta^{aa'}-\sum_{h=1}^{-1}\varepsilon^{a}(\vec{p},h)\varepsilon^{+a'}(\vec{p},h)$$

1.3 Mathematical analysis of Dirac basis

1.3.1 Equivalence relations between two dimensional spin bases

Pro. 1.3.1. $\begin{cases}\lambda^*(\hat{p},-\frac{\kappa}{2})=-i\kappa\sigma_y\lambda(\hat{p},\frac{\kappa}{2}),\lambda^+(\hat{p},-\frac{\kappa}{2})=i\kappa\lambda^T(\hat{p},\frac{\kappa}{2})\sigma_y\\ \lambda(\hat{p},\frac{\kappa}{2})=i\kappa\sigma_y\lambda^*(\hat{p},-\frac{\kappa}{2}),\lambda^T(\hat{p},\frac{\kappa}{2})=-i\kappa\lambda^+(\hat{p},-\frac{\kappa}{2})\sigma_y\end{cases}$

Pro. 1.3.2. $\begin{cases}\mu^*(\vec{p},-\frac{\kappa}{2})=\varsigma\sigma_x\mu(\vec{p},\frac{\kappa}{2}),\mu^+(\vec{p},-\frac{\kappa}{2})=\varsigma\mu^T(\vec{p},\frac{\kappa}{2})\sigma_x\\ \nu^*(\vec{p},-\frac{\kappa}{2})=-\varsigma\sigma_x\nu(\vec{p},\frac{\kappa}{2}),\nu^+(\vec{p},-\frac{\kappa}{2})=-\varsigma\nu^T(\vec{p},\frac{\kappa}{2})\sigma_x\end{cases}$

Pro. 1.3.3. $\begin{cases}\mu(\vec{p},\frac{\kappa}{2})=\varsigma\sigma_x\mu^*(\vec{p},-\frac{\kappa}{2}),\mu^T(\vec{p},\frac{\kappa}{2})=\varsigma\mu^+(\vec{p},-\frac{\kappa}{2})\sigma_x\\ \nu(\vec{p},\frac{\kappa}{2})=-\varsigma\sigma_x\nu^*(\vec{p},-\frac{\kappa}{2}),\nu^T(\vec{p},\frac{\kappa}{2})=-\varsigma\nu^+(\vec{p},-\frac{\kappa}{2})\sigma_x\end{cases}$

1.3.2 Equivalence relations between Dirac bases

Pro. 1.3.4. $\begin{cases} u(\vec{p},\frac{\kappa}{2}) = i\kappa\varsigma\sigma_y\otimes\sigma_x u^*(\vec{p},-\frac{\kappa}{2}) = \kappa\gamma_2\gamma_5 u^*(\vec{p},-\frac{\kappa}{2}) \\ v(\vec{p},\frac{\kappa}{2}) = -i\kappa\varsigma\sigma_y\otimes\sigma_x v^*(\vec{p},-\frac{\kappa}{2}) = -\kappa\gamma_2\gamma_5 v^*(\vec{p},-\frac{\kappa}{2}) \end{cases}$

Pro. 1.3.5. $\begin{cases} u^*(\vec{p},-\frac{\kappa}{2}) = -i\kappa\varsigma\sigma_y\otimes\sigma_x u(\vec{p},\frac{\kappa}{2}) = -\kappa\gamma_2\gamma_5 u(\vec{p},\frac{\kappa}{2}) \\ v^*(\vec{p},-\frac{\kappa}{2}) = i\kappa\varsigma\sigma_y\otimes\sigma_x v(\vec{p},\frac{\kappa}{2}) = \kappa\gamma_2\gamma_5 v(\vec{p},\frac{\kappa}{2}) \end{cases}$

Pro. 1.3.6. $\begin{cases} u^+(\vec{p},-\frac{\kappa}{2}) = i\kappa\varsigma u^T(\vec{p},\frac{\kappa}{2})\sigma_y\otimes\sigma_x = \kappa u^T(\vec{p},\frac{\kappa}{2})\gamma_2\gamma_5 \\ v^+(\vec{p},-\frac{\kappa}{2}) = -i\kappa\varsigma v^T(\vec{p},\frac{\kappa}{2})\sigma_y\otimes\sigma_x = -i\kappa v^T(\vec{p},\frac{\kappa}{2})\gamma_2\gamma_5 \end{cases}$

Pro. 1.3.7. $\begin{cases} u^T(\vec{p},\frac{\kappa}{2}) = -i\kappa\varsigma u^+(\vec{p},-\frac{\kappa}{2})\sigma_y\otimes\sigma_x = -\kappa u^+(\vec{p},-\frac{\kappa}{2})\gamma_2\gamma_5 \\ v^T(\vec{p},\frac{\kappa}{2}) = i\kappa\varsigma v^+(\vec{p},-\frac{\kappa}{2})\sigma_y\otimes\sigma_x = \kappa v^+(\vec{p},-\frac{\kappa}{2})\gamma_2\gamma_5 \end{cases}$

1.3.3 Completeness analysis of Dirac basis

Cor. 1.3.1. $\begin{cases} \mu(\vec{p},\frac{\kappa}{2})\mu^+(\vec{p},-\frac{\kappa}{2}) = \frac{1}{2m}\begin{bmatrix} m & \varsigma E-\kappa|\vec{p}| \\ \varsigma E+\kappa|\vec{p}| & m \end{bmatrix} = \frac{1}{2}(I+\varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y) \\ \mu(\vec{p},\frac{\kappa}{2})\mu^+(\vec{p},\frac{\kappa}{2}) = \frac{\varsigma}{2m}\begin{bmatrix} \varsigma E-\kappa|\vec{p}| & m \\ m & \varsigma E+\kappa|\vec{p}| \end{bmatrix} = \frac{\varsigma}{2}(I+\varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)\sigma_x \end{cases}$

Cor. 1.3.2. $\begin{cases} u(\vec{p},\frac{\kappa}{2})u^+(\vec{p},-\frac{\kappa}{2}) = \frac{1}{4}[\kappa(\sigma\cdot\hat{p}+I)i\sigma_y]\otimes(I+\varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y) \\ u(\vec{p},\frac{\kappa}{2})u^+(\vec{p},\frac{\kappa}{2}) = \frac{1}{4}[(\kappa\sigma\cdot\hat{p}+I)\otimes(I+\varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)](\varsigma I\otimes\sigma_x) \end{cases}$

Cor. 1.3.3. $\sum\limits_{h=1/2}^{-1/2} u(\vec{p},h)\bar{u}(\vec{p},h) - v(\vec{p},h)\bar{v}(\vec{p},h)] = I_4, \sum\limits_{h=1/2}^{-1/2} u(\vec{p},h)\bar{u}(\vec{p},h) + v(\vec{p},h)\bar{v}(\vec{p},h)] = \frac{-i\gamma^a p_a}{m}$

Cor. 1.3.4. $\sum\limits_{h=1/2}^{-1/2} u(\vec{p},h)u^+(\vec{p},h) + v(-\vec{p},h)v^+(-\vec{p},h)] = \frac{E}{m}$

1.3.4 Quasi projection operator for Dirac equation [4]

Def. 1.3.1. $\Lambda_+(\frac{1}{2}) := \sum\limits_{h=1/2}^{-1/2} u(\vec{p},h)u^+(\vec{p},h) = \frac{(m-i\gamma^a p_a)\gamma_4}{2m}, \Lambda_-(\frac{1}{2}) := \sum\limits_{h=1/2}^{-1/2} v(\vec{p},h)v^+(\vec{p},h) = \frac{(-m-i\gamma^a p_a)\gamma_4}{2m}$

1.3.5 Orthogonal properties of two dimensional spin basis

Def. 1.3.2. $\hat{p}_a := (\hat{p}, i)$

Pro. 1.3.8.

$\begin{cases} \lambda^+(\hat{p},\frac{\kappa}{2})(\sigma,i\kappa)_a\lambda(\hat{p},\frac{\kappa}{2}) = \kappa\hat{p}_a \\ \mu^+(\vec{p},\frac{\kappa}{2})(\sigma,I)_a\mu(\vec{p},\frac{\kappa}{2}) = \frac{1}{m}(\varsigma m,0,-\kappa\varsigma|\vec{p}|,E)_a \\ \nu^+(\vec{p},\frac{\kappa}{2})(\sigma,I)_a\nu(\vec{p},\frac{\kappa}{2}) = \frac{1}{m}(-\varsigma m,0,-\kappa\varsigma|\vec{p}|,E)_a \end{cases}$ $\begin{cases} \lambda^+(\hat{p},-\frac{\kappa}{2})(\sigma,i\kappa)_a\lambda(\hat{p},\frac{\kappa}{2}) = -i\kappa\sqrt{2}[\lambda_m(\hat{p},\kappa),0]_a \\ \mu^+(\vec{p},-\frac{\kappa}{2})(\sigma,I)_a\mu(\vec{p},\frac{\kappa}{2}) = \frac{1}{m}(\varsigma E,-i\kappa|\vec{p}|,0,m)_a \\ \nu^+(\vec{p},-\frac{\kappa}{2})(\sigma,I)_a\nu(\vec{p},\frac{\kappa}{2}) = \frac{1}{m}(-\varsigma E,i\kappa|\vec{p}|,0,m)_a \end{cases}$

1.3.6 Orthogonal properties of Dirac basis

Pro. 1.3.9.

$$\begin{cases} u^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_x]u(\vec{p},\frac{\kappa}{2}) = -v^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_x]v(\vec{p},\frac{\kappa}{2}) = \kappa\varsigma\hat{p}_a \\ u^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_y]u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_y]v(\vec{p},\frac{\kappa}{2}) = 0 \\ u^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_z]u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_z]v(\vec{p},\frac{\kappa}{2}) = -\frac{\varsigma p_a}{m} \\ u^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes I]u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes I]v(\vec{p},\frac{\kappa}{2}) = \frac{\kappa E\hat{p}_a}{m} \end{cases}$$

Pro. 1.3.10.

$$\begin{cases} u^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_x]u(\vec{p},\frac{\kappa}{2}) = -v^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_x]v(\vec{p},\frac{\kappa}{2}) = -\kappa\varsigma\sqrt{2}\frac{E}{m}\varepsilon_a(\vec{p},\kappa) \\ u^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_y]u(\vec{p},\frac{\kappa}{2}) = -v^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_y]v(\vec{p},\frac{\kappa}{2}) = i\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_a(\vec{p},\kappa) \\ u^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_z]u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes\sigma_z]v(\vec{p},\frac{\kappa}{2}) = 0 \\ u^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes I]u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},-\frac{\kappa}{2})[(\sigma,i\kappa)_a\otimes I]v(\vec{p},\frac{\kappa}{2}) = -\kappa\sqrt{2}\varepsilon_a(\vec{p},\kappa) \end{cases}$$

Cor. 1.3.5.

$$\begin{cases} \bar{u}(\vec{p},h)u(\vec{p},h') = \delta_{hh'}, \bar{v}(\vec{p},h)v(\vec{p},h') = -\delta_{hh'}, \bar{u}(\vec{p},h)v(\vec{p},h') = 0, \bar{v}(\vec{p},h)u(\vec{p},h') = 0 \\ u^+(\vec{p},h)u(\vec{p},h') = \frac{E}{m}\delta_{hh'}, v^+(\vec{p},h)v(\vec{p},h') = \frac{E}{m}\delta_{hh'}, u^+(\vec{p},h)v(-\vec{p},h') = 0, v^+(\vec{p},h)u(-\vec{p},h') = 0 \\ \Lambda_+(\vec{p},\frac{1}{2}) := \sum\limits_{h=1/2}^{-1/2} u(\vec{p},h)\bar{u}(\vec{p},h) = \frac{m-i\gamma^a p_a}{2m}, \Lambda_-(\vec{p},\frac{1}{2}) := \sum\limits_{h=1/2}^{-1/2} v(\vec{p},h)\bar{v}(\vec{p},h) = \frac{-m-i\gamma^a p_a}{2m} \end{cases}$$

1.3.7 Corollaries I of Dirac basis properties

Pro. 1.3.11.

$$\begin{cases} u^+(\vec{p},\frac{\kappa}{2})\gamma_a u(\vec{p},\frac{\kappa}{2}) = -v^+(\vec{p},\frac{\kappa}{2})\gamma_a v(\vec{p},\frac{\kappa}{2}) = (\vec{0},I) \\ u^+(\vec{p},\frac{\kappa}{2})\gamma_i\gamma_j u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},\frac{\kappa}{2})\gamma_i\gamma_j v(\vec{p},\frac{\kappa}{2}) = \frac{E}{m}(\delta_{ij}+i\kappa\varepsilon_{ijk}\hat{p}^k) \\ u^+(\vec{p},\frac{\kappa}{2})\gamma_4\gamma_a u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},\frac{\kappa}{2})\gamma_4\gamma_a v(\vec{p},\frac{\kappa}{2}) = -i\frac{p_a}{m} \\ u^+(\vec{p},\frac{\kappa}{2})\gamma_a\gamma_4 u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},\frac{\kappa}{2})\gamma_a\gamma_4 v(\vec{p},\frac{\kappa}{2}) = i\frac{p_a^*}{m} \\ u^+(\vec{p},\frac{\kappa}{2})\gamma_4\gamma_5 u(\vec{p},\frac{\kappa}{2}) = v^+(\vec{p},\frac{\kappa}{2})\gamma_4\gamma_5 v(\vec{p},\frac{\kappa}{2}) = 0 \end{cases}$$

Pro. 1.3.12.

$$\begin{cases} u^+(\vec{p},\frac{\kappa}{2})\gamma_a u(\vec{p},\frac{\kappa}{2}) = -v^+(\vec{p},\frac{\kappa}{2})\gamma_a v(\vec{p},\frac{\kappa}{2}) = (\vec{0},I) \\ u^+(\vec{p},-\frac{\kappa}{2})\gamma_a u(\vec{p},\frac{\kappa}{2}) = -v^+(\vec{p},-\frac{\kappa}{2})\gamma_a v(\vec{p},\frac{\kappa}{2}) = i\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_a(\vec{p},\kappa) \end{cases}$$

1.3.8 Corollaries II of Dirac basis properties

Pro. 1.3.13.

$$\begin{cases} \bar{u}(\vec{p},\frac{\kappa}{2})u(\vec{p},\frac{\kappa}{2}) = -\bar{v}(\vec{p},\frac{\kappa}{2})v(\vec{p},\frac{\kappa}{2}) = I \\ \bar{u}(\vec{p},\frac{\kappa}{2})\gamma_a u(\vec{p},\frac{\kappa}{2}) = \bar{v}(\vec{p},\frac{\kappa}{2})\gamma_a v(\vec{p},\frac{\kappa}{2}) = -i\frac{p_a}{m} \\ \bar{u}(\vec{p},\frac{\kappa}{2})\gamma_i\gamma_j u(\vec{p},\frac{\kappa}{2}) = -\bar{v}(\vec{p},\frac{\kappa}{2})\gamma_i\gamma_j v(\vec{p},\frac{\kappa}{2}) = \delta_{ij}+i\kappa\varepsilon_{ijk}\hat{p}^k \\ \bar{u}(\vec{p},\frac{\kappa}{2})\gamma_4\gamma_a u(\vec{p},\frac{\kappa}{2}) = -\bar{v}(\vec{p},\frac{\kappa}{2})\gamma_4\gamma_a v(\vec{p},\frac{\kappa}{2}) = (\vec{0},I) \\ \bar{u}(\vec{p},\frac{\kappa}{2})\gamma_a\gamma_4 u(\vec{p},\frac{\kappa}{2}) = -\bar{v}(\vec{p},\frac{\kappa}{2})\gamma_a\gamma_4 v(\vec{p},\frac{\kappa}{2}) = (\vec{0},I) \end{cases}$$

Pro. 1.3.14.

$$\begin{cases} \bar{u}(\vec{p},\frac{\kappa}{2})u(\vec{p},\frac{\kappa}{2}) = -\bar{v}(\vec{p},\frac{\kappa}{2})v(\vec{p},\frac{\kappa}{2}) = I \\ \bar{u}(\vec{p},\frac{\kappa}{2})\gamma_a u(\vec{p},\frac{\kappa}{2}) = \bar{v}(\vec{p},\frac{\kappa}{2})\gamma_a v(\vec{p},\frac{\kappa}{2}) = -i\frac{p_a}{m} \\ \bar{u}(\vec{p},\frac{\kappa}{2})\gamma_a\gamma_b u(\vec{p},\frac{\kappa}{2}) = -\bar{v}(\vec{p},\frac{\kappa}{2})\gamma_a\gamma_b v(\vec{p},\frac{\kappa}{2}) = \delta_{ab}+i\kappa\varepsilon_{abc4}\hat{p}^c \\ \bar{u}(\vec{p},\frac{\kappa}{2})S_{ab}(e,\varsigma)u(\vec{p},\frac{\kappa}{2}) = -\bar{v}(\vec{p},\frac{\kappa}{2})S_{ab}(e,\varsigma)v(\vec{p},\frac{\kappa}{2}) = \frac{\kappa}{2}\varepsilon_{abc4}\hat{p}^c \end{cases}$$

1.4 Analysis of relations between Dirac basis and 4D vector basis

1.4.1 Equivalent transformation between Dirac basis $u(\vec{p},\frac{\kappa}{2})$ and 4D vector basis $\varepsilon_a(\vec{p},h)$

Def. 1.4.1. $\mathbb{X}_a = [im\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)\partial^b]C, \mathbb{X}_a(p) = i[m\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)p^b]C$

Pro. 1.4.1.

$$\begin{cases} u^+(\vec{p},-\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p},-\frac{\kappa}{2}) = 2\sqrt{2}\frac{E^2}{m}\varepsilon_a^+(\vec{p},-\kappa) \\ u^+(\vec{p},-\frac{\kappa}{2})im\gamma_a C u^*(\vec{p},-\frac{\kappa}{2}) = \sqrt{2}m\varepsilon_a^+(\vec{p},-\kappa) \\ u^+(\vec{p},-\frac{\kappa}{2})[-2iS_{ab}(e,\varsigma)p^b]Cu^*(\vec{p},-\frac{\kappa}{2}) = \sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a^+(\vec{p},-\kappa) \end{cases}$$

Proof: $u^+(\vec{p},-\frac{\kappa}{2})im\gamma_a Cu^*(\vec{p},-\frac{\kappa}{2})$
$= -i\kappa\varsigma u^+(\vec{p},-\frac{\kappa}{2})im\gamma_a C\sigma_y\otimes\sigma_x u(\vec{p},\frac{\kappa}{2})$
$= \kappa u^+(\vec{p},-\frac{\kappa}{2})m\gamma_a(I\otimes\sigma_y)u(\vec{p},\frac{\kappa}{2})$
$= -i\sqrt{2}[m\lambda_m(\vec{p},\kappa),0]_a$
$= -\sqrt{2}m\varepsilon_a(\vec{p},\kappa) = \sqrt{2}m\varepsilon_a^+(\vec{p},\kappa)$ □

Proof: $u^+(\vec{p},-\frac{\kappa}{2})[-2iS_{ab}(e,\varsigma)p^b]Cu^*(\vec{p},-\frac{\kappa}{2})$
$= -i\kappa\varsigma u^+(\vec{p},-\frac{\kappa}{2})[-2iS_{ab}(e,\varsigma)p^b]C\sigma_y\otimes\sigma_x u(\vec{p},\frac{\kappa}{2})$
$= \kappa u^+(\vec{p},-\frac{\kappa}{2})i\gamma_a\gamma_b p^b(I\otimes\sigma_y)u(\vec{p},\frac{\kappa}{2})$
$= [\kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\vec{p},\kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\vec{p},\kappa),0]_a$
$= [i\kappa\sqrt{2}\frac{\vec{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\vec{p},0)\lambda_m^k(\vec{p},\kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\vec{p},\kappa),0]_a$
$= -i\sqrt{2}\frac{\vec{p}^2}{m}[\lambda_m(\vec{p},\kappa),0]_a - i\sqrt{2}\frac{E^2}{m}[\lambda_m(\vec{p},\kappa),0]_a$
$= -i\sqrt{2}\frac{E^2+\vec{p}^2}{m}[\lambda_m(\vec{p},\kappa),0]_a = -\sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a(\vec{p},\kappa) = \sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a^+(\vec{p},\kappa)$ □

Proof: $u^+(\vec{p},-\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p},-\frac{\kappa}{2})$
$= u^+(\vec{p},-\frac{\kappa}{2})i[m\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)p^b]Cu^*(\vec{p},-\frac{\kappa}{2})$
$= -i\kappa\varsigma u^+(\vec{p},-\frac{\kappa}{2})i[m\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)p^b]C\sigma_y\otimes\sigma_x u(\vec{p},\frac{\kappa}{2})$
$= \kappa u^+(\vec{p},-\frac{\kappa}{2})[m\gamma_a(\varsigma) + i\gamma_a\gamma_b p^b](I\otimes\sigma_y)u(\vec{p},\frac{\kappa}{2})$
$= -i\sqrt{2}[m\lambda_m(\hat{p},\kappa),0]_a + \kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\hat{p},\kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$= -i\sqrt{2}[m\lambda_m(\hat{p},\kappa),0]_a + i\kappa\sqrt{2}\frac{\vec{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\hat{p},0)\lambda_m^k(\hat{p},\kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$= -i\sqrt{2}[m\lambda_m(\hat{p},\kappa),0]_a - i\sqrt{2}\frac{\vec{p}^2}{m}[\lambda_m(\hat{p},\kappa),0]_a - i\sqrt{2}\frac{E^2}{m}[\lambda_m(\hat{p},\kappa),0]_a$
$= -i2\sqrt{2}\frac{E^2}{m}[\lambda_m(\hat{p},\kappa),0]_a = -2\sqrt{2}\frac{E^2}{m}\varepsilon_a(\vec{p},\kappa) = 2\sqrt{2}\frac{E^2}{m}\varepsilon_a^+(\vec{p},\kappa)$ □

Cor. 1.4.1.
$$\begin{cases}\varepsilon_a^+(\vec{p},\kappa)=\frac{i}{\sqrt{2}}u^+(\vec{p},\frac{\kappa}{2})\gamma_a Cu^*(\vec{p},\frac{\kappa}{2})=\frac{m}{2\sqrt{2}E^2}u^+(\vec{p},\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p},\frac{\kappa}{2})\\ \varepsilon_a(\vec{p},\kappa)=-\frac{i}{\sqrt{2}}u^T(\vec{p},\frac{\kappa}{2})\bar{C}\gamma_a u(\vec{p},\frac{\kappa}{2})=\frac{m}{2\sqrt{2}E^2}u^T(\vec{p},\frac{\kappa}{2})\mathbb{X}_a^+(p)u(\vec{p},\frac{\kappa}{2})\end{cases}$$

Cor. 1.4.2.
$$\begin{cases}\varepsilon^{+a}(\vec{p},\kappa)=\frac{i}{\sqrt{2}}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)(\gamma^a C)_{\lambda_\varsigma\mu_\varsigma}=\frac{m}{2\sqrt{2}E^2}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\\ \varepsilon_a(\vec{p},\kappa)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)=\frac{m}{2\sqrt{2}E^2}\mathbb{X}_a^{+\lambda_\varsigma\mu_\varsigma}(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)\end{cases}$$

Pro. 1.4.2. $u^+(\vec{p},\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p},-\frac{\kappa}{2})=2iE[\lambda_m(\hat{p},0),0]$

Proof: $u^+(\vec{p},\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p},-\frac{\kappa}{2})$
$=u^+(\vec{p},\frac{\kappa}{2})i[m\gamma_a(\varsigma)-2S_{ab}(e,\varsigma)p^b]Cu^*(\vec{p},-\frac{\kappa}{2})$
$=-i\kappa\varsigma u^+(\vec{p},\frac{\kappa}{2})i[m\gamma_a(\varsigma)-2S_{ab}(e,\varsigma)p^b]C\sigma_y\otimes\sigma_x u(\vec{p},\frac{\kappa}{2})$
$=\kappa u^+(\vec{p},\frac{\kappa}{2})[m\gamma_a(\varsigma)+i\gamma_a\gamma_b p^b](I\otimes\sigma_y)u(\vec{p},\frac{\kappa}{2})$
$=(E\hat{p},-i|\vec{p}|)+(E\hat{p},i|\vec{p}|)$
$=(2E\hat{p},0)=2iE[\lambda_m(\hat{p},0),0]$ □

Cor. 1.4.3. $u^+(\vec{p},\frac{\kappa}{2})im\gamma_a Cu^*(\vec{p},-\frac{\kappa}{2})=m\varepsilon_a^+(\vec{p},0), u^+(\vec{p},\frac{\kappa}{2})[-2iS_{ab}(e,\varsigma)p^bC]u^*(\vec{p},-\frac{\kappa}{2})=m\varepsilon_a(\vec{p},0)$

Cor. 1.4.4. $u^+(\vec{p},\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p},-\frac{\kappa}{2})=[2E\hat{p},0], u^+(\vec{p},\frac{\kappa}{2})\mathbb{X}_a(-p)u^*(\vec{p},-\frac{\kappa}{2})=[0,-2i|\vec{p}|]$

Cor. 1.4.5.
$$\begin{cases}\varepsilon_a^+(\vec{p},0)=iu^+(\vec{p},\frac{\kappa}{2})\gamma_a Cu^*(\vec{p},-\frac{\kappa}{2}),[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2E}u^+(\vec{p},\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p},-\frac{\kappa}{2})\\ \varepsilon_a(\vec{p},0)=-iu^T(\vec{p},-\frac{\kappa}{2})\bar{C}\gamma_a u(\vec{p},\frac{\kappa}{2}),[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2E}u^T(\vec{p},-\frac{\kappa}{2})\mathbb{X}_a^+(p)u(\vec{p},\frac{\kappa}{2})\end{cases}$$

Cor. 1.4.6.
$$\begin{cases}\varepsilon_a^+(\vec{p},0)=\frac{i}{\sqrt{2}}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)(\gamma^a C)_{\lambda_\varsigma\mu_\varsigma},[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2\sqrt{2}E}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\\ \varepsilon_a(\vec{p},0)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0),[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2\sqrt{2}E}\mathbb{X}_a^{+\lambda_\varsigma\mu_\varsigma}(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\end{cases}$$

Cor. 1.4.7.
$$\begin{cases}\varepsilon^{+a}(\vec{p},h)=\frac{i}{\sqrt{2}}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)(\gamma^a C)_{\lambda_\varsigma\mu_\varsigma},[-i\lambda_m^+(\hat{p},h),0]_a=(\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\\ \varepsilon_a(\vec{p},h)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h),[i\lambda_m(\hat{p},h),0]_a=(\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}\mathbb{X}_a^{+\lambda_\varsigma\mu_\varsigma}(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 1.4.8.
$$\begin{cases}\lambda_m^+(\hat{p},h)=(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}(p)\\ \lambda_m(\hat{p},h)=-(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}\qquad\begin{cases}0=(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\mathbb{X}^\pi_{\lambda_\varsigma\mu_\varsigma}(p)\\ 0=-(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}\mathbb{X}_\pi^{+\lambda_\varsigma\mu_\varsigma}(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 1.4.9.
$$\begin{cases}0=U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}(-p)\\ 0=\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(-p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\end{cases}\qquad\begin{cases}|\vec{p}|=\frac{i}{2\sqrt{2}}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\mathbb{X}^\pi_{\lambda_\varsigma\mu_\varsigma}(-p)\\ |\vec{p}|=-\frac{i}{2\sqrt{2}}\mathbb{X}_\pi^{+\lambda_\varsigma\mu_\varsigma}(-p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\end{cases}$$

1.4.2 Equivalent transformation between Dirac basis $v(\vec{p},\frac{\kappa}{2})$ and 4D vector basis $\varepsilon_a(\vec{p},h)$

Pro. 1.4.3.
$$\begin{cases}v^+(\hat{p},-\frac{\kappa}{2})\mathbb{X}_a(-p)v^*(\vec{p},-\frac{\kappa}{2})=-2\sqrt{2}\frac{E^2}{m}\varepsilon_a^+(\vec{p},-\kappa)\\ v^+(\hat{p},-\frac{\kappa}{2})im\gamma_a(\varsigma)Cv^*(\vec{p},-\frac{\kappa}{2})=-\sqrt{2}m\varepsilon_a^+(\vec{p},-\kappa)\\ v^+(\hat{p},-\frac{\kappa}{2})2iS_{ab}(e,\varsigma)p^bCv^*(\vec{p},-\frac{\kappa}{2})=-\sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a^+(\vec{p},-\kappa)\end{cases}$$

Proof: $v^+(\hat{p},-\frac{\kappa}{2})im\gamma_a(\varsigma)Cv^*(\vec{p},-\frac{\kappa}{2})$
$=i\kappa\varsigma v^+(\hat{p},-\frac{\kappa}{2})im\gamma_a(\varsigma)C\sigma_y\otimes\sigma_x v(\vec{p},\frac{\kappa}{2})$
$=-\kappa v^+(\hat{p},-\frac{\kappa}{2})m\gamma_a(\varsigma)(I\otimes\sigma_y)v(\vec{p},\frac{\kappa}{2})$
$=i\sqrt{2}[m\lambda_m(\hat{p},\kappa),0]_a=\sqrt{2}m\varepsilon_a(\vec{p},\kappa)=-\sqrt{2}m\varepsilon_a^+(\vec{p},-\kappa)$ □

Proof: $v^+(\hat{p},-\frac{\kappa}{2})2iS_{ab}(e,\varsigma)p^bCv^*(\vec{p},-\frac{\kappa}{2})$
$=i\kappa\varsigma v^+(\hat{p},-\frac{\kappa}{2})2iS_{ab}(e,\varsigma)p^bC\sigma_y\otimes\sigma_x v(\vec{p},\frac{\kappa}{2})$
$=-\kappa v^+(\hat{p},-\frac{\kappa}{2})[-i\gamma_a\gamma_b p^b](I\otimes\sigma_y)v(\vec{p},\frac{\kappa}{2})$
$=-\kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\hat{p},\kappa)+i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$=-i\kappa\sqrt{2}\frac{\vec{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\hat{p},0)\lambda_m^k(\hat{p},\kappa)+i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$=+i\sqrt{2}\frac{\vec{p}^2}{m}\lambda_m(\hat{p},\kappa)+i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$=i\sqrt{2}\frac{E^2+\vec{p}^2}{m}[\lambda_m(\hat{p},\kappa),0]_a=\sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a(\vec{p},\kappa)=-\sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a^+(\vec{p},-\kappa)$ □

Proof: $v^{+}(\hat{p},-\frac{\kappa}{2})\mathbb{X}_a(-p)v^{*}(\vec{p},-\frac{\kappa}{2})$
$= v^{+}(\hat{p},-\frac{\kappa}{2})i[m\gamma_a(\varsigma)+2S_{ab}(e,\varsigma)p^b]Cv^{*}(\vec{p},-\frac{\kappa}{2})$
$= i\kappa\varsigma v^{+}(\hat{p},-\frac{\kappa}{2})i[m\gamma_a(\varsigma)+2S_{ab}(e,\varsigma)p^b]C\sigma_y\otimes\sigma_x v(\vec{p},\frac{\kappa}{2})$
$= -\kappa v^{+}(\hat{p},-\frac{\kappa}{2})[m\gamma_a(\varsigma)-i\gamma_a\gamma_b p^b](I\otimes\sigma_y)v(\vec{p},\frac{\kappa}{2})$
$= i\sqrt{2}[m\lambda_m(\hat{p},\kappa),0]_a-\kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\hat{p},\kappa)+i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$= i\sqrt{2}[m\lambda_m(\hat{p},\kappa),0]_a-i\kappa\sqrt{2}\frac{\vec{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\hat{p},0)\lambda_m^k(\hat{p},\kappa)+i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$= i\sqrt{2}[m\lambda_m(\hat{p},\kappa),0]_a+i\sqrt{2}\frac{\vec{p}^2}{m}\lambda_m(\hat{p},\kappa)+i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p},\kappa)$
$= i2\sqrt{2}\frac{E^2}{m}[\lambda_m(\hat{p},\kappa),0]_a=2\sqrt{2}\frac{E^2}{m}\varepsilon_a(\vec{p},\kappa)=-2\sqrt{2}\frac{E^2}{m}\varepsilon_a^{+}(\vec{p},-\kappa)$ □

Cor. 1.4.10.
$$\begin{cases}-\varepsilon_a^{+}(\vec{p},\kappa)=\frac{i}{\sqrt{2}}v^{+}(\hat{p},\frac{\kappa}{2})\gamma_a Cv^{*}(\vec{p},\frac{\kappa}{2})=\frac{m}{2\sqrt{2}E^2}v^{+}(\hat{p},\frac{\kappa}{2})\mathbb{X}_a(p)v^{*}(\vec{p},\frac{\kappa}{2})\\ -\varepsilon_a(\vec{p},\kappa)=-\frac{i}{\sqrt{2}}v^{T}(\vec{p},\frac{\kappa}{2})\bar{C}\gamma_a v(\vec{p},\frac{\kappa}{2})=\frac{m}{2\sqrt{2}E^2}v^{T}(\vec{p},\frac{\kappa}{2})\mathbb{X}_a^{+}(p)v(\vec{p},\frac{\kappa}{2})\end{cases}$$

Cor. 1.4.11.
$$\begin{cases}-\varepsilon^{+a}(\vec{p},\kappa)=\frac{i}{\sqrt{2}}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)(\gamma^a C)_{\lambda_\varsigma\mu_\varsigma}=\frac{m}{2\sqrt{2}E^2}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\\ -\varepsilon_a(\vec{p},\kappa)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)=\frac{m}{2\sqrt{2}E^2}\mathbb{X}_a^{+\lambda_\varsigma\mu_\varsigma}(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)\end{cases}$$

Pro. 1.4.4. $v^{+}(\hat{p},\frac{\kappa}{2})\mathbb{X}_a(-p)v^{*}(\vec{p},-\frac{\kappa}{2})=-2iE[\lambda_m(\hat{p},0),0]_a$

Proof: $v^{+}(\hat{p},\frac{\kappa}{2})\mathbb{X}_a(-p)v^{*}(\vec{p},-\frac{\kappa}{2})$
$= v^{+}(\hat{p},\frac{\kappa}{2})i[m\gamma_a(\varsigma)+2S_{ab}(e,\varsigma)p^b]Cv^{*}(\vec{p},-\frac{\kappa}{2})$
$= i\kappa\varsigma v^{+}(\hat{p},\frac{\kappa}{2})i[m\gamma_a(\varsigma)+2S_{ab}(e,\varsigma)p^b]C\sigma_y\otimes\sigma_x v(\vec{p},\frac{\kappa}{2})$
$= -\kappa v^{+}(\hat{p},\frac{\kappa}{2})[m\gamma_a(\varsigma)-i\gamma_a\gamma_b p^b](I\otimes\sigma_y)v(\vec{p},\frac{\kappa}{2})$
$= -[E\hat{p},-i|\vec{p}|]-[E\hat{p},i|\vec{p}|]$
$= -2iE[\lambda_m(\hat{p},0),0]_a$ □

Cor. 1.4.12. $v^{+}(\hat{p},\frac{\kappa}{2})im\gamma_a Cv^{*}(\vec{p},-\frac{\kappa}{2})=-m\varepsilon_a^{+}(\vec{p},0), v^{+}(\hat{p},\frac{\kappa}{2})[2iS_{ab}(e,\varsigma)p^b C]v^{*}(\vec{p},-\frac{\kappa}{2})=-m\varepsilon_a(\vec{p},0)$

Cor. 1.4.13. $v^{+}(\hat{p},\frac{\kappa}{2})\mathbb{X}_a(-p)v^{*}(\vec{p},-\frac{\kappa}{2})=-[2E\hat{p},0], v^{+}(\hat{p},\frac{\kappa}{2})\mathbb{X}_a(p)v^{*}(\vec{p},-\frac{\kappa}{2})=-[0,-2i|\vec{p}|]$

Cor. 1.4.14.
$$\begin{cases}-\varepsilon_a^{+}(\vec{p},0)=iv^{+}(\hat{p},\frac{\kappa}{2})\gamma_a Cv^{*}(\vec{p},-\frac{\kappa}{2}),-[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2E}v^{+}(\hat{p},\frac{\kappa}{2})\mathbb{X}_a(-p)v^{*}(\vec{p},-\frac{\kappa}{2})\\ -\varepsilon_a(\vec{p},0)=-iv^{T}(\vec{p},-\frac{\kappa}{2})\bar{C}\gamma_a v(\vec{p},\frac{\kappa}{2}),-[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2E}v^{T}(\vec{p},-\frac{\kappa}{2})\mathbb{X}_a^{+}(-p)v(\vec{p},\frac{\kappa}{2})\end{cases}$$

Cor. 1.4.15.
$$\begin{cases}-\varepsilon_a^{+}(\vec{p},0)=\frac{i}{\sqrt{2}}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)(\gamma^a C)_{\lambda_\varsigma\mu_\varsigma},-[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2\sqrt{2}E}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\\ -\varepsilon_a(\vec{p},0)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0),-[i\lambda_m(\hat{p},0),0]_a=\frac{1}{2\sqrt{2}E}\mathbb{X}_a^{+\lambda_\varsigma\mu_\varsigma}(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\end{cases}$$

Cor. 1.4.16.
$$\begin{cases}-\varepsilon^{+a}(\vec{p},h)=\frac{i}{\sqrt{2}}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)(\gamma^a C)_{\lambda_\varsigma\mu_\varsigma},-[-i\lambda_m^{+}(\hat{p},h),0]_a=(\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\\ -\varepsilon_a(\vec{p},h)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h),-[i\lambda_m(\hat{p},h),0]_a=(\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}\mathbb{X}_a^{+\lambda_\varsigma\mu_\varsigma}(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 1.4.17.
$$\begin{cases}-\lambda_m^{+}(\hat{p},h)=(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}(-p)\\ -\lambda_m(\hat{p},h)=-(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}\quad\begin{cases}0=(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\mathbb{X}^{\pi}_{\lambda_\varsigma\mu_\varsigma}(-p)\\ 0=-(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}\mathbb{X}_{\pi}^{+\lambda_\varsigma\mu_\varsigma}(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 1.4.18.
$$\begin{cases}0=V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}(p)\\ 0=\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\end{cases}\quad\begin{cases}-|\vec{p}|=\frac{i}{2\sqrt{2}}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\mathbb{X}^{\pi}_{\lambda_\varsigma\mu_\varsigma}(p)\\ -|\vec{p}|=-\frac{i}{2\sqrt{2}}\mathbb{X}_{\pi}^{+\lambda_\varsigma\mu_\varsigma}(p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)\end{cases}$$

1.4.3 Wonderful relations between Dirac basis and 4D vector basis

Pro. 1.4.5. $[\sigma\cdot\lambda_m(\hat{p},\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0,[\sigma\cdot\lambda_m(\hat{p},-\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\lambda(\hat{p},-\frac{\kappa}{2}),[\sigma\cdot\lambda_m(\hat{p},0)]\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\lambda(\hat{p},\frac{\kappa}{2})$

Proof: $\lambda^{+}(\hat{p},\frac{\kappa}{2})(\sigma,i\kappa)_a\lambda(\hat{p},\frac{\kappa}{2})=\kappa\hat{p}_a,\lambda^{+}(\hat{p},-\frac{\kappa}{2})(\sigma,i\kappa)_a\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}[\lambda_m(\hat{p},\kappa),0]_a$
$\Rightarrow \lambda^{+}(\hat{p},\frac{\kappa}{2})\sigma_k\lambda(\hat{p},\frac{\kappa}{2})=\kappa\hat{p}_k,\lambda^{+}(\hat{p},-\frac{\kappa}{2})\sigma_k\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\lambda_{mk}(\hat{p},\frac{\kappa}{2})$
$\Rightarrow \lambda^{+}(\hat{p},\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0,\lambda^{+}(\hat{p},-\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0$
$\Rightarrow \lambda(\hat{p},\frac{\kappa}{2})\lambda^{+}(\hat{p},\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0,\lambda(\hat{p},-\frac{\kappa}{2})\lambda^{+}(\hat{p},-\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0$
$\Rightarrow [\lambda(\hat{p},\frac{\kappa}{2})\lambda^{+}(\hat{p},\frac{\kappa}{2})+\lambda(\hat{p},-\frac{\kappa}{2})\lambda^{+}(\hat{p},-\frac{\kappa}{2})][\sigma\cdot\lambda_m(\hat{p},\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0$
$\Rightarrow [\sigma\cdot\lambda_m(\hat{p},\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0$ □

Proof: $\lambda^{+}(\hat{p},\frac{\kappa}{2})(\sigma,i\kappa)_a\lambda(\hat{p},\frac{\kappa}{2})=\kappa\hat{p}_a,\lambda^{+}(\hat{p},-\frac{\kappa}{2})(\sigma,i\kappa)_a\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}[\lambda_m(\hat{p},\kappa),0]_a$
$\Rightarrow\lambda^{+}(\hat{p},\frac{\kappa}{2})\sigma_k\lambda(\hat{p},\frac{\kappa}{2})=\kappa\hat{p}_k,\lambda^{+}(\hat{p},-\frac{\kappa}{2})\sigma_k\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\lambda_{mk}(\hat{p},\frac{\kappa}{2})$
$\Rightarrow\lambda^{+}(\hat{p},\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},-\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0,\lambda^{+}(\hat{p},-\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},-\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}$
$\Rightarrow\lambda(\hat{p},\frac{\kappa}{2})\lambda^{+}(\hat{p},-\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},-\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=0,\lambda(\hat{p},-\frac{\kappa}{2})\lambda^{+}(\hat{p},-\frac{\kappa}{2})[\sigma\cdot\lambda_m(\hat{p},-\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\lambda(\hat{p},-\frac{\kappa}{2})$
$\Rightarrow[\lambda(\hat{p},\frac{\kappa}{2})\lambda^{+}(\hat{p},\frac{\kappa}{2})+\lambda(\hat{p},-\frac{\kappa}{2})\lambda^{+}(\hat{p},-\frac{\kappa}{2})][\sigma\cdot\lambda_m(\hat{p},-\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\lambda(\hat{p},-\frac{\kappa}{2})$
$\Rightarrow[\sigma\cdot\lambda_m(\hat{p},-\frac{\kappa}{2})]\lambda(\hat{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\lambda(\hat{p},-\frac{\kappa}{2})$ □

Pro. 1.4.6.

$$\begin{cases}[\gamma\cdot\lambda_m(\hat{p},\kappa)]u(\vec{p},\frac{\kappa}{2})=0,[\gamma\cdot\lambda_m(\hat{p},-\kappa)]u(\vec{p},\frac{\kappa}{2})=-\kappa\sqrt{2}\gamma_5u(\vec{p},-\frac{\kappa}{2})\\ [\gamma\cdot\lambda_m(\hat{p},0)]u(\vec{p},\frac{\kappa}{2})=-i\kappa(I\otimes\sigma_y)u(\vec{p},\frac{\kappa}{2})\\ [\gamma\cdot\lambda_m(\hat{p},\kappa)]v(\vec{p},\frac{\kappa}{2})=0,[\gamma\cdot\lambda_m(\hat{p},-\kappa)]v(\vec{p},\frac{\kappa}{2})=\kappa\sqrt{2}\gamma_5v(\vec{p},-\frac{\kappa}{2})\\ [\gamma\cdot\lambda_m(\hat{p},0)]v(\vec{p},\frac{\kappa}{2})=-i\kappa(I\otimes\sigma_y)v(\vec{p},\frac{\kappa}{2})\end{cases}$$

Pro. 1.4.7.

$$\begin{cases}[\gamma^a\varepsilon_a(\vec{p},\kappa)]u(\vec{p},\frac{\kappa}{2})=0,[\gamma^a\varepsilon_a(\vec{p},-\kappa)]u(\vec{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\gamma_5u(\vec{p},-\frac{\kappa}{2}),[\gamma^a\varepsilon_a(\vec{p},0)]u(\vec{p},\frac{\kappa}{2})=-i\kappa\gamma_5u(\vec{p},\frac{\kappa}{2})\\ [\gamma^a\varepsilon_a(\vec{p},\kappa)]v(\vec{p},\frac{\kappa}{2})=0,[\gamma^a\varepsilon_a(\vec{p},-\kappa)]v(\vec{p},\frac{\kappa}{2})=i\kappa\sqrt{2}\gamma_5v(\vec{p},-\frac{\kappa}{2}),[\gamma^a\varepsilon_a(\vec{p},0)]v(\vec{p},\frac{\kappa}{2})=i\kappa\gamma_5v(\vec{p},\frac{\kappa}{2})\end{cases}$$

1.5 Relations between second order B-W basis and 4D vector basis

1.5.1 Second order B-W basis decomposition into 4D vector bases

Thm. 1.5.1. $U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_a(\vec{p},h),V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\tilde{\varepsilon}_a(\vec{p},h)$

Proof: $\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_a(\vec{p},\kappa)$
$=\frac{1}{2\sqrt{2}m}\mathbb{X}^a(p)\varepsilon_a(\vec{p},\kappa)=\frac{i\varsigma}{2\sqrt{2}m}\mathbb{X}(p)\cdot\lambda_m(\vec{p},\kappa)$
$=\frac{-\varsigma}{2\sqrt{2}m}\{m\gamma-\frac{i}{2}[\gamma^ap_a,\gamma]\}C\cdot\lambda_m(\vec{p},\kappa)$
$=\frac{-\varsigma}{2\sqrt{2}m}\{m\gamma_j-\frac{i}{2}[\gamma_ip^i+\gamma_4iE,\gamma_j]\}C\lambda^j_m(\vec{p},\kappa)$
$=\frac{-\varsigma}{2\sqrt{2}m}[(m+E\gamma_4)\gamma_j+\varepsilon_{ijk}p^i\sigma^k\otimes I]C\lambda^j_m(\vec{p},\kappa)$
$=\frac{i\varsigma}{2\sqrt{2}m}[i(m+E\gamma_4)\sigma_j\sigma_y\lambda^j_m(\vec{p},\kappa)\otimes\sigma_x-i\kappa|\vec{p}|\sigma_j\sigma_y\lambda^j_m(\vec{p},\kappa)\otimes\sigma_z]$
$=-\frac{1}{\sqrt{2}}\sigma_j\sigma_y\lambda^j_m(\vec{p},\kappa)\otimes\frac{\varsigma}{2m}[(m\sigma_x+\varsigma E)-\kappa|\vec{p}|\sigma_z]$
$=\lambda(\hat{p},\frac{\varsigma}{2})\lambda^T(\hat{p},\frac{\varsigma}{2})\otimes\mu(\vec{p},\frac{\kappa}{2})\mu^T(\vec{p},\frac{\kappa}{2})$
$=u(\vec{p},\frac{\kappa}{2})u^T(\vec{p},\frac{\kappa}{2})$
$=U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},\kappa)$ □

Proof: $\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_a(\vec{p},0)$
$=\frac{1}{2\sqrt{2}m}\mathbb{X}^a(p)\varepsilon_a(\vec{p},0)=\frac{i\varsigma}{2\sqrt{2}m}\mathbb{X}(p)\cdot\frac{E}{m}\lambda_m(\vec{p},0)+\frac{i\varsigma}{2\sqrt{2}m}\mathbb{X}^\pi(p)\frac{|\vec{p}|}{m}$
$=\frac{-\varsigma}{2\sqrt{2}m}\{m\gamma-\frac{i}{2}[\gamma^ap_a,\gamma]\}C\cdot\frac{E}{m}\lambda_m(\vec{p},0)+\frac{-\varsigma}{2\sqrt{2}m}(m\gamma_4-i\gamma^jp_j\gamma_4)C\frac{|\vec{p}|}{m}$
$=\frac{-\varsigma}{2\sqrt{2}m}\{m\gamma_j-\frac{i}{2}[\gamma_ip^i+\gamma_4iE,\gamma_j]\}C\frac{E}{m}\lambda^j_m(\vec{p},0)+\frac{1}{2\sqrt{2}m}(m-i\gamma_ip^i)\gamma_2\frac{|\vec{p}|}{m}$
$=\frac{-\varsigma}{2\sqrt{2}m}[(m+E\gamma_4)\gamma_j+\varepsilon_{ijk}p^i\sigma^k\otimes I]C\frac{E}{m}\lambda^j_m(\vec{p},0)+\frac{1}{2\sqrt{2}m}(m\sigma_y\otimes\sigma_y-i\sigma_i\sigma_yp^i\otimes I)\frac{|\vec{p}|}{m}$
$=\frac{-\varsigma}{2\sqrt{2}m}(m+E\gamma_4)\sigma_j\sigma_y\frac{E}{m}\lambda^j_m(\vec{p},0)\otimes\sigma_x+\frac{1}{2\sqrt{2}m}(m\sigma_y\otimes\sigma_y-i\sigma_i\sigma_yp^i\otimes I)\frac{|\vec{p}|}{m}$
$=-\frac{1}{\sqrt{2}}\sigma_j\sigma_y\lambda^j_m(\vec{p},0)\otimes\frac{\varsigma}{2}(\frac{E}{m}\sigma_x+\varsigma\frac{E^2-\vec{p}^2}{m^2})+\frac{1}{2\sqrt{2}}\frac{|\vec{p}|}{m}\sigma_y\otimes\sigma_y$
$=-\frac{1}{2\sqrt{2}}[\sigma_j\sigma_y\lambda^j_m(\vec{p},0)\otimes(\varsigma\frac{E}{m}\sigma_x+I)-\frac{|\vec{p}|}{m}\sigma_y\otimes\sigma_y]$
$=U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)$ □

Proof: $U(\vec{p},0)=\frac{1}{\sqrt{2}}[u(\vec{p},\frac{\kappa}{2})u^T(\vec{p},-\frac{\kappa}{2})+u(\vec{p},-\frac{\kappa}{2})u^T(\vec{p},\frac{\kappa}{2})]$
$=\frac{1}{\sqrt{2}}[\lambda(\hat{p},\frac{\kappa}{2})\lambda^T(\hat{p},-\frac{\kappa}{2})\otimes\mu(\vec{p},\frac{\kappa}{2})\mu^T(\vec{p},-\frac{\kappa}{2})+\lambda(\hat{p},-\frac{\kappa}{2})\lambda^T(\hat{p},\frac{\kappa}{2})\otimes\mu(\vec{p},-\frac{\kappa}{2})\mu^T(\vec{p},\frac{\kappa}{2})]$
$=\frac{1}{\sqrt{2}}[\frac{i}{2}(\sigma\cdot\hat{p}+\kappa I)\sigma_y\otimes\frac{1}{2}(I+\varsigma\frac{E}{m}\sigma_x-i\kappa\frac{|\vec{p}|}{m}\sigma_y)+\frac{i}{2}(\sigma\cdot\hat{p}-\kappa I)\sigma_y\otimes\frac{1}{2}(I+\varsigma\frac{E}{m}\sigma_x-i\kappa\frac{|\vec{p}|}{m}\sigma_y)$
$=\frac{i}{4\sqrt{2}}[(\sigma\cdot\hat{p}+\kappa I)\sigma_y\otimes(I+\varsigma\frac{E}{m}\sigma_x-i\kappa\frac{|\vec{p}|}{m}\sigma_y)+(\sigma\cdot\hat{p}-\kappa I)\sigma_y\otimes(I+\varsigma\frac{E}{m}\sigma_x+i\kappa\frac{|\vec{p}|}{m}\sigma_y)$
$=\frac{i}{2\sqrt{2}}[(\sigma\sigma_y\cdot\hat{p})\otimes(I+\varsigma\frac{E}{m}\sigma_x)+\sigma_y\otimes(-i\frac{|\vec{p}|}{m}\sigma_y)$
$=-\frac{1}{2\sqrt{2}}\{[\sigma\sigma_y\cdot\lambda(\hat{p},0)]\otimes(I+\varsigma\frac{E}{m}\sigma_x)-\frac{|\vec{p}|}{m}\sigma_y\otimes\sigma_y\}$ □

1.5.2 Summary of equivalent relations between Dirac basis and 4D vector basis

Cor. 1.5.1.

$$\begin{cases}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_a(\vec{p},h),V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=-\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\varepsilon_a(\vec{p},h)\\ \varepsilon_a(\vec{p},h)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 1.5.2.
$$\begin{cases}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=-\frac{i}{4m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\bar{C}\gamma_a)^{\lambda'_\varsigma\mu'_\varsigma}U_{\lambda'_\varsigma\mu'_\varsigma}(\hat{p},h),V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=-\frac{i}{4m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)(\bar{C}\gamma_a)^{\lambda'_\varsigma\mu'_\varsigma}V_{\lambda'_\varsigma\mu'_\varsigma}(\hat{p},h)\\ \varepsilon_a(\vec{p},h)=-\frac{i}{4m}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_b(\vec{p},h)=-\frac{i}{4m}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(-p)\varepsilon_b(\vec{p},h)\end{cases}$$

Cor. 1.5.3.
$$\begin{cases}[i\lambda_m(\hat{p},h),0]_a=(\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=-(\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\\ [i\lambda_m(\hat{p},h),0]_a=(\frac{m}{E})^{|h|}\frac{1}{8mE}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(p)\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_b(\vec{p},h)=(\frac{m}{E})^{|h|}\frac{1}{8mE}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(-p)\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(-p)\varepsilon_b(\vec{p},h)\end{cases}$$

Cor. 1.5.4.
$$\begin{cases}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_\pi(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_\pi(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=0\\ \lambda_m(\hat{p},h)=-(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=(\frac{m}{E})^{|h|}\frac{i}{2\sqrt{2}E}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(-p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 1.5.5.
$$\begin{cases}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(-p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)=\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}(p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)=0\\ -\frac{i}{2\sqrt{2}}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_\pi(-p)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)=\frac{i}{2\sqrt{2}}\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_\pi(p)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)=|\vec{p}|\end{cases}$$

Thm. 1.5.2. $(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(p)=(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(-p)=4im\delta^b_a$

Thm. 1.5.3. $\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(p)\mathbb{X}_{b\lambda_\varsigma\mu_\varsigma}(p)=\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(-p)\mathbb{X}_{b\lambda_\varsigma\mu_\varsigma}(-p)=8E^2\delta_{ab}-4p_ap^+_b$

Proof: $\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(p)\mathbb{X}_{b\lambda_\varsigma\mu_\varsigma}(p)$
$=tr[\mathbb{X}_a(p)\mathbb{X}_b(p)]$
$=tr\{\bar{C}[m\gamma_a-2S_{ac}(e,\varsigma)p^{+c}][m\gamma_b-2S_{bd}(e,\varsigma)p^d]C\}$
$=tr\{[m\gamma_a-2S_{ac}(e,\varsigma)p^{+c}][m\gamma_b-2S_{bd}(e,\varsigma)p^d]\}$
$=m^2tr(\gamma_a\gamma_b)+4tr[S_{ac}(e,\varsigma)S_{bd}(e,\varsigma)p^{+c}p^d]$
$=4m^2\delta_{ab}+4(\delta_{ab}\delta_{dc}-\delta_{ad}\delta_{bc})p^{+c}p^d$
$=4m^2\delta_{ab}+4(\delta_{ab}p^+_cp^c-p_ap^+_b)$
$=8E^2\delta_{ab}-4p_ap^+_b$ □

Cor. 1.5.6. $\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(p)\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(p)=\mathbb{X}^{+\lambda_\varsigma\mu_\varsigma}_a(-p)\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(-p)=8E^2\delta^b_a-4p_ap^{+b}$

Ass. 1.5.1. $-\frac{i}{4m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\bar{C}\gamma_a)^{\lambda'_\varsigma\mu'_\varsigma}=-\frac{i}{4m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)(\bar{C}\gamma_a)^{\lambda'_\varsigma\mu'_\varsigma}=\frac{1}{(2!)^2}\delta^{(\lambda'_\varsigma}_{\{\lambda_\varsigma}\delta^{\mu'_\varsigma)}_{\mu_\varsigma\}}$

1.5.3 Quasi projection operator decomposes into Dirac ones for spin-1 particles with mass

Thm. 1.5.4.
$$\begin{cases}\Lambda_+(\vec{p},1):=\sum_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)=\frac{1}{(2!)^2}\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\}\mu'_\varsigma)}(\vec{p},\frac{1}{2})\\ \Lambda_-(\vec{p},1):=\sum_{h=1}^{-1}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)V^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)=\frac{1}{(2!)^2}\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\mu_\varsigma\}\mu'_\varsigma)}(\vec{p},\frac{1}{2})\end{cases}$$

Proof: $\Lambda_+(\vec{p},1):=\sum_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)$
$=$
$u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})$
$+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})][u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})+u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})$
$=$
$\frac{1}{4}\{[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})][u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})]$
$+[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})][u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})+u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})][u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})+u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})][u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})+u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]\}$
$=$
$\frac{1}{4}\{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})$
$+[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})$
$+[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})$
$+[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})$

$=$
$\frac{1}{4}\{[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})][u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})][u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})][u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})][u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$=\frac{1}{4}[\Lambda_{+\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})+\Lambda_{+\lambda_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\lambda'_\varsigma}(\vec{p},\frac{1}{2})+\Lambda_{+\mu_\varsigma\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\lambda_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})+\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\lambda_{+\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\frac{1}{2})\}$
$=\frac{1}{(2!)^2}\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\}\mu'_\varsigma)}(\vec{p},\frac{1}{2})$ □

1.5.4 Analytically proving an important theorem

Thm. 1.5.5. $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)=\frac{1}{2}[(m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}$

Proof: $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)$
$=\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\sum_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)$
$=8m^2\sum_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)$
$=8m^2\frac{1}{(2!)^2}\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\}\mu'_\varsigma)}(\vec{p},\frac{1}{2})$
$=\frac{1}{2}[(m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}$ □

1.6 Covariant anticommutation rules for Dirac equation

1.6.1 Dirac equation and its separated form [4, 5]

Def. 1.6.1. $\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),-\gamma^a\gamma^4=i\varsigma(\sigma\otimes\sigma_z,i\varsigma),\gamma^4\gamma^a=i\varsigma(\sigma\otimes\sigma_z,-i\varsigma),\gamma^4\prec\gamma^4_{\lambda_\varsigma\lambda'_\varsigma},\gamma_4\prec\gamma_4^{\lambda'_\varsigma\lambda_\varsigma}$

Cor. 1.6.1. $(\gamma^a\partial_a+m)\psi_{\lambda_\varsigma}(x)=0\Leftrightarrow[(\sigma\otimes\sigma_z,-i\varsigma)^a\partial_a-imI\otimes\sigma_x]\psi(x)=0$

Cor. 1.6.2. $\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma}(x)=0\\ \psi_{\lambda_\varsigma}(x)=[\lambda_{A_\varsigma}(x),\eta^{A'_\varsigma}(x)]^T\end{cases}\Leftrightarrow\begin{cases}(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\lambda_{A_\varsigma}(x)=im\eta^{A'_\varsigma}(x)\\ (\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a\eta^{A'_\varsigma}(x)=-im\lambda_{A_\varsigma}(x)\end{cases}$

1.6.2 Covariant anticommutation rules for Dirac equation

Cor. 1.6.3.
$\{\psi_{\lambda_\varsigma}(x),\bar{\psi}^{\mu_\varsigma}(x')\}=i(m-\gamma^a\partial_a)_{\lambda_\varsigma}{}^{\mu_\varsigma}\Delta(x-x')\Leftrightarrow\{\psi_{\lambda_\varsigma}(x),\psi^+_{\lambda'_\varsigma}(x')\}=i[(m-\gamma^a\partial_a)\gamma^4]_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')$

Cor. 1.6.4.

$$\begin{cases}\{\psi_{\lambda_\varsigma}(x),\psi^+_{\lambda'_\varsigma}(x')\}=i[(m-\gamma^a\partial_a)\gamma^4]_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')\\ \{\psi_{\lambda_\varsigma}(x),\psi_{\lambda'_\varsigma}(x')\}=0,\{\psi^+_{\lambda_\varsigma}(x),\psi^+_{\lambda'_\varsigma}(x')\}=0\\ \psi_{\lambda_\varsigma}(x)=[\lambda_{A_\varsigma}(x),\eta^{A'_\varsigma}(x)]^T\\ \gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)\\ S_{ab}(e,\varsigma)=-\frac{i}{4}[\gamma_a,\gamma_b]=S_{ab}(\varsigma)\oplus S_{ab}(-\varsigma)\\ S_{ab}(\varsigma)=\frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma}=-\frac{i}{4}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\end{cases}\Leftrightarrow\begin{cases}\{\lambda_{A_\varsigma}(x),\lambda^+_{A'_\varsigma}(x')\}=-\varsigma(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a\Delta(x-x')\\ \{\eta^{A'_\varsigma}(x),\eta^{A_\varsigma}_+(x')\}=\varsigma(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\Delta(x-x')\\ \{\lambda_{A_\varsigma}(x),\eta^{B_\varsigma}_+(x')\}=i\varsigma m\delta_{A_\varsigma}{}^{B_\varsigma}\Delta(x-x')\\ \{\eta^{A'_\varsigma}(x),\lambda^+_{B'_\varsigma}(x')\}=i\varsigma m\delta^{A'_\varsigma}{}_{B'_\varsigma}\Delta(x-x')\\ \{\lambda_{A_\varsigma}(x),\lambda_{B_\varsigma}(x')\}=0,\{\eta^{A'_\varsigma}(x),\eta^{B'_\varsigma}(x')\}=0\\ \{\lambda^+_{A'_\varsigma}(x),\lambda^+_{B'_\varsigma}(x')\}=0,\{\eta^{A_\varsigma}_+(x),\eta^{B_\varsigma}_+(x')\}=0\\ \{\lambda_{A_\varsigma}(x),\eta^{A'_\varsigma}(x')\}=0,\{\lambda^+_{A'_\varsigma}(x),\eta^{A_\varsigma}_+(x')\}=0\end{cases}$$

The above content is the basic part. And its reasoning and conclusions apply to all chapters, especially the following chapters and the next chapter.

1.7 Basic identities

Cor. 1.7.1. $C^m_n=C^{m-1}_{n-1}+C^m_{n-1}$

Cor. 1.7.2.
$$\begin{cases}\sum_{k=1}^n k=\frac{1}{2!}n(n+1),\sum_{k=1}^n k^3=[\frac{1}{2!}n(n+1)]^2\\ \sum_{k=1}^n k^2=\frac{1}{3!}n(n+1)(2n+1),\sum_{k=1}^n k^4=\frac{1}{5!}2n(2n+1)(2n+2)(3n^2+3n-1)\end{cases}$$

Pro. 1.7.1.
$$\begin{cases}\sum_{i=0}^n i^p=\frac{1}{p+1}\sum_{k=0}^p(-1)^kC^k_{p+1}B_kn^{p+1-k},B_k=\delta_{k0}-\frac{1}{k+1}\sum_{j=0}^{k-1}C^j_{k+1}B_j,\frac{z}{e^z-1}=\sum_{k=0}^\infty B_k\frac{z^k}{k!}\\ B_0=1,B_1=-\frac{1}{2},B_2=\frac{1}{6},B_4=-\frac{1}{30},B_6=\frac{1}{42},B_8=-\frac{1}{30},B_{2k+1}=0(k\geq1)\end{cases}$$

1.8 A combinatorial identity and its proof

Pro. 1.8.1. $\sum\limits_{h'=n'}^{-n'} C_{n+h}^{n'+h'} C_{n-h}^{n'-h'} = C_{2n}^{2n'}$

Proof: $(1+x)^{n+h}(1+x)^{n-h}|2n' = \sum\limits_{h'=n'}^{-n'} C_{n+h}^{n'+h'} x^{n'+h'} C_{n-h}^{n'-h'} x^{n'-h'} = (1+x)^{2n}|2n' = C_{2n}^{2n'} x^{2n'}$

$\Leftrightarrow \sum\limits_{h'=n'}^{-n'} C_{n+h}^{n'+h'} C_{n-h}^{n'-h'} = C_{2n}^{2n'}$ □

1.9 An important combinatorial identity and its corollaries

Lem. 1.9.1. $\sum\limits_{a+b=k} C_a^c C_b^d = C_{k+1}^{c+d+1}$

The above lemma must be correct. I have seen this formula in books. But I have not yet seen a suitable and understandable proof method.

Cor. 1.9.1. $\sum\limits_{h'=n'}^{-n'} C_{n+h'+h}^{n'+h'} C_{n-h'-h}^{n'-h'} = \sum\limits_{h'=n'}^{-n'} C_{n+h'+h}^{n-n'+h} C_{n-h'-h}^{n-n'-h} = C_{2n+1}^{2n'}; n'-n \le h \le n-n', n \ge n'$

Cor. 1.9.2. $\sum\limits_{h'=s'}^{-s'} C_{s+h'+h}^{s'+h'} C_{s-h'-h}^{s'-h'} = \sum\limits_{h'=s'}^{-s'} C_{s+h'+h}^{s-s'+h} C_{s-h'-h}^{s-s'-h} = C_{2s+1}^{2s'}; s'-s \le h \le s-s', s \ge s'$

Cor. 1.9.3. $\sum\limits_{h'=s'}^{-s'} C_{s+h'}^{s'+h'} C_{s-h'}^{s'-h'} = C_{2s+1}^{2s'}$

Cor. 1.9.4.

$\sum\limits_{h'=s}^{-s} C_{s+h'}^{s+h'} C_{s-h'}^{s-h'} = C_{2s+1}^{2s} [\Leftrightarrow] \sum\limits_{h'=s}^{-s} C_{s+h'}^{0} C_{s-h'}^{0} = C_{2s+1}^{1} [\Leftrightarrow] \sum\limits_{k=0}^{2s} C_k^0 C_{2s-k}^0 = C_{2s+1}^1$

$\sum\limits_{h'=s-1}^{1-s} C_{s+h'}^{s-1+h'} C_{s-h'}^{s-1-h'} = C_{2s+1}^{2s-2} [\Leftrightarrow] \sum\limits_{h'=s-1}^{1-s} C_{s+h'}^{1} C_{s-h'}^{1} = C_{2s+1}^{3} [\Leftrightarrow] \sum\limits_{k=1}^{2s-1} C_k^1 C_{2s-k}^1 = C_{2s+1}^3$

$\sum\limits_{h'=s-2}^{2-s} C_{s+h'}^{s-2+h'} C_{s-h'}^{s-2-h'} = C_{2s+1}^{2s-4} [\Leftrightarrow] \sum\limits_{h'=s-2}^{2-s} C_{s+h'}^{2} C_{s-h'}^{2} = C_{2s+1}^{5} [\Leftrightarrow] \sum\limits_{k=2}^{2s-2} C_k^2 C_{2s-k}^2 = C_{2s+1}^5$

$\cdots$

$\sum\limits_{h'=s-l}^{l-s} C_{s+h'}^{s-l+h'} C_{s-h'}^{s-l-h'} = C_{2s+1}^{2s-2l} [\Leftrightarrow] \sum\limits_{h'=s-l}^{l-s} C_{s+h'}^{l} C_{s-h'}^{l} = C_{2s+1}^{2l+1} [\Leftrightarrow] \sum\limits_{k=l}^{2s-l} C_k^l C_{2s-k}^l = C_{2s+1}^{2l+1}$

Cor. 1.9.5. $\sum\limits_{a+b=n} C_a^c C_b^d = C_{n+1}^{c+d+1} \Rightarrow \sum\limits_{k=l}^{n-l} C_k^l C_{n-k}^l = C_{n+1}^{2l+1}$

Cor. 1.9.6.

$$\begin{cases} C_{n+h}^2 + C_{n+h-1}^1 C_{n-h+1}^1 + C_{n-h+2}^2 = \sum\limits_{h'=1}^{-1} C_{(n+h')+(h-1)}^{1+h'} C_{(n-h')-(h-1)}^{1-h'} = C_{2n+1}^2 \\ C_{n+h}^{2n'} C_{n-h}^0 + C_{n+h-1}^{2n'-1} C_{n-h+1}^1 + C_{n+h-2}^{2n'-2} C_{n-h+2}^2 + \cdots + C_{n+h-2n'}^0 C_{n-h+2n'}^{2n'} = \sum\limits_{h'=n'}^{-n'} C_{(n+h')+(h-n')}^{n'+h'} C_{(n-h')-(h-n')}^{n'-h'} = C_{2n+1}^{2n'} \end{cases}$$

Cor. 1.9.7.

$$\begin{cases} \sum\limits_{h'=1}^{-1} C_{(n+h')+h}^{1+h'} C_{(n-h')-h}^{1-h'} = C_{2n+1}^2, \sum\limits_{h'=2}^{-2} C_{(n+h')+h}^{2+h'} C_{(n-h')-h}^{2-h'} = C_{2n+1}^4, \sum\limits_{h'=n'}^{-n'} C_{(n+h')+h}^{n'+h'} C_{(n-h')-h}^{n'-h'} = C_{2n+1}^{2n'} \\ \sum\limits_{h'=1}^{-1} C_{(n-h')+h}^{1-h'} C_{(n+h')-h}^{1+h'} = C_{2n+1}^2, \sum\limits_{h'=2}^{-2} C_{(n-h')+h}^{2-h'} C_{(n+h')-h}^{2+h'} = C_{2n+1}^4, \sum\limits_{h'=n'}^{-n'} C_{(n-h')+h}^{n'-h'} C_{(n+h')-h}^{n'+h'} = C_{2n+1}^{2n'} \end{cases}$$

2 Spin basis and plane wave solutions of Bargmann-Wigner equation [16]

2.1 Generalized binomial theorem and its corollaries of Dirac equation spin basis

Thm. 2.1.1.

$$\overbrace{\sum_{h=s}^{-s} C_{2s}^{s-h} \underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2}) u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2}) \cdots}_{s+h} \underbrace{\cdots u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2}) u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h} \underbrace{u^+_{(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2}) u^+_{\mu'_\varsigma}(\vec{p},\tfrac{1}{2}) \cdots}_{s+h} \underbrace{\cdots u^+_{\sigma'_\varsigma}(\vec{p},-\tfrac{1}{2}) u^+_{\tau'_\varsigma)}(\vec{p},-\tfrac{1}{2})}_{s-h}}^{2s}$$

$$= \sum_{h=1/2}^{-1/2} u_{\{\lambda_\varsigma}(\vec{p},h) u^+_{(\lambda'_\varsigma}(\vec{p},h) \sum_{h=1/2}^{-1/2} u_{\mu_\varsigma}(\vec{p},h) u^+_{\mu'_\varsigma}(\vec{p},h) \cdot\cdot \sum_{h=1/2}^{-1/2} u_{\sigma_\varsigma}(\vec{p},h) u^+_{\sigma'_\varsigma}(\vec{p},h)] \sum_{h=1/2}^{-1/2} u_{\tau_\varsigma\}}(\vec{p},h) u^+_{\tau'_\varsigma)}(\vec{p},h)]$$

Cor. 2.1.1.

$$\sum_{h=s}^{-s} C_{2s}^{s-h} \underbrace{u_{\{1_\varsigma}(\vec{p},\tfrac12)u_{1_\varsigma}(\vec{p},\tfrac12)\cdots}_{s+h}\underbrace{\cdots u_{1_\varsigma}(\vec{p},-\tfrac12)u_{1_\varsigma\}}(\vec{p},-\tfrac12)}_{s-h}\underbrace{u^+_{(1'_\varsigma}(\vec{p},\tfrac12)u^+_{1'_\varsigma}(\vec{p},\tfrac12)\cdots}_{s+h}\underbrace{\cdots u^+_{1'_\varsigma}(\vec{p},-\tfrac12)u^+_{1'_\varsigma)}(\vec{p},-\tfrac12)}_{s-h}$$

$$=\overbrace{\sum_{h=1/2}^{-1/2} u_{\{1_\varsigma}(\vec{p},h)u^+_{(1'_\varsigma}(\vec{p},h)\sum_{h=1/2}^{-1/2}u_{1_\varsigma}(\vec{p},h)u^+_{1'_\varsigma}(\vec{p},h)\cdot\cdot\sum_{h=1/2}^{-1/2}u_{1_\varsigma}(\vec{p},h)u^+_{1'_\varsigma}(\vec{p},h)]\sum_{h=1/2}^{-1/2}u_{1_\varsigma\}}(\vec{p},h)u^+_{1'_\varsigma)}(\vec{p},h)]}^{2s}$$

$$\Leftrightarrow \sum_{h=s}^{-s}C_{2s}^{s-h}[u_{1_\varsigma}(\vec{p},\tfrac12)u^+_{1_\varsigma}(\vec{p},\tfrac12)]^{s+h}[u_{1_\varsigma}(\vec{p},-\tfrac12)u^+_{(1'_\varsigma}(\vec{p},-\tfrac12)]^{s-h} = [u_{1_\varsigma}(\vec{p},\tfrac12)u^+_{1'_\varsigma}(\vec{p},\tfrac12)+u_{1_\varsigma}(\vec{p},-\tfrac12)u^+_{1'_\varsigma}(\vec{p},-\tfrac12)]^{2s}$$

Cor. 2.1.2.

$$\sum_{h=s}^{-s} C_{2s}^{s-h} \underbrace{u_{\{2_\varsigma}(\vec{p},\tfrac12)u_{2_\varsigma}(\vec{p},\tfrac12)\cdots}_{s+h}\underbrace{\cdots u_{2_\varsigma}(\vec{p},-\tfrac12)u_{2_\varsigma\}}(\vec{p},-\tfrac12)}_{s-h}\underbrace{u^+_{(2'_\varsigma}(\vec{p},\tfrac12)u^+_{2'_\varsigma}(\vec{p},\tfrac12)\cdots}_{s+h}\underbrace{\cdots u^+_{2'_\varsigma}(\vec{p},-\tfrac12)u^+_{2'_\varsigma)}(\vec{p},-\tfrac12)}_{s-h}$$

$$=\overbrace{\sum_{h=1/2}^{-1/2} u_{\{2_\varsigma}(\vec{p},h)u^+_{(2'_\varsigma}(\vec{p},h)\sum_{h=1/2}^{-1/2}u_{2_\varsigma}(\vec{p},h)u^+_{2'_\varsigma}(\vec{p},h)\cdot\cdot\sum_{h=1/2}^{-1/2}u_{2_\varsigma}(\vec{p},h)u^+_{2'_\varsigma}(\vec{p},h)]\sum_{h=1/2}^{-1/2}u_{2_\varsigma\}}(\vec{p},h)u^+_{2'_\varsigma)}(\vec{p},h)]}^{2s}$$

$$\Leftrightarrow \sum_{h=s}^{-s}C_{2s}^{s-h}[u_{2_\varsigma}(\vec{p},\tfrac12)u^+_{2_\varsigma}(\vec{p},\tfrac12)]^{s+h}[u_{2_\varsigma}(\vec{p},-\tfrac12)u^+_{(2'_\varsigma}(\vec{p},-\tfrac12)]^{s-h} = [u_{2_\varsigma}(\vec{p},\tfrac12)u^+_{2'_\varsigma}(\vec{p},\tfrac12)+u_{2_\varsigma}(\vec{p},-\tfrac12)u^+_{2'_\varsigma}(\vec{p},-\tfrac12)]^{2s}$$

The above two corollaries are just the binomial expansion theorem.

Lem. 2.1.1.

$$[u_{\lambda_\varsigma}(\vec{p},\tfrac12)u^+_{\lambda'_\varsigma}(\vec{p},\tfrac12)+u_{\lambda_\varsigma}(\vec{p},-\tfrac12)u^+_{\lambda'_\varsigma}(\vec{p},-\tfrac12)][u_{\mu_\varsigma}(\vec{p},\tfrac12)u^+_{\mu'_\varsigma}(\vec{p},\tfrac12)+u_{\mu_\varsigma}(\vec{p},-\tfrac12)u^+_{\mu'_\varsigma}(\vec{p},-\tfrac12)]$$
$$\neq$$
$$[u_{\mu_\varsigma}(\vec{p},\tfrac12)u^+_{\lambda'_\varsigma}(\vec{p},\tfrac12)+u_{\mu_\varsigma}(\vec{p},-\tfrac12)u^+_{\lambda'_\varsigma}(\vec{p},-\tfrac12)][u_{\lambda_\varsigma}(\vec{p},\tfrac12)u^+_{\mu'_\varsigma}(\vec{p},\tfrac12)+u_{\lambda_\varsigma}(\vec{p},-\tfrac12)u^+_{\mu'_\varsigma}(\vec{p},-\tfrac12)]$$

Lem. 2.1.2.

$$[v_{\lambda_\varsigma}(\vec{p},\tfrac12)v^+_{\lambda'_\varsigma}(\vec{p},\tfrac12)+v_{\lambda_\varsigma}(\vec{p},-\tfrac12)v^+_{\lambda'_\varsigma}(\vec{p},-\tfrac12)][v_{\mu_\varsigma}(\vec{p},\tfrac12)v^+_{\mu'_\varsigma}(\vec{p},\tfrac12)+v_{\mu_\varsigma}(\vec{p},-\tfrac12)v^+_{\mu'_\varsigma}(\vec{p},-\tfrac12)]$$
$$\neq$$
$$[v_{\mu_\varsigma}(\vec{p},\tfrac12)v^+_{\lambda'_\varsigma}(\vec{p},\tfrac12)+v_{\mu_\varsigma}(\vec{p},-\tfrac12)v^+_{\lambda'_\varsigma}(\vec{p},-\tfrac12)][v_{\lambda_\varsigma}(\vec{p},\tfrac12)v^+_{\mu'_\varsigma}(\vec{p},\tfrac12)+v_{\lambda_\varsigma}(\vec{p},-\tfrac12)v^+_{\mu'_\varsigma}(\vec{p},-\tfrac12)]$$

Cor. 2.1.3.

$$\Lambda_{+\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)\neq\Lambda_{+\mu_\varsigma\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{+\lambda_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)$$
$$\Lambda_{-\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)\neq\Lambda_{-\mu_\varsigma\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{-\lambda_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)$$

Cor. 2.1.4.

$$\begin{cases}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)\cdot\cdot\Lambda_{+\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\tfrac12)}_{2s}\neq\underbrace{\Lambda_{+\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)\cdot\cdot\Lambda_{+\tau_\varsigma\tau'_\varsigma}(\vec{p},\tfrac12)}_{2s}\\ \frac{1}{[(2s)!]^2}\underbrace{\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)\cdot\cdot\Lambda_{-\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\tfrac12)}_{2s}\neq\underbrace{\Lambda_{-\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\tfrac12)\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac12)\cdot\cdot\Lambda_{-\tau_\varsigma\tau'_\varsigma}(\vec{p},\tfrac12)}_{2s}\end{cases}$$

2.2 Reasonable conjecture for Bargmann-Wigner equation plane wave solutions (Strict proof will be provided later in this chapter.)

Thm. 2.2.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}}_{2s}}(\vec{r},t)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=s}^{-s}E^{s-\frac12}\sqrt{\tfrac{m}{E}}^{2s}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{(2s)!}\sqrt{C_{2s}^{s-h}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac12)u_{\mu_\varsigma}(\vec{p},\tfrac12)\cdots}_{s+h}\underbrace{\cdots u_{\sigma_\varsigma}(\vec{p},-\tfrac12)u_{\tau_\varsigma\}}(\vec{p},-\tfrac12)}_{s-h}$$

$$V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{(2s)!}\sqrt{C_{2s}^{s-h}}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},\tfrac12)v_{\mu_\varsigma}(\vec{p},\tfrac12)\cdots}_{s+h}\underbrace{\cdots v_{\sigma_\varsigma}(\vec{p},-\tfrac12)v_{\tau_\varsigma\}}(\vec{p},-\tfrac12)}_{s-h}$$

Cor. 2.2.1.

$$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac12)}\sqrt{\tfrac{m}{E}}^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ b^+(\vec{p},s)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac12)}\sqrt{\tfrac{m}{E}}^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Cor. 2.2.2. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}}_{2s}}(\vec{r},t)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=s}^{-s}E^{s-\frac{1}{2}}[a(\vec{p},h)\tilde{U}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{V}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\tilde{U}^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ b^+(\vec{p},s)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\tilde{V}^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Self comment: The above expression is very similar to the plane wave solutions form of massless particles, so physics has a unified expression in essence. However, it is important to note that plane wave solutions of massless particles cannot be obtained through $m\to 0$ for the plane wave solutions of massless particles mentioned above. This shows that there is an essential difference between massless particles and massive particles.

2.3 Plane wave solutions of Bargmann-Wigner equation under real representation

Thm. 2.3.1. $(\gamma_s^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}}_{2s}}(\vec{r},t)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)$$

$$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=s}^{-s}\sqrt{\frac{m^{2s}}{E}}[a(\vec{p},h)U_{s\,\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}-(-1)^{s+h}\varsigma^{2s}b^+(\vec{p},-h)U^+_{s\,\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$U_{s\,\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{u_{s\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{s\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot u_{s\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{s\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}$$

$$V_{s\,\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=-(-1)^{s-h}\varsigma^{2s}U^+_{s\,\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)$$

Cor. 2.3.1.

$$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^{2s}}{E}}U_s^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ b^+(\vec{p},s)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^{2s}}{E}}U_s^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

2.4 Bargmann-Wigner equation basis

Def. 2.4.1.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h):=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h):=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})v_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot v_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})v_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}\end{cases}$$

Def. 2.4.2.

$$\begin{cases}\tilde{U}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h):=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{\tilde{u}_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})\tilde{u}_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot\tilde{u}_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})\tilde{u}_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}\\ \tilde{V}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h):=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{\tilde{v}_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})\tilde{v}_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot\tilde{v}_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})\tilde{v}_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}\end{cases}$$

Cor. 2.4.1.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sqrt{C_{2s}^{h-s}}[\underbrace{u_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})}_{s-h}+\cdot\cdot]\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sqrt{C_{2s}^{h-s}}[\underbrace{v_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})v_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot v_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})v_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})}_{s-h}+\cdot\cdot]\end{cases}$$

Cor. 2.4.2. $u(\vec{p},h)=-\varsigma\gamma_5v(\vec{p},h),v(\vec{p},h)=-\varsigma\gamma_5u(\vec{p},h),h=-\frac{1}{2},\frac{1}{2}$

Cor. 2.4.3.
$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=(-\varsigma)^{2s}\overbrace{\gamma_5\otimes\gamma_5\cdots}^{2s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=(-\varsigma)^{2s}\overbrace{\gamma_5\otimes\gamma_5\cdots}^{2s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) \end{cases}$$

2.5 Relations between two spin bases of Bargmann-Wigner equation

Cor. 2.5.1.
$$\begin{cases} U^{+}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=(-1)^{s+h}\varsigma^{2s}\overbrace{\sigma_y\otimes\sigma_y\cdots}^{4s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h) \\ V^{+}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=(-1)^{s-h}\varsigma^{2s}\overbrace{\sigma_y\otimes\sigma_y\cdots}^{4s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h) \end{cases}$$

Proof: $U^{+}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{(2s)!}\sqrt{C^{s-h}_{2s}}\underbrace{u^{+}_{\{\lambda_\varsigma}(\vec{p},\tfrac12)u^{+}_{\mu_\varsigma}(\vec{p},\tfrac12)\cdots}_{s+h}\underbrace{\cdots u^{+}_{\sigma_\varsigma}(\vec{p},-\tfrac12)u^{+}_{\tau_\varsigma\}}(\vec{p},-\tfrac12)}_{s-h}$

$=(-1)^{s+h}\varsigma^{2s}\overbrace{\sigma_y\otimes\sigma_y\cdots}^{4s}\frac{1}{(2s)!}\sqrt{C^{s-h}_{2s}}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},-\tfrac12)v_{\mu_\varsigma}(\vec{p},-\tfrac12)\cdots}_{s+h}\underbrace{\cdots v_{\sigma_\varsigma}(\vec{p},\tfrac12)v_{\tau_\varsigma\}}(\vec{p},\tfrac12)}_{s-h}$

$=(-1)^{s+h}\varsigma^{2s}\overbrace{\sigma_y\otimes\sigma_y\cdots}^{4s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)$ □

Cor. 2.5.2.
$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)=(-1)^{s-h}\varsigma^{2s}\overbrace{\sigma_y\otimes\sigma_y\cdots}^{4s}V^{+}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)=(-1)^{s+h}\varsigma^{2s}\overbrace{\sigma_y\otimes\sigma_y\cdots}^{4s}U^{+}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) \end{cases}$$

Cor. 2.5.3.
$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)=(-1)^{s-h}\varsigma^{2s}\overbrace{(C\gamma_4)\otimes(C\gamma_4)\cdots}^{2s}V^{+}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)=(-1)^{s+h}\varsigma^{2s}\overbrace{(C\gamma_4)\otimes(C\gamma_4)\cdots}^{2s}U^{+}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) \end{cases}$$

2.6 Orthogonal properties of Bargmann-Wigner equation basis(Can be seen directly.)

Cor. 2.6.1.
$$\begin{cases} \bar{U}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h')=\delta_{hh'},\bar{U}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h')=0 \\ \bar{V}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h')=\delta_{hh'},\bar{V}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h')=0 \end{cases}$$

Cor. 2.6.2.
$$\begin{cases} U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h')=(\tfrac{E}{m})^{2s}\delta_{hh'},U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(-\vec{p},h')=0 \\ V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h')=(\tfrac{E}{m})^{2s}\delta_{hh'},V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(-\vec{p},h')=0 \end{cases}$$

3 Recursive relations of Bargmann-Wigner equation basis

3.1 Recursive relations of Bargmann-Wigner equation basis(Enumeration heuristic method.)

Thm. 3.1.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\tfrac12-l)$

$=\frac{1}{\sqrt{C^{l}_{2s+1}}}[\sqrt{C^{l-1}_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l+1)u_{\tau_\varsigma}(\vec{p},-\tfrac12)+\sqrt{C^{l}_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l)u_{\tau_\varsigma}(\vec{p},\tfrac12)]$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2})$$
$$=\frac{1}{\sqrt{(2s+1)!(2s+1)!(0)!}}u_{\{\lambda_\varsigma}\underbrace{(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{2s+1}\underbrace{\cdot u_{\rho_\varsigma}(\vec{p},-\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{0}$$
$$=\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2s+1}$$
$$=C^{\phi}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}u_{\tau_\varsigma}(\vec{p},\frac{1}{2})$$
$$=\frac{1}{\sqrt{C^0_{2s+1}}}C^0_{2s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})\qquad\square$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s-\frac{1}{2})$$
$$=\frac{1}{\sqrt{(2s+1)!(2s)!(1)!}}u_{\{\lambda_\varsigma}\underbrace{(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{2s}\underbrace{\cdot u_{\rho_\varsigma}(\vec{p},-\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{1}$$
$$=\frac{1}{\sqrt{(2s+1)!(2s)!(1)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{2s+1}$$
$$=\frac{1}{\sqrt{C^1_{2s+1}}}$$
$$\{[C^{\phi}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+C^{(\vec{p},-\frac{1}{2})}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]\}$$
$$=\frac{1}{\sqrt{C^1_{2s+1}}}[\sqrt{C^0_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^1_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-1)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]\qquad\square$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s-\frac{3}{2})$$
$$=\frac{1}{\sqrt{(2s+1)!(2s-1)!(2)!}}u_{\{\lambda_\varsigma}\underbrace{(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{2s-1}\underbrace{\cdot u_{\rho_\varsigma}(\vec{p},-\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{2}$$
$$=\frac{1}{\sqrt{(2s+1)!(2s-1)!(2)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{2s+1}$$
$$=\frac{1}{\sqrt{C^2_{2s+1}}}C^{(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2})}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2s+1}}$$
$$=\frac{1}{\sqrt{C^2_{2s+1}}}$$
$$[C^{(\vec{p},-\frac{1}{2})}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+C^{(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2})}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]$$
$$=\frac{1}{\sqrt{C^2_{2s+1}}}[\sqrt{C^1_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^2_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-2)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]\qquad\square$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}-l)$$
$$=\frac{1}{\sqrt{(2s+1)!(2s+1-l)!(l)!}}u_{\{\lambda_\varsigma}\underbrace{(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{2s+1-l}\underbrace{\cdot u_{\rho_\varsigma}(\vec{p},-\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{l}$$
$$=\frac{1}{\sqrt{(2s+1)!(2s-l+1)!(l)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{2s+1}$$
$$=\frac{1}{\sqrt{C^l_{2s+1}}}C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{l}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2s+1}}$$
$$=\frac{1}{\sqrt{C^2_{2s+1}}}$$

$$[C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{l-1}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{l}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\rho_\varsigma}(\vec{p},\frac{1}{2})u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]$$

$$=\frac{1}{\sqrt{C^l_{2s+1}}}[\sqrt{C^{l-1}_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l+1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^l_{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})] \quad \square$$

Thm. 3.1.2. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}-l)$

$$=\frac{1}{\sqrt{C^l_{2s+1}}}[\sqrt{C^{l-1}_{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l+1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^l_{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})]$$

Proof:

$V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}-l)$

$$=\frac{1}{\sqrt{(2s+1)!(2s+1-l)!(l)!}}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{2s+1-l}\underbrace{\cdot\cdot v_{\rho_\varsigma}(\vec{p},-\frac{1}{2})v_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{l}$$

$$=\frac{1}{\sqrt{(2s+1)!(2s-l+1)!(l)!}}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot v_{\rho_\varsigma}(\vec{p},\frac{1}{2})v_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{2s+1}$$

$$=\frac{1}{\sqrt{C^l_{2s+1}}}C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{l}}_{\underbrace{v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot v_{\rho_\varsigma}(\vec{p},\frac{1}{2})v_{\sigma_\varsigma}(\vec{p},\frac{1}{2})v_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2s+1}}$$

$$=\frac{1}{\sqrt{C^2_{2s+1}}}[C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{l-1}}_{\underbrace{v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot v_{\rho_\varsigma}(\vec{p},\frac{1}{2})v_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{l}}_{\underbrace{v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot v_{\rho_\varsigma}(\vec{p},\frac{1}{2})v_{\sigma_\varsigma}(\vec{p},\frac{1}{2})}_{2s}}v_{\tau_\varsigma}(\vec{p},\frac{1}{2})]$$

$$=\frac{1}{\sqrt{C^l_{2s+1}}}[\sqrt{C^{l-1}_{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l+1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^l_{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})] \quad \square$$

3.2 Decomposition of Bargmann-Wigner equation U-basis(Combinatorial method.)

Thm. 3.2.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p},h)=\sum_{h'=s'}^{-s'}\frac{\sqrt{C^{s'+h'}_{s+h}C^{s'-h'}_{s-h}}}{\sqrt{C^{2s'}_{2s}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h')$

Proof: $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}$

$$\sum_{h'=s'}^{-s'}C^{s'+h'}_{s+h}C^{s'-h'}_{s-h}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{(s-s')+(h-h')}\underbrace{\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{(s-s')-(h-h')}\underbrace{u_{(\lambda'_\varsigma}(\vec{p},\frac{1}{2})u_{\mu'_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{(s'+h')}\underbrace{\cdot\cdot u_{\sigma'_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau'_\varsigma)}(\vec{p},-\frac{1}{2})}_{(s'-h')}$$

$$=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}$$

$$\sum_{h'=s'}^{-s'}C^{s'+h'}_{s+h}C^{s'-h'}_{s-h}\sqrt{[2(s-s')]![(s-s')+(h-h')]![(s-s')-(h-h')]!}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')$$

$$\sqrt{(2s')!(s'+h')!(s'-h')!}U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h')$$

$$=\frac{\sqrt{[2(s-s')]!}\sqrt{(2s')!}}{\sqrt{(2s)!}}$$

$$\sum_{h'=s'}^{-s'}C^{s'+h'}_{s+h}C^{s'-h'}_{s-h}\frac{\sqrt{[(s-s')+(h-h')]![(s-s')-(h-h')]!}\sqrt{(s'+h')!(s'-h')!}}{\sqrt{(s+h)!(s-h)!}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h')$$

$$=\sum_{h'=s'}^{-s'}\frac{\sqrt{C^{s'+h'}_{s+h}C^{s'-h'}_{s-h}}}{\sqrt{C^{2s'}_{2s}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h') \quad \square$$

Cor. 3.2.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n}}(\vec{p},h)=\sum_{h'=n'}^{-n'}\frac{\sqrt{C^{n'+h'}_{n+h}C^{n'-h'}_{n-h}}}{\sqrt{C^{2n'}_{2n}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(n-n')}}(\vec{p},h-h')U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n'}}(\vec{p},h')$

Cor. 3.2.2. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},h)=\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n-2}}(\vec{p},h-1)U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)$
$+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n-2}}(\vec{p},h)U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n-2}}(\vec{p},h+1)U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},-1)$

Cor. 3.2.3. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-h')U_{\tau_\varsigma}(\vec{p},h')$
$=\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

3.3 Decomposition of Bargmann-Wigner equation V-basis(Combinatorial method.)

Thm. 3.3.1. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p},h)=\sum_{h'=s'}^{-s'}\frac{\sqrt{C_{s+h}^{s'+h'}C_{s-h}^{s'-h'}}}{\sqrt{C_{2s}^{2s'}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h')$

Proof: $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}$
$\sum_{h'=s'}^{-s'}C_{s+h}^{s'+h'}C_{s-h}^{s'-h'}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{(s-s')+(h-h')}\underbrace{\cdots v_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{(s-s')-(h-h')}\underbrace{v_{(\lambda'_\varsigma}(\vec{p},\frac{1}{2})v_{\mu'_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{(s'+h')}\underbrace{\cdots v_{\sigma'_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau'_\varsigma)}(\vec{p},-\frac{1}{2})}_{(s'-h')}$
$=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}$
$\sum_{h'=s'}^{-s'}C_{s+h}^{s'+h'}C_{s-h}^{s'-h'}\sqrt{[2(s-s')]![(s-s')+(h-h')]![(s-s')-(h-h')]!}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')$
$\sqrt{(2s')!(s'+h')!(s'-h')!}V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h')$
$=\frac{\sqrt{[2(s-s')]!}\sqrt{(2s')!}}{\sqrt{(2s)!}}$
$\sum_{h'=s'}^{-s'}C_{s+h}^{s'+h'}C_{s-h}^{s'-h'}\frac{\sqrt{[(s-s')+(h-h')]![(s-s')-(h-h')]!}\sqrt{(s'+h')!(s'-h')!}}{\sqrt{(s+h)!(s-h)!}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h')$
$=\sum_{h'=s'}^{-s'}\frac{\sqrt{C_{s+h}^{s'+h'}C_{s-h}^{s'-h'}}}{\sqrt{C_{2s}^{2s'}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p},h-h')V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p},h')$ □

Cor. 3.3.1. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n}}(\vec{p},h)=\sum_{h'=n'}^{-n'}\frac{\sqrt{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(n-n')}}(\vec{p},h-h')V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n'}}(\vec{p},h')$

Cor. 3.3.2. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},h)=\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n-2}}(\vec{p},h-1)V_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)$
$+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n-2}}(\vec{p},h)V_{\sigma_\varsigma\tau_\varsigma}(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma\xi_\varsigma}_{2n-2}}(\vec{p},h+1)V_{\sigma_\varsigma\tau_\varsigma}(\vec{p},-1)$

Cor. 3.3.3. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-h')V_{\tau_\varsigma}(\vec{p},h')$
$=\frac{\sqrt{s+h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})v_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

3.4 Synthesis of Bargmann-Wigner equation basis

Cor. 3.4.1. $\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')=U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}}(\vec{p},h)\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}^{2s'}}(\vec{p},h'),-s-s'\leq h\leq s+s'$

Cor. 3.4.2. $\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')=V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}}(\vec{p},h)\bar{V}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}^{2s'}}(\vec{p},h'),-s-s'\leq h\leq s+s'$

3.5 Corollaries of Bargmann-Wigner equation basis synthesis

Cor. 3.5.1.
$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}-l)u^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})=\frac{\sqrt{l}}{\sqrt{2s+1}}\frac{E}{m}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l+1) \\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}-l)u^{+\tau_\varsigma}(\vec{p},\frac{1}{2})=\frac{\sqrt{2s+1-l}}{\sqrt{2s+1}}\frac{E}{m}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l) \end{cases}$$

Cor. 3.5.2.
$$\begin{cases} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}-l)v^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})=\frac{\sqrt{l}}{\sqrt{2s+1}}\frac{E}{m}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l+1) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}-l)v^{+\tau_\varsigma}(\vec{p},\frac{1}{2})=\frac{\sqrt{2s+1-l}}{\sqrt{2s+1}}\frac{E}{m}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},s-l) \end{cases}$$

4 Quasi projection operator for Bargmann-Wigner equation

4.1 Definition and properties of quasi projection operator for Bargmann-Wigner equation

Def. 4.1.1.
$$\begin{cases} \Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},s):=\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p},h) \\ \Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},s):=\sum_{h=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p},h) \end{cases}$$

Cor. 4.1.1.
$$\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},s)=\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{\pm\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})\Lambda_{\pm\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot\Lambda_{\pm\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\tfrac{1}{2})}_{2s}$$

The above corollary can be directly obtained from the generalized binomial theorem for symmetric indicators.

Cor. 4.1.2.
$$\begin{cases} \sum_{h=s}^{-s}(-1)^{s-h}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p},-h) \\ =\frac{\varsigma^{2s}}{[(2s)!]^2}\underbrace{(\Lambda_+\bar{C}\gamma_4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})(\Lambda_+\bar{C}\gamma_4)_{\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot(\Lambda_+\bar{C}\gamma_4)_{\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\tfrac{1}{2})}_{2s} \\ \sum_{h=s}^{-s}(-1)^{s+h}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p},-h) \\ =\frac{\varsigma^{2s}}{[(2s)!]^2}\underbrace{(\Lambda_-\bar{C}\gamma_4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})(\Lambda_-\bar{C}\gamma_4)_{\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot(\Lambda_-\bar{C}\gamma_4)_{\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\tfrac{1}{2})}_{2s} \end{cases}$$

Cor. 4.1.3.
$$\begin{cases} \Lambda_{\pm\underbrace{\lambda_\varsigma\cdot\cdot}_{2n_1}\underbrace{\mu_\varsigma\cdot\cdot}_{2n_2}\cdot\cdot\underbrace{\tau_\varsigma\cdot\cdot}_{2n_k}\underbrace{\lambda'_\varsigma\cdot\cdot}_{2n_1}\underbrace{\mu'_\varsigma\cdot\cdot}_{2n_2}\cdot\cdot\underbrace{\tau'_\varsigma\cdot\cdot}_{2n_k}}(\vec{p},s)=\frac{1}{[(2s)!]^2}\Lambda_{\pm\{\underbrace{\lambda_\varsigma\cdot\cdot}_{2n_1}(\underbrace{\lambda'_\varsigma\cdot\cdot}_{2n_1}}(\vec{p},n_1)\Lambda_{\pm\underbrace{\mu_\varsigma\cdot\cdot}_{2n_2}\underbrace{\mu'_\varsigma\cdot\cdot}_{2n_2}}(\vec{p},n_2)\cdot\cdot\Lambda_{\pm\underbrace{\tau_\varsigma\cdot\cdot\}}_{2n_k}\underbrace{\tau'_\varsigma\cdot\cdot)}_{2n_k}}(\vec{p},n_k) \\ s=n_1+n_2+\cdot\cdot+n_k \end{cases}$$

Cor. 4.1.4.
$$\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{p},n)$$
$$=\frac{1}{[(2n)!]^2}\underbrace{\Lambda_{\pm\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})\Lambda_{\pm\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot_{\})}}_{2n}=\frac{1}{(2\sqrt{2}m)^{2n}}\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(\pm p)\cdot\cdot_{\}}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(\pm p)\cdot\cdot_{)}}_{n}\underbrace{\Lambda_{maa'}(\vec{p},1)\cdot\cdot}_{n}$$

Cor. 4.1.5.
$$\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{p},n)=\frac{1}{(2m)^{2n}}\frac{1}{[(2n)!]^2}\underbrace{[(\pm m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(\pm m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}_{2n}$$
$$=\frac{1}{(2\sqrt{2}m)^{2n}}\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(\pm p)\cdot\cdot_{\}}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(\pm p)\cdot\cdot_{)}}_{n}\underbrace{(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})\cdot\cdot}_{n}$$

Cor. 4.1.6.
$$\begin{cases} \sum_{h=s}^{-s}(-1)^{s-h}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},-h)=\frac{1}{(2m)^{2s}}\frac{\varsigma^{2s}}{[(2s)!]^2}\underbrace{[(m-i\gamma^a p_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}_{2s} \\ \sum_{h=s}^{-s}(-1)^{s+h}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},-h)=\frac{1}{(2m)^{2s}}\frac{\varsigma^{2s}}{[(2s)!]^2}\underbrace{[(-m-i\gamma^a p_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}_{2s} \end{cases}$$

4.2 Genuine two classes of projection operators for Bargmann-Wigner equation

Def. 4.2.1. $\tilde{\Lambda}_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s) := (\frac{m}{E})^{2s}\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)$

Def. 4.2.2. $\bar{\Lambda}_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}{}^{\overbrace{\eta_\varsigma\xi_\varsigma\cdots}^{2s}}(\vec{p},s) := \Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)\,\overbrace{\gamma^4_{\lambda'_\varsigma}{}^{\eta_\varsigma}\gamma^4_{\mu'_\varsigma}{}^{\xi_\varsigma}\cdots}^{2s}$

4.3 Recursive relations of quasi projection operator for B-W equation(Strict proof.)

Thm. 4.3.1. $\sum\limits_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h)$

$= \frac{2s+1}{2s+2s'+1}\sum\limits_{h''=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h'')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h'')]\sum\limits_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$

Proof: $\frac{C^{s'+h'}_{s+s'+h}C^{s'-h'}_{s+s'-h}}{C^{2s'}_{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h-h')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s_1}}(\vec{p},h-h')$

$= U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h)U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h'); -s-s'\leq h\leq s+s', -s'\leq h'\leq s'$

$\Rightarrow \sum\limits_{h=s+s'}^{-s-s'}\sum\limits_{h'=s'}^{-s'}\frac{C^{s'+h'}_{s+s'+h}C^{s'-h'}_{s+s'-h}}{C^{2s'}_{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h-h')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h-h')$

$= \sum\limits_{h=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h)]\sum\limits_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$

$\Leftrightarrow \sum\limits_{h'=s'}^{-s'}\sum\limits_{h''=s+s'-h'}^{-s-s'-h'}\frac{C^{s'+h'}_{s+s'+h'+h''}C^{s'-h'}_{s+s'-h'-h''}}{C^{2s'}_{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h'')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h'')$

$= \sum\limits_{h=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h)]\sum\limits_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$

$\Leftrightarrow \sum\limits_{h'=s'}^{-s'}\sum\limits_{h''=s}^{-s}\frac{C^{s'+h'}_{s+s'+h'+h''}C^{s'-h'}_{s+s'-h'-h''}}{C^{2s'}_{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h'')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h'')$

$= \sum\limits_{h=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h)]\sum\limits_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$

$\Leftrightarrow \sum\limits_{h''=s}^{-s}\sum\limits_{h'=s'}^{-s'}\frac{C^{s'+h'}_{s+s'+h'+h''}C^{s'-h'}_{s+s'-h'-h''}}{C^{2s'}_{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h'')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h'')$

$= \sum\limits_{h=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h)]\sum\limits_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$

$\Leftrightarrow \sum\limits_{h''=s}^{-s}\frac{C^{2s'}_{2(s+s')+1}}{C^{2s'}_{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h'')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h'')$

$= \sum\limits_{h=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h)]\sum\limits_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$

$\Leftrightarrow \sum\limits_{h''=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h'')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h'')$

$= \frac{2s+1}{2s+2s'+1}\sum\limits_{h=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h)]\sum\limits_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$

$$\Leftrightarrow \sum_{h=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h)$$

$$= \frac{2s+1}{2s+2s'+1}\sum_{h''=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h'')\bar{U}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h'')]\sum_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')] \qquad \square$$

Thm. 4.3.2. $\sum_{h=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h)$

$$= \frac{2s+1}{2s+2s'+1}(\frac{m}{E})^{4s'}\sum_{h''=s+s'}^{-s-s'}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdots}_{2s'}}(\vec{p},h'')U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}\underbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}_{2s'}}(\vec{p},h'')]\sum_{h'=s'}^{-s'}[U^{\overbrace{\rho'_\varsigma\sigma'_\varsigma\cdots}^{2s'}}(\vec{p},h')U^{+\overbrace{\rho_\varsigma\sigma_\varsigma\cdots}^{2s'}}(\vec{p},h')]$$

Self comment: A conjecture has finally been rigorously proven after many years. The trick lies in the use of a special combinatorial formula.

4.4 Relations between quasi projection operators for Bargmann-Wigner equation

Thm. 4.4.1.
$$\begin{cases}\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s) = \frac{2s+1}{2s+2}(\frac{m}{E})^2\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s+1}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2})\Lambda_\pm^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\frac{1}{2})\\ \Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s+1}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}) = \frac{1}{[(2s+1)!]^2}\Lambda_{\pm\{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}(\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)\Lambda_{\pm\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\frac{1}{2})\end{cases}$$

Thm. 4.4.2.
$$\begin{cases}\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s) = \frac{2s+1}{2(s+l)+1}(\frac{m}{E})^{4l}\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\rho_\varsigma\cdots\tau_\varsigma}_{2(s+l)}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\rho'_\varsigma\cdots\tau'_\varsigma}_{2(s+l)}}(\vec{p},s+l)\Lambda_\pm^{\overbrace{\rho'_\varsigma\cdots\tau'_\varsigma}^{2l}\overbrace{\rho_\varsigma\cdots\tau_\varsigma}^{2l}}(\vec{p},l)\\ \Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2(s+l)}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2(s+l)}}(\vec{p},s+l) = \frac{1}{[2(s+l)]!^2}\Lambda_{\pm\{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}(\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)\Lambda_{\pm\underbrace{\rho_\varsigma\cdots\tau_\varsigma\}}_{2l}\underbrace{\rho'_\varsigma\cdots\tau'_\varsigma)}_{2l}}(\vec{p},l)\end{cases}$$

5 Commutation rules for Bargmann-Wigner equation

5.1 Covariant commutation rules for Bargmann-Wigner equation

Thm. 5.1.1. $[a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[b(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[rest]_{-^{2s+1}} = 0$

$$\Rightarrow\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}\Delta(x-x')\\ [\psi^{(+)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^{(+)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}\Delta^{(+)}(x-x')\\ [\psi^{(-)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^{(-)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}\Delta^{(-)}(x-x')\\ [rest]_{-^{2s+1}} = 0\end{cases}$$

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}$

$$[[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot x},[a^+(\vec{p}',h')U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',h)e^{-ip'\cdot x'}+b(\vec{p}',h')V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',h')e^{ip'\cdot x'}]$$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}$$

$$[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',h')[a(\vec{p},h),a^+(\vec{p}',h')]e^{i(p\cdot x-p'\cdot x')}+V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',h')[b^+(\vec{p},h),b(\vec{p}',h')]e^{-i(p\cdot x-p'\cdot x')}$$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}$$

$$[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{i(p\cdot x-p'\cdot x')}+(-1)^{2s+1}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{-i(p\cdot x-p'\cdot x')}]$$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}[\sum_{h=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot(x-x')} + (-1)^{2s+1}\sum_{h=s}^{-s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot(x-x')}]$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}[\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},s)e^{ip\cdot(x-x')} + (-1)^{2s+1}\Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},s)e^{-ip\cdot(x-x')}]$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}[\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{2s} e^{ip\cdot(x-x')}$

$+ (-1)^{2s+1}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{2s} e^{-ip\cdot(x-x')}]$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s} e^{ip\cdot(x-x')}$

$+ (-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m+\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m+\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s} e^{-ip\cdot(x-x')}\}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\frac{-i}{(2\pi)^3}\int d^3\vec{p}\frac{1}{2E}[[e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}]$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')$

$= \frac{i(2m)^{2s}}{2^{2s-1}}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(-i\partial,s)\Delta(x-x')$

□

Proof: $[\psi^{(+)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^{(+)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot x}, a^+(\vec{p}',h')U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h)e^{-ip'\cdot x'}]$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')[a(\vec{p},h),a^+(\vec{p}',h')]e^{i(p\cdot x-p'\cdot x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{i(p\cdot x-p'\cdot x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},s)e^{ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{2s} e^{ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s} e^{ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\frac{-i}{(2\pi)^3}\int d^3\vec{p}\frac{1}{2E}e^{ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta^{(+)}(x-x')$

$= \frac{i(2m)^{2s}}{2^{2s-1}}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(-i\partial,s)\Delta^{(+)}(x-x')$

□

Proof: $[\psi^{(-)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^{(-)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}[b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot x}, b(\vec{p}',h')V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')e^{ip'\cdot x'}]$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')[b^+(\vec{p},h),b(\vec{p}',h')]e^{-i(p\cdot x-p'\cdot x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}' \sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}(-1)^{2s+1}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{-i(p\cdot x-p'\cdot x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\sum_{h=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)e^{-ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdots}_{2s}e^{-ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m+\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m+\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\frac{i}{(2\pi)^3}\int d^3\vec{p}\frac{1}{2E}e^{-ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(-)}(x-x')$

$= \frac{i(2m)^{2s}}{2^{2s-1}}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(-i\partial,s)\Delta^{(-)}(x-x')$ □

5.2 Reverse reasoning of covariant commutation rules for Bargmann-Wigner equation

Thm. 5.2.1.

$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')\\ [rest]_{-^{2s+1}}=0\end{cases}$$

$\Rightarrow [a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[b(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[rest]_{-^{2s+1}}=0$

The following has given a detailed proof process for several main commutative brackets.

Proof: $[a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}e^{-i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$

$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')e^{-i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$

$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\{\frac{-i}{(2\pi)^3}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec{p}_0\}e^{-i(p\cdot x-p'\cdot x')}$

$= [\frac{1}{(2\pi)^3}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$

$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$

$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}\}d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0$

$= [\frac{1}{(2\pi)^3}]^2\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$

$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}$

$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}\}$

$= \int d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$

$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)\delta^3(\vec{p}_0-\vec{p})\delta^3(\vec{p}_0-\vec{p}')$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{2iE_0(t-t')}\delta^3(\vec{p}_0+\vec{p})\delta^3(\vec{p}_0+\vec{p}')\}$$

$$=\delta^3(\vec{p}-\vec{p}')(\frac{m}{E})^{4s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p},h')$$

$$\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p},h_0)+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(-\vec{p},h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(-\vec{p},h_0)e^{2iE(t-t')}\}$$

$$=\delta^3(\vec{p}-\vec{p}')(\sum_{h_0=s}^{-s}\delta_{hh_0}\delta_{h'h_0}+0)$$

$$=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')$$

□

Proof: $[b^+(\vec{p},h),b(\vec{p}',h')]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}e^{i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$=\frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')$$

$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')e^{i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')$$

$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\{\frac{-i}{(2\pi)^3}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec{p}_0\}e^{i(p\cdot x-p'\cdot x')}$$

$$=[\frac{1}{(2\pi)^3}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')$$

$$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}e^{ip_0\cdot(x-x')}e^{i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}e^{-ip_0\cdot(x-x')}e^{i(p\cdot x-p'\cdot x')}\}d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0$$

$$=[\frac{1}{(2\pi)^3}]^2\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{i(p_0+p)\cdot x}e^{-i(p_0+p')\cdot x'}$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{-i(p_0-p)\cdot x}e^{i(p_0-p')\cdot x'}\}$$

$$=\int d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{-2iE_0(t-t')}\delta^3(\vec{p}_0+\vec{p})\delta^3(\vec{p}_0+\vec{p}')$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)\delta^3(\vec{p}_0-\vec{p})\delta^3(\vec{p}_0-\vec{p}')\}$$

$$=\delta^3(\vec{p}-\vec{p}')(\frac{m}{E})^{4s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p},h')$$

$$\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(-\vec{p},h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(-\vec{p},h_0)e^{-2iE(t-t')}+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p},h_0)\}$$

$$=(-1)^{2s+1}\delta^3(\vec{p}-\vec{p}')(0+\sum_{h_0=s}^{-s}\delta_{hh_0}\delta_{h'h_0})$$

$$=(-1)^{2s+1}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')$$

□

Proof: $[a(\vec{p},h),b(\vec{p}',h')]_{-2s+1}$

$= \frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-2s+1}e^{-i(p\cdot x+p'\cdot x')}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$

$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')e^{-i(p\cdot x+p'\cdot x')}d^3\vec{r}d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$

$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\{\frac{-i}{(2\pi)^3}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec{p}_0\}e^{-i(p\cdot x+p'\cdot x')}$

$= [\frac{1}{(2\pi)^3}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$

$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip_0\cdot(x-x')}e^{-i(p\cdot x+p'\cdot x')}d^3\vec{r}d^3\vec{r}'$

$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip_0\cdot(x-x')}e^{-i(p\cdot x+p'\cdot x')}\}d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0$

$= [\frac{1}{(2\pi)^3}]^2\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$

$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{i(p_0-p)\cdot x}e^{-i(p_0+p')\cdot x'}$

$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{-i(p_0+p)\cdot x}e^{i(p_0-p')\cdot x'}\}$

$= \int d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$

$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{-iE_0t'}\delta^3(\vec{p}_0-\vec{p})\delta^3(\vec{p}_0+\vec{p}')$

$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{iE_0t}\delta^3(\vec{p}_0+\vec{p})\delta^3(\vec{p}_0-\vec{p}')\}$

$= \delta^3(\vec{p}+\vec{p}')(\frac{m}{E})^{4s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$

$\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p},h_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p},h_0)+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}',h_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}',h_0)e^{2iE(t-t')}\}$

$= 0+0=0$ □

Self comment: The above proof method is similar to the case of Penrose equation. And it is no longer based on the isochronous commutation rule, but directly based on the covariant commutation rule. It seems more difficult, but it's actually simpler. Because the isochronous commutation rule is not easy to calculate. Even if it is calculated out, it is still difficult to use. The covariant commutation rule itself is known and very regular and can also be decomposed into the product of spin bases. The entire proof process basically depends on the properties of the spin base and hasn't complex calculations. The three most difficult proofs of commutative parentheses are given above. While the other several commutative parentheses can be easily proved, and will not be listed in detail. In fact, the isochronous commutation rule is a special case of the covariant commutation rule.Therefore, the above proof method can also be used for the isochronous commutation rule ($t=t'$ is taken).

5.3 Summary of covariant commutation rules for Bargmann-Wigner equation

By combining the proofs in the above two sections, the following important theorems are obtained.

Thm. 5.3.1.

$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')\\ [rest]_{-^{2s+1}}=0\end{cases}$$

$$\Leftrightarrow[a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[b(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[rest]_{-^{2s+1}}=0$$

Thm. 5.3.2. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=2im^{2s}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau_\varsigma}_{2s}}(-i\partial,s)$

5.4 Commutative function, causal function and Feynman propagator of B-W equation

Lem. 5.4.1. $\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$

$$=\sum_{n=0}^{2s}C^n_{2s}\overbrace{[-(\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[-(\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots}^{n}\overbrace{[m\gamma^4]_{\eta_\varsigma\eta'_\varsigma}[m\gamma^4]_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}$$

$$=\sum_{n=0}^{2s}(-1)^n m^{2s-n}C^n_{2s}\overbrace{(\gamma^a\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^b\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{n}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}\overbrace{\partial_a\partial_b\cdots}^{n}$$

Lem. 5.4.2. $\overbrace{[\theta(t),[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})]}^{2s}_{-^{2s+1}}$

$$=\sum_{n=0}^{2s}(-1)^n m^{2s-n}C^n_{2s}[\theta(t),\overbrace{(\gamma^a\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^b\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{n}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}\overbrace{\partial_a\partial_b\cdots}^{n}$$

$$=\sum_{n=0}^{2s}(-1)^n m^{2s-n}C^n_{2s}\sum_{l=0}^{n-1}C^l_n\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}[\partial^{n-l}_\pi\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{l}$$

$$=\sum_{n=0}^{2s}\sum_{l=0}^{n-1}(-1)^n m^{2s-n}C^n_{2s}C^l_n\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}[\partial^{n-l}_\pi\theta(t)]\overbrace{\partial_i\partial_j\cdots}^{l}$$

$$=\sum_{n=0}^{2s}\sum_{l=0}^{n-1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}[\partial^{n-1-l}_t\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{l}$$

Cor. 5.4.1.

$$\begin{cases}\Delta_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x)\\ \Delta^{(+)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(+)}(x)\\ \Delta^{(-)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(-)}(x)\\ \Delta^{(l)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(l)}(x)\end{cases}$$

Cor. 5.4.2.

$$\begin{cases}\Delta^{(c)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(c)}(x)\\ +\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}[\partial_t^{n-1-l}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{l}\Delta(x)\\ \Delta^{(F)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta_F(x)\\ +i\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}[\partial_t^{n-1-l}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{l}\Delta(x)\\ =i\Delta^{(c)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)\end{cases}$$

Cor. 5.4.3.

$$\begin{cases}\Delta^{(ret)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(ret)}(x)\\ +\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}[\partial_t^{n-1-l}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{l}\Delta(x)\\ \Delta^{(adv)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(adv)}(x)\\ +\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}[\partial_t^{n-1-l}\delta(t)]\overbrace{\partial_i\partial_j\cdots}^{l}\Delta(x)\end{cases}$$

Lem. 5.4.3. $\Delta(x)\partial_t^n\delta(t)=\sum_{l=0}^{[(n-1)/2]}C_n^{2l+1}(\nabla^2-m^2)^l\partial_t^{n-2l-1}\delta^4(x)$

Cor. 5.4.4. $\Delta(x)\partial_t^{n-1-l}\delta(t)=\sum_{r=0}^{[(n-l-2)/2]}C_{n-1-l}^{2r+1}(\nabla^2-m^2)^r\partial_t^{n-l-2-2r}\delta^4(x)$

Cor. 5.4.5.

$$\begin{cases}\Delta^{(c)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(c)}(x)+\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-2}\sum_{r=0}^{[(n-l-2)/2]}\\ C_{n-l-1}^{2r+1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}\overbrace{\partial_i\partial_j\cdots}^{l}(\nabla^2-m^2)^r\partial_t^{n-l-2-2r}\delta^4(x)\\ \Delta^{(F)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta_F(x)+i\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-2}\sum_{r=0}^{[(n-l-2)/2]}\\ C_{n-l-1}^{2r+1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}\overbrace{\partial_i\partial_j\cdots}^{l}(\nabla^2-m^2)^r\partial_t^{n-l-2-2r}\delta^4(x)\\ =i\Delta^{(c)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)\\ \Delta^{(F)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;p)=-i\frac{2^{1-2s}}{[(2s)!]^2}\frac{\overbrace{[(m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}}{p^2+m^2-i\varepsilon}+\cdots\end{cases}$$

Cor. 5.4.6.

$$\begin{cases}\Delta^{(ret)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(ret)}(x)+\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-2}\sum_{r=0}^{[(n-l-2)/2]}\\ C^{2r+1}_{n-l-1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}\overbrace{\partial_i\partial_j\cdots}^{l}(\nabla^2-m^2)^r\partial_t^{n-l-2-2r}\delta^4(x)\\ \Delta^{(adv)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x):=\frac{2^{1-2s}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(adv)}(x)+\frac{2^{1-2s}}{[(2s)!]^2}\sum_{n=0}^{2s}\sum_{l=0}^{n-2}\sum_{r=0}^{[(n-l-2)/2]}\\ C^{2r+1}_{n-l-1}\frac{i^{n+l}m^{2s-n}(2s)!}{l!(n-l)!(2s-n)!}\overbrace{(\gamma^i\gamma^4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\gamma^j\gamma^4)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{n-l}\overbrace{(\gamma^4)_{\eta_\varsigma\eta'_\varsigma}(\gamma^4)_{\xi_\varsigma\xi'_\varsigma}\cdots\})}^{2s-n}\overbrace{\partial_i\partial_j\cdots}^{l}(\nabla^2-m^2)^r\partial_t^{n-l-2-2r}\delta^4(x)\end{cases}$$

Lem. 5.4.4. $\Delta_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)|_{t=0}$

$$=\frac{-i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\sum_{l=0}^{[s-\frac{1}{2}]}[C^{2l+1}_{2s}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\delta_{\tau_\varsigma\}\tau'_\varsigma)}](m^2-\nabla^2)^l\delta^3(\vec{r})$$

Cor. 5.4.7.

$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=0\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(+)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=0\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(-)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=0\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(l)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=0\end{cases}\quad\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(c)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=-i\gamma^4\delta(t)\Delta_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)|_{t=0}\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(ret)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=-i\gamma^4\delta(t)\Delta_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)|_{t=0}\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(adv)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=-i\gamma^4\delta(t)\Delta_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)|_{t=0}\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(F)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)=\gamma^4\delta(t)\Delta_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(s;x)|_{t=0}\end{cases}$$

Cor. 5.4.8.

$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=0\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(+)}_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=0\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(-)}_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=0\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(l)}_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=0\end{cases}\quad\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(c)}_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=-\gamma^4\delta^4(x)\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(ret)}_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=-\gamma^4\delta^4(x)\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(adv)}_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=-\gamma^4\delta^4(x)\\ (\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\Delta^{(F)}_{\lambda_\varsigma\lambda'_\varsigma}(\frac{1}{2};x)=-i\gamma^4\delta^4(x)\end{cases}$$

5.5 Corollaries of B-W covariant quantization rules under separable representation

Def. 5.5.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}=\Gamma^{K_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}\psi_{K_\varsigma}(s)$

Cor. 5.5.1.

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')$$

$$\Rightarrow[\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x-x')$$

Proof:

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')$$

$$\Leftrightarrow[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}},\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$$

$$=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[-imI\otimes\sigma(x)+(\sigma\otimes\sigma_z,i\varsigma)^a\partial_a]_{\{\lambda_\varsigma(\lambda'_\varsigma}[-imI\otimes\sigma(x)+(\sigma\otimes\sigma_z,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')$$

$$\Rightarrow[\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x-x')$$

$$\Leftrightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma \cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = i\frac{(i\varsigma)^{2s}}{2^{2s-1}} \overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s} \overbrace{\partial_a\partial_b\cdot\cdot}^{2s} \Delta(x-x') \qquad \square$$

5.6 Equivalence proof on two descriptions of commutation rules for B-W equation

Lem. 5.6.1. $2\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'} + \frac{p_a p^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p) = [(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m - i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}$

Lem. 5.6.2. $2\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)(\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\Delta(x-x') = [(m - \gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m - \gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$

Thm. 5.6.1.

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{2n-1}}\frac{1}{[(2n)!]^2} \underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$$

$\Leftrightarrow$

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{3n-1}}\frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot\}}_{n} \underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot)}_{n} \underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

Proof:

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]$$

$$= \frac{i}{2^{2n-1}[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$$

$$= \frac{i}{2^{4n-1}[(2n)!]^2}\underbrace{\{[(m-\gamma^a\partial_a)\gamma^4]_{\{\{\lambda_\varsigma((\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\}\cdot\cdot\})}_{n}\Delta(x-x')$$

$$= \frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}][\eta_{bb'} - \frac{\partial_b\partial^+_{b'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

$$= \frac{i}{2^{3n-1}(n!)^2[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{bb'} - \frac{\partial_b\partial^+_{b'}}{m^2}]\cdot\cdot\})}_{n}\Delta(x-x') \qquad \square$$

Thm. 5.6.2.

$$\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$$

$$= \frac{i}{2^{2n}}\frac{1}{[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')$$

$\Leftrightarrow$

$$\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$$

$$= \frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

Proof:

$$\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$$

$$= \frac{i}{2^{2n}[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')$$

$$= \frac{i}{2^{4n}[(2n+1)!]^2}\underbrace{\{[(m-\gamma^a\partial_a)\gamma^4]_{\{\{\lambda_\varsigma((\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\}\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\Delta(x-x')$$

$$= \frac{i}{2^{3n}[(2n+1)!]^2}$$

$$\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{bb'} - \frac{\partial_b\partial^+_{b'}}{m^2}]\cdot\cdot\})}_{n}\Delta(x-x')$$

$$= \frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x') \qquad \square$$

5.7 Summary of massive boson commutation rules

Thm. 5.7.1. $n \geq 0$

$[a(\vec{p},h;n),a^{+}(\vec{p}',h';n)] = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[b(\vec{p},h;n),b^{+}(\vec{p}',h';n)] = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[rest]=0$

$\Leftrightarrow [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{2n-1}}\frac{1}{[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x'),[rest]=0$

$\Leftrightarrow [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{3n-1}}\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^{+}_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x'),[rest]=0$

5.8 Summary of massive fermion anticommutation rules

Thm. 5.8.1. $n \geq 0$

$\{a(\vec{p},h;n+\frac{1}{2}),a^{+}(\vec{p}',h';n+\frac{1}{2})\} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),\{b(\vec{p},h;n+\frac{1}{2}),b^{+}(\vec{p}',h';n+\frac{1}{2})\} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'),\{rest\}=0$

$\Leftrightarrow \{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$

$= \frac{i}{2^{2n}}\frac{1}{[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x'),\{rest\}=0$

$\Leftrightarrow$

$\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$

$= \frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^{+}_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x'),\{rest\}=0$

6 Extraction of various quantum operators for Bargmann-Wigner equation [16]

6.1 Isochronous commutation rules for Bargmann-Wigner equation

Thm. 6.1.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}}_{2s}}(\vec{r},t)$

$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^{+}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$

$$\begin{cases} a(\vec{p},h) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\frac{m}{E}}^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r} \\ b^{+}(\vec{p},s) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\frac{m}{E}}^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{ip\cdot x}d^3\vec{r} \end{cases}$$

Thm. 6.1.2. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{r}',t)]_{-^{2s+1}}$

$= \frac{1}{2^{2s-1}}\frac{1}{[(2s)!]^2}\sum\limits_{l=0}^{[s-\frac{1}{2}]}[C^{2l+1}_{2s}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2s-2l-1}\delta_{\tau_\varsigma\}\tau'_\varsigma)}](m^2-\nabla^2)^l\delta^3(\vec{r}-\vec{r}')$

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{r}',t)]_{-^{2s+1}}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^b\partial_b)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}^{2s}\Delta(x-x')|_{t=t'}$

$= \frac{1}{2^{2s-1}}\frac{1}{[(2s)!]^2}\sum\limits_{l=0}^{[s-\frac{1}{2}]}[C^{2l+1}_{2s}\overbrace{(m\gamma^4-\vec{\gamma}\gamma^4\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4-\vec{\gamma}\gamma^4\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2s-2l-1}\delta_{\tau_\varsigma\}\tau'_\varsigma)}](m^2-\nabla^2)^l\delta^3(\vec{r}-\vec{r}')$

$= \frac{1}{2^{2s-1}}\frac{1}{[(2s)!]^2}\sum\limits_{l=0}^{[s-\frac{1}{2}]}[C^{2l+1}_{2s}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2s-2l-1}\delta_{\tau_\varsigma\}\tau'_\varsigma)}](m^2-\nabla^2)^l\delta^3(\vec{r}-\vec{r}')$ □

6.2 Extraction of energy operators for Bargmann-Wigner equation

Lem. 6.2.1. $\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}$

$= \sum\limits_{l=0}^{2s}C^l_{2s}E^l\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2s-l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdot\cdot\})}^{l}$

Lem. 6.2.2. $\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$

$$=\sum_{l=0}^{2s}(-1)^l C_{2s}^l E^l\,\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{l}$$

Lem. 6.2.3.

$$\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$$

$$+\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$$

$$=2\sum_{l=0}^{[s]}C_{2s}^{2l}E^{2l}\,\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l}$$

Lem. 6.2.4.

$$\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$$

$$-\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$$

$$=2\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}E^{2l+1}\,\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}$$

Thm. 6.2.1.

$$H(s)=\int\sum_{h=s}^{-s}E[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s}}{(m^2-\nabla^2)^{2s-l}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$$

Proof: $\int\sum_{h=s}^{-s}E[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$

$$=\frac{1}{(2\pi)^3}\int\frac{m^{2s}}{E^{4s-2}}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$

$$\sum_{h=s}^{-s}[U^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}+(-1)^{2s}V^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h)e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$

$$=\frac{1}{(2\pi)^3}\int\frac{m^{2s}}{E^{4s-2}}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$

$$\sum_{h=s}^{-s}[U^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h)+(-1)^{2s}V^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(-\vec{p},h)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}\frac{m^{2s}}{E^{4s-2}}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)\frac{1}{(2m)^{2s}[(2s)!]^2}$$

$$\{\overbrace{[(m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}+\overbrace{[(m-i\gamma^a p_a^+)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b^+)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$

$$=\frac{1}{2^{2s}[(2s)!]^2}\frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)\frac{1}{E^{4s-2}}$$

$$\{\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$$

$$+\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$

$$=\frac{1}{2^{2s-1}[(2s)!]^2}\frac{1}{(2\pi)^3}\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)\frac{1}{E^{4s-2}}$$

$$\sum_{l=0}^{[s]}C_{2s}^{2l}E^{2l}\,\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$

$$=\frac{1}{2^{2s-1}[(2s)!]^2}\frac{1}{(2\pi)^3}\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$

$$\sum_{l=0}^{[s]}C_{2s}^{2l}\frac{1}{(E^2)^{2s-1-l}}\,\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$

$$= \frac{1}{2^{2s-1}[(2s)!]^2}\frac{1}{(2\pi)^3}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$
$$\sum_{l=0}^{[s]} C_{2s}^{2l}\frac{1}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$
$$= \frac{1}{2^{2s-1}[(2s)!]^2}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$
$$\sum_{l=0}^{[s]} C_{2s}^{2l}\frac{1}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$$
$$= \frac{1}{2^{2s-1}[(2s)!]^2}$$
$$\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s]} C_{2s}^{2l}\frac{\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s]} C_{2s}^{2l}\frac{\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\lambda_\varsigma\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{2l}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s]} C_{2s}^{2l}\frac{(i\partial_t)^{2s-2l}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{2l}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s]} C_{2s}^{2l}\frac{(i\partial_t)^{2s-2l}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\sum_{l=0}^{[s]} C_{2s}^{2l}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-2l}}{(m^2-\nabla^2)^{2s-1-l}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\sum_{l=0}^{[s]} C_{2s}^{2l}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} \qquad \square$$

Thm. 6.2.2.

$$H(n) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2n}}(\vec{r},t)\frac{1}{(m^2-\nabla^2)^{n-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{r},t)d^3\vec{r},\ H(n+\tfrac{1}{2}) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2n+1}}(\vec{r},t)\frac{i\partial_t}{(m^2-\nabla^2)^{n}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(\vec{r},t)d^3\vec{r}$$

6.3 Extraction of momentum operators for Bargmann-Wigner equation

Thm. 6.3.1.

$$P(s) = \int\sum_{h=s}^{-s}\vec{p}[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p} = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-l}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$$

Proof: $\int\sum_{h=s}^{-s}\vec{p}[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$

$$= \frac{1}{(2\pi)^3}\int\frac{m^{2s}}{E^{4s-1}}\vec{p}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$
$$\sum_{h=s}^{-s}[U^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}+(-1)^{2s}V^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h)e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$
$$= \frac{1}{(2\pi)^3}\int\frac{m^{2s}}{E^{4s-1}}\vec{p}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$
$$\sum_{h=s}^{-s}[U^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)U^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h)-(-1)^{2s}V^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(-\vec{p},h)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$
$$= \frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}\frac{m^{2s}}{E^{4s-1}}\vec{p}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)\frac{1}{(2m)^{2s}[(2s)!]^2}$$
$$\{\overbrace{[(m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}-\overbrace{[(m-i\gamma^a p_a^+)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b^+)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$

$$= \frac{1}{2^{2s}[(2s)!]^2}\frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)\frac{1}{E^{4s-1}}\vec{p}$$
$$\{\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})+E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}$$
$$-\overbrace{[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})-E]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$
$$= \frac{1}{2^{2s-1}[(2s)!]^2}\frac{1}{(2\pi)^3}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)\frac{1}{E^{4s-1}}\vec{p}$$
$$\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}E^{2l+1}\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$
$$= \frac{1}{2^{2s-1}[(2s)!]^2}\frac{1}{(2\pi)^3}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$
$$\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{\vec{p}}{(E^2)^{2s-1-l}}\overbrace{(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$
$$= \frac{1}{2^{2s-1}[(2s)!]^2}\frac{1}{(2\pi)^3}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$
$$\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{-i\nabla}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$$
$$= \frac{1}{2^{2s-1}[(2s)!]^2}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)$$
$$\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{-i\nabla}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$$
$$= \frac{1}{2^{2s-1}[(2s)!]^2}$$
$$\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{-i\nabla\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{-i\nabla\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\lambda_\varsigma\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{2l+1}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{-i\nabla(i\partial_t)^{2s-2l-1}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots}^{2l+1}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{-i\nabla(i\partial_t)^{2s-2l-1}}{(m^2-\nabla^2)^{2s-1-l}}\overbrace{\delta_{\lambda_\varsigma\lambda'_\varsigma}\delta_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\nabla(i\partial_t)^{2s-2l-1}}{(m^2-\nabla^2)^{2s-1-l}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \frac{1}{2^{2s-1}}\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$$
$$= \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} \qquad \square$$

Thm. 6.3.2.
$$P(n)=\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2n}}(\vec{r},t)\frac{(-i\nabla)(i\partial_t)}{(m^2-\nabla^2)^n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{r},t)d^3\vec{r},\ P(n+\tfrac{1}{2})=\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2n+1}}(\vec{r},t)\frac{-i\nabla}{(m^2-\nabla^2)^n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(\vec{r},t)d^3\vec{r}$$

6.4 Summary of energy momentum operators for Bargmann-Wigner equation

Thm. 6.4.1. $P_u(s)=\int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$

6.5 Various physical operators for Bargmann-Wigner equation

Thm. 6.5.1.

$$P_u(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} = \int\sum_h p_u[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$$

Proof: $P_u(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$

$$= \int \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{h'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec{p}',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'$$

$$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_h E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{p_uE^{2s-1}}{(E^2)^{2s-1}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s}b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}d^3\vec{r}$$

$$= \frac{1}{(2\pi)^3}\int\sum_{h,h'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}[a^+(\vec{p}',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$$

$$p_u[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s}b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}'d^3\vec{p}d^3\vec{r}$$

$$= \int d^3\vec{p}'d^3\vec{p}\sum_{h,h'}(\frac{m}{E})^{2s}p_u$$

$$\{\delta^3(\vec{p}-\vec{p}')[a^+(\vec{p},h')a(\vec{p},h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)+(-1)^{2s}b(\vec{p},h')b^+(\vec{p},h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)]$$

$$+\delta^3(\vec{p}+\vec{p}')[(-1)^{2s}e^{2iEt}a^+(-\vec{p},h')b^+(\vec{p},h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)$$

$$+e^{-2iEt}b(\vec{p},h')a(\vec{p},h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)]\}$$

$$= \int\sum_h p_u[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$$ □

Thm. 6.5.2.

$$Q(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} = \int\sum_h[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s-1}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$$

Proof: $Q(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$

$$= \int \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{h'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec{p}',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'$$

$$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_h E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{E^{2s-1}}{(E^2)^{2s-1}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s-1}b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}d^3\vec{r}$$

$$= \frac{1}{(2\pi)^3}\int\sum_{h,h'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}[a^+(\vec{p}',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$$

$$[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s-1}b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}'d^3\vec{p}d^3\vec{r}$$

$$= \int d^3\vec{p}'d^3\vec{p}\sum_{h,h'}(\frac{m}{E})^{2s}$$

$$\{\delta^3(\vec{p}-\vec{p}')[a^+(\vec{p},h')a(\vec{p},h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)+(-1)^{2s-1}b(\vec{p},h')b^+(\vec{p},h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)]$$

$$+\delta^3(\vec{p}+\vec{p}')[(-1)^{2s-1}e^{2iEt}a^+(-\vec{p},h')b^+(\vec{p},h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)$$

$$+e^{-2iEt}b(\vec{p},h')a(\vec{p},h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)]\}$$

$$= \int\sum_h[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s-1}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$$ □

Thm. 6.5.3.

$$N(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r=\int\sum_h[a^+(\vec p,h)a(\vec p,h)+(-1)^{2s}b(\vec p,h)b^+(\vec p,h)]d^3\vec p$$

Proof: $N(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec p'=-\infty}^{+\infty}\sum_{h'}E'^{s-\frac12}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec p',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{-i(\vec p'\cdot\vec r-E't)}+b(\vec p',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{i(\vec p'\cdot\vec r-E't)}]d^3\vec p'$$
$$\frac{1}{(2\pi)^{N/2}}\int_{\vec p=-\infty}^{+\infty}\sum_h E^{s-\frac12}\sqrt{\frac{m}{E}}^{2s}\frac{E^{2s}}{E^{4s-1}}[a(\vec p,h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{ip\cdot x}+(-1)^{2s}b^+(\vec p,h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{-ip\cdot x}]d^3\vec p d^3\vec r$$
$$=\frac{1}{(2\pi)^3}\int\sum_{h,h'}(\frac{E'}{E})^{s-\frac12}\sqrt{\frac{m^2}{E'E}}^{2s}[a^+(\vec p',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{-i(\vec p'\cdot\vec r-E't)}+b(\vec p',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{i(\vec p'\cdot\vec r-E't)}]$$
$$[a(\vec p,h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{ip\cdot x}+(-1)^{2s}b^+(\vec p,h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{-ip\cdot x}]d^3\vec p'd^3\vec p d^3\vec r$$
$$=\int d^3\vec p'd^3\vec p\sum_{h,h'}(\frac{m}{E})^{2s}$$
$$\{\delta^3(\vec p-\vec p')[a^+(\vec p,h')a(\vec p,h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)+(-1)^{2s}b(\vec p,h')b^+(\vec p,h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)]$$
$$+\delta^3(\vec p+\vec p')[(-1)^{2s}e^{2iEt}a^+(-\vec p,h')b^+(\vec p,h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec p,h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)$$
$$+e^{-2iEt}b(\vec p,h')a(\vec p,h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec p,h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)]\}$$
$$=\int\sum_h[a^+(\vec p,h)a(\vec p,h)+(-1)^{2s}b(\vec p,h)b^+(\vec p,h)]d^3\vec p$$ □

Thm. 6.5.4.

$$\vec S(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r=\int\sum_h\hat p[a^+(\vec p,h)a(\vec p,h)+(-1)^{2s}b(\vec p,h)b^+(\vec p,h)]d^3\vec p$$

Proof: $\vec S(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec p'=-\infty}^{+\infty}\sum_{h'}E'^{s-\frac12}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec p',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{-i(\vec p'\cdot\vec r-E't)}+b(\vec p',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{i(\vec p'\cdot\vec r-E't)}]d^3\vec p'$$
$$\frac{1}{(2\pi)^{N/2}}\int_{\vec p=-\infty}^{+\infty}\sum_h E^{s-\frac12}\sqrt{\frac{m}{E}}^{2s}\frac{\hat pE^{2s-1}}{(E^2)^{2s-1}}[a(\vec p,h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{ip\cdot x}+(-1)^{2s}b^+(\vec p,h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{-ip\cdot x}]d^3\vec p d^3\vec r$$
$$=\frac{1}{(2\pi)^3}\int\sum_{h,h'}(\frac{E'}{E})^{s-\frac12}\sqrt{\frac{m^2}{E'E}}^{2s}\hat p[a^+(\vec p',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{-i(\vec p'\cdot\vec r-E't)}+b(\vec p',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p',h')e^{i(\vec p'\cdot\vec r-E't)}]$$
$$[a(\vec p,h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{ip\cdot x}+(-1)^{2s}b^+(\vec p,h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)e^{-ip\cdot x}]d^3\vec p'd^3\vec p d^3\vec r$$
$$=\int d^3\vec p'd^3\vec p\sum_{h,h'}(\frac{m}{E})^{2s}\hat p$$
$$\{\delta^3(\vec p-\vec p')[a^+(\vec p,h')a(\vec p,h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)+(-1)^{2s}b(\vec p,h')b^+(\vec p,h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)]$$
$$+\delta^3(\vec p+\vec p')[(-1)^{2s}e^{2iEt}a^+(-\vec p,h')b^+(\vec p,h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec p,h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)$$
$$+e^{-2iEt}b(\vec p,h')a(\vec p,h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec p,h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p,h)]\}$$
$$=\int\sum_h\hat p[a^+(\vec p,h)a(\vec p,h)+(-1)^{2s}b(\vec p,h)b^+(\vec p,h)]d^3\vec p$$ □

Thm. 6.5.5.

$$\vec{M}(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}=\int\sum_h\hat{p}[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s-1}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$$

Proof: $\vec{M}(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$

$$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{h'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec{p}',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^3\vec{p}'$$

$$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_h E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{\hat{p}E^{2s}}{E^{4s-1}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s-1}b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}d^3\vec{r}$$

$$=\frac{1}{(2\pi)^3}\int\sum_{h,h'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}\hat{p}[a^+(\vec{p}',h')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',h')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',h')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$$

$$[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s-1}b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}'d^3\vec{p}d^3\vec{r}$$

$$=\int d^3\vec{p}'d^3\vec{p}\sum_{h,h'}(\frac{m}{E})^{2s}\hat{p}$$

$$\{\delta^3(\vec{p}-\vec{p}')[a^+(\vec{p},h')a(\vec{p},h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)+(-1)^{2s-1}b(\vec{p},h')b^+(\vec{p},h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)]$$

$$+\delta^3(\vec{p}+\vec{p}')[(-1)^{2s-1}e^{2iEt}a^+(-\vec{p},h')b^+(\vec{p},h)U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)$$

$$+e^{-2iEt}b(\vec{p},h')a(\vec{p},h)V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},h)]\}$$

$$=\int\sum_h\hat{p}[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s-1}b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$$ □

6.6 Quantum equation of Bargmann-Wigner equation

Thm. 6.6.1.

$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{r},t)=0\\ P_u(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}\end{cases}\Rightarrow -i\partial_u\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{r},t)=[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{r},t),P_u]$$

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t),P_u(s)]$

$$=[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t),\int\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{r}',t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)d^3\vec{r}']$$

$$=\int[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{r}',t)]_{-2s+1}\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)d^3\vec{r}'$$

$$=\int\frac{1}{2^{2s-1}}\frac{1}{[(2s)!]^2}\sum_{l=0}^{[s-\frac{1}{2}]}[C_{2s}^{2l+1}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}]$$

$$(m^2-\nabla^2)^l\delta^3(\vec{r}-\vec{r}')\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-l}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r}',t)d^3\vec{r}'$$

$$=\frac{1}{2^{2s-1}}\frac{1}{[(2s)!]^2}\sum_{l=0}^{[s-\frac{1}{2}]}[C_{2s}^{2l+1}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdots}^{2s-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdots\})}^{2l+1}]\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-l-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{r},t)$$

$$=\frac{1}{2^{2s-1}}\frac{1}{[(2s)!]^2}\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}(i\partial_t)^{2s-2l-1}\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-l-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)$$

$$=\frac{1}{2^{2s-1}}\sum_{l=0}^{[s-\frac{1}{2}]}C_{2s}^{2l+1}\frac{-i\partial_u(i\partial_t)^{4s-2}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)$$

$= -i\partial_u\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)$ □

6.7 Boson energy momentum operators for Bargmann-Wigner equation

Thm. 6.7.1.

$P_u(n)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2n}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)}{(m^2-\nabla^2)^n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t)d^3\vec{r},P_u(n+\frac{1}{2})=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2n+1}}(\vec{r},t)\frac{-i\partial_u}{(m^2-\nabla^2)^n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n+1}}(\vec{r},t)d^3\vec{r}$

Thm. 6.7.2.
$$\begin{cases}P_u(n)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2n}}(\vec{r},t)\frac{[-i\nabla,i\gamma^4(\vec{\gamma}\cdot\nabla+m)]\gamma^4(\vec{\gamma}\cdot\nabla+m)}{(m^2-\nabla^2)^n}{}_{\lambda_\varsigma}{}^{\eta_\varsigma}\psi_{\underbrace{\eta_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t)d^3\vec{r}\\ =\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2n}}(\vec{r},t)\frac{[-i\nabla\gamma^4(\vec{\gamma}\cdot\nabla+m),i(m^2-\nabla^2)]}{(m^2-\nabla^2)^n}{}_{\lambda_\varsigma}{}^{\eta_\varsigma}\psi_{\underbrace{\eta_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t)d^3\vec{r}\\ P_u(n+\frac{1}{2})=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2n+1}}(\vec{r},t)\frac{[-i\nabla,i\gamma^4(\vec{\gamma}\cdot\nabla+m)]}{(m^2-\nabla^2)^n}{}_{\lambda_\varsigma}{}^{\eta_\varsigma}\psi_{\underbrace{\eta_\varsigma\mu_\varsigma\cdot\cdot}_{2n+1}}(\vec{r},t)d^3\vec{r}\end{cases}$$

6.8 Boson quantum equation of Bargmann-Wigner equation

Thm. 6.8.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n}}(\vec{r},t)=0\Rightarrow-i\partial_u\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t)=[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),P_u(n)]$

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),H]$

$=[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),\int\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{r}',t)\frac{1}{(m^2-\nabla'^2)^{n-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)d^3\vec{r}']$

$=\int[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{r}',t)\frac{1}{(m^2-\nabla'^2)^{n-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)]d^3\vec{r}'$

$=\int[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{r}',t)]\frac{1}{(m^2-\nabla'^2)^{n-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)d^3\vec{r}'$

$=\int\frac{1}{2^{2n-1}}\frac{1}{[(2n)!]^2}\sum_{l=0}^{[n-\frac{1}{2}]}[C_{2n}^{2l+1}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2n-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdot\cdot\})}^{2l+1}]$

$(m^2-\nabla^2)^l\delta^3(\vec{r}-\vec{r}')\frac{1}{(m^2-\nabla'^2)^{n-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)d^3\vec{r}'$

$=\frac{1}{2^{2n-1}}\frac{1}{[(2n)!]^2}\sum_{l=0}^{[n-\frac{1}{2}]}[C_{2n}^{2l+1}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2n-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdot\cdot\})}^{2l+1}]\frac{1}{(m^2-\nabla^2)^{n-l-1}}\psi^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r},t)$

$=i\partial_t\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t)$ □

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),P]$

$=[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),\int\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{r}',t)\frac{-i\nabla'\gamma^4(\vec{\gamma}\cdot\nabla'+m)}{(m^2-\nabla'^2)^n}{}^{\lambda'_\varsigma}{}_{\eta'_\varsigma}\psi^{\overbrace{\eta'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)d^3\vec{r}']$

$=\int[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{r}',t)\frac{-i\nabla'\gamma^4(\vec{\gamma}\cdot\nabla'+m)}{(m^2-\nabla'^2)^n}{}^{\lambda'_\varsigma}{}_{\eta'_\varsigma}\psi^{\overbrace{\eta'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)]d^3\vec{r}'$

$=\int[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{r}',t)]\frac{-i\nabla'\gamma^4(\vec{\gamma}\cdot\nabla'+m)}{(m^2-\nabla'^2)^n}{}^{\lambda'_\varsigma}{}_{\eta'_\varsigma}\psi^{\overbrace{\eta'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)d^3\vec{r}'$

$=\int\frac{1}{2^{2n-1}}\frac{1}{[(2n)!]^2}\sum_{l=0}^{[n-\frac{1}{2}]}[C_{2n}^{2l+1}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2n-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdot\cdot\})}^{2l+1}]$

$(m^2-\nabla^2)^l\delta^3(\vec{r}-\vec{r}')\frac{-i\nabla'\gamma^4(\vec{\gamma}\cdot\nabla'+m)}{(m^2-\nabla'^2)^n}{}^{\lambda'_\varsigma}{}_{\eta'_\varsigma}\psi^{\overbrace{\eta'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r}',t)d^3\vec{r}'$

$$= \frac{1}{2^{2n-1}}\frac{1}{[(2n)!]^2}$$

$$\sum_{l=0}^{[n-\frac{1}{2}]}[C_{2n}^{2l+1}\overbrace{(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2n-2l-1}\overbrace{\delta_{\rho_\varsigma\rho'_\varsigma}\delta_{\tau_\varsigma\tau'_\varsigma}\cdot\cdot\})}^{2l+1}]\frac{-i\nabla\gamma^4(\vec{\gamma}\cdot\nabla+m)}{(m^2-\nabla^2)^{n-l}}{\lambda'_\varsigma}_{\eta'_\varsigma}\psi^{\overbrace{\eta'_\varsigma\mu'_\varsigma\cdot\cdot}^{2n}}(\vec{r},t)$$

$$= -i\nabla\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t) \qquad \square$$

Chapter25 Potential Covariation Scheme for Complex Particles with Mass

1 Commutation rules for Klein-Gordon equation

1.1 B-W equation is equivalent to K-G equation for spin-n particles with mass [16, 20, 21]

Def. 1.1.1. $\mathbb{X}_a = [im\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)\partial^b]C, \mathbb{X}_a(p) = i[m\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)p^b]C$

Thm. 1.1.1.

$$\begin{cases}[\gamma^a(\varsigma)\partial_a + m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(x) = 0\\ \psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(x)\ \textit{fully symmetric}\end{cases} \Leftrightarrow \begin{cases}(-\partial^c\partial_c + m^2)A_{\underbrace{ab\cdot\cdot}_{n}}(x) = 0\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}}(x) = 0, \partial^a A_{\underbrace{ab\cdot\cdot}_{n}}(x) = 0, A_{\underbrace{ab\cdot\cdot}_{n}}(x)\ \textit{fully symmetric}\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(x) = \frac{1}{2^n}\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n} A_{\underbrace{ab\cdot\cdot}_{n}}(x)\end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n} E^{n-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2n}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)e^{ip\cdot x} + b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$A_{\underbrace{ab\cdot\cdot}_{n}}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^n E}}[a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{ip\cdot x} + b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

Self comment: By substituting their respective plane wave solutions into the above two equivalent equations and using Fourier component equivalence, the following two corollaries can be easily obtained. This above equation is a macroscopic structure, while the below equation is a microscopic structure, which is a mathematical atom.

Cor. 1.1.1.

$$\begin{cases}(i\gamma^a p_a + m)U_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h) = 0\\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)\ \textit{fully symmetric}, \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\\ = \frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)\end{cases} \Leftrightarrow \begin{cases}(p^c p_c + m^2)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h) = 0, \delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h) = 0\\ p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h) = 0, \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\ \textit{fully symmetric}\\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)\\ = \frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot}_{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\end{cases}$$

Cor. 1.1.2.

$$\begin{cases}(-i\gamma^a p_a + m)V_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h) = 0\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)\ \textit{fully symmetric}, \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\\ = \frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)\end{cases} \Leftrightarrow \begin{cases}(p^c p_c + m^2)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h) = 0, \delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h) = 0\\ p^a\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h) = 0, \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\ \textit{fully symmetric}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)\\ = \frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(-p)\cdot\cdot}_{n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\end{cases}$$

1.2 B-W equation bose basis decomposes into spin-1 bases

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n)$$
$$= \frac{1}{\sqrt{(2n)!(2n)!(0)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{2n}\underbrace{\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{0}$$
$$= \underbrace{u_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})}_{2n}$$

$$= \underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n} = \frac{1}{\sqrt{(n!n!0!}}(\frac{1}{2\sqrt{2}m})^n \underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\cdot\cdot\mathbb{X}^d_{\sigma_\varsigma\tau_\varsigma}(p)}_{n}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot\varepsilon_{d\}}(\vec{p},1)}_{n}$$
$$= \frac{1}{n!\sqrt{C^0_{2n}}}(\frac{1}{2\sqrt{2}m})^n \underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\cdot\cdot\mathbb{X}^d_{\sigma_\varsigma\tau_\varsigma}(p)}_{n}\sqrt{2^0}C^0_nC^0_{n-0}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot\varepsilon_{d\}}(\vec{p},1)}_{n} \qquad \square$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n-1)$$
$$= \frac{1}{\sqrt{(2n)!(2n-1)!(1)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{2n-1}\underbrace{\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{1}$$
$$= \frac{1}{\sqrt{(2n)!(2n-1)!(1)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{2n}$$
$$= \frac{1}{\sqrt{C^1_{2n}}}\{[\underbrace{u_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})}_{2n}+\underbrace{u_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})}_{2n}]$$
$$+\cdot\cdot$$
$$+[\underbrace{u_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},-\tfrac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})}_{2n}+\underbrace{u_{\lambda_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})}_{2n}]\}$$
$$= \frac{1}{\sqrt{C^1_n}}[\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)U_{\eta_\varsigma\xi_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},0)}_{n}+\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)U_{\eta_\varsigma\xi_\varsigma}(\vec{p},0)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}$$
$$+\cdot\cdot+\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)U_{\eta_\varsigma\xi_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}]$$
$$= \frac{1}{\sqrt{n!(n-1)!1!}}(\frac{1}{2\sqrt{2}m})^n\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot\mathbb{X}^d_{\sigma_\varsigma\tau_\varsigma}(p)}_{n}\underbrace{\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},1)\varepsilon_{d\}}(\vec{p},0)}_{n}$$
$$= \frac{1}{n!\sqrt{C^1_{2n}}}(\frac{1}{2\sqrt{2}m})^n\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot\mathbb{X}^d_{\sigma_\varsigma\tau_\varsigma}(p)}_{n}\sqrt{2^1}C^1_nC^0_{n-1}\underbrace{\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},1)\varepsilon_{d\}}(\vec{p},0)}_{n} \qquad \square$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n-2)$$
$$= \frac{1}{\sqrt{(2n)!(2n-2)!(2)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{2n-2}\underbrace{\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{2}$$
$$= \frac{1}{\sqrt{(2n)!(2n-2)!(2)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{2n}$$
$$= \frac{1}{\sqrt{C^2_{2n}}}C^{(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2})}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}$$
$$= \frac{1}{\sqrt{C^2_{2n}}}[C^{(\vec{p},-\frac{1}{2})(\vec{p},-\frac{1}{2})_{1,3,5,\cdot\cdot}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}+C^{(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2})|_{rest}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}]$$
$$= \frac{1}{\sqrt{C^2_{2n}}}[\sqrt{2^0}C^{(\vec{p},-1)}_{\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}}+\sqrt{2^2}C^{(\vec{p},0),(\vec{p},0)}_{\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}}]$$
$$= \frac{1}{n!\sqrt{C^2_{2n}}}(\frac{1}{2\sqrt{2}m})^n\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot\mathbb{X}^d_{\sigma_\varsigma\tau_\varsigma}(p)}_{n}$$
$$[\sqrt{2^0}C^0_nC^1_{n-0}\underbrace{\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},1)\varepsilon_{d\}}(\vec{p},-1)}_{n}+\sqrt{2^2}C^2_nC^0_{n-2}\underbrace{\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},0)\varepsilon_{d\}}(\vec{p},0)}_{n}] \qquad \square$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n-3)$$
$$= \frac{1}{\sqrt{(2n)!(2n-3)!(3)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{2n-3}\underbrace{\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{3}$$
$$= \frac{1}{\sqrt{(2n)!(2n-3)!(3)!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot u_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})u_{\xi_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{2n}$$
$$= \frac{1}{\sqrt{C^3_{2n}}}C^{(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2})}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}$$

$$= \frac{1}{\sqrt{C_{2n}^{3}}}[C^{(\vec{p},-\frac{1}{2})(\vec{p},-\frac{1}{2})_{1,3,5,\cdot\cdot},(\vec{p},-\frac{1}{2})}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}} + C^{(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2}),(\vec{p},-\frac{1}{2})|_{rest}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}]$$

$$= \frac{1}{\sqrt{C_{2n}^{3}}}[\sqrt{2}P^{(\vec{p},-1),(\vec{p},0)}_{\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}} + \sqrt{2^3}C^{(\vec{p},0),(\vec{p},0),(\vec{p},0)}_{\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}}]$$

$$= \frac{1}{n!\sqrt{C_{2n}^{3}}}(\frac{1}{2\sqrt{2}m})^n \underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot\mathbb{X}^d_{\sigma_\varsigma\tau_\varsigma}(p)}_{n}$$

$$[\sqrt{2^1}C_n^1C_{n-1}^1 \underbrace{\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},0)\varepsilon_{d\}}(\vec{p},-1)}_{n} + \sqrt{2^3}C_n^3C_{n-3}^0 \underbrace{\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},0)\cdot\cdot\varepsilon_c(\vec{p},0)\varepsilon_{d\}}(\vec{p},0)}_{n}] \qquad \square$$

General case:

Thm. 1.2.1.

$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n-2k) = \frac{1}{n!\sqrt{C_{2n}^{2k}}}(\frac{1}{2\sqrt{2}m})^n \sum_{l=0}^{k|(n-k)} \sqrt{2^{2l}}C_n^{2l}C_{n-2l}^{k-l} \underbrace{\mathbb{X}^{a_1}_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^{a_2}_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot\mathbb{X}^{a_n}_{\sigma_\varsigma\tau_\varsigma}(p)}_{n} \\ \varepsilon_{\{a_1}(\vec{p},-1)\cdot\cdot\varepsilon_{a_{k-l}}(\vec{p},-1)|\varepsilon_{a_{k-l+1}}(\vec{p},0)\cdot\cdot\varepsilon_{a_k}(\vec{p},0)|\varepsilon_{a_{k+1}}(\vec{p},1)\cdot\cdot\varepsilon_{a_n\}}(\vec{p},1) \\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n-2k-1) = \frac{1}{n!\sqrt{C_{2n}^{2k+1}}}(\frac{1}{2\sqrt{2}m})^n \sum_{l=0}^{k|(n-1-k)} \sqrt{2^{2l+1}}C_n^{2l+1}C_{n-2l-1}^{k-l} \underbrace{\mathbb{X}^{a_1}_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^{a_2}_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot\mathbb{X}^{a_n}_{\sigma_\varsigma\tau_\varsigma}(p)}_{n} \\ \varepsilon_{\{a_1}(\vec{p},-1)\cdot\cdot\varepsilon_{a_{k-l}}(\vec{p},-1)|\varepsilon_{a_{k-l+1}}(\vec{p},0)\cdot\cdot\varepsilon_{a_k}(\vec{p},0)|\varepsilon_{a_{k+1}}(\vec{p},1)\cdot\cdot\varepsilon_{a_n\}}(\vec{p},1) \end{cases}$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n-2k)$$

$$= \frac{1}{\sqrt{(2n)!(2n-2k)!(2k)!}} \underbrace{u_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{2n-2k}\underbrace{\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{2k}$$

$$= \frac{1}{\sqrt{C_{2n}^{2k}}} C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{2k}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}$$

$$= \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{l=0}^{k|(n-k)} C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{2k-2l}|^{D}_{1,3,\cdot\cdot}\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{2l}|^{S}_{1,3,\cdot\cdot}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}$$

$$= \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{l=0}^{k|(n-k)} \sqrt{2^{2l}} C^{\overbrace{(\vec{p},-1),\cdot\cdot,(\vec{p},-1)}^{k-l}|^{D}_{1,3,\cdot\cdot}\overbrace{(\vec{p},0),\cdot\cdot,(\vec{p},0)}^{2l}|^{S}_{1,3,\cdot\cdot}}_{\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}}$$

$$= \frac{1}{n!\sqrt{C_{2n}^{2k}}}(\frac{1}{2\sqrt{2}m})^n \sum_{l=0}^{k|(n-k)} \sqrt{2^{2l}}C_n^{2l}C_{n-2l}^{k-l} \underbrace{\mathbb{X}^{a_1}_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^{a_2}_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot\mathbb{X}^{a_n}_{\sigma_\varsigma\tau_\varsigma}(p)}_{n}$$

$$\varepsilon_{\{a_1}(\vec{p},-1)\cdot\cdot\varepsilon_{a_{k-l}}(\vec{p},-1)|\varepsilon_{a_{k-l+1}}(\vec{p},0)\cdot\cdot\varepsilon_{a_k}(\vec{p},0)|\varepsilon_{a_{k+1}}(\vec{p},1)\cdot\cdot\varepsilon_{a_n\}}(\vec{p},1) \qquad \square$$

Proof:

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},n-2k-1)$$

$$= \frac{1}{\sqrt{(2n)!(2n-2k-1)!(2k+1)!}} \underbrace{u_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot}_{2n-2k-1}\underbrace{\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{2k+1}$$

$$= \frac{1}{\sqrt{C_{2n}^{2k+1}}} C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{2k+1}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}$$

$$= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{l=0}^{k|(n-1-k)} C^{\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{2k-2l}|^{D}_{1,3,\cdot\cdot}\overbrace{(\vec{p},-\frac{1}{2}),\cdot\cdot,(\vec{p},-\frac{1}{2})}^{2l+1}|^{S}_{1,3,\cdot\cdot}}_{\underbrace{u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\cdot\cdot u_{\sigma_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})}_{2n}}$$

$$= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{l=0}^{k|(n-1-k)} \sqrt{2^{2l+1}} C^{\overbrace{(\vec{p},-1),\cdot\cdot,(\vec{p},-1)}^{k-l}|^{D}_{1,3,\cdot\cdot}\overbrace{(\vec{p},0),\cdot\cdot,(\vec{p},0)}^{2l+1}|^{S}_{1,3,\cdot\cdot}}_{\underbrace{U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)\cdot\cdot U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1)}_{n}}$$

$= \frac{1}{n!\sqrt{C_{2n}^{2k+1}}}(\frac{1}{2\sqrt{2m}})^n \sum_{l=0}^{k|(n-1-k)} \sqrt{2^{2l+1}} C_n^{2l+1} C_{n-2l-1}^{k-l} \underbrace{\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^{a_1}(p)\mathbb{X}_{\eta_\varsigma\xi_\varsigma}^{a_2}(p)\cdot\cdot\mathbb{X}_{\sigma_\varsigma\tau_\varsigma}^{a_n}(p)}_{n}$

$\varepsilon_{\{a_1}(\vec{p},-1)\cdot\cdot\varepsilon_{a_{k-l}}(\vec{p},-1)|\varepsilon_{a_{k-l+1}}(\vec{p},0)\cdot\cdot\varepsilon_{a_k}(\vec{p},0)|\varepsilon_{a_{k+1}}(\vec{p},1)\cdot\cdot\varepsilon_{a_n\}}(\vec{p},1)$ □

1.3 Klein-Gordon equation basis decomposition

Thm. 1.3.1. $\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} \varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h-h')\varepsilon_{\underbrace{c\cdot\cdot d}_{n'}}(\vec{p},h')$

Proof: $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n}}(\vec{p},h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(n-n')}}(\vec{p},h-h')U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n'}}(\vec{p},h')$

$\Rightarrow \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot(\bar{C}\gamma_b)^{\sigma_\varsigma\tau_\varsigma}(\bar{C}\gamma_c)^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot(\bar{C}\gamma_d)^{\sigma'_\varsigma\tau'_\varsigma}}^{n} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n}}(\vec{p},h)$

$= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot(\bar{C}\gamma_b)^{\sigma_\varsigma\tau_\varsigma}(\bar{C}\gamma_c)^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot(\bar{C}\gamma_d)^{\sigma'_\varsigma\tau'_\varsigma}}^{n} \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(n-n')}}(\vec{p},h-h')U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2n'}}(\vec{p},h')$

$\Leftrightarrow \varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} \varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h-h')\varepsilon_{\underbrace{c\cdot\cdot d}_{n'}}(\vec{p},h')$ □

Cor. 1.3.1.
$\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h) = \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1) + \frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)$

Thm. 1.3.2.
$\varepsilon_{a_1\cdot\cdot a_n}(\vec{p},h) = \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1}\cdot\cdot\sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)!\cdot\cdot(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)!\cdot\cdot(1-h_n)!}} \varepsilon_{a_1}(\vec{p},h_1)\cdot\cdot\varepsilon_{a_n}(\vec{p},h_n); h_1 := h - \sum_{i=2}^{n} h_i$

Proof: $\varepsilon_{a_1a_2\cdot\cdot a_n}(\vec{p},h) = \sum_{h_n=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h_n}C_{n-h}^{1-h_n}}}{\sqrt{C_{2n}^2}} \varepsilon_{a_1a_2\cdot\cdot a_{n-1}}(\vec{p},h-h_n)\varepsilon_{a_n}(\vec{p},h_n)$

$= \sum_{h_n=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h_n}C_{n-h}^{1-h_n}}}{\sqrt{C_{2n}^2}} \sum_{h_{n-1}=1}^{-1} \frac{\sqrt{C_{(n-1)+(h-h_n)}^{1+h_{n-1}}C_{(n-1)-(h-h_n)}^{1-h_{n-1}}}}{\sqrt{C_{2(n-1)}^2}} \varepsilon_{a_1a_2\cdot\cdot a_{n-2}}(\vec{p},h-h_n-h_{n-1})\varepsilon_{a_{n-1}}(\vec{p},h_{n-1})\varepsilon_{a_n}(\vec{p},h_n)$

$= \frac{\sqrt{2!2!(2n-4)!}}{\sqrt{(2n)!}} \sum_{h_n=1}^{-1}\sum_{h_{n-1}=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_n)!(1+h_{n-1})![(n+h)-(1+h_n)-(1+h_{n-1})]!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_n)!(1-h_{n-1})![(n-h)-(1-h_n)-(1-h_{n-1})]!}}$

$\varepsilon_{a_1a_2\cdot\cdot a_{n-2}}(\vec{p},h-h_n-h_{n-1})\varepsilon_{a_{n-1}}(\vec{p},h_{n-1})\varepsilon_{a_n}(\vec{p},h_n)$

$= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_n=1}^{-1}\sum_{h_{n-1}=1}^{-1}\cdot\cdot\sum_{h_2=1}^{-1} \varepsilon_{a_1}(\vec{p},h-h_n-h_{n-1}\cdot\cdot-h_2)\varepsilon_{a_2}(\vec{p},h_2)\cdot\cdot\varepsilon_{a_{n-1}}(\vec{p},h_{n-1})\varepsilon_{a_n}(\vec{p},h_n)$

$\frac{\sqrt{(n+h)!}}{\sqrt{(1+h_n)!(1+h_{n-1})!\cdot\cdot(1+h_2)![(n+h)-(1+h_n)-(1+h_{n-1})\cdot\cdot-(1+h_2)]!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_n)!(1-h_{n-1})!\cdot\cdot(1-h_2)![(n-h)-(1-h_n)-(1-h_{n-1})\cdot\cdot-(1-h_2)]!}}$

$= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1}\cdot\cdot\sum_{h_{n-1}=1}^{-1}\sum_{h_n=1}^{-1} \varepsilon_{a_1}(\vec{p},h_1)\varepsilon_{a_2}(\vec{p},h_2)\cdot\cdot\varepsilon_{a_{n-1}}(\vec{p},h_{n-1})\varepsilon_{a_n}(\vec{p},h_n)$

$\frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)!(1+h_2)!\cdot\cdot(1+h_{n-1})!(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)!(1-h_2)!\cdot\cdot(1-h_{n-1})!(1-h_n)!}}; h_1 := h - \sum_{i=2}^{n} h_i$

$= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1}\cdot\cdot\sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)!\cdot\cdot(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)!\cdot\cdot(1-h_n)!}} \varepsilon_{a_1}(\vec{p},h_1)\cdot\cdot\varepsilon_{a_n}(\vec{p},h_n); h_1 := h - \sum_{i=2}^{n} h_i$ □

Cor. 1.3.2. $\varepsilon_{a_1\cdot\cdot a_n}(\vec{p},h); h_1 := h - \sum_{i=2}^{n} h_i$

$= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1}\cdot\cdot\sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)!\cdot\cdot(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)!\cdot\cdot(1-h_n)!}} \varepsilon_{a_1}(\vec{p},h_1)\cdot\cdot\varepsilon_{a_n}(\vec{p},h_n)[\delta(h_1-1)+\delta(h_1)+\delta(h_1+1)]$

$= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_1=1}^{-1}\sum_{h_2=1}^{-1}\cdot\cdot\sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)!\cdot\cdot(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)!\cdot\cdot(1-h_n)!}} \varepsilon_{a_1}(\vec{p},h_1)\varepsilon_{a_2}(\vec{p},h_2)\cdot\cdot\varepsilon_{a_n}(\vec{p},h_n)\delta(h - \sum_{i=1}^{n} h_i)$

1.4 Strictly prove complete decomposition of K-G equation basis by mathematical induction

Thm. 1.4.1.

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n}}(\vec{p},n-2k)=\frac{1}{\sqrt{C_{2n}^{2k}}}\sum_{i=0}^{\min(k,n-k)}\frac{2^i}{(n-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}\\ \varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n}}(\vec{p},n-2k-1)=\frac{1}{\sqrt{C_{2n}^{2k+1}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^i\sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}\end{cases}$$

Proof: Using mathematical induction to prove this theorem.

Step 1: When $n'=1$, the following is established.

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{1}}(\vec{p},1-2k)=\frac{1}{\sqrt{C_{2}^{2k}}}\sum_{i=0}^{\min(k,1-k)}\frac{2^i}{(1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{1-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}\\ \varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{1}}(\vec{p},1-2k-1)=\frac{1}{\sqrt{C_{2}^{2k+1}}}\sum_{i=0}^{\min(k,1-1-k)}\frac{2^i\sqrt{2}}{(1-k-i-1)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{1-k-i-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}\end{cases}$$

Step 2: Assume when $n'=n-1$, the following is established.

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n-1}}(\vec{p},(n-1)-2k)=\frac{1}{\sqrt{C_{2n-2}^{2k}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^i}{(n-1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}\\ \varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n-1}}(\vec{p},(n-1)-2k-1)=\frac{1}{\sqrt{C_{2n-2}^{2k+1}}}\sum_{i=0}^{\min(k,n-1-1-k)}\frac{2^i\sqrt{2}}{(n-1-k-i-1)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1-k-i-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}\end{cases}$$

Step 3: When $n'=n$, 1:

$$n!\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot cd}_{n}}(\vec{p},h)$$

$$=\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},h-1)\varepsilon_{d\}}(\vec{p},1)+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},h)\varepsilon_{d\}}(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},h+1)\varepsilon_{d\}}(\vec{p},-1)$$

$$\Rightarrow n!\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot cd}_{n}}(\vec{p},n-2k)=\frac{\sqrt{C_{2n-2k}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},(n-1)-2k)\varepsilon_{d\}}(\vec{p},1)$$

$$+\frac{\sqrt{C_{2n-2k}^1C_{2k}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},(n-1)-2(k-1)-1)\varepsilon_{d\}}(\vec{p},0)+\frac{\sqrt{C_{2k}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},(n-1)-2(k-1))\varepsilon_{d\}}(\vec{p},-1)$$

$$\Leftrightarrow n!\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot cd}_{n}}(\vec{p},n-2k)$$

$$=\frac{\sqrt{C_{2n-2k}^2}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^i}{(n-1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{(a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c)}(\vec{p},-1)}_{k-i}\varepsilon_{d\}}(\vec{p},1)$$

$$+\frac{\sqrt{C_{2n-2k}^1C_{2k}^1}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k-1}}}\sum_{i=0}^{\min(k-1,n-1-1-k+1)}\frac{2^i\sqrt{2}}{(n-1-k+1-i-1)!(2i+1)!(k-1-i)!}\underbrace{\varepsilon_{\{(a}(\vec{p},1)\cdot\cdot}_{n-1-k+1-i-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c)}(\vec{p},-1)}_{k-1-i}\varepsilon_{d\}}(\vec{p},0)$$

$$+\frac{\sqrt{C_{2k}^2}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k-2}}}\sum_{i=0}^{\min(k-1,n-1-k+1)}\frac{2^i}{(n-1-k+1-i)!(2i)!(k-1-i)!}\underbrace{\varepsilon_{\{(a}(\vec{p},1)\cdot\cdot}_{n-1-k+1-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c)}(\vec{p},-1)}_{k-1-i}\varepsilon_{d\}}(\vec{p},-1)$$

$$n\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot cd}_{n}}(\vec{p},n-2k)=\frac{\sqrt{C_{2n-2k}^2}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^i}{(n-1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c}(\vec{p},-1)}_{k-i}\varepsilon_{d\}}(\vec{p},1)$$

$$+\frac{\sqrt{C_{2n-2k}^1C_{2k}^1}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k-1}}}\sum_{i=0}^{\min(k-1,n-1-k)}\frac{2^i\sqrt{2}}{(n-1-k-i)!(2i+1)!(k-1-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c}(\vec{p},-1)}_{k-1-i}\varepsilon_{d\}}(\vec{p},0)$$

$$+\frac{\sqrt{C_{2k}^2}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k-2}}}\sum_{i=0}^{\min(k-1,n-k)}\frac{2^i}{(n-k-i)!(2i)!(k-1-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c}(\vec{p},-1)}_{k-1-i}\varepsilon_{d\}}(\vec{p},-1)$$

$$\Leftrightarrow n\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n}}(\vec{p},n-2k)=\frac{\sqrt{C_{2n-2k}^2}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^i}{(n-1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$+\frac{\sqrt{C_{2n-2k}^1C_{2k}^1}}{\sqrt{C_{2n}^2}}\frac{1}{\sqrt{C_{2n-2}^{2k-1}}}\sum_{i=0}^{\min(k-1,n-1-k)}\frac{2^i\sqrt{2}}{(n-1-k-i)!(2i+1)!(k-1-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+2}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-1-i}$$

$$+\frac{\sqrt{C_{2k}^{2}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k-2}}}\sum_{i=0}^{\min(k-1,n-k)}\frac{2^{i}}{(n-k-i)!(2i)!(k-1-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$\Leftrightarrow n\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n}}(\vec{p},n-2k)=\frac{\sqrt{C_{2n-2k}^{2}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^{i}}{(n-1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$+\frac{\sqrt{C_{2n-2k}^{1}C_{2k}^{1}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k-1}}}\sum_{i=1}^{\min(k,n-k)}\frac{2^{i-1}\sqrt{2}}{(n-k-i)!(2i-1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$+\frac{\sqrt{C_{2k}^{2}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k-2}}}\sum_{i=0}^{\min(k-1,n-k)}\frac{2^{i}}{(n-k-i)!(2i)!(k-1-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$=\frac{n}{\sqrt{C_{2n}^{2k}}}\sum_{i=0}^{\min(k,n-k)}\frac{2^{i}}{(n-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$\Leftrightarrow \varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n}}(\vec{p},n-2k)=\frac{1}{\sqrt{C_{2n}^{2k}}}\sum_{i=0}^{\min(k,n-k)}\frac{2^{i}}{(n-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

Similarly when $n'=n$, 2:

$$n!\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot cd}_{n}}(\vec{p},h)$$

$$=\frac{\sqrt{C_{n+h}^{2}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{\{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},h-1)\varepsilon_{d\}}(\vec{p},1)+\frac{\sqrt{C_{n+h}^{1}C_{n-h}^{1}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{\{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},h)\varepsilon_{d\}}(\vec{p},0)+\frac{\sqrt{C_{n-h}^{2}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{\{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},h+1)\varepsilon_{d\}}(\vec{p},-1)$$

$$\Rightarrow n!\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot cd}_{n}}(\vec{p},n-2k-1)$$

$$=\frac{\sqrt{C_{n+n-2k-1}^{2}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{\{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},(n-1)-2k-1)\varepsilon_{d\}}(\vec{p},1)+\frac{\sqrt{C_{n+n-2k-1}^{1}C_{n-n+2k+1}^{1}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{\{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},(n-1)-2k)\varepsilon_{d\}}(\vec{p},0)+\frac{\sqrt{C_{n-n+2k+1}^{2}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{\{a\cdot\cdot b\cdot\cdot c}_{n-1}}(\vec{p},(n-1)-2(k-1)-1)\varepsilon_{d\}}(\vec{p},-1)$$

$$=\frac{\sqrt{C_{n+n-2k-1}^{2}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k+1}}}\sum_{i=0}^{\min(k,n-1-1-k)}\frac{2^{i}\sqrt{2}}{(n-1-k-i-1)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{(a}(\vec{p},1)\cdot\cdot}_{n-1-k-i-1}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c)}(\vec{p},-1)}_{k-i}\varepsilon_{d\}}(\vec{p},1)$$

$$+\frac{\sqrt{C_{n+n-2k-1}^{1}C_{n-n+2k+1}^{1}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^{i}}{(n-1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{(a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c)}(\vec{p},-1)}_{k-i}\varepsilon_{d\}}(\vec{p},0)$$

$$+\frac{\sqrt{C_{n-n+2k+1}^{2}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k-1}}}\sum_{i=0}^{\min(k-1,n-1-1-k+1)}\frac{2^{i}\sqrt{2}}{(n-1-k+1-i-1)!(2i+1)!(k-1-i)!}\underbrace{\varepsilon_{\{(a}(\vec{p},1)\cdot\cdot}_{n-1-k+1-i-1}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c)}(\vec{p},-1)}_{k-1-i}\varepsilon_{d\}}(\vec{p},-1)$$

$$\Leftrightarrow n\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n}}(\vec{p},n-2k-1)$$

$$=\frac{\sqrt{C_{2n-2k-1}^{2}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k+1}}}\sum_{i=0}^{\min(k,n-2-k)}\frac{2^{i}\sqrt{2}}{(n-2-k-i)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$+\frac{\sqrt{C_{2n-2k-1}^{1}C_{2k+1}^{1}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^{i}}{(n-1-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$+\frac{\sqrt{C_{2k+1}^{2}}}{\sqrt{C_{2n}^{2}}}\frac{1}{\sqrt{C_{2n-2}^{2k-1}}}\sum_{i=0}^{\min(k-1,n-1-k)}\frac{2^{i}\sqrt{2}}{(n-1-k-i)!(2i+1)!(k-1-i)!}\underbrace{\varepsilon_{\{(a}(\vec{p},1)\cdot\cdot}_{n-1-k-i}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$=\frac{n}{\sqrt{C_{2n}^{2k+1}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^{i}\sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i-1}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

$$\Leftrightarrow \varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n}}(\vec{p},n-2k-1)=\frac{1}{\sqrt{C_{2n}^{2k+1}}}\sum_{i=0}^{\min(k,n-1-k)}\frac{2^{i}\sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i-1}\underbrace{\varepsilon_{b}(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$$

This step proves that when $n'=n$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Previously, this theorem was obtained only through intuition, speculation, trial and verification. After more than two years (2019-2022), I finally rigorously proved the above theorem by using mathematical induction. It seems that long-term persistence and continuous in-depth thinking are very important, sometimes more important than interest. This lays the foundation for completely proving

the Behrends-Frontsdal projection operator formula.

1.5 Recursive relations of quasi projection operator for K-G equation

Cor. 1.5.1.

$$\begin{cases} \varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)\bar{\varepsilon}^{c}(\vec{p},1)=\frac{\sqrt{C_{n+h}^{2}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1) \\ \varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)\bar{\varepsilon}^{c}(\vec{p},0)=\frac{\sqrt{C_{n+h}^{1}C_{n-h}^{1}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h) \\ \varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)\bar{\varepsilon}^{c}(\vec{p},-1)=\frac{\sqrt{C_{n-h}^{2}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1) \end{cases}$$

$$[\Rightarrow]\sum_{h=(n-1)}^{-(n-1)}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-1}}(\vec{p},h)=\tfrac{2(n-1)+1}{2n+1}[\sum_{h=n}^{-n}\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'c'}_{n}}(\vec{p},h)][\sum_{h'=1}^{-1}\varepsilon^{c'}(\vec{p},h')\bar{\varepsilon}^{c}(\vec{p},h')]$$

Cor. 1.5.2. $\sum_{h=(n-n')}^{-(n-n')}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},h)$

$$=\tfrac{2(n-n')+1}{2n+1}[\sum_{h=n}^{-n}\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'c'\cdot\cdot d'}_{n}}(\vec{p},h)]\overbrace{[\sum_{h'=1}^{-1}\varepsilon^{c'}(\vec{p},h')\bar{\varepsilon}^{c}(\vec{p},h')]\cdot\cdot[\sum_{h'=1}^{-1}\varepsilon^{d'}(\vec{p},h')\bar{\varepsilon}^{d}(\vec{p},h')]}^{n'}$$

Cor. 1.5.3.

$$\begin{cases} \sum_{h=(n-n')}^{-(n-n')}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},h)=\frac{2(n-n')+1}{2n+1}[\sum_{h=n}^{-n}\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'c'\cdot\cdot d'}_{n}}(\vec{p},h)]\overbrace{\delta^{cc'}\cdot\cdot\delta^{dd'}}^{n'} \\ \sum_{h=(n-n')}^{-(n-n')}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},h)=\frac{2(n-n')+1}{2n+1}[\sum_{h=n}^{-n}\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'\cdot\cdot b'c'\cdot\cdot d'}_{n}}(\vec{p},h)]\overbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}^{n'} \end{cases}$$

Thm. 1.5.1.

$$\sum_{h=(n-n')}^{-(n-n')}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},h)=\tfrac{2n+1-2n'}{2n+1}[\sum_{h=n}^{-n}\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'c'\cdot\cdot d'}_{n}}(\vec{p},h)][\sum_{h'=n'}^{-n'}\varepsilon^{\overbrace{c'\cdot\cdot d'}^{n'}}(\vec{p},h')\bar{\varepsilon}^{\overbrace{c\cdot\cdot d}^{n'}}(\vec{p},h')]$$

Proof:

$$\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h)\bar{\varepsilon}^{\overbrace{c\cdot\cdot d}^{n'}}(\vec{p},h')=\frac{\sqrt{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h-h')$$

$$\Rightarrow\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'c'\cdot\cdot d'}_{n}}(\vec{p},h)\varepsilon^{\overbrace{c'\cdot\cdot d'}^{n'}}(\vec{p},h')\bar{\varepsilon}^{\overbrace{c\cdot\cdot d}^{n'}}(\vec{p},h')=\frac{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}{C_{2n}^{2n'}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h-h')\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},h-h')$$

$$\Rightarrow\sum_{h=n}^{-n}\sum_{h'=n'}^{-n'}\varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p},h)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'c'\cdot\cdot d'}_{n}}(\vec{p},h)\varepsilon^{\overbrace{c'\cdot\cdot d'}^{n'}}(\vec{p},h')\bar{\varepsilon}^{\overbrace{c\cdot\cdot d}^{n'}}(\vec{p},h')$$

$$=\sum_{h=n}^{-n}\sum_{h'=n'}^{-n'}\frac{C_{n+h}^{n'+h'}C_{n-h}^{n'-h'}}{C_{2n}^{2n'}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h-h')\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},h-h')$$

$$=\sum_{l=0}^{2n'}\frac{C_{n+(n-l)}^{n'+(n'-l)}C_{n-(n-l)}^{n'-(n'-l)}}{C_{2n}^{2n'}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},n-n')\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},n-n')$$

$$+\sum_{l=0}^{2n'}\frac{C_{n+(n-1-l)}^{n'+(n'-l)}C_{n-(n-1-l)}^{n'-(n'-l)}}{C_{2n}^{2n'}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},n-n'-1)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},n-n'-1)$$

$$+\sum_{l=0}^{2n'}\frac{C_{n+(n-2-l)}^{n'+(n'-l)}C_{n-(n-2-l)}^{n'-(n'-l)}}{C_{2n}^{2n'}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},n-n'-2)\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},n-n'-2)$$

$$+\cdot\cdot$$

$$+\sum_{l=0}^{2n'}\frac{C_{n+[n-2(n-n')-l]}^{n'+(n'-l)}C_{n-[n-2(n-n')-l]}^{n'-(n'-l)}}{C_{2n}^{2n'}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},n-n'-2(n-n'))\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},n-n'-2(n-n'))$$

$$=\sum_{h=n}^{2n'-n}\sum_{l=0}^{2n'}\frac{C_{n+(h-l)}^{n'+(n'-l)}C_{n-(h-l)}^{n'-(n'-l)}}{C_{2n}^{2n'}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p},h-n')\bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p},h-n')$$

$$= \sum_{h=n}^{2n'-n} \sum_{h'=n'}^{-n'} \frac{C_{n+(h-n'+h')}^{n'+h'} C_{n-(h-n'+h')}^{n'-h'}}{C_{2n}^{2n'}} \varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p}, h-n') \bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p}, h-n')$$

$$= \sum_{h=n-n'}^{n'-n} \sum_{h'=n'}^{-n'} \frac{C_{(n+h')+h}^{n'+h'} C_{(n-h')-h}^{n'-h'}}{C_{2n}^{2n'}} \varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p}, h) \bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p}, h)$$

$$\Rightarrow \sum_{h=(n-n')}^{-(n-n')} \varepsilon_{\underbrace{a\cdot\cdot b}_{n-n'}}(\vec{p}, h) \bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'}_{n-n'}}(\vec{p}, h) = \frac{2n+1-2n'}{2n+1} [\sum_{h=n}^{-n} \varepsilon_{\underbrace{a\cdot\cdot bc\cdot\cdot d}_{n}}(\vec{p}, h) \bar{\varepsilon}_{\underbrace{a'\cdot\cdot b'c'\cdot\cdot d'}_{n}}(\vec{p}, h)][\sum_{h'=n'}^{-n'} \varepsilon^{\overbrace{c'\cdot\cdot d'}^{n'}}(\vec{p}, h') \bar{\varepsilon}^{\overbrace{c\cdot\cdot d}^{n'}}(\vec{p}, h')] \quad \square$$

1.6 Derived to plane wave solutions of spin-n particles K-G equation

Thm. 1.6.1. $(-\partial^c \partial_c + m^2) A_{\underbrace{ab\cdot\cdot}_{n}}(x) = 0, A_{\underbrace{ab\cdot\cdot}_{n}}(x) = (\frac{1}{2im})^n \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma \mu_\varsigma} (\bar{C}\gamma_b)^{\eta_\varsigma \xi_\varsigma} \cdot\cdot}^{n} \psi_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \cdot\cdot}_{2n}}(x)$

$$A_{\underbrace{ab\cdot\cdot}_{n}}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h) \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) e^{ip\cdot x} + b^+(\vec{p}, h) \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) e^{-ip\cdot x}] d^3\vec{p}$$

$$\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma \mu_\varsigma} (\bar{C}\gamma_b)^{\eta_\varsigma \xi_\varsigma} \cdot\cdot}^{n} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \cdot\cdot}_{2n}}(\vec{p}, h)$$

$$\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma \mu_\varsigma} (\bar{C}\gamma_b)^{\eta_\varsigma \xi_\varsigma} \cdot\cdot}^{n} V_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \cdot\cdot}_{2n}}(\vec{p}, h)$$

Proof: $[A_{\underbrace{ab\cdot\cdot}_{n}}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')]$

$$= \frac{1}{(2\pi)^{3/2}} \int \sum_{h,h'=n}^{-n} d^3\vec{p} d^3\vec{p}' \frac{1}{\sqrt{2^n E}} \frac{1}{\sqrt{2^n E'}}$$

$$[a(\vec{p}, h) \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) e^{ip\cdot x} + b^+(\vec{p}, h) \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) e^{-ip\cdot x}, a^+(\vec{p}', h') \varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}', h') e^{-ip'\cdot x'} + b(\vec{p}', h') \tilde{\varepsilon}^+_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}', h') e^{ip'\cdot x'}]$$

$$= \frac{1}{(2\pi)^{3/2}} \int \sum_{h,h'=n}^{-n} d^3\vec{p} d^3\vec{p}' \frac{1}{\sqrt{2^n E}} \frac{1}{\sqrt{2^n E'}}$$

$$\{\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) \varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}', h') [a(\vec{p}, h), a^+(\vec{p}', h')] e^{ip\cdot x} e^{-ip'\cdot x'} + \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) \tilde{\varepsilon}^+_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}', h') [b^+(\vec{p}, h), b(\vec{p}', h')] e^{-ip\cdot x} e^{ip'\cdot x'}\}$$

$$= \frac{1}{(2\pi)^{3/2}} \int \sum_{h,h'=n}^{-n} d^3\vec{p} d^3\vec{p}' \frac{1}{\sqrt{2^n E}} \frac{1}{\sqrt{2^n E'}} \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) \varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}', h') \delta_{hh'} \delta(\vec{p} - \vec{p}') (e^{ip\cdot x} e^{-ip'\cdot x'} - e^{-ip\cdot x} e^{ip'\cdot x'})$$

$$= \frac{i}{2^{n-1}} \int [\sum_{h=n}^{-n} \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) \varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h)] \{\frac{1}{(2\pi)^{3/2}} \frac{-i}{2E} [e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}] d^3\vec{p}\}$$

$$= \frac{i}{2^{n-1}} [\sum_{h=n}^{-n} \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(-i\partial, h) \varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}}(-i\partial, h)] \int \{\frac{1}{(2\pi)^{3/2}} \frac{-i}{2E} [e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}] d^3\vec{p}\}$$

$$= \frac{i}{2^{n-1}} [\sum_{h=n}^{-n} \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(-i\partial, h) \varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}}(-i\partial, h)] \Delta(x - x') \quad \square$$

1.7 Correct conjecture of K-G equation basis for spin-n particles(The original idea remains.)

Thm. 1.7.1.

$$\begin{cases} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \cdot\cdot}_{2n}}(\vec{p}, h) = \frac{1}{(2\sqrt{2}m)^n} \underbrace{\mathbb{X}^a_{\lambda_\varsigma \mu_\varsigma}(p) \mathbb{X}^b_{\eta_\varsigma \xi_\varsigma}(p) \cdot\cdot}_{n} \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) \\ [\Rightarrow] \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma \mu_\varsigma} (\bar{C}\gamma_b)^{\eta_\varsigma \xi_\varsigma} \cdot\cdot}^{n} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \cdot\cdot}_{2n}}(\vec{p}, h) \\ V_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \cdot\cdot}_{2n}}(\vec{p}, h) = \frac{1}{(2\sqrt{2}m)^n} \underbrace{\mathbb{X}^a_{\lambda_\varsigma \mu_\varsigma}(-p) \mathbb{X}^b_{\eta_\varsigma \xi_\varsigma}(-p) \cdot\cdot}_{n} \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) \\ [\Rightarrow] \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma \mu_\varsigma} (\bar{C}\gamma_b)^{\eta_\varsigma \xi_\varsigma} \cdot\cdot}^{n} V_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \cdot\cdot}_{2n}}(\vec{p}, h) \end{cases}$$

Cor. 1.7.1.

$$\begin{cases}\sum\limits_{h=n}^{-n} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p},h) = \frac{1}{(2\sqrt{2}m)^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}(p)\cdots}_{n}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdots}_{n}}(\vec{p},h)\\ \sum\limits_{h=n}^{-n} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p},h) = \frac{1}{(2\sqrt{2}m)^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}(p)\cdots}_{n}\sum\limits_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdots}_{n}}(\vec{p},h)\end{cases}$$

Cor. 1.7.2.

$$\begin{cases}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdots}_{n}}(\vec{p},h) = \frac{1}{2^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}\sum\limits_{h=n}^{-n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p},h)\\ \sum\limits_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdots}_{n}}(\vec{p},h) = \frac{1}{2^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}\sum\limits_{h=n}^{-n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p},h)\end{cases}$$

Thm. 1.7.2. $\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h) = (-1)^n\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p},h)$

Proof: $\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h) = \frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)$

$= (-\varsigma)^{2n}\frac{1}{(i\sqrt{2})^n}\overbrace{(\gamma_5\bar{C}\gamma_a\gamma_5)^{\lambda_\varsigma\mu_\varsigma}(\gamma_5\bar{C}\gamma_b\gamma_5)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)$

$= (-1)^n\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)$

$= (-1)^n\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p},h)$ □

Proof: $\varepsilon^{+\overbrace{ab\cdots}^{n}}(\vec{p},h')\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h)$

$= \frac{1}{(-i\sqrt{2})^n}\overbrace{(\gamma^aC)_{\lambda'_\varsigma\mu'_\varsigma}(\gamma^bC)_{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}U^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdots}^{2n}}(\vec{p},h')\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)$

$= \frac{1}{2^n}\overbrace{(\gamma^aC)_{\lambda'_\varsigma\mu'_\varsigma}(\gamma^bC)_{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdots}^{2n}}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)$ □

Thm. 1.7.3. $\varepsilon^+_{\underbrace{ab\cdots}_{n}}(\vec{p},h) = (-1)^h\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}\varepsilon_{\underbrace{a'b'\cdots}_{n}}(\vec{p},-h)$

Proof: $\varepsilon^+_{\underbrace{ab\cdots}_{n}}(\vec{p},h) = \frac{1}{(-i\sqrt{2})^n}\overbrace{(\bar{C}\gamma^*_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma^*_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)$

$= \frac{1}{(-i\sqrt{2})^n}\overbrace{(\bar{C}\gamma^*_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma^*_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}(-1)^{n+h}\varsigma^{2n}\overbrace{\sigma_y\otimes\sigma_y\cdots}^{4n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},-h)$

$= \frac{(-1)^{n+h}}{(-i\sqrt{2})^n}\overbrace{(\gamma_2\bar{C}\gamma^*_a\gamma_2)^{\lambda_\varsigma\mu_\varsigma}(\gamma_2\bar{C}\gamma^*_b\gamma_2)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},-h)$

$= \frac{(-1)^{n+h}}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_{a'}\eta^{a'}_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{b'}\eta^{b'}_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2n}}(\vec{p},-h)$

$= \frac{(-1)^{n+h}}{(i\sqrt{2})^n}\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}\overbrace{(\bar{C}\gamma_{a'})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{b'})^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},-h)$

$= (-1)^h\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}\varepsilon_{\underbrace{a'b'\cdots}_{n}}(\vec{p},-h)$ □

Ass. 1.7.1. $\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n}}(\vec{p},n-2k)=(-1)^n\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n}}(\vec{p},n-2k)$

$=\frac{1}{\sqrt{C_{2n}^{2k}}}\sum\limits_{i=0}^{\min(k,n-k)}\frac{2^i}{(n-k-i)!(2i)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$

$=\frac{1}{\sqrt{C_{2n}^{2k}}}\frac{1}{n!}\sum\limits_{i=0}^{\min(k,n-k)}2^iC_n^{2i}C_{n-2i}^{k-i}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$

$=\frac{1}{\sqrt{C_{2n}^{2k}}}\frac{1}{n!}[\sqrt{2^0}C_n^0C_{n-0}^k\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{0}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k}+\sqrt{2^2}C_n^2C_{n-2}^{k-1}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-1}$

$+\sqrt{2^4}C_n^4C_{n-4}^{k-2}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-2}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{4}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-2}+\sqrt{2^6}C_n^6C_{n-6}^{k-3}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-3}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{6}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-3}+\cdot\cdot]$

Ass. 1.7.2. $\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n}}(\vec{p},n-2k-1)=(-1)^n\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n}}(\vec{p},n-2k-1)$

$=\frac{1}{\sqrt{C_{2n}^{2k+1}}}\sum\limits_{i=0}^{\min(k,n-1-k)}\frac{2^i\sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$

$=\frac{1}{\sqrt{C_{2n}^{2k+1}}}\frac{1}{n!}\sum\limits_{i=0}^{\min(k,n-k-1)}\sqrt{2^{2i+1}}C_n^{2i+1}C_{n-2i-1}^{k-i}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-i-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2i+1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-i}$

$=\frac{1}{\sqrt{C_{2n}^{2k+1}}}\frac{1}{n!}[\sqrt{2^1}C_n^1C_{n-1}^k\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k}+\sqrt{2^3}C_n^3C_{n-3}^{k-1}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-2}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{3}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-1}$

$+\sqrt{2^5}C_n^5C_{n-5}^{k-2}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-3}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{5}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-2}+\sqrt{2^7}C_n^7C_{n-7}^{k-3}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-k-4}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{7}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{k-3}+\cdot\cdot]$

Cor. 1.7.3. $\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}$ *fully symmetric*

1.8 Plane wave solutions of K-G equation for spin-n particles

Cor. 1.8.1. $A_{ab\cdot\cdot}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}\varepsilon_{ab\cdot\cdot}(\vec{p},h)[a(\vec{p},h)e^{ip\cdot x}+(-1)^nb^+(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$

2 Several examples of spin basis and quasi projection operator for K-G equation

2.1 Quasi projection operator of K-G equation for spin-1 particles

Cor. 2.1.1. $\varepsilon_{a'}^+(\vec{p},h)\eta_a^{a'}=(-1)^h\varepsilon_a(\vec{p},-h)$

Thm. 2.1.1. $\begin{cases}\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon_{a'}^+(\vec{p},h)=\eta_{aa'}+\frac{p_ap_{a'}^+}{m^2}\\ \sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon_{a'}^+(\vec{p},h)\eta_b^{a'}=\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_a(\vec{p},h)\varepsilon_b(\vec{p},-h)=\delta_{ab}+\frac{p_ap_b}{m^2}\end{cases}$

Cor. 2.1.2. $[-\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},-1)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},0)-\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},1)]=\delta_{ab}+\frac{p_ap_b}{m^2}$

Cor. 2.1.3. $p^a\varepsilon_a(\vec{p},h)=0$

2.2 Relations of K-G and 1-spin bases for spin-1 particles

Lem. 2.2.1.

$\begin{cases}[\sigma_+^a\varepsilon_a(\vec{p},\kappa)]\lambda(\hat{p},\kappa)=0,[\sigma_+^a\varepsilon_a(\vec{p},-\kappa)]\lambda(\hat{p},\kappa)=-i\kappa\sqrt{2}\gamma_5u(\vec{p},-\frac{\kappa}{2}),[\sigma_+^a\varepsilon_a(\vec{p},0)]\lambda(\hat{p},\kappa)=-i\kappa\gamma_5\lambda(\hat{p},\kappa)\\ [\gamma^a\varepsilon_a(\vec{p},\kappa)]v(\vec{p},\frac{\kappa}{2})=0,[\gamma^a\varepsilon_a(\vec{p},-\kappa)]v(\vec{p},\frac{\kappa}{2})=i\kappa\sqrt{2}\gamma_5v(\vec{p},-\frac{\kappa}{2}),[\gamma^a\varepsilon_a(\vec{p},0)]v(\vec{p},\frac{\kappa}{2})=i\kappa\gamma_5v(\vec{p},\frac{\kappa}{2})\end{cases}$

2.3 Quasi projection operator of K-G equation for spin-2 particles

Pro. 2.3.1.

$\begin{cases}|2,2\rangle=|1\rangle\otimes|1\rangle;\\ |2,1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle+|1\rangle\otimes|0\rangle);\\ |2,0\rangle=\frac{1}{\sqrt{6}}(|-1\rangle\otimes|1\rangle+2|0\rangle\otimes|0\rangle+|1\rangle\otimes|-1\rangle);\\ |2,-1\rangle=\frac{1}{\sqrt{2}}(|-1\rangle\otimes|0\rangle+|0\rangle\otimes|-1\rangle);\\ |2,-2\rangle=|-1\rangle\otimes|-1\rangle;\end{cases}$

Pro. 2.3.2.

$$\begin{cases}\varepsilon_{ab}(\vec{p},2)=\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},1)\\ \varepsilon_{ab}(\vec{p},1)=\frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},1)]\\ \varepsilon_{ab}(\vec{p},0)=\frac{1}{\sqrt{6}}[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},-1)+\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},1)+2\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},0)]\\ \varepsilon_{ab}(\vec{p},-1)=\frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},-1)]\\ \varepsilon_{ab}(\vec{p},-2)=\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},-1)\\ \delta^{ab}\varepsilon_{ab}(\vec{p},h)=0,p^a\varepsilon_{ab}(\vec{p},h)=0,\varepsilon_{ab}(\vec{p},h)=\varepsilon_{ba}(\vec{p},h)\end{cases}$$

Cor. 2.3.1. $\varepsilon^+_{a'b'}(\vec{p},h)\eta^{a'}_a\eta^{b'}_b=(-1)^h\varepsilon_{ab}(\vec{p},-h)$

Thm. 2.3.1. $\sum\limits_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)=\frac{1}{4}\{[\eta_{\{a(a'}+\frac{p_{\{a}p^+_{(a'}}{m^2}][\eta_{b\}b')}+\frac{p_{b\}}p^+_{b')}}{m^2}]-\frac{1}{3}[\delta_{\{ab\}}+\frac{p_{\{a}p_{b\}}}{m^2}][\delta_{(a'b')}+\frac{p^+_{(a'}p^+_{b')}}{m^2}]\}$

Proof: $\sum\limits_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)$

$=\frac{1}{12}\{12\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},1)\varepsilon^+_{a'}(\vec{p},1)\varepsilon^+_{b'}(\vec{p},1)$
$+6[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},1)][\varepsilon^+_{a'}(\vec{p},1)\varepsilon^+_{b'}(\vec{p},0)+\varepsilon^+_{a'}(\vec{p},0)\varepsilon^+_{b'}(\vec{p},1)]$
$+2[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},-1)+\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},1)+2\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},0)]$
$[\varepsilon^+_{a'}(\vec{p},1)\varepsilon^+_{b'}(\vec{p},-1)+\varepsilon^+_{a'}(\vec{p},-1)\varepsilon^+_{b'}(\vec{p},1)+2\varepsilon_{a'}(\vec{p},0)\varepsilon_{b'}(\vec{p},0)]$
$+6[\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},-1)][\varepsilon^+_a(\vec{p},-1)\varepsilon^+_b(\vec{p},0)+\varepsilon^+_{a'}(\vec{p},0)\varepsilon^+_{b'}(\vec{p},-1)]$
$+12\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},-1)\varepsilon^+_{a'}(\vec{p},-1)\varepsilon^+_{b'}(\vec{p},-1)\}$
$=\frac{1}{12}\{$
$3[\varepsilon_a(\vec{p},1)\varepsilon^+_{a'}(\vec{p},1)+\varepsilon_a(\vec{p},0)\varepsilon^+_{a'}(\vec{p},0)+\varepsilon_a(\vec{p},-1)\varepsilon^+_{a'}(\vec{p},-1)][\varepsilon_b(\vec{p},1)\varepsilon^+_{b'}(\vec{p},1)+\varepsilon_b(\vec{p},0)\varepsilon^+_{b'}(\vec{p},0)+\varepsilon_b(\vec{p},-1)\varepsilon^+_{b'}(\vec{p},-1)]$
$+$
$3[\varepsilon_a(\vec{p},1)\varepsilon^+_{b'}(\vec{p},1)+\varepsilon_a(\vec{p},0)\varepsilon^+_{b'}(\vec{p},0)+\varepsilon_a(\vec{p},-1)\varepsilon^+_{b'}(\vec{p},-1)][\varepsilon_b(\vec{p},1)\varepsilon^+_{a'}(\vec{p},1)+\varepsilon_b(\vec{p},0)\varepsilon^+_{a'}(\vec{p},0)+\varepsilon_b(\vec{p},-1)\varepsilon^+_{a'}(\vec{p},-1)]$
$+$
$3[\varepsilon_b(\vec{p},1)\varepsilon^+_{a'}(\vec{p},1)+\varepsilon_b(\vec{p},0)\varepsilon^+_{a'}(\vec{p},0)+\varepsilon_b(\vec{p},-1)\varepsilon^+_{a'}(\vec{p},-1)][\varepsilon_a(\vec{p},1)\varepsilon^+_{b'}(\vec{p},1)+\varepsilon_a(\vec{p},0)\varepsilon^+_{b'}(\vec{p},0)+\varepsilon_a(\vec{p},-1)\varepsilon^+_{b'}(\vec{p},-1)]$
$+$
$3[\varepsilon_b(\vec{p},1)\varepsilon^+_{b'}(\vec{p},1)+\varepsilon_b(\vec{p},0)\varepsilon^+_{b'}(\vec{p},0)+\varepsilon_b(\vec{p},-1)\varepsilon^+_{b'}(\vec{p},-1)][\varepsilon_a(\vec{p},1)\varepsilon^+_{a'}(\vec{p},1)+\varepsilon_a(\vec{p},0)\varepsilon^+_{a'}(\vec{p},0)+\varepsilon_a(\vec{p},-1)\varepsilon^+_{a'}(\vec{p},-1)]$
$-$
$4[-\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},-1)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},0)-\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},1)]$
$[-\varepsilon^+_{a'}(\vec{p},1)\varepsilon^+_{b'}(\vec{p},-1)+\varepsilon^+_{a'}(\vec{p},0)\varepsilon^+_{b'}(\vec{p},0)-\varepsilon^+_{a'}(\vec{p},-1)\varepsilon^+_{b'}(\vec{p},1)]\}$
$=\frac{1}{4}\{[\eta_{\{a(a'}+\frac{p_{\{a}p^+_{(a'}}{m^2}][\eta_{b\}b')}+\frac{p_{b\}}p^+_{b')}}{m^2}]-\frac{4}{3}[\delta_{ab}+\frac{p_ap_b}{m^2}][\delta_{a'b'}+\frac{p^+_{a'}p^+_{b'}}{m^2}]\}$
$=\frac{1}{4}\{[\eta_{\{a(a'}+\frac{p_{\{a}p^+_{(a'}}{m^2}][\eta_{b\}b')}+\frac{p_{b\}}p^+_{b')}}{m^2}]-\frac{1}{3}[\delta_{\{ab\}}+\frac{p_{\{a}p_{b\}}}{m^2}][\delta_{(a'b')}+\frac{p^+_{(a'}p^+_{b')}}{m^2}]\}$ □

Cor. 2.3.2.
$\sum\limits_{h=2}^{-2}\varepsilon_{a_1a_2}(\vec{p},h)\varepsilon^+_{b'_1b'_2}(\vec{p},h)\eta^{b'_1}_{b_1}\eta^{b'_2}_{b_2}=\frac{1}{4}\{[\delta_{\{a_1(b_1}+\frac{p_{\{a_1}p_{(b_1}}{m^2}][\delta_{a_2\}b_2)}+\frac{p_{a_2\}}p_{b_2)}}{m^2}]-\frac{1}{3}[\delta_{\{a_1a_2\}}+\frac{p_{\{a_1}p_{a_2\}}}{m^2}][\delta_{(b_1b_2)}+\frac{p_{(b_1}p_{b_2)}}{m^2}]\}$

2.4 Another proof method

Pro. 2.4.1.

$$\begin{cases}\varepsilon_{a_1a_2}(\vec{p},2)=\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},1)\\ \varepsilon_{a_1a_2}(\vec{p},1)=\frac{1}{\sqrt{2}}[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},0)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},1)]\\ \varepsilon_{a_1a_2}(\vec{p},0)=\frac{1}{\sqrt{6}}[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)+2\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)]\\ \varepsilon_{a_1a_2}(\vec{p},-1)=\frac{1}{\sqrt{2}}[\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},0)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},-1)]\\ \varepsilon_{a_1a_2}(\vec{p},-2)=\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},-1)\\ \delta^{a_1a_2}\varepsilon_{a_1a_2}(\vec{p},h)=0,p^{a_1}\varepsilon_{a_1a_2}(\vec{p},h)=0,\varepsilon_{a_1a_2}(\vec{p},h)=\varepsilon_{a_2a_1}(\vec{p},h)\end{cases}$$

Pro. 2.4.2.

$$\begin{cases}\varepsilon_{b_1b_2}(\vec{p},2)=\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},1)\\ \varepsilon_{b_1b_2}(\vec{p},1)=\frac{1}{\sqrt{2}}[\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},0)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},1)]\\ \varepsilon_{b_1b_2}(\vec{p},0)=\frac{1}{\sqrt{6}}[\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)+2\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)]\\ \varepsilon_{b_1b_2}(\vec{p},-1)=\frac{1}{\sqrt{2}}[\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},0)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},-1)]\\ \varepsilon_{b_1b_2}(\vec{p},-2)=\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},-1)\\ \delta^{b_1b_2}\varepsilon_{b_1b_2}(\vec{p},h)=0,p^{b_1}\varepsilon_{b_1b_2}(\vec{p},h)=0,\varepsilon_{b_1b_2}(\vec{p},h)=\varepsilon_{b_2b_1}(\vec{p},h)\end{cases}$$

Cor. 2.4.1. $\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},1):=\frac{1}{\sqrt{2}}[-1,-i,0,0]_a,\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},0):=\frac{1}{m}[0,0,E,i|\vec{p}|]_a,\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},1):=\frac{1}{\sqrt{2}}[1,-i,0,0]_a$

Proof:
$2[\varepsilon_{a_1a_2}(\vec{p},1)\varepsilon_{b_1b_2}(\vec{p},1)+\varepsilon_{a_1a_2}(\vec{p},-1)\varepsilon_{b_1b_2}(\vec{p},-1)]$
$=[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},0)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},1)][\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},0)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},-1)]$
$+[\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},0)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},-1)][\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},0)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},1)]$
$=[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)$
$+[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$=Q_{\{a_1(b_1}[\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)]$
$=-P_{\{a_1(b_1}[\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)]+\varepsilon_{\{a_1}(\vec{p},0)\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{(b_1}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)$ □

Proof: $\sum\limits_{h=2}^{-2}\varepsilon_{a_1a_2}(\vec{p},h)\bar{\varepsilon}_{b_1b_2}(\vec{p},h)=\sum\limits_{h=2}^{-2}(-1)^h\varepsilon_{a_1a_2}(\vec{p},h)\varepsilon_{b_1b_2}(\vec{p},-h)$
$=\frac{1}{12}$
$\{12\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},-1)$
$-6[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},0)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},1)][\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},0)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},-1)]$
$+2[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)+2\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)]$
$[\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)+2\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)]$
$-6[\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},0)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},-1)][\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},0)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},1)]$
$+12\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},1)\}$
$=\frac{1}{12}$
$\{12\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},-1)+12\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},1)$
$+12\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$-4[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)][\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$-4\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+4[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)]\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+4[\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)$
$+6[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)][\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$-6\{[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)$
$+[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)\}\}$
$=\frac{1}{12}$
$\{12\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+6[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)][\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$+6[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)][\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$
$-6\{[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)$
$+[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)$
$+[\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)\}$
$-$
$4[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)]$
$[-\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]\}$
$=\frac{1}{12}\{$
$6[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$
$[-\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$+$
$6[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$[-\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)-\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$
$-$
$4[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)]$
$[-\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$=\frac{1}{12}\{$
$3[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$
$[-\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$+$
$3[-\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)-\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$
$[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$+$
$3[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$[-\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)-\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$

$+$
$3[-\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{b_1}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$
$-$
$4[-\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},0)\varepsilon_{a_2}(\vec{p},0)-\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)]$
$[-\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},0)\varepsilon_{b_2}(\vec{p},0)-\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$=\frac{1}{4}\{[\delta_{\{a_1(b_1}+\frac{p_{\{a_1}p_{(b_1}}{m^2}][\delta_{a_2\}b_2)}+\frac{p_{a_2\}}p_{b_2)}}{m^2}]-\frac{1}{3}[\delta_{\{a_1a_2\}}+\frac{p_{\{a_1}p_{a_2\}}}{m^2}][\delta_{(b_1b_2)}+\frac{p_{(b_1}p_{b_2)}}{m^2}]\}$ □

Cor. 2.4.2. $2[\varepsilon_{a_1a_2}(\vec{p},1)\varepsilon_{b_1b_2}(\vec{p},1)+\varepsilon_{a_1a_2}(\vec{p},-1)\varepsilon_{b_1b_2}(\vec{p},-1)]$
$=-P_{\{a_1(b_1}[\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)]+\varepsilon_{\{a_1}(\vec{p},0)\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{(b_1}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)$

Cor. 2.4.3.
$[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)][\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$
$+$
$[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)][\varepsilon_{a_2}(\vec{p},1)\varepsilon_{b_1}(\vec{p},-1)+\varepsilon_{a_2}(\vec{p},-1)\varepsilon_{b_1}(\vec{p},1)]$
$=2[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},1)][\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},-1)]+2[\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},-1)][\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},1)]$
$+[\varepsilon_{a_1}(\vec{p},1)\varepsilon_{a_2}(\vec{p},-1)+\varepsilon_{a_1}(\vec{p},-1)\varepsilon_{a_2}(\vec{p},1)][\varepsilon_{b_1}(\vec{p},1)\varepsilon_{b_2}(\vec{p},-1)+\varepsilon_{b_1}(\vec{p},-1)\varepsilon_{b_2}(\vec{p},1)]$

Cor. 2.4.4. $2\varepsilon_{\{a_1}(\vec{p},1)\varepsilon_{b_1\}}(\vec{p},-1)\varepsilon_{(a_2}(\vec{p},1)\varepsilon_{b_2)}(\vec{p},-1)+2\varepsilon_{\{a_1}(\vec{p},1)\varepsilon_{b_2\}}(\vec{p},-1)\varepsilon_{(a_2}(\vec{p},1)\varepsilon_{b_1)}(\vec{p},-1)$
$=\varepsilon_{\{a_1}(\vec{p},1)\varepsilon_{a_2\}}(\vec{p},1)\varepsilon_{\{b_1}(\vec{p},-1)\varepsilon_{b_2\}}(\vec{p},-1)+\varepsilon_{\{a_1}(\vec{p},-1)\varepsilon_{a_2\}}(\vec{p},-1)\varepsilon_{\{b_1}(\vec{p},1)\varepsilon_{b_2\}}(\vec{p},1)$
$+2\varepsilon_{\{a_1}(\vec{p},1)\varepsilon_{a_2\}}(\vec{p},-1)\varepsilon_{\{b_1}(\vec{p},1)\varepsilon_{b_2\}}(\vec{p},-1)$

Cor. 2.4.5. $Q_{\{a_1(b_1}Q_{a_2\}b_2)}$
$=\varepsilon_{\{a_1}(\vec{p},1)\varepsilon_{a_2\}}(\vec{p},1)\varepsilon_{(b_1}(\vec{p},-1)\varepsilon_{b_2)}(\vec{p},-1)+\varepsilon_{\{a_1}(\vec{p},-1)\varepsilon_{a_2\}}(\vec{p},-1)\varepsilon_{(b_1}(\vec{p},1)\varepsilon_{b_2)}(\vec{p},1)+2Q_{a_1a_2}Q_{b_1b_2}$

Cor. 2.4.6.
$P_{\{a_1(b_1}P_{a_2\}b_2)}=[Q_{\{a_1(b_1}-\varepsilon_{\{a_1}(\vec{p},0)\varepsilon_{(b_1}(\vec{p},0)][Q_{a_2\}b_2)}-\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)]$
$=Q_{\{a_1(b_1}Q_{a_2\}b_2)}-2Q_{\{a_1(b_1}[\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)]+\varepsilon_{\{a_1}(\vec{p},0)\varepsilon_{a_2\}}(\vec{p},0)\varepsilon_{(b_1}(\vec{p},0)\varepsilon_{b_2)}(\vec{p},0)$

2.5 CG coefficients and spin bases of K-G equation for spin-3 particles

Cor. 2.5.1.
$$\begin{cases}\langle 2,2;1,1|2,1;3,3\rangle=1\\ \langle 2,2;1,0|2,1;3,2\rangle=\frac{1}{\sqrt{3}},\langle 2,1;1,1|2,1;3,2\rangle=\frac{\sqrt{2}}{\sqrt{3}}\\ \langle 2,2;1,-1|2,1;3,1\rangle=\frac{1}{\sqrt{15}},\langle 2,1;1,0|2,1;3,1\rangle=\frac{\sqrt{8}}{\sqrt{15}},\langle 2,0;1,1|2,1;3,1\rangle=\frac{\sqrt{6}}{\sqrt{15}}\\ \langle 2,1;1,-1|2,1;3,0\rangle=\frac{1}{\sqrt{5}},\langle 2,0;1,0|2,1;3,0\rangle=\frac{\sqrt{3}}{\sqrt{15}},\langle 2,-1;1,1|2,1;3,0\rangle=\frac{1}{\sqrt{5}}\\ \langle 2,-2;1,1|2,1;3,-1\rangle=\frac{1}{\sqrt{15}},\langle 2,-1;1,0|2,1;3,-1\rangle=\frac{\sqrt{8}}{\sqrt{15}},\langle 2,0;1,-1|2,1;3,-1\rangle=\frac{\sqrt{6}}{\sqrt{15}}\\ \langle 2,-2;1,0|2,1;3,-2\rangle=\frac{1}{\sqrt{3}},\langle 2,-1;1,-1|2,1;3,-2\rangle=\frac{\sqrt{2}}{\sqrt{3}}\\ \langle 2,-2;1,-1|2,1;3,-3\rangle=1\end{cases}$$

Cor. 2.5.2.
$$\begin{cases}\varepsilon_{abc}(\vec{p},3)=\varepsilon_{ab}(\vec{p},2)\varepsilon_c(\vec{p},1)=\frac{1}{3!}\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\varepsilon_{c\}}(\vec{p},1)\\ \varepsilon_{abc}(\vec{p},2)=\frac{1}{\sqrt{3}}\varepsilon_{ab}(\vec{p},2)\varepsilon_c(\vec{p},0)+\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p},1)\varepsilon_c(\vec{p},1)=\frac{\sqrt{3}}{3!}\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\varepsilon_{c\}}(\vec{p},0)\\ \varepsilon_{abc}(\vec{p},1)=\frac{1}{\sqrt{15}}\varepsilon_{ab}(\vec{p},2)\varepsilon_c(\vec{p},-1)+\frac{\sqrt{8}}{\sqrt{15}}\varepsilon_{ab}(\vec{p},1)\varepsilon_c(\vec{p},0)+\frac{\sqrt{6}}{\sqrt{15}}\varepsilon_{ab}(\vec{p},0)\varepsilon_c(\vec{p},1)\\ =\frac{6}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},0)\varepsilon_{c\}}(\vec{p},0)+\frac{3}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},1)\varepsilon_{c\}}(\vec{p},-1)\\ \varepsilon_{abc}(\vec{p},0)=\frac{1}{\sqrt{5}}\varepsilon_{ab}(\vec{p},1)\varepsilon_c(\vec{p},-1)+\frac{\sqrt{3}}{\sqrt{5}}\varepsilon_{ab}(\vec{p},0)\varepsilon_c(\vec{p},0)+\frac{1}{\sqrt{5}}\varepsilon_{ab}(\vec{p},-1)\varepsilon_c(\vec{p},1)\\ =\frac{6}{3!\sqrt{10}}\varepsilon_{\{a}(\vec{p},1)\varepsilon_b(\vec{p},0)\varepsilon_{c\}}(\vec{p},-1)+\frac{2}{3!\sqrt{10}}\varepsilon_{\{a}(\vec{p},0)\varepsilon_b(\vec{p},0)\varepsilon_{c\}}(\vec{p},0)\\ \varepsilon_{abc}(\vec{p},-1)=\frac{1}{\sqrt{15}}\varepsilon_{ab}(\vec{p},-2)\varepsilon_c(\vec{p},1)+\frac{\sqrt{8}}{\sqrt{15}}\varepsilon_{ab}(\vec{p},-1)\varepsilon_c(\vec{p},0)+\frac{\sqrt{6}}{\sqrt{15}}\varepsilon_{ab}(\vec{p},0)\varepsilon_c(\vec{p},-1)\\ =\frac{6}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p},-1)\varepsilon_b(\vec{p},0)\varepsilon_{c\}}(\vec{p},0)+\frac{3}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p},-1)\varepsilon_b(\vec{p},-1)\varepsilon_{c\}}(\vec{p},1)\\ \varepsilon_{abc}(\vec{p},-2)=\frac{1}{\sqrt{3}}\varepsilon_{ab}(\vec{p},-2)\varepsilon_c(\vec{p},0)+\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p},-1)\varepsilon_c(\vec{p},-1)=\frac{\sqrt{3}}{3!}\varepsilon_{\{a}(\vec{p},-1)\varepsilon_b(\vec{p},-1)\varepsilon_{c\}}(\vec{p},0)\\ \varepsilon_{abc}(\vec{p},-3)=\varepsilon_{ab}(\vec{p},-2)\varepsilon_c(\vec{p},-1)=\frac{1}{3!}\varepsilon_{\{a}(\vec{p},-1)\varepsilon_b(\vec{p},-1)\varepsilon_{c\}}(\vec{p},-1)\end{cases}$$

2.6 Quasi projection operator of K-G equation for spin-n particles

Def. 2.6.1. $\bar{\varepsilon}_a(\vec{p},h):=\varepsilon^+_{a'}(\vec{p},h)\eta_a^{a'},\bar{\varepsilon}_{ab}(\vec{p},h):=\varepsilon^+_{a'b'}(\vec{p},h)\eta_a^{a'}\eta_b^{b'},P_{ab}:=\delta_{ab}+\frac{p_ap_b}{m^2}$

Cor. 2.6.1.
$$\begin{cases}\sum\limits_{h=1}^{-1}\varepsilon_{a_1}(\vec{p},h)\bar{\varepsilon}_{b_1}(\vec{p},h)=P_{a_1b_1},\sum\limits_{h=1}^{-1}-|h|\varepsilon_{a_1}(\vec{p},h)\bar{\varepsilon}_{b_1}(\vec{p},h):=Q_{a_1b_1}=\varepsilon_{\{a_1}(\vec{p},1)\varepsilon_{b_1\}}(\vec{p},-1)\\ \sum\limits_{h=2}^{-2}\varepsilon_{a_1a_2}(\vec{p},h)\bar{\varepsilon}_{b_1b_2}(\vec{p},h)=\frac{1}{(2!)^2}[P_{\{a_1(b_1}P_{a_2\}b_2)}-\frac{1}{3}P_{\{a_1a_2\}}P_{(b_1b_2)}]\end{cases}$$

Ass. 2.6.1.

$$\begin{cases}\sum\limits_{h=n}^{-n}\varepsilon_{a_1a_2\cdot\cdot a_n}(\vec{p},h)\bar{\varepsilon}_{b_1b_2\cdot\cdot b_n}(\vec{p},h)=\frac{1}{(n!)^2}\sum\limits_{r=0}^{[n/2]}A_{n,r}[P_{\{a_1a_2}P_{(b_1b_2}\cdot\cdot P_{a_{2r-1}a_{2r}}P_{b_{2r-1}b_{2r}}][P_{a_{2r+1}b_{2r+1}}\cdot\cdot P_{a_n\}b_n)}] \\ A_{n,r}=(-\frac{1}{2})^r\frac{n!(2n-2r-1)!!}{r!(n-2r)!(2n-1)!!}=(-\frac{1}{2})^r\frac{1}{r!}\frac{n(n-1)\cdot\cdot(n-2r+1)}{(2n-1)(2n-3)\cdot\cdot(2n-2r+1)}=(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}=(-1)^rC_{2n}^{-n}C_n^rC_{2n-2r}^n \\ A_{n,0}=1,A_{n,1}=-\frac{n(n-1)}{2(2n-1)},A_{n,2}=\frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)},\cdots\end{cases}$$

The above formula constructed by Behrends and Fronsdal has not been strictly proven, and is essentially a conjecture. It is a prerequisite for many latter important conclusions. It need to be proved strictly, but I can't finish the proof yet.

Def. 2.6.2. $P_{a_1a_2\cdot\cdot a_nb_1b_2\cdot\cdot b_n}(n):=\sum\limits_{h=n}^{-n}\varepsilon_{a_1a_2\cdot\cdot a_n}(\vec{p},h)\bar{\varepsilon}_{b_1b_2\cdot\cdot b_n}(\vec{p},h)$

Ass. 2.6.2.

$$P_{a_1a_2\cdot\cdot a_nb_1b_2\cdot\cdot b_n}(n)=\frac{1}{(n!)^2}\sum_{r=0}^{[n/2]}(-\frac{1}{2})^r\frac{1}{r!}\frac{n(n-1)\cdot\cdot(n-2r+1)}{(2n-1)(2n-3)\cdot\cdot(2n-2r+1)}[P_{\{a_1a_2}P_{(b_1b_2}\cdot\cdot P_{a_{2r-1}a_{2r}}P_{b_{2r-1}b_{2r}}][P_{a_{2r+1}b_{2r+1}}\cdot\cdot P_{a_n\}b_n)}]$$

$$=\frac{1}{(2n)!}\sum_{r=0}^{[n/2]}(-1)^rC_n^rC_{2n-2r}^n[P_{\{a_1a_2}P_{(b_1b_2}\cdot\cdot P_{a_{2r-1}a_{2r}}P_{b_{2r-1}b_{2r}}][P_{a_{2r+1}b_{2r+1}}\cdot\cdot P_{a_n\}b_n)}]$$

Ass. 2.6.3. $P_{a_1a_2\cdot\cdot a_nb_1b_2\cdot\cdot b_n}(n)=P_{\{a_1a_2\cdot\cdot a_{n-1},(b_1b_2\cdot\cdot b_{n-1}}(n-1)P_{a_n\}b_n)}$
$+P_{\{a_1a_2\cdot\cdot a_{n-1},a_n\}(b_1b_2\cdot\cdot b_{n-2}}(n-1)P_{b_{n-1}b_n)}+P_{\{a_1a_2\cdot\cdot a_{n-2}(b_n,b_1b_2\cdot\cdot b_{n-1})}(n-1)P_{a_{n-1}a_n\}}$

Ass. 2.6.4.

$$\begin{cases}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h)=\frac{n+1}{2n+3}\sum\limits_{h=n+1}^{-(n+1)}\sum\limits_{h'=1}^{-1}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\sigma^c_+\lambda_m(\vec{p},h')[\varepsilon_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\sigma^{c'}_+\lambda_m(\vec{p},h')]^+ \\ \sum\limits_{h=n+1/2}^{-(n+1/2)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},h)=\frac{n+1}{2n+3}\sum\limits_{h=n+1}^{-(n+1)}\sum\limits_{h'=1/2}^{-1/2}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\gamma^cu(\vec{p},h')[\varepsilon_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^{c'}u(\vec{p},h')]^+\end{cases}$$

3 Anticommutation rules for Rarita-Schwinger equation

3.1 B-W equation is equivalent to R-S equation for $s=n+\frac{1}{2}$ particles with mass [16, 17, 20]

Thm. 3.1.1.

$$\begin{cases}(\gamma^a\partial_a+m)\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)=0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)\ \textit{fully symmetric}\end{cases}\Leftrightarrow\begin{cases}(\gamma^c\partial_c+m)A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0 \\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0,\gamma^aA_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)\ \textit{fully symmetric} \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)=\frac{1}{2^n}\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x)\end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{r},t)$$

$$=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n+1/2}^{-(n+1/2)}E^n\sqrt{\frac{m}{E}}^{2n+1}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$A_{ab\cdot\cdot\tau_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}[\varepsilon_{ab\cdot\cdot\tau_\varsigma}(\vec{p},h)a(\vec{p},h)e^{ip\cdot x}+\tilde{\varepsilon}_{ab\cdot\cdot\tau_\varsigma}(\vec{p},h)b^+(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

Self comment: Treat it the same way as the boson case. The following two corollaries can also be easily obtained by substituting the respective plane wave solutions into the above two equivalent equations and using the Fourier component equivalence.

Cor. 3.1.1.

$$\begin{cases}(i\gamma^ap_a+m)U_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)=0 \\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)\ \textit{fully symmetric},\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h) \\ =\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)\end{cases}\Leftrightarrow\begin{cases}(i\gamma^cp_c+m)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)=0 \\ \varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\ \textit{fully symmetric},\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)=0, \\ \gamma^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)=0 \\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h) \\ =\frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdot\cdot}_{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 3.1.2.

$$\begin{cases}(-i\gamma^a p_a+m)V_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)=0\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)\ fully\ symmetric,\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\\ =\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)\end{cases}\Leftrightarrow\begin{cases}(-i\gamma^c p_c+m)\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p},h)=0\\ \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p},h)\ fully\ symmetric,\delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p},h)=0\\ \gamma^a\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p},h)=0,p^a\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p},h)=0\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)\\ =\frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(-p)\cdots}_{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 3.1.3. $\Lambda_-(\vec{p},\frac{1}{2})\gamma_4\varepsilon_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p},h)=0,\Lambda_+(\vec{p},\frac{1}{2})\gamma_4\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p},h)=0$

3.2 Spin basis of Rarita-Schwinger equation for $s=n+\frac{1}{2}$ particles

Thm. 3.2.1.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\\ [\Rightarrow]\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\\ [\Rightarrow]\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)\end{cases}$$

Thm. 3.2.2.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\\ [\Rightarrow]\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\\ [\Rightarrow]\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)\end{cases}$$

Cor. 3.2.1.

$$\begin{cases}\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p},h)\\ =\frac{1}{(2\sqrt{2}m)^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}(p)\cdots}_{n}\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdots}_{n}\tau'_\varsigma}(\vec{p},h)\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p},h)\\ =\frac{1}{(2\sqrt{2}m)^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}(p)\cdots}_{n}\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdots}_{n}\tau'_\varsigma}(\vec{p},h)\end{cases}$$

Cor. 3.2.2.

$$\begin{cases}\sum\limits_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\\ =\frac{1}{2^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\sum\limits_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec{p},h)\\ \sum\limits_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\\ =\frac{1}{2^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\sum\limits_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec{p},h)\end{cases}$$

Thm. 3.2.3. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=-\varsigma(-1)^n\gamma_{5\tau_\varsigma}{}^{\sigma_\varsigma}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot\sigma_\varsigma}_{n}}(\vec{p},h)$

Proof: $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\tau_\varsigma}(\vec{p},h)$

$=(-\varsigma)^{2n+1}\frac{1}{(i\sqrt{2})^n}\overbrace{(\gamma_5\bar{C}\gamma_a\gamma_5)^{\lambda_\varsigma\mu_\varsigma}(\gamma_5\bar{C}\gamma_b\gamma_5)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\gamma_{5\tau_\varsigma}{}^{\sigma_\varsigma}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\sigma_\varsigma}(\vec{p},h)$

$=-\varsigma(-1)^n\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\gamma_{5\tau_\varsigma}{}^{\sigma_\varsigma}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\sigma_\varsigma}(\vec{p},h)$

$=-\varsigma(-1)^n\gamma_{5\tau_\varsigma}{}^{\sigma_\varsigma}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot\sigma_\varsigma}_{n}}(\vec{p},h)$ □

Thm. 3.2.4. $\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=(-1)^{h-\frac{1}{2}}(\gamma_2\gamma_5)_{\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{\eta^a_{a'}\eta^b_{b'}\cdot\cdot}^{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},-h)$

Proof: $\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=\frac{1}{(-i\sqrt{2})^n}\overbrace{(\bar{C}\gamma^*_{a'})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma^*_{b'})^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}U^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\tau'_\varsigma}(\vec{p},h)$

$=\frac{1}{(-i\sqrt{2})^n}\overbrace{(\bar{C}\gamma^*_{a'})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma^*_{b'})^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}(-1)^{n+\frac{1}{2}+h}\varsigma^{2n+1}\overbrace{\sigma_y\otimes\sigma_y\cdot\cdot}^{4n+2}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\tau'_\varsigma}(\vec{p},-h)$

$=\varsigma\frac{(-1)^{n+\frac{1}{2}+h}}{(-i\sqrt{2})^n}\gamma_{2\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{(\gamma_2\bar{C}\gamma^*_{a'}\gamma_2)^{\lambda_\varsigma\mu_\varsigma}(\gamma_2\bar{C}\gamma^*_{b'}\gamma_2)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\tau_\varsigma}(\vec{p},-h)$

$=\varsigma\frac{(-1)^{n+\frac{1}{2}+h}}{(i\sqrt{2})^n}\gamma_{2\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{(\bar{C}\gamma_a\eta^a_{a'})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b\eta^b_{b'})^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\tau_\varsigma}(\vec{p},-h)$

$=\varsigma\frac{(-1)^{n+\frac{1}{2}+h}}{(i\sqrt{2})^n}\gamma_{2\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{\eta^a_{a'}\eta^b_{b'}\cdot\cdot}^{n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}\tau_\varsigma}(\vec{p},-h)$

$=\varsigma(-1)^{n+\frac{1}{2}+h}\gamma_{2\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{\eta^a_{a'}\eta^b_{b'}\cdot\cdot}^{n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},-h)$

$=(-1)^{h-\frac{1}{2}}(\gamma_2\gamma_5)_{\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{\eta^a_{a'}\eta^b_{b'}\cdot\cdot}^{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},-h)$ □

3.3 Plane wave solutions of R-S equation for $s=n+\frac{1}{2}$ particles

Cor. 3.3.1.

$$A_{ab\cdot\cdot\tau_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}[\varepsilon_{ab\cdot\cdot\tau_\varsigma}(\vec{p},h)a(\vec{p},h)e^{ip\cdot x}+\tilde{\varepsilon}_{ab\cdot\cdot\tau_\varsigma}(\vec{p},h)b^+(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

Proof: $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\}$

$$= \frac{1}{(2\pi)^{3/2}}\int\sum_{h,h'=n}^{-n} d^3\vec{p}d^3\vec{p}'\frac{\sqrt{m}}{\sqrt{2^nE}}\frac{\sqrt{m}}{\sqrt{2^nE'}}$$

$$\{a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)e^{-ip\cdot x}, a^+(\vec{p}',h')\varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}',h')e^{-ip'\cdot x'}+b(\vec{p}',h')\tilde{\varepsilon}^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}',h')e^{ip'\cdot x'}\}$$

$$= \frac{1}{(2\pi)^{3/2}}\int\sum_{h,h'=n}^{-n} d^3\vec{p}d^3\vec{p}'\frac{\sqrt{m}}{\sqrt{2^nE}}\frac{\sqrt{m}}{\sqrt{2^nE'}}e^{-ip\cdot x}e^{ip'\cdot x'}$$

$$\{\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}',h')\{a(\vec{p},h),a^+(\vec{p}',h')\}e^{ip\cdot x}e^{-ip'\cdot x'}+\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}',h')\{b^+(\vec{p},h),b(\vec{p}',h')\}\}$$

$$= \frac{1}{(2\pi)^{3/2}}\int\sum_{h,h'=n}^{-n} d^3\vec{p}d^3\vec{p}'\frac{\sqrt{m}}{\sqrt{2^nE}}\frac{\sqrt{m}}{\sqrt{2^nE'}}\delta_{hh'}\delta(\vec{p}-\vec{p}')$$

$$[\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}',h')e^{ip\cdot x}e^{-ip'\cdot x'}+\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}',h')e^{-ip\cdot x}e^{ip'\cdot x'}]$$

$$= \frac{im}{2^{n-1}}\int\sum_{h=n}^{-n}\{\frac{1}{(2\pi)^{3/2}}\frac{-i}{2E}[\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)e^{ip\cdot(x-x')}+\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)e^{-ip\cdot(x-x')}]d^3\vec{p}\}$$

$$= \frac{im}{2^{n-1}}[\sum_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(-i\partial,h)\varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(-i\partial,h)]\int\{\frac{1}{(2\pi)^{3/2}}\frac{-i}{2E}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]d^3\vec{p}\}$$

$$= \frac{im}{2^{n-1}}[\sum_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(-i\partial,h)\varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(-i\partial,h)]\Delta(x-x')$$ □

3.4 Spin basis properties of Rarita-Schwinger equation for $s=n+\frac{1}{2}$ particles

Thm. 3.4.1.

$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n+1}}(\vec{p},n+\frac{1}{2}-l) \\ =\frac{1}{\sqrt{C^l_{2n+1}}}[\sqrt{C^{l-1}_{2n}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l+1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^l_{2n}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})] \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n+1}}(\vec{p},n+\frac{1}{2}-l) \\ =\frac{1}{\sqrt{C^l_{2n+1}}}[\sqrt{C^{l-1}_{2n}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l+1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^l_{2n}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})] \end{cases}$$

Cor. 3.4.1.

$$\begin{cases} \frac{m}{E}u^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n+1}}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^l_{2n+1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l+1)=\sqrt{\frac{l}{2n+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l+1) \\ \frac{m}{E}u^{+\tau_\varsigma}(\vec{p},\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n+1}}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l}_{2n}}}{\sqrt{C^l_{2n+1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l)=\sqrt{\frac{2n+1-l}{2n+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l) \end{cases}$$

$$\begin{cases} \frac{m}{E}v^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n+1}}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^l_{2n+1}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l+1)=\sqrt{\frac{l}{2n+1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l+1) \\ \frac{m}{E}v^{+\tau_\varsigma}(\vec{p},\frac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2n+1}}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l}_{2n}}}{\sqrt{C^l_{2n+1}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l)=\sqrt{\frac{2n+1-l}{2n+1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\rho_\varsigma\sigma_\varsigma}_{2n}}(\vec{p},n-l) \end{cases}$$

Thm. 3.4.2. $\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{p},n)=\frac{2n+1}{2n+2}(\frac{m}{E})^2\Lambda_{\pm\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec{p},n+\frac{1}{2})\Lambda^{\tau'_\varsigma\tau_\varsigma}_{\pm}(\vec{p},\frac{1}{2})$

Cor. 3.4.2.

$$\begin{cases} \varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{1}{\sqrt{C^l_{2n+1}}}[\sqrt{C^{l-1}_{2n}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^l_{2n}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})] \\ \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{1}{\sqrt{C^l_{2n+1}}}[\sqrt{C^{l-1}_{2n}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^l_{2n}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})] \end{cases}$$

Cor. 3.4.3.

$$\begin{cases} \frac{m}{E}u^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^l_{2n+1}}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)=\sqrt{\frac{l}{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1) \\ \frac{m}{E}u^{+\tau_\varsigma}(\vec{p},\frac{1}{2})\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l}_{2n}}}{\sqrt{C^l_{2n+1}}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)=\sqrt{\frac{2n+1-l}{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l) \end{cases}$$

$$\begin{cases}\frac{m}{E}v^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})\tilde{\varepsilon}_{\underbrace{ab\,..}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C_{2n}^{l-1}}}{\sqrt{C_{2n+1}^{l}}}\tilde{\varepsilon}_{\underbrace{ab\,..}_{n}}(\vec{p},n-l+1)=\sqrt{\frac{l}{2n+1}}\tilde{\varepsilon}_{\underbrace{ab\,..}_{n}}(\vec{p},n-l+1)\\ \frac{m}{E}v^{+\tau_\varsigma}(\vec{p},\frac{1}{2})\tilde{\varepsilon}_{\underbrace{ab\,..}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C_{2n}^{l}}}{\sqrt{C_{2n+1}^{l}}}\tilde{\varepsilon}_{\underbrace{ab\,..}_{n}}(\vec{p},n-l)=\sqrt{\frac{2n+1-l}{2n+1}}\tilde{\varepsilon}_{\underbrace{ab\,..}_{n}}(\vec{p},n-l)\end{cases}$$

Thm. 3.4.3. $\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}}(\vec{p},h)=\frac{2n+1}{2n+2}(\frac{m}{E})^2\sum\limits_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\,..}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}\tau'_\varsigma}(\vec{p},h)\Lambda_{+}^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\frac{1}{2})$

Proof: $(\frac{m}{E})^2\sum\limits_{l=0}^{2n+1}[u^{\tau'_\varsigma}(\vec{p},-\frac{1}{2})u^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})+u^{\tau'_\varsigma}(\vec{p},\frac{1}{2})u^{+\tau_\varsigma}(\vec{p},\frac{1}{2})]\varepsilon_{\underbrace{ab\,..}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2}-l)$

$=(\frac{m}{E})^2\sum\limits_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\,..}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}\tau'_\varsigma}(\vec{p},h)\Lambda_{+}^{\tau'_\varsigma\tau_\varsigma}$

$=\sum\limits_{l=0}^{2n+1}[\frac{l}{2n+1}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},n-l+1)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}}(\vec{p},n-l+1)+\frac{2n+1-l}{2n+1}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},n-l)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}}(\vec{p},n-l)]$

$=\frac{2n+2}{2n+1}\sum\limits_{l=1}^{2n+1}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},n-l+1)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}}(\vec{p},n-l+1)$

$=\frac{2n+2}{2n+1}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}}(\vec{p},h)$ □

Proof: $\sum\limits_{l=0}^{2n+1}\varepsilon_{\underbrace{ab\,..}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2}-l)$

$=\sum\limits_{l=0}^{2n+1}\frac{1}{C_{2n+1}^{l}}[\sqrt{C_{2n}^{l-1}}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},n-l+1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C_{2n}^{l}}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},n-l)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]$

$[\sqrt{C_{2n}^{l-1}}\varepsilon_{\underbrace{a'b'\,..}_{n}}(\vec{p},n-l+1)u_{\tau'_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C_{2n}^{l}}\varepsilon_{\underbrace{a'b'\,..}_{n}}(\vec{p},n-l)u_{\tau'_\varsigma}(\vec{p},\frac{1}{2})]^{+}$

$??=\sum\limits_{l=0}^{2n+1}\frac{1}{C_{2n+1}^{l}}[C_{2n}^{l-1}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},n-l+1)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}}(\vec{p},n-l+1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})u^{+}_{\tau'_\varsigma}(\vec{p},-\frac{1}{2})$

$+\sum\limits_{l=0}^{2n+1}\frac{1}{C_{2n+1}^{l}}C_{2n}^{l}\varepsilon_{\underbrace{ab\,..}_{n}}(\vec{p},n-l)\varepsilon^{+}_{\underbrace{a'b'\,..}_{n}}(\vec{p},n-l)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})u^{+}_{\tau'_\varsigma}(\vec{p},\frac{1}{2})$ □

4 Several examples of R-S equation spin basis and quasi projection operators

4.1 Relations of R-S spin basis and Dirac basis for spin-1 particles

Lem. 4.1.1.

$$\begin{cases}[\gamma^a\varepsilon_a(\vec{p},\kappa)]u(\vec{p},\frac{\kappa}{2})=0,[\gamma^a\varepsilon_a(\vec{p},-\kappa)]u(\vec{p},\frac{\kappa}{2})=-i\kappa\sqrt{2}\gamma_5u(\vec{p},-\frac{\kappa}{2}),[\gamma^a\varepsilon_a(\vec{p},0)]u(\vec{p},\frac{\kappa}{2})=-i\kappa\gamma_5u(\vec{p},\frac{\kappa}{2})\\ [\gamma^a\varepsilon_a(\vec{p},\kappa)]v(\vec{p},\frac{\kappa}{2})=0,[\gamma^a\varepsilon_a(\vec{p},-\kappa)]v(\vec{p},\frac{\kappa}{2})=i\kappa\sqrt{2}\gamma_5v(\vec{p},-\frac{\kappa}{2}),[\gamma^a\varepsilon_a(\vec{p},0)]v(\vec{p},\frac{\kappa}{2})=i\kappa\gamma_5v(\vec{p},\frac{\kappa}{2})\end{cases}$$

Pro. 4.1.1.

$$\begin{cases}u(\vec{p},\frac{1}{2})=-\frac{i\varsigma}{\sqrt{2}}[\gamma^a\varepsilon_a(\vec{p},1)]v(\vec{p},-\frac{1}{2})=i\varsigma[\gamma^a\varepsilon_a(\vec{p},0)]v(\vec{p},\frac{1}{2})\\ u(\vec{p},-\frac{1}{2})=\frac{i\varsigma}{\sqrt{2}}[\gamma^a\varepsilon_a(\vec{p},-1)]v(\vec{p},\frac{1}{2})=-i\varsigma[\gamma^a\varepsilon_a(\vec{p},0)]v(\vec{p},-\frac{1}{2})\end{cases}$$

$$\begin{cases}v(\vec{p},\frac{1}{2})=\frac{i\varsigma}{\sqrt{2}}[\gamma^a\varepsilon_a(\vec{p},1)]u(\vec{p},-\frac{1}{2})=-i\varsigma[\gamma^a\varepsilon_a(\vec{p},0)]u(\vec{p},\frac{1}{2})\\ v(\vec{p},-\frac{1}{2})=-\frac{i\varsigma}{\sqrt{2}}[\gamma^a\varepsilon_a(\vec{p},-1)]u(\vec{p},\frac{1}{2})=i\varsigma[\gamma^a\varepsilon_a(\vec{p},0)]u(\vec{p},-\frac{1}{2})\end{cases}$$

Thm. 4.1.1.

$$\begin{cases}\sum\limits_{h=1/2}^{-1/2}u_{\tau_\varsigma}(\vec{p},h)u^{+}_{\tau'_\varsigma}(\vec{p},h)=\frac{1}{3}\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^{+}_{a'}(\vec{p},h)\gamma^a\Lambda_{-}(\vec{p},\frac{1}{2})\gamma^{a'}\\ \sum\limits_{h=1/2}^{-1/2}v_{\tau_\varsigma}(\vec{p},h)v^{+}_{\tau'_\varsigma}(\vec{p},h)=\frac{1}{3}\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^{+}_{a'}(\vec{p},h)\gamma^a\Lambda_{+}(\vec{p},\frac{1}{2})\gamma^{a'}\end{cases}$$

Proof: $\sum\limits_{h=1/2}^{-1/2}u_{\tau_\varsigma}(\vec{p},h)u^{+}_{\tau'_\varsigma}(\vec{p},h)$

$=\frac{1}{3}\sum\limits_{h=2}^{-2}\{[\varepsilon_a(\vec{p},h)\gamma^a v(\vec{p},\frac{1}{2})][\varepsilon_{a'}(\vec{p},h)\gamma^{a'}v(\vec{p},\frac{1}{2})]^{+}+[\varepsilon_a(\vec{p},h)\gamma^a v(\vec{p},-\frac{1}{2})][\varepsilon_{a'}(\vec{p},h)\gamma^{a'}v(\vec{p},-\frac{1}{2})]^{+}\}$

$=\frac{1}{3}\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^{+}_{a'}(\vec{p},h)\gamma^a\Lambda_{-}(\vec{p},\frac{1}{2})\gamma^{a'}$ □

Proof: $\sum_{h=1/2}^{-1/2} v_{\tau_\varsigma}(\vec{p},h)v^+_{\tau'_\varsigma}(\vec{p},h)$

$= \frac{1}{3}\sum_{h=2}^{-2}\{[\varepsilon_a(\vec{p},h)\gamma^a u(\vec{p},\frac{1}{2})][\varepsilon_{a'}(\vec{p},h)\gamma^{a'}u(\vec{p},\frac{1}{2})]^+ + [\varepsilon_a(\vec{p},h)\gamma^a u(\vec{p},-\frac{1}{2})][\varepsilon_{a'}(\vec{p},h)\gamma^{a'}u(\vec{p},-\frac{1}{2})]^+\}$

$= \frac{1}{3}\sum_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h)\gamma^a\Lambda_+(\vec{p},\frac{1}{2})\gamma^{a'}$ □

The direct verification of the above theorem is also valid.

4.2 Relations of R-S spin basis and Dirac basis for spin-2 particles

Lem. 4.2.1.

$$\begin{cases}\varepsilon_{ab}(\vec{p},2)\gamma^b u(\vec{p},\frac{1}{2}) = 0\\ \varepsilon_{ab}(\vec{p},1)\gamma^b u(\vec{p},\frac{1}{2}) = -\frac{i}{\sqrt{2}}\varepsilon_a(\vec{p},1)\gamma_5 u(\vec{p},\frac{1}{2})\\ \varepsilon_{ab}(\vec{p},0)\gamma^b u(\vec{p},\frac{1}{2}) = -\frac{i}{\sqrt{3}}[\varepsilon_a(\vec{p},1)\gamma_5 u(\vec{p},-\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 u(\vec{p},\frac{1}{2})]\\ \varepsilon_{ab}(\vec{p},-1)\gamma^b u(\vec{p},\frac{1}{2}) = -\frac{i}{\sqrt{2}}[\varepsilon_a(\vec{p},-1)\gamma_5 u(\vec{p},\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 u(\vec{p},-\frac{1}{2})]\\ \varepsilon_{ab}(\vec{p},-2)\gamma^b u(\vec{p},\frac{1}{2}) = -i\sqrt{2}\varepsilon_a(\vec{p},-1)\gamma_5 u(\vec{p},-\frac{1}{2})\end{cases}$$

Lem. 4.2.2.

$$\begin{cases}\varepsilon_{ab}(\vec{p},2)\gamma^b u(\vec{p},-\frac{1}{2}) = i\sqrt{2}\varepsilon_a(\vec{p},1)\gamma_5 u(\vec{p},\frac{1}{2})\\ \varepsilon_{ab}(\vec{p},1)\gamma^b u(\vec{p},-\frac{1}{2}) = \frac{i}{\sqrt{2}}[\varepsilon_a(\vec{p},1)\gamma_5 u(\vec{p},-\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 u(\vec{p},\frac{1}{2})\\ \varepsilon_{ab}(\vec{p},0)\gamma^b u(\vec{p},-\frac{1}{2}) = \frac{i}{\sqrt{3}}[\varepsilon_a(\vec{p},-1)\gamma_5 u(\vec{p},\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 u(\vec{p},-\frac{1}{2})]\\ \varepsilon_{ab}(\vec{p},-1)\gamma^b u(\vec{p},-\frac{1}{2}) = \frac{i}{\sqrt{2}}\varepsilon_a(\vec{p},-1)\gamma_5 u(\vec{p},-\frac{1}{2})\\ \varepsilon_{ab}(\vec{p},-2)\gamma^b u(\vec{p},-\frac{1}{2}) = 0\end{cases}$$

Lem. 4.2.3.

$$\begin{cases}\varepsilon_{ab}(\vec{p},2)\gamma^b v(\vec{p},\frac{1}{2}) = 0\\ \varepsilon_{ab}(\vec{p},1)\gamma^b v(\vec{p},\frac{1}{2}) = \frac{i}{\sqrt{2}}\varepsilon_a(\vec{p},1)\gamma_5 v(\vec{p},\frac{1}{2})\\ \varepsilon_{ab}(\vec{p},0)\gamma^b v(\vec{p},\frac{1}{2}) = \frac{i}{\sqrt{3}}[\varepsilon_a(\vec{p},1)\gamma_5 v(\vec{p},-\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 v(\vec{p},\frac{1}{2})]\\ \varepsilon_{ab}(\vec{p},-1)\gamma^b v(\vec{p},\frac{1}{2}) = \frac{i}{\sqrt{2}}[\varepsilon_a(\vec{p},-1)\gamma_5 v(\vec{p},\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 v(\vec{p},-\frac{1}{2})]\\ \varepsilon_{ab}(\vec{p},-2)\gamma^b v(\vec{p},\frac{1}{2}) = i\sqrt{2}\varepsilon_a(\vec{p},-1)\gamma_5 v(\vec{p},-\frac{1}{2})\end{cases}$$

Lem. 4.2.4.

$$\begin{cases}\varepsilon_{ab}(\vec{p},2)\gamma^b v(\vec{p},-\frac{1}{2}) = -i\sqrt{2}\varepsilon_a(\vec{p},1)\gamma_5 v(\vec{p},\frac{1}{2})\\ \varepsilon_{ab}(\vec{p},1)\gamma^b v(\vec{p},-\frac{1}{2}) = -\frac{i}{\sqrt{2}}[\varepsilon_a(\vec{p},1)\gamma_5 v(\vec{p},-\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 v(\vec{p},\frac{1}{2})]\\ \varepsilon_{ab}(\vec{p},0)\gamma^b v(\vec{p},-\frac{1}{2}) = -\frac{i}{\sqrt{3}}[\varepsilon_a(\vec{p},-1)\gamma_5 vu(\vec{p},\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)\gamma_5 v(\vec{p},-\frac{1}{2})]\\ \varepsilon_{ab}(\vec{p},-1)\gamma^b v(\vec{p},-\frac{1}{2}) = -\frac{i}{\sqrt{2}}\varepsilon_a(\vec{p},-1)\gamma_5 v(\vec{p},-\frac{1}{2})\\ \varepsilon_{ab}(\vec{p},-2)\gamma^b v(\vec{p},-\frac{1}{2}) = 0\end{cases}$$

4.3 Quasi projection operator of R-S equation for $s = \frac{3}{2}$ particles

Pro. 4.3.1.

$$\begin{cases}\varepsilon_{a\tau_\varsigma}(\vec{p},\frac{3}{2}) = \varepsilon_a(\vec{p},1)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})\\ \varepsilon_{a\tau_\varsigma}(\vec{p},\frac{1}{2}) = \frac{1}{\sqrt{3}}[\varepsilon_a(\vec{p},1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]\\ \varepsilon_{a\tau_\varsigma}(\vec{p},-\frac{1}{2}) = \frac{1}{\sqrt{3}}[\varepsilon_a(\vec{p},-1)u_{\tau_\varsigma}(\vec{p},\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})]\\ \varepsilon_{a\tau_\varsigma}(\vec{p},-\frac{3}{2}) = \varepsilon_a(\vec{p},-1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})\\ \gamma^a\varepsilon_{a[\tau_\varsigma]}(\vec{p},h) = 0\end{cases}$$

Pro. 4.3.2.

$$\begin{cases}\tilde{\varepsilon}_{a\tau_\varsigma}(\vec{p},\frac{3}{2}) = -\varepsilon_a(\vec{p},1)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})\\ \tilde{\varepsilon}_{a\tau_\varsigma}(\vec{p},\frac{1}{2}) = -\frac{1}{\sqrt{3}}[\varepsilon_a(\vec{p},1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})]\\ \tilde{\varepsilon}_{a\tau_\varsigma}(\vec{p},-\frac{1}{2}) = -\frac{1}{\sqrt{3}}[\varepsilon_a(\vec{p},-1)v_{\tau_\varsigma}(\vec{p},\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p},0)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})]\\ \tilde{\varepsilon}_{a\tau_\varsigma}(\vec{p},-\frac{3}{2}) = -\varepsilon_a(\vec{p},-1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})\\ \gamma^a\tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},h) = 0\end{cases}$$

Cor. 4.3.1.

$$\begin{cases}\varepsilon_{a[\tau_\varsigma]}(\vec{p},\frac{3}{2})=-\frac{i\varsigma}{\sqrt{2}}\varepsilon_{ab}(\vec{p},2)\gamma^b v(\vec{p},-\frac{1}{2})=i\varsigma\sqrt{2}\varepsilon_a(\vec{p},1)\gamma_5 v(\vec{p},\frac{1}{2})\\ \varepsilon_{a[\tau_\varsigma]}(\vec{p},\frac{1}{2})=-i\varsigma\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p},1)\gamma^b v(\vec{p},-\frac{1}{2})=i\varsigma\varepsilon_{ab}(\vec{p},0)\gamma^b v(\vec{p},\frac{1}{2})\\ \varepsilon_{a[\tau_\varsigma]}(\vec{p},-\frac{1}{2})=i\varsigma\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p},-1)\gamma^b v(\vec{p},\frac{1}{2})=-i\varsigma\varepsilon_{ab}(\vec{p},0)\gamma^b v(\vec{p},-\frac{1}{2})\\ \varepsilon_{a[\tau_\varsigma]}(\vec{p},-\frac{3}{2})=\frac{i\varsigma}{\sqrt{2}}\varepsilon_{ab}(\vec{p},-2)\gamma^b v(\vec{p},\frac{1}{2})=-i\varsigma\sqrt{2}\varepsilon_a(\vec{p},-1)\gamma_5 v(\vec{p},-\frac{1}{2})\\ \gamma^a\varepsilon_{a[\tau_\varsigma]}(\vec{p},h)=0\end{cases}$$

Cor. 4.3.2.

$$\begin{cases}\tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},\frac{3}{2})=-\frac{i\varsigma}{\sqrt{2}}\varepsilon_{ab}(\vec{p},2)\gamma^b u(\vec{p},-\frac{1}{2})=i\varsigma\sqrt{2}\varepsilon_a(\vec{p},1)\gamma_5 u(\vec{p},\frac{1}{2})\\ \tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},\frac{1}{2})=-i\varsigma\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p},1)\gamma^b u(\vec{p},-\frac{1}{2})=i\varsigma\varepsilon_{ab}(\vec{p},0)\gamma^b u(\vec{p},\frac{1}{2})\\ \tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},-\frac{1}{2})=i\varsigma\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p},-1)\gamma^b u(\vec{p},\frac{1}{2})=-i\varsigma\varepsilon_{ab}(\vec{p},0)\gamma^b u(\vec{p},-\frac{1}{2})\\ \tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},-\frac{3}{2})=\frac{i\varsigma}{\sqrt{2}}\varepsilon_{ab}(\vec{p},-2)\gamma^b u(\vec{p},\frac{1}{2})=-i\varsigma\sqrt{2}\varepsilon_a(\vec{p},-1)\gamma_5 u(\vec{p},-\frac{1}{2})\\ \gamma^a\tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},h)=0\end{cases}$$

Thm. 4.3.1.

$$\begin{cases}\sum_{h=3/2}^{-3/2}\varepsilon_{a[\tau_\varsigma]}(\vec{p},h)\varepsilon^+_{a'[\tau'_\varsigma]}(\vec{p},h)=\frac{2}{5}\sum_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)\gamma^b\Lambda_-(\vec{p},\frac{1}{2})\gamma^{b'}\\ \sum_{h=3/2}^{-3/2}\tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},h)\tilde{\varepsilon}^+_{a'[\tau'_\varsigma]}(\vec{p},h)=\frac{2}{5}\sum_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)\gamma^b\Lambda_+(\vec{p},\frac{1}{2})\gamma^{b'}\end{cases}$$

Proof: $\sum_{h=3/2}^{-3/2}\varepsilon_{a[\tau_\varsigma]}(\vec{p},h)\varepsilon^+_{a'[\tau'_\varsigma]}(\vec{p},h)$

$$=\frac{2}{5}\sum_{h=2}^{-2}\{[\varepsilon_{ab}(\vec{p},h)\gamma^b v(\vec{p},\frac{1}{2})][\varepsilon_{a'b'}(\vec{p},h)\gamma^{b'}v(\vec{p},\frac{1}{2})]^+ + [\varepsilon_{ab}(\vec{p},h)\gamma^b v(\vec{p},-\frac{1}{2})][\varepsilon_{a'b'}(\vec{p},h)\gamma^{b'}v(\vec{p},-\frac{1}{2})]^+\}$$

$$=\frac{2}{5}\sum_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)\gamma^b\Lambda_-(\vec{p},\frac{1}{2})\gamma^{b'}$$ □

Proof: $\sum_{h=3/2}^{-3/2}\tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p},h)\tilde{\varepsilon}^+_{a'[\tau'_\varsigma]}(\vec{p},h)$

$$=\frac{2}{5}\sum_{h=2}^{-2}\{[\varepsilon_{ab}(\vec{p},h)\gamma^b u(\vec{p},\frac{1}{2})][\varepsilon_{a'b'}(\vec{p},h)\gamma^{b'}u(\vec{p},\frac{1}{2})]^+ + [\varepsilon_{ab}(\vec{p},h)\gamma^b u(\vec{p},-\frac{1}{2})][\varepsilon_{a'b'}(\vec{p},h)\gamma^{b'}u(\vec{p},-\frac{1}{2})]^+\}$$

$$=\frac{2}{5}\sum_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)\gamma^b\Lambda_+(\vec{p},\frac{1}{2})\gamma^{b'}$$ □

4.4 Conjecture proof on quasi projection operator of R-S equation for $s=n+\frac{1}{2}$ particles

Lem. 4.4.1.

$$\begin{cases}[\gamma^a\varepsilon_a(\vec{p},\kappa)]u(\vec{p},\frac{\kappa}{2})=0,[\gamma^a\varepsilon_a(\vec{p},-\kappa)]u(\vec{p},\frac{\kappa}{2})=i\sqrt{2}\kappa\varsigma v(\vec{p},-\frac{\kappa}{2}),[\gamma^a\varepsilon_a(\vec{p},0)]u(\vec{p},\frac{\kappa}{2})=i\kappa\varsigma v(\vec{p},\frac{\kappa}{2})\\ [\gamma^a\varepsilon_a(\vec{p},\kappa)]v(\vec{p},\frac{\kappa}{2})=0,[\gamma^a\varepsilon_a(\vec{p},-\kappa)]v(\vec{p},\frac{\kappa}{2})=-i\sqrt{2}\kappa\varsigma u(\vec{p},-\frac{\kappa}{2}),[\gamma^a\varepsilon_a(\vec{p},0)]v(\vec{p},\frac{\kappa}{2})=-i\kappa\varsigma u(\vec{p},\frac{\kappa}{2})\end{cases}$$

Lem. 4.4.2. $\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n+1}}(\vec{p},n+1)\gamma^c u(\vec{p},\frac{1}{2})$

$$=\frac{1}{\sqrt{C^0_{2(n+1)}}}\frac{1}{(n+1)!}\sqrt{2^0}C^0_{n+1}C^0_{n+1-0}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot}_{n+1}\underbrace{\cdot\varepsilon_b(\vec{p},0)\cdot}_{0}\underbrace{\cdot\varepsilon_{c\}}(\vec{p},-1)}_{0}\gamma^c u(\vec{p},\frac{1}{2})$$

$$=\underbrace{\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot}_{n}\varepsilon_c(\vec{p},1)\gamma^c u(\vec{p},\frac{1}{2})=0$$

Lem. 4.4.3. $\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n+1}}(\vec{p},n+1)\gamma^c u(\vec{p},-\frac{1}{2})$

$$=\frac{1}{\sqrt{C^0_{2(n+1)}}}\frac{1}{(n+1)!}\sqrt{2^0}C^0_{n+1}C^0_{n+1-0}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot}_{n+1}\underbrace{\cdot\varepsilon_b(\vec{p},0)\cdot}_{0}\underbrace{\cdot\varepsilon_{c\}}(\vec{p},-1)}_{0}\gamma^c u(\vec{p},-\frac{1}{2})$$

$$=\underbrace{\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot}_{n}\varepsilon_c(\vec{p},1)\gamma^c u(\vec{p},\frac{1}{2})$$

$$=-i\varsigma\sqrt{2}(-1)^n\underbrace{\tilde{\varepsilon}_a(\vec{p},1)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot}_{n}v(\vec{p},\frac{1}{2})$$

$$=-i\varsigma\sqrt{2}(-1)^n\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},n+\frac{1}{2})$$

Lem. 4.4.4. $\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n+1}}(\vec{p},n)\gamma^c u(\vec{p},\frac{1}{2})$

$= \frac{1}{\sqrt{C^1_{2(n+1)}}}\frac{1}{(n+1)!}\sqrt{2^1}C^1_{n+1}C^0_n \underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{0}\gamma^c u(\vec{p},\frac{1}{2})$

$= \frac{1}{\sqrt{n+1}}$
$\underbrace{[\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},0)+\cdot\cdot+\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},0)\cdot\cdot\varepsilon_c(\vec{p},1)+\cdot\cdot+\varepsilon_a(\vec{p},0)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},1)]}_{n+1}\gamma^c u(\vec{p},\frac{1}{2})$

$= \frac{1}{\sqrt{n+1}}\underbrace{\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot}_{n}\varepsilon_c(\vec{p},0)\gamma^c u(\vec{p},\frac{1}{2})$

$= \frac{i\varsigma}{\sqrt{n+1}}(-1)^n\underbrace{\tilde{\varepsilon}_a(\vec{p},1)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot}_{n}v(\vec{p},\frac{1}{2})$

$= \frac{i\varsigma}{\sqrt{n+1}}(-1)^n\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},n+\frac{1}{2})$

Lem. 4.4.5. $\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c\cdot\cdot}_{n+1}}(\vec{p},n)\gamma^c u(\vec{p},-\frac{1}{2})$

$= \frac{1}{\sqrt{C^1_{2(n+1)}}}\frac{1}{(n+1)!}\sqrt{2^1}C^1_{n+1}C^0_n \underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{1}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{0}\gamma^c u(\vec{p},-\frac{1}{2})$

$= \frac{1}{\sqrt{n+1}}$
$\underbrace{[\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},0)+\cdot\cdot+\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},0)\cdot\cdot\varepsilon_c(\vec{p},1)+\cdot\cdot+\varepsilon_a(\vec{p},0)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot\varepsilon_c(\vec{p},1)]}_{n+1}\gamma^c u(\vec{p},-\frac{1}{2})$

$= \frac{1}{\sqrt{n+1}}$
$\{\underbrace{\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot}_{n}\varepsilon_c(\vec{p},0)\gamma^c u(\vec{p},-\frac{1}{2})+\underbrace{[\varepsilon_a(\vec{p},0)\cdot\cdot\varepsilon_b(\vec{p},1)\cdot\cdot+\cdot\cdot+\varepsilon_a(\vec{p},1)\cdot\cdot\varepsilon_b(\vec{p},0)\cdot\cdot]}_{n}\varepsilon_c(\vec{p},1)\gamma^c u(\vec{p},-\frac{1}{2})\}$

$= -\frac{i\varsigma}{\sqrt{n+1}}(-1)^n\{\underbrace{\tilde{\varepsilon}_a(\vec{p},1)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot}_{n}v(\vec{p},-\frac{1}{2})+\sqrt{2}\underbrace{[\tilde{\varepsilon}_a(\vec{p},0)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot+\cdot\cdot+\tilde{\varepsilon}_a(\vec{p},1)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},0)\cdot\cdot]}_{n}v(\vec{p},\frac{1}{2})\}$

$= -\frac{i\varsigma}{\sqrt{n+1}}(-1)^n\frac{1}{n!}[\sqrt{2^0}C^0_nC^0_{n-0}\underbrace{\tilde{\varepsilon}_{\{a}(\vec{p},1)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot_{\}}}_{n}v(\vec{p},-\frac{1}{2})+\sqrt{2^1}C^1_nC^0_{n-1}\underbrace{\tilde{\varepsilon}_{\{a}(\vec{p},0)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot_{\}}}_{n}v(\vec{p},\frac{1}{2})]$

$= -\frac{i\varsigma\sqrt{2n+1}}{\sqrt{n+1}}(-1)^n\frac{1}{\sqrt{C^1_{2n+1}}}[\sqrt{C^0_{2n}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}}v(\vec{p},-\frac{1}{2})+\sqrt{C^1_{2n}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}}v(\vec{p},\frac{1}{2})]$

$= -\frac{i\varsigma\sqrt{2n+1}}{\sqrt{n+1}}(-1)^n\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},n-\frac{1}{2})$

Lem. 4.4.6. $\varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n+1}}(\vec{p},n-1)\gamma^c u(\vec{p},\frac{1}{2})$

$= \frac{1}{\sqrt{C^2_{2(n+1)}}}\frac{1}{(n+1)!}$
$[\sqrt{2^0}C^0_{n+1}C^1_{n+1-0}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{0}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{1}+\sqrt{2^2}C^2_{n+1}C^0_{n-1}\underbrace{\varepsilon_{\{a}(\vec{p},1)\cdot\cdot}_{n-1}\underbrace{\varepsilon_b(\vec{p},0)\cdot\cdot}_{2}\underbrace{\varepsilon_{c\}}(\vec{p},-1)}_{0}]\gamma^c u(\vec{p},\frac{1}{2})$

$= \frac{1}{\sqrt{C^2_{2(n+1)}}}(-1)^n[\underbrace{\tilde{\varepsilon}_a(\vec{p},1)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot}_{n}\varepsilon_c(\vec{p},-1)\gamma^c u(\vec{p},\frac{1}{2})+\sqrt{2}\sqrt{2^1}\frac{1}{(n-1)!}\underbrace{\tilde{\varepsilon}_{\{a}(\vec{p},0)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot_{\}}}_{n}\varepsilon_c(\vec{p},0)\gamma^c u(\vec{p},\frac{1}{2})]$

$= \frac{i\sqrt{2(2n+1)}\varsigma}{\sqrt{C^2_{2(n+1)}}}(-1)^n\frac{1}{\sqrt{C^1_{2n+1}}}$
$[\sqrt{2^0}C^0_nC^0_{n-0}\frac{1}{n!}\underbrace{\tilde{\varepsilon}_{\{a}(\vec{p},1)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot_{\}}}_{n}v(\vec{p},-\frac{1}{2})+\sqrt{2^1}C^1_nC^0_{n-1}\frac{1}{n!}\underbrace{\tilde{\varepsilon}_{\{a}(\vec{p},0)\cdot\cdot\tilde{\varepsilon}_b(\vec{p},1)\cdot\cdot_{\}}}_{n}v(\vec{p},\frac{1}{2})]$

$= \frac{i\varsigma\sqrt{2}}{\sqrt{n+1}}(-1)^n\frac{1}{\sqrt{C^1_{2n+1}}}[\sqrt{C^0_{2n}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}}v(\vec{p},-\frac{1}{2})+\sqrt{C^1_{2n}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}}v(\vec{p},\frac{1}{2})]$

$= \frac{i\varsigma\sqrt{2}}{\sqrt{n+1}}(-1)^n\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},n-\frac{1}{2})$

Thm. 4.4.1.

$$\begin{cases}\gamma^c u(\vec{p},\frac{1}{2})\tilde{\varepsilon}_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)=-i\varsigma\frac{\sqrt{n+1-h}}{\sqrt{n+1}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h+\frac{1}{2})\\ \gamma^c u(\vec{p},-\frac{1}{2})\tilde{\varepsilon}_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)=i\varsigma\frac{\sqrt{n+1+h}}{\sqrt{n+1}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h-\frac{1}{2})\end{cases}\quad\begin{cases}\gamma^c v(\vec{p},\frac{1}{2})\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)=-i\varsigma\frac{\sqrt{n+1-h}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h+\frac{1}{2})\\ \gamma^c v(\vec{p},-\frac{1}{2})\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)=i\varsigma\frac{\sqrt{n+1+h}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h-\frac{1}{2})\end{cases}$$

Proof: $\gamma^c u(\vec{p},\frac{1}{2})\tilde{\varepsilon}_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)$

$= \frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h-1)\gamma^c u(\vec{p},\frac{1}{2})\tilde{\varepsilon}_c(\vec{p},1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)\gamma^c u(\vec{p},\frac{1}{2})\tilde{\varepsilon}_c(\vec{p},0)$

$$+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h+1)\gamma^c u(\vec{p},\tfrac{1}{2})\tilde{\varepsilon}_c(\vec{p},-1)$$
$$=-\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)i\varsigma v(\vec{p},\tfrac{1}{2})-\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h+1)i\sqrt{2}\varsigma v(\vec{p},-\tfrac{1}{2})$$
$$=-i\varsigma\frac{\sqrt{n+1-h}}{\sqrt{n+1}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h+\tfrac{1}{2})$$ □

Proof: $\gamma^c u(\vec{p},-\frac{1}{2})\tilde{\varepsilon}_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)$

$$=\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h-1)\gamma^c u(\vec{p},-\tfrac{1}{2})\tilde{\varepsilon}_c(\vec{p},1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)\gamma^c u(\vec{p},-\tfrac{1}{2})\tilde{\varepsilon}_c(\vec{p},0)$$
$$+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h+1)\gamma^c u(\vec{p},-\tfrac{1}{2})\tilde{\varepsilon}_c(\vec{p},-1)$$
$$=\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h-1)i\sqrt{2}\varsigma v(\vec{p},\tfrac{1}{2})+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)i\varsigma v(\vec{p},-\tfrac{1}{2})$$
$$=i\varsigma\frac{\sqrt{n+1+h}}{\sqrt{n+1}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h-\tfrac{1}{2})$$ □

Proof: $\gamma^c v(\vec{p},\frac{1}{2})\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)$

$$=\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h-1)\gamma^c v(\vec{p},\tfrac{1}{2})\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)\gamma^c v(\vec{p},\tfrac{1}{2})\varepsilon_c(\vec{p},0)$$
$$+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h+1)\gamma^c v(\vec{p},\tfrac{1}{2})\varepsilon_c(\vec{p},-1)$$
$$=-\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)i\varsigma u(\vec{p},\tfrac{1}{2})-\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h+1)i\sqrt{2}\varsigma u(\vec{p},-\tfrac{1}{2})$$
$$=-i\varsigma\frac{\sqrt{n+1-h}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h+\tfrac{1}{2})$$ □

Proof: $\gamma^c v(\vec{p},-\frac{1}{2})\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)$

$$=\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h-1)\gamma^c v(\vec{p},-\tfrac{1}{2})\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)\gamma^c v(\vec{p},-\tfrac{1}{2})\varepsilon_c(\vec{p},0)$$
$$+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h+1)\gamma^c v(\vec{p},-\tfrac{1}{2})\varepsilon_c(\vec{p},-1)$$
$$=\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h-1)i\sqrt{2}\varsigma u(\vec{p},\tfrac{1}{2})+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)i\varsigma u(\vec{p},-\tfrac{1}{2})$$
$$=i\varsigma\frac{\sqrt{n+1+h}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h-\tfrac{1}{2})$$ □

Cor. 4.4.1.

$$\begin{cases}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n+1}}(\vec{p},n+1-l)\gamma^c u(\vec{p},\frac{1}{2})=-i\varsigma\frac{\sqrt{l}}{\sqrt{n+1}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n-l+\frac{3}{2})\\ \tilde{\varepsilon}_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n+1}}(\vec{p},n+1-l)\gamma^c u(\vec{p},-\frac{1}{2})=i\varsigma\frac{\sqrt{2n+2-l}}{\sqrt{n+1}}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n-l+\frac{1}{2})\\ \varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n+1}}(\vec{p},n+1-l)\gamma^c v(\vec{p},\frac{1}{2})=-i\varsigma\frac{\sqrt{l}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n-l+\frac{3}{2})\\ \varepsilon_{\underbrace{a\cdot\cdot b\cdot\cdot c}_{n+1}}(\vec{p},n+1-l)\gamma^c v(\vec{p},-\frac{1}{2})=i\varsigma\frac{\sqrt{2n+2-l}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n-l+\frac{1}{2})\end{cases}$$

Cor. 4.4.2.

$$\begin{cases}\sum_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},h)=\frac{n+1}{2n+3}\sum_{h=n+1}^{-(n+1)}\sum_{h'=1/2}^{-1/2}[\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\gamma^c v(\vec{p},h')][\varepsilon_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^{c'}v(\vec{p},h')]^+\\ \sum_{h=n+1/2}^{-(n+1/2)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},h)=\frac{n+1}{2n+3}\sum_{h=n+1}^{-(n+1)}\sum_{h'=1/2}^{-1/2}[\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\gamma^c u(\vec{p},h')][\tilde{\varepsilon}_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^{c'}u(\vec{p},h')]^+\end{cases}$$

Cor. 4.4.3.

$$\begin{cases}\sum\limits_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},h)=\frac{1}{2}\frac{2n+2}{2n+3}\sum\limits_{h=n+1}^{-(n+1)}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^{c}\Lambda_{-}(\vec{p},\frac{1}{2})\gamma^{c'}\\ \sum\limits_{h=n+1/2}^{-(n+1/2)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},h)=\frac{1}{2}\frac{2n+2}{2n+3}\sum\limits_{h=n+1}^{-(n+1)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^{c}\Lambda_{+}(\vec{p},\frac{1}{2})\gamma^{c'}\end{cases}$$

4.5 Synthesis of Rarita-Schwinger equation basis for $s=n+\frac{1}{2}$ particles

Cor. 4.5.1.

$$\begin{cases}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{1}{\sqrt{C^{l}_{2n+1}}}[\sqrt{C^{l-1}_{2n}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^{l}_{2n}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})]\\ \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{1}{\sqrt{C^{l}_{2n+1}}}[\sqrt{C^{l-1}_{2n}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})+\sqrt{C^{l}_{2n}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})]\end{cases}$$

Cor. 4.5.2.

$$\begin{cases}\frac{m}{E}u^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)=\sqrt{\frac{l}{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)\\ \frac{m}{E}u^{+\tau_\varsigma}(\vec{p},\frac{1}{2})\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)=\sqrt{\frac{2n+1-l}{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)\end{cases}$$

$$\begin{cases}\frac{m}{E}v^{+\tau_\varsigma}(\vec{p},-\frac{1}{2})\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)=\sqrt{\frac{l}{2n+1}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l+1)\\ \frac{m}{E}v^{+\tau_\varsigma}(\vec{p},\frac{1}{2})\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},n+\frac{1}{2}-l)=\frac{\sqrt{C^{l}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)=\sqrt{\frac{2n+1-l}{2n+1}}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-l)\end{cases}$$

Cor. 4.5.3. $\varepsilon_a(\vec{p},\kappa)=-\frac{i}{\sqrt{2}}u^{T}(\vec{p},\frac{\kappa}{2})\bar{C}\gamma_a u(\vec{p},\frac{\kappa}{2}),\varepsilon_a(\vec{p},0)=-iu^{T}(\vec{p},\frac{\kappa}{2})\bar{C}\gamma_a u(\vec{p},\frac{\kappa}{2})$

Cor. 4.5.4. $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h)$

$$=\frac{\sqrt{C^{2}_{n+1+h}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^{1}_{n+1+h}C^{1}_{n+1-h}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^{2}_{n+1-h}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)$$

Thm. 4.5.1. $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},n+1-l)$

$$=-\frac{i}{\sqrt{2}}u^{T}(\vec{p},\frac{1}{2})\bar{C}\gamma_c\frac{\sqrt{2n+2-l}}{\sqrt{2n+2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n+\frac{1}{2}-l)-\frac{i}{\sqrt{2}}u^{T}(\vec{p},-\frac{1}{2})\bar{C}\gamma_c\frac{\sqrt{l}}{\sqrt{2n+2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n+\frac{3}{2}-l)$$

Proof:

$$\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},n+1-l)$$

$$=\frac{\sqrt{C^{2}_{2n+2-l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n-l)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^{1}_{2n+2-l}C^{1}_{l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^{2}_{l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+2-l)\varepsilon_c(\vec{p},-1)$$

$$=\frac{\sqrt{C^{2}_{2n+2-l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n-l)[-\frac{i}{\sqrt{2}}u^{T}(\vec{p},\frac{1}{2})\bar{C}\gamma_c u(\vec{p},\frac{1}{2})]$$

$$+\frac{\sqrt{C^{1}_{2n+2-l}C^{1}_{l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)\{[-\frac{i}{2}u^{T}(\vec{p},-\frac{1}{2})\bar{C}\gamma_c u(\vec{p},\frac{1}{2})]+[-\frac{i}{2}u^{T}(\vec{p},\frac{1}{2})\bar{C}\gamma_c u(\vec{p},-\frac{1}{2})]\}$$

$$+\frac{\sqrt{C^{2}_{l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+2-l)[-\frac{i}{\sqrt{2}}u^{T}(\vec{p},-\frac{1}{2})\bar{C}\gamma_c u(\vec{p},-\frac{1}{2})]$$

$$=-\frac{i}{\sqrt{2}}u^{T}(\vec{p},\frac{1}{2})\bar{C}\gamma_c\{\frac{\sqrt{C^{2}_{2n+2-l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n-l)u(\vec{p},\frac{1}{2})+\frac{\sqrt{C^{1}_{2n+2-l}C^{1}_{l}}}{\sqrt{2C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)u(\vec{p},-\frac{1}{2})\}$$

$$-\frac{i}{\sqrt{2}}u^{T}(\vec{p},-\frac{1}{2})\bar{C}\gamma_c\{\frac{\sqrt{C^{1}_{2n+2-l}C^{1}_{l}}}{\sqrt{2C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)u(\vec{p},\frac{1}{2})+\frac{\sqrt{C^{2}_{l}}}{\sqrt{C^{2}_{2n+2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+2-l)u(\vec{p},-\frac{1}{2})\}$$

$$=-\frac{i}{\sqrt{2}}u^{T}(\vec{p},\frac{1}{2})\bar{C}\gamma_c\{\frac{\sqrt{C^{2}_{2n+2-l}}}{\sqrt{C^{2}_{2n+2}}}\frac{\sqrt{C^{l}_{2n+1}}}{\sqrt{C^{l}_{2n}}}\frac{\sqrt{C^{l}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n-l)u(\vec{p},\frac{1}{2})+\frac{\sqrt{C^{1}_{2n+2-l}C^{1}_{l}}}{\sqrt{2C^{2}_{2n+2}}}\frac{\sqrt{C^{l}_{2n+1}}}{\sqrt{C^{l-1}_{2n}}}\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)u(\vec{p},-\frac{1}{2})\}$$

$$-\frac{i}{\sqrt{2}}u^{T}(\vec{p},-\frac{1}{2})\bar{C}\gamma_c\{\frac{\sqrt{C^{1}_{2n+2-l}C^{1}_{l}}}{\sqrt{2C^{2}_{2n+2}}}\frac{\sqrt{C^{l-l}_{2n+1}}}{\sqrt{C^{l-1}_{2n}}}\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^{l-1}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)u(\vec{p},\frac{1}{2})+\frac{\sqrt{C^{2}_{l}}}{\sqrt{C^{2}_{2n+2}}}\frac{\sqrt{C^{l-l}_{2n+1}}}{\sqrt{C^{l-2}_{2n}}}\frac{\sqrt{C^{l-2}_{2n}}}{\sqrt{C^{l-1}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+2-l)u(\vec{p},-\frac{1}{2})\}$$

$$=-\frac{i}{\sqrt{2}}u^{T}(\vec{p},\frac{1}{2})\bar{C}\gamma_c\frac{\sqrt{C^{2}_{2n+2-l}}}{\sqrt{C^{2}_{2n+2}}}\frac{\sqrt{C^{l}_{2n+1}}}{\sqrt{C^{l}_{2n}}}[\frac{\sqrt{C^{l}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n-l)u(\vec{p},\frac{1}{2})+\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^{l}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)u(\vec{p},-\frac{1}{2})]$$

$$-\frac{i}{\sqrt{2}}u^{T}(\vec{p},-\frac{1}{2})\bar{C}\gamma_c\frac{\sqrt{C^{2}_{l}}}{\sqrt{C^{2}_{2n+2}}}\frac{\sqrt{C^{l-1}_{2n+1}}}{\sqrt{C^{l-2}_{2n}}}\{\frac{\sqrt{C^{l-1}_{2n}}}{\sqrt{C^{l-1}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+1-l)u(\vec{p},\frac{1}{2})+\frac{\sqrt{C^{l-2}_{2n}}}{\sqrt{C^{l-1}_{2n+1}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}}(\vec{p},n+2-l)u(\vec{p},-\frac{1}{2})\}$$

$$= -\frac{i}{\sqrt{2}}u^T(\vec{p},\tfrac{1}{2})\bar{C}\gamma_c\frac{\sqrt{2n+2-l}}{\sqrt{2n+2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n+\tfrac{1}{2}-l) - \frac{i}{\sqrt{2}}u^T(\vec{p},-\tfrac{1}{2})\bar{C}\gamma_c\frac{\sqrt{l}}{\sqrt{2n+2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n+\tfrac{3}{2}-l)$$

$$= -\frac{i}{\sqrt{2}}u^T(\vec{p},\tfrac{1}{2})\gamma_4\gamma_2\gamma_c\frac{\sqrt{2n+2-l}}{\sqrt{2n+2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n+\tfrac{1}{2}-l) - \frac{i}{\sqrt{2}}u^T(\vec{p},-\tfrac{1}{2})\gamma_4\gamma_2\gamma_c\frac{\sqrt{l}}{\sqrt{2n+2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},n+\tfrac{3}{2}-l) \qquad \square$$

Cor. 4.5.5. $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h) = -\frac{i}{2}[\frac{\sqrt{n+1+h}}{\sqrt{n+1}}u^T(\vec{p},\tfrac{1}{2})\bar{C}\gamma_c\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h-\tfrac{1}{2}) + \frac{\sqrt{n+1-h}}{\sqrt{n+1}}u^T(\vec{p},-\tfrac{1}{2})\bar{C}\gamma_c\varepsilon_{\underbrace{a\cdot\cdot b}_{n}[\tau_\varsigma]}(\vec{p},h+\tfrac{1}{2})]$

Cor. 4.5.6. $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n+1}}(\vec{p},h) = -\frac{i}{2}[\frac{\sqrt{n+1+h}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}\tau_\varsigma}(\vec{p},h-\tfrac{1}{2})u_{\sigma_\varsigma}(\hat{p},\tfrac{1}{2}) + \frac{\sqrt{n+1-h}}{\sqrt{n+1}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n}\tau_\varsigma}(\vec{p},h+\tfrac{1}{2})u_{\sigma_\varsigma}(\hat{p},-\tfrac{1}{2})](\bar{C}\gamma_c)^{\tau_\varsigma\sigma_\varsigma}$

5 Reduction mode for various potential quasi projection operators

5.1 Formal definition of various potential quasi projection operators

Def. 5.1.1.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n) := \sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h) = \sum\limits_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h)\\ \Lambda_{+m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\tfrac{1}{2}) := \sum\limits_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\\ \Lambda_{-m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\tfrac{1}{2}) := \sum\limits_{h=n+1/2}^{-(n+1/2)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\end{cases}$$

5.2 Relations between various potential quasi projection operators-Minimal reduction mode

Thm. 5.2.1.

$$\begin{cases}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h) = \frac{2n+1}{2n+2}(\frac{m}{E})^2\sum\limits_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\Lambda_+^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\tfrac{1}{2})\\ \sum\limits_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},h) = \frac{2n+2}{2n+3}\frac{1}{2}\sum\limits_{h=n+1}^{-(n+1)}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^c\Lambda_-(\vec{p},\tfrac{1}{2})\gamma^{c'}\end{cases}$$

Thm. 5.2.2.

$$\begin{cases}\sum\limits_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h) = \frac{2n+1}{2n+2}(\frac{m}{E})^2\sum\limits_{h=n+1/2}^{-(n+1/2)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\Lambda_-^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\tfrac{1}{2})\\ \sum\limits_{h=n+1/2}^{-(n+1/2)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},h) = \frac{2n+2}{2n+3}\frac{1}{2}\sum\limits_{h=n+1}^{-(n+1)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^c\Lambda_+(\vec{p},\tfrac{1}{2})\gamma^{c'}\end{cases}$$

Cor. 5.2.1.

$$\begin{cases}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h) = \frac{2n+1}{2n+3}\sum\limits_{h=n+1}^{-(n+1)}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\eta^{cc'}\\ \sum\limits_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h) = \frac{2n+2}{2n+4}\sum\limits_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n}\tau'_\varsigma}(\vec{p},h)\eta^{cc'}\end{cases}$$

Proof: $\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h)$

$$= \frac{2n+1}{2n+2}(\frac{m}{E})^2\sum\limits_{h=n+1/2}^{-(n+1/2)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\Lambda_+^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$= \frac{2n+1}{4n+6}(\frac{m}{E})^2\sum\limits_{h=n+1}^{-(n+1)}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)[\gamma^c\Lambda_-(\vec{p},\tfrac{1}{2})\gamma^{c'}]_{\tau_\varsigma\tau'_\varsigma}\Lambda_+^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$= \frac{2n+1}{4n+6}(\frac{m}{E})^2\sum\limits_{h=n+1}^{-(n+1)}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)tr[\gamma^c\Lambda_-(\vec{p},\tfrac{1}{2})\gamma^{c'}\Lambda_+(\vec{p},\tfrac{1}{2})]$$

$$= \frac{2n+1}{2n+3}\sum\limits_{h=n+1}^{-(n+1)}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\eta^{cc'} \qquad \square$$

Proof: $\sum_{h=n+1/2}^{-(n+1/2)} \varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)$

$$= \tfrac{2n+2}{2n+3}\tfrac{1}{2}\sum_{h=n+1}^{-(n+1)} \varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\gamma^{c}\Lambda_{-}(\vec{p},\tfrac{1}{2})\gamma^{c'}$$

$$= \tfrac{2n+2}{2n+3}\tfrac{1}{2}\tfrac{2n+3}{2n+5}\sum_{h=n+2}^{-(n+2)} \varepsilon_{\underbrace{ab\cdot\cdot cd}_{n+2}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot c'd'}_{n+2}}(\vec{p},h)\gamma^{c}\Lambda_{-}(\vec{p},\tfrac{1}{2})\gamma^{c'}\eta^{dd'}$$

$$= \tfrac{2n+2}{2n+5}\tfrac{2n+5}{2n+4}\tfrac{1}{2}\tfrac{2n+4}{2n+5}\sum_{h=n+2}^{-(n+2)} \varepsilon_{\underbrace{ab\cdot\cdot cd}_{n+2}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot c'd'}_{n+2}}(\vec{p},h)\gamma^{c}\Lambda_{-}(\vec{p},\tfrac{1}{2})\gamma^{c'}\eta^{dd'}$$

$$= \tfrac{2n+2}{2n+4}\tfrac{1}{2}\tfrac{2n+4}{2n+5}\sum_{h=n+2}^{-(n+2)} \varepsilon_{\underbrace{ab\cdot\cdot cd}_{n+2}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot c'd'}_{n+2}}(\vec{p},h)\gamma^{d}\Lambda_{-}(\vec{p},\tfrac{1}{2})\gamma^{d'}\eta^{cc'}$$

$$= \tfrac{2n+2}{2n+4}\sum_{h=n+3/2}^{-(n+3/2)} \varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}\tau_\varsigma}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'b'\cdot\cdot c'}_{n+1}\tau'_\varsigma}(\vec{p},h)\eta^{cc'} \qquad \square$$

Cor. 5.2.2.

$$\begin{cases}\sum_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h)=\tfrac{2n+1}{2n+3}\sum_{h=n+1}^{-(n+1)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)\eta^{cc'}\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=\tfrac{2n+2}{2n+4}\sum_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot c}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'b'\cdot\cdot c'}_{n}\tau'_\varsigma}(\vec{p},h)\eta^{cc'}\end{cases}$$

Cor. 5.2.3.

$$\begin{cases}\sum_{h=n}^{-n}\varepsilon_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'\cdot\cdot}_{n}}(\vec{p},h)=\tfrac{2n+1}{2(n+m)+1}\sum_{h=n+m}^{-(n+m)}\varepsilon_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n+m}}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n+m}}(\vec{p},h)\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{m}\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{a\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=\tfrac{2n+2}{2(n+m)+2}\sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})}\varepsilon_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^{+}_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n}\tau'_\varsigma}(\vec{p},h)\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{m}\end{cases}$$

Cor. 5.2.4.

$$\begin{cases}\sum_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'\cdot\cdot}_{n}}(\vec{p},h)=\tfrac{2n+1}{2(n+m)+1}\sum_{h=n+m}^{-(n+m)}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n+m}}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n+m}}(\vec{p},h)\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{m}\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=\tfrac{2n+2}{2(n+m)+2}\sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^{+}_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n}\tau'_\varsigma}(\vec{p},h)\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{m}\end{cases}$$

5.3 Reorganization of minimal reduction mode for various potential quasi projection operators

Thm. 5.3.1.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\tfrac{2n+1}{2n+2}(\tfrac{m}{E})^{2}\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\tfrac{1}{2})\Lambda_{\pm}^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\tfrac{1}{2})\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},n+\tfrac{1}{2})=\tfrac{2n+2}{2n+3}\tfrac{1}{2}\Lambda_{m\underbrace{ab\cdot\cdot c}_{n+1}\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},n+1)\gamma^{c}\Lambda_{\mp}(\vec{p},\tfrac{1}{2})\gamma^{c'}\end{cases}$$

Cor. 5.3.1.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\tfrac{2n+1}{2n+3}\Lambda_{m\underbrace{ab\cdot\cdot c}_{n+1}\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},n+1)\eta^{cc'}\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\tfrac{1}{2})=\tfrac{2n+2}{2n+4}\Lambda_{\pm m\underbrace{ab\cdot\cdot c}_{n+1}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'}_{n+1}\tau'_\varsigma}(\vec{p},n+\tfrac{3}{2})\eta^{cc'}\end{cases}$$

Cor. 5.3.2.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\tfrac{2n+1}{2(n+l)+1}\Lambda_{m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}}(\vec{p},n+l)\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{l}\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\tfrac{1}{2})=\tfrac{2(n+\frac{1}{2})+1}{2(n+l+\frac{1}{2})+1}\Lambda_{\pm m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}\tau'_\varsigma}(\vec{p},n+l+\tfrac{1}{2})\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{l}\end{cases}$$

Cor. 5.3.3.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{2n+1}{2(n+l+\frac{1}{2})+1}(\frac{m}{E})^2\Lambda_{\pm m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}\tau'_\varsigma}(\vec{p},n+l+\frac{1}{2})\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{l}\Lambda_{\pm}^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\frac{1}{2})\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]\underbrace{a'b'\cdot\cdot}_{n}[\tau'_\varsigma]}(\vec{p},n+\frac{1}{2})=\frac{2(n+\frac{1}{2})+1}{2(n+1+l)+1}\frac{1}{2}\Lambda_{m\underbrace{ab\cdot\cdot c\cdot\cdot de}_{n+1+l}\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'e'}_{n+1+l}}(\vec{p},n+1+l)\underbrace{\eta^{cc'}\cdot\cdot\eta^{dd'}}_{l}\gamma^e\Lambda_{\mp}(\vec{p},\frac{1}{2})\gamma^{e'}\end{cases}$$

5.4 Relations between various potential quasi projection operators-Physical reduction mode

Cor. 5.4.1.

$$\begin{cases}\sum_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h)=\frac{2n+1}{2n+3}\sum_{h=n+1}^{-(n+1)}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=\frac{2n+2}{2n+4}\sum_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot c'}_{n}\tau'_\varsigma}(\vec{p},h)(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\end{cases}$$

Cor. 5.4.2.

$$\begin{cases}\sum_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h)=\frac{2n+1}{2n+3}\sum_{h=n+1}^{-(n+1)}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot c}_{n+1}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},h)(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=\frac{2n+2}{2n+4}\sum_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot c}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot c'}_{n}\tau'_\varsigma}(\vec{p},h)(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\end{cases}$$

Cor. 5.4.3.

$$\begin{cases}\sum_{h=n}^{-n}\varepsilon_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h)\varepsilon^+_{\underbrace{a'\cdot\cdot}_{n}}(\vec{p},h)=\frac{2n+1}{2(n+m)+1}\sum_{h=n+m}^{-(n+m)}\varepsilon_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n+m}}(\vec{p},h)\varepsilon^+_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n+m}}(\vec{p},h)\underbrace{(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\cdot\cdot(\eta^{dd'}+\frac{p^dp^{+d'}}{m^2})}_{m}\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{a\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\\ =\frac{2n+2}{2(n+m)+2}\sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})}\varepsilon_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n}\tau'_\varsigma}(\vec{p},h)\underbrace{(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\cdot\cdot(\eta^{dd'}+\frac{p^dp^{+d'}}{m^2})}_{m}\end{cases}$$

Cor. 5.4.4.

$$\begin{cases}\sum_{h=n}^{-n}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'\cdot\cdot}_{n}}(\vec{p},h)=\frac{2n+1}{2(n+m)+1}\sum_{h=n+m}^{-(n+m)}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n+m}}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n+m}}(\vec{p},h)\underbrace{(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\cdot\cdot(\eta^{dd'}+\frac{p^dp^{+d'}}{m^2})}_{m}\\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)\\ =\frac{2n+2}{2(n+m)+2}\sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})}\tilde{\varepsilon}_{\underbrace{a\cdot\cdot c\cdot\cdot d}_{n}\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{\underbrace{a'\cdot\cdot c'\cdot\cdot d'}_{n}\tau'_\varsigma}(\vec{p},h)\underbrace{(\eta^{cc'}+\frac{p^cp^{+c'}}{m^2})\cdot\cdot(\eta^{dd'}+\frac{p^dp^{+d'}}{m^2})}_{m}\end{cases}$$

5.5 Reorganization of physical reduction mode for various potential quasi projection operators

Thm. 5.5.1.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{2n+1}{2n+3}\Lambda_{m\underbrace{ab\cdot\cdot c}_{n+1}\underbrace{a'b'\cdot\cdot c'}_{n+1}}(\vec{p},n+1)\Lambda_m^{cc'}(\vec{p},1)\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=\frac{2n+2}{2n+4}\Lambda_{\pm m\underbrace{ab\cdot\cdot c}_{n+1}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'}_{n+1}\tau'_\varsigma}(\vec{p},n+\frac{3}{2})\Lambda_m^{cc'}(\vec{p},1)\end{cases}$$

Cor. 5.5.1.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{2n+1}{2(n+l)+1}\Lambda_{m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}}(\vec{p},n+l)\underbrace{\Lambda_m^{cc'}(\vec{p},1)\cdot\cdot\Lambda_m^{dd'}(\vec{p},1)}_{l}\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=\frac{2(n+\frac{1}{2})+1}{2(n+l+\frac{1}{2})+1}\Lambda_{\pm m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}\tau'_\varsigma}(\vec{p},n+l+\frac{1}{2})\underbrace{\Lambda_m^{cc'}(\vec{p},1)\cdot\cdot\Lambda_m^{dd'}(\vec{p},1)}_{l}\end{cases}$$

Cor. 5.5.2.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{2n+1}{2(n+2)+1}\Lambda_{m\underbrace{ab\cdot\cdot cd}_{n+2}\underbrace{a'b'\cdot\cdot c'd'}_{n+2}}(\vec{p},n+2)\Lambda_m^{cdc'd'}(\vec{p},2)\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=\frac{2(n+\frac{1}{2})+1}{2(n+2+\frac{1}{2})+1}\Lambda_{\pm m\underbrace{ab\cdot\cdot cd}_{n+2}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'd'}_{n+2}\tau'_\varsigma}(\vec{p},n+2+\frac{1}{2})\Lambda_m^{cdc'd'}(\vec{p},2)\end{cases}$$

Cor. 5.5.3.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{2n+1}{2(n+l)+1}\Lambda_{m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}}(\vec{p},n+l)\Lambda_{m}^{\overbrace{c\cdot\cdot d}^{l}\overbrace{c'\cdot\cdot d'}^{l}}(\vec{p},l)\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=\frac{2(n+\frac{1}{2})+1}{2(n+l+\frac{1}{2})+1}\Lambda_{\pm m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}\tau'_\varsigma}(\vec{p},n+l+\frac{1}{2})\Lambda_{m}^{\overbrace{c\cdot\cdot d}^{l}\overbrace{c'\cdot\cdot d'}^{l}}(\vec{p},l)\end{cases}$$

Cor. 5.5.4.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{2n+1}{2(n+l+\frac{1}{2})+1}(\frac{m}{E})^2\Lambda_{\pm m\underbrace{ab\cdot\cdot c\cdot\cdot d}_{n+l}\tau_\varsigma\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'}_{n+l}\tau'_\varsigma}(\vec{p},n+l+\frac{1}{2})\Lambda_{m}^{\overbrace{c\cdot\cdot d}^{l}\overbrace{c'\cdot\cdot d'}^{l}}(\vec{p},l)\Lambda_{\pm}^{\tau'_\varsigma\tau_\varsigma}(\vec{p},\frac{1}{2})\\ \Lambda_{\pm m\underbrace{ab\cdot\cdot[\tau_\varsigma]}_{n}\underbrace{a'b'\cdot\cdot[\tau'_\varsigma]}_{n}}(\vec{p},n+\frac{1}{2})=\frac{2(n+\frac{1}{2})+1}{2(n+1+l)+1}\frac{1}{2}\Lambda_{m\underbrace{ab\cdot\cdot c\cdot\cdot de}_{n+1+l}\underbrace{a'b'\cdot\cdot c'\cdot\cdot d'e'}_{n+1+l}}(\vec{p},n+1+l)\Lambda_{m}^{\overbrace{c\cdot\cdot d}^{l}\overbrace{c'\cdot\cdot d'}^{l}}(\vec{p},l)\gamma^e\Lambda_{\mp}(\vec{p},\frac{1}{2})\gamma^{e'}\end{cases}$$

5.6 Universal properties of various potential quasi projection operators

Pro. 5.6.1.

$$\begin{cases}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{1}{n!}\Lambda_{m\underbrace{\{ab\cdot\cdot\}}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\frac{1}{n!}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{(a'b'\cdot\cdot)}_{n}}(\vec{p},n)=\frac{1}{(n!)^2}\Lambda_{m\underbrace{\{ab\cdot\cdot\}}_{n}\underbrace{(a'b'\cdot\cdot)}_{n}}(\vec{p},n)\\ \delta^{ab}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=\delta^{a'b'}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=0,p^a\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=p^{+a'}\Lambda_{m\underbrace{ab\cdot\cdot}_{n}\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},n)=0\end{cases}$$

Pro. 5.6.2.

$$\begin{cases}\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=\frac{1}{n!}\Lambda_{\pm m\underbrace{\{ab\cdot\cdot\}}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})\\ =\frac{1}{n!}\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{(a'b'\cdot\cdot)}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=\frac{1}{(n!)^2}\Lambda_{\pm m\underbrace{\{ab\cdot\cdot\}}_{n}\tau_\varsigma\underbrace{(a'b'\cdot\cdot)}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})\\ \delta^{ab}\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=\delta^{a'b'}\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=0\\ p^a\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=p^{+a'}\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=0\\ \gamma^a\Lambda_{\pm m\underbrace{ab\cdot\cdot[\tau_\varsigma]}_{n}\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=0,\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot[\tau'_\varsigma]}_{n}}(\vec{p},n+\frac{1}{2})\gamma^{a'}=0\\ (\pm i\gamma^c p_c+m)\Lambda_{\pm m\underbrace{ab\cdot\cdot[\tau_\varsigma]}_{n}\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},n+\frac{1}{2})=0,\Lambda_{\pm m\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma\underbrace{a'b'\cdot\cdot[\tau'_\varsigma]}_{n}}(\vec{p},n+\frac{1}{2})(\pm i\gamma^{c'}p^+_{c'}-m)=0\end{cases}$$

6 Direct solution to commutation rules for potential

(Equivalent transformation method.)

6.1 Lemma

Lem. 6.1.1. $\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}(p)[\delta_{ab}+\frac{p_ap_b}{m^2}]=0$

6.2 Transformation solving method of commutation rules for potential $A_{abc\cdot\cdot}$

Thm. 6.2.1.

$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]=\frac{i}{2^{3n-1}}\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')\\ A_{\underbrace{ab\cdot\cdot}_{n}}(x)=\frac{1}{(i2m)^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(x),\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x)=\frac{1}{(2n)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\}}_{2n}}(x)\\ A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x)=\frac{1}{(-i2m)^n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x)=\frac{1}{(2n)!}\psi^+_{\underbrace{\{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\}}_{2n}}(x)\end{cases}$$

$$\Rightarrow[A_{\underbrace{ab\cdot\cdot}_{n}}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')]=i\frac{1}{2^{5n-1}m^{2n}}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$

$$\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+c'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+d'}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{cc'}-\frac{\partial_c\partial^+_{c'}}{m^2}][\eta_{dd'}-\frac{\partial_d\partial^+_{d'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

$$=i\frac{1}{2^{4n-1}m^{2n}}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$

$$\frac{1}{[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$$

Proof: $[A_{\underbrace{ab\cdot\cdot}_{n}}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')]$

$$=\frac{1}{(2m)^{2n}}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(x')]$$

$$=\frac{1}{(2m)^{2n}}\frac{i}{2^{3n-1}[(2n)!]^2}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$
$$\underbrace{\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+c'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+d'}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{cc'}-\frac{\partial_c\partial^+_{c'}}{m^2}][\eta_{dd'}-\frac{\partial_d\partial^+_{d'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

$$=i\frac{1}{2^{5n-1}m^{2n}}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$
$$\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+c'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+d'}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{cc'}-\frac{\partial_c\partial^+_{c'}}{m^2}][\eta_{dd'}-\frac{\partial_d\partial^+_{d'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

$$=i\frac{1}{2^{4n-1}m^{2n}}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$
$$\frac{1}{[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')\qquad\square$$

6.3 Transformation solving method of anticommutation rules for potential $A_{abc\cdot\cdot\tau_\varsigma}$

Thm. 6.3.1.

$$\begin{cases}\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}\\ =\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')\\ A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x)=\frac{1}{(i2m)^n}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)=\frac{1}{(2n+1)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}}_{2n+1}}(x)\\ A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x)=\frac{1}{(-i2m)^n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x)=\frac{1}{(2n+1)!}\psi^+_{\underbrace{\{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau_\varsigma\}}_{2n+1}}(x)\end{cases}$$

$$\Rightarrow\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\}$$

$$=i\frac{1}{2^{5n}m^{2n}}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$
$$\frac{1}{[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

$$=i\frac{1}{2^{4n}m^{2n}}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$
$$\frac{1}{[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')$$

Proof: $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\}$

$$=\frac{1}{(2m)^{2n}}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$$

$$=i\frac{1}{2^{5n}m^{2n}}\overbrace{(\bar C\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar C\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$$
$$\frac{1}{[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

$= i\frac{1}{2^{4n}m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$
$\frac{1}{[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')$ □

6.4 Isochronous quantization rules for fully symmetric B-W equation

Thm. 6.4.1.

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-2s+1} = i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma\otimes\sigma_z,i\varsigma)^a_{\{\lambda_\varsigma(\lambda'_\varsigma}(\sigma\otimes\sigma_z,i\varsigma)^b_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta(x-x')$$

$$\Rightarrow [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\xi_\varsigma\eta_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\xi'_\varsigma\eta'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{r}',t)]_{-2s+1}$$

$$= -\frac{(i\varsigma)^{2s+1}}{2^{2s-1}}\sum_{k=0}^{[s-\frac{1}{2}]}\frac{(2s)!}{(2s-2k-1)!(2k)!1!}\overbrace{[(\sigma\cdot\nabla)\otimes\sigma_z]_{\lambda_\varsigma\lambda'_\varsigma}[(\sigma\cdot\nabla)\otimes\sigma_z]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2s-2k-1}\overbrace{\delta_{\xi_\varsigma\xi'_\varsigma}\delta_{\eta_\varsigma\eta'_\varsigma}\cdot\cdot}^{2k}\nabla^{2k}\delta_{\tau_\varsigma\tau'_\varsigma}\delta^3(\vec{r}-\vec{r}')$$

7 Summary and carding of covariant commutation rules for massive particles

7.1 Carding of covariant commutation rules for massive boson

Def. 7.1.1. $\begin{cases}\hat{P}_{a_1\cdot\cdot a_n a'_1\cdot\cdot a'_n}(n) = \frac{1}{(n!)^2}\sum_{P(a)}^{P(b)}\sum_{r=0}^{[n/2]}k_r\hat{P}_{a_1a_2}\hat{P}_{a'_1a'_2}\cdot\cdot\hat{P}_{a_{2r-1}a_{2r}}\hat{P}_{a'_{2r-1}a'_{2r}}\prod_{i=2r+1}^{n}\hat{P}_{a_ia'_i}\\ k_r = (-\frac{1}{2})^r\frac{n!(2n-2r-1)!!}{r!(n-2r)!(2n-1)!!}\end{cases}$

Def. 7.1.2. $\begin{cases}\hat{P}_{a_1\cdot\cdot a_n b_1\cdot\cdot b_n}(n) = \frac{1}{(n!)^2}\sum_{P(a)}^{P(b)}\sum_{r=0}^{[n/2]}k_r\hat{P}_{a_1a_2}\hat{P}_{b_1b_2}\cdot\cdot\hat{P}_{a_{2r-1}a_{2r}}\hat{P}_{b_{2r-1}b_{2r}}\prod_{i=2r+1}^{n}\hat{P}_{a_ib_i}\\ \hat{P}_{a_1\cdot\cdot a_n b_1\cdot\cdot b_n}(n) := \eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdot\cdot\eta^{a'_n}_{b_n}\hat{P}_{a_1\cdot\cdot a_n a'_1\cdot\cdot a'_n}(n)\end{cases}$

Thm. 7.1.1. $[A_{a_1a_2\cdot\cdot a_n}(x),\bar{A}_{b_1b_2\cdot\cdot b_n}(x')] = i\hat{P}_{a_1a_2\cdot\cdot a_n b_1b_2\cdot\cdot b_n}(n)\Delta(x-x'),\ \bar{A}_{b_1b_2\cdot\cdot b_n} := \eta^{b'_1}_{b_1}\eta^{b'_2}_{b_2}\cdot\cdot\eta^{b'_n}_{b_n}A^+_{b'_1b'_2\cdot\cdot b'_n}$

$[\Updownarrow]$

Thm. 7.1.2. $[A_{a_1a_2\cdot\cdot a_n}(x),A^+_{a'_1a'_2\cdot\cdot a'_n}(x')] = i\hat{P}_{a_1a_2\cdot\cdot a_n a'_1a'_2\cdot\cdot a'_n}(n)\Delta(x-x')$

$[\Updownarrow]$

Thm. 7.1.3. $[A_{\underbrace{ab\cdot\cdot}_{n}}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')] = \frac{1}{m^{2n}}\frac{i}{2^{5n-1}[(2n)!]^2}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$

$\underbrace{\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+c'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+d'}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{cc'}-\frac{\partial_c\partial^+_{c'}}{m^2}][\eta_{dd'}-\frac{\partial_d\partial^+_{d'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 7.1.4. $[A_{\underbrace{ab\cdot\cdot}_{n}}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')] = \frac{1}{m^{2n}}\frac{i}{2^{4n-1}[(2n)!]^2}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$

$\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 7.1.5. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 7.1.6. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{2n-1}[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$

$[\Downarrow]$

Thm. 7.1.7. $[\psi_{\underbrace{A_\varsigma B_\varsigma \cdot\cdot}_{2n}}(x), \psi^+_{\underbrace{A'_\varsigma B'_\varsigma \cdot\cdot}_{2n}}(x')] = i\frac{(i\varsigma)^{2n}}{2^{2n-1}[(2n)!]^2}\overbrace{(\sigma, i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma, i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot\})}^{2n}\overbrace{\partial_a\partial_b\cdot\cdot}^{2n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 7.1.8. $[\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')] = i\frac{(-1)^{2n}}{2^{n-1}}\Gamma^{\overbrace{abc\cdot\cdot}^{2n}}_{k_\varsigma k'_\varsigma}(n)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\Delta(x-x')$

There are two equivalent expressions for the commutative relationship between potential A and field ψ, and they are mutually premises and causal each other. You can deduce everything from potential commutation relations, or you can deduce everything from field commutation relations too. This shows that the two descriptions of potential and field are completely equivalent for massive particles. And it can be deduced from the massive particle commutation rules which are completely similar to the massless particle commutation rules. But not vice versa.

7.2 Relations between commutation rules and quasi projection operators for massive boson

Cor. 7.2.1. $\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}, h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}, h)$

$= \frac{1}{2^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\bar{C}\gamma_{a'})^{+\lambda'_\varsigma\mu'_\varsigma}(\bar{C}\gamma_{b'})^{+\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\sum\limits_{h=n}^{-n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p}, h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(\vec{p}, h)$

Thm. 7.2.1. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-2s+1} = 2im^{2s}\sum\limits_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(-i\partial, h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(-i\partial, h)$

Thm. 7.2.2. $[A_{\underbrace{ab\cdot\cdot}_{n}}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')] = \frac{i}{2^{n-1}}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(-i\partial, h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(-i\partial, h)$

Proof: $[A_{\underbrace{ab\cdot\cdot}_{n}}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')] = \frac{1}{m^{2n}}\frac{i}{2^{4n-1}[(2n)!]^2}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$

$\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$

$= \frac{i}{2^{2n-1}}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\sum\limits_{h=n}^{-n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(-i\partial, h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(-i\partial, h)\Delta(x-x')$

$= \frac{i}{2^{n-1}}\sum\limits_{h=n}^{-n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(-i\partial, h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(-i\partial, h)$ □

7.3 Carding of covariant anticommutation rules for massive fermion

Def. 7.3.1. $\hat{P}_{a_1\cdot\cdot a_n\tau_\varsigma a'_1\cdot\cdot a'_n\tau'_\varsigma}(n+\frac{1}{2}) = \frac{n+1}{2n+3}\hat{P}_{aa_1\cdot\cdot a_n a' a'_1\cdot\cdot a'_n}(n+1)[\gamma^a(-m-\gamma^c\partial_c)\gamma^4\gamma^{a'}]_{\tau_\varsigma\tau'_\varsigma}, \gamma^{a'} = \gamma^a\eta^{a'}_a$

Def. 7.3.2. $\begin{cases}\hat{P}_{a_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2}) = \frac{n+1}{2n+3}\hat{P}_{aa_1\cdot\cdot a_n bb_1\cdot\cdot b_n}(n+1)[\gamma^a(m+\gamma^c\partial_c)\gamma^b\gamma^4]_{\tau_\varsigma\tau'_\varsigma} \\ \hat{P}_{a_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2}) := \eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdot\cdot\eta^{a'_n}_{b_n}\hat{P}_{a_1\cdot\cdot a_n\tau_\varsigma a'_1\cdot\cdot a'_n\tau'_\varsigma}(n+\frac{1}{2})\end{cases}$

Cor. 7.3.1. $\hat{P}_{a_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2}) = \frac{n+1}{2n+3}\hat{P}_{aa_1\cdot\cdot a_n bb_1\cdot\cdot b_n}(n+1)[(m-\gamma^c\partial_c)\gamma^a\gamma^b\gamma^4]_{\tau_\varsigma\tau'_\varsigma}$

Thm. 7.3.1. $\{A_{a_1a_2\cdot\cdot a_n\tau_\varsigma}(x), \bar{A}_{b_1b_2\cdot\cdot b_n\tau'_\varsigma}(x')\} = i\hat{P}_{a_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2})\Delta(x-x')$

$[\Updownarrow]$

Thm. 7.3.2. $\{A_{a_1a_2\cdot\cdot a_n\tau_\varsigma}(x), A^+_{a'_1a'_2\cdot\cdot a'_n\tau'_\varsigma}(x')\} = i\hat{P}_{a_1\cdot\cdot a_n\tau_\varsigma a'_1\cdot\cdot a'_n\tau'_\varsigma}(n+\frac{1}{2})\Delta(x-x')$

$[\Updownarrow]$

Thm. 7.3.3. $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\} = i\frac{1}{m^{2n}}\frac{1}{2^{5n}[(2n+1)!]^2}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$

$\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 7.3.4. $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\} = \frac{1}{m^{2n}}\frac{i}{2^{4n}[(2n+1)!]^2}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$
$\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 7.3.5. $\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n+1}}(x')\}$
$= \frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 7.3.6. $\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n+1}}(x')\} = \frac{i}{2^{2n}[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n+1}\Delta(x-x')$

$$[\Downarrow]$$

Thm. 7.3.7. $\{\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2n+1}}(x), \psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2n+1}}(x')\} = i\frac{(i\varsigma)^{2n+1}}{2^{2n}[(2n+1)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot\})}^{2n+1}\overbrace{\partial_a\partial_b\cdot\cdot}^{2n+1}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 7.3.8. $\{\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')\} = i\frac{(-1)^{2n+1}}{2^{n-1/2}}\Gamma^{\overbrace{abc\cdot\cdot}^{2n+1}}_{k_\varsigma k'_\varsigma}(n+\frac{1}{2})\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2n+1}\Delta(x-x')$

There are two equivalent expressions for the commutative relationship between potential A and field ψ, and they are mutually premises and causal each other. You can deduce everything from potential commutation relations, or you can deduce everything from field commutation relations too. This shows that the two descriptions of potential and field are completely equivalent for massive particles. And it can be deduced from the massive particle commutation rules which are completely similar to the massless particle commutation rules. But not vice versa.

7.4 Relations between commutation rules and quasi projection operators for massive fermion

Cor. 7.4.1. $\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)$
$= \frac{1}{2^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec{p},h)$

Thm. 7.4.1. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-2s+1} = 2im^{2s}\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(-i\partial,h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(-i\partial,h)$

Thm. 7.4.2. $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\} = \frac{im}{2^{n-1}}\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(-i\partial,h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(-i\partial,h)$

Proof: $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\} = \frac{1}{m^{2n}}\frac{i}{2^{4n}[(2n+1)!]^2}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$
$\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')$
$= \frac{im}{2^{2n-1}}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(-i\partial,h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(-i\partial,h)\Delta(x-x')$
$= \frac{im}{2^{n-1}}\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(-i\partial,h)\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(-i\partial,h)$ □

Chapter26 Covariant Quantization Scheme for Real Particles with Mass

Self comment: Massive real particles are the Majorana particles etc. The positive and negative particles are the sam. In essence, mathematics can be completely described by real functions. It can be described by complex functions, but it must meet the Majorana condition. For particles described by the Bargmann-Wigner equation or Dirac equation, it is generally possible to describe both charged complex particles and uncharged Majorana particles. The principal commutation rule in both cases is consistent, but the rest are generally zero for charged complex particles. For uncharged Majorana particles, the rest of the commutative or anti commutative brackets are naturally derived from the principal commutative rule and Majorana conditions. And they are generally not zero. In this chapter, we discuss both the complex particle case and the Majorana particle case. The complete commutation rules for both cases are given. However, in latter chapters, we will generally not seek completeness, but only discuss the complex particle case and the principal commutation rule. The Majorana particle case is no longer specifically discussed. If we want to obtain the quantum field theory of the Majorana particle case, we only need to add the Majorana condition to the complex particle case. Then we will naturally obtain it. In this chapter, the corresponding quantum field theory is established for Majorana particles with any spin in a unified manner. Like complex particles, there is no need to know the hamiltonian. Then various massive spin particles can be quantized by using a unified new program. Unified quantization commutation rules and energy momentum operators for fields and potentials are given. And a partial quantum Poincare algebra is given too. Like complex particles, the angular momentum operator has only achieved partial success and has not been thoroughly resolved. Efforts are still needed. The problem of angular momentum operators is a difficult problem that needs to be solved urgently in the new quantization program.

1 Majorana equation

1.1 Majorana equation under real representation and Dirac separated representation [4]

Def. 1.1.1.

$$\begin{cases}(\gamma_s^a\partial_a+m)\psi_s=0,\gamma_s^a=(\sigma_{-\kappa}\sigma_{\kappa y},\varsigma\sigma_{\kappa x}),\psi_s^*=\psi_s\\(\gamma^a\partial_a+m)\psi=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\psi^*=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi\end{cases}$$

$$\begin{cases}\psi_s=S_s(\kappa,\theta)\psi,S_s(\kappa,\theta):=e^{i\theta}S_{em}(\kappa)\\S_s^T(\kappa,\theta)S_s(\kappa,\theta)=e^{2i\theta}S_{em}^T(\kappa)S_{em}(\kappa)=-e^{2i\theta}\sigma_y\otimes\sigma_y\end{cases},S_{em}(\kappa)=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&0&-i\\-1&0&0&-1\\0&-i&-i&0\\0&-\kappa&\kappa&0\end{bmatrix}$$

1.2 Majorana condition under real representation and Dirac separated representation

Cor. 1.2.1. $\psi_s=\psi_s^*\Leftrightarrow\psi^*=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi,-\sigma_y\otimes\sigma_y=\bar{C}\gamma_4$

θ is adjust phase parameters, generally take 0 or $\pi/2$.

2 Majorana B-W equation

2.1 Majorana B-W equation under real representation and Dirac separated representation [16]

Def. 2.1.1.

$$\begin{cases}(\gamma_s^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=0,\gamma_s^a=(\sigma_{-\kappa}\sigma_{\kappa y},\varsigma\sigma_{\kappa x}),\psi_s^*=\psi_s\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\psi^*=(-1)^{2s}e^{4si\theta}\overbrace{\sigma_y\otimes\cdot\cdot\otimes\sigma_y}^{4s}\psi\end{cases}$$

2.2 Majorana condition under real representation and Dirac separated representation

Cor. 2.2.1.

$$\psi_s=\psi_s^*\Leftrightarrow\psi^*=(-1)^{2s}e^{4si\theta}\overbrace{\sigma_y\otimes\cdot\cdot\otimes\sigma_y}^{4s}\psi=e^{4si\theta}\overbrace{(\bar{C}\gamma_4)\otimes(\bar{C}\gamma_4)\cdot\cdot}^{2s}\psi$$

θ is adjust phase parameters, generally take 0 or $\pi/2$.

3 Plane wave solutions of Majorana B-W equation under separated representation

3.1 Lemma

Lem. 3.1.1. $\sum\limits_{h=s}^{-s} b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)=(-1)^{2s}e^{-4si\theta}\sum\limits_{h=s}^{-s} a^+(\vec{p},h)\overbrace{\sigma_y\otimes\sigma_y\cdot\cdot}^{4s}U^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)$

$\Leftrightarrow b^+(\vec{p},h)=\varsigma^{2s}e^{-4si\theta}(-1)^{s+h}a^+(\vec{p},-h)\Leftrightarrow b(\vec{p},h)=\varsigma^{2s}e^{4si\theta}(-1)^{s+h}a(\vec{p},-h)$

Proof: $\sum\limits_{h=s}^{-s} b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)=(-1)^{2s}e^{-4si\theta}\sum\limits_{h=s}^{-s} a^+(\vec{p},h)\overbrace{\sigma_y\otimes\sigma_y\cdot\cdot}^{4s}U^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)$

$=\varsigma^{2s}e^{-4si\theta}\sum\limits_{h=s}^{-s}(-1)^{s+h}a^+(\vec{p},-h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h)$

$\Leftrightarrow b^+(\vec{p},h)=\varsigma^{2s}e^{-4si\theta}(-1)^{s+h}a^+(\vec{p},-h)\Leftrightarrow b(\vec{p},h)=\varsigma^{2s}e^{4si\theta}(-1)^{s+h}a(\vec{p},-h)$ □

Lem. 3.1.2.

$$\begin{cases}[a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ [a(\vec{p},h),a(\vec{p}',h')]_{-^{2s+1}}=0\\ [a^+(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}=0\\ b^+(\vec{p},h)=\varsigma^{2s}e^{-4si\theta}(-1)^{s+h}a^+(\vec{p},-h)\\ b(\vec{p},h)=\varsigma^{2s}e^{4si\theta}(-1)^{s+h}a(\vec{p},-h)\end{cases}\Rightarrow\begin{cases}[b(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ [b(\vec{p},h),b(\vec{p}',h')]_{-^{2s+1}}=0\\ [b^+(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}=0\\ [a(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}=\varsigma^{2s}e^{-4si\theta}(-1)^{s-h}\delta_{-h,h'}\delta^3(\vec{p}-\vec{p}')\\ [a^+(\vec{p},h),b(\vec{p}',h')]_{-^{2s+1}}=-\varsigma^{2s}e^{4si\theta}(-1)^{s-h}\delta_{-h,h'}\delta^3(\vec{p}-\vec{p}')\\ [a(\vec{p},h),b(\vec{p}',h')]_{-^{2s+1}}=0\\ [a^+(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}=0\end{cases}$$

3.2 Plane wave solutions of Majorana B-W equation [16] under separated representation (The proof needs to be supplemented.)

Thm. 3.2.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\}}_{2s}}(\vec{r},t),\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=(-1)^{2s}e^{4si\theta}\overbrace{\sigma_y\otimes\cdot\cdot\otimes\sigma_y}^{4s}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\sum\limits_{h=s}^{-s}\sqrt{\frac{m^{2s}}{E}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\ b^+(\vec{p},h)=\varsigma^{2s}e^{-4si\theta}(-1)^{s+h}a^+(\vec{p},-h)\end{cases}$$

$$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\frac{m}{E}}^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ b^+(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\frac{m}{E}}^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Cor. 3.2.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\}}_{2s}}(\vec{r},t),\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=(-1)^{2s}e^{4si\theta}\overbrace{\sigma_y\otimes\cdot\cdot\otimes\sigma_y}^{4s}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\sum\limits_{h=s}^{-s}\sqrt{\frac{m^{2s}}{E}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s}e^{-4si\theta}a^+(\vec{p},h)\overbrace{\sigma_y\otimes\sigma_y\cdot\cdot}^{4s}U^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\frac{m}{E}}^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}$$

Cor. 3.2.2. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\}}_{2s}}(\vec{r},t),\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=(-1)^{2s}e^{4si\theta}\overbrace{\sigma_y\otimes\cdot\cdot\otimes\sigma_y}^{4s}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\sum_{h=s}^{-s}E^{s-\frac{1}{2}}[a(\vec{p},h)\tilde{U}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot x}+(-1)^{2s}e^{-4si\theta}a^+(\vec{p},h)\overbrace{\sigma_y\otimes\sigma_y\cdot\cdot}^{4s}\tilde{U}^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\tilde{U}^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}$$

3.3 Covariant commutation rules for Majorana B-W equation under separated representation

Thm. 3.3.1.

$$\begin{cases}[a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[b(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ [a(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}=\varsigma^{2s}e^{-4si\theta}(-1)^{s-h}\delta_{-h,h'}\delta^3(\vec{p}-\vec{p}')\\ [a^+(\vec{p},h),b(\vec{p}',h')]_{-^{2s+1}}=-\varsigma^{2s}e^{4si\theta}(-1)^{s+h}\delta_{-h,h'}\delta^3(\vec{p}-\vec{p}')\\ [rest]_{-^{2s+1}}=0\end{cases}$$

$\Rightarrow\Rightarrow$

$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}^{2s}\Delta(x-x')\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}^{2s}\Delta(x-x')\\ [\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{e^{4si\theta}}{[(2s)!]^2}\overbrace{[C^+(m-\gamma^a\partial_a^+)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[C^+(m-\gamma^b\partial_b^+)]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}^{2s}\Delta(x-x')\end{cases}$$

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}$

$$[[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot x},[a^+(\vec{p}',h')U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h)e^{-ip'\cdot x'}+b(\vec{p}',h')V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')e^{ip'\cdot x'}]$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}\{[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')[a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}e^{i(p\cdot x-p'\cdot x')}$$

$$+V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')[b^+(\vec{p},h),b(\vec{p}',h')]_{-^{2s+1}}e^{-i(p\cdot x-p'\cdot x')}\}$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}$$

$$[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{i(p\cdot x-p'\cdot x')}+(-1)^{2s+1}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{-i(p\cdot x-p'\cdot x')}]$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}[\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot(x-x')}+(-1)^{2s+1}\sum_{h=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot(x-x')}]$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}[\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},s)e^{ip\cdot(x-x')}+(-1)^{2s+1}\Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},s)e^{-ip\cdot(x-x')}]$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}[\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{2s}e^{ip\cdot(x-x')}$$

$$+(-1)^{2s+1}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{2s}e^{-ip\cdot(x-x')}]$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}^{2s}e^{ip\cdot(x-x')}$$

$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m+\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m+\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}^{2s}e^{-ip\cdot(x-x')}\}$$

$$=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}^{2s}\frac{-i}{(2\pi)^3}\int d^3\vec{p}\frac{1}{2E}[[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]$$

$$=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}^{2s}\Delta(x-x')$$

$= \frac{i(2m)^{2s}}{2^{2s-1}}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(-i\partial,s)\Delta(x-x')$

□

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}$

$[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-ip\cdot x}$

$,a(\vec{p}',h')U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h)e^{ip'\cdot x'}+b^+(\vec{p}',h')V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')e^{-ip'\cdot x'}]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}$

$[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')[a(\vec{p},h),b^+(\vec{p}',h')]_{-^{2s+1}}e^{i(p\cdot x-p'\cdot x')}$

$+V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')[b^+(\vec{p},h),a(\vec{p}',h')]_{-^{2s+1}}e^{-i(p\cdot x-p'\cdot x')}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}d^3\vec{p}'\sum_{h,h'=s}^{-s}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}\varsigma^{2s}e^{-4si\theta}(-1)^{s-h}\delta_{-h,h'}$

$[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')\delta^3(\vec{p}-\vec{p}')e^{i(p\cdot x-p'\cdot x')}-V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p}',h')\delta^3(\vec{p}-\vec{p}')e^{-i(p\cdot x-p'\cdot x')}]$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\sum_{h=s}^{-s}\frac{m^{2s}}{E}\varsigma^{2s}e^{-4si\theta}$

$[(-1)^{s-h}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},-h)e^{ip\cdot(x-x')}+(-1)^{2s+1}(-1)^{s+h}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(\vec{p},-h)e^{-ip\cdot(x-x')}]$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}\sum_{h=s}^{-s}\frac{m^{2s}}{E}e^{-4si\theta}$

$[\frac{1}{[(2s)!]^2}\underbrace{(\Lambda_+\bar{C}\gamma_4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})(\Lambda_+\bar{C}\gamma_4)_{\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot(\Lambda_+\bar{C}\gamma_4)_{\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\tfrac{1}{2})}_{2s}e^{ip\cdot(x-x')}$

$+(-1)^{2s+1}\frac{1}{[(2s)!]^2}\underbrace{(\Lambda_-\bar{C}\gamma_4)_{\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})(\Lambda_-\bar{C}\gamma_4)_{\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot(\Lambda_-\bar{C}\gamma_4)_{\tau_\varsigma\}\tau'_\varsigma)}(\vec{p},\tfrac{1}{2})}_{2s}e^{-ip\cdot(x-x')}]$

$= e^{-4si\theta}\frac{1}{(2\pi)^3}\int d^3\vec{p}\frac{m^{2s}}{E}\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}e^{ip\cdot(x-x')}$

$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m+\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m+\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}e^{-ip\cdot(x-x')}\}$

$= e^{-4si\theta}\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\frac{-i}{(2\pi)^3}\int d^3\vec{p}\frac{1}{2E}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]$

$= e^{-4si\theta}\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')$

□

Cor. 3.3.1. $[a(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[a(\vec{p},h),a(\vec{p}',h')]_{-^{2s+1}}=0,[a^+(\vec{p},h),a^+(\vec{p}',h')]_{-^{2s+1}}=0$

$$\Rightarrow\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')\\ [\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{e^{4si\theta}}{[(2s)!]^2}\overbrace{[C^+(m-\gamma^a\partial^+_a)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[C^+(m-\gamma^b\partial^+_b)]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')\end{cases}$$

3.4 Reverse reasoning of Majorana B-W commutation rules under separated representation

Thm. 3.4.1.

$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')\\ [\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{e^{4si\theta}}{[(2s)!]^2}\overbrace{[C^+(m-\gamma^a\partial^+_a)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[C^+(m-\gamma^b\partial^+_b)]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')\end{cases}$$

$$\Rightarrow\begin{cases}[a(\vec p,h),a^+(\vec p',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec p-\vec p'),[b(\vec p,h),b^+(\vec p',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec p-\vec p')\\ [a(\vec p,h),b^+(\vec p',h')]_{-^{2s+1}}=\varsigma^{2s}e^{-4si\theta}(-1)^{s-h}\delta_{-h,h'}\delta^3(\vec p-\vec p')\\ [a^+(\vec p,h),b(\vec p',h')]_{-^{2s+1}}=-\varsigma^{2s}e^{4si\theta}(-1)^{s+h}\delta_{-h,h'}\delta^3(\vec p-\vec p')\\ [rest]_{-^{2s+1}}=0\end{cases}$$

The following is a detailed proof process for several main commutative brackets.

Proof: $[a(\vec p,h),a^+(\vec p',h')]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec p',h')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}e^{-i(p\cdot x-p'\cdot x')}d^3\vec rd^3\vec r'$$

$$=\frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec p',h')$$

$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')e^{-i(p\cdot x-p'\cdot x')}d^3\vec rd^3\vec r'$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec rd^3\vec r'\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec p',h')$$

$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\{\frac{-i}{(2\pi)^3}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec p_0\}e^{-i(p\cdot x-p'\cdot x')}$$

$$=[\frac{1}{(2\pi)^3}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec p',h')$$

$$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^ap_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}$$

$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^ap_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^bp_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}\}d^3\vec rd^3\vec r'd^3\vec p_0$$

$$=[\frac{1}{(2\pi)^3}]^2\int d^3\vec rd^3\vec r'd^3\vec p_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec p',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec p_0,h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec p_0,h_0)e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec p_0,h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec p_0,h_0)e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}\}$$

$$=\int d^3\vec p_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec p',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec p_0,h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec p_0,h_0)\delta^3(\vec p_0-\vec p)\delta^3(\vec p_0-\vec p')$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec p_0,h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec p_0,h_0)e^{2iE_0(t-t')}\delta^3(\vec p_0+\vec p)\delta^3(\vec p_0+\vec p')\}$$

$$=\delta^3(\vec p-\vec p')(\frac{m}{E})^{4s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec p,h)U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec p,h')$$

$$\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec p,h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec p,h_0)+(-1)^{2s+1}\sum_{h_0=s}^{-s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(-\vec p,h_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(-\vec p,h_0)e^{2iE(t-t')}\}$$

$= \delta^3(\vec{p}-\vec{p}')(\sum_{h_0=s}^{-s} \delta_{hh_0}\delta_{h'h_0} + 0)$
$= \delta_{hh'}\delta^3(\vec{p}-\vec{p}')$

□

Proof: $[b^+(\vec{p},h), b(\vec{p}',h')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int \sqrt{EE'}(\frac{m}{EE'})^{2s} V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} e^{i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int \sqrt{EE'}(\frac{m}{EE'})^{2s} V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$
$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s} \Delta(x-x') e^{i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int d^3\vec{r} d^3\vec{r}' \sqrt{EE'}(\frac{m}{EE'})^{2s} V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$
$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s} \{\frac{-i}{(2\pi)^3}\int \frac{1}{2E_0}[e^{ip_0\cdot(x-x')} - e^{-ip_0\cdot(x-x')}]d^3\vec{p}_0\} e^{i(p\cdot x - p'\cdot x')}$

$= [\frac{1}{(2\pi)^3}]^2 \int \frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s} V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$
$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s} e^{ip_0\cdot(x-x')} e^{i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}'$
$+ (-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s} e^{-ip_0\cdot(x-x')} e^{i(p\cdot x - p'\cdot x')}\} d^3\vec{r} d^3\vec{r}' d^3\vec{p}_0$

$= [\frac{1}{(2\pi)^3}]^2 \int d^3\vec{r} d^3\vec{r}' d^3\vec{p}_0 \frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$
$V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0) U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0) e^{i(p_0+p)\cdot x} e^{-i(p_0+p')\cdot x'}$
$+ (-1)^{2s+1}\sum_{h_0=s}^{-s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0) V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0) e^{-i(p_0-p)\cdot x} e^{i(p_0-p')\cdot x'}\}$

$= \int d^3\vec{p}_0 \frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$
$V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0) U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0) e^{-2iE_0(t-t')}\delta^3(\vec{p}_0+\vec{p})\delta^3(\vec{p}_0+\vec{p}')$
$+ (-1)^{2s+1}\sum_{h_0=s}^{-s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0) V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)\delta^3(\vec{p}_0-\vec{p})\delta^3(\vec{p}_0-\vec{p}')\}$

$= \delta^3(\vec{p}-\vec{p}')(\frac{m}{E})^{4s} V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h')$
$\{\sum_{h_0=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(-\vec{p},h_0) U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(-\vec{p},h_0) e^{-2iE(t-t')} + (-1)^{2s+1}\sum_{h_0=s}^{-s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p},h_0) V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p},h_0)\}$
$= (-1)^{2s+1}\delta^3(\vec{p}-\vec{p}')(0 + \sum_{h_0=s}^{-s} \delta_{hh_0}\delta_{h'h_0})$
$= (-1)^{2s+1}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')$

□

Proof: $[a(\vec{p},h), b^+(\vec{p}',h')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^3}\int \sqrt{EE'}(\frac{m}{EE'})^{2s} U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x), \psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} e^{-i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}'$

$= \frac{1}{(2\pi)^3}\int \sqrt{EE'}(\frac{m}{EE'})^{2s} U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h) V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$
$\frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s} \Delta(x-x') e^{-i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}'$

$$= \frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$$

$$\frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}\{\frac{-i}{(2\pi)^3}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec{p}_0\}e^{-i(p\cdot x-p'\cdot x')}$$

$$=[\frac{1}{(2\pi)^3}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$$

$$\{\frac{1}{(2m)^{2s}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})C]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}e^{ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})C]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}e^{-ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}\}d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0$$

$$=\varsigma^{2s}e^{-4si\theta}[\frac{1}{(2\pi)^3}]^2\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}(-1)^{s-h_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,-h_0)e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}(-1)^{s+h_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,-h_0)e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}\}$$

$$=\varsigma^{2s}e^{-4si\theta}\int d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}(-1)^{s-h_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,-h_0)\delta^3(\vec{p}_0-\vec{p})\delta^3(\vec{p}_0-\vec{p}')$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}(-1)^{s+h_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,-h_0)\delta^3(\vec{p}_0+\vec{p})\delta^3(\vec{p}_0+\vec{p}')\}$$

$$=\varsigma^{2s}e^{-4si\theta}(\frac{m}{E})^{4s}$$

$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{+\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},h')\{\sum_{h_0=s}^{-s}(-1)^{s-h_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p},h_0)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p},-h_0)\delta^3(\vec{p}-\vec{p}')$$

$$+(-1)^{2s+1}\sum_{h_0=s}^{-s}(-1)^{s+h_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(-\vec{p},h_0)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(-\vec{p},-h_0)\delta^3(\vec{p}+\vec{p}')\}$$

$$=\varsigma^{2s}e^{-4si\theta}(-1)^{s-h}\delta_{-h,h'}\delta^3(\vec{p}-\vec{p}')$$ □

Proof: $[a(\vec{p},h),b(\vec{p}',h')]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}e^{-i(p\cdot x+p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$=\frac{1}{(2\pi)^3}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$$

$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}\Delta(x-x')e^{-i(p\cdot x+p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$=\frac{1}{(2\pi)^3}\int d^3\vec{r}d^3\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$$

$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}\{\frac{-i}{(2\pi)^3}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec{p}_0\}e^{-i(p\cdot x+p'\cdot x')}$$

$$=[\frac{1}{(2\pi)^3}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')$$

$$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}e^{ip_0\cdot(x-x')}e^{-i(p\cdot x+p'\cdot x')}d^3\vec{r}d^3\vec{r}'$$

$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots_{\})}}^{2s}e^{-ip_0\cdot(x-x')}e^{-i(p\cdot x+p'\cdot x')}\}d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0$$

$$=[\frac{1}{(2\pi)^3}]^2\int d^3\vec{r}d^3\vec{r}'d^3\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},h)V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,h_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0)e^{i(p_0-p)\cdot x}e^{-i(p_0+p')\cdot x'}$$

$+ (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0) V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0) e^{-i(p_0+p)\cdot x} e^{i(p_0-p')\cdot x'}\}$

$= \int d^3\vec{p}_0 \frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$

$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')\{\sum_{h_0=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0) U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0) e^{-iE_0t'}\delta^3(\vec{p}_0-\vec{p})\delta^3(\vec{p}_0+\vec{p}')$

$+ (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}_0,h_0) V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}_0,h_0) e^{iE_0t}\delta^3(\vec{p}_0+\vec{p})\delta^3(\vec{p}_0-\vec{p}')\}$

$= \delta^3(\vec{p}+\vec{p}')(\frac{m}{E})^{4s} U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p},h) V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}^{2s}}(\vec{p}',h')$

$\{\sum_{h_0=s}^{-s} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},h_0) U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p},h_0) + (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p}',h_0) V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec{p}',h_0) e^{2iE(t-t')}\}$

$= 0 + 0 = 0$ □

Cor. 3.4.1.

$$\begin{cases} [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x') \\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x') \\ [\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{e^{4si\theta}}{[(2s)!]^2}\overbrace{[C^+(m-\gamma^a\partial^+_a)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[C^+(m-\gamma^b\partial^+_b)]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x') \end{cases}$$

$\Rightarrow [a(\vec{p},h), a^+(\vec{p}',h')]_{-^{2s+1}} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'), [a(\vec{p},h), a(\vec{p}',h')]_{-^{2s+1}} = 0, [a^+(\vec{p},h), a^+(\vec{p}',h')]_{-^{2s+1}} = 0$

3.5 Summary of Majorana B-W covariant commutation rules under separated representation

Thm. 3.5.1.

$$\begin{cases} [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x') \\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x') \\ [\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{e^{4si\theta}}{[(2s)!]^2}\overbrace{[C^+(m-\gamma^a\partial^+_a)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[C^+(m-\gamma^b\partial^+_b)]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x') \end{cases}$$

$\Leftrightarrow [a(\vec{p},h), a^+(\vec{p}',h')]_{-^{2s+1}} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}'), [a(\vec{p},h), a(\vec{p}',h')]_{-^{2s+1}} = 0, [a^+(\vec{p},h), a^+(\vec{p}',h')]_{-^{2s+1}} = 0$

3.6 Important corollary of Majorana B-W covariant rules under separated representation

Def. 3.6.1. $(\gamma^a\partial_a + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}} = 0, \gamma^a = (\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x), \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}} = \Gamma^{K_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}\psi_{K_\varsigma}(s)$

Cor. 3.6.1.

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')$$

$$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta(x-x')$$

Proof:

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')$$

$$\Leftrightarrow [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}},\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$$

$$=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[-imI\otimes\sigma(x)+(\sigma\otimes\sigma_z,i\varsigma)^a\partial_a]_{\{\lambda_\varsigma(\lambda'_\varsigma}[-imI\otimes\sigma(x)+(\sigma\otimes\sigma_z,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')$$

$$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot\})}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta(x-x')$$

$$\Leftrightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta(x-x')$$ □

Cor. 3.6.2.

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]=\frac{i}{2^{2s-1}}\frac{e^{-4si\theta}}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')$$

$$\Rightarrow [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\Delta(x-x')$$

Cor. 3.6.3.

$$[\psi^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]=\frac{i}{2^{2s-1}}\frac{e^{4si\theta}}{[(2s)!]^2}\overbrace{[C^+(m-\gamma^a\partial^+_a)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[C^+(m-\gamma^b\partial^+_b)]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\Delta(x-x')$$

$$\Rightarrow [\psi^+_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=\Delta(x-x')$$

4 Equivalent Majorana B-W commutation rules under separated representation

4.1 Equivalent Majorana B-W commutation rules under separated representation

Lem. 4.1.1.

$$\begin{cases}2\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)=[(m-i\gamma^ap_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\\2\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})(\bar C\gamma_4\mathbb{X}^{+a'}\bar C\gamma_4)_{\lambda'_\varsigma\mu'_\varsigma}(p)=[(m-i\gamma^ap_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp_b)C]_{\mu_\varsigma\}\mu'_\varsigma)}\end{cases}$$

$$\begin{cases}2\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\Delta(x-x')=[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')\\2\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})(\bar C\gamma_4\mathbb{X}^{+a'}\bar C\gamma_4)_{\lambda'_\varsigma\mu'_\varsigma}(x')\Delta(x-x')=[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')\end{cases}$$

Thm. 4.1.1.

$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]=\frac{i}{2^{2n-1}}\frac{1}{[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')\\\Leftrightarrow\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]=\frac{i}{2^{3n-1}}\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]=\frac{i}{2^{2n-1}}\frac{e^{-4ni\theta}}{[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')\\\Leftrightarrow\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]=\frac{i}{2^{3n-1}}\frac{e^{-4ni\theta}}{[(2s)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{(\bar C\gamma_4\mathbb{X}^{+a'}\bar C\gamma_4)_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')\end{cases}$$

Thm. 4.1.2.

$$\begin{cases}\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}\\=\frac{i}{2^{2n}}\frac{1}{[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')\\\Leftrightarrow\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}\\=\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')\end{cases}$$

$$\Leftrightarrow \begin{cases} \{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\} \\ =\frac{i}{2^{2n}}\frac{e^{-(4n+2)i\theta}}{[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)C]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x') \\ \Leftrightarrow \{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\} \\ =\frac{ie^{-(4n+2)i\theta}}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{(\bar{C}\gamma_4\mathbb{X}^{+a'}\bar{C}\gamma_4)_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)C]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x') \end{cases}$$

4.2 Summary of commutation rules for Majorana boson under separated representation

Thm. 4.2.1. $n\geq 0$

$[a(\vec{p},h;n),a^+(\vec{p}',h';n)]=\delta_{hh'}\delta^3(\vec{p}-\vec{p}'),[a(\vec{p},h;n),a(\vec{p}',h';n)]=0,[a^+(\vec{p},h;n),a^+(\vec{p}',h';n)]=0$

$$\Leftrightarrow [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]=\frac{i}{2^{2n-1}}\frac{1}{[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$$

$$\Leftrightarrow [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')]=\frac{i}{2^{3n-1}}\frac{1}{[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot)}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

4.3 Summary of anticommutation rules for Majorana fermion under separated representation

Thm. 4.3.1. $n\geq 0$

$\{a(\vec{p},h;n+\frac{1}{2}),a^+(\vec{p}',h';n+\frac{1}{2})\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}'),\{rest\}=0$

$$\Leftrightarrow \{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$$

$$=\frac{i}{2^{2n}}\frac{1}{[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')$$

$\Leftrightarrow$

$$\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$$

$$=\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')$$

4.4 Plane wave solutions of K-G equation with $s=n$ under separated representation

Thm. 4.4.1. $(-\partial^c\partial_c+m^2)A_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n}}(x)=(\frac{1}{2im})^n\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(x)$

$$A_{\underbrace{ab\cdot\cdot}_{n}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}[a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)$$

$$\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p},h)$$

5 Quantum operators for Majorana B-W equation under separated representation

5.1 Extraction of various Majorana B-W operators under separated representation

Thm. 5.1.1.

$$\begin{cases} P_u(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} = \int\sum_h p_u[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}a(\vec{p},h)a^+(\vec{p},h)]d^3\vec{p} \\ Q(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} = \int\sum_h [a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s-1}a(\vec{p},h)a^+(\vec{p},h)]d^3\vec{p} \\ N(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} = \int\sum_h [a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}a(\vec{p},h)a^+(\vec{p},h)]d^3\vec{p} \\ \vec{S}(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} = \int\sum_h \hat{p}[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s}a(\vec{p},h)a^+(\vec{p},h)]d^3\vec{p} \\ \vec{M}(s) = \int \psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r} = \int\sum_h \hat{p}[a^+(\vec{p},h)a(\vec{p},h)+(-1)^{2s-1}a(\vec{p},h)a^+(\vec{p},h)]d^3\vec{p} \end{cases}$$

6 Commutation rules for K-G equation

6.1 Majorana B-W equation is equivalent to K-G equation [16, 20, 21] for massive $s = n$ particles

Def. 6.1.1. $\mathbb{X}_a \equiv [im\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)\partial^b]C$

Thm. 6.1.1. $\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x) = 0 \\ \psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x) \ fully\ symmetric \end{cases} \Leftrightarrow \begin{cases} (-\partial^c\partial_c + m^2)A_{\underbrace{ab\cdots}_{n}}(x) = 0 \\ \delta^{ab}A_{\underbrace{ab\cdots}_{n}}(x) = 0, \partial^a A_{\underbrace{ab\cdots}_{n}}(x) = 0, A_{\underbrace{ab\cdots}_{n}}(x)\ fully\ symmetric \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x) = \frac{1}{2^n}\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdots}^{n} A_{\underbrace{ab\cdots}_{n}}(x) \end{cases}$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}E^{n-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2n}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$A_{\underbrace{ab\cdots}_{n}}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}[a(\vec{p},h)\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

6.2 Plane wave solutions of K-G equation with $s = n$

Cor. 6.2.1. $A_{ab\cdots}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}\varepsilon_{ab\cdots}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + (-1)^n b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 6.2.2. $A_{ab\cdots}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}\varepsilon_{ab\cdots}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + e^{-4ni\theta}(-1)^h a^+(\vec{p},-h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Pro. 6.2.1. $b^+(\vec{p},h) = \varsigma^{2s}e^{-4si\theta}(-1)^{s+h}a^+(\vec{p},-h) \Rightarrow b(\vec{p},h) = \varsigma^{2s}e^{4si\theta}(-1)^{s+h}a(\vec{p},-h)$

Cor. 6.2.3. $A_{ab\cdots}(\vec{r},t) = e^{-4ni\theta}\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}A^+_{a'b'\cdots}(\vec{r},t)$

Proof: $e^{-4ni\theta}\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}A^+_{a'b'\cdots}(\vec{r},t)$

$= \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}\varepsilon^+_{a'b'\cdots}(\vec{p},h)[e^{-4ni\theta}a^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h a(\vec{p},-h)e^{i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

$= \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}(-1)^h\varepsilon_{ab\cdots}(\vec{p},-h)[e^{-4ni\theta}a^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h a(\vec{p},-h)e^{i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

$= \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}\varepsilon_{ab\cdots}(\vec{p},-h)[a(\vec{p},-h)e^{i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h e^{-4ni\theta}a^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

$= \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}\varepsilon_{ab\cdots}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h e^{-4ni\theta}a^+(\vec{p},-h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

$= A_{ab\cdots}(\vec{r},t)$ □

From the above conclusion it can be seen that in order to maintain the real property of potential $A_{ab\cdots}$, it is appropriate to take θ=0 or $\pi/2$.

7 Anticommutation rules for R-S equation

7.1 Majorana B-W equation⇔R-S equation [16,17,20] for massive $s=n+\frac{1}{2}$ particles

Thm. 7.1.1.

$$\begin{cases}(\gamma^a\partial_a+m)\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)\ \textit{fully symmetric}\end{cases}\Leftrightarrow\begin{cases}(\gamma^c\partial_c+m)A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0\\ \delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0,\gamma^a A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)\ \textit{fully symmetric}\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)=\frac{1}{2^n}\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x)\end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec r,t)$$

$$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum_{h=n+1/2}^{-(n+1/2)}E^{n-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2n+1}[a(\vec p,h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec p,h)e^{i(\vec p\cdot\vec r-Et)}+b^+(\vec p,h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec p,h)e^{-i(\vec p\cdot\vec r-Et)}]d^3\vec p$$

$$A_{ab\cdot\cdot\tau_\varsigma}(\vec r,t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}[\varepsilon_{ab\cdot\cdot\tau_\varsigma}(\vec p,h)a(\vec p,h)e^{i(\vec p\cdot\vec r-Et)}+\tilde\varepsilon_{ab\cdot\cdot\tau_\varsigma}(\vec p,h)b^+(\vec p,h)e^{-i(\vec p\cdot\vec r-Et)}]d^3\vec p$$

7.2 Plane wave solutions of R-S equation with $s=n+\frac{1}{2}$ under separated representation

Cor. 7.2.1. $A_{ab\cdot\cdot\tau_\varsigma}(\vec r,t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}[\varepsilon_{ab\cdot\cdot\tau_\varsigma}(\vec p,h)a(\vec p,h)e^{i(\vec p\cdot\vec r-Et)}+\tilde\varepsilon_{ab\cdot\cdot\tau_\varsigma}(\vec p,h)b^+(\vec p,h)e^{-i(\vec p\cdot\vec r-Et)}]d^3\vec p$

Cor. 7.2.2. $A_{ab\cdot\cdot\tau_\varsigma}(\vec r,t)$

$$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}\varepsilon_{ab\cdot\cdot\sigma_\varsigma}(\vec p,h)[\delta_{\tau_\varsigma}{}^{\sigma_\varsigma}a(\vec p,h)e^{i(\vec p\cdot\vec r-Et)}+(-1)^{h-\frac{1}{2}}e^{-(4n+2)i\theta}\gamma_{5\tau_\varsigma}{}^{\sigma_\varsigma}a^+(\vec p,-h)e^{-i(\vec p\cdot\vec r-Et)}]d^3\vec p$$

Thm. 7.2.1. $A^+_{ab\cdot\cdot\tau_\varsigma}(\vec r,t)=-e^{(4n+2)i\theta}\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}(\sigma_y\otimes\sigma_y)_{\tau_\varsigma}{}^{\tau'_\varsigma}A_{a'b'\cdot\cdot\tau'_\varsigma}(\vec r,t)=e^{(4n+2)i\theta}\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}(\bar C\gamma_4)_{\tau_\varsigma}{}^{\tau'_\varsigma}A_{a'b'\cdot\cdot\tau'_\varsigma}(\vec r,t)$

Proof: $-e^{-(4n+2)i\theta}\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}(\sigma_y\otimes\sigma_y)_{\tau_\varsigma}{}^{\tau'_\varsigma}A^+_{a'b'\cdot\cdot\tau'_\varsigma}(\vec r,t)$

$$=-\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}$$

$$\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}\gamma_{2\tau_\varsigma}{}^{\tau'_\varsigma}\varepsilon^+_{a'b'\cdot\cdot\sigma'_\varsigma}(\vec p,h)[e^{-(4n+2)i\theta}\delta_{\tau'_\varsigma}{}^{\sigma'_\varsigma}a^+(\vec p,h)e^{-i(\vec p\cdot\vec r-Et)}+(-1)^{h-\frac{1}{2}}\gamma_{5\tau_\varsigma}{}^{\sigma_\varsigma}a(\vec p,-h)e^{i(\vec p\cdot\vec r-Et)}]d^3\vec p$$

$$=-\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau'_\varsigma}(\vec p,-h)[(-1)^{h-\frac{1}{2}}e^{-(4n+2)i\theta}\gamma_{5\tau_\varsigma}{}^{\tau'_\varsigma}a^+(\vec p,h)e^{-i(\vec p\cdot\vec r-Et)}-\delta_{\tau_\varsigma}{}^{\tau'_\varsigma}a(\vec p,-h)e^{i(\vec p\cdot\vec r-Et)}]d^3\vec p$$

$$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum_{h=n}^{-n}\frac{\sqrt{m}}{\sqrt{2^nE}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\sigma_\varsigma}(\vec p,h)[(-1)^{h-\frac{1}{2}}e^{-(4n+2)i\theta}\gamma_{5\tau_\varsigma}{}^{\sigma_\varsigma}a^+(\vec p,-h)e^{-i(\vec p\cdot\vec r-Et)}+\delta_{\tau_\varsigma}{}^{\sigma_\varsigma}a(\vec p,h)e^{i(\vec p\cdot\vec r-Et)}]d^3\vec p$$

$=A_{ab\cdot\cdot\tau_\varsigma}(\vec r,t)$ □

From the above conclusion it can be seen that in order to maintain the real property of potential $A_{ab\cdot\cdot}$, it is appropriate to take θ=0 or $\pi/2$. But taking $\theta=0$ is simpler.

7.3 Isochronous quantization rules for Majorana B-W equation under separated representation

Thm. 7.3.1.

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(\sigma\otimes\sigma_z,i\varsigma)^a_{\{\lambda_\varsigma(\lambda'_\varsigma}(\sigma\otimes\sigma_z,i\varsigma)^b_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}^{2s}\overbrace{\partial_a\partial_b\cdot\cdot}^{2s}\Delta(x-x')$$

$$\Rightarrow[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\xi_\varsigma\eta_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec r,t),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\xi'_\varsigma\eta'_\varsigma\cdot\cdot\tau'_\varsigma}_{2s}}(\vec r',t)]_{-^{2s+1}}$$

$$=-\frac{(i\varsigma)^{2s+1}}{2^{2s-1}}\sum_{k=0}^{[s-\frac{1}{2}]}\frac{(2s)!}{(2s-2k-1)!(2k)!1!}\overbrace{[(\sigma\cdot\nabla)\otimes\sigma_z]_{\lambda_\varsigma\lambda'_\varsigma}[(\sigma\cdot\nabla)\otimes\sigma_z]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}^{2s-2k-1}\overbrace{\delta_{\xi_\varsigma\xi'_\varsigma}\delta_{\eta_\varsigma\eta'_\varsigma}\cdot\cdot}^{2k}\nabla^{2k}\delta_{\tau_\varsigma\tau'_\varsigma}\delta^3(\vec r-\vec r')$$

8 Card commutation rules for Majorana particles under separated representation

8.1 Definition

Def. 8.1.1.

$$\Gamma^{\overbrace{b_1b_2\cdot\cdot}^{n}\overbrace{b'_1b'_2\cdot\cdot}^{n}}_{\underbrace{a_1a_2\cdot\cdot}_{n}\underbrace{a'_1a'_2\cdot\cdot}_{n}}(p;n)$$

$$:= \overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{a_2})^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'_1}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{a'_2}C)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}\underbrace{\mathbb{X}^{b_1}_{\{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^{b_2}_{\eta_\varsigma\xi_\varsigma}(p)\cdots\}}_{n}\underbrace{\mathbb{X}^{+b'_1}_{(\lambda'_\varsigma\mu'_\varsigma}(-p)\mathbb{X}^{+b'_2}_{\eta'_\varsigma\xi'_\varsigma}(-p)\cdots)}_{n}$$

$$\Gamma^{\overbrace{b_1b_2\cdots}^{n}\overbrace{b'_1b'_2\cdots}^{n}}_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x,x';n)$$

$$:= \overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{a_2})^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'_1}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{a'_2}C)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}\underbrace{\mathbb{X}^{b_1}_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{b_2}_{\eta_\varsigma\xi_\varsigma}(x)\cdots\}}_{n}\underbrace{\mathbb{X}^{+b'_1}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+b'_2}_{\eta'_\varsigma\xi'_\varsigma}(x')\cdots)}_{n}$$

Def. 8.1.2.

$$\Gamma_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(p;n)$$

$$:= \overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{a_2})^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'_1}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{a'_2}C)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}\underbrace{[(m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n}$$

$$\Gamma_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x;n)$$

$$:= \overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{a_2})^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'_1}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{a'_2}C)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n}$$

Cor. 8.1.1. $\Gamma_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x;n)=\frac{1}{2^n}\Gamma^{\overbrace{b_1b_2\cdots}^{n}\overbrace{b'_1b'_2\cdots}^{n}}_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x,x';n)\underbrace{[\eta_{b_1b'_1}-\frac{\partial_{b_1}\partial^+_{b'_1}}{m^2}][\eta_{b_2b'_2}-\frac{\partial_{b_2}\partial^+_{b'_2}}{m^2}]\cdots}_{n}$

8.2 Card commutation rules for Majorana boson under separated representation(take $\theta=0$)

Def. 8.2.1. $\hat{P}_{a_1\cdots a_n\tau_\varsigma b_1\cdots b_n}(n):=\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdots\eta^{a'_n}_{b_n}\hat{P}_{a_1\cdots a_n\tau_\varsigma a'_1\cdots a'_n}(n)$

Thm. 8.2.1.
$$\begin{cases}[A_{a_1a_2\cdots a_n}(x),A^+_{a'_1a'_2\cdots a'_n}(x')]=i\hat{P}_{a_1a_2\cdots a_na'_1a'_2\cdots a'_n}(n)\Delta(x-x')\\ [A_{a_1a_2\cdots a_n}(x),A_{b_1b_2\cdots b_n}(x')]=i\hat{P}_{a_1a_2\cdots a_na'_1a'_2\cdots a'_n}(n)\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdots\eta^{a'_n}_{b_n}\Delta(x-x')\\ A_{a_1a_2\cdots a_n}=A^+_{a'_1a'_2\cdots a'_n}\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdots\eta^{a'_n}_{a_n},A^+_{\underbrace{a'_1a'_2\cdots}_{n}}=A_{\underbrace{a_1a_2\cdots}_{n}}\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdots}_{n}\end{cases}$$

$$[\Updownarrow]$$

Thm. 8.2.2.
$$\begin{cases}[A_{\underbrace{a_1a_2\cdots}_{n}}(x),A^+_{\underbrace{a'_1a'_2\cdots}_{n}}(x')]=\frac{im^{-2n}}{2^{4n-1}[(2n)!]^2}\Gamma_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x;n)\Delta(x-x')\\ [A_{\underbrace{a_1a_2\cdots}_{n}}(x),A_{\underbrace{b_1b_2\cdots}_{n}}(x')]=\frac{im^{-2n}}{2^{4n-1}[(2n)!]^2}\Gamma_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x;n)\underbrace{\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdots}_{n}\Delta(x-x')\\ A_{\underbrace{a_1a_2\cdots}_{n}}=A^+_{\underbrace{a'_1a'_2\cdots}_{n}}\underbrace{\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdots}_{n},A^+_{\underbrace{a'_1a'_2\cdots}_{n}}=A_{\underbrace{a_1a_2\cdots}_{n}}\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdots}_{n}\end{cases}$$

$$[\Updownarrow]$$

Thm. 8.2.3.
$$\begin{cases}[A_{\underbrace{a_1a_2\cdots}_{n}}(x),A^+_{\underbrace{a'_1a'_2\cdots}_{n}}(x')]=\frac{im^{-2n}}{2^{5n-1}[(2n)!]^2}\Gamma^{\overbrace{b_1b_2\cdots}^{n}\overbrace{b'_1b'_2\cdots}^{n}}_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x,x';n)\underbrace{[\eta_{b_1b'_1}-\frac{\partial_{b_1}\partial^+_{b'_1}}{m^2}]\cdots}_{n}\Delta(x-x')\\ [A_{\underbrace{a_1a_2\cdots}_{n}}(x),A_{\underbrace{b_1b_2\cdots}_{n}}(x')]=\frac{im^{-2n}}{2^{5n-1}[(2n)!]^2}\Gamma^{\overbrace{c_1c_2\cdots}^{n}\overbrace{c'_1c'_2\cdots}^{n}}_{\underbrace{a_1a_2\cdots}_{n}\underbrace{a'_1a'_2\cdots}_{n}}(x,x';n)\underbrace{[\eta_{c_1c'_1}-\frac{\partial_{c_1}\partial^+_{c'_1}}{m^2}]\cdots}_{n}\underbrace{\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdots}_{n}\Delta(x-x')\\ A_{\underbrace{a_1a_2\cdots}_{n}}=A^+_{\underbrace{a'_1a'_2\cdots}_{n}}\underbrace{\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdots}_{n},A^+_{\underbrace{a'_1a'_2\cdots}_{n}}=A_{\underbrace{a_1a_2\cdots}_{n}}\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdots}_{n}\end{cases}$$

$$[\Updownarrow]$$

Thm. 8.2.4.
$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=\frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots\}}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots)}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=\frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots\}}_{n}\underbrace{\mathbb{X}^{a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots)}_{n}\underbrace{[\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}=\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}\underbrace{\gamma_2{}^{\lambda'_\varsigma}_{\lambda_\varsigma}\gamma_2{}^{\mu'_\varsigma}_{\mu_\varsigma}\cdots}_{2n},\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}\underbrace{\gamma_2{}^{\lambda_\varsigma}_{\lambda'_\varsigma}\gamma_2{}^{\mu_\varsigma}_{\mu'_\varsigma}\cdots}_{2n},\mathbb{X}^a=\gamma_2\mathbb{X}^{+a'}\gamma_2\eta^a_{a'}\end{cases}$$

[⇕]

Thm. 8.2.5.
$$\begin{cases}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=\frac{i}{2^{2n-1}[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n}\Delta(x-x')\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=\frac{i}{2^{2n-1}[(2n)!]^2}\underbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n}\Delta(x-x')\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}=\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}\underbrace{\gamma_2{}^{\lambda'_\varsigma}_{\lambda_\varsigma}\gamma_2{}^{\mu'_\varsigma}_{\mu_\varsigma}\cdots}_{2n},\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}\underbrace{\gamma_2{}^{\lambda_\varsigma}_{\lambda'_\varsigma}\gamma_2{}^{\mu_\varsigma}_{\mu'_\varsigma}\cdots}_{2n}\end{cases}$$

[⇓]

Thm. 8.2.6. $[\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2n}}(x')]=i\frac{(i\varsigma)^{2n}}{2^{2n-1}[(2n)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2n}\overbrace{\partial_a\partial_b\cdots}^{2n}\Delta(x-x')$

[⇕]

Thm. 8.2.7. $[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\frac{(-1)^{2n}}{2^{n-1}}\Gamma^{\overbrace{abc\cdots}^{2n}}_{k_\varsigma k'_\varsigma}(n)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x')$

There are two equivalent expressions for the commutative relationship between potential A and field ψ, and they are mutually premises and causal each other. You can deduce everything from potential commutation relations, or you can deduce everything from field commutation relations too. This shows that the two descriptions of potential and field are completely equivalent for massive particles. And it can be deduced from the massive particle commutation rules which are completely similar to the massless particle commutation rules. But not vice versa.

8.3 Card anticommutation rules for Majorana fermion under separated representation

Thm. 8.3.1. $\{A_{a_1a_2\cdots a_n\tau_\varsigma}(x),A^+_{a'_1a'_2\cdots a'_n\tau'_\varsigma}(x')\}=i\hat{P}_{a_1\cdots a_n\tau_\varsigma a'_1\cdots a'_n\tau'_\varsigma}(n+\frac{1}{2})\Delta(x-x')$

[⇕]

Thm. 8.3.2.
$$\begin{cases}\{A_{a_1a_2\cdots a_n\tau_\varsigma}(x),A_{b_1b_2\cdots b_n\sigma_\varsigma}(x')\}=-i\hat{P}_{a_1\cdots a_n\tau_\varsigma b_1\cdots b_n\tau'_\varsigma}(n+\frac{1}{2})\gamma_2{}^{\tau'_\varsigma}_{\sigma_\varsigma}\Delta(x-x')\\ A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(\vec{r},t)=-A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(\vec{r},t)\underbrace{\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdots}_{n}\gamma_2{}^{\tau'_\varsigma}_{\tau_\varsigma},A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(\vec{r},t)=-A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(\vec{r},t)\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdots}_{n}\gamma_2{}^{\tau_\varsigma}_{\tau'_\varsigma}\end{cases}$$

[⇕]

Thm. 8.3.3.
$$\begin{cases}\{A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(x')\}=\frac{im^{-2n}}{2^{5n}[(2n+1)!]^2}\\ \overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'_1}C)^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\underbrace{\mathbb{X}^{b_1}_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+b'_1}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{b_1b'_1}-\frac{\partial_{b_1}\partial^+_{b'_1}}{m^2}]\cdots}_{n}\Delta(x-x')\\ \{A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(x),A_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(x')\}=\frac{i(im)^{-2n}}{2^{5n}[(2n+1)!]^2}\\ \overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\bar{C}\gamma_{a'_1})^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\underbrace{\mathbb{X}^{b_1}_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{b'_1}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\delta_{b_1b'_1}-\frac{\partial_{b_1}\partial_{b'_1}}{m^2}]\cdots}_{n}\Delta(x-x')\\ A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(\vec{r},t)=-A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(\vec{r},t)\underbrace{\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdots}_{n}\gamma_2{}^{\tau'_\varsigma}_{\tau_\varsigma},A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(\vec{r},t)=-A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(\vec{r},t)\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdots}_{n}\gamma_2{}^{\tau_\varsigma}_{\tau'_\varsigma}\\ \bar{C}\gamma_a=-\gamma_2\gamma_{a'}C\gamma_2\eta^{a'}_a,\mathbb{X}^a=\gamma_2\mathbb{X}^{+a'}\gamma_2\eta^a_{a'}\end{cases}$$

$[\Updownarrow]$

Thm. 8.3.4.

$$\begin{cases}\{A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(x')\}\\=\frac{im^{-2n}}{2^{4n}[(2n+1)!]^2}\overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'_1}C)^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}\cdots[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')\\\{A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(x),A_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(x')\}\\=\frac{i(im)^{-2n}}{2^{4n}[(2n+1)!]^2}\overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\bar{C}\gamma_{a'_1})^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\underbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}\cdots[(m-\gamma^c\partial_c)C]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')\\A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(\vec{r},t)=-A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(\vec{r},t)\underbrace{\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdots}_{n}\gamma_2{}^{\tau'_\varsigma}_{\tau_\varsigma},A^+_{\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(\vec{r},t)=-A_{\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(\vec{r},t)\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdots}_{n}\gamma_2{}^{\tau_\varsigma}_{\tau'_\varsigma},\bar{C}\gamma_a=-\gamma_2\gamma_{a'}C\gamma_2\eta^{a'}_{a}\end{cases}$$

$[\Updownarrow]$

Thm. 8.3.5.

$$\begin{cases}\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}=\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}=\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c\partial_c)C]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}=-\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}\underbrace{\gamma_2{}^{\lambda'_\varsigma}_{\lambda_\varsigma}\gamma_2{}^{\mu'_\varsigma}_{\mu_\varsigma}\cdots}_{2n+1},\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}=-\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}\underbrace{\gamma_2{}^{\lambda_\varsigma}_{\lambda'_\varsigma}\gamma_2{}^{\mu_\varsigma}_{\mu'_\varsigma}\cdots}_{2n+1},\mathbb{X}^a=\gamma_2\mathbb{X}^{+a'}\gamma_2\eta^a_{a'}\end{cases}$$

$[\Updownarrow]$

Thm. 8.3.6.

$$\begin{cases}\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}=\frac{i}{2^{2n}[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n+1}\Delta(x-x')\\\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}=\frac{i}{2^{2n}[(2n+1)!]^2}\underbrace{[(m-\gamma^a\partial_a)C]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)C]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n+1}\Delta(x-x')\\\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}=-\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}\underbrace{\gamma_2{}^{\lambda'_\varsigma}_{\lambda_\varsigma}\gamma_2{}^{\mu'_\varsigma}_{\mu_\varsigma}\cdots}_{2n+1},\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}=-\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}\underbrace{\gamma_2{}^{\lambda_\varsigma}_{\lambda'_\varsigma}\gamma_2{}^{\mu_\varsigma}_{\mu'_\varsigma}\cdots}_{2n+1}\end{cases}$$

$[\Downarrow]$

Thm. 8.3.7. $\{\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n+1}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2n+1}}(x')\}=i\frac{(i\varsigma)^{2n+1}}{2^{2n}[(2n+1)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2n+1}\overbrace{\partial_a\partial_b\cdots}^{2n+1}\Delta(x-x')$

$[\Updownarrow]$

Thm. 8.3.8. $\{\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')\}=i\frac{(-1)^{2n+1}}{2^{n-1/2}}\Gamma^{\overbrace{abc\cdots}^{2n+1}}_{k_\varsigma k'_\varsigma}(n+\frac{1}{2})\overbrace{\partial_a\partial_b\partial_c\cdots}^{2n+1}\Delta(x-x')$

There are two equivalent expressions for the commutative relationship between potential A and field ψ, and they are mutually premises and causal each other. You can deduce everything from potential commutation relations, or you can deduce everything from field commutation relations too. This shows that the two descriptions of potential and field are completely equivalent for massive particles. And it can be deduced from the massive particle commutation rules which are completely similar to the massless particle commutation rules. But not vice versa.

8.4 Majorana equation under real representation and Dirac separated representation

Lem. 8.4.1. $S_y=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}$

$S_y(\sigma_x,\sigma_y,\sigma_z)S_y^+=(-\sigma_z,\sigma_y,\sigma_x),S_y^+(\sigma_x,\sigma_y,\sigma_z)S_y=(\sigma_z,\sigma_y,-\sigma_x)$

$I\otimes S_y[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_z),-\varsigma I\otimes\sigma_x]I\otimes S_y^+=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_z]$

$I\otimes S_y^+[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\varsigma I\otimes\sigma_x]I\otimes S_y=[(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_z),-\varsigma I\otimes\sigma_x]$

Def. 8.4.1.

$$\begin{cases}(\gamma_s^a\partial_a+m)\psi_s=0,\gamma_s^a=(\sigma_{-\kappa}\sigma_{\kappa y},\varsigma\sigma_{\kappa z}),\psi_s^*=\psi_s\\(\gamma^a\partial_a+m)\psi=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x),\psi^*=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi\end{cases}$$

$$\begin{cases}\psi_s=S_s(\kappa,\theta)\psi,S_s(\kappa,\theta):=e^{i\theta}S_{em}(\kappa)|(I\otimes S_y^+)\\S_s^T(\kappa,\theta)S_s(\kappa,\theta)=e^{2i\theta}S_{em}^T(\kappa)S_{em}(\kappa)=-e^{2i\theta}\sigma_y\otimes\sigma_y\end{cases},S_{em}(\kappa)=\frac{1}{\sqrt{2}}\begin{bmatrix}i&0&0&-i\\-1&0&0&-1\\0&-i&-i&0\\0&-\kappa&\kappa&0\end{bmatrix},S_y=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}$$

9 Bargmann-Wigner equation under real representation

9.1 B-W basis under real representation

Lem. 9.1.1.

$$\begin{cases}\overbrace{S_s\otimes S_s\cdots}^{2s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{[S_su_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})][S_su_{\mu_\varsigma}(\vec{p},\frac{1}{2})]\cdots}_{s+h}\underbrace{\cdots[S_su_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})][S_su_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})]}_{s-h}\\\overbrace{S_s\otimes S_s\cdots}^{2s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{[S_sv_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})][S_sv_{\mu_\varsigma}(\vec{p},\frac{1}{2})]\cdots}_{s+h}\underbrace{\cdots[S_sv_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})][S_sv_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})]}_{s-h}\end{cases}$$

Def. 9.1.1.

$$\begin{cases}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h):=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{u_{s\{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{s\mu_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{s+h}\underbrace{\cdots u_{s\sigma_\varsigma}(\vec{p},-\frac{1}{2})u_{s\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{s-h}\\V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h):=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{v_{s\{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{s\mu_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{s+h}\underbrace{\cdots v_{s\sigma_\varsigma}(\vec{p},-\frac{1}{2})v_{s\tau_\varsigma\}}(\vec{p},-\frac{1}{2})}_{s-h}\end{cases}$$

9.2 Relations between B-W bases under real representation

Cor. 9.2.1.
$$\begin{cases}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=(-\varsigma)^{2s}\overbrace{\gamma_{s5}\otimes\gamma_{s5}\cdots}^{2s}V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)\\V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=(-\varsigma)^{2s}\overbrace{\gamma_{s5}\otimes\gamma_{s5}\cdots}^{2s}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)\end{cases}$$

Cor. 9.2.2.
$$\begin{cases}U^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=-(-1)^{s+h}\varsigma^{2s}V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)\\V^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=-(-1)^{s-h}\varsigma^{2s}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)\end{cases}$$

Cor. 9.2.3.
$$\begin{cases}U^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=-(-1)^{s-h}\overbrace{\gamma_{s5}\otimes\gamma_{s5}\cdots}^{2s}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)\\V^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=-(-1)^{s+h}\varsigma^{2s}\overbrace{\gamma_{s5}\otimes\gamma_{s5}\cdots}^{2s}V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-h)\end{cases}$$

9.3 Relations between B-W basis and vector basis under real representation

Cor. 9.3.1.

$$\begin{cases}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{[S_s\mathbb{X}^a(p)S_s^T]_{\lambda_\varsigma\mu_\varsigma}[S_s\mathbb{X}^b(p)S_s^T]_{\eta_\varsigma\xi_\varsigma}\cdots}_{n}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h)\\{}[\Rightarrow]\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(S_s^*\bar{C}\gamma_aS_s^+)^{\lambda_\varsigma\mu_\varsigma}(S_s^*\bar{C}\gamma_bS_s^+)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)\\V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{[S_s\mathbb{X}^a(-p)S_s^T]_{\lambda_\varsigma\mu_\varsigma}[S_s\mathbb{X}^b(-p)S_s^T]_{\eta_\varsigma\xi_\varsigma}\cdots}_{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p},h)\\{}[\Rightarrow]\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(S_s^*\bar{C}\gamma_aS_s^+)^{\lambda_\varsigma\mu_\varsigma}(S_s^*\bar{C}\gamma_bS_s^+)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p},h)\end{cases}$$

Cor. 9.3.2.

$$\begin{cases} U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{[S_s\mathbb{X}^a(p)S_s^T]_{\lambda_\varsigma\mu_\varsigma}[S_s\mathbb{X}^b(p)S_s^T]_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}_{n}\varepsilon_{s\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h) \\ [\Rightarrow]\varepsilon_{s\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(S_s^*\bar{C}\gamma_aS_s^+)^{\lambda_\varsigma\mu_\varsigma}(S_s^*\bar{C}\gamma_bS_s^+)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h) \\ V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h)=\frac{1}{(2\sqrt{2}m)^n}\underbrace{[S_s\mathbb{X}^a(-p)S_s^T]_{\lambda_\varsigma\mu_\varsigma}[S_s\mathbb{X}^b(-p)S_s^T]_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}_{n}\tilde{\varepsilon}_{s\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h) \\ [\Rightarrow]\tilde{\varepsilon}_{s\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^n}\overbrace{(S_s^*\bar{C}\gamma_aS_s^+)^{\lambda_\varsigma\mu_\varsigma}(S_s^*\bar{C}\gamma_bS_s^+)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}V_{s\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},h) \end{cases}$$

9.4 vector basis under real representation

Thm. 9.4.1.

$$\begin{cases} \varepsilon^+_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=(-1)^h\overbrace{\eta_a^{a'}\eta_b^{b'}\cdot\cdot}^{n}\varepsilon_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},-h) \\ \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=(-1)^h\overbrace{\eta_a^{a'}\eta_b^{b'}\cdot\cdot}^{n}\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},-h) \\ \varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=(-1)^n\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h) \end{cases}$$

$$\begin{cases} \varepsilon^+_{\underbrace{sab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=(-1)^{h+\frac{1}{2}}e^{-2i\theta}\gamma_{s5\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdot\cdot}^{n}\varepsilon_{s\underbrace{a'b'\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},-h) \\ \varepsilon_{\underbrace{sab\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},h)=(-1)^{h-\frac{1}{2}}e^{2i\theta}\gamma_{s5\tau'_\varsigma}{}^{\tau_\varsigma}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdot\cdot}^{n}\varepsilon^+_{s\underbrace{a'b'\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},-h) \\ \varepsilon_{s\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=-\varsigma(-1)^n\gamma_{s5\tau_\varsigma}{}^{\sigma_\varsigma}\tilde{\varepsilon}_{s\underbrace{ab\cdot\cdot\sigma_\varsigma}_{n}}(\vec{p},h) \end{cases}$$

Proof: $\varepsilon^+_{\underbrace{sab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)$

$$=(-1)^{h-\frac{1}{2}}(S_s^*\gamma_2\gamma_5S_s^+)_{\tau_\varsigma}{}^{\tau'_\varsigma}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdot\cdot}^{n}\varepsilon_{s\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},-h)$$

$$=-(-1)^{h-\frac{1}{2}}e^{-2i\theta}(S_s^*S_s^TS_s\gamma_5S_s^+)_{\tau_\varsigma}{}^{\tau'_\varsigma}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdot\cdot}^{n}\varepsilon_{s\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},-h)$$

$$=(-1)^{h+\frac{1}{2}}e^{-2i\theta}\gamma_{s5\tau_\varsigma}{}^{\tau'_\varsigma}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdot\cdot}^{n}\varepsilon_{s\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p},-h)$$ □

9.5 Plane wave solutions of Majorana B-W equation [16] under real representation

Thm. 9.5.1.

$(\gamma_s^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=0,\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{s\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\}}_{2s}}(\vec{r},t),\psi^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)$

$$\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=s}^{-s}\sqrt{\frac{m^{2s}}{E}}[a(\vec{p},h)U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+a^+(\vec{p},h)U^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$U_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}}\underbrace{u_{s\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{s\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot u_{s\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{s\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}$$

$$a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^{2s}}{E}}U_s^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p},h)\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}$$

9.6 Extraction of various operators for Majorana B-W equation under real representation

Thm. 9.6.1.

$$\begin{cases} P_u(s)=\int \psi_s^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r=\int\sum\limits_h p_u[a^+(\vec p,h)a(\vec p,h)+(-1)^{2s}a(\vec p,h)a^+(\vec p,h)]d^3\vec p \\ Q(s)=\int \psi_s^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r=\int\sum\limits_h [a^+(\vec p,h)a(\vec p,h)+(-1)^{2s-1}a(\vec p,h)a^+(\vec p,h)]d^3\vec p \\ N(s)=\int \psi_s^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r=\int\sum\limits_h [a^+(\vec p,h)a(\vec p,h)+(-1)^{2s}a(\vec p,h)a^+(\vec p,h)]d^3\vec p \\ \vec S(s)=\int \psi_s^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r=\int\sum\limits_h \hat p[a^+(\vec p,h)a(\vec p,h)+(-1)^{2s}a(\vec p,h)a^+(\vec p,h)]d^3\vec p \\ \vec M(s)=\int \psi_s^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r=\int\sum\limits_h \hat p[a^+(\vec p,h)a(\vec p,h)+(-1)^{2s-1}a(\vec p,h)a^+(\vec p,h)]d^3\vec p \end{cases}$$

9.7 Plane wave solutions of K-G equation and R-S equation under real representation

Cor. 9.7.1.

$$\begin{cases} A_{\underbrace{ab\cdots}_{n}}(\vec r,t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}[a(\vec p,h)\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec p,h)e^{ip\cdot x}+e^{-4ni\theta}a^+(\vec p,h)\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}\varepsilon^+_{\underbrace{a'b'\cdots}_{n}}(\vec p,h)e^{-ip\cdot x}]d^3\vec p \\ A_{s\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec r,t) \\ =\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{\sqrt m}{\sqrt{2^nE}}[a(\vec p,h)\varepsilon_{s\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec p,h)e^{ip\cdot x}+e^{-4ni\theta}a^+(\vec p,h)\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}\varepsilon^+_{\underbrace{sa'b'\cdots}_{n}\tau_\varsigma}(\vec p,h)e^{-ip\cdot x}]d^3\vec p \end{cases}$$

Cor. 9.7.2.

$$\begin{cases} A_{\underbrace{ab\cdots}_{n}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec p,h)[a(\vec p,h)e^{i(\vec p\cdot\vec r-Et)}+e^{-4ni\theta}(-1)^h a^+(\vec p,-h)e^{-i(\vec p\cdot\vec r-Et)}]d^3\vec p \\ A_{s\underbrace{ab\cdots}_{n}\tau_\varsigma}(x) \\ =\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{\sqrt m}{\sqrt{2^nE}}\varepsilon_{s\underbrace{ab\cdots}_{n}\sigma_\varsigma}(\vec p,h)[a(\vec p,h)\delta_{\tau_\varsigma}{}^{\sigma_\varsigma}e^{ip\cdot x}+e^{-(4n+2)i\theta}(-1)^{h-\frac12}a^+(\vec p,-h)\gamma_{s5\tau_\varsigma}{}^{\sigma_\varsigma}e^{-ip\cdot x}]d^3\vec p \end{cases}$$

Cor. 9.7.3.

$$\begin{cases} A_{ab\cdots}(\vec r,t)=e^{-4ni\theta}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}A^+_{a'b'\cdots}(\vec r,t),A^+_{sab\cdots}(\vec r,t)=e^{4ni\theta}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}A_{sa'b'\cdots}(\vec r,t) \\ A_{sab\cdots\tau_\varsigma}(\vec r,t)=e^{-4ni\theta}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}A^+_{sa'b'\cdots\tau_\varsigma}(\vec r,t),A^+_{sab\cdots\tau_\varsigma}(\vec r,t)=e^{4ni\theta}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}A_{sa'b'\cdots\tau_\varsigma}(\vec r,t) \end{cases}$$

Proof: $A^+_{sab\cdots\tau_\varsigma}(\vec r,t)$

$$=-e^{(4n+2)i\theta}(S_s^*)_{\tau_\varsigma}{}^{\sigma_\varsigma}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}(\sigma_y\otimes\sigma_y)_{\sigma_\varsigma}{}^{\xi_\varsigma}(S_s^+)_{\xi_\varsigma}{}^{\eta_\varsigma}A_{sa'b'\cdots\eta_\varsigma}(\vec r,t)$$

$$=e^{4ni\theta}(S_s^*S_s^TS_sS_s^+)_{\tau_\varsigma}{}^{\sigma_\varsigma}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}A_{sa'b'\cdots\sigma_\varsigma}(\vec r,t)$$

$$=e^{4ni\theta}\overbrace{\eta_a^{a'}\eta_b^{b'}\cdots}^{n}A_{sa'b'\cdots\tau_\varsigma}(\vec r,t)$$ □

9.8 Covariant commutation rules for Majorana B-W equation under real representation

Cor. 9.8.1.

$$\begin{cases} [\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma_s^a\partial_a)\gamma_s^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma_s^b\partial_b)\gamma_s^4]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x') \\ [\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=[\psi^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=[\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} \\ \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x)=\psi^+_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x) \end{cases}$$

$$\Leftrightarrow [a(\vec p,h),a^+(\vec p',h')]_{-^{2s+1}}=\delta_{hh'}\delta^3(\vec p-\vec p'),[a(\vec p,h),a(\vec p',h')]_{-^{2s+1}}=0,[a^+(\vec p,h),a^+(\vec p',h')]_{-^{2s+1}}=0$$

10 Card commutation rules for Majorana particles under real representation

10.1 Card commutation rules for Majorana boson under real representation(take $\theta = 0$)

Thm. 10.1.1.
$$\begin{cases}[A_{a_1a_2\cdot\cdot a_n}(x), A^+_{a'_1a'_2\cdot\cdot a'_n}(x')] = i\hat{P}_{a_1a_2\cdot\cdot a_n a'_1a'_2\cdot\cdot a'_n}(n)\Delta(x-x') \\ [A_{a_1a_2\cdot\cdot a_n}(x), A_{b_1b_2\cdot\cdot b_n}(x')] = i\hat{P}_{a_1a_2\cdot\cdot a_n a'_1a'_2\cdot\cdot a'_n}(n)\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdot\cdot\eta^{a'_n}_{b_n}\Delta(x-x') \\ A_{a_1a_2\cdot\cdot a_n} = A^+_{a'_1a'_2\cdot\cdot a'_n}\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdot\cdot\eta^{a'_n}_{a_n}, A^+_{\underbrace{a'_1a'_2\cdot\cdot}_n} = A_{\underbrace{a_1a_2\cdot\cdot}_n}\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdot\cdot}_n\end{cases}$$

$$[\Updownarrow]$$

Thm. 10.1.2.
$$\begin{cases}[A_{\underbrace{a_1a_2\cdot\cdot}_n}(x), A^+_{\underbrace{a'_1a'_2\cdot\cdot}_n}(x')] = \frac{im^{-2n}}{2^{4n-1}[(2n)!]^2}\Gamma_{\underbrace{a_1a_2\cdot\cdot}_n\underbrace{a'_1a'_2\cdot\cdot}_n}(x;n)\Delta(x-x') \\ [A_{\underbrace{a_1a_2\cdot\cdot}_n}(x), A_{\underbrace{b_1b_2\cdot\cdot}_n}(x')] = \frac{im^{-2n}}{2^{4n-1}[(2n)!]^2}\Gamma_{\underbrace{a_1a_2\cdot\cdot}_n\underbrace{a'_1a'_2\cdot\cdot}_n}(x;n)\underbrace{\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdot\cdot}_n\Delta(x-x') \\ A_{\underbrace{a_1a_2\cdot\cdot}_n} = A^+_{\underbrace{a'_1a'_2\cdot\cdot}_n}\underbrace{\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdot\cdot}_n, A^+_{\underbrace{a'_1a'_2\cdot\cdot}_n} = A_{\underbrace{a_1a_2\cdot\cdot}_n}\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdot\cdot}_n\end{cases}$$

$$[\Updownarrow]$$

Thm. 10.1.3.
$$\begin{cases}[A_{\underbrace{a_1a_2\cdot\cdot}_n}(x), A^+_{\underbrace{a'_1a'_2\cdot\cdot}_n}(x')] = \frac{im^{-2n}}{2^{5n-1}[(2n)!]^2}\Gamma^{\overbrace{b_1b_2\cdot\cdot}^n\overbrace{b'_1b'_2\cdot\cdot}^n}_{\underbrace{a_1a_2\cdot\cdot}_n\underbrace{a'_1a'_2\cdot\cdot}_n}(x,x';n)\underbrace{[\eta_{b_1b'_1} - \frac{\partial_{b_1}\partial^+_{b'_1}}{m^2}]\cdot\cdot}_n\Delta(x-x') \\ [A_{\underbrace{a_1a_2\cdot\cdot}_n}(x), A_{\underbrace{b_1b_2\cdot\cdot}_n}(x')] = \frac{im^{-2n}}{2^{5n-1}[(2n)!]^2}\Gamma^{\overbrace{c_1c_2\cdot\cdot}^n\overbrace{c'_1c'_2\cdot\cdot}^n}_{\underbrace{a_1a_2\cdot\cdot}_n\underbrace{a'_1a'_2\cdot\cdot}_n}(x,x';n)\underbrace{[\eta_{c_1c'_1} - \frac{\partial_{c_1}\partial^+_{c'_1}}{m^2}]\cdot\cdot}_n\underbrace{\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdot\cdot}_n\Delta(x-x') \\ A_{\underbrace{a_1a_2\cdot\cdot}_n} = A^+_{\underbrace{a'_1a'_2\cdot\cdot}_n}\underbrace{\eta^{a'_1}_{a_1}\eta^{a'_2}_{a_2}\cdot\cdot}_n, A^+_{\underbrace{a'_1a'_2\cdot\cdot}_n} = A_{\underbrace{a_1a_2\cdot\cdot}_n}\underbrace{\eta^{a_1}_{a'_1}\eta^{a_2}_{a'_2}\cdot\cdot}_n\end{cases}$$

$$[\Updownarrow]$$

Thm. 10.1.4.
$$\begin{cases}[\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot_\}}_n\underbrace{\mathbb{X}_s{}^{+a}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot_)}_n\underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\cdot\cdot}_n\Delta(x-x') \\ [\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot_\}}_n\underbrace{\mathbb{X}_s{}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot_)}_n\underbrace{[\eta_{aa'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2}]\cdot\cdot}_n\Delta(x-x') \\ \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}} = \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}\underbrace{\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\delta^{\mu'_\varsigma}_{\mu_\varsigma}\cdot\cdot}_{2n}, \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}} = \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}\underbrace{\delta^{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma}_{\mu'_\varsigma}\cdot\cdot}_{2n}, \mathbb{X}^a_s := S_s\mathbb{X}^aS^T_s\end{cases}$$

$$[\Updownarrow]$$

Thm. 10.1.5.
$$\begin{cases}[\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot_\}}_n\underbrace{\mathbb{X}_s{}^{a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot_)}_n\underbrace{[\delta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2}]\cdot\cdot}_n\Delta(x-x') \\ [\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{3n-1}[(2n)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot_\}}_n\underbrace{\mathbb{X}_s{}^{a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot_)}_n\underbrace{[\delta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2}]\cdot\cdot}_n\Delta(x-x') \\ \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}} = \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}\underbrace{\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\delta^{\mu'_\varsigma}_{\mu_\varsigma}\cdot\cdot}_{2n}, \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}} = \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}\underbrace{\delta^{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma}_{\mu'_\varsigma}\cdot\cdot}_{2n}, \mathbb{X}^a_s := S_s\mathbb{X}^aS^T_s, \mathbb{X}^a_s = \mathbb{X}^{+a'}_s\eta^a_{a'}\end{cases}$$

$$[\Updownarrow]$$

Thm. 10.1.6.
$$\begin{cases}[\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{2n-1}[(2n)!]^2}\underbrace{[(m-\gamma^a_s\partial_a)\gamma^4_s]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b_s\partial_b)\gamma^4_s]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}_{2n}\Delta(x-x') \\ [\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{i}{2^{2n-1}[(2n)!]^2}\underbrace{[(m-\gamma^a_s\partial_a)\gamma^4_s]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b_s\partial_b)\gamma^4_s]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot_{\})}}_{2n}\Delta(x-x') \\ \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}} = \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}\underbrace{\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\delta^{\mu'_\varsigma}_{\mu_\varsigma}\cdot\cdot}_{2n}, \psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}} = \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}\underbrace{\delta^{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma}_{\mu'_\varsigma}\cdot\cdot}_{2n}\end{cases}$$

$$[\Downarrow]$$

Thm. 10.1.7. $[\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2n}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2n}}(x')]=i\frac{(i\varsigma)^{2n}}{2^{2n-1}[(2n)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot\})}^{2n}\overbrace{\partial_a\partial_b\cdot\cdot}^{2n}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 10.1.8. $[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\frac{(-1)^{2n}}{2^{n-1}}\Gamma^{\overbrace{abc\cdot\cdot}^{2n}}_{k_\varsigma k'_\varsigma}(n)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\Delta(x-x')$

Def. 10.1.1. $\mathbb{X}^a\equiv[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]C,\mathbb{X}^a(p)\equiv i[m\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)p_b]C$

Def. 10.1.2. $\mathbb{X}^a_s\equiv[im\gamma^a_s(\varsigma)-2S^{ab}_s(e,\varsigma)\partial_b]C_s,\mathbb{X}^a_s(p)\equiv i[m\gamma^a_s(\varsigma)-2S^{ab}_s(e,\varsigma)p_b]C_s,C_s:=-\gamma^2_s\gamma^4_s\gamma^2$

There are two equivalent expressions for the commutative relationship between potential A and field ψ, and they are mutually premises and causal each other. You can deduce everything from potential commutation relations, or you can deduce everything from field commutation relations too. This shows that the two descriptions of potential and field are completely equivalent for massive particles. And it can be deduced from the massive particle commutation rules which are completely similar to the massless particle commutation rules. But not vice versa.

10.2 Card anticommutation rules for Majorana fermion under real representation

Def. 10.2.1. $\hat{P}_{sa_1\cdot\cdot a_n\tau_\varsigma a'_1\cdot\cdot a'_n\tau'_\varsigma}(n+\frac{1}{2})=\frac{n+1}{2n+3}\hat{P}_{aa_1\cdot\cdot a_na'a'_1\cdot\cdot a'_n}(n+1)[\gamma^a_s(-m-\gamma^c_s\partial_c)\gamma^4_s\gamma^{a'}_s]_{\tau_\varsigma\tau'_\varsigma},\gamma^{a'}_s=\gamma^a_s\eta^{a'}_a$

Def. 10.2.2. $\begin{cases}\hat{P}_{sa_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2})=\frac{n+1}{2n+3}\hat{P}_{aa_1\cdot\cdot a_nbb_1\cdot\cdot b_n}(n+1)[\gamma^a_s(m+\gamma^c_s\partial_c)\gamma^b_s\gamma^4_s]_{\tau_\varsigma\tau'_\varsigma}\\ \hat{P}_{sa_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2}):=\eta^{a'_1}_{b_1}\eta^{a'_2}_{b_2}\cdot\cdot\eta^{a'_n}_{b_n}\hat{P}_{sa_1\cdot\cdot a_n\tau_\varsigma a'_1\cdot\cdot a'_n\tau'_\varsigma}(n+\frac{1}{2})\end{cases}$

Cor. 10.2.1. $\hat{P}_{sa_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2})=\frac{n+1}{2n+3}\hat{P}_{aa_1\cdot\cdot a_nbb_1\cdot\cdot b_n}(n+1)[(m-\gamma^c_s\partial_c)\gamma^a_s\gamma^b_s\gamma^4_s]_{\tau_\varsigma\tau'_\varsigma}$

Thm. 10.2.1. $\{A_{sa_1a_2\cdot\cdot a_n\tau_\varsigma}(x),A^+_{sa'_1a'_2\cdot\cdot a'_n\tau'_\varsigma}(x')\}=i\hat{P}_{sa_1\cdot\cdot a_n\tau_\varsigma a'_1\cdot\cdot a'_n\tau'_\varsigma}(n+\frac{1}{2})\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 10.2.2. $\begin{cases}\{A_{sa_1a_2\cdot\cdot a_n\tau_\varsigma}(x),A_{sb_1b_2\cdot\cdot b_n\tau'_\varsigma}(x')\}=i\hat{P}_{sa_1\cdot\cdot a_n\tau_\varsigma b_1\cdot\cdot b_n\tau'_\varsigma}(n+\frac{1}{2})\Delta(x-x')\\ A_{sab\cdot\cdot\tau_\varsigma}=\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}A^+_{sa'b'\cdot\cdot\tau_\varsigma},A^+_{sab\cdot\cdot\tau_\varsigma}=\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}A_{sa'b'\cdot\cdot\tau_\varsigma}\end{cases}$

$$[\Updownarrow]$$

Thm. 10.2.3.

$$\begin{cases}\{A_{s\underbrace{a_1a_2\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{s\underbrace{a'_1a'_2\cdot\cdot}_{n}\tau'_\varsigma}(x')\}=\frac{im^{-2n}}{2^{5n}[(2n+1)!]^2}\\ \overbrace{(\gamma^4_s\gamma_{sa_1})^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma^4_s\gamma_{sa'_1})^{+\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}\underbrace{\mathbb{X}^{b_1}_{s\ \{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+b'_1}_{s\ (\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{b_1b'_1}-\frac{\partial_{b_1}\partial^+_{b'_1}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')\\ \{A_{s\underbrace{a_1a_2\cdot\cdot}_{n}\tau_\varsigma}(x),A_{s\underbrace{a'_1a'_2\cdot\cdot}_{n}\tau'_\varsigma}(x')\}=\frac{i(im)^{-2n}}{2^{5n}[(2n+1)!]^2}\\ \overbrace{(\gamma^4_s\gamma_{sa_1})^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma^4_s\gamma_{sa'_1})^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}\underbrace{\mathbb{X}^{b_1}_{s\ \{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{b'_1}_{s\ (\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\delta_{b_1b'_1}-\frac{\partial_{b_1}\partial_{b'_1}}{m^2}]\cdot\cdot}_{n}\Delta(x-x')\\ A_{sab\cdot\cdot\tau_\varsigma}=\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}A^+_{sa'b'\cdot\cdot\tau_\varsigma},A^+_{sab\cdot\cdot\tau_\varsigma}=\overbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}^{n}A_{sa'b'\cdot\cdot\tau_\varsigma}\\ S^*_s\bar{C}\gamma^aS^+_s=\gamma^4_s\gamma^a_s,S_s\gamma^{a'}CS^T_s=\gamma^{a'}_s\gamma^4_s,\mathbb{X}^a_s:=S_s\mathbb{X}^aS^T_s,\mathbb{X}^a_s=\mathbb{X}^{+a'}_s\eta^a_{a'}\end{cases}$$

$$[\Updownarrow]$$

Thm. 10.2.4.

$$\begin{cases}\{A_{s\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(x),A^+_{s\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(x')\}\\ =\frac{im^{-2n}}{2^{4n}[(2n+1)!]^2}\overbrace{(\gamma^4_s\gamma_{sa_1})^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\gamma_{sa'_1}\gamma^4_s)^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\underbrace{[(m-\gamma^a_s\partial_a)\gamma^4_s]_{\{\lambda_\varsigma(\lambda'_\varsigma}\cdots[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')\\ \{A_{s\underbrace{a_1a_2\cdots}_{n}\tau_\varsigma}(x),A_{s\underbrace{a'_1a'_2\cdots}_{n}\tau'_\varsigma}(x')\}\\ =\frac{i(im)^{-2n}}{2^{4n}[(2n+1)!]^2}\overbrace{(\gamma^4_s\gamma_{sa_1})^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\gamma^4_s\gamma_{sa'_1})^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\underbrace{[(m-\gamma^a_s\partial_a)\gamma^4_s]_{\{\lambda_\varsigma(\lambda'_\varsigma}\cdots[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}}_{2n+1}\Delta(x-x')\\ A_{sab\cdots\tau_\varsigma}=\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}A^+_{sa'b'\cdots\tau_\varsigma},A^+_{sab\cdots\tau_\varsigma}=\overbrace{\eta^{a'}_a\eta^{b'}_b\cdots}^{n}A_{sa'b'\cdots\tau_\varsigma},S^*_s\bar{C}\gamma^aS^+_s=\gamma^4_s\gamma^a_s,S_s\gamma^{a'}CS^T_s=\gamma^{a'}_s\gamma^4_s\end{cases}$$

[⇕]

Thm. 10.2.5.

$$\begin{cases}\{\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}\\ =\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}_s{}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\ \{\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}\\ =\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}_s{}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\ \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}=\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}\underbrace{\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\delta^{\mu'_\varsigma}_{\mu_\varsigma}\cdots}_{2n+1},\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}=\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}\underbrace{\delta^{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma}_{\mu'_\varsigma}\cdots}_{2n+1},\mathbb{X}^a_s:=S_s\mathbb{X}^aS^T_s\end{cases}$$

[⇕]

Thm. 10.2.6.

$$\begin{cases}\{\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}\\ =\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}_s{}^{a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\ \{\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}\\ =\frac{i}{2^{3n}[(2n+1)!]^2}\underbrace{\mathbb{X}_s{}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}_s{}^{a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}[(m-\gamma^c_s\partial_c)\gamma^4_s]_{\tau_\varsigma\}\tau'_\varsigma)}\underbrace{[\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')\\ \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}=\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}\underbrace{\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\delta^{\mu'_\varsigma}_{\mu_\varsigma}\cdots}_{2n+1},\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}=\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}\underbrace{\delta^{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma}_{\mu'_\varsigma}\cdots}_{2n+1},\mathbb{X}^a_s:=S_s\mathbb{X}^aS^T_s,\mathbb{X}^a_s=\mathbb{X}^{+a'}_s\eta^a_{a'}\end{cases}$$

[⇕]

Thm. 10.2.7.

$$\begin{cases}\{\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}=\frac{i}{2^{2n}[(2n+1)!]^2}\underbrace{[(m-\gamma^a_s\partial_a)\gamma^4_s]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b_s\partial_b)\gamma^4_s]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n+1}\Delta(x-x')\\ \{\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x),\psi_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')\}=\frac{i}{2^{2n}[(2n+1)!]^2}\underbrace{[(m-\gamma^a_s\partial_a)\gamma^4_s]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b_s\partial_b)\gamma^4_s]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}_{2n+1}\Delta(x-x')\\ \psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}=\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}\underbrace{\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\delta^{\mu'_\varsigma}_{\mu_\varsigma}\cdots}_{2n+1},\psi^+_{s\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}=\psi_{s\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}\underbrace{\delta^{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma}_{\mu'_\varsigma}\cdots}_{2n+1}\end{cases}$$

[⇓]

Thm. 10.2.8. $\{\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2n+1}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2n+1}}(x')\}=i\frac{(i\varsigma)^{2n+1}}{2^{2n}[(2n+1)!]^2}\overbrace{(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2n+1}\overbrace{\partial_a\partial_b\cdots}^{2n+1}\Delta(x-x')$

[⇕]

Thm. 10.2.9. $\{\psi_{k_\varsigma}(x), \psi^{+}_{k'_\varsigma}(x')\} = i\frac{(-1)^{2n+1}}{2^{n-1/2}}\Gamma^{\overbrace{abc\cdots}^{2n+1}}_{k_\varsigma k'_\varsigma}(n+\frac{1}{2})\overbrace{\partial_a\partial_b\partial_c\cdots}^{2n+1}\Delta(x-x')$

There are two equivalent expressions for the commutative relationship between potential A and field ψ, and they are mutually premises and causal each other. You can deduce everything from potential commutation relations, or you can deduce everything from field commutation relations too. This shows that the two descriptions of potential and field are completely equivalent for massive particles. And it can be deduced from the massive particle commutation rules which are completely similar to the massless particle commutation rules. But not vice versa.

Chapter27 Covariant quantization scheme for massive vector particles

Self comment: For particles described by the Bargmann Wigner equation, it is generally possible to describe both charged complex particles and uncharged Majorana particles. The principal commutation rule in both cases is consistent, but the rest are generally zero for charged complex particles. For uncharged Majorana particles, the rest of the commutative or anti commutative brackets are naturally derived from the principal commutative rule and Majorana conditions. And they are generally not zero. In this chapter, we discuss both the complex particle case and the Majorana particle case. The complete commutation rules for both cases are given. However, in latter chapters, we will generally not seek completeness, but only discuss the complex particle case and the principal commutation rule. The Majorana particle case is no longer specifically discussed. If we want to obtain the quantum field theory of the Majorana particle case, we only need to add the Majorana condition to the complex particle case. Then we will naturally obtain it.

1 Mutual conversion of two equivalent descriptions for massive vector particles

1.1 Two equivalent descriptions of B-W and K-G equation for spin-1 particles [16, 20, 21]

Def. 1.1.1. $\mathbb{X}_a(x) := [im\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)\partial^b]C, \mathbb{X}_a(p) := i[m\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)p^b]C, C = \gamma_2\gamma_4$
$\gamma_a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$

Thm. 1.1.1. $\begin{cases}[\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma\mu_\varsigma]} = 0, \psi_{\lambda_\varsigma\mu_\varsigma} = \psi_{\mu_\varsigma\lambda_\varsigma} \\ im\frac{A_a}{2} = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}], C = \gamma_y(\varsigma)\gamma_4(\varsigma)\end{cases} \Leftrightarrow \begin{cases}\partial^b F_{ab} + m^2 A_a = 0, F_{ab} = \partial_a A_b - \partial_b A_a \\ \psi_{\lambda_\varsigma\mu_\varsigma} = \mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\frac{A_a}{2}\end{cases}$

Thm. 1.1.2. $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'} + \frac{p_a p^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p) = \frac{1}{2}[(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m - i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}$

1.2 Plane wave solutions of Bargmann-Wigner equation for spin-1 particles [16]

Thm. 1.2.1. $(\gamma^a\partial_a + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t) = 0, \psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t) = \frac{1}{2!}\psi_{\{\lambda_\varsigma\mu_\varsigma\}}(\vec{r},t)$

$$\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{\frac{m^2}{E}}[a(\vec{p},h)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\begin{cases}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1) = u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2}), U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},-1) = u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2}) \\ U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0) = \frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2}) + u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})]\end{cases}$$

$$\begin{cases}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1) = v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2}), V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},-1) = v_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2}) \\ V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0) = \frac{1}{\sqrt{2}}[v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2}) + v_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})]\end{cases}$$

$$\begin{cases}a(\vec{p},h) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^2}{E^3}}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r} \\ b^+(\vec{p},s) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^2}{E^3}}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

Thm. 1.2.2. $[\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]$

$= \frac{i}{8}[(m - \gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m - \gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x - x') = \frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\Delta(x - x')$

Def. 1.2.1.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1) := \sum_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h) \\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1) := \sum_{h=1}^{-1}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)V^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)\end{cases}$$

Thm. 1.2.3.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1) = \frac{1}{8m^2}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\Lambda_{maa'}(\vec{p},1)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p) = \frac{1}{(2!)^2}\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\}\mu'_\varsigma)}(\vec{p},\frac{1}{2}) \\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1) = \frac{1}{8m^2}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\Lambda_{maa'}(\vec{p},1)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(-p) = \frac{1}{(2!)^2}\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\mu_\varsigma\}\mu'_\varsigma)}(\vec{p},\frac{1}{2})\end{cases}$$

Thm. 1.2.4.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1)=\frac{1}{8m^2}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)=\frac{1}{16m^2}[(m-i\gamma^ap_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1)=\frac{1}{8m^2}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(-p)=\frac{1}{16m^2}[(m+i\gamma^ap_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m+i\gamma^bp_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\end{cases}$$

$$\Downarrow$$

1.3 Derived to plane wave solutions of Klein-Gordon equation for spin-1 particles [25,37,38]

Thm. 1.3.1. $\partial^bF_{ab}+m^2A_a=0,F_{ab}=\partial_aA_b-\partial_bA_a,A_a=\frac{1}{2im}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}$

$$A_a(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}[a(\vec{p},h)\varepsilon_a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)\tilde{\varepsilon}_a(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\begin{cases}\varepsilon_a(\vec{p},1)=\frac{1}{i\sqrt{2}}u^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},\frac{1}{2}),\varepsilon_a(\vec{p},-1)=\frac{1}{i\sqrt{2}}u^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})\\ \varepsilon_a(\vec{p},0)=\frac{1}{i\sqrt{2}}\frac{1}{\sqrt{2}}[u^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})+u^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_au(\vec{p},\frac{1}{2})]\end{cases}$$

$$\begin{cases}\tilde{\varepsilon}_a(\vec{p},1)=\frac{1}{i\sqrt{2}}v^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},\frac{1}{2}),\tilde{\varepsilon}_a(\vec{p},-1)=\frac{1}{i\sqrt{2}}v^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})\\ \tilde{\varepsilon}_a(\vec{p},0)=\frac{1}{i\sqrt{2}}\frac{1}{\sqrt{2}}[v^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})+v^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_av(\vec{p},\frac{1}{2})]\end{cases}$$

Thm. 1.3.2. $\varepsilon^+(\vec{p},h)\varepsilon(\vec{p},h')=(\frac{E^2+p^2}{m^2})^{1-|h|}\delta_{hh'},\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h)=\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2},\sum\limits_{h=1}^{-1}h\varepsilon(\vec{p},h)\varepsilon^+(\vec{p},h)=R\cdot\hat{p}$

Lem. 1.3.1. $\gamma^a(\varsigma)C=\begin{bmatrix}0&(\sigma,i\varsigma)\sigma_y\\(\sigma,-i\varsigma)\sigma_y&0\end{bmatrix},\bar{C}\gamma^a(\varsigma)=\begin{bmatrix}0&\sigma_y(\sigma,i\varsigma)^a\\\sigma_y(\sigma,-i\varsigma)^a&0\end{bmatrix}$

Def. 1.3.1. $u^+(\vec{p},\frac{1}{2})=-i\varsigma u^T(\vec{p},\frac{1}{2})\sigma_y\otimes\sigma_x,u^+(\vec{p},-\frac{1}{2})=i\varsigma u^T(\vec{p},-\frac{1}{2})\sigma_y\otimes\sigma_x$

Pro. 1.3.1.

$$\begin{cases}\varepsilon_a(\vec{p},1)=-\frac{i}{\sqrt{2}}u^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},\frac{1}{2})=-\frac{i}{\sqrt{2}}\lambda^T(\vec{p},\frac{1}{2})\sigma_y(\sigma,0)_a\lambda(\vec{p},\frac{1}{2})=[i\lambda_m(\vec{p},1),0]_a\\ \varepsilon_a(\vec{p},0)=-iu^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})=-\frac{i}{m}\lambda^T(\vec{p},\frac{1}{2})\sigma_y(E\sigma,i|\vec{p}|)_a\lambda(\vec{p},-\frac{1}{2})=\frac{1}{m}[iE\lambda_m(\vec{p},0),i|\vec{p}|]_a\\ \varepsilon_a(\vec{p},-1)=-\frac{i}{\sqrt{2}}u^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})=-\frac{i}{\sqrt{2}}\lambda^T(\vec{p},-\frac{1}{2})\sigma_y(\sigma,0)_a\lambda(\vec{p},-\frac{1}{2})=[i\lambda_m(\vec{p},-1),0]_a\end{cases}$$

Pro. 1.3.2.

$$\begin{cases}\tilde{\varepsilon}_a(\vec{p},1)=-\frac{i}{\sqrt{2}}v^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},\frac{1}{2})=\frac{i}{\sqrt{2}}\lambda^T(\vec{p},\frac{1}{2})\sigma_y(\sigma,0)_a\lambda(\vec{p},\frac{1}{2})=-[i\lambda_m(\vec{p},1),0]_a\\ \tilde{\varepsilon}_a(\vec{p},0)=-iv^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})=\frac{i}{m}\lambda^T(\vec{p},\frac{1}{2})\sigma_y(E\sigma,i|\vec{p}|)_a\lambda(\vec{p},-\frac{1}{2})=-\frac{1}{m}[iE\lambda_m(\vec{p},0),i|\vec{p}|]_a\\ \tilde{\varepsilon}_a(\vec{p},-1)=-\frac{i}{\sqrt{2}}v^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})=\frac{i}{\sqrt{2}}\lambda^T(\vec{p},-\frac{1}{2})\sigma_y(\sigma,0)_a\lambda(\vec{p},-\frac{1}{2})=-[i\lambda_m(\vec{p},-1),0]_a\end{cases}$$

Cor. 1.3.1. $\tilde{\varepsilon}_a(\vec{p},1)=-\varepsilon_a(\vec{p},1),\tilde{\varepsilon}_a(\vec{p},0)=-\varepsilon_a(\vec{p},0),\tilde{\varepsilon}_a(\vec{p},-1)=-\varepsilon_a(\vec{p},-1)$

Cor. 1.3.2. $\partial^bF_{ab}+m^2A_a=0,F_{ab}=\partial_aA_b-\partial_bA_a,A_a=\frac{1}{2im}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}$

$$A_a(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\varepsilon_a(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}-b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$\varepsilon_a(\vec{p},1)=[i\lambda_m(\vec{p},1),0]_a,\varepsilon_a(\vec{p},0)=\frac{1}{m}[iE\lambda_m(\vec{p},0),i|\vec{p}|]_a,\varepsilon_a(\vec{p},-1)=[i\lambda_m(\vec{p},-1),0]_a$

Thm. 1.3.3. $[A_a(x),A^+_{a'}(x')]=i(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')$

Thm. 1.3.4. $\Lambda_{maa'}(\vec{p},1):=\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h)=\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2}$

Thm. 1.3.5. $\Lambda_{\pm\tau_\varsigma\tau'_\varsigma}(\vec{p},\frac{1}{2})=\frac{1}{3}\Lambda_{maa'}(\vec{p},1)\gamma^a\Lambda_{\mp}(\vec{p},\frac{1}{2})\gamma^{a'}$

$$\Downarrow$$

1.4 Back to plane wave solutions of Bargmann-Wigner equation for spin-1 particles [16]

Thm. 1.4.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=0,\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]\frac{A_a(\vec{r},t)}{2}$

$$\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{\frac{m^2}{E}}[a(\vec{p},h)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2m}}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_a(\vec{p},h),V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2m}}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\tilde{\varepsilon}_a(\vec{p},h)$

$$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^2}{E^3}}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ b^+(\vec{p},s)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^2}{E^3}}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

2 Third equivalent description of massive vector particle equation

2.1 Equivalent description of spin equation for massive vector particles

Thm. 2.1.1. $(\partial_a + iS_{ab}\partial^b)_{\beta_\varsigma}{}^{\alpha_\varsigma}\psi_{\alpha_\varsigma} = \frac{i}{\sqrt{2}}im^2\sigma^{ab}_{\varsigma\beta_\varsigma}A_b, \psi_{\alpha_\varsigma} := \frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}, S_{ab} := i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$A_a(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\varepsilon_a(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$F_{ab}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}[ip_a\varepsilon_b(\vec{p},h) - ip_b\varepsilon_a(\vec{p},h)][a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\psi_{\alpha_\varsigma}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\frac{-i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}p_a\varepsilon_b(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

2.2 Plane wave solutions and projection operators of massive vector particle field F_{ab}

Def. 2.2.1. $\lambda_{ab}(\vec{p},h) := [ip_a\varepsilon_b(\vec{p},h) - ip_b\varepsilon_a(\vec{p},h)]$

Cor. 2.2.1. $F_{ab}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\lambda_{ab}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Thm. 2.2.1. $\sum_{h=1}^{-1}\lambda_{ab}(\vec{p},h)\lambda^+_{a'b'}(\vec{p},h) = p_{[a}p^+_{[a'}\eta_{b]b']}$

Proof: $\sum_{h=1}^{-1}\lambda_{ab}(\vec{p},h)\lambda^+_{a'b'}(\vec{p},h)$

$= \sum_{h=1}^{-1}[ip_a\varepsilon_b(\vec{p},h) - ip_b\varepsilon_a(\vec{p},h)][ip_{a'}\varepsilon_{b'}(\vec{p},h) - ip_{b'}\varepsilon_{a'}(\vec{p},h)]^+$

$= p_ap^+_{a'}\sum_{h=1}^{-1}\varepsilon_b(\vec{p},h)\varepsilon^+_{b'}(\vec{p},h) + p_bp^+_{b'}\sum_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h) - p_ap^+_{b'}\sum_{h=1}^{-1}\varepsilon_b(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h) - p_bp^+_{a'}\sum_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{b'}(\vec{p},h)$

$= p_ap^+_{a'}(\eta_{bb'} + \frac{p_bp^+_{b'}}{m^2}) + p_bp^+_{b'}(\eta_{aa'} + \frac{p_ap^+_{a'}}{m^2}) - p_ap^+_{b'}(\eta_{ba'} + \frac{p_bp^+_{a'}}{m^2}) - p_bp^+_{a'}(\eta_{ab'} + \frac{p_ap^+_{b'}}{m^2})$

$= p_ap^+_{a'}\eta_{bb'} + p_bp^+_{b'}\eta_{aa'} - p_ap^+_{b'}\eta_{ba'} - p_bp^+_{a'}\eta_{ab'}$

$= p_{[a}p^+_{[a'}\eta_{b]b']}$ □

Thm. 2.2.2. $[F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a[a'}\partial_{b]}\partial^+_{b']}\Delta(x-x')$

2.3 Plane wave solutions and projection operators of massive vector particle field Ψ_{α_ς}

Def. 2.3.1. $\lambda_{\alpha_\varsigma}(\vec{p},h) := \frac{-i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}p_a\varepsilon_b(\vec{p},h)$

Cor. 2.3.1. $\psi_{\alpha_\varsigma}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\lambda_{\alpha_\varsigma}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Cor. 2.3.2. $\lambda_{\alpha_\varsigma}(\vec{p},h) = \frac{-\varsigma}{\sqrt{2}}h|\vec{p}|\lambda_{m\alpha_\varsigma}(\vec{p},h) - \frac{-\varsigma}{\sqrt{2}}p_{\alpha_\varsigma}\varepsilon_4(\vec{p},h) + \frac{-i\varsigma}{\sqrt{2}}E\varepsilon_{\alpha_\varsigma}(\vec{p},h)$

Proof: $-\varsigma\lambda_{\alpha_\varsigma}(\vec{p},h) := \frac{i\varsigma}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}p_a\varepsilon_b(\vec{p},h)$

$= \frac{i\varsigma}{\sqrt{2}}\sigma^{ij}_{\varsigma\alpha_\varsigma}p_i\varepsilon_j(\vec{p},h) + \frac{i\varsigma}{\sqrt{2}}\sigma^{i4}_{\varsigma\alpha_\varsigma}p_i\varepsilon_4(\vec{p},h) + \frac{i\varsigma}{\sqrt{2}}\sigma^{4j}_{\varsigma\alpha_\varsigma}p_4\varepsilon_j(\vec{p},h)$

$= \frac{1}{\sqrt{2}}h|\vec{p}|\lambda_{\alpha_\varsigma}(\vec{p},h) - \frac{\varsigma}{\sqrt{2}}\varsigma p_{\alpha_\varsigma}\varepsilon_4(\vec{p},h) + \frac{\varsigma}{\sqrt{2}}\varsigma p_4\varepsilon_{\alpha_\varsigma}(\vec{p},h)$

$= \frac{1}{\sqrt{2}}h|\vec{p}|\lambda_{m\alpha_\varsigma}(\vec{p},h) - \frac{1}{\sqrt{2}}p_{\alpha_\varsigma}\varepsilon_4(\vec{p},h) + \frac{i}{\sqrt{2}}E\varepsilon_{\alpha_\varsigma}(\vec{p},h)$ □

Cor. 2.3.3. $\lambda_{\alpha_\varsigma}(\vec{p},\kappa) = \frac{1}{\sqrt{2}}(E - \varsigma\kappa\vec{p}|)\lambda_{m\alpha_\varsigma}(\vec{p},\kappa), \lambda_{\alpha_\varsigma}(\vec{p},0) = \frac{1}{\sqrt{2}}m\lambda_{m\alpha_\varsigma}(\vec{p},0)$

Thm. 2.3.1. $\sum_{h=1}^{-1}\lambda_{\alpha_\varsigma}(\vec{p},h)\lambda^+_{\alpha'_\varsigma}(\vec{p},h) = -\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b$

Proof: $\sum_{h=1}^{-1}\lambda_{\alpha_\varsigma}(\vec{p},h)\lambda^+_{\alpha'_\varsigma}(\vec{p},h)$

$= \frac{1}{2}(E - \varsigma\vec{p}|)^2\lambda_{m\alpha_\varsigma}(\vec{p},1)\lambda^+_{m\alpha'_\varsigma}(\vec{p},1) + \frac{1}{2}(E + \varsigma\vec{p}|)^2\lambda_{m\alpha_\varsigma}(\vec{p},-1)\lambda^+_{m\alpha'_\varsigma}(\vec{p},-1) + \frac{1}{2}m^2\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma}$

$= -\frac{1}{4}(E - \varsigma|\vec{p}|)^2(\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma} + i\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\hat{p}_k) - \frac{1}{4}(E + \varsigma|\vec{p}|)^2(\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma} - \delta_{\alpha_\varsigma\alpha'_\varsigma} - i\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}\hat{p}_k) + \frac{1}{2}m^2\hat{p}_{\alpha_\varsigma}\hat{p}_{\alpha'_\varsigma}$

$= -p_{\alpha_\varsigma}p_{\alpha'_\varsigma} + \frac{1}{2}(E^2 + \vec{p}^2)\delta_{\alpha_\varsigma\alpha'_\varsigma} - i\varsigma E\varepsilon^k{}_{\alpha_\varsigma\alpha'_\varsigma}p_k$

$= -\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_b$ □

Proof: $\sum_{h=1}^{-1} \lambda_{\alpha_\varsigma}(\vec{p},h)\lambda^+_{\alpha'_\varsigma}(\vec{p},h)$
$= \sum_{h=1}^{-1} \frac{-i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma} p_a \varepsilon_b(\vec{p},h) \frac{-i}{\sqrt{2}}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma} p'^+_{a'} \varepsilon^+_{b'}(\vec{p},h)$
$= -\frac{1}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma} p_a p'^+_{a'} \sum_{h=1}^{-1} \varepsilon_b(\vec{p},h)\varepsilon^+_{b'}(\vec{p},h)$
$= -\frac{1}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma} p_a p'^+_{a'} (\eta_{bb'} + \frac{p_b p^+_{b'}}{m^2})$
$= -\frac{1}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{-\varsigma\alpha'_\varsigma} p_a p'_{a'} \delta_{bb'}$
$= -\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} p_a p_b$ □

Thm. 2.3.2. $[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$

2.4 Summary of third equivalent description for massive vector particle equation

Thm. 2.4.1. $(\partial_a + iS_{ab}\partial^b)_{\beta_\varsigma}{}^{\alpha_\varsigma}\psi_{\alpha_\varsigma} = \frac{i}{\sqrt{2}} im^2\sigma^{ab}_{\varsigma\beta_\varsigma}A_b, \psi_{\alpha_\varsigma} := \frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}, S_{ab} := i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$A_a(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}}\varepsilon_a(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$F_{ab}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}}\lambda_{ab}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\psi_{\alpha_\varsigma}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}}\lambda_{\alpha_\varsigma}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

Thm. 2.4.2. $\begin{cases} \sum_{h=1}^{-1} \varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h) = \eta_{aa'} + \frac{p_a p^+_{a'}}{m^2} \\ \sum_{h=1}^{-1} \lambda_{ab}(\vec{p},h)\lambda^+_{a'b'}(\vec{p},h) = p_{[a}p^+_{[a'}\eta_{b]b']} \\ \sum_{h=1}^{-1} \lambda_{\alpha_\varsigma}(\vec{p},h)\lambda^+_{\alpha'_\varsigma}(\vec{p},h) = -\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_a p_b \end{cases}$ $\begin{cases} [A_a(x), A^+_{a'}(x')] = i(\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x') \\ [F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a[a'}\partial_{b]}\partial^+_{b']}\Delta(x-x') \\ [\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x') \end{cases}$

2.5 $m \to 0$ formally derived photon case

Thm. 2.5.1. $(\partial_a + iS_{ab}\partial^b)_{\beta_\varsigma}{}^{\alpha_\varsigma}\psi_{\alpha_\varsigma} \to 0, \psi_{\alpha_\varsigma} := \frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}, S_{ab} := i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$A_a(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}}\varepsilon_a(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p} \to \infty$$

$$F_{ab}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}}\lambda_{ab}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p} \to \infty$$

$$\psi_{\alpha_\varsigma}(\vec{r},t) \to \frac{-\varsigma}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{|\vec{p}|}\lambda_{m\alpha_\varsigma}(\vec{p},-\varsigma)[a(\vec{p},-\varsigma)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},-\varsigma)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

Cor. 2.5.1. $\varepsilon_a(\vec{p},1) = [i\lambda_m(\vec{p},1),0]_a, \varepsilon_a(\vec{p},0) = \frac{1}{m}[iE\lambda_m(\vec{p},0), i|\vec{p}|]_a \to \infty, \varepsilon_a(\vec{p},-1) = [i\lambda_m(\vec{p},-1),0]_a$

Cor. 2.5.2. $\lambda_{\alpha_\varsigma}(\vec{p},-\varsigma) \to \sqrt{2}|\vec{p}|\lambda_{m\alpha_\varsigma}(\vec{p},-\varsigma), \lambda_{\alpha_\varsigma}(\vec{p},\varsigma) \to 0, \lambda_{\alpha_\varsigma}(\vec{p},0) \to 0$

Cor. 2.5.3. $\begin{cases} [A_a(x), A^+_{a'}(x')] = i(\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x') \to \infty \\ [F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a[a'}\partial_{b]}\partial^+_{b']}\Delta(x-x') \\ [\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x') \end{cases}$

From the above, we can be seen when $m \to 0$ then $A_a, F_{ab} \to \infty$. And it will become meaningless. But ψ_{α_ς} still makes sense and can be naturally transitioned. It is rewrote below. Of course, the strict approach still requires the method about massless particles. And here is only a formal derivation.

Cor. 2.5.4. $(\partial_a + iS_{ab}\partial^b)_{\beta_\varsigma}{}^{\alpha_\varsigma}\psi_{\alpha_\varsigma} \to 0$

$$\psi_{\alpha_\varsigma}(\vec{r},t) \to \frac{-\varsigma}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{|\vec{p}|}\lambda_{m\alpha_\varsigma}(\vec{p},-\varsigma)[a(\vec{p},-\varsigma)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},-\varsigma)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$\lambda_{\alpha_\varsigma}(\vec{p},-\varsigma)\lambda^+_{\alpha'_\varsigma}(\vec{p},-\varsigma) = -\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}p_a p_b, [\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$

3 Intuitive proof method of two representations equivalence for massive vector field

Self comment: The simple and ingenious analytical proof method has been given in the previous chapter, so why should other proof methods be given? here are three reasons: first, I was the first to prove it by using this complex and intuitive method. The second is that in this complex proof process, all commutative rules with mass vector field decomposition can be obtained incidentally. Third, we can obtain a set of useful identities. I have encountered such similar cases many times. Different proof methods often require completely different mathematical skills. Some abstract proofs are concise, but the details are unclear. Sometimes it is clearly demonstrated, and there is always some suspicion because it is not intuitive and too abstract. Constructive proofs are sometimes concise and sometimes cumbersome. But every step can be clearly seen.

3.1 Mathematical preparation

3.1.1 Introduction of constant invariant tensor $\Gamma_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}$

Def. 3.1.1. $\Gamma^a_{\lambda_\varsigma\mu_\varsigma} := [\gamma^a(\varsigma)C]_{\lambda_\varsigma\mu_\varsigma} = \begin{bmatrix} 0^a_{A_\varsigma B_\varsigma} & [(\sigma,i\varsigma)\sigma_y]^a_{A_\varsigma}{}^{B'_\varsigma} \\ [(\sigma,-i\varsigma)\sigma_y]_a{}^{A'_\varsigma}{}_{B_\varsigma} & 0_a^{A'_\varsigma B'_\varsigma} \end{bmatrix}$

Def. 3.1.2. $\Gamma^{ab}_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} := \Gamma^a_{\lambda_\varsigma\mu_\varsigma}\Gamma^b_{\eta_\varsigma\xi_\varsigma}, \Gamma_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} := \Gamma^a_{\lambda_\varsigma\mu_\varsigma}\delta_{ab}\Gamma^b_{\eta_\varsigma\xi_\varsigma}$

Pro. 3.1.1.
$\Gamma_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \Gamma_{\mu_\varsigma\lambda_\varsigma\eta_\varsigma\xi_\varsigma}, \Gamma_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \Gamma_{\lambda_\varsigma\mu_\varsigma\xi_\varsigma\eta_\varsigma}, \Gamma_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \Gamma_{\eta_\varsigma\xi_\varsigma\lambda_\varsigma\mu_\varsigma}$
$\Gamma_{1_\varsigma 3_\varsigma 2_\varsigma 4_\varsigma} = \Gamma_{3_\varsigma 1_\varsigma 2_\varsigma 4_\varsigma} = \Gamma_{1_\varsigma 3_\varsigma 4_\varsigma 2_\varsigma} = \Gamma_{3_\varsigma 1_\varsigma 4_\varsigma 2_\varsigma} = 1,$
$\Gamma_{2_\varsigma 4_\varsigma 1_\varsigma 3_\varsigma} = \Gamma_{2_\varsigma 4_\varsigma 3_\varsigma 1_\varsigma} = \Gamma_{4_\varsigma 2_\varsigma 1_\varsigma 3_\varsigma} = \Gamma_{4_\varsigma 2_\varsigma 3_\varsigma 1_\varsigma} = 1,$
$\Gamma_{1_\varsigma 4_\varsigma 2_\varsigma 3_\varsigma} = \Gamma_{4_\varsigma 1_\varsigma 2_\varsigma 3_\varsigma} = \Gamma_{1_\varsigma 4_\varsigma 3_\varsigma 2_\varsigma} = \Gamma_{4_\varsigma 1_\varsigma 3_\varsigma 2_\varsigma} = -1,$
$\Gamma_{2_\varsigma 3_\varsigma 1_\varsigma 4_\varsigma} = \Gamma_{2_\varsigma 3_\varsigma 4_\varsigma 1_\varsigma} = \Gamma_{3_\varsigma 2_\varsigma 1_\varsigma 4_\varsigma} = \Gamma_{3_\varsigma 2_\varsigma 4_\varsigma 1_\varsigma} = -1$
$\Gamma_{rest} = 0, \Gamma_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}} = 0$

Pro. 3.1.2. $\Gamma_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = (-1)^{\lambda_\varsigma+\mu_\varsigma}u(\lambda_\varsigma+\mu_\varsigma-3)u(\eta_\varsigma+\xi_\varsigma-3)|\varepsilon_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}|$

3.1.2 Matrix expansion of various quantities

Lem. 3.1.1. $\gamma^a = (\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x) = i\begin{bmatrix} 0 & -(\sigma,i\varsigma) \\ (\sigma,-i\varsigma) & 0 \end{bmatrix}, S_{ab}(\varsigma) = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\alpha_\varsigma} = -\frac{i}{4}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}$

Lem. 3.1.2. $im\gamma^a(\varsigma)C = \begin{bmatrix} 0 & im(\sigma,i\varsigma)\sigma_y \\ im(\sigma,-i\varsigma)\sigma_y & 0 \end{bmatrix}$

Proof: $im\gamma^a(\varsigma)C = im(\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x)\varsigma\gamma_y(\varsigma)\gamma_4(\varsigma) = im(\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x)(\sigma_y\otimes\sigma_y)(I\otimes\sigma_x)$
$= im(\sigma\sigma_y\otimes\sigma_x, -\varsigma\sigma_y\otimes\sigma_y) = \begin{bmatrix} 0 & im(\sigma,i\varsigma)\sigma_y \\ im(\sigma,-i\varsigma)\sigma_y & 0 \end{bmatrix}$ □

Lem. 3.1.3. $S_{ab}(e,\varsigma) = \begin{bmatrix} S_{ab}(\varsigma) & 0 \\ 0 & S_{ab}(-\varsigma) \end{bmatrix}$

Proof: $S_{ab}(e,\varsigma) = -\frac{i}{4}[\gamma_a(\varsigma), \gamma_b(\varsigma)]$
$= -\frac{i}{4}\{\begin{bmatrix} 0 & -i(\sigma,i\varsigma)_a \\ i(\sigma,-i\varsigma)_a & 0 \end{bmatrix}\begin{bmatrix} 0 & -i(\sigma,i\varsigma)_b \\ i(\sigma,-i\varsigma)_b & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i(\sigma,i\varsigma)_b \\ i(\sigma,-i\varsigma)_b & 0 \end{bmatrix}\begin{bmatrix} 0 & -i(\sigma,i\varsigma)_a \\ i(\sigma,-i\varsigma)_a & 0 \end{bmatrix}\}$
$= -\frac{i}{4}\begin{bmatrix} (\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]} & 0 \\ 0 & (\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]} \end{bmatrix} = \begin{bmatrix} S_{ab}(\varsigma) & 0 \\ 0 & S_{ab}(-\varsigma) \end{bmatrix}$ □

Lem. 3.1.4. $-2S^{ab}(e,\varsigma)C\partial_b = \begin{bmatrix} \frac{1}{2}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\sigma_y\partial^b & 0 \\ 0 & -\frac{1}{2}(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\sigma_y\partial^b \end{bmatrix}$

Proof: $-2S^{ab}(e,\varsigma)C\partial_b = -2\begin{bmatrix} S^{ab}(\varsigma)\partial_b & 0 \\ 0 & S^{ab}(-\varsigma)\partial_b \end{bmatrix}(\sigma_y\otimes\sigma_y)(I\otimes\sigma_x) = 2i\begin{bmatrix} S^{ab}(\varsigma)\partial_b\sigma_y & 0 \\ 0 & -S^{ab}(-\varsigma)\partial_b\sigma_y \end{bmatrix}$
$= \begin{bmatrix} \frac{1}{2}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\sigma_y\partial^b & 0 \\ 0 & -\frac{1}{2}(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\sigma_y\partial^b \end{bmatrix}$ □

Cor. 3.1.1. $\mathbb{X}_a = \begin{bmatrix} 2iS_{ab}(\varsigma)\sigma_y\partial^b & im(\sigma,i\varsigma)_a\sigma_y \\ im(\sigma,-i\varsigma)_a\sigma_y & -2iS_{ab}(-\varsigma)\sigma_y\partial^b \end{bmatrix} = \begin{bmatrix} 2\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b & im(\sigma,i\varsigma)_a\sigma_y \\ im(\sigma,-i\varsigma)_a\sigma_y & 2\varsigma S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b \end{bmatrix}$

Cor. 3.1.2. $\mathbb{X}_a = \begin{bmatrix} \frac{1}{2}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\sigma_y\partial^b & im(\sigma,i\varsigma)_a\sigma_y \\ im(\sigma,-i\varsigma)_a\sigma_y & -\frac{1}{2}(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\sigma_y\partial^b \end{bmatrix}, \mathbb{X}^+_a = \begin{bmatrix} -\frac{1}{2}\sigma_y(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\partial^{+b} & -im\sigma_y(\sigma,i\varsigma)_a \\ -im\sigma_y(\sigma,-i\varsigma)_a & \frac{1}{2}\sigma_y(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\partial^{+b} \end{bmatrix}$

Cor. 3.1.3. $\psi_{\lambda_\varsigma\mu_\varsigma} = \mathbb{X}_a\frac{A^a}{2}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma} = [im\gamma_a(\varsigma)C - 2S_{ab}(e,\varsigma)\partial^b C]_{\lambda_\varsigma\mu_\varsigma}\frac{A^a}{2}$
$= \frac{1}{2}\begin{bmatrix} 2\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b A_a & im[(\sigma,i\varsigma)^a\sigma_y]_{A_\varsigma}{}^{B'_\varsigma}A_a \\ im[(\sigma,-i\varsigma)_a\sigma_y]^{A_\varsigma}{}_{B'_\varsigma}A^a & 2\varsigma S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b A_a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}F_{ab} & im[(\sigma,i\varsigma)^a\sigma_y]_{A_\varsigma}{}^{B'_\varsigma}A_a \\ im[(\sigma,-i\varsigma)_a\sigma_y]^{A_\varsigma}{}_{B'_\varsigma}A^a & -\varsigma S_{ab}{}^{A'_\varsigma B'_\varsigma}F^{ab} \end{bmatrix}$
$= \begin{bmatrix} \psi_{A_\varsigma B_\varsigma} & \psi_{A_\varsigma}{}^{B'_\varsigma} \\ \psi^{A_\varsigma}{}_{B'_\varsigma} & \psi^{A'_\varsigma B'_\varsigma} \end{bmatrix}_{B-G} = -\varsigma\begin{bmatrix} \Psi_{A_\varsigma B_\varsigma} & \Psi_{A_\varsigma}{}^{B'_\varsigma} \\ \Psi^{A_\varsigma}{}_{B'_\varsigma} & \Psi^{A'_\varsigma B'_\varsigma} \end{bmatrix}_{Two} = \frac{i}{\sqrt{2}}\begin{bmatrix} \psi_{A_\varsigma B_\varsigma} & \psi_{A_\varsigma}{}^{B'_\varsigma} \\ \psi^{A_\varsigma}{}_{B'_\varsigma} & \psi^{A'_\varsigma B'_\varsigma} \end{bmatrix}_{One}$

3.1.3 An important lemma

Lem. 3.1.5. $[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')$

$$=-\frac{i}{4}\begin{bmatrix}2\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b & im[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\\ im[(\sigma,-i\varsigma)\sigma_y]_a{}^{A'_\varsigma}{}_{B_\varsigma} & 2\varsigma S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\end{bmatrix}_{\lambda_\varsigma\mu_\varsigma}\begin{bmatrix}2\varsigma S^{a'b'}{}_{A'_\varsigma B'_\varsigma}\partial_{b'} & im[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{A'_\varsigma}{}^{B_\varsigma}\\ im[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{A_\varsigma}{}_{B'_\varsigma} & 2\varsigma S_{a'b'}{}^{A_\varsigma B_\varsigma}\partial^{b'}\end{bmatrix}_{\lambda'_\varsigma\mu'_\varsigma}(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\Delta(x-x')$$

Proof: $[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}\Delta(x-x')$

$=\frac{i}{4}\{[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]C\}_{\lambda_\varsigma\mu_\varsigma}[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\{[im\gamma^{a'}(\varsigma)+2S^{a'b'}(e,\varsigma)\partial_{'b'}]C\}^+_{\lambda'_\varsigma\mu'_\varsigma}$

$\Delta(x-x')$

$=\frac{i}{4}\{[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]C\}_{\lambda_\varsigma\mu_\varsigma}[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\{C^+[-im\gamma^{a'}(\varsigma)-2S^{a'b'}(e,\varsigma)\partial^+_{'b'}]\}_{\lambda'_\varsigma\mu'_\varsigma}\Delta(x-x')$

$=\frac{i}{4}\{[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]C\}_{\lambda_\varsigma\mu_\varsigma}(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\{C^+[-im\gamma^{a'}(-\varsigma)-2S^{a'b'}(e,-\varsigma)\partial_{'b'}]\}_{\lambda'_\varsigma\mu'_\varsigma}\Delta(x-x')$

$=\frac{i}{4}\{[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]\varsigma\gamma_y(\varsigma)\gamma_4(\varsigma)\}_{\lambda_\varsigma\mu_\varsigma}$

$(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\{\varsigma\gamma_4(\varsigma)\gamma_y(\varsigma)[-im\gamma^{a'}(-\varsigma)-2S^{a'b'}(e,-\varsigma)\partial_{'b'}]\}_{\lambda'_\varsigma\mu'_\varsigma}\Delta(x-x')$

$=\frac{i}{4}\{[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b][-\varsigma\gamma_4(\varsigma)\gamma_y(\varsigma)]\}_{\lambda_\varsigma\mu_\varsigma}$

$(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\{[\varsigma\gamma_y(-\varsigma)\gamma_4(-\varsigma)][-im\gamma^{a'}(-\varsigma)-2S^{a'b'}(e,-\varsigma)\partial_{'b'}]\}_{\lambda'_\varsigma\mu'_\varsigma}\Delta(x-x')$

$=\frac{i}{4}\{im\begin{bmatrix}0 & (\sigma,i\varsigma)\sigma_y\\ (\sigma,-i\varsigma)\sigma_y & 0\end{bmatrix}+2i\begin{bmatrix}S^{ab}(\varsigma)\sigma_y & 0\\ 0 & -S^{ab}(-\varsigma)\sigma_y\end{bmatrix}\partial_b\}_{\lambda_\varsigma\mu_\varsigma}(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})$

$\{-im\begin{bmatrix}0 & \sigma_y(\sigma,-i\varsigma)\\ \sigma_y(\sigma,i\varsigma) & 0\end{bmatrix}-2i\begin{bmatrix}\sigma_yS^{ab}(-\varsigma) & 0\\ 0 & -\sigma_yS^{ab}(\varsigma)\end{bmatrix}\partial_{'b}\}_{\lambda'_\varsigma\mu'_\varsigma}\Delta(x-x')$

$=\frac{i}{4}\begin{bmatrix}2iS^{ab}(\varsigma)\sigma_y\partial_b & im(\sigma,i\varsigma)\sigma_y\\ im(\sigma,-i\varsigma)\sigma_y & -2iS^{ab}(-\varsigma)\sigma_y\partial_b\end{bmatrix}_{\lambda_\varsigma\mu_\varsigma}(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\begin{bmatrix}-2i\sigma_yS^{a'b'}(-\varsigma)\partial_{'b'} & -im\sigma_y(\sigma,-i\varsigma)\\ -im\sigma_y(\sigma,i\varsigma) & 2i\sigma_yS^{a'b'}(\varsigma)\partial_{'b'}\end{bmatrix}_{\lambda'_\varsigma\mu'_\varsigma}\Delta(x-x')$

$$=-\frac{i}{4}\begin{bmatrix}2\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b & im[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\\ im[(\sigma,-i\varsigma)\sigma_y]_a{}^{A'_\varsigma}{}_{B_\varsigma} & 2\varsigma S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\end{bmatrix}_{\lambda_\varsigma\mu_\varsigma}\begin{bmatrix}2\varsigma S^{a'b'}{}_{A'_\varsigma B'_\varsigma}\partial_{b'} & im[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{A'_\varsigma}{}^{B_\varsigma}\\ im[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{A_\varsigma}{}_{B'_\varsigma} & 2\varsigma S_{a'b'}{}^{A_\varsigma B_\varsigma}\partial^{b'}\end{bmatrix}_{\lambda'_\varsigma\mu'_\varsigma}(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\Delta(x-x')\quad\square$$

3.2 First intuitive proof of two representations equivalence for massive vector field

Lem. 3.2.1. $[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$

$=-[-imI\otimes\sigma_x+(\sigma\otimes\sigma_z,i\varsigma)^a\partial_a]_{\{\lambda_\varsigma(\lambda'_\varsigma}[-imI\otimes\sigma_x+(\sigma\otimes\sigma_z,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\Delta(x-x'),\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_x)$

Thm. 3.2.1. $[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{8}[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$

$$=-\frac{i}{8}\begin{bmatrix}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a & -im\delta_{A_\varsigma}{}^{B_\varsigma}\\ -im\delta^{A'_\varsigma}{}_{B'_\varsigma} & -(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\end{bmatrix}_{\{\lambda_\varsigma(\lambda'_\varsigma}\begin{bmatrix}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b & -im\delta_{A_\varsigma}{}^{B_\varsigma}\\ -im\delta^{A'_\varsigma}{}_{B'_\varsigma} & -(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}\partial^b\end{bmatrix}_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$$

$\Leftrightarrow[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')$

$$=-\frac{i}{4}\begin{bmatrix}2\varsigma S^{ab}{}_{A_\varsigma B_\varsigma}\partial_b & im[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}\\ im[(\sigma,-i\varsigma)\sigma_y]_a{}^{A'_\varsigma}{}_{B_\varsigma} & 2\varsigma S_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\end{bmatrix}_{\lambda_\varsigma\mu_\varsigma}\begin{bmatrix}2\varsigma S^{a'b'}{}_{A'_\varsigma B'_\varsigma}\partial_{b'} & im[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{A'_\varsigma}{}^{B_\varsigma}\\ im[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{A_\varsigma}{}_{B'_\varsigma} & 2\varsigma S_{a'b'}{}^{A_\varsigma B_\varsigma}\partial^{b'}\end{bmatrix}_{\lambda'_\varsigma\mu'_\varsigma}(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\Delta(x-x')$$

Proof:

$$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]=-\frac{i}{8}(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma\}B'_\varsigma)}\partial_a\partial_b\Delta(x-x')=iS^{ac}{}_{A_\varsigma B_\varsigma}\delta_{cd}S^{db}{}_{A'_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi^{A'_\varsigma B'_\varsigma}(x),\psi_+^{A_\varsigma B_\varsigma}(x')]=-\frac{i}{8}(\sigma,-i\varsigma)_a^{\{A'_\varsigma(A_\varsigma}(\sigma,-i\varsigma)_b^{B'_\varsigma\}B_\varsigma)}\partial^a\partial^b\Delta(x-x')=iS_{ac}{}^{A'_\varsigma B'_\varsigma}\delta^{cd}S_{db}{}^{A_\varsigma B_\varsigma}\partial^a\partial^b\Delta(x-x')\end{cases}$$

$$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi_+^{C_\varsigma D_\varsigma}(x')]=\frac{i}{8}m^2\delta^{(C_\varsigma}_{\{A_\varsigma}\delta^{D_\varsigma)}_{B_\varsigma\}}\Delta(x-x')=iS^{ac}{}_{A_\varsigma B_\varsigma}\delta^d_cS_{db}{}^{C_\varsigma D_\varsigma}\partial_a\partial^b\Delta(x-x')\\ [\psi^{A'_\varsigma B'_\varsigma}(x),\psi^+_{C'_\varsigma D'_\varsigma}(x')]=\frac{i}{8}m^2\delta^{\{A'_\varsigma}_{(C'_\varsigma}\delta^{B'_\varsigma\}}_{D'_\varsigma)}\Delta(x-x')=iS_{ac}{}^{A'_\varsigma B'_\varsigma}\delta^c_dS^{db}{}_{C'_\varsigma D'_\varsigma}\partial_a\partial^b\Delta(x-x')\end{cases}$$

$$\begin{cases}[\psi^{B'_\varsigma}_{A_\varsigma}(x),^+\psi^{B_\varsigma}_{A'_\varsigma}(x')]=\frac{i}{4}[(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma,-i\varsigma)_b^{B'_\varsigma B_\varsigma}\partial^b+m^2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}]\Delta(x-x')\\ =-\frac{i}{4}\{[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}[\sigma_y(\sigma,-i\varsigma)]^b{}_{A'_\varsigma}{}^{B_\varsigma}\partial_a\partial_b\Delta(x-x')-2m^2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}\}\\ [\psi^{A'_\varsigma}_{B_\varsigma}(x),^+\psi^{A_\varsigma}_{B'_\varsigma}(x')]=\frac{i}{4}[(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b+m^2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}]\Delta(x-x')\\ =-\frac{i}{4}\{[(\sigma,i\varsigma)\sigma_y]^a{}_{B_\varsigma}{}^{A'_\varsigma}[\sigma_y(\sigma,-i\varsigma)]^b{}_{B'_\varsigma}{}^{A_\varsigma}\partial_a\partial_b\Delta(x-x')-2m^2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}\}\end{cases}$$

$$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),^+\psi^{C_\varsigma}_{A'_\varsigma}(x')]=-\frac{1}{4}m(\sigma,i\varsigma)^a_{\{A_\varsigma A'_\varsigma}\delta^{C_\varsigma}_{B_\varsigma\}}\partial_a\Delta(x-x')=\frac{\varsigma}{2}mS^{ab}{}_{A_\varsigma B_\varsigma}\partial_b\delta_{aa'}[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{A'_\varsigma}{}^{C_\varsigma}\Delta(x-x')\\ [\psi^{A'_\varsigma B'_\varsigma}(x),^+\psi^{A_\varsigma}_{C'_\varsigma}(x')]=\frac{1}{4}m(\sigma,-i\varsigma)_a^{\{A'_\varsigma A_\varsigma}\delta^{B'_\varsigma\}}_{C'_\varsigma}\partial^a\Delta(x-x')=\frac{\varsigma}{2}mS_{ab}{}^{A'_\varsigma B'_\varsigma}\partial^b\delta^{aa'}[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{A_\varsigma}{}_{C'_\varsigma}\Delta(x-x')\end{cases}$$

$$\begin{cases}[\psi^{C'_\varsigma}_{A_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]=-\frac{1}{4}m(\sigma,i\varsigma)^a_{A_\varsigma\{A'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma\}}\partial_a\Delta(x-x')=\frac{\varsigma}{2}mS^{ab}{}_{A'_\varsigma B'_\varsigma}\partial_b\delta_{aa'}[(\sigma,-i\varsigma)\sigma_y]^{a'C'_\varsigma}{}_{A_\varsigma}\Delta(x-x')\\ [\psi^{A'_\varsigma}_{C_\varsigma}(x),\psi_+^{A_\varsigma B_\varsigma}(x')]=\frac{1}{4}m(\sigma,-i\varsigma)_a^{A'_\varsigma\{A_\varsigma}\delta^{B_\varsigma\}}_{C_\varsigma}\partial_a\Delta(x-x')=\frac{\varsigma}{2}mS_{ab}{}^{A_\varsigma B_\varsigma}\partial^b\delta^{aa'}[(\sigma,i\varsigma)\sigma_y]_{a'C_\varsigma}{}^{A'_\varsigma}\Delta(x-x')\end{cases}$$

$$\begin{cases}[\sigma_y(\sigma,-i\varsigma)]^a{}_{A'_\varsigma}{}^{B_\varsigma}=[\sigma_y(\sigma,i\varsigma)]^{aB_\varsigma}{}_{A'_\varsigma},[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}=[(\sigma,-i\varsigma)\sigma_y]^{aB'_\varsigma}{}_{A_\varsigma}\\ \{[\sigma_y(\sigma,-i\varsigma)]^a{}_{A'_\varsigma}{}^{B_\varsigma}\}^*=[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma},\{[\sigma_y(\sigma,i\varsigma)]_a{}^{B_\varsigma}{}_{A'_\varsigma}\}^*=[(\sigma,-i\varsigma)\sigma_y]_a{}^{B'_\varsigma}{}_{A_\varsigma}\end{cases}\quad\square$$

Cor. 3.2.1. $2\mathbb{X}^{a}_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')=[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$

3.3 Second intuitive proof of two representations equivalence for massive vector field

Lem. 3.3.1. $\begin{cases}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\delta_{ab}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}=-2\varepsilon_{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}\\(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\delta^{ab}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b=-2\varepsilon^{A_\varsigma B_\varsigma}\varepsilon^{A'_\varsigma B'_\varsigma}\end{cases}$ $\begin{cases}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\delta^b_a(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b=2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}\\(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\delta^a_b(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}=2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}\end{cases}$

Thm. 3.3.1. $[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\Delta(x-x')$

$=\begin{bmatrix}\frac{1}{2}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\sigma_y\partial^b & im(\sigma,i\varsigma)_a\sigma_y\\ im(\sigma,-i\varsigma)_a\sigma_y & -\frac{1}{2}(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\sigma_y\partial^b\end{bmatrix}_{\lambda_\varsigma\mu_\varsigma}\begin{bmatrix}\frac{1}{2}\sigma_y(\sigma,-i\varsigma)_{[a'}(\sigma,i\varsigma)_{b']}\partial^{b'} & -im\sigma_y(\sigma,-i\varsigma)_{a'}\\ -im\sigma_y(\sigma,i\varsigma)_{a'} & -\frac{1}{2}\sigma_y(\sigma,i\varsigma)_{[a'}(\sigma,-i\varsigma)_{b']}\partial^{b'}\end{bmatrix}_{\lambda'_\varsigma\mu'_\varsigma}$

$\frac{i}{4}(\delta^{aa'}-\frac{\partial^a\partial^{a'}}{m^2})\Delta(x-x')$

$\Leftrightarrow[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{8}[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$

$=-\frac{i}{8}\begin{bmatrix}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a & -im\delta_{A_\varsigma}{}^{B_\varsigma}\\ -im\delta^{A'_\varsigma}{}_{B'_\varsigma} & -(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a\end{bmatrix}_{\{\lambda_\varsigma(\lambda'_\varsigma}\begin{bmatrix}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b & -im\delta_{A_\varsigma}{}^{B_\varsigma}\\ -im\delta^{A'_\varsigma}{}_{B'_\varsigma} & -(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b\partial^b\end{bmatrix}_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$

Proof:

$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]=\frac{i}{16}\{(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\sigma_y\partial^b\}_{A_\varsigma B_\varsigma}\delta^{aa'}\{\sigma_y(\sigma,-i\varsigma)_{[a'}(\sigma,i\varsigma)_{b']}\partial^{b'}\}_{A'_\varsigma B'_\varsigma}\Delta(x-x')\\=-\frac{i}{8}(\sigma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma\}B'_\varsigma)}\partial_a\partial_b\Delta(x-x')\\ [\psi^{A'_\varsigma B'_\varsigma}(x),\psi^{A_\varsigma B_\varsigma}_+(x')]=\frac{i}{16}\{(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\sigma_y\partial^b\}^{A'_\varsigma B'_\varsigma}\delta^{aa'}\{\sigma_y(\sigma,i\varsigma)_{[a'}(\sigma,-i\varsigma)_{b']}\partial^{b'}\}^{A_\varsigma B_\varsigma}\Delta(x-x')\\=-\frac{i}{8}(\sigma,-i\varsigma)^{\{A'_\varsigma(A_\varsigma}_a(\sigma,-i\varsigma)^{B'_\varsigma\}B_\varsigma)}_b\partial^a\partial^b\Delta(x-x')\end{cases}$

$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^{C_\varsigma D_\varsigma}_+(x')]=-\frac{i}{16}\{(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\sigma_y\partial^b\}_{A_\varsigma B_\varsigma}\delta^{aa'}\{\sigma_y(\sigma,i\varsigma)_{[a'}(\sigma,-i\varsigma)_{b']}\partial^{b'}\}^{C_\varsigma D_\varsigma}\Delta(x-x')\\=\frac{i}{8}\delta^{(C_\varsigma}_{\{A_\varsigma}\delta^{D_\varsigma)}_{B_\varsigma\}}\partial^a\partial_a\Delta(x-x')\\ [\psi^{A'_\varsigma B'_\varsigma}(x),\psi^+_{C'_\varsigma D'_\varsigma}(x')]=-\frac{i}{16}\{(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\sigma_y\partial^b\}^{A'_\varsigma B'_\varsigma}\delta^{aa'}\{\sigma_y(\sigma,-i\varsigma)_{[a'}(\sigma,i\varsigma)_{b']}\partial^{b'}\}_{C'_\varsigma D'_\varsigma}\Delta(x-x')\\=\frac{i}{8}\delta^{\{A'_\varsigma}_{(C'_\varsigma}\delta^{B'_\varsigma\}}_{D'_\varsigma)}\partial^a\partial_a\Delta(x-x')\end{cases}$

$\begin{cases}[\psi^{B'_\varsigma}_{A_\varsigma}(x),^+\psi^{B_\varsigma}_{A'_\varsigma}(x')]=-\frac{i}{4}\{[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}[\sigma_y(\sigma,-i\varsigma)]^b{}_{A'_\varsigma}{}^{B_\varsigma}\partial_a\partial_b\Delta(x-x')-2m^2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}\}\\=\frac{i}{4}[(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b\partial^b+m^2\delta^{B_\varsigma}_{A_\varsigma}\delta^{B'_\varsigma}_{A'_\varsigma}]\Delta(x-x')\\ [\psi^{A'_\varsigma}_{B_\varsigma}(x),^+\psi^{A_\varsigma}_{B'_\varsigma}(x')]=-\frac{i}{4}\{[(\sigma,i\varsigma)\sigma_y]^a{}_{B_\varsigma}{}^{A'_\varsigma}[\sigma_y(\sigma,-i\varsigma)]^b{}_{B'_\varsigma}{}^{A_\varsigma}\partial_a\partial_b\Delta(x-x')-2m^2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}\}\\=\frac{i}{4}[(\sigma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b+m^2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}]\Delta(x-x')\end{cases}$

$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),^+\psi^{C_\varsigma}_{A'_\varsigma}(x')]=\frac{1}{8}m\{(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\sigma_y\partial^b\}_{A_\varsigma B_\varsigma}\delta^a_{a'}[\sigma_y(\sigma,-i\varsigma)]^{a'}{}_{A'_\varsigma}{}^{C_\varsigma}\Delta(x-x')\\=-\frac{1}{4}m(\sigma,i\varsigma)^a_{\{A_\varsigma A'_\varsigma}\delta^{C_\varsigma}_{B_\varsigma\}}\partial_a\Delta(x-x')\\ [\psi^{A'_\varsigma B'_\varsigma}(x),^+\psi^{A_\varsigma}_{C'_\varsigma}(x')]=-\frac{1}{8}m\{(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\sigma_y\partial^b\}^{A'_\varsigma B'_\varsigma}\delta^a_{a'}[\sigma_y(\sigma,i\varsigma)]_{a'}{}^{A_\varsigma}{}_{C'_\varsigma}\Delta(x-x')\\=\frac{1}{4}m(\sigma,-i\varsigma)^{\{A'_\varsigma A_\varsigma}_a\delta^{B'_\varsigma\}}_{C'_\varsigma}\partial^a\Delta(x-x')\end{cases}$

$\begin{cases}[\psi^{C'_\varsigma}_{A_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]=-\frac{1}{8}m\{\sigma_y(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{b]}\partial^b\}_{A'_\varsigma B'_\varsigma}\delta^a_{a'}[(\sigma,-i\varsigma)\sigma_y]^{a'C'_\varsigma}{}_{A_\varsigma}\Delta(x-x')\\=-\frac{1}{4}m(\sigma,i\varsigma)^a_{A_\varsigma\{A'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma\}}\partial_a\Delta(x-x')\\ [\psi^{A'_\varsigma}_{C_\varsigma}(x),\psi^{A_\varsigma B_\varsigma}_+(x')]=\frac{1}{8}m\{\sigma_y(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{b]}\partial^b\}^{A_\varsigma B_\varsigma}\delta^{aa'}[(\sigma,i\varsigma)\sigma_y]_{a'C_\varsigma}{}^{A'_\varsigma}\Delta(x-x')\\=\frac{1}{4}m(\sigma,-i\varsigma)^{A'_\varsigma\{A_\varsigma}_a\delta^{B_\varsigma\}}_{C_\varsigma}\partial_a\Delta(x-x')\end{cases}$

$\begin{cases}[\sigma_y(\sigma,-i\varsigma)]^a{}_{A'_\varsigma}{}^{B_\varsigma}=[\sigma_y(\sigma,i\varsigma)]^{aB_\varsigma}{}_{A'_\varsigma},[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma}=[(\sigma,-i\varsigma)\sigma_y]^{aB'_\varsigma}{}_{A_\varsigma}\\ \{[\sigma_y(\sigma,-i\varsigma)]^a{}_{A'_\varsigma}{}^{B_\varsigma}\}^*=[(\sigma,i\varsigma)\sigma_y]^a{}_{A_\varsigma}{}^{B'_\varsigma},\{[\sigma_y(\sigma,i\varsigma)]_a{}^{B_\varsigma}{}_{A'_\varsigma}\}^*=[(\sigma,-i\varsigma)\sigma_y]_a{}^{B'_\varsigma}{}_{A_\varsigma}\end{cases}$ □

4 Extraction of various quantum operators of B-W equation for spin-1 particles

4.1 Isochronous commutation rules of Bargmann-Wigner equation for spin-1 particles [16]

Thm. 4.1.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=0,\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=\frac{1}{2!}\psi_{\{\lambda_\varsigma\mu_\varsigma\}}(\vec{r},t)$

$\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{\frac{m^2}{E}}[a(\vec{p},h)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^2}{E^3}}U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ b^+(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^2}{E^3}}V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$

Thm. 4.1.2. $[\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)]=\frac{1}{4}[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}]\delta^3(\vec{r}-\vec{r}')$

Proof: $[\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)]$
$= \frac{i}{8}[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')|_{t=t'}$
$= \frac{i}{8}[(m\gamma^4-\vec{\gamma}\gamma^4\cdot\nabla)+i\partial_t]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4-\vec{\gamma}\gamma^4\cdot\nabla)+i\partial_t]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')|_{t=t'}$
$= \frac{i}{8}[(m\gamma^4-\vec{\gamma}\gamma^4\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}i\partial_t\delta_{\mu_\varsigma\}\mu'_\varsigma)}+i\partial_t\delta_{\{\lambda_\varsigma(\lambda'_\varsigma}(m\gamma^4-\vec{\gamma}\gamma^4\cdot\nabla)_{\mu_\varsigma\}\mu'_\varsigma)}]\Delta(x-x')|_{t=t'}$
$= \frac{1}{4}[(m\gamma^4-\vec{\gamma}\gamma^4\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}]\delta^3(\vec{r}-\vec{r}')$
$= \frac{1}{4}[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}]\delta^3(\vec{r}-\vec{r}')$ □

4.2 Extraction of Bargmann-Wigner equation energy operators for spin-1 particles

Thm. 4.2.1. $H=\int\sum_{h=1}^{-1}E[a^+(\vec{p},h)a(\vec{p},h)+b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}=\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)d^3\vec{r}$

Proof: $\int\sum_{h=1}^{-1}E[a^+(\vec{p},h)a(\vec{p},h)+b(\vec{p},h)b^+(\vec{p},h)]d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{m^2}{E^2}\psi^+_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\sum_{h=1}^{-1}[U^{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^{+\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}+V^{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)V^{+\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{m^2}{E^2}\psi^+_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\sum_{h=1}^{-1}[U^{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^{+\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h)+V^{\lambda_\varsigma\mu_\varsigma}(-\vec{p},h)V^{+\lambda'_\varsigma\mu'_\varsigma}(-\vec{p},h)]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{m^2}{E^2}\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\frac{1}{16m^2}[[(m-i\gamma^ap_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}+[(m-i\gamma^ap^+_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp^+_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\frac{m^2}{E^2}\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\frac{1}{16m^2}[[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})+E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})+E]_{\mu_\varsigma\}\mu'_\varsigma)}$
$+[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})-E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})-E]_{\mu_\varsigma\}\mu'_\varsigma)}]e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\frac{1}{16E^2}\{[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})+E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})+E]_{\mu_\varsigma\}\mu'_\varsigma)}$
$+[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})-E]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})-E]_{\mu_\varsigma\}\mu'_\varsigma)}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\frac{1}{8E^2}\{[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4-i\vec{\gamma}\gamma^4\cdot\vec{p})]_{\mu_\varsigma\}\mu'_\varsigma)}+E^2\delta_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{(2\pi)^3}\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\{\frac{1}{8E^2}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+i\gamma^4\vec{\gamma}\cdot\vec{p})]_{\mu_\varsigma\}\mu'_\varsigma)}+\frac{1}{2}\delta_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}\}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{2}\frac{1}{(2\pi)^3}\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)$
$\{[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\mu_\varsigma\}\mu'_\varsigma)}\frac{1}{E^2}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}+\delta_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}\}d^3\vec{r}d^3\vec{r}'d^3\vec{p}$
$= \frac{1}{8}\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r}',t)\{\frac{[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\mu_\varsigma\}\mu'_\varsigma)}}{m^2-\nabla^2}+\delta_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}\}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}'$
$= \frac{1}{8}\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\{\frac{[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\mu_\varsigma\}\mu'_\varsigma)}}{m^2-\nabla^2}+\delta_{\{\lambda_\varsigma(\lambda'_\varsigma}\delta_{\mu_\varsigma\}\mu'_\varsigma)}\}\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r},t)d^3\vec{r}$
$= \frac{1}{2}\int\{\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\frac{[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\lambda_\varsigma\lambda'_\varsigma}[(m\gamma^4+\gamma^4\vec{\gamma}\cdot\nabla)]_{\mu_\varsigma\mu'_\varsigma}}{m^2-\nabla^2}\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r},t)+\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\}d^3\vec{r}$
$= \frac{1}{2}\int\{\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\frac{(i\partial_t)^2}{m^2-\nabla^2}\psi^{\lambda'_\varsigma\mu'_\varsigma}(\vec{r},t)+\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\}d^3\vec{r}$
$= \int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)d^3\vec{r}$ □

Thm. 4.2.2. $H=\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)d^3\vec{r}=\int\{\frac{1}{2}F^{+ab}F_{ab}+m^2A^{+a}(\vec{r},t)A_a(\vec{r},t)\}d^3\vec{r}$

Proof: $H=\int\psi^{+\lambda_\varsigma\mu_\varsigma}(\vec{r},t)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)d^3\vec{r}$
$tr[S_{ab}(e,\varsigma)S_{cd}(e,\varsigma)]=S_{abcd}=\delta_{ac}\delta_{db}-\delta_{ad}\delta_{cb}$
$= \int\{\bar{C}[-im\gamma^{a'}(\varsigma)-2S^{a'b'}(e,\varsigma)\partial^+_{b'}]\}^{\lambda_\varsigma\mu_\varsigma}\frac{A^+_{a'}(\vec{r},t)}{2}[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}\frac{A_a(\vec{r},t)}{2}d^3\vec{r}$
$= \frac{1}{4}\int tr\{\bar{C}[-im\gamma^{a'}(\varsigma)-2S^{a'b'}(e,\varsigma)\partial^+_{b'}][im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]C\}A^+_{a'}(\vec{r},t)A_a(\vec{r},t)d^3\vec{r}$
$= \frac{1}{4}\int\{m^2tr[\gamma^{a'}(\varsigma)\gamma^a(\varsigma)]A^+_{a'}(\vec{r},t)A_a(\vec{r},t)+4tr[S^{a'b'}(e,\varsigma)S^{ab}(e,\varsigma)]\partial^+_{b'}A^+_{a'}(\vec{r},t)\partial_bA_a(\vec{r},t)\}d^3\vec{r}$
$= \int\{m^2\delta^{a'a}A^+_{a'}(\vec{r},t)A_a(\vec{r},t)+S^{a'b'ab}\partial^+_{b'}A^+_{a'}(\vec{r},t)\partial_bA_a(\vec{r},t)\}d^3\vec{r}$
$= \int\{m^2\delta^{a'a}A^+_{a'}(\vec{r},t)A_a(\vec{r},t)+(\delta^{a'a}\delta^{b'b}-\delta^{a'b}\delta^{b'a})\partial^+_{b'}A^+_{a'}(\vec{r},t)\partial_bA_a(\vec{r},t)\}d^3\vec{r}$
$= \int\{m^2A^{+a}(\vec{r},t)A_a(\vec{r},t)+\partial^{+b}A^{+a}(\vec{r},t)\partial_bA_a(\vec{r},t)-\partial^{+a}A^{+b}(\vec{r},t)\partial_bA_a(\vec{r},t)\}d^3\vec{r}$
$= \int\{\frac{1}{2}F^{+ab}F_{ab}+m^2A^{+a}(\vec{r},t)A_a(\vec{r},t)\}d^3\vec{r}$ □

5 Massive Majorana vector field(take $\theta = 0$)

5.1 Comparison between massive Majorana vector field and massive vector field

The above conclusions for massive vector fields in this chapter are also valid for massive Majorana vector fields, but the following additional conditions must be added.

Thm. 5.1.1. $\psi = \gamma_2 \otimes \gamma_2 \psi^*, A_a = A^+_{a'} \eta^{a'}_a (F_{ab} = F^+_{a'b'} \eta^{a'}_a \eta^{b'}_b), b^+(\vec{p}, h) = (-1)^{s+h} a^+(\vec{p}, -h)$

The following is a detailed comparison and discussion of the difference between a massive Majorana vector field and a massive complex vector field.

5.2 Derive equivalent commutation rules of various physical quantities for spin-1 particles

5.2.1 Redefinition

Def. 5.2.1. *Third definition of electromagnetic field vector*

$$\begin{cases} F_{ab} := \partial_a A_b - \partial_b A_a \\ \psi_{\alpha_\varsigma} := -\frac{1}{2\sqrt{2}} \sigma^{ab}_{\varsigma\alpha_\varsigma} F_{ab} = -\frac{\varsigma}{\sqrt{2}} (E - i\varsigma B)_{\alpha_\varsigma} \\ \psi_{A_\varsigma B_\varsigma} := \frac{i\varsigma}{\sqrt{2}} \sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma} \psi_{\alpha_\varsigma} \Leftrightarrow \psi_{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}} \sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma} \psi_{A_\varsigma B_\varsigma} \end{cases} \Rightarrow \begin{cases} \psi_{A_\varsigma B_\varsigma} = -\varsigma S^{ab}{}_{A_\varsigma B_\varsigma} F_{ab} \\ F_{ab} = \frac{1}{\sqrt{2}} (\sigma^{\alpha_{-\varsigma}}_{-\varsigma ab} \psi_{\alpha_{-\varsigma}} + \sigma^{\alpha_\varsigma}_{\varsigma ab} \psi_{\alpha_\varsigma}) \\ *F_{ab} = \frac{\varsigma}{\sqrt{2}} (\sigma^{\alpha_{-\varsigma}}_{-\varsigma ab} \psi_{\alpha_{-\varsigma}} - \sigma^{\alpha_\varsigma}_{\varsigma ab} \psi_{\alpha_\varsigma}) \end{cases}$$

5.3 Equivalent commutation rules for $\psi_{\lambda_\varsigma \mu_\varsigma}$ and (A_a, F_{ab})

5.3.1 Common commutation rules for complex and real fields

Thm. 5.3.1.

$$\begin{cases} [\psi_{\lambda_\varsigma \mu_\varsigma}(x), \psi^+_{\lambda'_\varsigma \mu'_\varsigma}(x')] = \frac{i}{4} \mathbb{X}^a_{\lambda_\varsigma \mu_\varsigma}(x) \mathbb{X}^{+a'}_{\lambda'_\varsigma \mu'_\varsigma}(x') (\eta_{aa'} - \frac{\partial_a \partial^+_{a'}}{m^2}) \Delta(x - x'), \psi_{\lambda_\varsigma \mu_\varsigma} = \psi_{\mu_\varsigma \lambda_\varsigma} \\ im \frac{A_a}{2}(x) = \frac{1}{4} tr\{\bar{C} \gamma_a(\varsigma) \psi_{[\lambda_\varsigma \mu_\varsigma]}(x)\}, i \frac{F_{ab}}{2}(x) = \frac{1}{2} tr\{\bar{C} S_{ab}(e, \varsigma) \psi_{[\lambda_\varsigma \mu_\varsigma]}(x)\} \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} [A_a(x), A^+_{a'}(x')] = i(\eta_{aa'} - \frac{\partial_a \partial^+_{a'}}{m^2}) \Delta(x - x'), [F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a<a'} \partial_{b]} \partial^+_{b'>} \Delta(x - x') \\ \psi_{[\lambda_\varsigma \mu_\varsigma]} = [im\gamma^a(\varsigma) C \frac{A_a}{2} + S^{ab}(e, \varsigma) C \frac{F_{ab}}{2}] \end{cases}$$

Proof: $m^2 [A_a(x), A^+_{a'}(x')] = [imA_a(x), -imA^+_{a'}(x')]$

$= [\frac{1}{2} tr[\bar{C} \gamma_a(\varsigma) \psi_{[\lambda_\varsigma \mu_\varsigma]}(x)], \frac{1}{2} tr^+[\bar{C} \gamma_a(\varsigma) \psi_{[\lambda_\varsigma \mu_\varsigma]}(x')]]$

$= [\frac{1}{2} [\bar{C} \gamma_a(\varsigma)]^{\lambda_\varsigma \mu_\varsigma} \psi_{\lambda_\varsigma \mu_\varsigma}(x), \frac{1}{2} [\gamma_{a'}(\varsigma) C]^{\lambda'_\varsigma \mu'_\varsigma} \psi^+_{\lambda'_\varsigma \mu'_\varsigma}(x')]$

$= \frac{1}{4} [\bar{C} \gamma_a(\varsigma)]^{\lambda_\varsigma \mu_\varsigma} [\gamma_{a'}(\varsigma) C]^{\lambda'_\varsigma \mu'_\varsigma} [\psi_{\lambda_\varsigma \mu_\varsigma}(x), \psi_{\lambda'_\varsigma \mu'_\varsigma}(x')]$

$= \frac{i}{16} [\bar{C} \gamma_a(\varsigma)]^{\lambda_\varsigma \mu_\varsigma} [\gamma_{a'}(\varsigma) C]^{\lambda'_\varsigma \mu'_\varsigma} \mathbb{X}^b_{\lambda_\varsigma \mu_\varsigma}(x) \mathbb{X}^{+b'}_{\lambda'_\varsigma \mu'_\varsigma}(x') (\eta_{bb'} - \frac{\partial_b \partial^+_{b'}}{m^2}) \Delta(x - x')$

$= \frac{i}{16} tr[\bar{C} \gamma_a(\varsigma) \mathbb{X}^b(x)] tr[\mathbb{X}^{+b'}(x') \gamma_{a'}(\varsigma) C] (\eta_{bb'} - \frac{\partial_b \partial^+_{b'}}{m^2}) \Delta(x - x')$

$= \frac{i}{16} tr\{\bar{C} \gamma_a(\varsigma) [im\gamma^b(\varsigma) - 2S^{bc}(e, \varsigma) \partial_c] C\} tr\{\bar{C} [-im\gamma^{b'}(\varsigma) - 2S^{b'c'}(e, \varsigma) \partial^+_{'c'}] \gamma_{a'}(\varsigma) C\} (\eta_{bb'} - \frac{\partial_b \partial^+_{b'}}{m^2}) \Delta(x - x')$

$= \frac{i}{16} tr\{\gamma_a(\varsigma) [im\gamma^b(\varsigma) - 2S^{bc}(e, \varsigma) \partial_c]\} tr\{[-im\gamma^{b'}(\varsigma) - 2S^{b'c'}(e, \varsigma) \partial^+_{'c'}] \gamma_{a'}(\varsigma)\} (\eta_{bb'} - \frac{\partial_b \partial^+_{b'}}{m^2}) \Delta(x - x')$

$= \frac{i}{16} m^2 tr[\gamma_a(\varsigma) \gamma^b(\varsigma)] tr[\gamma^{b'}(\varsigma) \gamma_{a'}(\varsigma)] (\eta_{bb'} - \frac{\partial_b \partial^+_{b'}}{m^2}) \Delta(x - x')$

$= im^2 \delta^b_a \delta^{b'}_{a'} (\eta_{bb'} - \frac{\partial_b \partial^+_{b'}}{m^2}) \Delta(x - x')$

$= i(m^2 \eta_{aa'} - \partial_a \partial^+_{a'}) \Delta(x - x')$ □

Proof: $[F_{ab}(x), F^+_{a'b'}(x')] = [iF_{ab}(x), -iF^+_{a'b'}(x')]$

$= [tr[\bar{C} S_{ab}(e, \varsigma) \psi_{[\lambda_\varsigma \mu_\varsigma]}(x)], tr^+[\bar{C} S_{a'b'}(e, \varsigma)(\varsigma) \psi_{[\lambda_\varsigma \mu_\varsigma]}(x')]]$

$= [[\bar{C} S_{ab}(e, \varsigma)]^{\lambda_\varsigma \mu_\varsigma} \psi_{\lambda_\varsigma \mu_\varsigma}(x), [S_{a'b'}(e, \varsigma)(\varsigma) C]^{\lambda'_\varsigma \mu'_\varsigma} \psi^+_{\lambda'_\varsigma \mu'_\varsigma}(x')]$

$= [\bar{C} S_{ab}(e, \varsigma)]^{\lambda_\varsigma \mu_\varsigma} [S_{a'b'}(e, \varsigma) C]^{\lambda'_\varsigma \mu'_\varsigma} [\psi_{\lambda_\varsigma \mu_\varsigma}(x), \psi_{\lambda'_\varsigma \mu'_\varsigma}(x')]$

$= \frac{i}{4} [\bar{C} S_{ab}(e, \varsigma)]^{\lambda_\varsigma \mu_\varsigma} [S_{a'b'}(e, \varsigma) C]^{\lambda'_\varsigma \mu'_\varsigma} \mathbb{X}^c_{\lambda_\varsigma \mu_\varsigma}(x) \mathbb{X}^{+c'}_{\lambda'_\varsigma \mu'_\varsigma}(x') (\eta_{cc'} - \frac{\partial_c \partial^+_{c'}}{m^2}) \Delta(x - x')$

$= \frac{i}{4} tr[\bar{C} S_{ab}(e, \varsigma) \mathbb{X}^c(x)] tr[\mathbb{X}^{+c'}(x') S_{a'b'}(e, \varsigma) C] (\eta_{cc'} - \frac{\partial_c \partial^+_{c'}}{m^2}) \Delta(x - x')$

$= \frac{i}{4} tr\{\bar{C} S_{ab}(e, \varsigma) [im\gamma^c(\varsigma) - 2S^{cd}(e, \varsigma) \partial_d] C\} tr\{\bar{C} [-im\gamma^{c'}(\varsigma) - 2S^{c'd'}(e, \varsigma) \partial^+_{'d'}] S_{a'b'}(e, \varsigma) C\}$

$(\eta_{cc'} - \frac{\partial_c \partial^+_{c'}}{m^2}) \Delta(x - x')$

$= \frac{i}{4} tr\{S_{ab}(e, \varsigma) [im\gamma^c(\varsigma) - 2S^{cd}(e, \varsigma) \partial_d]\} tr\{[-im\gamma^{c'}(\varsigma) - 2S^{c'd'}(e, \varsigma) \partial^+_{'d'}] S_{a'b'}(e, \varsigma)\} (\eta_{cc'} - \frac{\partial_c \partial^+_{c'}}{m^2}) \Delta(x - x')$

$= -itr[S_{ab}(e, \varsigma) S^{cd}(e, \varsigma)] tr[S^{c'd'}(e, \varsigma) S_{a'b'}(e, \varsigma)] \partial_d \partial^+_{d'} (\eta_{cc'} - \frac{\partial_c \partial^+_{c'}}{m^2}) \Delta(x - x')$

$= -itr[S_{ab}(e, \varsigma) S^{cd}(e, \varsigma)] tr[S^{c'd'}(e, \varsigma) S_{a'b'}(e, \varsigma)] \eta_{cc'} \partial_d \partial^+_{d'} \Delta(x - x')$

$= -iS_{abcd} S_{a'b'c'd'} \eta^{cc'} \partial^d \partial^{d'}_+ \Delta(x - x')$

$= -i(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc})(\delta_{a'c'} \delta_{b'd'} - \delta_{a'd'} \delta_{b'c'}) \eta^{cc'} \partial^d \partial^{d'}_+ \Delta(x - x')$

$= -i\eta_{[a<a'} \partial_{b]} \partial^+_{b'>} \Delta(x - x')$ □

5.3.2 Complex field condition

Thm. 5.3.2.

$$\begin{cases}[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi_{\eta_\varsigma\xi_\varsigma}(x')]=0,[\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x),\psi^+_{\eta'_\varsigma\xi'_\varsigma}(x')]=0,,\psi_{\lambda_\varsigma\mu_\varsigma}=\psi_{\mu_\varsigma\lambda_\varsigma}\\ im\frac{A_a}{2}(x)=\frac{1}{4}tr\{\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\},i\frac{F_{ab}}{2}(x)=\frac{1}{2}tr\{\bar{C}S_{ab}(e,\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\}\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}[A_a(x),A_b(x')]=0,[F_{ab}(x),F_{cd}(x')]=0;[A^+_{a'}(x),A^+_{b'}(x')]=0,[F^+_{a'b'}(x),F^+_{c'd'}(x')]=0\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}=[im\gamma^a(\varsigma)C\frac{A_a}{2}+S^{ab}(e,\varsigma)C\frac{F_{ab}}{2}]\end{cases}$$

5.3.3 Complete commutation rules for complex fields

Thm. 5.3.3.

$$\begin{cases}[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')\\ [\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi_{\eta_\varsigma\xi_\varsigma}(x')]=0,[\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x),\psi^+_{\eta'_\varsigma\xi'_\varsigma}(x')]=0,\psi_{\lambda_\varsigma\mu_\varsigma}=\psi_{\mu_\varsigma\lambda_\varsigma}\\ im\frac{A_a}{2}(x)=\frac{1}{4}tr\{\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\},i\frac{F_{ab}}{2}(x)=\frac{1}{2}tr\{\bar{C}S_{ab}(e,\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\}\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}[A_a(x),A^+_{a'}(x')]=i(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x'),[F_{ab}(x),F^+_{a'b'}(x')]=-i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')\\ [A_a(x),A_b(x')]=0,[F_{ab}(x),F_{cd}(x')]=0;[A^+_{a'}(x),A^+_{b'}(x')]=0,[F^+_{a'b'}(x),F^+_{c'd'}(x')]=0\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}=[im\gamma^a(\varsigma)C\frac{A_a}{2}+S^{ab}(e,\varsigma)C\frac{F_{ab}}{2}]\end{cases}$$

5.3.4 Majorana real field condition

Thm. 5.3.4.

$$\begin{cases}\psi=\gamma_2\psi^+\gamma_2,\psi_{\lambda_\varsigma\mu_\varsigma}=\psi_{\mu_\varsigma\lambda_\varsigma}\\ im\frac{A_a}{2}(x)=\frac{1}{4}tr\{\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\}\\ i\frac{F_{ab}}{2}(x)=\frac{1}{2}tr\{\bar{C}S_{ab}(e,\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\}\end{cases}\Leftrightarrow\begin{cases}A_a=A^+_{a'}\eta^{a'}_a,F_{ab}=F^+_{a'b'}\eta^{a'}_a\eta^{b'}_b\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}=[im\gamma^a(\varsigma)C\frac{A_a}{2}+S^{ab}(e,\varsigma)C\frac{F_{ab}}{2}]\end{cases}$$

5.3.5 Complete commutation rules for Majorana real fields

Thm. 5.3.5.

$$\begin{cases}[\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')\\ [\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\delta_{aa'}-\frac{\partial_a\partial_{a'}}{m^2})\Delta(x-x')\\ [\psi^+_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]=\frac{i}{4}\mathbb{X}^{+a}_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\delta_{aa'}-\frac{\partial^+_a\partial^+_{a'}}{m^2})\Delta(x-x')\\ \psi=\gamma_2\psi^+\gamma_2,\psi_{\lambda_\varsigma\mu_\varsigma}=\psi_{\mu_\varsigma\lambda_\varsigma}\\ im\frac{A_a}{2}(x)=\frac{1}{4}tr\{\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\},i\frac{F_{ab}}{2}(x)=\frac{1}{2}tr\{\bar{C}S_{ab}(e,\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\}\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}[A_a(x),A^+_{a'}(x')]=i(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x'),[F_{ab}(x),F^+_{a'b'}(x')]=-i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')\\ [A_a(x),A_b(x')]=i(\delta_{ab}-\frac{\partial_a\partial_b}{m^2})\Delta(x-x'),[F_{ab}(x),F_{cd}(x')]=-i\delta_{[a<c}\partial_{b]}\partial_{d>}\Delta(x-x')\\ [A^+_{a'}(x),A^+_{b'}(x')]=i(\delta_{a'b'}-\frac{\partial^+_{a'}\partial^+_{b'}}{m^2})\Delta(x-x'),[F^+_{a'b'}(x),F^+_{c'd'}(x')]=-i\delta_{[a'<c'}\partial^+_{b']}\partial^+_{d'>}\Delta(x-x')\\ A_a=A^+_{a'}\eta^{a'}_a,F_{ab}=F^+_{a'b'}\eta^{a'}_a\eta^{b'}_b\\ \psi_{[\lambda_\varsigma\mu_\varsigma]}=[im\gamma^a(\varsigma)C\frac{A_a}{2}+S^{ab}(e,\varsigma)C\frac{F_{ab}}{2}]\end{cases}$$

5.4 Derive F_{ab} commutative relation from A_a

5.4.1 Common commutation rules for complex and real fields

Thm. 5.4.1. $\begin{cases}[A_a(x),A^+_{a'}(x')]=i(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')\\ F_{ab}:=\partial_aA_b-\partial_bA_a\end{cases}\Rightarrow[F_{ab}(x),F^+_{a'b'}(x')]=-i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')$

5.4.2 Complex field condition

Thm. 5.4.2. $\begin{cases}[A_a(x),A_b(x')]=0,[A^+_{a'}(x),A^+_{b'}(x')]=0\\ F_{ab}:=\partial_aA_b-\partial_bA_a\end{cases}\Rightarrow\begin{cases}[F_{ab}(x),F_{cd}(x')]=0\\ [F^+_{a'b'}(x),F^+_{c'd'}(x')]=0\end{cases}$

5.4.3 Complete commutation rules for complex fields

Thm. 5.4.3. $\begin{cases}[A_a(x),A^+_{a'}(x')]=i(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')\\ [A_a(x),A_b(x')]=0,[A^+_{a'}(x),A^+_{b'}(x')]=0\\ F_{ab}:=\partial_aA_b-\partial_bA_a\end{cases}\Rightarrow\begin{cases}[F_{ab}(x),F^+_{a'b'}(x')]=-i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')\\ [F_{ab}(x),F_{cd}(x')]=0,[F^+_{a'b'}(x),F^+_{c'd'}(x')]=0\end{cases}$

5.4.4 Majorana real field condition

Thm. 5.4.4. $A_a=A^+_{a'}\eta^{a'}_a,F_{ab}:=\partial_aA_b-\partial_bA_a\Rightarrow F_{ab}=F^+_{a'b'}\eta^{a'}_a\eta^{b'}_b$

5.4.5 Complete commutation rules for Majorana real field

Thm. 5.4.5. $\begin{cases}[A_a(x), A^+_{a'}(x')] = i(\eta_{aa'} - \frac{\partial_a \partial^+_{a'}}{m^2})\Delta(x-x') \\ [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\partial_a \partial_b}{m^2})\Delta(x-x') \\ [A^+_{a'}(x), A^+_{b'}(x')] = i(\delta_{a'b'} - \frac{\partial^+_{a'}\partial^+_{b'}}{m^2})\Delta(x-x') \\ A_a = A^+_{a'}\eta^{a'}_a, F_{ab} := \partial_a A_b - \partial_b A_a \end{cases} \Rightarrow \begin{cases}[F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x') \\ [F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c}\partial_{b]}\partial_{d>}\Delta(x-x') \\ [F^+_{a'b'}(x), F^+_{c'd'}(x')] = -i\delta_{[a'<c'}\partial^+_{b']}\partial^+_{d'>}\Delta(x-x') \\ F_{ab} = F^+_{a'b'}\eta^{a'}_a\eta^{b'}_b \end{cases}$

5.5 Equivalence commutative relations of ψ_{α_ς} and F_{ab}

5.5.1 Common commutation rules for complex and real fields

Thm. 5.5.1.

$$\begin{cases}[F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x') \\ \psi_{\alpha_\varsigma} := -\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}\end{cases} \Leftrightarrow \begin{cases}[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x') \\ [\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_{-\varsigma}}(x')] = -\frac{i}{2}m^2\delta_{\alpha_\varsigma\alpha'_{-\varsigma}}\Delta(x-x') \\ F_{ab} = \frac{1}{\sqrt{2}}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma} + \sigma^{\alpha_{-\varsigma}}_{-\varsigma ab}\psi_{\alpha_{-\varsigma}})\end{cases}$$

Proof: $[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')]$
$= [-\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}(x), \frac{1}{2\sqrt{2}}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}F^+_{a'b'}(x')]$
$= -\frac{1}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}[F_{ab}(x), F^+_{a'b'}(x')]$
$= \frac{i}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')$
$= \frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}\eta_{aa'}\partial_b\partial^+_{b'}\Delta(x-x')$
$= \frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{-\varsigma\alpha'_\varsigma}\delta_{aa'}\partial_b\partial_{b'}\Delta(x-x')$
$= i\sigma^{bb'}_{\alpha_\varsigma\alpha'_\varsigma}\partial_b\partial_{b'}\Delta(x-x')$
$= i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

5.5.2 Complex field condition

Thm. 5.5.2.
$$\begin{cases}[F_{ab}(x), F_{cd}(x')] = 0, [F^+_{a'b'}(x), F^+_{c'd'}(x')] = 0 \\ \psi_{\alpha_\varsigma} := -\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}\end{cases} \Leftrightarrow \begin{cases}[\psi_{\alpha_\varsigma}(x), \psi_{\beta_\kappa}(x')] = 0, [\psi^+_{\alpha'_\varsigma}(x), \psi^+_{\beta'_\kappa}(x')] = 0 \\ F_{ab} = \frac{1}{\sqrt{2}}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma} + \sigma^{\alpha_{-\varsigma}}_{-\varsigma ab}\psi_{\alpha_{-\varsigma}})\end{cases}$$

5.5.3 Complete commutation rules for complex fields

Thm. 5.5.3.

$$\begin{cases}[F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x') \\ [F_{ab}(x), F_{cd}(x')] = 0, [F^+_{a'b'}(x), F^+_{c'd'}(x')] = 0 \\ \psi_{\alpha_\varsigma} := -\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}\end{cases} \Leftrightarrow \begin{cases}[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x') \\ [\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_{-\varsigma}}(x')] = -\frac{i}{2}m^2\delta_{\alpha_\varsigma\alpha'_{-\varsigma}}\Delta(x-x') \\ [\psi_{\alpha_\varsigma}(x), \psi_{\beta_\kappa}(x')] = 0, [\psi^+_{\alpha'_\varsigma}(x), \psi^+_{\beta'_\kappa}(x')] = 0 \\ F_{ab} = \frac{1}{\sqrt{2}}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma} + \sigma^{\alpha_{-\varsigma}}_{-\varsigma ab}\psi_{\alpha_{-\varsigma}})\end{cases}$$

Thm. 5.5.4.
$$\begin{cases}[F_{ab}(x), A^+_{c'}(x')] = -i\eta_{c'[a}\partial_{b]}\Delta(x-x') \\ [F^+_{a'b'}(x), A_c(x')] = -i\eta_{c[a'}\partial^+_{b']}\Delta(x-x') \\ [F_{ab}(x), A_c(x')] = 0, [F^+_{a'b'}(x), A^+_{c'}(x')] = 0\end{cases} \quad \begin{cases}[\psi_{\alpha_\varsigma}(x), A^+_{c'}(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\varsigma}, -i\varsigma)^b|_{c'\alpha_\varsigma}\partial_b\Delta(x-x') \\ [\psi^+_{\alpha'_\varsigma}(x), A_c(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\varsigma}, -i\varsigma)^b|_{c\alpha'_\varsigma}\partial_b\Delta(x-x') \\ [\psi_{\alpha_\varsigma}(x), A_c(x')] = 0, [\psi^+_{\alpha'_\varsigma}(x), A^+_{c'}(x')] = 0\end{cases}$$

Proof: $[F_{ab}(x), A^+_c(x')] = [\partial_a A_b(x) - \partial_b A_a(x), A^+_c(x')]$
$= i(\eta_{bc} - \frac{\partial_b\partial^+_c}{m^2})\partial_a\Delta(x-x') - i(\eta_{ac} - \frac{\partial_a\partial^+_c}{m^2})\partial_b\Delta(x-x')$
$= -i\eta_{c[a}\partial_{b]}\Delta(x-x')$ □

Proof: $[\psi_{\alpha_\varsigma}(x), A^+_c(x')] = \frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}[F_{ab}(x), A^+_c(x')]$
$= -i\frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\eta_{c[a}\partial_{b]}\Delta(x-x')$
$= \frac{i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}\eta_{ca}\partial_b\Delta(x-x')$
$= -\frac{i}{\sqrt{2}}(\sigma_\varsigma)_{\alpha_\varsigma}|^{ab}\eta_{ca}\partial_b\Delta(x-x'), [(\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}|_{ab} = (\sigma_{-\varsigma}, -i\varsigma)_a|_b{}^{\alpha_\varsigma}]$
$= -\frac{i}{\sqrt{2}}(\sigma_{-\varsigma}, -i\varsigma)^b|^a{}_{\alpha_\varsigma}\eta_{ca}\partial_b\Delta(x-x')$
$= -\frac{i}{\sqrt{2}}(\sigma_{+\varsigma}, -i\varsigma)^b|_{c\alpha_\varsigma}\partial_b\Delta(x-x')$ □

5.5.4 Partial isochronous commutation rules for complex fields

Cor. 5.5.1.

$$\begin{cases}[E_i(\vec{r},t),A^+_{c'}(\vec{r}',t)]=i\eta_{ic'}\delta^3(\vec{r}-\vec{r}'),[E^+_{i'}(\vec{r},t),A_c(\vec{r}',t)]=i\eta_{i'c}\delta^3(\vec{r}-\vec{r}')\\ [E_i(\vec{r},t),A_c(\vec{r}',t)]=0,[E^+_{i'}(\vec{r},t),A^+_{c'}(\vec{r}',t)]=0\\ [B_i(\vec{r},t),A^+_{c'}(\vec{r}',t)]=0,[B^+_{i'}(\vec{r},t),A_c(\vec{r}',t)]=0\\ [B_i(\vec{r},t),A_c(\vec{r}',t)]=0,[B^+_{i'}(\vec{r},t),A^+_{c'}(\vec{r}',t)]=0\end{cases}$$

Cor. 5.5.2.

$$\begin{cases}[E_i(\vec{r},t),A^+_j(\vec{r}',t)]=i\delta_{ij}\delta^3(\vec{r}-\vec{r}'),[E^+_i(\vec{r},t),A_j(\vec{r}',t)]=i\delta_{ij}\delta^3(\vec{r}-\vec{r}')\\ [E_i(\vec{r},t),A_j(\vec{r}',t)]=0,[E^+_i(\vec{r},t),A^+_j(\vec{r}',t)]=0\\ [E_i(\vec{r},t),E^+_j(\vec{r}',t)]=0,[E^+_i(\vec{r},t),E_j(\vec{r}',t)]=0,[E_i(\vec{r},t),E_j(\vec{r}',t)]=0,[E^+_i(\vec{r},t),E^+_j(\vec{r}',t)]=0\\ [A_i(\vec{r},t),A^+_j(\vec{r}',t)]=0,[A^+_i(\vec{r},t),A_j(\vec{r}',t)]=0,[A_i(\vec{r},t),A_j(\vec{r}',t)]=0,[A^+_i(\vec{r},t),A^+_j(\vec{r}',t)]=0\end{cases}$$

$[\Downarrow]$ $[\Downarrow]$

Cor. 5.5.3.

$$\begin{cases}[E_i(\vec{r},t),B^+_j(\vec{r}',t)]=-i\varepsilon_{ij}{}^k\partial_k\delta^3(\vec{r}-\vec{r}'),[E^+_i(\vec{r},t),B_j(\vec{r}',t)]=-i\varepsilon_{ij}{}^k\partial_k\delta^3(\vec{r}-\vec{r}')\\ [E_i(\vec{r},t),B_j(\vec{r}',t)]=0,[E^+_i(\vec{r},t),B^+_j(\vec{r}',t)]=0\\ [B_i(\vec{r},t),B^+_j(\vec{r}',t)]=0,[B^+_i(\vec{r},t),B_j(\vec{r}',t)]=0,[B_i(\vec{r},t),B_j(\vec{r}',t)]=0,[B^+_i(\vec{r},t),B^+_j(\vec{r}',t)]=0\\ [B_i(\vec{r},t),A^+_j(\vec{r}',t)]=0,[B^+_i(\vec{r},t),A_j(\vec{r}',t)]=0,[B_i(\vec{r},t),A_j(\vec{r}',t)]=0,[B^+_i(\vec{r},t),A^+_j(\vec{r}',t)]=0\end{cases}$$

5.5.5 Majorana real field condition

Thm. 5.5.5.

$$\begin{cases}F_{ab}=F^+_{a'b'}\eta^{a'}_a\eta^{b'}_b\\ \psi_{\alpha_\varsigma}:=-\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}\end{cases}\Leftrightarrow\begin{cases}\psi_{\alpha_{-\varsigma}}(x)=-\psi^+_{\alpha_\varsigma}(x)\\ F_{ab}=\frac{1}{\sqrt{2}}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}-\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\psi^+_{\alpha'_\varsigma})\end{cases}$$

5.5.6 Complete commutation rules for Majorana real field

Lem. 5.5.1.

$$2\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma}\partial_c\partial_{c'}=\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha'_\varsigma c'd'}\eta^{cc'}\partial^d\partial^{+d'}=(S_{abcd}-\varsigma\varepsilon_{abcd})(S_{a'b'c'd'}-\varsigma\varepsilon_{a'b'c'd'})\eta^{cc'}\partial^d\partial^{+d'}$$
$$2\sigma^{\alpha_\varsigma}_{-\varsigma ab}\sigma^{\alpha'_\varsigma}_{-\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma}\partial_c\partial_{c'}=\sigma^{\alpha_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma cd}\sigma^{\alpha'_\varsigma}_{-\varsigma a'b'}\sigma_{-\varsigma\alpha'_\varsigma c'd'}\eta^{cc'}\partial^d\partial^{+d'}=(S_{abcd}+\varsigma\varepsilon_{abcd})(S_{a'b'c'd'}+\varsigma\varepsilon_{a'b'c'd'})\eta^{cc'}\partial^d\partial^{+d'}$$

Proof: $=-\frac{i}{2}\{(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma}+\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{c'c}_{\alpha_\varsigma\alpha'_\varsigma})\partial_c\partial_{c'}\}\Delta(x-x')$
$=-\frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{-\varsigma\alpha'_\varsigma c'd'}\delta^{dd'}+\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd}\delta^{dd'})\partial^c\partial^{c'}]\Delta(x-x')$
$=-\frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{\varsigma\alpha'_\varsigma c'd'}+\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd})\eta^{dd'}\partial^c\partial^{+c'}]\Delta(x-x')$
$=-\frac{i}{4}\{[(-S_{abcd}+\varsigma\varepsilon_{abcd})(-S_{a'b'c'd'}+\varsigma\varepsilon_{a'b'c'd'})+(-S_{abcd}-\varsigma\varepsilon_{abcd})(-S_{a'b'c'd'}-\varsigma\varepsilon_{a'b'c'd'})]\eta^{dd'}\partial^c\partial^{+c'}\}$
$\Delta(x-x')$
$=-\frac{i}{2}[(S_{abcd}S_{a'b'c'd'}+\varepsilon_{abcd}\varepsilon_{a'b'c'd'})\eta^{dd'}\partial^c\partial^{+c'}]\Delta(x-x')$
$=-\frac{i}{2}\{[(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})(\delta_{a'c'}\delta_{b'd'}-\delta_{a'd'}\delta_{b'c'})+\varepsilon_{abcd}\varepsilon_{a'b'c'd'}]\eta^{dd'}\partial^c\partial^{+c'}\}\Delta(x-x')$
$=-\frac{i}{2}[(\delta_{a[c}\delta_{bd]}\delta_{a'[c'}\delta_{b'd']}+\delta_{a[a'}\delta_{bb'}\delta_{cc'}\delta_{dd']})\eta^{dd'}\partial^c\partial^{+c'}+m^2\delta_{a[c}\delta_{bd]}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$=-i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')$ □

Proof: $=-\frac{i}{2}\{(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma}+\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{c'c}_{\alpha_\varsigma\alpha'_\varsigma})\partial_c\partial_{c'}\}\Delta(x-x')$
$=-\frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{-\varsigma\alpha'_\varsigma c'd'}\delta^{dd'}+\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd}\delta^{dd'})\partial^c\partial^{c'}]\Delta(x-x')$
$=-\frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{\varsigma\alpha'_\varsigma c'd'}+\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd})\eta^{dd'}\partial^c\partial^{+c'}]\Delta(x-x')$
$=-\frac{i}{4}\{[(-S_{abcd}+\varsigma\varepsilon_{abcd})(-S_{a'b'c'd'}+\varsigma\varepsilon_{a'b'c'd'})+(-S_{abcd}-\varsigma\varepsilon_{abcd})(-S_{a'b'c'd'}-\varsigma\varepsilon_{a'b'c'd'})]\eta^{dd'}\partial^c\partial^{+c'}\}$
$\Delta(x-x')$
$=-\frac{i}{2}[(S_{abcd}S_{a'b'c'd'}+\varepsilon_{abcd}\varepsilon_{a'b'c'd'})\eta^{dd'}\partial^c\partial^{+c'}]\Delta(x-x')$
$=-\frac{i}{2}\{[(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})(\delta_{a'c'}\delta_{b'd'}-\delta_{a'd'}\delta_{b'c'})+\varepsilon_{abcd}\varepsilon_{a'b'c'd'}]\eta^{dd'}\partial^c\partial^{+c'}\}\Delta(x-x')$
$=-\frac{i}{2}[(\delta_{a[c}\delta_{bd]}\delta_{a'[c'}\delta_{b'd']}+\delta_{a[a'}\delta_{bb'}\delta_{cc'}\delta_{dd']})\eta^{dd'}\partial^c\partial^{+c'}+m^2\delta_{a[c}\delta_{bd]}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$=-i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')$ □

Thm. 5.5.6.

$$\begin{cases}[F_{ab}(x),F^+_{a'b'}(x')]=-i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')\\ [F_{ab}(x),F_{cd}(x')]=-i\delta_{[a<c}\partial_{b]}\partial_{d>}\Delta(x-x')\\ [F^+_{a'b'}(x),F^+_{c'd'}(x')]=-i\delta_{[a'<c'}\partial^+_{b']}\partial^+_{d'>}\Delta(x-x')\\ F_{ab}=F^+_{a'b'}\eta^{a'}_a\eta^{b'}_b,\psi_{\alpha_\varsigma}:=-\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}\end{cases}\Leftrightarrow\begin{cases}[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')]=\frac{i}{2}m^2\delta_{\alpha_\varsigma\beta_\varsigma}\Delta(x-x')\\ [\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]=\frac{i}{2}m^2\delta_{\alpha'_\varsigma\beta'_\varsigma}\Delta(x-x')\\ \psi_{\alpha_{-\varsigma}}(x)=-\psi^+_{\alpha_\varsigma}(x),F_{ab}=\frac{1}{\sqrt{2}}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}-\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\psi^+_{\alpha'_\varsigma})\end{cases}$$

Proof: $[\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')]$
$= [-\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}(x), -\frac{1}{2\sqrt{2}}\sigma^{cd}_{\varsigma\beta_\varsigma}F_{cd}(x')]$
$= \frac{1}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}[F_{ab}(x),F_{cd}(x')]$
$= -\frac{i}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}\delta_{[a<c}\partial_{b]}\partial_{d>}\Delta(x-x')$
$= -\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{cd}_{\varsigma\beta_\varsigma}\delta_{ac}\partial_b\partial_d\Delta(x-x')$
$= \frac{i}{2}[\delta^{bd}\delta_{\alpha_\varsigma\beta_\varsigma} - \sigma^{bd}_{\varsigma\gamma_\varsigma}\gamma^{\gamma_\varsigma}{}_{\alpha_\varsigma\beta_\varsigma}]\partial_b\partial_d\Delta(x-x')$
$= \frac{i}{2}\delta_{\alpha_\varsigma\beta_\varsigma}\partial_a\partial^a\Delta(x-x')$
$= \frac{i}{2}m^2\delta_{\alpha_\varsigma\beta_\varsigma}\Delta(x-x')$ □

Proof: $[\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]$
$= [\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha'_\varsigma}F^+_{ab}(x), \frac{1}{2\sqrt{2}}\sigma^{cd}_{\varsigma\beta'_\varsigma}F^+_{cd}(x')]$
$= [\frac{1}{2\sqrt{2}}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}F_{ab}(x), \frac{1}{2\sqrt{2}}\sigma^{cd}_{-\varsigma\beta'_\varsigma}F_{cd}(x')]$
$= \frac{1}{8}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}\sigma^{cd}_{-\varsigma\beta'_\varsigma}[F_{ab}(x),F_{cd}(x')]$
$= -\frac{i}{8}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}\sigma^{cd}_{-\varsigma\beta'_\varsigma}\delta_{[a<c}\partial_{b]}\partial_{d>}\Delta(x-x')$
$= -\frac{i}{2}\sigma^{ab}_{-\varsigma\alpha'_\varsigma}\sigma^{cd}_{-\varsigma\beta'_\varsigma}\delta_{ac}\partial_b\partial_d\Delta(x-x')$
$= \frac{i}{2}[\delta^{bd}\delta_{\alpha'_\varsigma\beta'_\varsigma} - \sigma^{bd}_{-\varsigma\gamma'_\varsigma}\gamma^{\gamma'_\varsigma}{}_{\alpha'_\varsigma\beta'_\varsigma}]\partial_b\partial_d\Delta(x-x')$
$= \frac{i}{2}\delta_{\alpha'_\varsigma\beta'_\varsigma}\partial_a\partial^a\Delta(x-x')$
$= \frac{i}{2}m^2\delta_{\alpha'_\varsigma\beta'_\varsigma}\Delta(x-x')$ □

Proof: $[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]$
$= [-\frac{1}{2\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{ab}(x), \frac{1}{2\sqrt{2}}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}F^+_{a'b'}(x')]$
$= -\frac{1}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}[F_{ab}(x),F^+_{a'b'}(x')]$
$= \frac{i}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')$
$= \frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}\eta_{aa'}\partial_b\partial^+_{b'}\Delta(x-x')$
$= \frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{-\varsigma\alpha'_\varsigma}\delta_{aa'}\partial_b\partial_{b'}\Delta(x-x')$
$= i\sigma^{bb'}_{\alpha_\varsigma\alpha'_\varsigma}\partial_b\partial_{b'}\Delta(x-x')$
$= i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

Proof: $[F_{ab}(x),F^+_{a'b'}(x')]$
$= [\frac{1}{\sqrt{2}}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}(x) - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\psi^+_{\alpha'_\varsigma}(x)), -\frac{1}{\sqrt{2}}(\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\psi^+_{\alpha'_\varsigma}(x') - \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\psi_{\alpha_\varsigma}(x'))]$
$= -\frac{1}{2}\{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')] + \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x),\psi_{\alpha_\varsigma}(x')]$
$- \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{-\varsigma a'b'}[\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')] - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]\}$
$= -\frac{1}{2}\{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')] - \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}[\psi_{\alpha_\varsigma}(x'),\psi^+_{\alpha'_\varsigma}(x)]$
$- \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{-\varsigma a'b'}[\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')] - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]\}$
$= -\frac{1}{2}\{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}i\sigma^{cd}_{\alpha_\varsigma\alpha'_\varsigma}\partial_c\partial_d + \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}i\sigma^{cd}_{\alpha_\varsigma\alpha'_\varsigma}\partial'_c\partial'_d - \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{-\varsigma a'b'}\frac{i}{2}m^2\delta_{\alpha_\varsigma\beta_\varsigma} - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{\varsigma a'b'}\frac{i}{2}m^2\delta_{\alpha'_\varsigma\beta'_\varsigma}\}\Delta(x-x')$
$= -\frac{i}{2}\{(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma} + \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{c'c}_{\alpha_\varsigma\alpha'_\varsigma})\partial_c\partial_{c'} - \frac{1}{2}m^2(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{\varsigma cd} + \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{-\varsigma cd})\eta^c_{a'}\eta^d_{b'}\}\Delta(x-x')$
$= -\frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{-\varsigma\alpha'_\varsigma c'd'}\delta^{dd'} + \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd}\delta^{dd'})\partial^c\partial^{c'} + 2m^2S_{abcd}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$= -\frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{\varsigma\alpha'_\varsigma c'd'} + \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd})\eta^{dd'}\partial^c\partial^{+c'} + 2m^2S_{abcd}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$= -\frac{i}{4}\{[(-S_{abcd}+\varsigma\varepsilon_{abcd})(-S_{a'b'c'd'}+\varsigma\varepsilon_{a'b'c'd'}) + (-S_{abcd}-\varsigma\varepsilon_{abcd})(-S_{a'b'c'd'}-\varsigma\varepsilon_{a'b'c'd'})]\eta^{dd'}\partial^c\partial^{+c'} + 2m^2S_{abcd}\eta^c_{a'}\eta^d_{b'}\}$
$\Delta(x-x')$
$= -\frac{i}{2}[(S_{abcd}S_{a'b'c'd'} + \varepsilon_{abcd}\varepsilon_{a'b'c'd'})\eta^{dd'}\partial^c\partial^{+c'} + m^2S_{abcd}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$= -\frac{i}{2}\{[(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})(\delta_{a'c'}\delta_{b'd'}-\delta_{a'd'}\delta_{b'c'}) + \varepsilon_{abcd}\varepsilon_{a'b'c'd'}]\eta^{dd'}\partial^c\partial^{+c'} + m^2(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})\eta^c_{a'}\eta^d_{b'}\}\Delta(x-x')$
$= -\frac{i}{2}[(\delta_{a[c}\delta_{bd]}\delta_{a'[c'}\delta_{b'd']} + \delta_{a[a'}\delta_{bb'}\delta_{cc'}\delta_{dd']})\eta^{dd'}\partial^c\partial^{+c'} + m^2\delta_{a[c}\delta_{bd]}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$= -i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')$ □

Thm. 5.5.7.

$$\begin{cases}[F_{ab}(x),A^+_{c'}(x')] = -i\eta_{c'[a}\partial_{b]}\Delta(x-x')\\ [F^+_{a'b'}(x),A_c(x')] = -i\eta_{c[a'}\partial^+_{b']}\Delta(x-x')\\ [F_{ab}(x),A_c(x')] = -i\delta_{c[a}\partial_{b]}\Delta(x-x')\\ [F^+_{a'b'}(x),A^+_{c'}(x')] = -i\delta_{c'[a'}\partial^+_{b']}\Delta(x-x')\end{cases}
\qquad
\begin{cases}[\psi_{\alpha_\varsigma}(x),A^+_{c'}(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\varsigma},-i\varsigma)^b|_{c'\alpha_\varsigma}\partial_b\Delta(x-x')\\ [\psi^+_{\alpha'_\varsigma}(x),A_c(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\varsigma},-i\varsigma)^b|_{c\alpha'_\varsigma}\partial_b\Delta(x-x')\\ [\psi_{\alpha_\varsigma}(x),A_c(x')] = -\frac{i}{\sqrt{2}}(\sigma_{-\varsigma},-i\varsigma)^b|_{c\alpha_\varsigma}\partial_b\Delta(x-x')\\ [\psi^+_{\alpha'_\varsigma}(x),A^+_{c'}(x')] = -\frac{i}{\sqrt{2}}(\sigma_{-\varsigma},-i\varsigma)^b|_{c'\alpha'_\varsigma}\partial_b\Delta(x-x')\end{cases}$$

Proof: $[F_{ab}(x), A_c(x')] = [\partial_a A_b(x) - \partial_b A_a(x), A_c(x')]$
$= i(\delta_{bc} - \frac{\partial_b \partial_c}{m^2})\partial_a \Delta(x-x') - i(\delta_{ac} - \frac{\partial_a \partial_c}{m^2})\partial_b \Delta(x-x')$
$= -i\delta_{c[a}\partial_{b]}\Delta(x-x')$ □

Proof: $[\psi_{\alpha_\varsigma}(x), A_c(x')] = \frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}[F_{ab}(x), A_c(x')]$
$= -i\frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\delta_{c[a}\partial_{b]}\Delta(x-x')$
$= \frac{i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}\delta_{ca}\partial_b\Delta(x-x')$
$= -\frac{i}{\sqrt{2}}(\sigma_\varsigma)_{\alpha_\varsigma}|^b{}_c\partial_b\Delta(x-x'), [(\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma}|_{ab} = (\sigma_{-\varsigma}, -i\varsigma)_a|_b{}^{\alpha_\varsigma}]$
$= -\frac{i}{\sqrt{2}}(\sigma_{-\varsigma}, -i\varsigma)^b|_{c\alpha_\varsigma}\partial_b\Delta(x-x')$ □

5.5.7 Partial isochronous commutation rules for Majorana real fields

Cor. 5.5.4.
$$\begin{cases}[E_i(\vec r,t), A^+_{c'}(\vec r',t)] = i\eta_{ic'}\delta^3(\vec r-\vec r'), [E_i(\vec r,t), A_c(\vec r',t)] = i\delta_{ic}\delta^3(\vec r-\vec r')\\ [B_i(\vec r,t), A^+_{c'}(\vec r',t)] = 0, [B_i(\vec r,t), A_c(\vec r',t)] = 0\end{cases}$$

Cor. 5.5.5.
$$\begin{cases}[E_i(\vec r,t), A_j(\vec r',t)] = i\delta_{ij}\delta^3(\vec r-\vec r')\\ [E_i(\vec r,t), E_j(\vec r',t)] = 0, [A_i(\vec r,t), A_j(\vec r',t)] = 0\end{cases} [\Rightarrow] \begin{cases}[E_i(\vec r,t), B_j(\vec r',t)] = -i\varepsilon_{ij}{}^k\partial_k\delta^3(\vec r-\vec r')\\ [B_i(\vec r,t), A_j(\vec r',t)] = 0\end{cases}$$

5.6 Equivalent commutative relations of ψ_{α_ς} and $\psi_{A_\varsigma B_\varsigma}$

5.6.1 Common commutation rules for complex and real fields

Thm. 5.6.1.
$$\begin{cases}[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ \psi_{A_\varsigma B_\varsigma} := \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma}\end{cases} \Leftrightarrow \begin{cases}[\psi_{A_\varsigma B_\varsigma}(x), \psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ = -\frac{i}{2}(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ \psi_{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}\end{cases}$$

Proof: $[\psi_{A_\varsigma B_\varsigma}(x), \psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= [\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma}(x), \frac{-i\varsigma}{\sqrt{2}}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\psi_{\alpha'_\varsigma}(x')]$
$= \frac{1}{2}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}[\psi_{\alpha_\varsigma}(x), \psi_{\alpha'_\varsigma}(x')]$
$= \frac{1}{2}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= \frac{i}{2}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^a{}_{C_\varsigma C'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\frac{-i\varsigma}{\sqrt{2}}\sigma^{C'_\varsigma D'_\varsigma}_{\alpha'_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{8}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\alpha_\varsigma}\sigma^{\alpha'_\varsigma}_{A'_\varsigma B'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\alpha'_\varsigma}(\sigma, i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{8}\delta^{C_\varsigma}_{\{A_\varsigma}\delta^{D_\varsigma}_{B_\varsigma\}}\delta^{C'_\varsigma}_{(A'_\varsigma}\delta^{D'_\varsigma}_{B'_\varsigma)}(\sigma, i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma, i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{8}(\sigma, i\varsigma)^a{}_{\{A_\varsigma(A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma\}B'_\varsigma)}\partial_a\partial_b\Delta(x-x')$
$= -\frac{i}{2}(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

Proof: $[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')]$
$= [\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}(x), \frac{-i\varsigma}{\sqrt{2}}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= \frac{1}{2}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}[\psi_{A_\varsigma B_\varsigma}(x), \psi^+_{A'_\varsigma B'_\varsigma}(x')]$
$= -\frac{i}{4}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$= i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

5.6.2 Complex field condition

Thm. 5.6.2.
$$\begin{cases}[\psi_{\alpha_\varsigma}(x), \psi_{\beta_\varsigma}(x')] = 0, [\psi^+_{\alpha'_\varsigma}(x), \psi^+_{\beta'_\varsigma}(x')] = 0\\ \psi_{A_\varsigma B_\varsigma} := \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma}\end{cases} \Leftrightarrow \begin{cases}[\psi_{A_\varsigma B_\varsigma}(x), \psi_{C_\varsigma D_\varsigma}(x')] = 0, [\psi^+_{A'_\varsigma B'_\varsigma}(x), \psi^+_{C'_\varsigma D'_\varsigma}(x')] = 0\\ \psi_{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}\end{cases}$$

5.6.3 Complete commutation rules for complex fields

Thm. 5.6.3.
$$\begin{cases}[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] = i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{\alpha_\varsigma}(x), \psi_{\beta_\varsigma}(x')] = 0, [\psi^+_{\alpha'_\varsigma}(x), \psi^+_{\beta'_\varsigma}(x')] = 0\\ \psi_{A_\varsigma B_\varsigma} := \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma}\end{cases} \Leftrightarrow \begin{cases}[\psi_{A_\varsigma B_\varsigma}(x), \psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ = -\frac{i}{2}(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{A_\varsigma B_\varsigma}(x), \psi_{C_\varsigma D_\varsigma}(x')] = 0, [\psi^+_{A'_\varsigma B'_\varsigma}(x), \psi^+_{C'_\varsigma D'_\varsigma}(x')] = 0\\ \psi_{\alpha_\varsigma} = \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}\end{cases}$$

5.6.4 Majorana real field condition

Thm. 5.6.4.

$$\begin{cases}\psi_{\alpha_{-\varsigma}}(x)=-\psi^+_{\alpha_\varsigma}(x)\\ \psi_{A_\varsigma B_\varsigma}:=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma}\end{cases}\Leftrightarrow\begin{cases}\psi??=-\sigma_y\psi^+\sigma_y\\ \psi_{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}\end{cases}$$

5.6.5 Complete commutation rules for Majorana real fields

Thm. 5.6.5.

$$\begin{cases}[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')]=\frac{i}{2}m^2\delta_{\alpha_\varsigma\beta_\varsigma}\Delta(x-x')\\ [\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]=\frac{i}{2}m^2\delta_{\alpha'_\varsigma\beta'_\varsigma}\Delta(x-x')\\ \psi_{\alpha_{-\varsigma}}(x)=-\psi^+_{\alpha_\varsigma}(x),\psi_{A_\varsigma B_\varsigma}:=\frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_{\alpha_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ =-\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{A_\varsigma B_\varsigma}(x),\psi_{C_\varsigma D_\varsigma}(x')]=\frac{i}{8}m^2\varepsilon_{\{A_\varsigma(C_\varsigma}\varepsilon_{B_\varsigma\}D_\varsigma)}\Delta(x-x')\\ [\psi^+_{A'_\varsigma B'_\varsigma}(x),\psi^+_{C'_\varsigma D'_\varsigma}(x')]=\frac{i}{8}m^2\varepsilon_{\{A'_\varsigma(C'_\varsigma}\varepsilon_{B'_\varsigma\}D'_\varsigma)}\Delta(x-x')\\ \psi_{\alpha_\varsigma}=\frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\psi_{A_\varsigma B_\varsigma}\end{cases}$$

5.7 Equivalent commutative relations of ψ_{k_ς} and $\psi_{A_\varsigma B_\varsigma}$

5.7.1 Common commutation rules for complex and real fields

Thm. 5.7.1.

$$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ =-\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ \psi_{k_\varsigma}=\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi_{A_\varsigma B_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ \psi_{A_\varsigma B_\varsigma}=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

Proof: $[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]$

$$\begin{aligned}&=[\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi_{A_\varsigma B_\varsigma}(x),\Gamma^{A'_\varsigma B'_\varsigma}_{k'_\varsigma}(1)\psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ &=\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\Gamma^{A'_\varsigma B'_\varsigma}_{k'_\varsigma}(1)[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ &=-\frac{i}{2}\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\Gamma^{A'_\varsigma B'_\varsigma}_{k'_\varsigma}(1)(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ &=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\end{aligned}$$ □

Proof: $[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]$

$$\begin{aligned}&=[\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}(x),\Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)\psi^+_{k'_\varsigma}(x')]\\ &=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]\\ &=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ &=-\frac{i}{2}\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\Gamma^{k'_\varsigma}_{A'_\varsigma B'_\varsigma}(1)\Gamma^{C_\varsigma D_\varsigma}_{k_\varsigma}(1)\Gamma^{C'_\varsigma D'_\varsigma}_{k'_\varsigma}(1)(\sigma,i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ &=-\frac{i}{8}\delta^{\{C_\varsigma}_{A_\varsigma}\delta^{D_\varsigma\}}_{B_\varsigma}\delta^{(C'_\varsigma}_{A'_\varsigma}\delta^{D'_\varsigma)}_{B'_\varsigma}(\sigma,i\varsigma)^a{}_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^b{}_{D_\varsigma D'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ &=-\frac{i}{8}(\sigma,i\varsigma)^a{}_{\{A_\varsigma(A'_\varsigma}(\sigma,i\varsigma)^b{}_{B\}_\varsigma B'_\varsigma)}\partial_a\partial_b\Delta(x-x')\\ &=-\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\end{aligned}$$ □

5.7.2 Complex field condition

Thm. 5.7.2.

$$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi_{C_\varsigma D_\varsigma}(x')]=0,[\psi^+_{A'_\varsigma B'_\varsigma}(x),\psi^+_{C'_\varsigma D'_\varsigma}(x')]=0\\ \psi_{k_\varsigma}=\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi_{A_\varsigma B_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0,[\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\\ \psi_{A_\varsigma B_\varsigma}=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

5.7.3 Complete commutation rules for complex fields

Thm. 5.7.3.

$$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ =-\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{A_\varsigma B_\varsigma}(x),\psi_{C_\varsigma D_\varsigma}(x')]=0,[\psi^+_{A'_\varsigma B'_\varsigma}(x),\psi^+_{C'_\varsigma D'_\varsigma}(x')]=0\\ \psi_{k_\varsigma}=\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi_{A_\varsigma B_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0,[\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\\ \psi_{A_\varsigma B_\varsigma}=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

5.7.4 Majorana real field condition

5.7.5 Complete commutation rules for Majorana real fields

Thm. 5.7.4.

$$\begin{cases}[\psi_{A_\varsigma B_\varsigma}(x),\psi^+_{A'_\varsigma B'_\varsigma}(x')]\\ =-\frac{i}{2}(\sigma,i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{A_\varsigma B_\varsigma}(x),\psi_{C_\varsigma D_\varsigma}(x')]=\frac{i}{8}m^2\varepsilon_{\{A_\varsigma(C_\varsigma}\varepsilon_{B_\varsigma\}D_\varsigma)}\Delta(x-x')\\ [\psi^+_{A'_\varsigma B'_\varsigma}(x),\psi^+_{C'_\varsigma D'_\varsigma}(x')]=\frac{i}{8}m^2\varepsilon_{\{A'_\varsigma(C'_\varsigma}\varepsilon_{B'_\varsigma\}D'_\varsigma)}\Delta(x-x')\\ \psi_{k_\varsigma}=\Gamma^{A_\varsigma B_\varsigma}_{k_\varsigma}(1)\psi_{A_\varsigma B_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=\frac{i}{2}m^2\varepsilon_{k_\varsigma l_\varsigma}(1)\Delta(x-x')\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=\frac{i}{2}m^2\varepsilon_{k'_\varsigma l'_\varsigma}(1)\Delta(x-x')\\ \psi_{A_\varsigma B_\varsigma}=\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

5.8 Equivalent commutative relations of ψ_{α_ς} and ψ_{k_ς}

5.8.1 Common commutation rules for complex and real fields

Lem. 5.8.1. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)\Gamma^{ab}_{k_\varsigma k'_\varsigma},\Gamma^{ab}_{k_\varsigma k'_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}$

Thm. 5.8.1.

$$\begin{cases}[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ \psi_{k_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi_{\alpha_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ \psi_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

Proof: $[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]$
$=[\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi_{\alpha_\varsigma}(x),\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)\psi^+_{\alpha'_\varsigma}(x')]$
$=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]$
$=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\Gamma^{\alpha'_\varsigma}_{k'_\varsigma}(1)i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

Proof: $[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]$
$=[\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi_{k_\varsigma}(x),\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)\psi^+_{k'_\varsigma}(x')]$
$=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]$
$=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\Gamma^{k'_\varsigma}_{\alpha'_\varsigma}(1)i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')$
$=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')$ □

5.8.2 Complex field condition

Thm. 5.8.2.

$$\begin{cases}[\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')]=0,[\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]=0\\ \psi_{k_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi_{\alpha_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0,[\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\\ \psi_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

5.8.3 Complete commutation rules for complex fields

Thm. 5.8.3.

$$\begin{cases}[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')]=0,[\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]=0\\ \psi_{k_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi_{\alpha_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=0,[\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=0\\ \psi_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

5.8.4 Majorana real field condition

5.8.5 Complete commutation rules for Majorana real fields

Thm. 5.8.4.

$$\begin{cases}[\psi_{\alpha_\varsigma}(x),\psi^+_{\alpha'_\varsigma}(x')]=i\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{\alpha_\varsigma}(x),\psi_{\beta_\varsigma}(x')]=\frac{i}{2}m^2\delta_{\alpha_\varsigma\beta_\varsigma}\Delta(x-x')\\ [\psi^+_{\alpha'_\varsigma}(x),\psi^+_{\beta'_\varsigma}(x')]=\frac{i}{2}m^2\delta_{\alpha'_\varsigma\beta'_\varsigma}\Delta(x-x')\\ \psi_{\alpha_{-\varsigma}}(x)=-\psi^+_{\alpha_\varsigma}(x),\psi_{k_\varsigma}=\Gamma^{\alpha_\varsigma}_{k_\varsigma}(1)\psi_{\alpha_\varsigma}\end{cases}\Leftrightarrow\begin{cases}[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\Gamma^{ab}_{k_\varsigma k'_\varsigma}\partial_a\partial_b\Delta(x-x')\\ [\psi_{k_\varsigma}(x),\psi_{l_\varsigma}(x')]=\frac{i}{2}m^2\varepsilon_{k_\varsigma l_\varsigma}(1)\Delta(x-x')\\ [\psi^+_{k'_\varsigma}(x),\psi^+_{l'_\varsigma}(x')]=\frac{i}{2}m^2\varepsilon_{k'_\varsigma l'_\varsigma}(1)\Delta(x-x')\\ \psi_{\alpha_\varsigma}=\Gamma^{k_\varsigma}_{\alpha_\varsigma}(1)\psi_{k_\varsigma}\end{cases}$$

5.9 Equivalent commutative relations of massive complex field $\psi_{\lambda_\varsigma\mu_\varsigma}$ and complex potential A_a

Thm. 5.9.1. $[\psi_{A_\varsigma}{}^{B'_\varsigma}(x),\psi^+_{A'_\varsigma}{}^{B_\varsigma}(x')]=\frac{i}{4}[(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,-i\varsigma)^{B'_\varsigma B_\varsigma}_b\partial_a\partial^b+m^2\delta_{A_\varsigma}{}^{B_\varsigma}\delta_{A'_\varsigma}{}^{B'_\varsigma}]\Delta(x-x')$
$\Leftrightarrow[\psi_{A_\varsigma B'_\varsigma}(x),\psi^+_{A'_\varsigma B_\varsigma}(x')]=\frac{i}{4}[-(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_a\partial_b+m^2\varepsilon_{A'_\varsigma B'_\varsigma}\varepsilon_{A_\varsigma B_\varsigma}]\Delta(x-x')$
$\psi_{[\lambda_\varsigma\mu_\varsigma]}=[im\gamma^a(\varsigma)C\frac{A_a}{2}+S^{ab}(e,\varsigma)C\frac{F_{ab}}{2}]=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]\frac{A_a}{2}$

Thm. 5.9.2.
$[A_a(x), A^+_{a'}(x')] = i(\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x'), \psi_{[\lambda_\varsigma\mu_\varsigma]} = [im\gamma^a(\varsigma)C - 2S^{ab}(e,\varsigma)C\partial_b]\frac{A_a}{2}$
$\Leftrightarrow [\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')] = \frac{i}{8}[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$
$\Rightarrow [\psi_{A_\varsigma B_\varsigma}(x), \psi^+_{A'_\varsigma B'_\varsigma}(x')] = -\frac{i}{2}(\sigma, i\varsigma)^a{}_{A_\varsigma A'_\varsigma}(\sigma, i\varsigma)^b{}_{B_\varsigma B'_\varsigma}\partial_a\partial_b\Delta(x-x')$

5.10 Deriving various commutation rules from real potential for spin-1 Majorana particles

The following is true for real field cases.

Lem. 5.10.1. $F_{ab} = \frac{1}{\sqrt{2}}(-\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\psi^+_{\alpha'_\varsigma} + \sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}), *F_{ab} = \frac{\varsigma}{\sqrt{2}}(-\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\psi^+_{\alpha'_\varsigma} - \sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma})$

Thm. 5.10.1.
$[A_a(x), A^+_{a'}(x')] = i(\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x'), A^+_{a'} = A_a\eta^a_{a'} \Leftrightarrow [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\partial_a\partial_b}{m^2})\Delta(x-x')$

Thm. 5.10.2. $[A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\partial_a\partial_b}{m^2}) \Rightarrow [F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c}\partial_{b]}\partial_{d>}\Delta(x-x')$

Thm. 5.10.3. $[F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c}\partial_{b]}\partial_{d>}\Delta(x-x') \Leftrightarrow [F_{ab}(x), F^+_{a'b'}(x')] = -i\eta_{[a<a'}\partial_{b]}\partial_{b'>}\Delta(x-x')$

Thm. 5.10.4. $[*F_{ab}(x), *F^+_{a'b'}(x')] = -i\eta_{[a<a'}(\partial_{b]}\partial^+_{b'>} - \frac{1}{2}m^2\eta_{b]b'>})\Delta(x-x')$

Proof: $[*F_{ab}(x), *F^+_{a'b'}(x')]$
$= [\frac{1}{\sqrt{2}}(-\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}(x) - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\psi^+_{\alpha'_\varsigma}(x)), -\frac{1}{\sqrt{2}}(-\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\psi^+_{\alpha'_\varsigma}(x') - \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\psi_{\alpha_\varsigma}(x'))]$
$= -\frac{1}{2}\{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] + \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x), \psi_{\alpha_\varsigma}(x')]$
$+ \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{-\varsigma a'b'}[\psi_{\alpha_\varsigma}(x), \psi_{\beta_\varsigma}(x')] + \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x), \psi^+_{\beta'_\varsigma}(x')]\}$
$= -\frac{i}{2}\{[(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})(\delta_{a'c'}\delta_{b'd'} - \delta_{a'd'}\delta_{b'c'}) + \varepsilon_{abcd}\varepsilon_{a'b'c'd'}]\eta^{dd'}\partial^c\partial^{+c'} - m^2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\eta^c_{a'}\eta^d_{b'}\}\Delta(x-x')$
$= -\frac{i}{2}[(\delta_{a[c}\delta_{bd]}\delta_{a'[c'}\delta_{b'd']} + \delta_{a[a'}\delta_{bb'}\delta_{cc'}\delta_{dd']})\eta^{dd'}\partial^c\partial^{+c'} + m^2\delta_{a[c}\delta_{bd]}\eta^c_{a'}\eta^d_{b'} - 2m^2\delta_{a[c}\delta_{bd]}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$= -i\eta_{[a<a'}(\partial_{b]}\partial^+_{b'>} - \frac{1}{2}m^2\eta_{b]b'>})\Delta(x-x')$ □

Thm. 5.10.5. $[*F_{ab}(x), *F^+_{a'b'}(x')] = -i\eta_{[a<a'}(\partial_{b]}\partial^+_{b'>} - \frac{1}{2}m^2\eta_{b]b'>})\Delta(x-x')$
$\Leftrightarrow [*F_{ab}(x), *F_{cd}(x')] = -i\delta_{[a<c}(\partial_{b]}\partial_{d>} - \frac{1}{2}m^2\delta_{b]d>})\Delta(x-x')$

Proof: $[F_{ab}(x), *F^+_{a'b'}(x')]$
$= [\frac{1}{\sqrt{2}}(\sigma^{\alpha_\varsigma}_{\varsigma ab}\psi_{\alpha_\varsigma}(x) - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\psi^+_{\alpha'_\varsigma}(x)), \frac{1}{\sqrt{2}}(\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\psi^+_{\alpha'_\varsigma}(x') + \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\psi_{\alpha_\varsigma}(x'))]$
$= \frac{1}{2}\{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\alpha_\varsigma}_{-\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x), \psi_{\alpha_\varsigma}(x')]$
$+ \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{-\varsigma a'b'}[\psi_{\alpha_\varsigma}(x), \psi_{\beta_\varsigma}(x')] - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x), \psi^+_{\beta'_\varsigma}(x')]\}$
$= \frac{1}{2}\{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}[\psi_{\alpha_\varsigma}(x), \psi^+_{\alpha'_\varsigma}(x')] + \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}[\psi_{\alpha_\varsigma}(x'), \psi^+_{\alpha'_\varsigma}(x)]$
$+ \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{-\varsigma a'b'}[\psi_{\alpha_\varsigma}(x), \psi_{\beta_\varsigma}(x')] - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{\varsigma a'b'}[\psi^+_{\alpha'_\varsigma}(x), \psi^+_{\beta'_\varsigma}(x')]\}$
$= \frac{1}{2}\{\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}i\sigma^{cd}_{\alpha_\varsigma\alpha'_\varsigma}\partial_c\partial_d - \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}i\sigma^{cd}_{\alpha_\varsigma\alpha'_\varsigma}\partial'_c\partial'_d + \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{-\varsigma a'b'}\frac{i}{2}m^2\delta_{\alpha_\varsigma\beta_\varsigma} - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{\varsigma a'b'}\frac{i}{2}m^2\delta_{\alpha'_\varsigma\beta'_\varsigma}\}\Delta(x-x')$
$= \frac{i}{2}\{(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma} - \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{c'c}_{\alpha_\varsigma\alpha'_\varsigma})\partial_c\partial_{c'} + \frac{1}{2}m^2(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\beta_\varsigma}_{\varsigma cd} - \sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma^{\beta'_\varsigma}_{-\varsigma cd})\eta^c_{a'}\eta^d_{b'}\}\Delta(x-x')$
$= \frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{-\varsigma\alpha'_\varsigma c'd'}\delta^{dd'} - \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd}\delta^{dd'})\partial^c\partial^{c'} + 2m^2\varsigma\varepsilon_{abcd}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$= \frac{i}{4}[(\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha_\varsigma cd}\sigma_{\varsigma\alpha'_\varsigma c'd'} - \sigma^{\alpha_\varsigma}_{-\varsigma a'b'}\sigma^{\alpha'_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma c'd'}\sigma_{-\varsigma\alpha'_\varsigma cd})\eta^{dd'}\partial^c\partial^{+c'} + 2m^2\varsigma\varepsilon_{abcd}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$
$= \frac{i}{4}\{[(-S_{abcd} + \varsigma\varepsilon_{abcd})(-S_{a'b'c'd'} + \varsigma\varepsilon_{a'b'c'd'}) - (-S_{abcd} - \varsigma\varepsilon_{abcd})(-S_{a'b'c'd'} - \varsigma\varepsilon_{a'b'c'd'})]\eta^{dd'}\partial^c\partial^{+c'} + 2m^2\varsigma\varepsilon_{abcd}\eta^c_{a'}\eta^d_{b'}\}$
$\Delta(x-x')$
$= \frac{-i\varsigma}{2}[(S_{abcd}\varepsilon_{a'b'c'd'} + \varepsilon_{abcd}S_{a'b'c'd'})\eta^{dd'}\partial^c\partial^{+c'} - m^2\varepsilon_{abcd}\eta^c_{a'}\eta^d_{b'}]\Delta(x-x')$ □

5.11 Commutation relations of massive Majorana vector fields

Def. 5.11.1. $\psi := \begin{bmatrix} \lambda & \xi \\ \eta & \varphi \end{bmatrix} = \begin{bmatrix} \lambda_{A_\varsigma B_\varsigma} & \xi_{A_\varsigma}{}^{B'_\varsigma} \\ \eta^{A'_\varsigma}{}_{B_\varsigma} & \varphi^{A'_\varsigma B'_\varsigma} \end{bmatrix}$

Thm. 5.11.1.
$\psi = \gamma_2\psi^+\gamma_2, \psi = \psi^T \Leftrightarrow \begin{bmatrix} \lambda & \xi \\ \eta & \varphi \end{bmatrix} = \begin{bmatrix} \lambda^T & \eta^T \\ \xi^T & \varphi^T \end{bmatrix}, \begin{bmatrix} \lambda^* & \xi^* \\ \eta^* & \varphi^* \end{bmatrix} = \begin{bmatrix} \sigma_y\varphi\sigma_y & -\sigma_y\eta\sigma_y \\ -\sigma_y\xi\sigma_y & \sigma_y\lambda\sigma_y \end{bmatrix}, \begin{bmatrix} \lambda^+ & \eta^+ \\ \xi^+ & \varphi^+ \end{bmatrix} = \begin{bmatrix} \sigma_y\varphi\sigma_y & -\sigma_y\eta\sigma_y \\ -\sigma_y\xi\sigma_y & \sigma_y\lambda\sigma_y \end{bmatrix}$

Thm. 5.11.2. $\lambda^+ = \sigma_y\varphi\sigma_y, \varphi^+ = \sigma_y\lambda\sigma_y, \eta^+ = -\sigma_y\eta\sigma_y, \xi^+ = -\sigma_y\xi\sigma_y, \eta^T = \xi, \lambda^T = \lambda$

Thm. 5.11.3. $\psi = \gamma_2\psi^+\gamma_2, \psi = \psi^T \Leftrightarrow \lambda^+ = \sigma_y\varphi\sigma_y, \eta^+ = -\sigma_y\eta\sigma_y, \eta^T = \xi, \lambda^T = \lambda$

Thm. 5.11.4. $\psi := \begin{bmatrix} \lambda & \eta^T \\ \eta & \sigma_y\lambda^*\sigma_y \end{bmatrix} = \begin{bmatrix} \lambda_{A_\varsigma B_\varsigma} & \eta^*_{A_\varsigma}{}^{B'_\varsigma} \\ \eta^{A'_\varsigma}{}_{B_\varsigma} & \lambda^{*A'_\varsigma B'_\varsigma} \end{bmatrix}$
$\lambda^{*A'_\varsigma B'_\varsigma} = (\varsigma\varepsilon^{A'_\varsigma C'_\varsigma})(\varsigma\varepsilon^{B'_\varsigma D'_\varsigma})\lambda^*_{C'_\varsigma D'_\varsigma}, \eta^{B'_\varsigma}{}_{A_\varsigma} = \eta^T_{A_\varsigma}{}^{B'_\varsigma} = \eta^*_{A_\varsigma}{}^{B'_\varsigma} := (-\varsigma\varepsilon_{A_\varsigma C_\varsigma})(\varsigma\varepsilon^{B'_\varsigma D'_\varsigma})\eta^{*C_\varsigma}{}_{D'_\varsigma}$

Proof:

$$\begin{cases} [\lambda_{A_\varsigma B_\varsigma}(x), \lambda^{+}_{A'_\varsigma B'_\varsigma}(x')] = -\frac{i}{8}(\sigma, i\varsigma)^{a}_{\{A_\varsigma (A'_\varsigma}(\sigma, i\varsigma)^{b}_{B_\varsigma\} B'_\varsigma)}\partial_a\partial_b\Delta(x-x') \\ [\lambda_{A_\varsigma B_\varsigma}(x), \lambda_{C_\varsigma D_\varsigma}(x')] = \frac{i}{8}m^2\varepsilon_{\{A_\varsigma (C_\varsigma}\varepsilon_{B_\varsigma\} D_\varsigma)}\Delta(x-x') \\ [\lambda^{+}_{A'_\varsigma B'_\varsigma}(x), \lambda^{+}_{C'_\varsigma D'_\varsigma}(x')] = \frac{i}{8}m^2\varepsilon_{\{A'_\varsigma (C'_\varsigma}\varepsilon_{B'_\varsigma\} D'_\varsigma)}\Delta(x-x') \end{cases}$$

$$\begin{cases} [\eta^{A'_\varsigma}{}_{B_\varsigma}(x), \eta^{+A_\varsigma}{}_{B'_\varsigma}(x')] = \frac{i}{8}[(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}(\sigma, i\varsigma)^{b}_{B_\varsigma B'_\varsigma}\partial^a\partial_b + m^2\delta^{A_\varsigma}_{B_\varsigma}\delta^{A'_\varsigma}_{B'_\varsigma}]\Delta(x-x') \\ [\eta^{A'_\varsigma}{}_{B_\varsigma}(x), \eta^{B'_\varsigma}{}_{A_\varsigma}(x')] = \frac{i}{8}\{[(\sigma, -i\varsigma)\sigma_y]_a^{A'_\varsigma}{}_{A_\varsigma}[(\sigma, -i\varsigma)\sigma_y]_b^{B'_\varsigma}{}_{B_\varsigma}\partial^a\partial^b + m^2\varepsilon^{A'_\varsigma B'_\varsigma}\varepsilon_{A_\varsigma B_\varsigma}\}\Delta(x-x') \\ [\eta^{+A_\varsigma}{}_{B'_\varsigma}(x), \eta^{+B_\varsigma}{}_{A'_\varsigma}(x')] = \frac{i}{8}\{[\sigma_y(\sigma, -i\varsigma)]_{aA'_\varsigma}{}^{A_\varsigma}[\sigma_y(\sigma, -i\varsigma)]_{bB'_\varsigma}{}^{B_\varsigma}\partial^a\partial^b + m^2\varepsilon^{A_\varsigma B_\varsigma}\varepsilon_{A'_\varsigma B'_\varsigma}\}\Delta(x-x') \end{cases}$$

$$\begin{cases} [\lambda_{A_\varsigma B_\varsigma}(x), \eta^{+C_\varsigma}{}_{A'_\varsigma}(x')] = -\frac{1}{4}m(\sigma, i\varsigma)^{a}_{\{A_\varsigma A'_\varsigma}\delta^{C_\varsigma}_{B_\varsigma\}}\partial_a\Delta(x-x') \\ [\lambda^{+}_{A'_\varsigma B'_\varsigma}(x), \eta^{C'_\varsigma}{}_{A_\varsigma}(x')] = \frac{1}{4}m(\sigma, i\varsigma)^{a}_{\{A_\varsigma A'_\varsigma}\delta^{C'_\varsigma}_{B'_\varsigma\}}\partial_a\Delta(x-x') \\ [\lambda_{A_\varsigma B_\varsigma}(x), \eta^{A'_\varsigma}{}_{C_\varsigma}(x')] = \frac{i}{4}m[(\sigma, -i\varsigma)\sigma_y]_a^{A'_\varsigma}{}_{\{A_\varsigma}\varepsilon_{B_\varsigma\}C_\varsigma}\partial^a\Delta(x-x') \\ [\lambda^{+}_{A'_\varsigma B'_\varsigma}(x), \eta^{+A_\varsigma}{}_{C'_\varsigma}(x')] = \frac{i}{4}m[\sigma_y(\sigma, i\varsigma)]_a^{A_\varsigma}{}_{\{A'_\varsigma}\varepsilon_{B'_\varsigma\}C'_\varsigma}\partial^a\Delta(x-x') \end{cases} \quad \square$$

Chapter28 Covariant Quantization Scheme for Massive Gravitino

Self comment: For particles described by the Bargmann Wigner equation, it is generally possible to describe both charged complex particles and uncharged Mayorana particles. The principal commutation rule in both cases is consistent, but the rest are generally zero for charged complex particles. For uncharged Mayorana particles, the rest of the commutative or anti commutative brackets are naturally derived from the principal commutative rule and Mayorana conditions. And they are generally not zero. In this chapter, we only discuss the case of complex particles and generally only give the principal commutation rule. The Mayorana particle case is no longer specifically discussed. If we want to obtain the quantum field theory of the Mayorana particle case, we only need to add the Mayorana condition to the complex particle case. Then we will naturally obtain it.

1 Mutual conversion of two equivalent descriptions for massive gravitino

1.1 Two equivalent descriptions of B-W equation and R-S equation for spin-$\frac{3}{2}$ particles [16, 17, 20]

Thm. 1.1.1. $\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{1}{3!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\}}\\ im\frac{A_{a\eta_\varsigma}}{2}=\frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}]\end{cases}\Leftrightarrow\begin{cases}[\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}=0\\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\frac{A_{a\eta_\varsigma}}{2}\end{cases}$

Thm. 1.1.2. $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)=\frac{1}{2}[(m-i\gamma^ap_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp_b)\gamma^4]_{\mu_\varsigma\}\mu'_\varsigma)}$

1.2 Plane wave solutions of B-W equation for spin-$\frac{3}{2}$ particles [16]

Thm. 1.2.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)=\frac{1}{3!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\}}(\vec{r},t)$

$$\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=3/2}^{-3/2}\sqrt{\frac{m^3}{E}}[a(\vec{p},h)U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\begin{cases}U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},\frac{3}{2})=u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})\\ U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},\frac{1}{2})\\ =\frac{1}{\sqrt{3}}[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]\\ U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},-\frac{1}{2})\\ =\frac{1}{\sqrt{3}}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})]\\ U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},-\frac{3}{2})=u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})\end{cases}$$

$$\begin{cases}V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},\frac{3}{2})=v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})\\ V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},\frac{1}{2})\\ =\frac{1}{\sqrt{3}}[v_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]\\ V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},-\frac{1}{2})\\ =\frac{1}{\sqrt{3}}[v_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+v_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+v_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})]\\ V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},-\frac{3}{2})=v_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})\end{cases}$$

$$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^3}{E^5}}U^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ b^+(\vec{p},s)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^3}{E^5}}V^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

Thm. 1.2.2. $[\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(x')]$

$=\frac{i}{4}\frac{1}{(3!)^2}[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(m-\gamma^c\partial_c)\gamma^4]_{\eta_\varsigma\}\eta'_\varsigma)}\Delta(x-x')$

$=\frac{i}{8}\frac{1}{(3!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')[(m-\gamma^c\partial_c)\gamma^4]_{\eta_\varsigma\}\eta'_\varsigma)}(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')$

Def. 1.2.1.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},\frac{3}{2}):=\sum\limits_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)U^{+}_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},h)\\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\eta_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},\frac{3}{2}):=\sum\limits_{h=1}^{-1}V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)V^{+}_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},h)\end{cases}$$

Thm. 1.2.3.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},\frac{3}{2})\\ =\frac{1}{8m^2}\frac{1}{(3!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(p)\Lambda_{maa'}(\vec{p},1)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(p)\Lambda_{+\eta_\varsigma\}\eta'_\varsigma)}(\vec{p},\frac{1}{2})=\frac{1}{(3!)^2}\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\eta_\varsigma\}\eta'_\varsigma)}(\vec{p},\frac{1}{2})\\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\eta_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},\frac{3}{2})\\ =\frac{1}{8m^2}\frac{1}{(3!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(-p)\Lambda_{maa'}(\vec{p},1)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(-p)\Lambda_{-\eta_\varsigma\}\eta'_\varsigma)}(\vec{p},\frac{1}{2})=\frac{1}{(3!)^2}\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\eta_\varsigma\}\eta'_\varsigma)}(\vec{p},\frac{1}{2})\end{cases}$$

Thm. 1.2.4.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},\frac{3}{2})=\frac{1}{16m^3}\frac{1}{(3!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(p)[(m-i\gamma^bp_b)\gamma^4]_{\eta_\varsigma\}\eta'_\varsigma)}\\ =\frac{1}{8m^3}\frac{1}{(3!)^2}[(m-i\gamma^ap_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^bp_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(m-i\gamma^bp_b)\gamma^4]_{\eta_\varsigma\}\eta'_\varsigma)}\\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\eta_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma}(\vec{p},\frac{3}{2})=-\frac{1}{16m^3}\frac{1}{(3!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(-p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(-p)[(m+i\gamma^bp_b)\gamma^4]_{\eta_\varsigma\}\eta'_\varsigma)}\\ =-\frac{1}{8m^3}\frac{1}{(3!)^2}[(m+i\gamma^ap_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m+i\gamma^bp_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(m+i\gamma^bp_b)\gamma^4]_{\eta_\varsigma\}\eta'_\varsigma)}\end{cases}$$

$$\Downarrow$$

1.3 Derived to plane wave solutions of R-S equation for spin-$\frac{3}{2}$ particles

1.3.1 Derived to plane wave solutions of R-S equation for spin-$\frac{3}{2}$ particles [16]

Thm. 1.3.1. $[\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}=0,A_a=\frac{1}{2im}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}$

$$A_{a\eta_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\sqrt{\frac{m}{2E}}[a(\vec{p},h)\varepsilon_a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)\tilde{\varepsilon}_a(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\begin{cases}\varepsilon_{a\eta_\varsigma}(\vec{p},\frac{3}{2})=\frac{1}{i\sqrt{2}}u^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})\\ \varepsilon_{a\eta_\varsigma}(\vec{p},\frac{1}{2})\\ =\frac{1}{i\sqrt{6}}[u^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_au(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+u^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+u^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]\\ \varepsilon_{a\eta_\varsigma}(\vec{p},-\frac{1}{2})\\ =\frac{1}{i\sqrt{6}}[u^T(\vec{p},\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+u^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_au(\vec{p},\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+u^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})]\\ \varepsilon_{a\eta_\varsigma}(\vec{p},-\frac{3}{2})=\frac{1}{i\sqrt{2}}u^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_au(\vec{p},-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})\end{cases}$$

$$\begin{cases}\tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},\frac{3}{2})=\frac{1}{i\sqrt{2}}v^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})\\ \tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},\frac{1}{2})\\ =\frac{1}{i\sqrt{6}}[v^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_av(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+v^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+v^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]\\ \tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},-\frac{1}{2})\\ =\frac{1}{i\sqrt{6}}[v^T(\vec{p},\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+v^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_av(\vec{p},\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+v^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})]\\ \tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},-\frac{3}{2})=\frac{1}{i\sqrt{2}}v^T(\vec{p},-\frac{1}{2})\bar{C}\gamma_av(\vec{p},-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})\end{cases}$$

Lem. 1.3.1. $\frac{1}{2\sqrt{2}m}U^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\hat{p},h)\mathbb{X}^{+a}_{\lambda_\varsigma\mu_\varsigma}(p)=\frac{1}{i\sqrt{2}}U^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\hat{p},h)(\bar{C}\gamma_a)_{\lambda_\varsigma\mu_\varsigma}=\varepsilon_{a\eta_\varsigma}(\vec{p},h)$

Cor. 1.3.1. $[\gamma^b(\varsigma)\partial_b+m]A_{a[\eta_\varsigma]}=0,\gamma^a(\varsigma)A_{a[\eta_\varsigma]}=0,A_a=\frac{1}{2im}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}$

$$A_{a\eta_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\sqrt{\frac{m}{E}}[a(\vec{p},h)\varepsilon_a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)\tilde{\varepsilon}_a(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\begin{cases}\varepsilon_{a\eta_\varsigma}(\vec{p},\frac{3}{2})=\varepsilon_a(\vec{p},1)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})\\ \varepsilon_{a\eta_\varsigma}(\vec{p},\frac{1}{2})=\frac{1}{\sqrt{3}}[\sqrt{2}\varepsilon_a(\vec{p},0)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\varepsilon_a(\vec{p},1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]\\ \varepsilon_{a\eta_\varsigma}(\vec{p},-\frac{1}{2})=\frac{1}{\sqrt{3}}[\sqrt{2}\varepsilon_a(\vec{p},0)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+\varepsilon_a(\vec{p},-1)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})]\\ \varepsilon_{a\eta_\varsigma}(\vec{p},-\frac{3}{2})=\varepsilon_a(\vec{p},-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})\end{cases}$$

$$\begin{cases}\tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},\frac{3}{2})=-\varepsilon_a(\vec{p},1)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})\\ \tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},\frac{1}{2})=-\frac{1}{\sqrt{3}}[\sqrt{2}\varepsilon_a(\vec{p},0)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\varepsilon_a(\vec{p},1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]\\ \tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},-\frac{1}{2})=-\frac{1}{\sqrt{3}}[\sqrt{2}\varepsilon_a(\vec{p},0)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})+\varepsilon_a(\vec{p},-1)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})]\\ \tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},-\frac{3}{2})=-\varepsilon_a(\vec{p},-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})\end{cases}$$

1.3.2 Derivate R-S anticommutative relations for spin-$\frac{3}{2}$ particles under classical conventions

Thm. 1.3.2. $\{A_{a_1a_2\cdots a_n\tau_\varsigma}(x),\bar{A}_{b_1b_2\cdots b_n\tau'_\varsigma}(x')\}=i\hat{P}_{a_1\cdots a_n\tau_\varsigma b_1\cdots b_n\tau'_\varsigma}(n+\frac{1}{2})\Delta(x-x')$

Lem. 1.3.2.

$$\begin{cases}(m-\kappa\gamma^b\partial_b)(\gamma_a+\kappa\frac{\partial_a}{m})=(\gamma_a-\kappa\frac{\partial_a}{m})(m+\kappa\gamma^b\partial_b)\\(m-\gamma^b\partial_b)(\gamma_a+\frac{\partial_a}{m})=(\gamma_a-\frac{\partial_a}{m})(m+\gamma^b\partial_b)\\(m+\gamma^b\partial_b)(\gamma_a-\frac{\partial_a}{m})=(\gamma_a+\frac{\partial_a}{m})(m-\gamma^b\partial_b)\end{cases}$$

Proof: $(m-\kappa\gamma^b\partial_b)(\gamma_a+\kappa\frac{\partial_a}{m})$
$=m(\gamma_a+\kappa\frac{\partial_a}{m})-\kappa\gamma^b\gamma_a\partial_b-\frac{\kappa\kappa\gamma^b\partial_b\partial_a}{m}$
$=m(\gamma_a+\kappa\frac{\partial_a}{m})-\kappa\{\gamma^b,\gamma_a\}\partial_b+\kappa\gamma_a\gamma^b\partial_b-\kappa\kappa\frac{\partial_a}{m}\gamma^b\partial_b$
$=m(\gamma_a+\kappa\frac{\partial_a}{m})-2\kappa\delta_a^b\partial_b+\kappa(\gamma_a-\kappa\frac{\partial_a}{m})\gamma^b\partial_b$
$=m(\gamma_a-\kappa\frac{\partial_a}{m})+(\gamma_a-\kappa\frac{\partial_a}{m})\kappa\gamma^b\partial_b$
$=(\gamma_a-\kappa\frac{\partial_a}{m})(m+\kappa\gamma^b\partial_b)$ □

Thm. 1.3.3.

$$\begin{cases}\hat{P}_{a_1\tau_\varsigma b_1\tau'_\varsigma}(\frac{3}{2})=\frac{2}{5}\hat{P}_{aa_1bb_1}(2)[(m-\gamma^c\partial_c)\gamma^a\gamma^b\gamma^4]_{\tau_\varsigma\tau'_\varsigma}\\\hat{P}_{aa_1bb_1}(2)=\frac{1}{8}\{[\delta_{\{a(b}-\frac{\partial_{\{a}\partial_{(b}}{m^2}][\delta_{a_1\}b_1)}-\frac{\partial_{a_1\}}\partial_{b_1)}}{m^2}]-\frac{1}{3}[\delta_{\{aa_1\}}-\frac{\partial_{\{a}\partial_{a_1\}}}{m^2}][\delta_{(bb_1)}-\frac{\partial_{(b}\partial_{b_1)}}{m^2}]\}\Delta(x-x')\end{cases}$$

[⇒]

$$\hat{P}_{a\tau_\varsigma b\tau'_\varsigma}(\tfrac{3}{2})=\tfrac{1}{2}\{[(\delta_{a_1b_1}-\tfrac{\partial_{a_1}\partial_{b_1}}{m^2})-\tfrac{1}{3}(\gamma_{a_1}-\tfrac{\partial_{a_1}}{m})(\gamma_{b_1}+\tfrac{\partial_{b_1}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$$

Proof: $\hat{P}_{a_1\tau_\varsigma b_1\tau'_\varsigma}(\frac{3}{2})=\frac{2}{5}\hat{P}_{aa_1bb_1}(2)[(m-\gamma^c\partial_c)\gamma^a\gamma^b\gamma^4]_{\tau_\varsigma\tau'_\varsigma}$
$=\frac{2}{5}\frac{1}{8}\{[\delta_{\{a(b}-\frac{\partial_{\{a}\partial_{(b}}{m^2}][\delta_{a_1\}b_1)}-\frac{\partial_{a_1\}}\partial_{b_1)}}{m^2}]-\frac{1}{3}[\delta_{\{aa_1\}}-\frac{\partial_{\{a}\partial_{a_1\}}}{m^2}][\delta_{(bb_1)}-\frac{\partial_{(b}\partial_{b_1)}}{m^2}]\}[(m-\gamma^c\partial_c)\gamma^a\gamma^b\gamma^4]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{i}{10}\{[\delta_{ab}-\frac{\partial_a\partial_b}{m^2}][\delta_{a_1b_1}-\frac{\partial_{a_1}\partial_{b_1}}{m^2}]+[\delta_{ab_1}-\frac{\partial_a\partial_{b_1}}{m^2}][\delta_{a_1b}-\frac{\partial_{a_1}\partial_b}{m^2}]-\frac{2}{3}[\delta_{aa_1}-\frac{\partial_a\partial_{a_1}}{m^2}][\delta_{bb_1}-\frac{\partial_b\partial_{b_1}}{m^2}]\}$
$\{(m-\gamma^c\partial_c)\gamma^a\gamma^b\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{10}\{(m-\gamma^c\partial_c)[3(\delta_{a_1b_1}-\frac{\partial_{a_1}\partial_{b_1}}{m^2})+(\gamma_{b_1}+\frac{\partial_{b_1}}{m})(\gamma_{a_1}-\frac{\gamma^c\partial_c\partial_{a_1}}{m^2})-\frac{2}{3}(\gamma_{a_1}+\frac{\partial_{a_1}}{m})(\gamma_{b_1}-\frac{\gamma^c\partial_c\partial_{b_1}}{m^2})]\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{10}\{[3(\delta_{a_1b_1}-\frac{\partial_{a_1}\partial_{b_1}}{m^2})(m-\gamma^c\partial_c)$
$+(\gamma_{b_1}-\frac{\partial_{b_1}}{m})(m+\gamma^c\partial_c)(\gamma_{a_1}-\frac{\partial_{a_1}}{m})-\frac{2}{3}(\gamma_{a_1}-\frac{\partial_{a_1}}{m})(m+\gamma^c\partial_c)(\gamma_{b_1}-\frac{\partial_{b_1}}{m})]\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{10}\{[3(\delta_{a_1b_1}-\frac{\partial_{a_1}\partial_{b_1}}{m^2})(m-\gamma^c\partial_c)$
$+(\gamma_{b_1}-\frac{\partial_{b_1}}{m})(\gamma_{a_1}+\frac{\partial_{a_1}}{m})(m-\gamma^c\partial_c)-\frac{2}{3}(\gamma_{a_1}-\frac{\partial_{a_1}}{m})(\gamma_{b_1}+\frac{\partial_{b_1}}{m})(m-\gamma^c\partial_c)]\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{10}\{[3(\delta_{a_1b_1}-\frac{\partial_{a_1}\partial_{b_1}}{m^2})+(\gamma_{b_1}-\frac{\partial_{b_1}}{m})(\gamma_{a_1}+\frac{\partial_{a_1}}{m})-\frac{2}{3}(\gamma_{a_1}-\frac{\partial_{a_1}}{m})(\gamma_{b_1}+\frac{\partial_{b_1}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{10}\{[3\delta_{a_1b_1}-\frac{10}{3}\frac{\partial_{a_1}\partial_{b_1}}{m^2}+\{\gamma_{b_1},\gamma_{a_1}\}-\frac{5}{3}\gamma_{a_1}\gamma_{b_1}-\frac{5}{3}(\gamma_{a_1}\frac{\partial_{b_1}}{m}-\gamma_{b_1}\frac{\partial_{a_1}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{2}\{[\delta_{a_1b_1}-\frac{2}{3}\frac{\partial_{a_1}\partial_{b_1}}{m^2}-\frac{1}{3}\gamma_{a_1}\gamma_{b_1}-\frac{1}{3}(\gamma_{a_1}\frac{\partial_{b_1}}{m}-\gamma_{b_1}\frac{\partial_{a_1}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{2}\{[(\delta_{a_1b_1}-\frac{\partial_{a_1}\partial_{b_1}}{m^2})-\frac{1}{3}(\gamma_{a_1}-\frac{\partial_{a_1}}{m})(\gamma_{b_1}+\frac{\partial_{b_1}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$
$=\frac{1}{2}\{(\delta_{a_1b_1}-\frac{\partial_{a_1}\partial_{b_1}}{m^2})[(m-\gamma^c\partial_c)\gamma^4]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')-\frac{1}{3}[(\gamma_{a_1}-\frac{\partial_{a_1}}{m})(m+\gamma^c\partial_c)(\gamma_{b_1}-\frac{\partial_{b_1}}{m})\gamma^4]_{\tau_\varsigma\tau'_\varsigma}\}\Delta(x-x')$ □

Cor. 1.3.2. $\hat{P}_{a\tau_\varsigma b'\tau'_\varsigma}(\frac{3}{2})=\frac{1}{2}\{[(\eta_{ab'}-\frac{\partial_a\partial^+_{b'}}{m^2})-\frac{1}{3}(\gamma_a-\frac{\partial_a}{m})(\gamma_b\eta^b_{b'}+\frac{\partial^+_{b'}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

Cor. 1.3.3.

$$\begin{cases}\{A_{a\tau_\varsigma}(x),\bar{A}_{b\tau'_\varsigma}(x')\}=\frac{i}{2}\{[(\delta_{ab}-\frac{\partial_a\partial_b}{m^2})-\frac{1}{3}(\gamma_a-\frac{\partial_a}{m})(\gamma_b+\frac{\partial_b}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')\\\{A_{a\tau_\varsigma}(x),A^+_{a'\tau'_\varsigma}(x')\}=\frac{i}{2}\{[(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})-\frac{1}{3}(\gamma_a-\frac{\partial_a}{m})(\gamma_b\eta^b_{a'}+\frac{\partial^+_{a'}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')\\{}[A_{ab}(x),A^+_{a'b'}(x')]=\frac{i}{8}\{[\eta_{\{a(a'}-\frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{b\}b')}-\frac{\partial_{b\}}\partial^+_{b')}}{m^2}]-\frac{1}{3}[\delta_{\{ab\}}-\frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{(a'b')}-\frac{\partial^+_{(a'}\partial^+_{b')}}{m^2}]\}\Delta(x-x')\end{cases}$$

1.3.3 Comparison of relations between quasi projection operators

Thm. 1.3.4.

$$\begin{cases}\Lambda_{+ma\tau_\varsigma a'\tau'_\varsigma}(\vec{p},\frac{3}{2}):=\sum\limits_{h=3/2}^{-3/2}\varepsilon_{a\tau_\varsigma}(\vec{p},h)\varepsilon^+_{a'\tau'_\varsigma}(\vec{p},h)=\frac{2}{5}\sum\limits_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)\gamma^b\Lambda_-(\vec{p},\frac{1}{2})\gamma^{b'}\\\Lambda_{-ma\tau_\varsigma a'\tau'_\varsigma}(\vec{p},\frac{3}{2}):=\sum\limits_{h=3/2}^{-3/2}\tilde{\varepsilon}_{a\tau_\varsigma}(\vec{p},h)\tilde{\varepsilon}^+_{a'\tau'_\varsigma}(\vec{p},h)=\frac{2}{5}\sum\limits_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)\gamma^b\Lambda_+(\vec{p},\frac{1}{2})\gamma^{b'}\\\Lambda_{\pm\tau_\varsigma\tau'_\varsigma}(\vec{p},\frac{1}{2})=\frac{1}{2}\Lambda_{\pm ma\tau_\varsigma a'\tau'_\varsigma}(\vec{p},\frac{3}{2})\eta^{aa'},\Lambda_{maa'}(\vec{p},1)=\frac{3}{4}(\frac{m}{E})^2\Lambda_{\pm ma\tau_\varsigma a'\tau'_\varsigma}(\vec{p},\frac{3}{2})\Lambda_\pm^{\tau'_\varsigma\tau_\varsigma}\end{cases}$$

Thm. 1.3.5.

$$\begin{cases}\sum\limits_{h=1}^{-1}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h)=\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2}\\\sum\limits_{h=3/2}^{-3/2}\varepsilon_{a\tau_\varsigma}(\vec{p},h)\varepsilon^+_{a'\tau'_\varsigma}(\vec{p},h)=\{[(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})-\frac{1}{3}(\gamma_a-\frac{ip_a}{m})(\gamma_b\eta^b_{a'}+\frac{ip^+_{a'}}{m})]\frac{(m-i\gamma^cp_c)\gamma^4}{2m}\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')\\\sum\limits_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)=\frac{1}{4}\{[\eta_{\{a(a'}+\frac{p_{\{a}p^+_{(a'}}{m^2}][\eta_{b\}b')}+\frac{p_{b\}}p^+_{b')}}{m^2}]-\frac{1}{3}[\delta_{\{ab\}}+\frac{p_{\{a}p_{b\}}}{m^2}][\delta_{(a'b')}+\frac{p^+_{(a'}p^+_{b')}}{m^2}]\}\end{cases}$$

⇓

1.4 Back to plane wave solutions of B-W equation for spin-$\frac{3}{2}$ particles [16]

Thm. 1.4.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)=0, \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)=[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]\frac{A_{a\eta_\varsigma}(\vec{r},t)}{2}$

$$\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=3/2}^{-3/2}\sqrt{\frac{m^3}{E}}[a(\vec{p},h)U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_{a\eta_\varsigma}(\vec{p},h), V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2}m}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\tilde{\varepsilon}_{a\eta_\varsigma}(\vec{p},h)$$

$$\begin{cases}a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^3}{E^5}}U^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ b^+(\vec{p},s)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^3}{E^5}}V^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

1.5 Anticommutative relation for spin-$\frac{3}{2}$ particle field $F_{ab\tau_\varsigma},\psi_{\alpha_\kappa\tau_\varsigma}$

Thm. 1.5.1.

$$\begin{cases}\{A_{a\tau_\varsigma}(x),A^+_{a'\tau'_\varsigma}(x')\}=\frac{i}{2}\{[(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})-\frac{1}{3}(\gamma_a-\frac{\partial_a}{m})(\gamma_b\eta^b_{a'}+\frac{\partial^+_{a'}}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')\\ F_{ab\tau_\varsigma}=\partial_aA_{b\tau_\varsigma}-\partial_bA_{a\tau_\varsigma}\end{cases}$$

$$\Rightarrow\{F_{ab\tau_\varsigma}(x),F^+_{a'b'\tau'_\varsigma}(x')\}=-\frac{i}{2}[(\eta_{[a<a'}-\frac{1}{3}\gamma_{[a}\eta^d_{<a'}\gamma_d)\gamma^c\gamma^4]_{\tau_\varsigma\tau'_\varsigma}\partial_{b]}\partial^+_{b'>}\partial_c\Delta(x-x')$$

Thm. 1.5.2.

$$\begin{cases}\{F_{ab\tau_\varsigma}(x),F^+_{a'b'\tau'_\varsigma}(x')\}=-\frac{i}{2}[(\eta_{[a<a'}-\frac{1}{3}\gamma_{[a}\eta^d_{<a'}\gamma_d)\gamma^c\gamma^4]_{\tau_\varsigma\tau'_\varsigma}\partial_{b]}\partial^+_{b'>}\partial_c\Delta(x-x')\\ \psi_{\alpha_\kappa\tau_\varsigma}:=-\frac{1}{2\sqrt{2}}\sigma^{ab}_{\kappa\alpha_\kappa}F_{ab\tau_\varsigma}\end{cases}$$

$$\Rightarrow\{\psi_{\alpha_\kappa\tau_\varsigma}(x),\psi^+_{\alpha'_\kappa\tau'_\varsigma}(x')\}=\frac{i}{2}[(\sigma^{ab}_{\alpha_\kappa\alpha'_\kappa}+\frac{1}{6}\sigma^{aa'}_{\kappa\alpha_\kappa}\gamma_{a'}\gamma_{b'}\sigma^{b'b}_{-\kappa\alpha_\kappa})\gamma^c\gamma^4]_{\tau_\varsigma\tau'_\varsigma}\partial_a\partial_b\partial_c\Delta(x-x')$$

1.6 Extraction of energy momentum operator for massive gravitino field

Thm. 1.6.1. $P_u(\frac{3}{2})=\int\psi^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)d^3\vec{r}$

Thm. 1.6.2. $P_u(\frac{3}{2})=\int[\frac{1}{2}F^{+ab\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}F_{ab\eta_\varsigma}(\vec{r},t)+m^2A^{+a\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}A_{a\eta_\varsigma}(\vec{r},t)]d^3\vec{r}$

Proof: $P_u(\frac{3}{2})=\int\psi^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(\vec{r},t)d^3\vec{r}$

$$=\int\{\bar{C}[-im\gamma^{a'}(\varsigma)-2S^{a'b'}(e,\varsigma)\partial^+_{b'}]\}^{\lambda_\varsigma\mu_\varsigma}\frac{A^{+\eta_\varsigma}_{a'}(\vec{r},t)}{2}\frac{-i\partial_u}{m^2-\nabla^2}[im\gamma^a(\varsigma)C-2S^{ab}(e,\varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma}\frac{A_{a\eta_\varsigma}(\vec{r},t)}{2}d^3\vec{r}$$
$$=\frac{1}{4}\int tr\{\bar{C}[-im\gamma^{a'}(\varsigma)-2S^{a'b'}(e,\varsigma)\partial^+_{b'}]A^{+\eta_\varsigma}_{a'}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]CA_{a\eta_\varsigma}(\vec{r},t)\}d^3\vec{r}$$
$$=\frac{1}{4}\int tr\{[-im\gamma^{a'}(\varsigma)-2S^{a'b'}(e,\varsigma)\partial^+_{b'}]A^{+\eta_\varsigma}_{a'}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]A_{a\eta_\varsigma}(\vec{r},t)\}d^3\vec{r}$$
$$=\frac{1}{4}\int tr\{[-im\gamma^{a'}(\varsigma)-2S^{a'b'}(e,\varsigma)\partial^+_{b'}]A^{+\eta_\varsigma}_{a'}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}[im\gamma^a(\varsigma)-2S^{ab}(e,\varsigma)\partial_b]A_{a\eta_\varsigma}(\vec{r},t)\}d^3\vec{r}$$
$$=\int m^2A^{+a\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}A_{a\eta_\varsigma}(\vec{r},t)d^3\vec{r}+\int S^{a'b'ab}\partial^+_{b'}A^{+\eta_\varsigma}_{a'}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}\partial_bA_{a\eta_\varsigma}(\vec{r},t)\}d^3\vec{r}$$
$$=\int m^2A^{+a\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}A_{a\eta_\varsigma}(\vec{r},t)d^3\vec{r}$$
$$+\frac{1}{4}\int S^{a'b'ab}[\partial^+_{a'}A^{+\eta_\varsigma}_{b'}(\vec{r},t)-\partial^+_{b'}A^{+\eta_\varsigma}_{a'}(\vec{r},t)]\frac{-i\partial_u}{m^2-\nabla^2}[\partial_aA_{b\eta_\varsigma}(\vec{r},t)-\partial_bA_{a\eta_\varsigma}(\vec{r},t)]\}d^3\vec{r}$$
$$=\int m^2A^{+a\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}A_{a\eta_\varsigma}(\vec{r},t)d^3\vec{r}+\frac{1}{4}\int(\delta^{a'a}\delta^{b'b}-\delta^{a'b}\delta^{b'a})F^{+\eta_\varsigma}_{a'b'}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}F_{ab\eta_\varsigma}(\vec{r},t)d^3\vec{r}$$
$$=\int[\frac{1}{2}F^{+ab\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}F_{ab\eta_\varsigma}(\vec{r},t)+m^2A^{+a\eta_\varsigma}(\vec{r},t)\frac{-i\partial_u}{m^2-\nabla^2}A_{a\eta_\varsigma}(\vec{r},t)]d^3\vec{r}$$ □

Chapter29 Covariant quantization scheme for massive graviton

Self comment: For particles described by the Bargmann Wigner equation, it is generally possible to describe both charged complex particles and uncharged Mayorana particles. The principal commutation rule in both cases is consistent, but the rest are generally zero for charged complex particles. For uncharged Mayorana particles, the rest of the commutative or anti commutative brackets are naturally derived from the principal commutative rule and Mayorana conditions. And they are generally not zero. In this chapter, we only discuss the case of complex particles and generally only give the principal commutation rule. The Mayorana particle case is no longer specifically discussed. If we want to obtain the quantum field theory of the Mayorana particle case, we only need to add the Mayorana condition to the complex particle case. Then we will naturally obtain it.

1 Mutual conversion of two equivalent descriptions for massive graviton

1.1 Two equivalent descriptions of B-W equation for spin-2 particles and K-G equation [16,20,21]

Def. 1.1.1. $\mathbb{X}_a(x) := [im\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)\partial^b]C, \mathbb{X}_a(p) := i[m\gamma_a(\varsigma) - 2S_{ab}(e,\varsigma)p^b]C, C = \gamma_2\gamma_4$

Thm. 1.1.1.

$$\begin{cases}(\gamma^a\partial_a + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = 0, \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}} \\ A_{ab} = \frac{1}{(2im)^2}[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta_\varsigma\xi_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} \\ C = \gamma_2(\varsigma)\gamma_4(\varsigma)\end{cases} \Leftrightarrow \begin{cases}\partial^c F_{c|ab} + m^2 A_{ab} = 0, F_{c|ab} = \partial_c A_{ab} - \partial_a A_{cb} \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{1}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(x)A_{ab} \\ A_{ab} = A_{ba}, \delta^{ab}A_{ab} = 0\end{cases}$$

1.2 Plane wave solutions of B-W equation for spin-2 particles [16]

Thm. 1.2.1. $(\gamma^a\partial_a + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t) = 0, \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t) = \frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}}(\vec{r},t)$

$$\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{\frac{m^4}{E}}[a(\vec{p},h)U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\begin{cases}a(\vec{p},h) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^4}{E^7}}U^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r} \\ b^+(\vec{p},s) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m^4}{E^7}}V^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

Thm. 1.2.2. $[\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x), \psi^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(x')]$

$$= \frac{i}{2^3}\frac{1}{(4!)^2}[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(m-\gamma^a\partial_a)\gamma^4]_{\eta_\varsigma\eta'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\xi_\varsigma\}\xi'_\varsigma)}\Delta(x-x')$$

$$= \frac{i}{2^5}\frac{1}{(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}(x)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}(x')(\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2})(\eta_{bb'} - \frac{\partial_b\partial^+_{b'}}{m^2})\Delta(x-x')$$

Def. 1.2.1.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},2) := \sum_{h=2}^{-2}U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},h) \\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},2) := \sum_{h=2}^{-2}V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)V^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},h)\end{cases}$$

Thm. 1.2.3.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},2) = \frac{1}{(4!)^2}\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\eta_\varsigma\eta'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\xi_\varsigma\}\xi'_\varsigma)}(\vec{p},\frac{1}{2}) \\ = \frac{1}{2^6m^4}\frac{1}{(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}(p)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}(p)\Lambda_{maa'}(\vec{p},1)\Lambda_{mbb'}(\vec{p},1) \\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},2) = \frac{1}{(4!)^2}\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\eta_\varsigma\eta'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\xi_\varsigma\}\xi'_\varsigma)}(\vec{p},\frac{1}{2}) \\ = \frac{1}{2^6m^4}\frac{1}{(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}(-p)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}(-p)\Lambda_{maa'}(\vec{p},1)\Lambda_{mbb'}(\vec{p},1)\end{cases}$$

Thm. 1.2.4.

$$\begin{cases}\Lambda_{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},2)\\ =\frac{1}{(2m)^4}\frac{1}{(4!)^2}[(m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(m-i\gamma^a p_a)\gamma^4]_{\eta_\varsigma\eta'_\varsigma}[(m-i\gamma^b p_b)\gamma^4]_{\xi_\varsigma\}\xi'_\varsigma)}\\ =\frac{1}{(2\sqrt{2}m)^4}\frac{1}{(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}(p)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}(p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})(\eta_{bb'}+\frac{p_b p^+_{b'}}{m^2})\\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(\vec{p},2)\\ =\frac{1}{(2m)^4}\frac{1}{(4!)^2}[(-m-i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(-m-i\gamma^a p_a)\gamma^4]_{\eta_\varsigma\eta'_\varsigma}[(-m-i\gamma^b p_b)\gamma^4]_{\xi_\varsigma\}\xi'_\varsigma)}\\ =\frac{1}{(2\sqrt{2}m)^4}\frac{1}{(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}(-p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}(-p)\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}(-p)\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}(-p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})(\eta_{bb'}+\frac{p_b p^+_{b'}}{m^2})\end{cases}$$

$$\Downarrow$$

1.3 Plane wave solutions of derived to Klein-Gordon equation for spin-2 particles [16]

Thm. 1.3.1.

$$\begin{cases}\partial^c F_{c|ab}+m^2 A_{ab}=0, F_{c|ab}=\partial_c A_{ab}-\partial_a A_{cb}, \delta^{ab}A_{ab}=0, A_{ab}=A_{ba}\\ A_{ab}=(\frac{1}{2im})^2(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}\end{cases}$$

$$A_{ab}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\varepsilon_{ab}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\varepsilon_{ab}(\vec{p},h)=\frac{1}{(i\sqrt{2})^2}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)=\frac{1}{(i\sqrt{2})^2}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)$$

Cor. 1.3.1.

$$\begin{cases}\varepsilon_{ab}(\vec{p},2)=\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},1)\\ \varepsilon_{ab}(\vec{p},1)=\frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},1)]\\ \varepsilon_{ab}(\vec{p},0)=\frac{1}{\sqrt{6}}[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},-1)+\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},1)+2\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},0)]\\ \varepsilon_{ab}(\vec{p},-1)=\frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},-1)]\\ \varepsilon_{ab}(\vec{p},-2)=\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},-1)\end{cases}$$

Pro. 1.3.1. $\varepsilon_{ab}(\vec{p},h)=\varepsilon_{ba}(\vec{p},h), \delta^{ab}\varepsilon_{ab}(\vec{p},h)=0$

Thm. 1.3.2. $\sum_{h=2}^{-2}\varepsilon_{ab}(\vec{p},h)\varepsilon^+_{a'b'}(\vec{p},h)=\frac{1}{4}\{[\eta_{\{a(a'}+\frac{p_{\{a}p^+_{(a'}}{m^2}][\eta_{b\}b')}+\frac{p_{b\}}p^+_{b')}}{m^2}]-\frac{1}{3}[\delta_{\{ab\}}+\frac{p_{\{a}p_{b\}}}{m^2}][\delta_{(a'b')}+\frac{p^+_{(a'}p^+_{b')}}{m^2}]\}$

Thm. 1.3.3. $[A_{ab}(x),A^+_{a'b'}(x')]=\frac{i}{8}\{[\eta_{\{a(a'}-\frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{b\}b')}-\frac{\partial_{b\}}\partial^+_{b')}}{m^2}]-\frac{1}{3}[\delta_{\{ab\}}-\frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{(a'b')}-\frac{\partial^+_{(a'}\partial^+_{b')}}{m^2}]\}\Delta(x-x')$

Thm. 1.3.4. $\{A_{a\tau_\varsigma}(x),\bar{A}_{b\tau'_\varsigma}(x')\}=\frac{i}{2}\{[(\delta_{ab}-\frac{\partial_a\partial_b}{m^2})-\frac{1}{3}(\gamma_a-\frac{\partial_a}{m})(\gamma_b+\frac{\partial_b}{m})](m-\gamma^c\partial_c)\gamma^4\}_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

Thm. 1.3.5. $[A_a(x),\bar{A}_b(x')]=i(\delta_{ab}-\frac{\partial_a\partial_b}{m^2})\Delta(x-x')$

Lem. 1.3.1. $\eta^{bb'}=\delta^{bb'}-2\delta^{b4}\delta^{b'4}$

Thm. 1.3.6. $\Lambda_{\pm\tau_\varsigma\tau'_\varsigma}(\vec{p},\frac{1}{2})=\frac{1}{5}\Lambda_{maba'b'}(\vec{p},2)\eta^{bb'}\gamma^a\Lambda_{\mp}(\vec{p},\frac{1}{2})\gamma^{a'}$

Thm. 1.3.7. $\Lambda_{\pm ma\tau_\varsigma a'\tau'_\varsigma}(\vec{p},\frac{3}{2})=\frac{2}{5}\Lambda_{maba'b'}(\vec{p},2)\gamma^b\Lambda_{\mp}(\vec{p},\frac{1}{2})\gamma^{b'}$

Thm. 1.3.8. $\Lambda_{maa'}(\vec{p},1)=\frac{3}{5}\Lambda_{maba'b'}(\vec{p},2)\eta^{bb'}$

$$\Downarrow$$

1.4 Back to plane wave solution of B-W equation for spin-2 particles [16]

Thm. 1.4.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t)=0, \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t)=\frac{1}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(x)A_{ab}(\vec{r},t)$

$$\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\sqrt{\frac{m^4}{E}}[a(\vec{p},h)U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)=\frac{1}{8m^2}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(p)\varepsilon_{ab}(\vec{p},h), V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec{p},h)=\frac{1}{8m^2}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}(-p)\varepsilon_{ab}(\vec{p},h)$$

2 Third equivalent description of massive graviton equation

2.1 Equivalent description of massive graviton spin equation

Thm. 2.1.1. $(\partial_a+iS_{ab}\partial^b)_{\beta_\varsigma}{}^{\alpha_\varsigma}\psi_{\alpha_\varsigma c}=\frac{i}{\sqrt{2}}im^2\sigma^{ab}_{\varsigma\beta_\varsigma}A_{bc}, \psi_{\alpha_\varsigma c}:=\frac{i}{\sqrt{2}}\frac{i}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}F_{a|bc}, S_{ab}:=i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}$

$$A_{bc}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\varepsilon_{bc}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$F_{a|bc}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}[ip_a\varepsilon_{bc}(\vec{p},h)-ip_b\varepsilon_{ac}(\vec{p},h)][a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}-b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

$$\psi_{\alpha_\varsigma c}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\frac{-i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}p_a\varepsilon_{bc}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}-b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

2.2 Plane wave solutions and projection operators for massive graviton field F_{ab}

Def. 2.2.1. $\lambda_{abc}(\vec{p},h) := [ip_a\varepsilon_{bc}(\vec{p},h) - ip_b\varepsilon_{ac}(\vec{p},h)]$

Cor. 2.2.1. $F_{abc}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}[ip_a\varepsilon_{bc}(\vec{p},h) - ip_b\varepsilon_{ac}(\vec{p},h)][a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Thm. 2.2.1. $\sum\limits_{h=2}^{-2}\lambda_{abc}(\vec{p},h)\lambda^+_{a'b'c'}(\vec{p},h) = \frac{1}{2}p_{[a}p^+_{[a'}\eta_{b]c'}\eta_{b']c} + \frac{1}{2}p_{[a}p^+_{[a'}\eta_{b]b']}[\eta_{cc'} + \frac{p_cp^+_{c'}}{m^2}] - \frac{1}{3}p_{[a}\delta_{b]c}p^+_{[a'}\delta_{b']c'}$

Proof: $\sum\limits_{h=2}^{-2}\lambda_{abc}(\vec{p},h)\lambda^+_{a'b'c'}(\vec{p},h)$

$= \sum\limits_{h=2}^{-2}[ip_a\varepsilon_{bc}(\vec{p},h) - ip_b\varepsilon_{ac}(\vec{p},h)][ip_{a'}\varepsilon_{b'c'}(\vec{p},h) - ip_{b'}\varepsilon_{a'c'}(\vec{p},h)]^+$

$= p_ap^+_{a'}\sum\limits_{h=2}^{-2}\varepsilon_{bc}(\vec{p},h)\varepsilon^+_{b'c'}(\vec{p},h) + p_bp^+_{b'}\sum\limits_{h=2}^{-2}\varepsilon_{ac}(\vec{p},h)\varepsilon^+_{a'c'}(\vec{p},h)$

$- p_ap^+_{b'}\sum\limits_{h=2}^{-2}\varepsilon_{bc}(\vec{p},h)\varepsilon^+_{a'c'}(\vec{p},h) - p_bp^+_{a'}\sum\limits_{h=2}^{-2}\varepsilon_{ac}(\vec{p},h)\varepsilon^+_{b'c'}(\vec{p},h)$

$= p_ap^+_{a'}\frac{1}{4}\{[\eta_{\{b(b'} + \frac{p_{\{b}p^+_{(b'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{\{bc\}} + \frac{p_{\{b}p_{c\}}}{m^2}][\delta_{(b'c')} + \frac{p^+_{(b'}p^+_{c')}}{m^2}]\}$

$+ p_bp^+_{b'}\frac{1}{4}\{[\eta_{\{a(a'} + \frac{p_{\{a}p^+_{(a'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{\{ac\}} + \frac{p_{\{a}p_{c\}}}{m^2}][\delta_{(a'c')} + \frac{p^+_{(a'}p^+_{c')}}{m^2}]\}$

$- p_ap^+_{b'}\frac{1}{4}\{[\eta_{\{b(a'} + \frac{p_{\{b}p^+_{(a'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{\{bc\}} + \frac{p_{\{b}p_{c\}}}{m^2}][\delta_{(a'c')} + \frac{p^+_{(a'}p^+_{c')}}{m^2}]\}$

$- p_bp^+_{a'}\frac{1}{4}\{[\eta_{\{a(b'} + \frac{p_{\{a}p^+_{(b'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{\{ac\}} + \frac{p_{\{a}p_{c\}}}{m^2}][\delta_{(b'c')} + \frac{p^+_{(b'}p^+_{c')}}{m^2}]\}$

$= +p_ap^+_{a'}\{\frac{1}{4}[\eta_{\{b(b'} + \frac{p_{\{b}p^+_{(b'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{bc} + \frac{p_bp_c}{m^2}][\delta_{b'c'} + \frac{p^+_{b'}p^+_{c'}}{m^2}]\}$

$+ p_bp^+_{b'}\{\frac{1}{4}[\eta_{\{a(a'} + \frac{p_{\{a}p^+_{(a'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{ac} + \frac{p_ap_c}{m^2}][\delta_{a'c'} + \frac{p^+_{a'}p^+_{c'}}{m^2}]\}$

$- p_ap^+_{b'}\{\frac{1}{4}[\eta_{\{b(a'} + \frac{p_{\{b}p^+_{(a'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{bc} + \frac{p_bp_c}{m^2}][\delta_{a'c'} + \frac{p^+_{a'}p^+_{c'}}{m^2}]\}$

$- p_bp^+_{a'}\{\frac{1}{4}[\eta_{\{a(b'} + \frac{p_{\{a}p^+_{(b'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{ac} + \frac{p_ap_c}{m^2}][\delta_{b'c'} + \frac{p^+_{b'}p^+_{c'}}{m^2}]\}$

$= \frac{1}{2}(p_ap^+_{a'}\eta_{bc'}\eta_{cb'} + p_bp^+_{b'}\eta_{ac'}\eta_{a'c} - p_ap^+_{b'}\eta_{ca'}\eta_{bc'} - p_bp^+_{a'}\eta_{ac'}\eta_{cb'})$

$+ \frac{1}{2}(p_ap^+_{a'}\eta_{bb'} + p_bp^+_{b'}\eta_{aa'} - p_ap^+_{b'}\eta_{ba'} - p_bp^+_{a'}\eta_{ab'})[\eta_{cc'} + \frac{p_cp^+_{c'}}{m^2}]$

$- \frac{1}{3}(p_a\delta_{bc} - p_b\delta_{ac})(p^+_{a'}\delta_{b'c'} - p^+_{b'}\delta_{a'c'})$

$= \frac{1}{2}p_{[a}p^+_{[a'}\eta_{b]c'}\eta_{b']c} + \frac{1}{2}p_{[a}p^+_{[a'}\eta_{b]b']}[\eta_{cc'} + \frac{p_cp^+_{c'}}{m^2}] - \frac{1}{3}p_{[a}\delta_{b]c}p^+_{[a'}\delta_{b']c'}$ □

Thm. 2.2.2. $[F_{abc}(x), F^+_{a'b'c'}(x')] = -i\{\frac{1}{2}\partial_{[a}\partial^+_{[a'}\eta_{b]c'}\eta_{b']c} + \frac{1}{2}\partial_{[a}\partial^+_{[a'}\eta_{b]b']}[\eta_{cc'} - \frac{\partial_c\partial^+_{c'}}{m^2}] - \frac{1}{3}\partial_{[a}\delta_{b]c}\partial^+_{[a'}\delta_{b']c'}\}\Delta(x-x')$

2.3 Plane wave solutions and projection operators for massive graviton field Ψ_{α_ς}

Def. 2.3.1. $\lambda_{\alpha_\varsigma c}(\vec{p},h) := \frac{-i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}p_a\varepsilon_{bc}(\vec{p},h)$

Cor. 2.3.1. $\psi_{\alpha_\varsigma c}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}\lambda_{\alpha_\varsigma c}(\vec{p},h)[a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$

Thm. 2.3.1. $\sum\limits_{h=1}^{-1}\lambda_{\alpha_\varsigma c}(\vec{p},h)\lambda^+_{\alpha'_\varsigma c'}(\vec{p},h) = -\frac{1}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}p_ap^+_{a'}[\eta_{\{b(b'}\eta_{c\}c')} - \frac{1}{3}\delta_{\{bc\}}\delta_{(b'c')}] - \frac{1}{2m^2}\sigma^{aa'}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_{a'}p_cp^+_{c'}$

Proof: $\sum\limits_{h=1}^{-1}\lambda_{\alpha_\varsigma c}(\vec{p},h)\lambda^+_{\alpha'_\varsigma c'}(\vec{p},h)$

$= \sum\limits_{h=1}^{-1}\frac{-i}{\sqrt{2}}\sigma^{ab}_{\varsigma\alpha_\varsigma}p_a\varepsilon_{bc}(\vec{p},h)\frac{-i}{\sqrt{2}}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}p^+_{a'}\varepsilon^+_{b'c'}(\vec{p},h)$

$= -\frac{1}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}p_ap^+_{a'}\sum\limits_{h=1}^{-1}\varepsilon_{bc}(\vec{p},h)\varepsilon^+_{b'c'}(\vec{p},h)$

$= -\frac{1}{2}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}p_ap^+_{a'}\frac{1}{4}\{[\eta_{\{b(b'} + \frac{p_{\{b}p^+_{(b'}}{m^2}][\eta_{c\}c')} + \frac{p_{c\}}p^+_{c')}}{m^2}] - \frac{1}{3}[\delta_{\{bc\}} + \frac{p_{\{b}p_{c\}}}{m^2}][\delta_{(b'c')} + \frac{p^+_{(b'}p^+_{c')}}{m^2}]\}$

$= -\frac{1}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}p_ap^+_{a'}\{2[\eta_{bb'} + \frac{p_bp^+_{b'}}{m^2}][\eta_{cc'} + \frac{p_cp^+_{c'}}{m^2}] + 2[\eta_{cb'} + \frac{p_cp^+_{b'}}{m^2}][\eta_{bc'} + \frac{p_bp^+_{c'}}{m^2}] - \frac{4}{3}[\delta_{bc} + \frac{p_bp_c}{m^2}][\delta_{b'c'} + \frac{p^+_{b'}p^+_{c'}}{m^2}]\}$

$= -\frac{1}{4}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}p_ap^+_{a'}\{\eta_{bb'}[\eta_{cc'} + \frac{p_cp^+_{c'}}{m^2}] + \eta_{cb'}\eta_{bc'} - \frac{2}{3}\delta_{bc}\delta_{b'c'}]\}$

$= -\frac{1}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}p_ap^+_{a'}[\eta_{\{b(b'}\eta_{c\}c')} - \frac{1}{3}\delta_{\{bc\}}\delta_{(b'c')}] - \frac{1}{2m^2}\sigma^{aa'}_{\alpha_\varsigma\alpha'_\varsigma}p_ap_{a'}p_cp^+_{c'}$ □

Thm. 2.3.2. $[\psi_{\alpha_\varsigma c}(x), \psi^+_{\alpha'_\varsigma c'}(x')] = i\{\frac{1}{2m^2}\sigma^{aa'}_{\alpha_\varsigma\alpha'_\varsigma}\partial_a\partial_{a'}\partial_c\partial^+_{c'} - \frac{1}{8}\sigma^{ab}_{\varsigma\alpha_\varsigma}\sigma^{a'b'}_{\varsigma\alpha'_\varsigma}\partial_a\partial^+_{a'}[\eta_{\{b(b'}\eta_{c\}c')} - \frac{1}{3}\delta_{\{bc\}}\delta_{(b'c')}]\}\Delta(x-x')$

3 Fourth equivalent description of massive graviton equation

3.1 Definitions of various physical quantities formassive graviton

Def. 3.1.1. $\begin{cases} \textit{Weyl complex tensor } C^{\alpha_\varsigma\beta_\kappa} := \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}C^{ab\beta_\kappa} = \frac{i}{2}\sigma^{\alpha_\varsigma}_{\varsigma ab}\frac{i}{2}\sigma^{\beta_\kappa}_{\kappa cd}C^{abcd} \\ \psi_{\alpha_\varsigma\beta_\varsigma} = (\frac{i}{\sqrt{2}})^2 C_{\alpha_\varsigma\beta_\varsigma}, \psi^+_{\alpha_\varsigma\beta_\varsigma} = -\psi_{\alpha_{-\varsigma}\beta_{-\varsigma}} \end{cases}$

Def. 3.1.2. *Gravitational curvature spinor*
$\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa} := \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{C_\kappa D_\kappa}_{\beta_\kappa}\psi^{\alpha_\varsigma\beta_\kappa} = \frac{i}{\sqrt{2}}\frac{i\varsigma}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{C_\kappa D_\kappa}_{\beta_\kappa}C^{ab\beta_\kappa} = \frac{i}{\sqrt{2}}\frac{i\varsigma}{\sqrt{2}}S_{ab}{}^{A_\varsigma B_\varsigma}\frac{i}{\sqrt{2}}\frac{i\kappa}{\sqrt{2}}S_{cd}{}^{C_\kappa D_\kappa}C^{abcd}$

Cor. 3.1.1. $\psi^{\alpha_\varsigma\beta_\kappa} = \frac{i\varsigma}{\sqrt{2}}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\frac{i\kappa}{\sqrt{2}}\sigma^{\beta_\kappa}_{C_\kappa D_\kappa}\psi^{A_\varsigma B_\varsigma C_\kappa D_\kappa}$

Cor. 3.1.2. $\psi_{\alpha_\varsigma\beta_\varsigma} = \frac{i\varsigma}{\sqrt{2}}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\frac{i\varsigma}{\sqrt{2}}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma} = -\frac{1}{2}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}, [\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}]^* = \sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}$

Def. 3.1.3.

$$\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{2n}}(n) := \Gamma^{k_\varsigma}_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{2n}}(n)\Gamma^{\overbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}^{2n}}_{k_\varsigma}(n) = (\frac{i\varsigma}{\sqrt{2}})^n\frac{1}{(2n)!}\underbrace{\sigma^{(A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots)}_{n}$$

$$\Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{2n}}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n) := \Gamma^{\overbrace{\alpha_\varsigma\beta_\varsigma\cdots}^{2n}}_{k_\varsigma}(n)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(n) = (\frac{i\varsigma}{\sqrt{2}})^n\frac{1}{(2n)!}\overbrace{\sigma^{\alpha_\varsigma}_{(A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots)}^{n}$$

3.2 Two equivalent descriptions of general commutation rules for massive particles

Thm. 3.2.1.

$$\begin{cases} [\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(x), \psi^+_{\underbrace{A'_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma\cdots}_{2n}}(x')] = i\frac{(i\varsigma)^{2n}}{2^{2n-1}}\overbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^d_{D_\varsigma D'_\varsigma}\cdots}^{2n}\overbrace{\partial_a\partial_b\partial_c\partial_d\cdots}^{2n}\Delta(x-x') \\ \psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(x) = (\frac{i\varsigma}{\sqrt{2}})^n\underbrace{\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\cdots}_{n}\psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(x) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} [\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(x), \psi^+_{\underbrace{\alpha'_\varsigma\beta'_\varsigma\cdots}_{n}}(x')] = \frac{i}{2^{n-1}}\overbrace{\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\cdots}^{n}\overbrace{\partial_a\partial_b\partial_c\partial_d\cdots}^{2n}\Delta(x-x') \\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma D_\varsigma\cdots}_{2n}}(x) = (\frac{i\varsigma}{\sqrt{2}})^n\overbrace{\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\sigma^{\beta_\varsigma}_{C_\varsigma D_\varsigma}\cdots}^{n}\psi_{\underbrace{\alpha_\varsigma\beta_\varsigma\cdots}_{n}}(x) \end{cases}$$

3.3 Commutation rules for linear gravitational field $\psi_{\alpha_\varsigma\beta_\varsigma}$

Thm. 3.3.1.

$$\begin{cases} [\psi_{\alpha_\varsigma\beta_\varsigma}(x), \psi^+_{\alpha'_\varsigma\beta'_\varsigma}(x')] = \frac{i}{2}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x') \\ [\psi_{\alpha_\varsigma\beta_\varsigma}(x), \psi_{\rho_\varsigma\sigma_\varsigma}(x')] = \frac{i}{32}m^4\delta_{\{\alpha_\varsigma(\rho_\varsigma}\delta_{\beta_\varsigma\}\sigma_\varsigma)}\Delta(x-x') \\ [\psi_{\alpha'_\varsigma\beta'_\varsigma}(x), \psi_{\rho'_\varsigma\sigma'_\varsigma}(x')] = \frac{i}{32}m^4\delta_{\{\alpha'_\varsigma(\rho'_\varsigma}\delta_{\beta'_\varsigma\}\sigma'_\varsigma)}\Delta(x-x') \end{cases}$$

Proof: $[\psi_{\alpha_\varsigma\beta_\varsigma}(x), \psi^+_{\alpha'_\varsigma\beta'_\varsigma}(x')]$
$= \frac{1}{4}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\beta'_\varsigma}[\psi_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}, \psi^+_{A'_\varsigma B'_\varsigma C'_\varsigma D'_\varsigma}]$
$= \frac{i}{2^5}\sigma^{A_\varsigma B_\varsigma}_{\alpha_\varsigma}\sigma^{C_\varsigma D_\varsigma}_{\beta_\varsigma}\sigma^{A'_\varsigma B'_\varsigma}_{\alpha'_\varsigma}\sigma^{C'_\varsigma D'_\varsigma}_{\beta'_\varsigma}(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}(\sigma,i\varsigma)^c_{C_\varsigma C'_\varsigma}(\sigma,i\varsigma)^d_{D_\varsigma D'_\varsigma}\Delta(x-x')$
$= \frac{i}{2}\Gamma^{abcd}_{\alpha_\varsigma\alpha'_\varsigma\beta_\varsigma\beta'_\varsigma}(2)\partial_a\partial_b\partial_c\partial_d\Delta(x-x')$
$= \frac{i}{2}\Gamma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}(1)\Gamma^{cd}_{\beta_\varsigma\beta'_\varsigma}(1)\partial_a\partial_b\partial_c\partial_d\Delta(x-x')$
$= \frac{i}{2}\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma}\sigma^{cd}_{\beta_\varsigma\beta'_\varsigma}\partial_a\partial_b\partial_c\partial_d\Delta(x-x')$ □

3.4 Commutation rules for linear gravitational field C_{abcd}

Cor. 3.4.1. $\sigma^{ab}_{\alpha_\varsigma\alpha'_\varsigma} = -\frac{1}{2}\sigma^{ac}_{\varsigma\alpha_\varsigma}\delta_{cd}\sigma^{db}_{-\varsigma\alpha'_\varsigma}, \sigma^{\alpha'_\varsigma\alpha_\varsigma}_{ab} = -\frac{1}{2}\sigma^{\alpha_\varsigma}_{\varsigma ac}\delta^{cd}\sigma^{\alpha'_\varsigma}_{-\varsigma db}$

Lem. 3.4.1.
$2\sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma}\partial_c\partial_{c'} = \sigma^{\alpha_\varsigma}_{\varsigma ab}\sigma_{\varsigma\alpha_\varsigma cd}\sigma^{\alpha'_\varsigma}_{\varsigma a'b'}\sigma_{\varsigma\alpha'_\varsigma c'd'}\eta^{cc'}\partial^d\partial^{+d'} = (S_{abcd} - \varsigma\varepsilon_{abcd})(S_{a'b'c'd'} - \varsigma\varepsilon_{a'b'c'd'})\eta^{cc'}\partial^d\partial^{+d'}$
$2\sigma^{\alpha_\varsigma}_{-\varsigma ab}\sigma^{\alpha'_\varsigma}_{-\varsigma a'b'}\sigma^{cc'}_{\alpha_\varsigma\alpha'_\varsigma}\partial_c\partial_{c'} = \sigma^{\alpha_\varsigma}_{-\varsigma ab}\sigma_{-\varsigma\alpha_\varsigma cd}\sigma^{\alpha'_\varsigma}_{-\varsigma a'b'}\sigma_{-\varsigma\alpha'_\varsigma c'd'}\eta^{cc'}\partial^d\partial^{+d'} = (S_{abcd} + \varsigma\varepsilon_{abcd})(S_{a'b'c'd'} + \varsigma\varepsilon_{a'b'c'd'})\eta^{cc'}\partial^d\partial^{+d'}$

Cor. 3.4.2. $(S_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'} + \varepsilon_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'})\eta^{\tilde{c}\tilde{c}'}\partial^{\tilde{d}}\partial^{+\tilde{d}'}\Delta(x-x')$
$= [(\delta_{a\tilde{c}}\delta_{b\tilde{d}} - \delta_{a\tilde{d}}\delta_{b\tilde{c}})\varepsilon_{a'b'\tilde{c}'\tilde{d}'} + \varepsilon_{ab\tilde{c}\tilde{d}}(\delta_{a'\tilde{c}'}\delta_{b'\tilde{d}'} - \delta_{a'\tilde{d}'}\delta_{b'\tilde{c}'})]\eta^{\tilde{c}\tilde{c}'}\partial^{\tilde{d}}\partial^{+\tilde{d}'}\Delta(x-x')$
$= [(\eta^{\tilde{c}'}_a\partial_b\partial^{+\tilde{d}'} - \eta^{\tilde{c}'}_b\partial_a\partial^{+\tilde{d}'})\varepsilon_{a'b'\tilde{c}'\tilde{d}'} + \varepsilon_{ab\tilde{c}\tilde{d}}(\eta^{\tilde{c}}_{a'}\partial^+_{b'}\partial^{\tilde{d}} - \eta^{\tilde{c}}_{b'}\partial^+_{a'}\partial^{\tilde{d}})]\Delta(x-x')$

Cor. 3.4.3. $(S_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'})\eta^{\tilde{a}\tilde{a}'}\partial^{\tilde{b}}\partial^{+\tilde{b}'}\Delta(x-x')$
$=[(\eta_c^{\tilde{a}'}\partial_d\partial^{+\tilde{b}'}-\eta_d^{\tilde{a}'}\partial_c\partial^{+\tilde{b}'})\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{\tilde{a}\tilde{b}cd}(\eta_{c'}^{\tilde{a}}\partial_{d'}^{+}\partial^{\tilde{b}}-\eta_{d'}^{\tilde{a}}\partial_{c'}^{+}\partial^{\tilde{b}})]\Delta(x-x')$

Cor. 3.4.4.
$[(\eta_a^{\tilde{c}'}\partial_b\partial^{+\tilde{d}'}-\eta_b^{\tilde{c}'}\partial_a\partial^{+\tilde{d}'})\varepsilon_{a'b'\tilde{c}'\tilde{d}'}+\varepsilon_{ab\tilde{c}\tilde{d}}(\eta_{a'}^{\tilde{c}}\partial_{b'}^{+}\partial^{\tilde{d}}-\eta_{b'}^{\tilde{c}}\partial_{a'}^{+}\partial^{\tilde{d}})]$
$[(\eta_c^{\tilde{a}'}\partial_d\partial^{+\tilde{b}'}-\eta_d^{\tilde{a}'}\partial_c\partial^{+\tilde{b}'})\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{\tilde{a}\tilde{b}cd}(\eta_{c'}^{\tilde{a}}\partial_{d'}^{+}\partial^{\tilde{b}}-\eta_{d'}^{\tilde{a}}\partial_{c'}^{+}\partial^{\tilde{b}})]\Delta(x-x')$
$=(\eta_{[a}^{\tilde{c}'}\partial_{b]}\partial^{+\tilde{d}'}\varepsilon_{a'b'\tilde{c}'\tilde{d}'}+\varepsilon_{ab\tilde{c}\tilde{d}}\eta_{<a'}^{\tilde{c}}\partial_{b'>}^{+}\partial^{\tilde{d}})(\eta_{[c}^{\tilde{a}'}\partial_{d]}\partial^{+\tilde{b}'}\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{\tilde{a}\tilde{b}cd}(\eta_{<c'}^{\tilde{a}}\partial_{d'>}^{+}\partial^{\tilde{b}})\Delta(x-x')$

Cor. 3.4.5. $C_{abcd}=\frac{1}{2}(\sigma_{-\varsigma ab}^{\alpha_\varsigma'}\sigma_{-\varsigma cd}^{\beta_\varsigma'}\psi_{\alpha_\varsigma'\beta_\varsigma'}^{+}+\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\varsigma cd}^{\beta_\varsigma}\psi_{\alpha_\varsigma\beta_\varsigma})$

Thm. 3.4.1.
$$\begin{cases}[C_{abcd}(x),C_{a'b'c'd'}^{+}(x')]?=\frac{i}{4}\{\eta_{[a<a'}\partial_{b]}\partial_{b'>}^{+}\eta_{[c<c'}\partial_{d]}\partial_{d'>}^{+}+\eta_{[c<a'}\partial_{d]}\partial_{b'>}^{+}\eta_{[a<c'}\partial_{b]}\partial_{d'>}^{+}\}\Delta(x-x')\\ [C_{abcd}(x),C_{a'b'c'd'}(x')]=\frac{i}{4}\{\delta_{[a<a'}\partial_{b]}\partial_{b'>}\eta_{[c<c'}\partial_{d]}\partial_{d'>}+\delta_{[c<a'}\partial_{d]}\partial_{b'>}\eta_{[a<c'}\partial_{b]}\partial_{d'>}\}\Delta(x-x')\\ [C_{abcd}^{+}(x),C_{a'b'c'd'}^{+}(x')]=\frac{i}{4}\{\delta_{[a<a'}\partial_{b]}^{+}\partial_{b'>}^{+}\eta_{[c<c'}\partial_{d]}^{+}\partial_{d'>}^{+}+\delta_{[c<a'}\partial_{d]}^{+}\partial_{b'>}^{+}\eta_{[a<c'}\partial_{b]}^{+}\partial_{d'>}^{+}\}\Delta(x-x')\\ [C_{ab}^{\alpha_\varsigma}(x),C_{a'b'}^{\alpha_\varsigma'}(x')]=-\frac{i}{2}\{\eta_{[a<a'}\partial_{b]}\partial_{b'>}^{+}\sigma_{cd}^{\alpha_\varsigma'\alpha_\varsigma}\partial^c\partial^d+\frac{1}{2}(\sigma_{-\varsigma c[a}^{\alpha_\varsigma'}\partial_{b]}\partial^c)(\sigma_{-\varsigma c'<a'}^{\alpha_\varsigma}\partial_{b'>}^{+}\partial^{+c'})\}\Delta(x-x')\end{cases}$$

Proof: $[C_{abcd}(x),C_{a'b'c'd'}^{+}(x')]$
$=\frac{1}{4}[[\sigma_{-\varsigma ab}^{\alpha_\varsigma'}\sigma_{-\varsigma cd}^{\beta_\varsigma'}\psi_{\alpha_\varsigma'\beta_\varsigma'}^{+}(x)+\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\varsigma cd}^{\beta_\varsigma}\psi_{\alpha_\varsigma\beta_\varsigma}(x)],[\sigma_{-\varsigma a'b'}^{\alpha_\varsigma}\sigma_{-\varsigma c'd'}^{\beta_\varsigma}\psi_{\alpha_\varsigma\beta_\varsigma}(x')+\sigma_{\varsigma a'b'}^{\alpha_\varsigma'}\sigma_{\varsigma c'd'}^{\beta_\varsigma'}\psi_{\alpha_\varsigma'\beta_\varsigma'}^{+}(x')]]$
$=\frac{1}{4}\{\sigma_{-\varsigma ab}^{\alpha_\varsigma'}\sigma_{-\varsigma cd}^{\beta_\varsigma'}\sigma_{-\varsigma a'b'}^{\alpha_\varsigma}\sigma_{-\varsigma c'd'}^{\beta_\varsigma}[\psi_{\alpha_\varsigma'\beta_\varsigma'}^{+}(x),\psi_{\alpha_\varsigma\beta_\varsigma}(x')]+\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\varsigma cd}^{\beta_\varsigma}\sigma_{\varsigma a'b'}^{\alpha_\varsigma'}\sigma_{\varsigma c'd'}^{\beta_\varsigma'}[\psi_{\alpha_\varsigma\beta_\varsigma}(x),\psi_{\alpha_\varsigma'\beta_\varsigma'}^{+}(x')]\}$
$=\frac{i}{8}\{\sigma_{-\varsigma ab}^{\alpha_\varsigma'}\sigma_{-\varsigma cd}^{\beta_\varsigma'}\sigma_{-\varsigma a'b'}^{\alpha_\varsigma}\sigma_{-\varsigma c'd'}^{\beta_\varsigma}+\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\varsigma cd}^{\beta_\varsigma}\sigma_{\varsigma a'b'}^{\alpha_\varsigma'}\sigma_{\varsigma c'd'}^{\beta_\varsigma'}\}\sigma_{\alpha_\varsigma\alpha_\varsigma'}^{\tilde{a}\tilde{b}}\sigma_{\beta_\varsigma\beta_\varsigma'}^{\tilde{c}\tilde{d}}\partial_{\tilde{a}}\partial_{\tilde{b}}\partial_{\tilde{c}}\partial_{\tilde{d}}\Delta(x-x')$
$=\frac{i}{32}\{2\sigma_{-\varsigma ab}^{\alpha_\varsigma'}\sigma_{-\varsigma a'b'}^{\alpha_\varsigma}2\sigma_{-\varsigma cd}^{\beta_\varsigma'}\sigma_{-\varsigma c'd'}^{\beta_\varsigma}+2\sigma_{\varsigma ab}^{\alpha_\varsigma}\sigma_{\varsigma a'b'}^{\alpha_\varsigma'}2\sigma_{\varsigma cd}^{\beta_\varsigma}\sigma_{\varsigma c'd'}^{\beta_\varsigma'}\}\sigma_{\alpha_\varsigma\alpha_\varsigma'}^{\tilde{c}\tilde{d}}\partial_{\tilde{c}}\partial_{\tilde{d}}\sigma_{\beta_\varsigma\beta_\varsigma'}^{\tilde{a}\tilde{b}}\partial_{\tilde{a}}\partial_{\tilde{b}}\Delta(x-x')$
$=\frac{i}{32}\{(S_{ab\tilde{c}\tilde{d}}-\varsigma\varepsilon_{ab\tilde{c}\tilde{d}})(S_{a'b'\tilde{c}'\tilde{d}'}-\varsigma\varepsilon_{a'b'\tilde{c}'\tilde{d}'})(S_{cd\tilde{a}\tilde{b}}-\varsigma\varepsilon_{cd\tilde{a}\tilde{b}})(S_{c'd'\tilde{a}'\tilde{b}'}-\varsigma\varepsilon_{c'd'\tilde{a}'\tilde{b}'})$
$+(S_{ab\tilde{c}\tilde{d}}+\varsigma\varepsilon_{ab\tilde{c}\tilde{d}})(S_{a'b'\tilde{c}'\tilde{d}'}+\varsigma\varepsilon_{a'b'\tilde{c}'\tilde{d}'})(S_{cd\tilde{a}\tilde{b}}+\varsigma\varepsilon_{cd\tilde{a}\tilde{b}})(S_{c'd'\tilde{a}'\tilde{b}'}+\varsigma\varepsilon_{c'd'\tilde{a}'\tilde{b}'})\}\eta^{\tilde{c}\tilde{c}'}\partial^{\tilde{d}}\partial^{+\tilde{d}'}\eta^{\tilde{a}\tilde{a}'}\partial^{\tilde{b}}\partial^{+\tilde{b}'}\Delta(x-x')$
$=\frac{i}{32}\{(S_{ab\tilde{c}\tilde{d}}-\varsigma\varepsilon_{ab\tilde{c}\tilde{d}})(S_{a'b'\tilde{c}'\tilde{d}'}-\varsigma\varepsilon_{a'b'\tilde{c}'\tilde{d}'})(S_{\tilde{a}\tilde{b}cd}-\varsigma\varepsilon_{\tilde{a}\tilde{b}cd})(S_{\tilde{a}'\tilde{b}'c'd'}-\varsigma\varepsilon_{\tilde{a}'\tilde{b}'c'd'})$
$+(S_{ab\tilde{c}\tilde{d}}+\varsigma\varepsilon_{ab\tilde{c}\tilde{d}})(S_{a'b'\tilde{c}'\tilde{d}'}+\varsigma\varepsilon_{a'b'\tilde{c}'\tilde{d}'})(S_{\tilde{a}\tilde{b}cd}+\varsigma\varepsilon_{\tilde{a}\tilde{b}cd})(S_{\tilde{a}'\tilde{b}'c'd'}+\varsigma\varepsilon_{\tilde{a}'\tilde{b}'c'd'})\}\eta^{\tilde{a}\tilde{a}'}\eta^{\tilde{c}\tilde{c}'}\partial^{\tilde{b}}\partial^{+\tilde{b}'}\partial^{\tilde{d}}\partial^{+\tilde{d}'}\Delta(x-x')$
$=\frac{i}{16}\{(S_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'}S_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'}\varepsilon_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'})$
$+S_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'}\varepsilon_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+S_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'}S_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+S_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'}\varepsilon_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'}$
$+\varepsilon_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'}S_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'}\varepsilon_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'}S_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'}\}$
$\eta^{\tilde{a}\tilde{a}'}\eta^{\tilde{c}\tilde{c}'}\partial^{\tilde{b}}\partial^{+\tilde{b}'}\partial^{\tilde{d}}\partial^{+\tilde{d}'}\Delta(x-x')$
$=\frac{i}{16}\{(S_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'}+\varepsilon_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'})(S_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'})$
$+(S_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'}+\varepsilon_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'})(S_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'})\}$
$\eta^{\tilde{a}\tilde{a}'}\eta^{\tilde{c}\tilde{c}'}\partial^{\tilde{b}}\partial^{+\tilde{b}'}\partial^{\tilde{d}}\partial^{+\tilde{d}'}\Delta(x-x')$
$=\frac{i}{16}\{4\eta_{[a<a'}\partial_{b]}\partial_{b'>}^{+}\eta_{[c<c'}\partial_{d]}\partial_{d'>}^{+}\Delta(x-x')$
$+(S_{ab\tilde{c}\tilde{d}}\varepsilon_{a'b'\tilde{c}'\tilde{d}'}+\varepsilon_{ab\tilde{c}\tilde{d}}S_{a'b'\tilde{c}'\tilde{d}'})(S_{\tilde{a}\tilde{b}cd}\varepsilon_{\tilde{a}'\tilde{b}'c'd'}+\varepsilon_{\tilde{a}\tilde{b}cd}S_{\tilde{a}'\tilde{b}'c'd'})\eta^{\tilde{a}\tilde{a}'}\eta^{\tilde{c}\tilde{c}'}\partial^{\tilde{b}}\partial^{+\tilde{b}'}\partial^{\tilde{d}}\partial^{+\tilde{d}'}\Delta(x-x')\}$
$=\frac{i}{16}\{4\eta_{[a<a'}\partial_{b]}\partial_{b'>}^{+}\eta_{[c<c'}\partial_{d]}\partial_{d'>}^{+}+4\eta_{[c<a'}\partial_{d]}\partial_{b'>}^{+}\eta_{[a<c'}\partial_{b]}\partial_{d'>}^{+}\}\Delta(x-x')$
$=\frac{i}{4}\{\eta_{[a<a'}\partial_{b]}\partial_{b'>}^{+}\eta_{[c<c'}\partial_{d]}\partial_{d'>}^{+}+\eta_{[c<a'}\partial_{d]}\partial_{b'>}^{+}\eta_{[a<c'}\partial_{b]}\partial_{d'>}^{+}\}\Delta(x-x')$ □

4 Complex direct computation of covariant commutation rules for A_{ab}

4.1 Mathematical preparation

4.1.1 Reverse inference

Cor. 4.1.1. $[A_{ab}(x),A_{a'b'}^{+}(x')]$
$=\frac{1}{m^4}\frac{i}{2^9(4!)^2}\{64m^2(\delta_{ab}\delta_{cd}-2\delta_{c\{a}\delta_{b\}d})+64(\delta_{ab}S_{cedf}+\delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f\}\{64m^2(\delta_{a'b'}\delta_{c'd'}-2\delta_{c'(a'}\delta_{b')d'})+64(\delta_{a'b'}S_{c'e'd'f'}+\delta_{\{a'[c'}S_{e']b'\}d'f'})\partial'^{e'}\partial'^{f'}\}[\eta_{cc'}-\frac{\partial_c\partial_{c'}^{+}}{m^2}][\eta_{dd'}-\frac{\partial_d\partial_{d'}^{+}}{m^2}]\Delta(x-x')$

Lem. 4.1.1.
$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)]=4[\delta_{ab}\delta_{cd}-\delta_{a[c}\delta_{d]b}]$
$tr[\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)]=\delta_{ad}S_{bcef}+\delta_{a[b}S_{c]def}+\delta_{d[b}S_{c]aef}$
$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e,\varsigma)\gamma_d(\varsigma)S_{ef}(e,\varsigma)]=-\{\delta_{ad}\varepsilon_{bcef}+\delta_{a[b}\varepsilon_{c]def}+\delta_{d[b}\varepsilon_{c]aef}\}$

4.1.2 Series lemmas

Lem. 4.1.2. $tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}_b(x)]=4im\delta_{ab}$

Proof: $tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}_b(x)]$
$=tr\{\bar{C}\gamma_a(\varsigma)[im\gamma_b(\varsigma)C-2S_{bc}(e,\varsigma)C\partial^c]\}$
$=tr\{\gamma_a(\varsigma)[im\gamma_b(\varsigma)-2S_{bc}(e,\varsigma)\partial^c]\}$
$=imtr\{\gamma_a(\varsigma)\gamma_b(\varsigma)\}$
$=4im\delta_{ab}$ □

Lem. 4.1.3. $tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}_b(x)\bar{C}\gamma_c(\varsigma)\mathbb{X}_d(x)] = -4m^2(\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b}) + 4(\delta_{ac}S_{bedf} + \delta_{\{a[b}S_{e]c\}df})\partial^e\partial^f$

Proof: $tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}_b(x)\bar{C}\gamma_c(\varsigma)\mathbb{X}_d(x)]$
$= tr\{\bar{C}\gamma_a(\varsigma)[im\gamma_b(\varsigma)C - 2S_{be}(e,\varsigma)C\partial^e]\bar{C}\gamma_c(\varsigma)[im\gamma_d(\varsigma)C - 2S_{df}(e,\varsigma)C\partial^f]\}$
$= tr\{\gamma_a(\varsigma)[im\gamma_b(\varsigma) - 2S_{be}(e,\varsigma)\partial^e]\gamma_c(\varsigma)[im\gamma_d(\varsigma) - 2S_{df}(e,\varsigma)\partial^f]\}$
$= (im)^2 tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] + 4tr[\gamma_a(\varsigma)S_{be}(e,\varsigma)\gamma_c(\varsigma)S_{df}(e,\varsigma)]\partial^e\partial^f$
$= -4m^2(\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b}) + 4(\delta_{ac}S_{bedf} + \delta_{\{a[b}S_{e]c\}df})\partial^e\partial^f$ □

Lem. 4.1.4. $[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta_\varsigma\xi_\varsigma}\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma\}}(x)$
$= 64m^2(\delta_{ab}\delta_{cd} - 2\delta_{c\{a}\delta_{b\}d}) + 64(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f$

Proof: $[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta_\varsigma\xi_\varsigma}\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma\}}(x)$
$= 4tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}^c(x)]tr[\bar{C}\gamma_b(\varsigma)\mathbb{X}^d(x)]+4tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}^d(x)]tr[\bar{C}\gamma_b(\varsigma)\mathbb{X}^c(x)]+8tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}^c(x)\bar{C}\gamma_b(\varsigma)\mathbb{X}^d(x)]+8tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}^d(x)\bar{C}\gamma_b(\varsigma)\mathbb{X}$
$= 4tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}_c(x)]tr[\bar{C}\gamma_b(\varsigma)\mathbb{X}_d(x)] + 4tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}_d(x)]tr[\bar{C}\gamma_b(\varsigma)\mathbb{X}_c(x)] + 16tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}_c(x)\bar{C}\gamma_b(\varsigma)\mathbb{X}_d(x)]$
$= -64m^2\delta_{ac}\delta_{bd} - 64m^2\delta_{ad}\delta_{bc} - 64m^2\{\delta_{ac}\delta_{bd} - \delta_{a[b}\delta_{d]c}\} + 64(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f$
$= 64m^2(\delta_{ab}\delta_{cd} - 2\delta_{c\{a}\delta_{b\}d}) + 64(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f$ □

Lem. 4.1.5. $[\gamma_{a'}(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}[\gamma_{b'}(\varsigma)C]^{\eta'_\varsigma\xi'_\varsigma}\mathbb{X}^{+c'}_{\{\lambda'_\varsigma\mu'_\varsigma}(x)\mathbb{X}^{+d'}_{\eta'_\varsigma\xi'_\varsigma\}}(x')$
$= 64m^2(\delta_{a'b'}\delta_{c'd'} - 2\delta_{c'(a'}\delta_{b')d'}) + 64(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'}S_{e']b'\}d'f'})\partial'^{+e'}\partial'^{+f'}$

4.1.3 Series calculation I

Lem. 4.1.6. $\delta_{\{a[c}S_{e]b\}df}\delta_{(a'[c'}S_{e']b')d'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= [m^2\eta_{\{a(a'} - \partial_{\{a}\partial^+_{(a'}][m^2\eta_{b\}b')} - \partial_{b\}}\partial^+_{b')}] + 4(m^2\delta_{ab} + \partial_a\partial_b)(m^2\delta_{a'b'} + \partial^+_{a'}\partial^+_{b'}) - 4m^4\delta_{ab}\delta_{a'b'}$

Proof: $\delta_{ac}S_{ebdf}\delta_{a'c'}S_{e'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \delta_{ac}\delta_{a'c'}(\delta_{ed}\delta_{fb} - \delta_{ef}\delta_{db})(\delta_{e'd'}\delta_{f'b'} - \delta_{e'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \delta_{ac}\delta_{a'c'}(\partial_d\partial_b - m^2\delta_{db})(\partial^+_{d'}\partial^+_{b'} - m^2\delta_{d'b'})\eta^{cc'}\eta^{dd'}$
$= m^4(\eta_{aa'}\eta_{bb'} - \eta_{aa'}\frac{\partial_b\partial^+_{b'}}{m^2})$ □

Proof: $\delta_{ae}S_{cbdf}\delta_{a'e'}S_{c'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \delta_{ae}\delta_{a'e'}(\delta_{cd}\delta_{fb} - \delta_{cf}\delta_{db})(\delta_{c'd'}\delta_{f'b'} - \delta_{c'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \partial_a\partial^+_{a'}(\delta_{cd}\partial_b - \partial_c\delta_{db})(\delta_{c'd'}\partial^+_{b'} - \partial^+_{c'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}$
$= m^2\partial_a\partial^+_{a'}(\eta_{bb'} + \frac{2\partial_b\partial^+_{b'}}{m^2})$ □

Proof: $-\delta_{ac}S_{ebdf}\delta_{a'e'}S_{c'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= -\delta_{ac}\delta_{a'e'}(\delta_{ed}\delta_{fb} - \delta_{ef}\delta_{db})(\delta_{c'd'}\delta_{f'b'} - \delta_{c'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= -\delta_{ac}\partial^+_{a'}(\partial_d\partial_b - m^2\delta_{db})(\delta_{c'd'}\partial^+_{b'} - \partial^+_{c'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}$
$= m^2(\delta_{ab}\partial^+_{a'}\partial^+_{b'} - \eta_{bb'}\partial_a\partial^+_{a'})$ □

Proof: $-\delta_{ae}S_{cbdf}\delta_{a'c'}S_{e'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = m^2(\delta_{a'b'}\partial_a\partial_b - \eta_{bb'}\partial_a\partial^+_{a'})$ □

Proof: $\delta_{a[c}S_{e]bdf}\delta_{a'[c'}S_{e']b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= m^4(\eta_{aa'}\eta_{bb'} - \eta_{aa'}\frac{\partial_b\partial^+_{b'}}{m^2}) + m^2\partial_a\partial^+_{a'}(\eta_{bb'} + \frac{2\partial_b\partial^+_{b'}}{m^2}) + m^2(\delta_{ab}\partial^+_{a'}\partial^+_{b'} - \eta_{bb'}\partial_a\partial^+_{a'}) + m^2(\delta_{a'b'}\partial_a\partial_b - \eta_{bb'}\partial_a\partial^+_{a'})$
$= (m^4\eta_{aa'}\eta_{bb'} + 2\partial_a\partial^+_{a'}\partial_b\partial^+_{b'}) + m^2(\delta_{ab}\partial^+_{a'}\partial^+_{b'} - \eta_{bb'}\partial_a\partial^+_{a'}) + m^2(\delta_{a'b'}\partial_a\partial_b - \eta_{aa'}\partial_b\partial^+_{b'})$
$= (m^2\eta_{aa'} - \partial_a\partial^+_{a'})(m^2\eta_{bb'} - \partial_b\partial^+_{b'}) + (m^2\delta_{ab} + \partial_a\partial_b)(m^2\delta_{a'b'} + \partial^+_{a'}\partial^+_{b'}) - m^4\delta_{ab}\delta_{a'b'}$ □

4.1.4 Series calculation II

Lem. 4.1.7. $\delta_{ab}S_{cedf}\delta_{a'b'}S_{c'e'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = 3m^4\delta_{ab}\delta_{a'b'}$

Proof: $\delta_{ab}S_{cedf}\delta_{a'b'}S_{c'e'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \delta_{ab}\delta_{a'b'}(\delta_{cd}\delta_{fe} - \delta_{cf}\delta_{de})(\delta_{c'd'}\delta_{f'e'} - \delta_{c'f'}\delta_{d'e'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \delta_{ab}\delta_{a'b'}(\delta_{cd}m^2 - \partial_c\partial_d)(\delta_{c'd'}m^2 - \partial^+_{c'}\partial^+_{d'})\eta^{cc'}\eta^{dd'}$
$= 3m^4\delta_{ab}\delta_{a'b'}$ □

4.1.5 Series calculation III

Lem. 4.1.8.
$\delta_{ab}S_{cedf}\delta_{\{a'[c'}S_{e']b'\}d'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -2m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\frac{\partial^+_{a'}\partial^+_{b'}}{m^2})$
$\delta_{a'b'}S_{c'e'd'f'}\delta_{\{a[c}S_{e]b\}df}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -2m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{a'b'}\frac{\partial_a\partial_b}{m^2})$

Proof: $\delta_{ab}S_{cedf}\delta_{a'c'}S_{e'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \delta_{ab}\delta_{a'c'}(\delta_{cd}\delta_{fe} - \delta_{cf}\delta_{de})(\delta_{e'd'}\delta_{f'b'} - \delta_{e'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= \delta_{ab}\delta_{a'c'}(\delta_{cd}m^2 - \partial_c\partial_d)(\partial^+_{d'}\partial^+_{b'} - m^2\delta_{d'b'})\eta^{cc'}\eta^{dd'}$
$= -m^4(\delta_{ab}\delta_{a'b'} - \delta_{ab}\frac{\partial^+_{a'}\partial^+_{b'}}{m^2})$ □

Proof: $-\delta_{ab}S_{cedf}\delta_{a'e'}S_{c'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= -\delta_{ab}\delta_{a'e'}(\delta_{cd}\delta_{fe} - \delta_{cf}\delta_{de})(\delta_{c'd'}\delta_{f'b'} - \delta_{c'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= -\delta_{ab}(\delta_{cd}m^2 - \partial_c\partial_d)(\delta_{c'd'}\partial^+_{a'}\partial^+_{b'} - \partial^+_{a'}\partial^+_{c'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}$
$= -3m^2\delta_{ab}\partial^+_{a'}\partial^+_{b'}$ □

Proof: $\delta_{ab}S_{cedf}\delta_{a'[c'}S_{e']b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\frac{\partial^+_{a'}\partial^+_{b'}}{m^2})$ □

Proof: $\delta_{ab}S_{cedf}\delta_{b'[c'}S_{e']a'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\frac{\partial^+_{a'}\partial^+_{b'}}{m^2})$ □

4.1.6 Series calculation IV

Lem. 4.1.9. $(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'}S_{e']b'\}d'f'})\eta_{cc'}\eta_{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= -4m^4(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial^+_{a'}\partial^+_{b'}}{m^4}) + m^2[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2}]$

Proof: $(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'}S_{e']b'\}d'f'})\eta_{cc'}\eta_{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$
$= 3m^4\delta_{ab}\delta_{a'b'} - 2m^2(m^2\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\partial^+_{a'}\partial^+_{b'}) - 2m^2(m^2\delta_{ab}\delta_{a'b'} + 2\delta_{a'b'}\partial_a\partial_b) + [m^2\eta_{\{a(a'} - \partial_{\{a}\partial^+_{(a'}][m^2\eta_{b\}b')} - \partial_{b\}}\partial^+_{b')}] + 4(m^2\delta_{ab} + \partial_a\partial_b)(m^2\delta_{a'b'} + \partial^+_{a'}\partial^+_{b'}) - 4m^4\delta_{ab}\delta_{a'b'}$
$= -4m^4(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial^+_{a'}\partial^+_{b'}}{m^4}) + m^2(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2})(\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2})$ □

4.1.7 Proof of A_{ab} commutative relation

Thm. 4.1.1.

$$\begin{cases} [A_a(x), A^+_{a'}(x')] = i(\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x'), \bar{A}_{a'b'} := \eta^{c'}_{a'}\eta^{d'}_{b'}A^+_{c'd'} \\ [A_{ab}(x), A^+_{a'b'}(x')] = \frac{i}{8}\{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2}] - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{(a'b')} - \frac{\partial^+_{(a'}\partial^+_{b')}}{m^2}]\}\Delta(x-x') \\ [A_{ab}(x), \bar{A}_{a'b'}(x')] = \frac{i}{8}\{[\delta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}{m^2}][\delta_{b\}b')} - \frac{\partial_{b\}}\partial_{b')}}{m^2}] - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{(a'b')} - \frac{\partial_{(a'}\partial_{b')}}{m^2}]\}\Delta(x-x') \end{cases}$$

Proof: $[A_{ab}(x), A^+_{a'b'}(x')]$
$= \frac{1}{(i2m)^4}[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta_\varsigma\xi_\varsigma}[\gamma_{a'}(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}[\gamma_{b'}(\varsigma)C]^{\eta'_\varsigma\xi'_\varsigma}[\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x), \psi^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}(x')]$
$= \frac{1}{(i2m)^4}\frac{i}{2^5(4!)^2}[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta_\varsigma\xi_\varsigma}[\gamma_{a'}(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}[\gamma_{b'}(\varsigma)C]^{\eta'_\varsigma\xi'_\varsigma}\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma\}}(x)\mathbb{X}^{+c'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+d'}_{\eta'_\varsigma\xi'_\varsigma)}(x')$
$(\eta_{cc'} - \frac{\partial_c\partial^+_{c'}}{m^2})(\eta_{dd'} - \frac{\partial_d\partial^+_{d'}}{m^2})\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^{13}(3!)^2}[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta_\varsigma\xi_\varsigma}\mathbb{X}^c_{\{\lambda_\varsigma\mu_\varsigma}(x)\mathbb{X}^d_{\eta_\varsigma\xi_\varsigma\}}(x)[\gamma_{a'}(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}[\gamma_{b'}(\varsigma)C]^{\eta'_\varsigma\xi'_\varsigma}\mathbb{X}^{+c'}_{(\lambda'_\varsigma\mu'_\varsigma}(x')\mathbb{X}^{+d'}_{\eta'_\varsigma\xi'_\varsigma)}(x')$
$(\eta_{cc'} - \frac{\partial_c\partial^+_{c'}}{m^2})(\eta_{dd'} - \frac{\partial_d\partial^+_{d'}}{m^2})\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^{13}(3!)^2}\{64m^2(\delta_{ab}\delta_{cd} - 2\delta_{c\{a}\delta_{b\}d}) + 64(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f\}$
$\{64m^2(\delta_{a'b'}\delta_{c'd'} - 2\delta_{c'(a'}\delta_{b')d'}) + 64(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'}S_{e']b'\}d'f'})\partial'^{+e'}\partial'^{+f'}\}$
$(\eta_{cc'} - \frac{\partial_c\partial^+_{c'}}{m^2})(\eta_{dd'} - \frac{\partial_d\partial^+_{d'}}{m^2})\Delta(x-x')$
$= \frac{i}{72m^4}\{m^2(\delta_{ab}\delta_{cd} - 2\delta_{c\{a}\delta_{b\}d}) + (\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f\}$
$\{m^2(\delta_{a'b'}\delta_{c'd'} - 2\delta_{c'(a'}\delta_{b')d'}) + (\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'}S_{e']b'\}d'f'})\partial^{+e'}\partial^{+f'}\}$
$(\eta_{cc'} - \frac{\partial_c\partial^+_{c'}}{m^2})(\eta_{dd'} - \frac{\partial_d\partial^+_{d'}}{m^2})\Delta(x-x')$
$= \frac{1}{36}| - 2i(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial^+_{a'}\partial^+_{b'}}{m^4}) - 2i(\delta_{ab} - \frac{\partial_a\partial_b}{m^2})(\delta_{a'b'} - \frac{\partial^+_{a'}\partial^+_{b'}}{m^2}) + 2i(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2})(\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2})$
$+ 4i(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial^+_{a'}\partial^+_{b'}}{m^4}) - 4i(\delta_{ab} - \frac{\partial_a\partial_b}{m^2})(\delta_{a'b'} - \frac{\partial^+_{a'}\partial^+_{b'}}{m^2}) + 2i(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2})(\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2})$
$- 2i(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial^+_{a'}\partial^+_{b'}}{m^4}) + \frac{i}{2}(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2})(\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2})\Delta(x-x')$
$= \frac{i}{8}\{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2}] - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{(a'b')} - \frac{\partial^+_{(a'}\partial^+_{b')}}{m^2}]\}\Delta(x-x')$ □

4.2 Reverse verification of field commutative relation

Lem. 4.2.1. $\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}=\frac{1}{4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab},\psi^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}=\frac{1}{4}\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}A^+_{a'b'}$

Lem. 4.2.2. $\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}[\delta_{ab}-\frac{\partial_a\partial_b}{m^2}]=$

$$\begin{bmatrix}\frac{1}{2}(\sigma,i\varsigma)_{[a}(\sigma,-i\varsigma)_{c]}\sigma_y\partial^c & im(\sigma,i\varsigma)_a\sigma_y\\ im(\sigma,-i\varsigma)_a\sigma_y & -\frac{1}{2}(\sigma,-i\varsigma)_{[a}(\sigma,i\varsigma)_{c]}\sigma_y\partial^c\end{bmatrix}_{\{\lambda_\varsigma\mu_\varsigma}\begin{bmatrix}\frac{1}{2}(\sigma,i\varsigma)_{[b}(\sigma,-i\varsigma)_{d]}\sigma_y\partial^d & im(\sigma,i\varsigma)_b\sigma_y\\ im(\sigma,-i\varsigma)_b\sigma_y & -\frac{1}{2}(\sigma,-i\varsigma)_{[b}(\sigma,i\varsigma)_{d]}\sigma_y\partial^d\end{bmatrix}_{\eta_\varsigma\xi_\varsigma\}}[\delta^{ab}-\frac{\partial^a\partial^b}{m^2}]$$

$=0$

Thm. 4.2.1. $[\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma},\psi^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}]=\frac{i}{2^5(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}][\eta_{bb'}-\frac{\partial_b\partial^+_{b'}}{m^2}]\Delta(x-x')$

Proof: $[\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma},\psi^+_{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma}]$

$=\frac{1}{2^4}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma}[A_{ab},A^+_{a'b'}]$

$=\frac{1}{2^4(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}[A_{ab},A^+_{a'b'}]$

$=\frac{i}{2^5(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}\{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}][\eta_{bb'}-\frac{\partial_b\partial^+_{b'}}{m^2}]-\frac{1}{3}[\delta_{ab}-\frac{\partial_a\partial_b}{m^2}][\delta_{a'b'}-\frac{\partial^+_{a'}\partial^+_{b'}}{m^2}]\}\Delta(x-x')$

$=\frac{i}{2^5(4!)^2}\mathbb{X}^a_{\{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma\}}\mathbb{X}^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}^{+b'}_{\eta'_\varsigma\xi'_\varsigma)}[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}][\eta_{bb'}-\frac{\partial_b\partial^+_{b'}}{m^2}]\Delta(x-x')$ ☐

4.3 Extraction of energy momentum operator for massive graviton field

Thm. 4.3.1. $P_u(2)=\int\psi^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec r,t)d^3\vec r$

Thm. 4.3.2. $P_u(2)=\int m^4A^{+ab}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}A_{ab}(\vec r,t)\}d^3\vec r+\int m^2F^{+ab|c}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{ab|c}(\vec r,t)\}d^3\vec r$
$+\frac{1}{4}\int F^{+ab|cd}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{ab|cd}(\vec r,t)\}d^3\vec r$

Proof: $P_u(2)=\int\psi^{+\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(\vec r,t)d^3\vec r$

$=\int\{\bar C[-im\gamma^{a'}(\varsigma)-2S^{a'c'}(e,\varsigma)\partial^+_{c'}]\}^{\lambda_\varsigma\mu_\varsigma}\{\bar C[-im\gamma^{b'}(\varsigma)-2S^{b'd'}(e,\varsigma)\partial^+_{d'}]\}^{\eta_\varsigma\xi_\varsigma}\frac{A^+_{a'b'}(\vec r,t)}{4}$
$\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}[im\gamma^a(\varsigma)C-2S^{ac}(e,\varsigma)C\partial_c]_{\lambda_\varsigma\mu_\varsigma}[im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]_{\eta_\varsigma\xi_\varsigma}\frac{A_{ab}(\vec r,t)}{4}d^3\vec r$

$=\frac{1}{16}\int tr\{\bar C[-im\gamma^{a'}(\varsigma)-2S^{a'c'}(e,\varsigma)\partial^+_{c'}]\{\bar C[-im\gamma^{b'}(\varsigma)-2S^{b'd'}(e,\varsigma)\partial^+_{d'}]\}^{\eta_\varsigma\xi_\varsigma}A^+_{a'b'}(\vec r,t)$
$\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}[im\gamma^a(\varsigma)C-2S^{ac}(e,\varsigma)C\partial_c][im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]_{\eta_\varsigma\xi_\varsigma}A_{ab}(\vec r,t)\}d^3\vec r$

$=\frac{1}{16}\int tr\{\bar C[-im\gamma^{a'}(\varsigma)-2S^{a'c'}(e,\varsigma)\partial^+_{c'}]\{\bar C[-im\gamma^{b'}(\varsigma)-2S^{b'd'}(e,\varsigma)\partial^+_{d'}]\}^{\eta_\varsigma\xi_\varsigma}A^+_{a'b'}(\vec r,t)$
$\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}[im\gamma^a(\varsigma)C-2S^{ac}(e,\varsigma)C\partial_c][im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]_{\eta_\varsigma\xi_\varsigma}A_{ab}(\vec r,t)\}d^3\vec r$

$=\frac{1}{4}\int\{\bar C[-im\gamma^{b'}(\varsigma)-2S^{b'd'}(e,\varsigma)\partial^+_{d'}]\}^{\eta_\varsigma\xi_\varsigma}A^+_{a'b'}(\vec r,t)m^2\delta^{a'a}\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}[im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]_{\eta_\varsigma\xi_\varsigma}A_{ab}(\vec r,t)\}d^3\vec r$
$+\frac{1}{4}\int S^{a'c'ac}\partial^+_{c'}\{\bar C[-im\gamma^{b'}(\varsigma)-2S^{b'd'}(e,\varsigma)\partial^+_{d'}]\}^{\eta_\varsigma\xi_\varsigma}A^+_{a'b'}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\partial_c[im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]_{\eta_\varsigma\xi_\varsigma}A_{ab}(\vec r,t)\}d^3\vec r$

$=\frac{1}{4}\int tr\{\bar C[-im\gamma^{b'}(\varsigma)-2S^{b'd'}(e,\varsigma)\partial^+_{d'}]A^+_{a'b'}(\vec r,t)m^2\delta^{a'a}\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}[im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]A_{ab}(\vec r,t)\}d^3\vec r$
$+\frac{1}{4}\int S^{a'c'ac}\partial^+_{c'}tr\{\bar C[-im\gamma^{b'}(\varsigma)-2S^{b'd'}(e,\varsigma)\partial^+_{d'}]A^+_{a'b'}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\partial_c[im\gamma^b(\varsigma)C-2S^{bd}(e,\varsigma)C\partial_d]A_{ab}(\vec r,t)\}d^3\vec r$

$=\int m^4\delta^{a'a}\delta^{b'b}A^+_{a'b'}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}A_{ab}(\vec r,t)\}d^3\vec r$
$+\int m^2S^{b'd'bd}\delta^{a'a}\partial^+_{d'}A^+_{a'b'}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\partial_dA_{ab}(\vec r,t)\}d^3\vec r$
$+\int m^2S^{a'c'ac}\delta^{b'b}\partial^+_{c'}A^+_{a'b'}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\partial_cA_{ab}(\vec r,t)\}$
$+\int S^{a'c'ac}S^{b'd'bd}\partial^+_{c'}\partial^+_{d'}A^+_{a'b'}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\partial_c\partial_dA_{ab}(\vec r,t)\}d^3\vec r$

$=\int m^4A^{+ab}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}A_{ab}(\vec r,t)\}d^3\vec r$
$+\int\frac{1}{2}m^2F^{+bc|a}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{bc|a}(\vec r,t)\}d^3\vec r+\int\frac{1}{2}m^2F^{+ac|b}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{ac|b}(\vec r,t)\}$
$+\int S^{b'd'bd}\partial^+_{d'}F^{+ac}_{b'}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}\partial_dF_{ac|b}(\vec r,t)\}d^3\vec r$

$=\int m^4A^{+ab}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}A_{ab}(\vec r,t)\}d^3\vec r$
$+\int\frac{1}{2}m^2F^{+bc|a}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{bc|a}(\vec r,t)\}d^3\vec r+\int\frac{1}{2}m^2F^{+ac|b}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{ac|b}(\vec r,t)\}$
$+\int\frac{1}{4}F^{+ac|bd}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{ac|bd}(\vec r,t)\}d^3\vec r$

$=\int m^4A^{+ab}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}A_{ab}(\vec r,t)\}d^3\vec r+\int m^2F^{+ab|c}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{ab|c}(\vec r,t)\}d^3\vec r$
$+\frac{1}{4}\int F^{+ab|cd}(\vec r,t)\frac{(-i\partial_u)(i\partial_t)}{(m^2-\nabla^2)^2}F_{ab|cd}(\vec r,t)\}d^3\vec r$ ☐

Pro. 4.3.1. $F_{ab|cd}=\partial_aF_{b|cd}-\partial_bF_{a|cd}=\partial_cF_{ab|d}-\partial_dF_{ab|c}=\partial_a\partial_cA_{bd}+\partial_b\partial_dA_{ac}-\partial_a\partial_dA_{bc}-\partial_b\partial_cA_{ad}$

Proof: $F_{ab|cd}=\partial_aF_{b|cd}-\partial_bF_{a|cd}$
$=\partial_a(\partial_cA_{bd}-\partial_dA_{bc})-\partial_b(\partial_cA_{ad}-\partial_dA_{ac})$
$=\partial_a\partial_cA_{bd}+\partial_b\partial_dA_{ac}-\partial_a\partial_dA_{bc}-\partial_b\partial_cA_{ad}$ ☐

Proof: $F_{ab|cd} = \partial_c F_{ab|d} - \partial_d F_{ab|c}$
$= \partial_c(\partial_a A_{bd} - \partial_b A_{ad}) - \partial_d(\partial_a A_{bc} - \partial_b A_{ac})$
$= \partial_a\partial_c A_{bd} + \partial_b\partial_d A_{ac} - \partial_a\partial_d A_{bc} - \partial_b\partial_c A_{ad}$ □

Pro. 4.3.2. $F_{ab|cd} = F_{cd|ab}, F_{ab|cd} = -F_{ba|cd}, F_{ab|cd} = -F_{ab|dc}, F_{ab|cd} = F_{ba|dc}$

5 New nolving methods for massive $s = 2$ potential commutation relations in 4D (trial and error)

5.1 Potential commutation relations of massive $s = 2$ B-W equation in 4D

Lem. 5.1.1. $K := (m - \gamma_a\partial^a)\gamma_0, \tilde{K} := CK^T\bar{C} = -\gamma_0(m + \gamma_a\partial^a), Q := (m - \gamma_a\partial^a), \tilde{Q} := (m + \gamma_a\partial^a)$

Lem. 5.1.2. $\begin{cases} tr[\gamma_a Q\gamma_{a'}\tilde{Q}] = -tr[\gamma_5\gamma_a Q\gamma_5\gamma_{a'}Q] = -tr[\gamma_a\gamma_5\tilde{Q}\gamma_{a'}\gamma_5\tilde{Q}] \\ tr[\gamma_a Q\gamma_{a'}\tilde{Q}\gamma_b Q\gamma_{b'}\tilde{Q}] = tr[\gamma_5\gamma_a Q\gamma_5\gamma_{a'}Q\gamma_5\gamma_b Q\gamma_5\gamma_{b'}Q] = tr[\gamma_a\gamma_5\tilde{Q}\gamma_{a'}\gamma_5\tilde{Q}\gamma_b\gamma_5\tilde{Q}\gamma_{b'}\gamma_5\tilde{Q}] \end{cases}$

Lem. 5.1.3. $tr[\gamma_a Q\gamma_{a'}\tilde{Q}] = tr[\gamma_a(m - \gamma_{a_1}\partial^{a_1})\gamma_{a'}(m + \gamma_{a_2}\partial^{a_2})]$
$= m^2 tr(\gamma_a\gamma_{a'}) - tr(\gamma_a\gamma_{a_1}\gamma_{a'}\gamma_{a_2})\partial^{a_1}\partial^{a_2}$
$= 4m^2\delta_{aa'} - 4(\delta_{aa_1}\delta_{a'a_2} - \delta_{aa'}\delta_{a_1a_2} + \delta_{aa_2}\delta_{a_1a'})\partial^{a_1}\partial^{a_2}$
$= 4m^2\delta_{aa'} - 4(2\partial_a\partial_{a'} - \delta_{aa'}m^2)$
$= 8(m^2\delta_{aa'} - \partial_a\partial_{a'})$

Proof: $tr[\gamma_a Q\gamma_{a'}\tilde{Q}\gamma_b Q\gamma_{b'}\tilde{Q}] = tr[\gamma_a(m - \gamma_{a_1}\partial^{a_1})\gamma_{a'}(m + \gamma_{a_2}\partial^{a_2})\gamma_b(m - \gamma_{a_3}\partial^{a_3})\gamma_{b'}(m + \gamma_{a_4}\partial^{a_4})]$
$= tr[\gamma_a\gamma_5(m + \gamma_{a_1}\partial^{a_1})\gamma_{a'}\gamma_5(m + \gamma_{a_2}\partial^{a_2})\gamma_b\gamma_5(m + \gamma_{a_3}\partial^{a_3})\gamma_{b'}\gamma_5(m + \gamma_{a_4}\partial^{a_4})]$
$= tr\{$
$[(\gamma_a\gamma_5 m\gamma_{a'}\gamma_5 m + \gamma_a\gamma_5\gamma_{a_1}\partial^{a_1}\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2}) + (\gamma_a\gamma_5\gamma_{a_1}\partial^{a_1}\gamma_{a'}\gamma_5 m + \gamma_a\gamma_5 m\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2})]$
$[(\gamma_b\gamma_5 m\gamma_{b'}\gamma_5 m + \gamma_b\gamma_5\gamma_{a_3}\partial^{a_3}\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4}) + (\gamma_b\gamma_5\gamma_{a_3}\partial^{a_3}\gamma_{b'}\gamma_5 m + \gamma_b\gamma_5 m\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4})]\}$

$= tr[(\gamma_a\gamma_5 m\gamma_{a'}\gamma_5 m + \gamma_a\gamma_5\gamma_{a_1}\partial^{a_1}\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_b\gamma_5 m\gamma_{b'}\gamma_5 m + \gamma_b\gamma_5\gamma_{a_3}\partial^{a_3}\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ tr[(\gamma_a\gamma_5\gamma_{a_1}\partial^{a_1}\gamma_{a'}\gamma_5 m + \gamma_a\gamma_5 m\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_b\gamma_5\gamma_{a_3}\partial^{a_3}\gamma_{b'}\gamma_5 m + \gamma_b\gamma_5 m\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4})]$

$= tr[(\gamma_a\gamma_5 m)(\gamma_{a'}\gamma_5 m)(\gamma_b\gamma_5 m)(\gamma_{b'}\gamma_5 m)] + tr[(\gamma_a\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_b\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ tr[(\gamma_a\gamma_5 m)(\gamma_{a'}\gamma_5 m)(\gamma_b\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4})] + tr[(\gamma_a\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_b\gamma_5 m)(\gamma_{b'}\gamma_5 m)]$
$+ tr[(\gamma_a\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{a'}\gamma_5 m)(\gamma_b\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b'}\gamma_5 m)] + tr[(\gamma_a\gamma_5 m)(\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_b\gamma_5 m)(\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ tr[(\gamma_a\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{a'}\gamma_5 m)(\gamma_b\gamma_5 m)(\gamma_{b'}\gamma_5\gamma_{a_4}\partial^{a_4})] + tr[(\gamma_a\gamma_5 m)(\gamma_{a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_b\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b'}\gamma_5 m)]$ □

Proof: $tr[\gamma_{\{a}Q\gamma_{(a'}\tilde{Q}\gamma_{b\}}Q\gamma_{b')}\tilde{Q}]$
$= tr[(\gamma_{\{a}\gamma_5 m)(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5 m)(\gamma_{b')}\gamma_5 m)] + tr[(\gamma_{\{a}\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{(a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_{b\}}\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ tr[(\gamma_{\{a}\gamma_5 m)(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})] + tr[(\gamma_{\{a}\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{(a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_{b\}}\gamma_5 m)(\gamma_{b')}\gamma_5 m)]$
$+ tr[(\gamma_{\{a}\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b')}\gamma_5 m)] + tr[(\gamma_{\{a}\gamma_5 m)(\gamma_{(a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_{b\}}\gamma_5 m)(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ tr[(\gamma_{\{a}\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5 m)(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})] + tr[(\gamma_{\{a}\gamma_5 m)(\gamma_{(a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_{b\}}\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b')}\gamma_5 m)]$
$= tr[(\gamma_{\{a}\gamma_5 m)(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5 m)(\gamma_{b')}\gamma_5 m)] + tr[(\gamma_{\{a}\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{(a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_{b\}}\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ 2tr[(\gamma_{\{a}\gamma_5 m)(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ tr[(\gamma_{\{a}\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5\gamma_{a_3}\partial^{a_3})(\gamma_{b')}\gamma_5 m)] + tr[(\gamma_{\{a}\gamma_5 m)(\gamma_{(a'}\gamma_5\gamma_{a_2}\partial^{a_2})(\gamma_{b\}}\gamma_5 m)(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$+ 2tr[(\gamma_{\{a}\gamma_5\gamma_{a_1}\partial^{a_1})(\gamma_{(a'}\gamma_5 m)(\gamma_{b\}}\gamma_5 m)(\gamma_{b')}\gamma_5\gamma_{a_4}\partial^{a_4})]$
$= m^4 tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}] + tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{a_2}\gamma_{b\}}\gamma_{a_3}\gamma_{b')}\gamma_{a_4}]\partial^{a_1}\partial^{a_2}\partial^{a_3}\partial^{a_4}$
$- 2m^2 tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{a_3}\gamma_{b')}\gamma_{a_4}]\partial^{a_3}\partial^{a_4}$
$+ m^2 tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{a_3}\gamma_{b')}]\partial^{a_1}\partial^{a_3} + m^2 tr[\gamma_{\{a}\gamma_{(a'}\gamma_{a_2}\gamma_{b\}}\gamma_{b')}\gamma_{a_4}]\partial^{a_2}\partial^{a_4}$
$- 2m^2 tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}\gamma_{a_4}]\partial^{a_1}\partial^{a_4}$
$= m^4 tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}] + tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{a_2}\gamma_{b\}}\gamma_{a_3}\gamma_{b')}\gamma_{a_4}]\partial^{a_1}\partial^{a_2}\partial^{a_3}\partial^{a_4}$
$- 2m^2 tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{a_1}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$+ m^2 tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{a_2}\gamma_{b')}]\partial^{a_1}\partial^{a_2} + m^2 tr[\gamma_{\{a}\gamma_{(a'}\gamma_{a_1}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$- 2m^2 tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$ □

Lem. 5.1.4. $tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{a_2}\gamma_{b\}}\gamma_{a_3}\gamma_{b')}\gamma_{a_4}]\partial^{a_1}\partial^{a_2}\partial^{a_3}\partial^{a_4}$
$= [\gamma_{\{a_1}\gamma_{a_2}\gamma_{(a'_3}\gamma_{a_4}\gamma_{a_5\}}\gamma_{a_6}\gamma_{a'_7)}\gamma_{a_8}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \delta_{a_1a_2}[\delta_{a_3a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_3a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_3a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$- \delta_{a_1a_3}[\delta_{a_2a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$+ \delta_{a_1a_4}[\delta_{a_2a_3}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$- \delta_{a_1a_5}[\delta_{a_2a_3}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7}) - \delta_{a_2a_4}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$

$+ \delta_{a_1a_6}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$- \delta_{a_1a_7}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_8} - \delta_{a_4a_6}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_6}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$+ \delta_{a_1a_8}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_7} - \delta_{a_4a_6}\delta_{a_5a_7} + \delta_{a_4a_7}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_5a_7} + \delta_{a_3a_7}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_4a_7} + \delta_{a_3a_7}\delta_{a_4a_6})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$

Thm. 5.1.1. $[A_{\underbrace{ab\cdot\cdot}_{n}}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')] = \frac{1}{m^{2n}}\frac{i}{2^{4n-1}[(2n)!]^2}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$
$\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$

Thm. 5.1.2. $[A_{ab}(x), A^+_{a'b'}(x')] = \frac{1}{m^4}\frac{i}{2^7(4!)^2}[(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}][(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}]$
$[(m-\gamma^{a_1}\partial_{a_1})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^{a_2}\partial_{a_2})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(m-\gamma^{a_3}\partial_{a_3})\gamma^4]_{\eta_\varsigma\eta'_\varsigma}[(m-\gamma^{a_4}\partial_{a_4})\gamma^4]_{\xi_\varsigma\}\xi'_\varsigma)}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}\eta^{a''}_{a'}\eta^{b''}_{b'}\{\frac{3!}{1!2!}\frac{2!}{2!0!0!}tr[\gamma_{\{a}Q\gamma_{(a''}\tilde{Q}]tr[\gamma_{b\}}Q\gamma_{b'')}\tilde{Q}] + \frac{3!}{2!1!}\frac{2!}{1!1!0!}tr[\gamma_{\{a}Q\gamma_{(a''}\tilde{Q}\gamma_{b\}}Q\gamma_{b'')}\tilde{Q}]\}\Delta(x-x')$

Proof: $[A_{ab}(x), A^+_{a'b'}(x')] = \frac{1}{m^4}\frac{i}{2^7(4!)^2}[(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}][(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}]$
$[(m-\gamma^{a_1}\partial_{a_1})\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^{a_2}\partial_{a_2})\gamma^4]_{\mu_\varsigma\mu'_\varsigma}[(m-\gamma^{a_3}\partial_{a_3})\gamma^4]_{\eta_\varsigma\eta'_\varsigma}[(m-\gamma^{a_4}\partial_{a_4})\gamma^4]_{\xi_\varsigma\}\xi'_\varsigma)}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^7(4!)^2}[(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma}][(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma}]$
$[(m-\gamma_{a_1}\partial^{a_1})\gamma_0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma_{a_2}\partial^{a_2})\gamma_0]_{\mu_\varsigma\mu'_\varsigma}[(m-\gamma_{a_3}\partial^{a_3})\gamma_0]_{\eta_\varsigma\eta'_\varsigma}[(m-\gamma_{a_4}\partial^{a_4})\gamma_0]_{\xi_\varsigma\}\xi'_\varsigma)}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^7(4!)^2}[(\bar{C}\gamma_a)^{\{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma\}}][(\gamma_{a'}C)^{(\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma)}]$
$[(m-\gamma_{a_1}\partial^{a_1})\gamma_0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma_{a_2}\partial^{a_2})\gamma_0]_{\mu_\varsigma\mu'_\varsigma}[(m-\gamma_{a_3}\partial^{a_3})\gamma_0]_{\eta_\varsigma\eta'_\varsigma}[(m-\gamma_{a_4}\partial^{a_4})\gamma_0]_{\xi_\varsigma\xi'_\varsigma}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}$
$[(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma} + (\bar{C}\gamma_a)^{\eta_\varsigma\xi_\varsigma}(\bar{C}\gamma_b)^{\lambda_\varsigma\mu_\varsigma} + (\bar{C}\gamma_a)^{\lambda_\varsigma\eta_\varsigma}(\bar{C}\gamma_b)^{\mu_\varsigma\xi_\varsigma} + (\bar{C}\gamma_a)^{\mu_\varsigma\xi_\varsigma}(\bar{C}\gamma_b)^{\lambda_\varsigma\eta_\varsigma}$
$+ (\bar{C}\gamma_a)^{\lambda_\varsigma\xi_\varsigma}(\bar{C}\gamma_b)^{\mu_\varsigma\eta_\varsigma} + (\bar{C}\gamma_a)^{\mu_\varsigma\eta_\varsigma}(\bar{C}\gamma_b)^{\lambda_\varsigma\xi_\varsigma}]$
$[(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma} + (\gamma_{a'}C)^{\eta'_\varsigma\xi'_\varsigma}(\gamma_{b'}C)^{\lambda'_\varsigma\mu'_\varsigma} + (\gamma_{a'}C)^{\lambda'_\varsigma\eta'_\varsigma}(\gamma_{b'}C)^{\mu'_\varsigma\xi'_\varsigma} + (\gamma_{a'}C)^{\mu'_\varsigma\xi'_\varsigma}(\gamma_{b'}C)^{\lambda'_\varsigma\eta'_\varsigma}$
$+ (\gamma_{a'}C)^{\lambda'_\varsigma\xi'_\varsigma}(\gamma_{b'}C)^{\mu'_\varsigma\eta'_\varsigma} + (\gamma_{a'}C)^{\mu'_\varsigma\eta'_\varsigma}(\gamma_{b'}C)^{\lambda'_\varsigma\xi'_\varsigma}]$
$[(m-\gamma_{a_1}\partial^{a_1})\gamma_0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma_{a_2}\partial^{a_2})\gamma_0]_{\mu_\varsigma\mu'_\varsigma}[(m-\gamma_{a_3}\partial^{a_3})\gamma_0]_{\eta_\varsigma\eta'_\varsigma}[(m-\gamma_{a_4}\partial^{a_4})\gamma_0]_{\xi_\varsigma\xi'_\varsigma}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}$
$[(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma} + (\bar{C}\gamma_a)^{\eta_\varsigma\xi_\varsigma}(\bar{C}\gamma_b)^{\lambda_\varsigma\mu_\varsigma} + (\bar{C}\gamma_a)^{\lambda_\varsigma\eta_\varsigma}(\bar{C}\gamma_b)^{\mu_\varsigma\xi_\varsigma} + (\bar{C}\gamma_a)^{\mu_\varsigma\xi_\varsigma}(\bar{C}\gamma_b)^{\lambda_\varsigma\eta_\varsigma}$
$+ (\bar{C}\gamma_a)^{\lambda_\varsigma\xi_\varsigma}(\bar{C}\gamma_b)^{\mu_\varsigma\eta_\varsigma} + (\bar{C}\gamma_a)^{\mu_\varsigma\eta_\varsigma}(\bar{C}\gamma_b)^{\lambda_\varsigma\xi_\varsigma}]$
$[(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}C)^{\eta'_\varsigma\xi'_\varsigma} + (\gamma_{a'}C)^{\eta'_\varsigma\xi'_\varsigma}(\gamma_{b'}C)^{\lambda'_\varsigma\mu'_\varsigma} + (\gamma_{a'}C)^{\lambda'_\varsigma\eta'_\varsigma}(\gamma_{b'}C)^{\mu'_\varsigma\xi'_\varsigma} + (\gamma_{a'}C)^{\mu'_\varsigma\xi'_\varsigma}(\gamma_{b'}C)^{\lambda'_\varsigma\eta'_\varsigma}$
$+ (\gamma_{a'}C)^{\lambda'_\varsigma\xi'_\varsigma}(\gamma_{b'}C)^{\mu'_\varsigma\eta'_\varsigma} + (\gamma_{a'}C)^{\mu'_\varsigma\eta'_\varsigma}(\gamma_{b'}C)^{\lambda'_\varsigma\xi'_\varsigma}]K_{\lambda_\varsigma\lambda'_\varsigma}K_{\mu_\varsigma\mu'_\varsigma}K_{\eta_\varsigma\eta'_\varsigma}K_{\xi_\varsigma\xi'_\varsigma}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}$
$[(\bar{C}\gamma_{\{a})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{b\}})^{\eta_\varsigma\xi_\varsigma} + (\bar{C}\gamma_{\{a})^{\lambda_\varsigma\eta_\varsigma}(\bar{C}\gamma_{b\}})^{\mu_\varsigma\xi_\varsigma} + (\bar{C}\gamma_{\{a})^{\lambda_\varsigma\xi_\varsigma}(\bar{C}\gamma_{b\}})^{\mu_\varsigma\eta_\varsigma}]$
$[(\gamma_{(a'}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b')}C)^{\eta'_\varsigma\xi'_\varsigma} + (\gamma_{(a'}C)^{\lambda'_\varsigma\eta'_\varsigma}(\gamma_{b')}C)^{\mu'_\varsigma\xi'_\varsigma} + (\gamma_{(a'}C)^{\lambda'_\varsigma\xi'_\varsigma}(\gamma_{b')}C)^{\mu'_\varsigma\eta'_\varsigma}]$
$K_{\lambda_\varsigma\lambda'_\varsigma}K_{\mu_\varsigma\mu'_\varsigma}K_{\eta_\varsigma\eta'_\varsigma}K_{\xi_\varsigma\xi'_\varsigma}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}$
$\{6tr[(\bar{C}\gamma_a)K(\gamma_{a'}C)K^T]tr[(\bar{C}\gamma_b)K(\gamma_{b'}C)K^T] + 6tr[(\bar{C}\gamma_a)K(\gamma_{b'}C)K^T]tr[(\bar{C}\gamma_b)K(\gamma_{a'}C)K^T]$
$+ 12tr[(\bar{C}\gamma_a)K(\gamma_{a'}C)K^T(\bar{C}\gamma_b)K(\gamma_{b'}C)K^T] + 12tr[(\bar{C}\gamma_a)K(\gamma_{b'}C)K^T(\bar{C}\gamma_b)K(\gamma_{a'}C)K^T]\}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}\{3tr[\bar{C}\gamma_{\{a}K\gamma_{(a'}CK^T]tr[\bar{C}\gamma_{b\}}K\gamma_{b')}CK^T] + 6tr[\bar{C}\gamma_{\{a}K\gamma_{(a'}CK^T\bar{C}\gamma_{b\}}K\gamma_{b')}CK^T]\}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}\{3tr[\gamma_{\{a}K\gamma_{(a'}\tilde{K}]tr[\gamma_{b\}}K\gamma_{b')}\tilde{K}] + 6tr[\gamma_{\{a}K\gamma_{(a'}\tilde{K}\gamma_{b\}}K\gamma_{b')}\tilde{K}]\}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}\eta^{a''}_{a'}\eta^{b''}_{b'}\{\frac{3!}{1!2!}\frac{2!}{2!0!0!}tr[\gamma_{\{a}Q\gamma_{(a''}\tilde{Q}]tr[\gamma_{b\}}Q\gamma_{b'')}\tilde{Q}] + \frac{3!}{2!1!}\frac{2!}{1!1!0!}tr[\gamma_{\{a}Q\gamma_{(a''}\tilde{Q}\gamma_{b\}}Q\gamma_{b'')}\tilde{Q}]\}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}\eta^{a''}_{a'}\eta^{b''}_{b'}$
$\{c^1_3\frac{2!}{2!0!0!}tr[\gamma_{\{a}\gamma_5\tilde{Q}\gamma_{(a''}\gamma_5\tilde{Q}]tr[\gamma_{b\}}\gamma_5\tilde{Q}\gamma_{b'')}\gamma_5\tilde{Q}] + c^2_3\frac{2!}{1!1!0!}tr[\gamma_{\{a}\gamma_5\tilde{Q}\gamma_{(a''}\gamma_5\tilde{Q}\gamma_{b\}}\gamma_5\tilde{Q}\gamma_{b'')}\gamma_5\tilde{Q}]\}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}\eta^{a''}_{a'}\eta^{b''}_{b'}$
$\{c^1_3\frac{2!}{2!0!0!}tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q]tr[\gamma_5\gamma_{b\}}Q\gamma_5\gamma_{b'')}Q] + c^2_3\frac{2!}{1!1!0!}tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q\gamma_5\gamma_{b\}}Q\gamma_5\gamma_{b'')}Q]\}\Delta(x-x')$
$= \frac{1}{m^4}\frac{i}{2^3(4!)^2}\eta^{a''}_{a'}\eta^{b''}_{b'}$
$\{\frac{3!}{1!2!}\frac{2!}{2!0!0!}tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q]tr[\gamma_5\gamma_{b\}}Q\gamma_5\gamma_{b'')}Q] + \frac{3!}{2!1!}\frac{2!}{1!1!0!}tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q\gamma_5\gamma_{b\}}Q\gamma_5\gamma_{b'')}Q]\}\Delta(x-x')$ □

Ass. 5.1.1. $[A_{\underbrace{a_1a_2\cdot\cdot}_{n}}(x), A^+_{\underbrace{a'_1a'_2\cdot\cdot}_{n}}(x')] = \frac{1}{m^{2n}}\frac{i}{2^{4n-1}[(2n)!]^2}\overbrace{(\bar{C}\gamma_{a_1})^{\lambda_\varsigma\mu_\varsigma}(\bar{C}\gamma_{a_2})^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'_1}C)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{a'_2}C)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}$
$\underbrace{[(m-\gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot\})}_{2n}\Delta(x-x')$

$= \frac{1}{m^{2n}} \frac{i}{2^{2n-1}[(2n)!]^2} \eta_{a_1'}^{a_1''} \eta_{a_2'}^{a_2''} \cdot \cdot \eta_{a_n'}^{a_n''} \sum_{j_1 \cdot \cdot j_n} \frac{[3^{n-2}(2n-1)]!}{j_1! \cdot \cdot j_n! [3^{n-2}(2n-1)-(j_1+\cdot\cdot+j_n)]!} \frac{n!}{(1!)^{j_1} \cdot \cdot (n!)^{j_n}}$

$tr^{j_1 \cdot \cdot j_n} [\gamma_{\{a_1} Q \gamma_{(a_1''} \tilde{Q} | \gamma_{a_2} Q \gamma_{a_2''} \tilde{Q} | \gamma_{a_3} Q \gamma_{a_3''} \tilde{Q} | \cdot \cdot | \gamma_{a_n\}} Q \gamma_{a_n'')} \tilde{Q}] \Delta(x-x'), 1j_1 + 2j_2 + \cdot + \cdot + nj_n = n$

5.2 Itemized solution 1

Proof:
$\delta_{a_1a_2}[\delta_{a_3a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_3a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_3a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \delta_{a_1a_2}[\delta_{a_3a_4}(2\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8}) - \delta_{a_3a_5}(\delta_{a_4a_6}\delta_{a_7a_8}) + \delta_{a_3a_6}(\delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \partial_{a_1}[\partial_{a_3}(2\partial_{a_5}\partial_{a_7} - \delta_{a_5a_7}m^2) - \delta_{a_3a_5}(m^2\partial_{a_7}) + \partial_{a_3}(m^2\delta_{a_5a_7})]$
$= 2\partial_{a_1}\partial_{a_3}\partial_{a_5}\partial_{a_7} - m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}$ □

Proof:
$-\delta_{a_1a_3}[\delta_{a_2a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= -\delta_{a_1a_3}[\delta_{a_2a_4}(2\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8}) - \delta_{a_2a_5}(\delta_{a_4a_6}\delta_{a_7a_8}) + \delta_{a_2a_6}(\delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= -\delta_{a_1a_3}[m^2(2\partial_{a_5}\partial_{a_7} - \delta_{a_5a_7}m^2) - \partial_{a_5}m^2\partial_{a_7} + m^4\delta_{a_5a_7}]$
$= -m^2\delta_{a_1a_3}\partial_{a_5}\partial_{a_7}$ □

Proof:
$+\delta_{a_1a_4}[\delta_{a_2a_3}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \delta_{a_1a_4}[\delta_{a_2a_3}(2\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8}) - \delta_{a_2a_5}(2\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8})$
$+ \delta_{a_2a_6}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= +\partial_{a_1}[\partial_{a_3}(2\partial_{a_5}\partial_{a_7} - \delta_{a_5a_7}m^2) - \partial_{a_5}(2\partial_{a_3}\partial_{a_7} - \delta_{a_3a_7}m^2) + m^2(\delta_{a_3a_5}\partial_{a_7} - \delta_{a_3a_7}\partial_{a_5} + \partial_{a_3}\delta_{a_5a_7})]$
$= m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}$ □

Proof:
$-\delta_{a_1a_5}[\delta_{a_2a_3}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7}) - \delta_{a_2a_4}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= -\delta_{a_1a_5}[\delta_{a_2a_3}(\delta_{a_4a_6}\delta_{a_7a_8}) - \delta_{a_2a_4}(2\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8}) + \delta_{a_2a_6}(2\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= -\delta_{a_1a_5}[\partial_{a_3}(m^2\partial_{a_7}) - m^2(2\partial_{a_3}\partial_{a_7} - \delta_{a_3a_7}m^2) + m^2(2\partial_{a_3}\partial_{a_7} - \delta_{a_3a_7}m^2)]$
$= -m^2\delta_{a_1a_5}\partial_{a_3}\partial_{a_7}$ □

Proof:
$+\delta_{a_1a_6}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \delta_{a_1a_6}[\delta_{a_2a_3}(\delta_{a_4a_8}\delta_{a_5a_7}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7}) + \delta_{a_2a_5}(2\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \partial_{a_1}[\partial_{a_3}(m^2\delta_{a_5a_7}) - m^2(\delta_{a_3a_5}\partial_{a_7} - \delta_{a_3a_7}\partial_{a_5} + \partial_{a_3}\delta_{a_5a_7}) + \partial_{a_5}(2\partial_{a_3}\partial_{a_7} - \delta_{a_3a_7}m^2)]$
$= \partial_{a_1}[-m^2(\delta_{a_3a_5}\partial_{a_7}) + \partial_{a_5}(2\partial_{a_3}\partial_{a_7})]$
$= 2\partial_{a_1}\partial_{a_3}\partial_{a_5}\partial_{a_7} - m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}$ □

Proof:
$-\delta_{a_1a_7}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_8} - \delta_{a_4a_6}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_6})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= -\delta_{a_1a_7}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_8}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_8}) + \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_8})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= -\delta_{a_1a_7}[\partial_{a_3}(\partial_{a_5}m^2) - m^2(\delta_{a_3a_5}m^2) + \partial_{a_5}(\partial_{a_3}m^2)]$
$= m^4\delta_{a_1a_7}\delta_{a_3a_5} - 2m^2\delta_{a_1a_7}\partial_{a_3}\partial_{a_5}$ □

Proof:
$+\delta_{a_1a_8}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_7} - \delta_{a_4a_6}\delta_{a_5a_7} + \delta_{a_4a_7}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_5a_7} + \delta_{a_3a_7}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_4a_7} + \delta_{a_3a_7}\delta_{a_4a_6})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \delta_{a_1a_8}[\delta_{a_2a_3}(2\delta_{a_4a_5}\delta_{a_6a_7} - \delta_{a_4a_6}\delta_{a_5a_7}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_5a_7} + \delta_{a_3a_7}\delta_{a_5a_6}) + \delta_{a_2a_5}(\delta_{a_3a_7}\delta_{a_4a_6})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \partial_{a_1}[\partial_{a_3}(2\partial_{a_5}\partial_{a_7} - m^2\delta_{a_5a_7}) - m^2(\delta_{a_3a_5}\partial_{a_7} - \partial_{a_3}\delta_{a_5a_7} + \delta_{a_3a_7}\partial_{a_5}) + \partial_{a_5}(\delta_{a_3a_7}m^2)]$
$= \partial_{a_1}[\partial_{a_3}(2\partial_{a_5}\partial_{a_7}) - m^2(\delta_{a_3a_5}\partial_{a_7})]$
$= 2\partial_{a_1}\partial_{a_3}\partial_{a_5}\partial_{a_7} - m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}$ □

Proof: $[\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6}\gamma_{a_7}\gamma_{a_8}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= \delta_{a_1a_2}[\delta_{a_3a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_3a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_3a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$-\delta_{a_1a_3}[\delta_{a_2a_4}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$+\delta_{a_1a_4}[\delta_{a_2a_3}(\delta_{a_5a_6}\delta_{a_7a_8} - \delta_{a_5a_7}\delta_{a_6a_8} + \delta_{a_5a_8}\delta_{a_6a_7}) - \delta_{a_2a_5}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$

$+ \delta_{a_2a_6}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$- \delta_{a_1a_5}[\delta_{a_2a_3}(\delta_{a_4a_6}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_6a_8} + \delta_{a_4a_8}\delta_{a_6a_7}) - \delta_{a_2a_4}(\delta_{a_3a_6}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_6a_8} + \delta_{a_3a_8}\delta_{a_6a_7})$
$+ \delta_{a_2a_6}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$+ \delta_{a_1a_6}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_7a_8} - \delta_{a_4a_7}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_7}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_7})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_7a_8} - \delta_{a_3a_7}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_7})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$- \delta_{a_1a_7}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_8} - \delta_{a_4a_6}\delta_{a_5a_8} + \delta_{a_4a_8}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_5a_8} + \delta_{a_3a_8}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_8} - \delta_{a_3a_6}\delta_{a_4a_8} + \delta_{a_3a_8}\delta_{a_4a_6}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$+ \delta_{a_1a_8}[\delta_{a_2a_3}(\delta_{a_4a_5}\delta_{a_6a_7} - \delta_{a_4a_6}\delta_{a_5a_7} + \delta_{a_4a_7}\delta_{a_5a_6}) - \delta_{a_2a_4}(\delta_{a_3a_5}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_5a_7} + \delta_{a_3a_7}\delta_{a_5a_6})$
$+ \delta_{a_2a_5}(\delta_{a_3a_4}\delta_{a_6a_7} - \delta_{a_3a_6}\delta_{a_4a_7} + \delta_{a_3a_7}\delta_{a_4a_6})]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= 2\partial_{a_1}\partial_{a_3}\partial_{a_5}\partial_{a_7} - m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}| - m^2\delta_{a_1a_3}\partial_{a_5}\partial_{a_7}| + m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}| - m^2\delta_{a_1a_5}\partial_{a_3}\partial_{a_7}|$
$+ 2\partial_{a_1}\partial_{a_3}\partial_{a_5}\partial_{a_7} - m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}| + m^4\delta_{a_1a_7}\delta_{a_3a_5} - 2m^2\delta_{a_1a_7}\partial_{a_3}\partial_{a_5}| + 2\partial_{a_1}\partial_{a_3}\partial_{a_5}\partial_{a_7} - m^2\partial_{a_1}\delta_{a_3a_5}\partial_{a_7}$
$= 6\partial_{a_1}\partial_{a_3}\partial_{a_5}\partial_{a_7}| - 2m^2\delta_{a_3a_5}\partial_{a_1}\partial_{a_7} - 2m^2\delta_{a_1a_7}\partial_{a_3}\partial_{a_5}| - m^2\delta_{a_1a_3}\partial_{a_5}\partial_{a_7} - m^2\delta_{a_1a_5}\partial_{a_3}\partial_{a_7}| + m^4\delta_{a_1a_7}\delta_{a_3a_5}$ □

Proof: $[\gamma_a\gamma_{a_2}\gamma_{a'}\gamma_{a_4}\gamma_b\gamma_{a_6}\gamma_{b'}\gamma_{a_8}]\partial^{a_2}\partial^{a_4}\partial^{a_6}\partial^{a_8}$
$= 6\partial_a\partial_{a'}\partial_b\partial_{b'} - m^2\delta_{aa'}\partial_b\partial_{b'} - m^2\delta_{ab}\partial_{a'}\partial_{b'} - 2m^2\delta_{a'b}\partial_a\partial_{b'} - 2m^2\delta_{ab'}\partial_{a'}\partial_b + m^4\delta_{ab'}\delta_{a'b}$ □

Proof: $\frac{1}{2^2}tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{a_2}\gamma_{b\}}\gamma_{a_3}\gamma_{b')}\gamma_{a_4}]\partial^{a_1}\partial^{a_2}\partial^{a_3}\partial^{a_4}$
$= 24\partial_a\partial_{a'}\partial_b\partial_{b'} - m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 4m^2\delta_{ab}\partial_{a'}\partial_{b'} - 2m^2\delta_{(a'\{b}\partial_{a\}}\partial_{b')} - 2m^2\delta_{\{a(b'}\partial_{a')}\partial_{b\}} + m^4\delta_{\{a(b'}\delta_{a')b\}}$
$= 24\partial_a\partial_{a'}\partial_b\partial_{b'} - m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 4m^2\delta_{ab}\partial_{a'}\partial_{b'} - 2m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 2m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + m^4\delta_{\{a(a'}\delta_{b\}b')}$
$= 24\partial_a\partial_{a'}\partial_b\partial_{b'} - 5m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 4m^2\delta_{ab}\partial_{a'}\partial_{b'} + m^4\delta_{\{a(a'}\delta_{b\}b')}$ □

5.3 Itemized solution 2

Proof: $\frac{1}{2^2}tr[\gamma_{\{a}Q\gamma_{(a'}\tilde{Q}\gamma_{b\}}Q\gamma_{b')}\tilde{Q}]$
$= m^4tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}] + tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{a_2}\gamma_{b\}}\gamma_{a_3}\gamma_{b')}\gamma_{a_4}]\partial^{a_1}\partial^{a_2}\partial^{a_3}\partial^{a_4}$
$+ m^2tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{a_2}\gamma_{b')}]\partial^{a_1}\partial^{a_2} + m^2tr[\gamma_{\{a}\gamma_{(a'}\gamma_{a_1}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$- 2m^2tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{a_1}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2} - 2m^2tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$ □

Proof: $\frac{1}{2^2}m^4tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}]$
$= m^4tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}]$
$= m^4(\delta_{\{a(a'}\delta_{b\}b')} - 4\delta_{ab}\delta_{a'b'} + \delta_{\{a(b'}\delta_{b\}a')})$
$= 2m^4\delta_{\{a(a'}\delta_{b\}b')} - 4m^4\delta_{ab}\delta_{a'b'}$ □

5.4 Itemized solution 3

Lem. 5.4.1. $\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6}$ *constant terms*
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5}) - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})$
$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5}) - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})$

Proof: $\frac{1}{2^2}tr(\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6})\partial^{a_2}\partial^{a_5}$
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_5} - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_5}$
$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5})\partial^{a_2}\partial^{a_5} - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})\partial^{a_2}\partial^{a_5}$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_2}\partial^{a_5}$
$= \partial_{a_1}(\delta_{a_3a_4}\partial_{a_6} - \partial_{a_3}\delta_{a_4a_6} + \delta_{a_3a_6}\partial_{a_4}) - \delta_{a_1a_3}(2\partial_{a_4}\partial_{a_6} - m^2\delta_{a_4a_6})$
$+ \delta_{a_1a_4}(2\partial_{a_3}\partial_{a_6} - m^2\delta_{a_3a_6}) - \partial_{a_1}(\partial_{a_3}\delta_{a_4a_6} - \partial_{a_4}\delta_{a_3a_6} + \partial_{a_6}\delta_{a_3a_4})$
$+ \delta_{a_1a_6}(m^2\delta_{a_3a_4})$
$= 2\partial_{a_1}(-\partial_{a_3}\delta_{a_4a_6} + \delta_{a_3a_6}\partial_{a_4}) - \delta_{a_1a_3}(2\partial_{a_4}\partial_{a_6} - m^2\delta_{a_4a_6}) + \delta_{a_1a_4}(2\partial_{a_3}\partial_{a_6} - m^2\delta_{a_3a_6}) + \delta_{a_1a_6}(m^2\delta_{a_3a_4})$
$= -2\partial_{a_1}\partial_{a_3}\delta_{a_4a_6} + 2\delta_{a_3a_6}\partial_{a_1}\partial_{a_4} - 2\delta_{a_1a_3}\partial_{a_4}\partial_{a_6} + 2\delta_{a_1a_4}\partial_{a_3}\partial_{a_6} + m^2\delta_{a_1a_3}\delta_{a_4a_6} - m^2\delta_{a_1a_4}\delta_{a_3a_6} + m^2\delta_{a_1a_6}\delta_{a_3a_4}$
$\frac{1}{2^2}m^2|tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{a_2}\gamma_{b')}]\partial^{a_1}\partial^{a_2}$
$= m^2| - 2\partial_{\{a}\partial_{(a'}\delta_{b\}b')} + 8\delta_{a'b'}\partial_a\partial_b - 2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 8\delta_{ab}\partial_{a'}\partial_{b'} + m^2\delta_{\{a(a'}\delta_{b\}b')} - 4m^2\delta_{ab}\delta_{a'b'} + m^2\delta_{\{a(b'}\delta_{b\}a')}$
$= 8m^2\delta_{a'b'}\partial_a\partial_b - 4m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 8m^2\delta_{ab}\partial_{a'}\partial_{b'} + 2m^4\delta_{\{a(a'}\delta_{b\}b')} - 4m^4\delta_{ab}\delta_{a'b'}$ □

Proof: $\frac{1}{2^2}tr(\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6})\partial^{a_3}\partial^{a_6}$
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_3}\partial^{a_6} - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})\partial^{a_3}\partial^{a_6}$
$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5})\partial^{a_3}\partial^{a_6} - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})\partial^{a_3}\partial^{a_6}$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_3}\partial^{a_6}$
$= \delta_{a_1a_2}(m^2\delta_{a_4a_5}) - 2\partial_{a_1}(\delta_{a_2a_4}\partial_{a_5} - \delta_{a_2a_5}\partial_{a_4})$
$+ \delta_{a_1a_4}(2\partial_{a_2}\partial_{a_5} - \delta_{a_2a_5}m^2) - \delta_{a_1a_5}(2\partial_{a_2}\partial_{a_4} - \delta_{a_2a_4}m^2)$
$= -2\partial_{a_1}\delta_{a_2a_4}\partial_{a_5} + 2\partial_{a_1}\delta_{a_2a_5}\partial_{a_4} + 2\delta_{a_1a_4}\partial_{a_2}\partial_{a_5} - 2\delta_{a_1a_5}\partial_{a_2}\partial_{a_4} + m^2\delta_{a_1a_2}\delta_{a_4a_5} - m^2\delta_{a_1a_4}\delta_{a_2a_5} + m^2\delta_{a_1a_5}\delta_{a_2a_4}$
$\frac{1}{2^2}m^2tr[\gamma_{\{a}\gamma_{(a'}\gamma_{a_1}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$= m^2| - 2\partial_{\{a}\delta_{(a'b\}}\partial_{b')} + 8\partial_a\delta_{a'b'}\partial_b + 8\delta_{ab}\partial_{a'}\partial_{b'} - 2\delta_{\{a(b'}\partial_{a')}\partial_{b\}} + m^2\delta_{\{a(a'}\delta_{b\}b')} - 4m^2\delta_{ab}\delta_{a'b'} + m^2\delta_{\{a(b'}\delta_{a')b\}}$
$= 8m^2\delta_{a'b'}\partial_a\partial_b - 4m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 8m^2\delta_{ab}\partial_{a'}\partial_{b'} + 2m^4\delta_{\{a(a'}\delta_{b\}b')} - 4m^4\delta_{ab}\delta_{a'b'}$ □

Proof: $\frac{1}{2^2}tr(\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6})\partial^{a_4}\partial^{a_6}$
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_4}\partial^{a_6} - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})\partial^{a_4}\partial^{a_6}$
$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5})\partial^{a_4}\partial^{a_6} - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})\partial^{a_4}\partial^{a_6}$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_4}\partial^{a_6}$
$= \delta_{a_1a_2}(2\partial_{a_3}\partial_{a_5} - \delta_{a_3a_5}m^2) - \delta_{a_1a_3}(2\partial_{a_2}\partial_{a_5} - \delta_{a_2a_5}m^2)$
$+ \partial_{a_1}(\delta_{a_2a_3}\partial_{a_5} - \delta_{a_2a_5}\partial_{a_3} + \partial_{a_2}\delta_{a_3a_5}) - \delta_{a_1a_5}(\delta_{a_2a_3}m^2)$
$+ \partial_{a_1}(\delta_{a_2a_3}\partial_{a_5} - \partial_{a_2}\delta_{a_3a_5} + \delta_{a_2a_5}\partial_{a_3})$
$= \delta_{a_1a_2}(2\partial_{a_3}\partial_{a_5} - \delta_{a_3a_5}m^2) - \delta_{a_1a_3}(2\partial_{a_2}\partial_{a_5} - \delta_{a_2a_5}m^2) + 2\delta_{a_2a_3}\partial_{a_1}\partial_{a_5} - \delta_{a_1a_5}(\delta_{a_2a_3}m^2)$
$= 2\delta_{a_1a_2}\partial_{a_3}\partial_{a_5} - 2\delta_{a_1a_3}\partial_{a_2}\partial_{a_5} + 2\delta_{a_2a_3}\partial_{a_1}\partial_{a_5} - m^2\delta_{a_1a_5}\delta_{a_2a_3} + m^2\delta_{a_1a_3}\delta_{a_2a_5} - m^2\delta_{a_1a_2}\delta_{a_3a_5}$
$- 2m^2\frac{1}{2^2}tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{a_1}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$= -2m^2|2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 8\delta_{ab}\partial_{a'}\partial_{b'} + 2\delta_{(a'\{b}\partial_{a\}}\partial_{b')} - m^2\delta_{\{a(b'}\delta_{a')b\}} + 4m^2\delta_{ab}\delta_{a'b'} - m^2\delta_{\{a(a'}\delta_{b\}b')}$
$= -2m^2|4\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 8\delta_{ab}\partial_{a'}\partial_{b'} + 4m^2\delta_{ab}\delta_{a'b'} - 2m^2\delta_{\{a(a'}\delta_{b\}b')}$
$= -8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{ab}\partial_{a'}\partial_{b'} - 8m^4\delta_{ab}\delta_{a'b'} + 4m^4\delta_{\{a(a'}\delta_{b\}b')}$ □

Proof: $\frac{1}{2^2}tr(\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6})\partial^{a_2}\partial^{a_6}$
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_6}$
$+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})\partial^{a_2}\partial^{a_6}$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_2}\partial^{a_6}$
$= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_3}(m^2\delta_{a_4a_5})\partial^{a_2}\partial^{a_6}$
$+ \delta_{a_1a_4}(m^2\delta_{a_3a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_5}(m^2\delta_{a_3a_4})\partial^{a_2}\partial^{a_6}$
$+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_2}\partial^{a_6}$
$= 2\partial_{a_1}(\delta_{a_3a_4}\partial_{a_5} - \delta_{a_3a_5}\partial_{a_4} + \partial_{a_3}\delta_{a_4a_5}) - \delta_{a_1a_3}(m^2\delta_{a_4a_5})$
$+ \delta_{a_1a_4}(m^2\delta_{a_3a_5}) - \delta_{a_1a_5}(m^2\delta_{a_3a_4})$
$= 2\delta_{a_3a_4}\partial_{a_1}\partial_{a_5} - 2\delta_{a_3a_5}\partial_{a_1}\partial_{a_4} + 2\partial_{a_1}\partial_{a_3}\delta_{a_4a_5} - m^2\delta_{a_1a_3}\delta_{a_4a_5} + m^2\delta_{a_1a_4}\delta_{a_3a_5} - m^2\delta_{a_1a_5}\delta_{a_3a_4}$
$- 2m^2\frac{1}{2^2}tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$= -2m^2|2\delta_{(a'\{b}\partial_{a\}}\partial_{b')} - 8\delta_{a'b'}\partial_a\partial_b + 2\partial_{\{a}\partial_{(a'}\delta_{b\}b')} - m^2\delta_{\{a(a'}\delta_{b\}b')} + 4m^2\delta_{ab}\delta_{a'b'} - m^2\delta_{\{a(b'}\delta_{a')b\}}$
$= -8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{a'b'}\partial_a\partial_b + 4m^4\delta_{\{a(a'}\delta_{b\}b')} - 8m^4\delta_{ab}\delta_{a'b'}$ □

Proof: $\frac{1}{2^2}|m^2tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{a_2}\gamma_{b')}]\partial^{a_1}\partial^{a_2} + m^2tr[\gamma_{\{a}\gamma_{(a'}\gamma_{a_1}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$- 2m^2tr[\gamma_{\{a}\gamma_{(a'}\gamma_{b\}}\gamma_{a_1}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2} - 2m^2tr[\gamma_{\{a}\gamma_{a_1}\gamma_{(a'}\gamma_{b\}}\gamma_{b')}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$
$= 8m^2\delta_{a'b'}\partial_a\partial_b - 4m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 8m^2\delta_{ab}\partial_{a'}\partial_{b'} + 2m^4\delta_{\{a(a'}\delta_{b\}b')} - 4m^4\delta_{ab}\delta_{a'b'}$
$+ 8m^2\delta_{a'b'}\partial_a\partial_b - 4m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 8m^2\delta_{ab}\partial_{a'}\partial_{b'} + 2m^4\delta_{\{a(a'}\delta_{b\}b')} - 4m^4\delta_{ab}\delta_{a'b'}$
$- 8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{ab}\partial_{a'}\partial_{b'} - 8m^4\delta_{ab}\delta_{a'b'} + 4m^4\delta_{\{a(a'}\delta_{b\}b')}$
$- 8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{a'b'}\partial_a\partial_b + 4m^4\delta_{\{a(a'}\delta_{b\}b')} - 8m^4\delta_{ab}\delta_{a'b'}$
$= 32m^2\delta_{a'b'}\partial_a\partial_b + 32m^2\delta_{ab}\partial_{a'}\partial_{b'} - 24m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 12m^4\delta_{\{a(a'}\delta_{b\}b')} - 24m^4\delta_{ab}\delta_{a'b'}$ □

Thm. 5.4.1. $[A_{ab}(x), A^+_{a'b'}(x')] = \frac{i}{8}\{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{b\}b')} - \frac{\partial_{b\}}\partial^+_{b')}}{m^2}] - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{(a'b')} - \frac{\partial^+_{(a'}\partial^+_{b')}}{m^2}]\}\Delta(x-x')$

Proof: $2m^4\delta_{\{a(a'}\delta_{b\}b')} - 4m^4\delta_{ab}\delta_{a'b'}$
$+ 24\partial_a\partial_{a'}\partial_b\partial_{b'} - 5m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 4m^2\delta_{ab}\partial_{a'}\partial_{b'} + m^4\delta_{\{a(a'}\delta_{b\}b')}$
$+ 32m^2\delta_{a'b'}\partial_a\partial_b + 32m^2\delta_{ab}\partial_{a'}\partial_{b'} - 24m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 12m^4\delta_{\{a(a'}\delta_{b\}b')} - 24m^4\delta_{ab}\delta_{a'b'}$
$= +24\partial_a\partial_{a'}\partial_b\partial_{b'} - 5m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 4m^2\delta_{ab}\partial_{a'}\partial_{b'}$
$+ 32m^2\delta_{a'b'}\partial_a\partial_b + 32m^2\delta_{ab}\partial_{a'}\partial_{b'} - 24m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 15m^4\delta_{\{a(a'}\delta_{b\}b')} - 28m^4\delta_{ab}\delta_{a'b'}$ □

Chapter30 Mathematical Analysis of Spin Bases and CG Coefficients

Self comment: This chapter conducts general mathematical analysis and logical deduction for various spin bases. And I finds that the coefficients of the transformation relationship between spin bases is just the CG coefficients of the spin coupling system. This also provides a new method for solving CG coefficients in general. This new method is more intuitive, specific, and simple than traditional methods for solving spin coupled eigenstates. Because the new method is completely constructive. And the selected spin basis is more general, universal and rigorous than the traditional one. So it is more convenient and useful to use. It may provide some help to thoroughly clarify quantum entanglement.

1 Bargmann-WignerReorganization and analysis of equation spin basis

1.1 Dirac spin basis is a common eigenstate of spin, helicity and charge three operators

Def. 1.1.1. $\hat{Q}(\vec{p}) := \frac{i\gamma^a p_a}{m}, \hat{q}(\vec{p},\kappa) := \frac{-\varsigma E\sigma_x + i\kappa|\vec{p}|\sigma_y}{m}$

Pro. 1.1.1.

$$\begin{cases}\sigma^2(\frac{1}{2})\otimes Iu(\vec{p},\frac{\kappa}{2}) = \frac{1}{2}(\frac{1}{2}+1)u(\vec{p},\frac{\kappa}{2})\\ \sigma(\frac{1}{2})\cdot\hat{p}\otimes Iu(\vec{p},\frac{\kappa}{2}) = \frac{\kappa}{2}u(\vec{p},\frac{\kappa}{2})\\ \hat{Q}(\vec{p})u(\vec{p},\frac{\kappa}{2}) = -u(\vec{p},\frac{\kappa}{2})\\ \textit{Describe electron: } (s,h;Q) = (\frac{1}{2};\frac{\kappa}{2},-1)\end{cases} \quad \begin{cases}\sigma^2(\frac{1}{2})\otimes Iv(\vec{p},\frac{\kappa}{2}) = \frac{1}{2}(\frac{1}{2}+1)v(\vec{p},\frac{\kappa}{2})\\ \sigma(\frac{1}{2})\cdot\hat{p}\otimes Iv(\vec{p},\frac{\kappa}{2}) = \frac{\kappa}{2}v(\vec{p},\frac{\kappa}{2})\\ \hat{Q}(\vec{p})v(\vec{p},\frac{\kappa}{2}) = v(\vec{p},\frac{\kappa}{2})\\ \textit{Describe positive electron: } (s,h;Q) = (\frac{1}{2};\frac{\kappa}{2},1)\end{cases}$$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $s' = \frac{1}{2}$, the following is established.

Step 2: Assume when $s' = s - \frac{1}{2}$, the following is established.

Step 3: When $s' = s$,
This step proves that when $s' = s$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

1.2 Quasi projection operator of Dirac equation $\gamma^a = (\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x)$

Cor. 1.2.1. $\mu(\vec{p},\frac{\kappa}{2})\mu^+(\vec{p},-\frac{\kappa}{2}) = \frac{1}{2m}\begin{bmatrix} m & \varsigma E-\kappa|\vec{p}| \\ \varsigma E+\kappa|\vec{p}| & m\end{bmatrix} = \frac{1}{2}(I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)$

Cor. 1.2.2. $\mu(\vec{p},\frac{\kappa}{2})\mu^+(\vec{p},\frac{\kappa}{2}) = \frac{\varsigma}{2m}\begin{bmatrix} \varsigma E-\kappa|\vec{p}| & m \\ m & \varsigma E+\kappa|\vec{p}|\end{bmatrix} = \frac{\varsigma}{2}(I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)\sigma_x$

Cor. 1.2.3. $u(\vec{p},\frac{\kappa}{2}) = \frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix} m \\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}, v(\vec{p},\frac{\kappa}{2}) = \frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix} -m \\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}$

Cor. 1.2.4. $u(\vec{p},\frac{\kappa}{2})u^+(\vec{p},\frac{\kappa}{2}) = \frac{1}{4}[(\kappa\sigma\cdot\hat{p}+I)\otimes(I+\varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)](\varsigma I\otimes\sigma_x), \gamma^a = (\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x)$

1.3 Corresponding spin basis under special representation $\gamma^a = (\sigma\otimes\sigma_y, I\otimes\sigma_x; I\otimes\sigma_z)$

Cor. 1.3.1. $\gamma^a = (\sigma\otimes\sigma_y, I\otimes\sigma_x)$

$u(\vec{p},\frac{\kappa}{2}) = \frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa|\vec{p}|)}}\otimes\begin{bmatrix} m \\ E+\kappa|\vec{p}|\end{bmatrix}, v(\vec{p},\frac{\kappa}{2}) = \frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa|\vec{p}|)}}\otimes\begin{bmatrix} -m \\ E+\kappa|\vec{p}|\end{bmatrix}$

1.4 Spin basis combinatorial properties $(\gamma_a,\gamma_5) := [(\sigma\otimes\sigma_y, I\otimes\sigma_z), -I\otimes\sigma_x]$

Cor. 1.4.1.

$$\begin{cases}\lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2}) = \frac{1}{2}(\sigma\cdot\hat{p}+I) = \frac{1}{2}(\sigma,-i)^a\hat{p}_a, \hat{p}_a := (\hat{p},i)\\ \lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2}) = -\frac{1}{2}(\sigma\cdot\hat{p}-I) = -\frac{1}{2}(\sigma,i)^a\hat{p}_a\\ \lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2}) = \frac{1}{2}(\sigma\cdot\hat{p}+I)i\sigma_y = \frac{1}{2}(\sigma,i)^a\hat{p}_a i\sigma_y\\ \lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2}) = -\frac{1}{2}i\sigma_y(\sigma\cdot\hat{p}+I) = -\frac{1}{2}i\sigma_y(\sigma,i)^a\hat{p}_a\end{cases}$$

Cor. 1.4.2. $u(\vec{p},\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p},\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}, u(\vec{p},-\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p},-\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}$

Proof: $u(\vec{p},\frac{1}{2})u^+(\vec{p},\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p},\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}\frac{1}{\sqrt{2m}}\lambda^+(\hat{p},\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}^+$
$= \frac{1}{2m}[\lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2})]\otimes[\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}^+]$
$= \frac{1}{4m}(\sigma,-i)^a\hat{p}_a\otimes(E+m\sigma_z+|\vec{p}|\sigma_x)$ □

Proof: $u(\vec{p},-\frac{1}{2})u^+(\vec{p},-\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p},-\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}\frac{1}{\sqrt{2m}}\lambda^+(\hat{p},-\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}^+$
$= \frac{1}{2m}[\lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2})]\otimes[\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}^+]$
$= -\frac{1}{4m}(\sigma,i)^a\hat{p}_a\otimes(E+m\sigma_z-|\vec{p}|\sigma_x)$ □

Proof: $u(\vec{p},\frac{1}{2})u^+(\vec{p},-\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p},\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}\frac{1}{\sqrt{2m}}\lambda^+(\hat{p},-\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}^+$
$= \frac{1}{2m}[\lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2})]\otimes[\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}^+]$
$= \frac{1}{4m}[(\sigma,i)^a\hat{p}_a i\sigma_y]\otimes(E\sigma_z+m-i|\vec{p}|\sigma_y)$ □

Proof: $u(\vec{p},-\frac{1}{2})u^+(\vec{p},\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p},-\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}\frac{1}{\sqrt{2m}}\lambda^+(\hat{p},\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}^+$
$= \frac{1}{2m}[\lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2})]\otimes[\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}^+]$
$= \frac{1}{2m}[\lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2})]\otimes[\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}^+]$
$= -\frac{1}{4m}[i\sigma_y(\sigma,i)^a\hat{p}_a]\otimes(E\sigma_z+m+i|\vec{p}|\sigma_y)$ □

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Proof: $u(\vec{p},\frac{1}{2})u^+(\vec{p},\frac{1}{2})+u(\vec{p},-\frac{1}{2})u^+(\vec{p},-\frac{1}{2})$
$= \frac{1}{4m}(\sigma,-i)^a\hat{p}_a\otimes(E+m\sigma_z+|\vec{p}|\sigma_x)-\frac{1}{4m}(\sigma,i)^a\hat{p}_a\otimes(E+m\sigma_z-|\vec{p}|\sigma_x)$
$= \frac{1}{2m}(\sigma\cdot\vec{p}\otimes\sigma_x+E+mI\otimes\sigma_z) = \frac{(m-i\gamma^a p_a)\gamma_4}{2m}$ □

Proof: $u(\vec{p},\frac{1}{2})u^+(\vec{p},\frac{1}{2})-u(\vec{p},-\frac{1}{2})u^+(\vec{p},-\frac{1}{2})$
$= \frac{1}{4m}(\sigma,-i)^a\hat{p}_a\otimes(E+m\sigma_z+|\vec{p}|\sigma_x)+\frac{1}{4m}(\sigma,i)^a\hat{p}_a\otimes(E+m\sigma_z-|\vec{p}|\sigma_x)$
$= \frac{1}{2m}[\sigma\cdot\hat{p}\otimes(E+m\sigma_z)+|\vec{p}|I\otimes\sigma_x]$
$= \frac{1}{2m}[i(\gamma\cdot\hat{p})(E\gamma_4-m)-|\vec{p}|]\gamma_5$ □

Def. 1.4.1. $\Lambda_+(\frac{1}{2}) := \sum_{h=1/2}^{-1/2}u(\vec{p},h)u^+(\vec{p},h) = \frac{(m-i\gamma^a p_a)\gamma_4}{2m}, \Lambda_-(\frac{1}{2}) := \sum_{h=1/2}^{-1/2}v(\vec{p},h)v^+(\vec{p},h) = \frac{(-m-i\gamma^a p_a)\gamma_4}{2m}$

Cor. 1.4.3. $u(\vec{p},\frac{\kappa}{2}) = \lambda(\hat{p},\frac{\kappa}{2})\otimes\mu(\vec{p},\frac{\kappa}{2}), v(\vec{p},\frac{\kappa}{2}) = \lambda(\hat{p},\frac{\kappa}{2})\otimes\nu(\vec{p},\frac{\kappa}{2})$

1.5 Definition-Spin basis decomposition: $s = (s-\frac{1}{2})\oplus\frac{1}{2}$

Def. 1.5.1. $U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})U_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})U_{\tau_\varsigma}(\vec{p},-\frac{1}{2}), -s\le h\le s$

Cor. 1.5.1. $U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-h')U_{\tau_\varsigma}(\vec{p},h'), -s\le h\le s$

Cor. 1.5.2. $U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \hat{\Gamma}_{\tau_\varsigma}(h)U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)$
$= \sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}}U_{\tau_\varsigma}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-h'), -s\le h\le s$

Def. 1.5.2. $\hat{\Gamma}_{\tau_\varsigma}(h) := \sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}}U_{\tau_\varsigma}(\vec{p},h')[](\vec{p},h-h'), -s\le h\le s$

1.6 Corollary-$U_{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}(\vec{p},h)$ is a spin eigenstate

Def. 1.6.1. $\Omega(s;\sigma(\frac{1}{2})\otimes I):=[\sigma(\frac{1}{2})\otimes I]\otimes I_{4^{2s-1}}+I_4\otimes[\sigma(\frac{1}{2})\otimes I]\otimes I_{4^{2s-2}}+\cdot\cdot+I_{4^{2s-1}}\otimes[\sigma(\frac{1}{2})\otimes I]$

Thm. 1.6.1. $[\Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot\hat{p}]U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h)=hU_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h),-s\leq h\leq s$

Proof: $[\Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot\hat{p}]U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h)$

$=\{\Omega(s-\frac{1}{2};\sigma(\frac{1}{2})\otimes I)\otimes I_4+I_{4^{2s-1}}\otimes[\sigma(\frac{1}{2})\otimes I]\}\cdot\hat{p}$

$[\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},-\frac{1}{2})],-s\leq h\leq s$

$=[\frac{\sqrt{s+h}}{\sqrt{2s}}hU_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}hU_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},-\frac{1}{2})],-s\leq h\leq s$

$=hU_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h),-s\leq h\leq s$ □

Thm. 1.6.2. $\Omega^2(s;\sigma(\frac{1}{2})\otimes I)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h)=s(s+1)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h),-s\leq h\leq s$

he above theorem can be easily proved using a fully symmetric representation transformation method. From the above, it can be seen that $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)$ is a spin eigenstate. Therefore, the expansion coefficients are CG coefficients, and the actual calculation results also indicate that they are indeed the corresponding CG coefficients. This also provides a unified, standardized, intuitive and complete new method for calculating CG coefficients.

1.7 Raising and lowering operator of Dirac spin basis under special representation

Dirac spinor boost transformation$[\gamma^a=(\sigma\otimes\sigma_y,I\otimes\sigma_z;-I\otimes\sigma_x)]$:

Cor. 1.7.1. $D_{\vec{v}}=e^{-ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)}=\frac{1+\gamma_v-i\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_4}{\sqrt{2(\gamma_v+1)}}=\frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}}=\frac{m-i\gamma^a p_a\gamma_4}{\sqrt{2m(E+m)}}$

Thm. 1.7.1. $e^{-ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)}(\sigma\otimes I)e^{ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)}=\frac{1}{m(E+m)}\begin{bmatrix}E(E+m)\sigma_i-2p_i(\sigma\cdot\vec{p}) & (E+m)[(\sigma\cdot\vec{p})\sigma_i-p_i]\\(E+m)[(\sigma\cdot\vec{p})\sigma_i-p_i] & E(E+m)\sigma_i-2p_i(\sigma\cdot\vec{p})\end{bmatrix}$

Proof: $e^{-ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)}(\sigma\otimes I)e^{ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)}$

$=\frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}}(\sigma\otimes I)\frac{E+m+i\vec{p}\cdot\vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}}$

$=\frac{1}{\sqrt{2m(E+m)}}\begin{bmatrix}E+m & \sigma\cdot\vec{p}\\\sigma\cdot\vec{p} & E+m\end{bmatrix}\begin{bmatrix}\sigma & 0\\0 & \sigma\end{bmatrix}\frac{1}{\sqrt{2m(E+m)}}\begin{bmatrix}E+m & -\sigma\cdot\vec{p}\\-\sigma\cdot\vec{p} & E+m\end{bmatrix}$

$=\frac{1}{2m(E+m)}\begin{bmatrix}E+m & \sigma\cdot\vec{p}\\\sigma\cdot\vec{p} & E+m\end{bmatrix}\begin{bmatrix}(E+m)\sigma & -\sigma(\sigma\cdot\vec{p})\\-\sigma(\sigma\cdot\vec{p}) & (E+m)\sigma\end{bmatrix}$

$=\frac{1}{2m(E+m)}\begin{bmatrix}(E+m)^2\sigma-(\sigma\cdot\vec{p})\sigma(\sigma\cdot\vec{p}) & (E+m)[(\sigma\cdot\vec{p})\sigma-\sigma(\sigma\cdot\vec{p})]\\(E+m)[(\sigma\cdot\vec{p})\sigma-\sigma(\sigma\cdot\vec{p})] & (E+m)^2\sigma-(\sigma\cdot\vec{p})\sigma(\sigma\cdot\vec{p})\end{bmatrix}$

$=\frac{1}{2m(E+m)}\begin{bmatrix}(E+m)^2\sigma_i+\vec{p}^2\sigma_i-2p_i(\sigma\cdot\vec{p}) & 2(E+m)[(\sigma\cdot\vec{p})\sigma_i-p_i]\\2(E+m)[(\sigma\cdot\vec{p})\sigma_i-p_i] & (E+m)^2\sigma_i+\vec{p}^2\sigma_i-2p_i(\sigma\cdot\vec{p})\end{bmatrix}$

$=\frac{1}{m(E+m)}\begin{bmatrix}E(E+m)\sigma_i-2p_i(\sigma\cdot\vec{p}) & (E+m)[(\sigma\cdot\vec{p})\sigma_i-p_i]\\(E+m)[(\sigma\cdot\vec{p})\sigma_i-p_i] & E(E+m)\sigma_i-2p_i(\sigma\cdot\vec{p})\end{bmatrix}$ □

Cor. 1.7.2. $e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}(\sigma\otimes I)e^{ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}=$

$$\begin{cases}x:\frac{E}{m}\sigma_x\otimes I+\frac{i|\vec{p}|}{m}\sigma_y\otimes\sigma_x=\frac{E}{m}(-i\gamma_y+\frac{|\vec{p}|}{E}\gamma_x\gamma_5)\gamma_z\\y:\frac{E}{m}\sigma_y\otimes I-\frac{i|\vec{p}|}{m}\sigma_x\otimes\sigma_x=\frac{1}{m}(i\gamma_x+\frac{|\vec{p}|}{E}\gamma_y\gamma_5)\gamma_z\\z:\sigma_z\otimes I=-i\gamma_x\gamma_y\end{cases}$$

Cor. 1.7.3.

$$\begin{cases}u(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},\frac{1}{2})=e^{-ln[\gamma_v(1+v)](\frac{i}{2}\gamma_z\gamma_4)}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\sqrt{2m}}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}\sqrt{E+m}\\\sqrt{E-m}\end{bmatrix}\\u(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},-\frac{1}{2})=e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\sqrt{2m}}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}\sqrt{E+m}\\-\sqrt{E-m}\end{bmatrix}\end{cases}$$

$$\begin{cases}v(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},\frac{1}{2})=e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{2m}}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}\sqrt{E-m}\\\sqrt{E+m}\end{bmatrix}\\v(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},-\frac{1}{2})=e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{2m}}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}-\sqrt{E-m}\\\sqrt{E+m}\end{bmatrix}\end{cases}$$

Proof: $u(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},\frac{1}{2})$

$=e^{-ln[\gamma_v(1+v)](\frac{i}{2}\gamma_z\gamma_4)}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$

$= \frac{E+m-i|\vec{p}|\gamma_z\gamma_4}{\sqrt{2m(E+m)}}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{1}{\sqrt{2m(E+m)}}\begin{bmatrix}E+m & \sigma_z|\vec{p}|\\ \sigma_z|\vec{p}| & E+m\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{1}{\sqrt{2m}}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix}$ □

Proof: $u(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},-\frac{1}{2})$
$= e^{-ln[\gamma_v(1+v)](\frac{i}{2}\gamma_z\gamma_4)}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{E+m-i|\vec{p}|\gamma_z\gamma_4}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{1}{\sqrt{2m(E+m)}}\begin{bmatrix}E+m & \sigma_z|\vec{p}|\\ \sigma_z|\vec{p}| & E+m\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{1}{\sqrt{2m}}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix}$ □

Proof: $v(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},\frac{1}{2})$
$= e^{-ln[\gamma_v(1+v)](\frac{i}{2}\gamma_z\gamma_4)}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{E+m-i|\vec{p}|\gamma_z\gamma_4}{\sqrt{2m(E+m)}}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{1}{\sqrt{2m(E+m)}}\begin{bmatrix}E+m & \sigma_z|\vec{p}|\\ \sigma_z|\vec{p}| & E+m\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{1}{\sqrt{2m}}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}\sqrt{E-m}\\ \sqrt{E+m}\end{bmatrix}$ □

Proof: $v(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},-\frac{1}{2})$
$= e^{-ln[\gamma_v(1+v)](\frac{i}{2}\gamma_z\gamma_4)}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{E+m-i|\vec{p}|\gamma_z\gamma_4}{\sqrt{2m(E+m)}}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{1}{\sqrt{2m(E+m)}}\begin{bmatrix}E+m & \sigma_z|\vec{p}|\\ \sigma_z|\vec{p}| & E+m\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{1}{\sqrt{2m}}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}-\sqrt{E-m}\\ \sqrt{E+m}\end{bmatrix}$ □

Cor. 1.7.4. $e^{i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}}e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}(\sigma\otimes I)e^{ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}e^{-i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}} =$

$$\begin{cases} x: \frac{E}{m}[\sigma_x(\frac{1}{2})-\hat{p}_x\frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}]\otimes I+\frac{i|\vec{p}|}{m}[\sigma_y(\frac{1}{2})-\hat{p}_y\frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}]\otimes\sigma_x \\ =\frac{iE}{2m}[-\gamma_y\gamma_z+\frac{\hat{p}_x}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]-\frac{|\vec{p}|}{2m}[-\gamma_z\gamma_x+\frac{\hat{p}_y}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\gamma_5 \\ y: \frac{E}{m}[\sigma_y(\frac{1}{2})-\hat{p}_y\frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}]\otimes I-\frac{i|\vec{p}|}{m}[\sigma_x(\frac{1}{2})-\hat{p}_x\frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}]\otimes\sigma_x \\ =\frac{iE}{2m}[-\gamma_z\gamma_x+\frac{\hat{p}_y}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]+\frac{|\vec{p}|}{2m}[-\gamma_y\gamma_z+\frac{\hat{p}_x}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\gamma_5 \\ z: [\sigma(\frac{1}{2})\cdot\hat{p}]\otimes I=-\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k=\frac{1}{2}\varepsilon^{ijk}\hat{p}_iS_{jk}(e,\frac{1}{2}) \end{cases}$$

Def. 1.7.1.
$$\begin{cases} \hat{J}_x(\vec{0},\frac{1}{2};\gamma_a):=-\frac{i}{2}\gamma_y\gamma_z \\ \hat{J}_y(\vec{0},\frac{1}{2};\gamma_a):=-\frac{i}{2}\gamma_z\gamma_x \\ \hat{J}_z(\vec{0},\frac{1}{2};\gamma_a):=-\frac{i}{2}\gamma_x\gamma_y \end{cases}$$

Def. 1.7.2.
$$\begin{cases} x+iy: \hat{J}_+(\vec{p},\frac{1}{2};m):=\{[\sigma_x(\frac{1}{2})+i\sigma_y(\frac{1}{2})]-\frac{(\hat{p}_x+i\hat{p}_y)}{1+\hat{p}_z}[\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})]\}\otimes I(\frac{E}{m}+\frac{|\vec{p}|}{m}\otimes\sigma_x) \\ =[i(\gamma_x+i\gamma_y)\gamma_z+\frac{(\hat{p}_x+i\hat{p}_y)}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\frac{i(E-|\vec{p}|\gamma_5)}{2m} \\ x-iy: \hat{J}_-(\vec{p},\frac{1}{2};m):=\{[\sigma_x(\frac{1}{2})-i\sigma_y(\frac{1}{2})]-\frac{(\hat{p}_x-i\hat{p}_y)}{1+\hat{p}_z}[\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})]\}(\frac{E}{m}-\frac{|\vec{p}|}{m}\otimes\sigma_x) \\ =[-i(\gamma_x-i\gamma_y)\gamma_z+\frac{(\hat{p}_x-i\hat{p}_y)}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\frac{i(E+|\vec{p}|\gamma_5)}{2m} \\ z: \hat{J}_z(\vec{p},\frac{1}{2};m):=[\sigma(\frac{1}{2})\cdot\hat{p}]\otimes I=-\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k=\frac{1}{2}\varepsilon^{ijk}\hat{p}_iS_{jk}(e,\frac{1}{2}),\hat{Q}(\vec{p},\frac{1}{2};m):=\frac{i\gamma^a p_a}{m} \end{cases}$$

Cor. 1.7.5.
$$\begin{cases} u(\vec{p},\frac{1}{2})=e^{i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}}e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\sqrt{2m}}\lambda(\hat{p},\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ \sqrt{E-m}\end{bmatrix} \\ u(\vec{p},-\frac{1}{2})=e^{i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}}e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\sqrt{2m}}\lambda(\hat{p},-\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E+m}\\ -\sqrt{E-m}\end{bmatrix} \end{cases}$$
$$\begin{cases} v(\vec{p},\frac{1}{2})=e^{i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}}e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{2m}}\lambda(\hat{p},\frac{1}{2})\otimes\begin{bmatrix}\sqrt{E-m}\\ \sqrt{E+m}\end{bmatrix} \\ v(\vec{p},-\frac{1}{2})=e^{i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}}e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)}\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{2m}}\lambda(\hat{p},-\frac{1}{2})\otimes\begin{bmatrix}-\sqrt{E-m}\\ \sqrt{E+m}\end{bmatrix} \end{cases}$$

1.8 Raising and lowering operator of Dirac spin basis under general representation

Def. 1.8.1.

$$\begin{cases}\hat{J}_x(\vec{p},\frac{1}{2};\gamma_a):=\frac{iE}{2m}[-\gamma_y\gamma_z+\frac{\hat{p}_x}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]-\frac{|\vec{p}|}{2m}[-\gamma_z\gamma_x+\frac{\hat{p}_y}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\gamma_5\\ \hat{J}_y(\vec{p},\frac{1}{2};\gamma_a):=\frac{iE}{2m}[-\gamma_z\gamma_x+\frac{\hat{p}_y}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]+\frac{|\vec{p}|}{2m}[-\gamma_y\gamma_z+\frac{\hat{p}_x}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\gamma_5\\ \hat{J}_z(\vec{p},\frac{1}{2};\gamma_a):=-\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k=\frac{1}{2}\varepsilon^{ijk}\hat{p}_iS_{jk}(e,\frac{1}{2}),\hat{Q}(\vec{p},\frac{1}{2};\gamma_a):=\frac{i\gamma^ap_a}{m}\end{cases}$$

Cor. 1.8.1.

$$\begin{cases}\hat{J}_x^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{4},\hat{J}_y^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{4},\hat{J}_z^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{4}\\ [\hat{J}_i(\vec{p},\frac{1}{2};\gamma_a),\hat{J}_j(\vec{p},\frac{1}{2};\gamma_a)]=\varepsilon_{ij}{}^k\hat{J}_k(\vec{p},\frac{1}{2};\gamma_a),\hat{J}^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{2}(\frac{1}{2}+1)\end{cases}$$

Cor. 1.8.2. $\frac{1}{2}\delta^{\lambda''_\varsigma}_{\lambda_\varsigma}\frac{1}{2}\delta^{\mu''_\varsigma}_{\mu_\varsigma}=$

$$[\tfrac{1}{2}\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\tfrac{1}{2}\delta^{\mu'_\varsigma}_{\mu_\varsigma}+\hat{J}_x{}^{\lambda'_\varsigma}_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)\hat{J}_x{}^{\mu'_\varsigma}_{\mu_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)+\hat{J}_y{}^{\lambda'_\varsigma}_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)\hat{J}_y{}^{\mu'_\varsigma}_{\mu_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)+\hat{J}_z{}^{\lambda'_\varsigma}_{\lambda_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)\hat{J}_z{}^{\mu'_\varsigma}_{\mu_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)]$$
$$[\tfrac{1}{2}\delta^{\lambda''_\varsigma}_{\lambda'_\varsigma}\tfrac{1}{2}\delta^{\mu''_\varsigma}_{\mu'_\varsigma}+\hat{J}_x{}^{\lambda''_\varsigma}_{\lambda'_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)\hat{J}_x{}^{\mu''_\varsigma}_{\mu'_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)+\hat{J}_y{}^{\lambda''_\varsigma}_{\lambda'_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)\hat{J}_y{}^{\mu''_\varsigma}_{\mu'_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)+\hat{J}_z{}^{\lambda''_\varsigma}_{\lambda'_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)\hat{J}_z{}^{\mu''_\varsigma}_{\mu'_\varsigma}(\vec{p},\tfrac{1}{2};\gamma_a)]$$

Cor. 1.8.3.

$$\begin{cases}\hat{J}_+(\vec{p},\frac{1}{2};\gamma_a):=[i(\gamma_x+i\gamma_y)\gamma_z+\frac{\hat{p}_x+i\hat{p}_y}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\frac{i(E-|\vec{p}|\gamma_5)}{2m}\\ \hat{J}_-(\vec{p},\frac{1}{2};\gamma_a):=[-i(\gamma_x-i\gamma_y)\gamma_z+\frac{\hat{p}_x-i\hat{p}_y}{1+\hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k+\gamma_x\gamma_y)]\frac{i(E+|\vec{p}|\gamma_5)}{2m}\\ \hat{J}_z(\vec{p},\frac{1}{2};\gamma_a):=-\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k=\frac{1}{2}\varepsilon^{ijk}\hat{p}_iS_{jk}(e,\frac{1}{2}),\hat{Q}(\vec{p},\frac{1}{2};\gamma_a):=\frac{i\gamma^ap_a}{m}\end{cases}$$

Cor. 1.8.4.

$$\begin{cases}\hat{J}_+(\vec{p},\frac{1}{2};\gamma_a)u(\vec{p},-\frac{1}{2})=u(\vec{p},\frac{1}{2});\hat{J}_+(\vec{p},\frac{1}{2};\gamma_a)u(\vec{p},\frac{1}{2})=0\\ \hat{J}_-(\vec{p},\frac{1}{2};\gamma_a)u(\vec{p},\frac{1}{2})=u(\vec{p},-\frac{1}{2});\hat{J}_-(\vec{p},\frac{1}{2};\gamma_a)u(\vec{p},-\frac{1}{2})=0\\ \hat{J}_z(\vec{p},\frac{1}{2};\gamma_a)u(\vec{p},h)=hu(\vec{p},h),\hat{Q}(\vec{p},\frac{1}{2};\gamma_a)u(\vec{p},h)=u(\vec{p},h),-\frac{1}{2}\le h\le\frac{1}{2}\end{cases}$$

Cor. 1.8.5.

$$\begin{cases}\hat{J}_+(\vec{p},\frac{1}{2};\gamma_a)v(\vec{p},-\frac{1}{2})=v(\vec{p},\frac{1}{2});\hat{J}_+(\vec{p},\frac{1}{2};\gamma_a)v(\vec{p},\frac{1}{2})=0\\ \hat{J}_-(\vec{p},\frac{1}{2};\gamma_a)v(\vec{p},\frac{1}{2})=v(\vec{p},-\frac{1}{2});\hat{J}_-(\vec{p},\frac{1}{2};\gamma_a)v(\vec{p},-\frac{1}{2})=0\\ \hat{J}_z(\vec{p},\frac{1}{2};\gamma_a)v(\vec{p},h)=hv(\vec{p},h),\hat{Q}(\vec{p},\frac{1}{2};\gamma_a)v(\vec{p},h)=-v(\vec{p},h),-\frac{1}{2}\le h\le\frac{1}{2}\end{cases}$$

1.9 Raising and lowering operator of Bargmann-Wigner equation spin basis

Def. 1.9.1.

$$\begin{cases}\hat{J}(\vec{p},s;\gamma_a):=\underbrace{\hat{J}(\vec{p},\frac{1}{2};\gamma_a)\otimes I_4\otimes\cdot\cdot\otimes I_4}_{2s}+\underbrace{I_4\otimes\hat{J}(\vec{p},\frac{1}{2};\gamma_a)\otimes\cdot\cdot\otimes I_4}_{2s}+\cdots+\underbrace{I_4\otimes\cdot\cdot\otimes I_4\otimes\hat{J}(\vec{p},\frac{1}{2};\gamma_a)}_{2s}\\ \hat{Q}(\vec{p},s;\gamma_a):=\underbrace{\hat{Q}(\vec{p},\frac{1}{2};\gamma_a)\otimes I_4\otimes\cdot\cdot\otimes I_4}_{2s}+\underbrace{I_4\otimes\hat{Q}(\vec{p},\frac{1}{2};\gamma_a)\otimes\cdot\cdot\otimes I_4}_{2s}+\cdots+\underbrace{I_4\otimes\cdot\cdot\otimes I_4\otimes\hat{Q}(\vec{p},\frac{1}{2};\gamma_a)}_{2s}\end{cases}$$

Cor. 1.9.1.

$$\begin{cases}\hat{J}_x^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{4},\hat{J}_y^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{4},\hat{J}_z^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{4},\hat{J}^2(\vec{p},\frac{1}{2};\gamma_a)=\frac{1}{2}(\frac{1}{2}+1)\\ [\hat{J}_i(\vec{p},s;\gamma_a),\hat{J}_j(\vec{p},s;\gamma_a)]=\varepsilon_{ij}{}^k\hat{J}_k(\vec{p},s;\gamma_a)\end{cases}$$

Thm. 1.9.1. $\hat{J}_+(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h+1)}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h+1)$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $s'=\frac{1}{2}$, the following is established.
$\hat{J}_+(\vec{p},\frac{1}{2};\gamma_a)U_{\otimes\tau_\varsigma}(\vec{p},h)=\sqrt{\frac{3}{4}-h(h+1)}U_{\otimes\tau_\varsigma}(\vec{p},h+1),-\frac{1}{2}\le h\le\frac{1}{2}$
Step 2: Assume when $s'=s-\frac{1}{2}$, the following is established.
$\hat{J}_+(\vec{p},s-\frac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h)=\sqrt{(s-\frac{1}{2})(s+\frac{1}{2})-h(h+1)}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h),-s+\frac{1}{2}\le h\le s-\frac{1}{2}$
Step 3: When $s'=s,\ -s\le h\le s,\hat{J}_+(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h)$

$$=\frac{\sqrt{s+h}}{\sqrt{2s}}[\hat{J}_+(\vec{p},s-\tfrac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})]U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}[\hat{J}_+(\vec{p},s-\tfrac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})]U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$
$$+\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})\hat{J}_+(\vec{p},\tfrac{1}{2};\gamma_a)U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})\hat{J}_+(\vec{p},\tfrac{1}{2};\gamma_a)U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$
$$=\frac{\sqrt{s+h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+\frac{\sqrt{s-h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h+1-\frac{1}{2})(h+1+\frac{1}{2})}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{3}{2})]U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\frac{\sqrt{(s-h)(s+h+1)}\sqrt{s+h+1}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{(s-h)(s+h+1)}\sqrt{s-h-1}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{3}{2})U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\sqrt{s(s+1)-h(h+1)}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h+1)$$

This step proves that when $s'=s$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Thm. 1.9.2. $\hat{J}_-(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h-1)}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h-1)$

Proof: Using mathematical induction to prove this theorem.

Step 1: When $s'=\frac{1}{2}$, the following is established.

$\hat{J}_-(\vec{p},\frac{1}{2};\gamma_a)U_{\otimes\tau_\varsigma}(\vec{p},h)=\sqrt{\frac{3}{4}-h(h-1)}U_{\otimes\tau_\varsigma}(\vec{p},h-1),-\frac{1}{2}\le h\le\frac{1}{2}$

Step 2: Assume when $s'=s-\frac{1}{2}$, the following is established.

$\hat{J}_-(\vec{p},s-\frac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h)=\sqrt{(s-\frac{1}{2})(s+\frac{1}{2})-h(h-1)}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-1),-s+\frac{1}{2}\le h\le s-\frac{1}{2}$

Step 3: When $s'=s,\ -s\le h\le s,\hat{J}_-(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h)$

$$=\frac{\sqrt{s+h}}{\sqrt{2s}}[\hat{J}_-(\vec{p},s-\tfrac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})]U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}[\hat{J}_-(\vec{p},s-\tfrac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})]U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$+\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})\hat{J}_-(\vec{p},\tfrac{1}{2};\gamma_a)U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\hat{J}_-(\vec{p},s-\tfrac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})\hat{J}_-(\vec{p},\tfrac{1}{2};\gamma_a)U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\frac{\sqrt{s+h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h-1-\frac{1}{2})(h-1+\frac{1}{2})}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{3}{2})U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+\frac{\sqrt{s-h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})]U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})+\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\frac{\sqrt{s+h}\sqrt{(s+h-1)(s-h+1)}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{3}{2})U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+\frac{\sqrt{s-h}\sqrt{(s+h)(s-h)}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})]U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})+\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\frac{\sqrt{(s+h)(s-h+1)}\sqrt{(s+h-1)}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{3}{2})U_{\otimes\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+\frac{\sqrt{(s+h)(s-h+1)}\sqrt{s-h+1}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})U_{\otimes\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\sqrt{s(s+1)-h(h-1)}U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{p},h-1)$$

This step proves that when $s'=s$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Cor. 1.9.2.

$$\begin{cases}\hat{J}_+(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h+1)}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h+1),-s\le h\le s\\ \hat{J}_-(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h-1)}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h-1),-s\le h\le s\\ \hat{J}_z(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=hU_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),-s\le h\le s\\ \hat{Q}(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=-2sU_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),-s\le h\le s\end{cases}$$

Cor. 1.9.3.

$$\begin{cases}\hat{J}_+(\vec{p},s;\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h+1)}V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h+1),-s\le h\le s\\ \hat{J}_-(\vec{p},s;\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h-1)}V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h-1),-s\le h\le s\\ \hat{J}_z(\vec{p},s;\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=hV_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),-s\le h\le s\\ \hat{Q}(\vec{p},s;\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=2sV_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),-s\le h\le s\end{cases}$$

Cor. 1.9.4.

$$\begin{cases}\hat{J}^2(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=s(s+1)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),\hat{J}^2(\vec{p},*\frac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=\frac{3}{4}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)\\ \hat{J}_z(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=hU_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),\hat{Q}(\vec{p},*\frac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=-U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)\\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\,..}_{2s}}(\vec{p},h)=\frac{1}{(2s)!}U_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\,..\}}_{2s}}(\vec{p},h),\hat{J}^2=\hat{J}_z^2+\frac{1}{2}(\hat{J}_+\hat{J}_-+\hat{J}_+\hat{J}_-),-s\le h\le s\end{cases}$$

Cor. 1.9.5.

$$\begin{cases}\hat{J}^2(\vec{p},s;\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=s(s+1)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),\hat{J}^2(\vec{p},*\frac{1}{2};\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=\frac{3}{4}V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)\\ \hat{J}_z(\vec{p},s;\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=hV_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h),\hat{Q}(\vec{p},*\frac{1}{2};\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)=V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes..}_{2s}}(\vec{p},h)\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\,..}_{2s}}(\vec{p},h)=\frac{1}{(2s)!}V_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\,..\}}_{2s}}(\vec{p},h),\hat{J}^2=\hat{J}_z^2+\frac{1}{2}(\hat{J}_+\hat{J}_-+\hat{J}_+\hat{J}_-),-s\le h\le s\end{cases}$$

1.10 Corollary-$U_{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}(\vec{p},h)$ orthogonality

Def. 1.10.1. $\bar{U}^{\tau_\varsigma}(\vec{p},h')U_{\tau_\varsigma}(\vec{p},h)=\delta_{hh'},-\frac{1}{2}\le h,h'\le\frac{1}{2}$

Thm. 1.10.1. $\bar{U}^{\overbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\delta_{hh'},-s\le h',h\le s$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $s'=\frac{1}{2}$, the following is established.
$\bar{U}^{\lambda_\varsigma}(\vec{p},h')U_{\lambda_\varsigma}(\vec{p},h)=\delta_{hh'},-\frac{1}{2}\le h,h'\le\frac{1}{2}$
Step 2: Assume when $s'=s-\frac{1}{2}$, the following is established.
$\bar{U}^{\overbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}^{2s-1}}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)=\delta_{hh'},-s+\frac{1}{2}\le h',h\le s-\frac{1}{2}$

Step 3: When $s'=s$, $\bar{U}^{\overbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma\tau_\varsigma}^{2s}}(\vec{p},h')U_{\underbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h),-s\le h',h\le s$

$$=[\sum_{\bar{h}'=1/2}^{-1/2}\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'}C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^1}}\bar{U}^{\overbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}^{2s-1}}(\vec{p},h'-\bar{h}')\bar{U}^{\tau_\varsigma}(\vec{p},\bar{h}')][\sum_{\bar{h}=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}U_{\underbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\bar{h})U_{\tau_\varsigma}(\vec{p},\bar{h})]$$

$$=\sum_{\bar{h}',\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'}C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^1}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}\bar{U}^{\overbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}^{2s-1}}(\vec{p},h'-\bar{h}')U_{\underbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\bar{h})\delta_{\bar{h}\bar{h}'}]$$

$$=\sum_{\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}}C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}\bar{U}^{\overbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}^{2s-1}}(\vec{p},h'-\bar{h})U_{\underbrace{\lambda_\varsigma\,\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\bar{h})]$$

$$=\sum_{\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}}C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}\delta_{hh'}]$$

$$=\sum_{\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}}]\delta_{hh'}$$

$=\delta_{hh'}$
This step proves that when $s'=s$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

1.11 Corollary-Spin basis decomposition: $1 = \frac{1}{2} \oplus \frac{1}{2}$

Cor. 1.11.1. $U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h) = \frac{\sqrt{1+h}}{\sqrt{2}}U_{\lambda_\varsigma}(\vec{p},h-\frac{1}{2})U_{\mu_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{1-h}}{\sqrt{2}}U_{\lambda_\varsigma}(\vec{p},h+\frac{1}{2})U_{\mu_\varsigma}(\vec{p},-\frac{1}{2})$

$$= \begin{cases} U_{\lambda_\varsigma}(\vec{p},\frac{1}{2})U_{\mu_\varsigma}(\vec{p},\frac{1}{2}), h=1 \\ \frac{1}{\sqrt{2}}U_{\{\lambda_\varsigma}(\vec{p},\frac{1}{2})U_{\mu_\varsigma\}}(\vec{p},-\frac{1}{2}), h=0 \\ U_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})U_{\mu_\varsigma}(\vec{p},-\frac{1}{2}), h=-1 \end{cases}$$

Cor. 1.11.2. $U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h) = U_{\mu_\varsigma\lambda_\varsigma}(\vec{p},h), -1 \le h \le 1$

1.12 Corollary-Spin basis decomposition: $0 = \frac{1}{2} \ominus \frac{1}{2}$

Cor. 1.12.1. $F_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h) = \frac{1}{\sqrt{2}}u_{[\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma]}(\vec{p},-\frac{1}{2}), h=0$

Cor. 1.12.2. $[(\sigma\otimes I)\cdot(I\otimes\sigma)][\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})] = -3[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})]$

Proof: $\sigma\cdot[\lambda(\hat{p},\frac{\varsigma}{2})\lambda^T(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\lambda^T(\hat{p},\frac{\varsigma}{2})]\sigma^T$
$= \frac{i}{2}\sigma\cdot[(\sigma,-i\varsigma)^a\hat{p}_a\sigma_y - (\sigma,i\varsigma)^a\hat{p}_a\sigma_y]\sigma^T$
$= \sigma\cdot(i\varsigma\sigma_y)\sigma^T$
$= \sigma_x(i\varsigma\sigma_y)\sigma_x^T + \sigma_y(i\varsigma\sigma_y)\sigma_y^T + \sigma_z(i\varsigma\sigma_y)\sigma_z^T$
$= -3(i\varsigma\sigma_y) = -3[\lambda(\hat{p},\frac{\varsigma}{2})\lambda^T(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\lambda^T(\hat{p},\frac{\varsigma}{2})]$ □

Cor. 1.12.3.

$$\begin{cases} [\sigma(\frac{1}{2})\otimes I + I\otimes\sigma(\frac{1}{2})]^2[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})] = 0[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})] \\ [\sigma(\frac{1}{2})\otimes I + I\otimes\sigma(\frac{1}{2})]\cdot\hat{p}[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})] = 0[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2}) - \lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})] \end{cases}$$

Thm. 1.12.1.

$$\begin{cases} F = -[C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi] \\ F = \frac{1}{\sqrt{2}}[u(\vec{p},\frac{1}{2})u^T(\vec{p},-\frac{1}{2}) - u(\vec{p},-\frac{1}{2})u^T(\vec{p},\frac{1}{2})] = -\frac{m-i\gamma^a p_a}{2\sqrt{2}m}\gamma_5 C \\ \varepsilon_a(\vec{p},0;0) := \frac{1}{i\sqrt{2}}(\bar{C}\gamma_a\gamma_5)^{\lambda_\varsigma\mu_\varsigma}F_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h) = \frac{p_a}{m} \end{cases}$$

Proof: $-2i\sqrt{2}(-im\mathbf{A}_a) = \varepsilon_a(\vec{p},0;0) = \frac{1}{i\sqrt{2}}(\bar{C}\gamma_a\gamma_5)^{\lambda_\varsigma\mu_\varsigma}F_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)$
$= \frac{1}{2i}(\bar{C}\gamma_a\gamma_5)^{\lambda_\varsigma\mu_\varsigma}u_{[\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma]}(\vec{p},-\frac{1}{2})$
$= -i(\bar{C}\gamma_a\gamma_5)^{\lambda_\varsigma\mu_\varsigma}u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})$
$= -iu^T(\vec{p},\frac{1}{2})(\bar{C}\gamma_a\gamma_5)u(\vec{p},-\frac{1}{2})$
$= iu^T(\vec{p},\frac{1}{2})\gamma_2\gamma_5\gamma_4\gamma_a u(\vec{p},-\frac{1}{2})$
$= iu^+(\vec{p},-\frac{1}{2})\gamma_4\gamma_a u(\vec{p},-\frac{1}{2})$
$= \frac{p_a}{m}$ □

Proof: $2i\sqrt{2}(-\Phi) = \frac{1}{i\sqrt{2}}(\bar{C}\gamma_5)^{\lambda_\varsigma\mu_\varsigma}F_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)$
$= \frac{1}{2i}(\bar{C}\gamma_5)^{\lambda_\varsigma\mu_\varsigma}u_{[\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma]}(\vec{p},-\frac{1}{2})$
$= -i(\bar{C}\gamma_5)^{\lambda_\varsigma\mu_\varsigma}u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})$
$= -iu^T(\vec{p},\frac{1}{2})(\bar{C}\gamma_5)u(\vec{p},-\frac{1}{2})$
$= -iu^T(\vec{p},\frac{1}{2})\gamma_2\gamma_5\gamma_4 u(\vec{p},-\frac{1}{2})$
$= -iu^+(\vec{p},-\frac{1}{2})\gamma_4 u(\vec{p},-\frac{1}{2})$
$= -i$ □

Proof: $2i\sqrt{2}(-\phi) = \frac{1}{i\sqrt{2}}(\bar{C})^{\lambda_\varsigma\mu_\varsigma}F_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)$
$= \frac{1}{2i}(\bar{C})^{\lambda_\varsigma\mu_\varsigma}u_{[\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma]}(\vec{p},-\frac{1}{2})$
$= -i(\bar{C})^{\lambda_\varsigma\mu_\varsigma}u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})$
$= -iu^T(\vec{p},\frac{1}{2})\bar{C}u(\vec{p},-\frac{1}{2})$
$= -iu^T(\vec{p},\frac{1}{2})\gamma_2\gamma_5\gamma_4\gamma_5 u(\vec{p},-\frac{1}{2})$
$= -iu^+(\vec{p},-\frac{1}{2})\gamma_4\gamma_5 u(\vec{p},-\frac{1}{2})$
$= 0$ □

Proof: $F = \frac{1}{\sqrt{2}}[u(\vec{p},\frac{1}{2})u^T(\vec{p},-\frac{1}{2}) - u(\vec{p},-\frac{1}{2})u^T(\vec{p},\frac{1}{2})]$
$= \frac{1}{\sqrt{2}}[u(\vec{p},\frac{1}{2})u^+(\vec{p},\frac{1}{2}) + u(\vec{p},-\frac{1}{2})u^+(\vec{p},-\frac{1}{2})]\gamma_2\gamma_5 = \frac{m-i\gamma^a p_a}{2\sqrt{2}m}\gamma_4\gamma_2\gamma_5$
$= \frac{ip^a}{2m\sqrt{2}}\gamma_a\gamma_5 C - \frac{1}{2\sqrt{2}}\gamma_5 C = -\frac{m-i\gamma^a p_a}{2\sqrt{2}m}\gamma_5 C$ □

Proof: $[u(\vec{p},\frac{1}{2})u^T(\vec{p},-\frac{1}{2}) - u(\vec{p},-\frac{1}{2})u^T(\vec{p},\frac{1}{2})] = \frac{ip^a}{2m}\gamma_a\gamma_5 C - \frac{1}{2}\gamma_5 C$ □

1.13 Corollary-Spin basis decomposition: $s = (s-1) \oplus 1$

Thm. 1.13.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \sum\limits_{h'=1}^{-1}\frac{\sqrt{C_{s+h}^{1+h'}C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2(s-1)}}(\vec{p},h-h')U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},h'), s\geq 1, -s\leq h\leq s$

Proof: $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)$

$= \frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})U_{\tau_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})U_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

$= \frac{\sqrt{s+h}}{\sqrt{2s}}[\frac{\sqrt{s+h-1}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h-1)U_{\sigma_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-1}}(\vec{p},h)U_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})]U_{\tau_\varsigma}(\vec{p},\frac{1}{2})$

$+ \frac{\sqrt{s-h}}{\sqrt{2s}}[\frac{\sqrt{s+h}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h)U_{\sigma_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s-h-1}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-1}}(\vec{p},h+1)U_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})]U_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

$= [\frac{\sqrt{s+h}}{\sqrt{2s}}\frac{\sqrt{s+h-1}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h-1)U_{\sigma_\varsigma}(\vec{p},\frac{1}{2})U_{\tau_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}}\frac{\sqrt{s-h}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-1}}(\vec{p},h)U_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})U_{\tau_\varsigma}(\vec{p},\frac{1}{2})]$

$+ [\frac{\sqrt{s-h}}{\sqrt{2s}}\frac{\sqrt{s+h}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h)U_{\sigma_\varsigma}(\vec{p},\frac{1}{2})U_{\tau_\varsigma}(\vec{p},-\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}}\frac{\sqrt{s-h-1}}{\sqrt{2s-1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-1}}(\vec{p},h+1)U_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})U_{\tau_\varsigma}(\vec{p},-\frac{1}{2})]$

$= \frac{\sqrt{C_{s+h}^{2}C_{s-h}^{0}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h-1)U_{\sigma_\varsigma}(\vec{p},\frac{1}{2})U_{\tau_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{C_{s+h}^{1}C_{s-h}^{1}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h)\frac{1}{\sqrt{2}}U_{\{\sigma_\varsigma}(\vec{p},\frac{1}{2})U_{\tau_\varsigma\}}(\vec{p},-\frac{1}{2})$

$+ \frac{\sqrt{C_{s+h}^{0}C_{s-h}^{2}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-1}}(\vec{p},h+1)U_{\sigma_\varsigma}(\vec{p},-\frac{1}{2})U_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

$= \frac{\sqrt{C_{s+h}^{2}C_{s-h}^{0}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h-1)U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},1) + \frac{\sqrt{C_{s+h}^{1}C_{s-h}^{1}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-2}}(\vec{p},h)U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},0)$

$+ \frac{\sqrt{C_{s+h}^{0}C_{s-h}^{2}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s-1}}(\vec{p},h+1)U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},-1)$

$= \sum\limits_{h'=1}^{-1}\frac{\sqrt{C_{s+h}^{1+h'}C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2(s-1)}}(\vec{p},h-h')U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},h')$ □

Cor. 1.13.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\sigma_\varsigma}_{2s}}(\vec{p},h), s\geq 1, -s\leq h\leq s$

1.14 Corollary-Spin basis decomposition: $s + s' = s \oplus s'$

Thm. 1.14.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}}(\vec{p},h) = \sum\limits_{h'=s'}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}}(\vec{p},h'), -s-s'\leq h\leq s+s'$

Proof: For s' using mathematical induction to prove this theorem.
Step 1: When $s'' = \frac{1}{2}$, the following is established.

$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\tau_\varsigma}_{1}}(\vec{p},h) = \sum\limits_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+1/2+h}^{1/2+h'}C_{s+1/2-h}^{1/2-h'}}}{\sqrt{C_{2(s+1/2)}^{1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')U_{\underbrace{\tau_\varsigma}_{1}}(\vec{p},h'), -s-\frac{1}{2}\leq h\leq s+\frac{1}{2}$

Step 2: Assume when $s'' = s'-\frac{1}{2}$, the following is established.

$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h) = \sum\limits_{h'=s'-1/2}^{-s'+1/2}\frac{\sqrt{C_{s+s'-1/2+h}^{s'-1/2+h'}C_{s+s'-1/2-h}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2s'-1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h')$

$-s-s'+\frac{1}{2}\leq h\leq s+s'-\frac{1}{2}$

Step 3: When $s'' = s'$, $-s-s'\leq h\leq s+s'$, $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}}(\vec{p},h)$

$= \frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h-\frac{1}{2})U_{\tau_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s+s'-h}}{\sqrt{2(s+s')}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h+\frac{1}{2})U_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

$= \frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}}[\sum\limits_{h'=s'-1/2}^{-s'+1/2}\frac{\sqrt{C_{s+s'-1+h}^{s'-1/2+h'}C_{s+s'-h}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2s'-1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-\frac{1}{2}-h')U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h')]U_{\tau_\varsigma}(\vec{p},\frac{1}{2})$

$+ [\sum\limits_{h'=s'-1/2}^{-s'+1/2}\frac{\sqrt{C_{s+s'+h}^{s'-1/2+h'}C_{s+s'-1-h}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2s'-1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h+\frac{1}{2}-h')U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h')]U_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

$= [\sum\limits_{h'=s'}^{-s'+1}\frac{\sqrt{C_{s+s'-1+h}^{s'-1+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')\frac{\sqrt{s+s'+h}}{\sqrt{2s'}}U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h'-\frac{1}{2})]U_{\tau_\varsigma}(\vec{p},\frac{1}{2})$

$$+[\sum_{h'=s'-1}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-1-h}^{s'-1-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')\frac{\sqrt{s+s'-h}}{\sqrt{2s'}}U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h'+\tfrac{1}{2})]U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=[\sum_{h'=s'}^{-s'+1}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')\frac{\sqrt{s'+h'}}{\sqrt{2s'}}U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h'-\tfrac{1}{2})]U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+[\sum_{h'=s'-1}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')\frac{\sqrt{s'-h'}}{\sqrt{2s'}}U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h'+\tfrac{1}{2})]U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=[\sum_{h'=s'}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')\frac{\sqrt{s'+h'}}{\sqrt{2s'}}U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h'-\tfrac{1}{2})]U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+[\sum_{h'=s'}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')\frac{\sqrt{s'-h'}}{\sqrt{2s'}}U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s'-1}}(\vec{p},h'+\tfrac{1}{2})]U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\sum_{h'=s'}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}}(\vec{p},h'),-s-s'\leq h\leq s+s'$$

This step proves that when $s''=s'$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Cor. 1.14.1. $-s_1-s_2\leq h\leq s_1+s_2$

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h)=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\frac{\sqrt{C_{s_1+s_2+h}^{s_2+h_2}C_{s_1+s_2-h}^{s_2-h_2}}}{\sqrt{C_{2(s_1+s_2)}^{2s_2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_2)\delta(h-h_1-h_2)\\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h)=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\frac{\sqrt{C_{s_1+s_2+h}^{s_1+h_1}C_{s_1+s_2-h}^{s_1-h_1}}}{\sqrt{C_{2(s_1+s_2)}^{2s_1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_2)\delta(h-h_1-h_2)\end{cases}$$

Cor. 1.14.2. $-s_1-s_2\leq h\leq s_1+s_2, U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h)$

$$=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}[\frac{(2s_1)!(2s_2)!}{(2s_1+2s_2)!}\frac{(s_1+h_1+s_2+h_2)!}{(s_1+h_1)!(s_2+h_2)!}\frac{(s_1-h_1+s_2-h_2)!}{(s_1-h_1)!(s_2-h_2)!}]^{1/2}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_2)\delta(h-h_1-h_2)$$

1.15 Corollary-Spin basis reverse synthesis

Cor. 1.15.1. $\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')=U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}}(\vec{p},h)\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}^{2s'}}(\vec{p},h'),-s-s'\leq h\leq s+s'$

Cor. 1.15.2. $\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h-h')=\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}^{2s'}}(\vec{p},h')U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot\tau_\varsigma}_{2s'}\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},h),-s-s'\leq h\leq s+s'$

Cor. 1.15.3. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)=\frac{\sqrt{C_{2(s_1+s_2)}^{2s_2}}}{\sqrt{C_{s_1+h_1+s_2+h_2}^{s_2+h_2}C_{s_1-h_1+s_2-h_2}^{s_2-h_2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_1+h_2)\bar{U}^{\overbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}^{2s_2}}(\vec{p},h_2)$

Cor. 1.15.4. $\begin{cases}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)\bar{U}^{\tau_\varsigma}(\vec{p},\tfrac{1}{2})=\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2}),-s\leq h\leq s\\ U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)\bar{U}^{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})=\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2}),-s\leq h\leq s\end{cases}$

1.16 Corollary-Spin basis decomposition: $s_1+s_2+s_3=s_1\oplus s_2\oplus s_3$

Cor. 1.16.1. $-s_1-s_2-s_3\leq h\leq s_1+s_2+s_3, U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_3}}(\vec{p},h)$

$$=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\sum_{h_3=s_3}^{-s_3}[\frac{(2s_1)!(2s_2)!(2s_3)!}{(2s_1+2s_2+2s_3)!}\frac{(s_1+h_1+s_2+h_2+s_3+h_3)!}{(s_1+h_1)!(s_2+h_2)!(s_3+h_3)!}\frac{(s_1-h_1+s_2-h_2+s_3-h_3)!}{(s_1-h_1)!(s_2-h_2)!(s_3-h_3)!}]^{1/2}$$

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_2)U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_3}}(\vec{p},h_3)\delta(h-h_1-h_2-h_3)$$

Proof: $-s_1-s_2-s_3\leq h\leq s_1+s_2+s_3, U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_3}}(\vec{p},h)$

$$=\sum_{h_1=s_1}^{-s_1}\sum_{h_{23}=s_2+s_3}^{-s_2-s_3}[\frac{(2s_1)!(2s_2+2s_3)!}{(2s_1+2s_2+2s_3)!}\frac{(s_1+h_1+s_2+s_3+h_{23})!}{(s_1+h_1)!(s_2+s_3+h_{23})!}\frac{(s_1-h_1+s_2+s_3-h_{23})!}{(s_1-h_1)!(s_2+s_3-h_{23})!}]^{1/2}$$

$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_3}}(\vec{p},h_{23})\delta(h-h_1-h_{23})$

$= \sum_{h_1=s_1}^{-s_1}\sum_{h_{23}=s_2+s_3}^{-s_2-s_3}[\frac{(2s_1)!(2s_2+2s_3)!}{(2s_1+2s_2+2s_3)!}\frac{(s_1+h_1+s_2+s_3+h_{23})!}{(s_1+h_1)!(s_2+s_3+h_{23})!}\frac{(s_1-h_1+s_2+s_3-h_{23})!}{(s_1-h_1)!(s_2+s_3-h_{23})!}]^{1/2}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)\delta(h-h_1-h_{23})$

$\sum_{h_2=s_2}^{-s_2}\sum_{h_3=s_3}^{-s_3}[\frac{(2s_2)!(2s_3)!}{(2s_2+2s_3)!}\frac{(s_2+h_2+s_3+h_3)!}{(s_2+h_2)!(s_3+h_3)!}\frac{(s_2-h_2+s_3-h_3)!}{(s_2-h_2)!(s_3-h_3)!}]^{1/2}U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_2)U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_3}}(\vec{p},h_3)\delta(h_{23}-h_2-h_3)$

$= \sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\sum_{h_3=s_3}^{-s_3}[\frac{(2s_1)!(2s_2)!(2s_3)!}{(2s_1+2s_2+2s_3)!}\frac{(s_1+h_1+s_2+h_2+s_3+h_3)!}{(s_1+h_1)!(s_2+h_2)!(s_3+h_3)!}\frac{(s_1-h_1+s_2-h_2+s_3-h_3)!}{(s_1-h_1)!(s_2-h_2)!(s_3-h_3)!}]^{1/2}$

$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_2)U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_3}}(\vec{p},h_3)\delta(h-h_1-h_2-h_3)$ □

1.17 Corollary-Spin basis decomposition: $s_1+s_2\cdot\cdot+s_n = s_1\oplus s_2\cdot\cdot\oplus s_n$

Cor. 1.17.1. $-\sum_{i=1}^{n}s_i \le h \le \sum_{i=1}^{n}s_i, U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec{p},h)$

$= \sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p},h_1)U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec{p},h_2)\cdot\cdot U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec{p},h_n)\delta(h-\sum_{i=1}^{n}h_i)$

1.18 Corollary-$U_{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}(\vec{p},h)$ full symmetry

Thm. 1.18.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \frac{1}{(2s)!}U_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\sigma_\varsigma\}}_{2s}}(\vec{p},h), -s\le h\le s$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $s' = \frac{1}{2}, 1$, the following is established.
$U_{\lambda_\varsigma}(\vec{p},h) = \frac{1}{1!}U_{\lambda_\varsigma}(\vec{p},h), -\frac{1}{2}\le h\le\frac{1}{2}; U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h) = \frac{1}{2!}U_{\{\lambda_\varsigma\mu_\varsigma\}}(\vec{p},h), -1\le h\le 1$
Step 2: Assume when $s' = s-\frac{1}{2}$, the following is established.
$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h) = \frac{1}{(2s-1)!}U_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}}_{2s-1}}(\vec{p},h), -s+\frac{1}{2}\le h\le s-\frac{1}{2}$
Step 3: When $1\le s' = s,\ -s\le h\le s, U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)$

$= \sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^{1}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-h')U_{\tau_\varsigma}(\vec{p},h') = \sum_{h'=1}^{-1}\frac{\sqrt{C_{s+h}^{1+h'}C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^{2}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2(s-1)}}(\vec{p},h-h')U_{\sigma_\varsigma\tau_\varsigma}(\vec{p},h')$

$\Rightarrow U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \frac{1}{(2s-1)!}U_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}\sigma_\varsigma}_{2s}}(\vec{p},h), U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h), -s\le h\le s$

$\Leftrightarrow U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \frac{1}{(2s)!}U_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\sigma_\varsigma\}}_{2s}}(\vec{p},h), -s\le h\le s$

This step proves that when $s' = s$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

1.19 Quasi projection operators for spin-s particles with mass

Lem. 1.19.1.

$$\begin{cases}\hat{J}_+(\vec{p},s-\frac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s-1}}(\vec{p},h-\frac{1}{2}) = \sqrt{(s-h)(s+h)}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s-1}}(\vec{p},h+\frac{1}{2}), -s\le h\le s\\ \hat{J}_-(\vec{p},s-\frac{1}{2};\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s-1}}(\vec{p},h+\frac{1}{2}) = \sqrt{(s+h)(s-h)}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s-1}}(\vec{p},h-\frac{1}{2}), -s\le h\le s\\ \hat{J}_z(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) = hU_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h), -s\le h\le s\\ \hat{J}(s) :\prec \hat{J}(\vec{p},s;\gamma_a)\end{cases}$$

Thm. 1.19.1. $\Lambda_+(s) = [\frac{2s+1}{4s}+\frac{1}{s}\hat{J}_x(s-\frac{1}{2})\otimes\hat{J}_x(\frac{1}{2})+\frac{1}{s}\hat{J}_y(s-\frac{1}{2})\otimes\hat{J}_y(\frac{1}{2})+\frac{1}{s}\hat{J}_z(s-\frac{1}{2})\otimes\hat{J}_z(\frac{1}{2})][\Lambda_+(s-\frac{1}{2})\otimes\Lambda_+(\frac{1}{2})]$

Proof: $\Lambda_+(s) \prec \sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)U^+_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p},h)$

$= \sum_{h=s}^{-s}[\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})U_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})U_{\tau_\varsigma}(\vec{p},-\frac{1}{2})]$

$[\frac{\sqrt{s+h}}{\sqrt{2s}}U^+_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})U^+_{\tau'_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U^+_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})U^+_{\tau'_\varsigma}(\vec{p},-\frac{1}{2})]$

$$= \sum_{h=s}^{-s}[\tfrac{s+h}{2s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$+ \sum_{h=s}^{-s}[\tfrac{s-h}{2s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ \sum_{h=s}^{-s}[\tfrac{\sqrt{(s+h)(s-h)}}{2s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ \sum_{h=s}^{-s}[\tfrac{\sqrt{(s-h)(s+h)}}{2s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$= [\tfrac{s+\frac{1}{2}+\hat{J}_z(s-\frac{1}{2})}{2s}\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$+ [\tfrac{s+\frac{1}{2}-\hat{J}_z(s-\frac{1}{2})}{2s}\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ [\tfrac{\hat{J}_-(s-\frac{1}{2})}{2s}\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h+\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ [\tfrac{\hat{J}_+(s-\frac{1}{2})}{2s}\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h-\tfrac{1}{2})][U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$= [\tfrac{s+\frac{1}{2}+\hat{J}_z(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$+ [\tfrac{s+\frac{1}{2}-\hat{J}_z(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ [\tfrac{\hat{J}_-(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ [\tfrac{\hat{J}_+(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$= [\tfrac{s+\frac{1}{2}+\hat{J}_z(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$+ [\tfrac{\hat{J}_+(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][\hat{J}_-(\tfrac{1}{2})U_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},\tfrac{1}{2})]$$

$$+ [\tfrac{s+\frac{1}{2}-\hat{J}_z(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ [\tfrac{\hat{J}_-(s-\frac{1}{2})}{2s}\sum_{h=s-1/2}^{-s+1/2}U_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h)U^{+}_{\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}}(\vec{p},h)][\hat{J}_+(\tfrac{1}{2})U_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})U^{+}_{\tau'_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$\succ [\tfrac{2s+1}{4s}+\tfrac{1}{s}\hat{J}_x(s-\tfrac{1}{2})\otimes\hat{J}_x(\tfrac{1}{2})+\tfrac{1}{s}\hat{J}_y(s-\tfrac{1}{2})\otimes\hat{J}_y(\tfrac{1}{2})+\tfrac{1}{s}\hat{J}_z(s-\tfrac{1}{2})\otimes\hat{J}_z(\tfrac{1}{2})][\Lambda_+(s-\tfrac{1}{2})\otimes\Lambda_+(\tfrac{1}{2})] \qquad \square$$

Thm. 1.19.2. $\Lambda_{+\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}\underbrace{\lambda''_\varsigma\cdot\cdot\sigma''_\varsigma\tau''_\varsigma}_{2s}}(s)-\frac{3}{4}\Lambda_{+\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}\underbrace{\lambda''_\varsigma\cdot\cdot\sigma''_\varsigma}_{2s-1}}(s-\frac{1}{2})\Lambda_{+\tau_\varsigma\tau''_\varsigma}(\frac{1}{2})$

$$= [\hat{J}_x{}^{\overbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}^{2s-1}}_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(s-\tfrac{1}{2})\hat{J}_x{}^{\tau'_\varsigma}_{\tau_\varsigma}(\tfrac{1}{2})+\hat{J}_y{}^{\overbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}^{2s-1}}_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(s-\tfrac{1}{2})\hat{J}_y{}^{\tau'_\varsigma}_{\tau_\varsigma}(\tfrac{1}{2})+\hat{J}_z{}^{\overbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}^{2s-1}}_{\underbrace{\lambda_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(s-\tfrac{1}{2})\hat{J}_z{}^{\tau'_\varsigma}_{\tau_\varsigma}(\tfrac{1}{2})]\tfrac{1}{s}\Lambda_{+\underbrace{\lambda'_\varsigma\cdot\cdot\sigma'_\varsigma}_{2s-1}\underbrace{\lambda''_\varsigma\cdot\cdot\sigma''_\varsigma}_{2s-1}}(s-\tfrac{1}{2})\Lambda_{+\tau'_\varsigma\tau''_\varsigma}(\tfrac{1}{2})$$

1.20 Quasi projection operators for spin-1 particles with mass

Thm. 1.20.1. $\Lambda_+(1)=[\frac{3}{4}+\hat{J}_x(\frac{1}{2})\otimes\hat{J}_x(\frac{1}{2})+\hat{J}_y(\frac{1}{2})\otimes\hat{J}_y(\frac{1}{2})+\hat{J}_z(\frac{1}{2})\otimes\hat{J}_z(\frac{1}{2})][\Lambda_+(\frac{1}{2})\otimes\Lambda_+(\frac{1}{2})]$

Cor. 1.20.1. $\Lambda_{+\lambda_\varsigma\mu_\varsigma\lambda''_\varsigma\mu''_\varsigma}(1)-\frac{1}{2}\Lambda_{+\lambda_\varsigma\lambda''_\varsigma}(\frac{1}{2})\Lambda_{+\mu_\varsigma\mu''_\varsigma}(\frac{1}{2})=$
$\frac{1}{2}\Lambda_{+\lambda_\varsigma\mu''_\varsigma}(\frac{1}{2})\Lambda_{+\mu_\varsigma\lambda''_\varsigma}(\frac{1}{2})=[\frac{1}{2}\delta^{\lambda'_\varsigma}_{\lambda_\varsigma}\frac{1}{2}\delta^{\mu'_\varsigma}_{\mu_\varsigma}+\hat{J}_x{}^{\lambda'_\varsigma}_{\lambda_\varsigma}(\frac{1}{2})\hat{J}_x{}^{\mu'_\varsigma}_{\mu_\varsigma}(\frac{1}{2})+\hat{J}_y{}^{\lambda'_\varsigma}_{\lambda_\varsigma}(\frac{1}{2})\hat{J}_y{}^{\mu'_\varsigma}_{\mu_\varsigma}(\frac{1}{2})+\hat{J}_z{}^{\lambda'_\varsigma}_{\lambda_\varsigma}(\frac{1}{2})\hat{J}_z{}^{\mu'_\varsigma}_{\mu_\varsigma}(\frac{1}{2})]\Lambda_{+\lambda'_\varsigma\lambda''_\varsigma}(\frac{1}{2})\Lambda_{+\mu'_\varsigma\mu''_\varsigma}(\frac{1}{2})$

1.21 Operator expression of plane wave solutions for Bargmann-Wigner equation

Thm. 1.21.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\}}_{2s}}(x)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{m^s}{\sqrt{E}}\sum_{h=s}^{-s}\frac{\hat{J}_-^{s-h}(\vec{p},s;\gamma_a)}{(s-h)!\sqrt{C_{2s}^{s-h}}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},s)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},s)e^{-ip\cdot x}]d^3\vec{p}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{m^s}{\sqrt{E}}\sum_{h=s}^{-s}\frac{\hat{J}_+^{s+h}(\vec{p},s;\gamma_a)}{(s+h)!\sqrt{C_{2s}^{s+h}}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s)e^{-ip\cdot x}]d^3\vec{p}$$

2 Reorganization and analysis of Klein-Gordon equation spin basis

2.1 Vector spin basis with mass in static system

Cor. 2.1.1. $(R\cdot\hat{p})\varepsilon(\vec{p},h)=h\varepsilon(\vec{p},h),(R\cdot\hat{p})\frac{p_{[a]}}{m}=0;R^2\varepsilon(\vec{p},h)=1(1+1)\varepsilon(\vec{p},h)$

Cor. 2.1.2. $\begin{cases}\lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},1)=\frac{1}{\sqrt{2}}\begin{bmatrix}i\\-1\\0\end{bmatrix}\\ \lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},0)=\begin{bmatrix}0\\0\\-i\end{bmatrix}\\ \lambda_m(\begin{bmatrix}0\\0\\1\end{bmatrix},-1)=\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\-1\\0\end{bmatrix}\end{cases}$

Cor. 2.1.3. $\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},1):=\frac{1}{\sqrt{2}}[-1,-i,0,0]_a,\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},0):=\frac{1}{m}[0,0,E,i|\vec{p}|]_a,\varepsilon_a(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},-1):=\frac{1}{\sqrt{2}}[1,-i,0,0]_a$

Cor. 2.1.4. $\varepsilon_a(\vec{0},1):=\frac{1}{\sqrt{2}}[-1,-i,0,0]_a,\varepsilon_a(\vec{0},0):=[0,0,1,0]_a,\varepsilon_a(\vec{0},-1):=\frac{1}{\sqrt{2}}[1,-i,0,0]_a$

Cor. 2.1.5. $\begin{cases}(R_x+iR_y)\varepsilon_a(\vec{0},h)=\varepsilon_a(\vec{0},h+1),-1\le h<1;(R_x+iR_y)\varepsilon_a(\vec{0},1)=0\\(R_x-iR_y)\varepsilon_a(\vec{0},h)=\varepsilon_a(\vec{0},h-1),-1<h\le 1;(R_x-iR_y)\varepsilon_a(\vec{0},-1)=0\end{cases}$

2.2 Vector spin basis with mass in z-axis

Cor. 2.2.1.
$L_{\vec{v}}=e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L}=1-\gamma_v(\vec{v}\cdot L)+\frac{\gamma_v-1}{v^2}(\vec{v}\cdot L)^2=\gamma_v(1-\vec{v}\cdot L)-\frac{\gamma_v-1}{v^2}(\vec{v}\cdot R)^2,L_{\vec{v}}L_{-\vec{v}}=L_{-\vec{v}}L_{\vec{v}}=I$

Thm. 2.2.1. $\varepsilon(\begin{bmatrix}0\\0\\|\vec{p}|\end{bmatrix},h)=e^{-ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0},h)$

Proof: $e^{-ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0},1)=\frac{1}{m}\begin{bmatrix}m&0&0&0\\0&m&0&0\\0&0&E&-i|\vec{p}|\\0&0&i|\vec{p}|&E\end{bmatrix}\frac{1}{\sqrt{2}}[-1,-i,0,0]^T=\frac{1}{\sqrt{2}}[-1,-i,0,0]^T$ □

Proof: $e^{-ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0},0)=\frac{1}{m}\begin{bmatrix}m&0&0&0\\0&m&0&0\\0&0&E&-i|\vec{p}|\\0&0&i|\vec{p}|&E\end{bmatrix}[0,0,1,0]^T=\frac{1}{m}[0,0,E,i|\vec{p}|]^T$ □

Proof: $e^{-ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0},-1)=\frac{1}{m}\begin{bmatrix}m&0&0&0\\0&m&0&0\\0&0&E&-i|\vec{p}|\\0&0&i|\vec{p}|&E\end{bmatrix}\frac{1}{\sqrt{2}}[1,-i,0,0]^T=\frac{1}{\sqrt{2}}[1,-i,0,0]^T$ □

Thm. 2.2.2.
$\begin{cases}e^{-ln[\gamma_v(1+v)]L_z}R_xe^{ln[\gamma_v(1+v)]L_z}=\frac{E}{m}R_x-\frac{i|\vec{p}|}{m}L_y\\e^{-ln[\gamma_v(1+v)]L_z}R_ye^{ln[\gamma_v(1+v)]L_z}=\frac{E}{m}R_y+\frac{i|\vec{p}|}{m}L_x\\e^{-ln[\gamma_v(1+v)]L_z}R_ze^{ln[\gamma_v(1+v)]L_z}=R_z\end{cases}$

Proof: $e^{-ln[\gamma_v(1+v)]L_z}R_ze^{ln[\gamma_v(1+v)]L_z}$
$=[1-\gamma_vvL_z+(\gamma_v-1)L_z^2]R_z[1+\gamma_vvL_z+(\gamma_v-1)L_z^2]$
$=\frac{1}{m^2}[m-|\vec{p}|L_z+(E-m)L_z^2]R_z[m+|\vec{p}|L_z+(E-m)L_z^2]$
$=\frac{1}{m^2}\begin{bmatrix}m&0&0&0\\0&m&0&0\\0&0&E&-i|\vec{p}|\\0&0&i|\vec{p}|&E\end{bmatrix}R_z\begin{bmatrix}m&0&0&0\\0&m&0&0\\0&0&E&i|\vec{p}|\\0&0&-i|\vec{p}|&E\end{bmatrix}=R_z$ □

Proof: $e^{-ln[\gamma_v(1+v)]L_z}R_xe^{ln[\gamma_v(1+v)]L_z}$
$=\frac{1}{m^2}\begin{bmatrix}m&0&0&0\\0&m&0&0\\0&0&E&-i|\vec{p}|\\0&0&i|\vec{p}|&E\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}m&0&0&0\\0&m&0&0\\0&0&E&i|\vec{p}|\\0&0&-i|\vec{p}|&E\end{bmatrix}$

$= \frac{1}{m^2}\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -iE & |\vec{p}| \\ 0 & im & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= \frac{1}{m}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -iE & |\vec{p}| \\ 0 & iE & 0 & 0 \\ 0 & -|\vec{p}| & 0 & 0 \end{bmatrix} = \frac{E}{m}R_x - \frac{i|\vec{p}|}{m}L_y$ □

Proof: $e^{-ln[\gamma_v(1+v)]L_z}R_y e^{ln[\gamma_v(1+v)]L_z}$

$= \frac{1}{m^2}\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix}\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & i|\vec{p}| \\ 0 & 0 & -i|\vec{p}| & E \end{bmatrix}$

$= \frac{1}{m^2}\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix}\begin{bmatrix} 0 & 0 & iE & -|\vec{p}| \\ 0 & 0 & 0 & 0 \\ -im & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= \frac{1}{m}\begin{bmatrix} 0 & 0 & iE & -|\vec{p}| \\ 0 & 0 & 0 & 0 \\ -iE & 0 & 0 & 0 \\ |\vec{p}| & 0 & 0 & 0 \end{bmatrix} = \frac{E}{m}R_y + \frac{i|\vec{p}|}{m}L_x$ □

Cor. 2.2.2.
$$\begin{cases} e^{-ln[\gamma_v(1+v)]L_z}(R_x+iR_y)e^{ln[\gamma_v(1+v)]L_z} = \frac{E}{m}(R_x+iR_y) - \frac{|\vec{p}|}{m}(L_x+iL_y) \\ e^{-ln[\gamma_v(1+v)]L_z}(R_x-iR_y)e^{ln[\gamma_v(1+v)]L_z} = \frac{E}{m}(R_x-iR_y) + \frac{|\vec{p}|}{m}(L_x-iL_y) \\ e^{-ln[\gamma_v(1+v)]L_z}R_z e^{ln[\gamma_v(1+v)]L_z} = R_z \end{cases}$$

2.3 $e^{i\vec{\omega}\cdot R}Re^{-i\vec{\omega}\cdot R}$ properties

Lem. 2.3.1. $e^{i\vec{\omega}\cdot R} = 1 + i(R\times\hat{p})_z - (R\times\hat{p})_z^2/(1+\hat{p}_z) = 1 + i(R_x\hat{p}_y - R_y\hat{p}_x) - (R_x\hat{p}_y - R_y\hat{p}_x)_z^2/(1+\hat{p}_z)$

$= 1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)$

Thm. 2.3.1. $e^{i\vec{\omega}\cdot R}R_x e^{-i\vec{\omega}\cdot R} = R_x - \frac{\hat{p}_x}{1+\hat{p}_z}(R\cdot\hat{p}+R_z), e^{i\vec{\omega}\cdot R}R_y e^{-i\vec{\omega}\cdot R} = R_y - \frac{\hat{p}_y}{1+\hat{p}_z}(R\cdot\hat{p}+R_z), e^{i\vec{\omega}\cdot R}R_z e^{-i\vec{\omega}\cdot R} = R\cdot\hat{p}$

Proof: $e^{i\vec{\omega}\cdot R}R_x e^{-i\vec{\omega}\cdot R}$

$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)]\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}[1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)]$

$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)][\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ i(1+\hat{p}_z)\hat{p}_x & i(1+\hat{p}_z)\hat{p}_y & -i+i\hat{p}_z^2 & 0 \\ i\hat{p}_y\hat{p}_x & i\hat{p}_y^2 & i\hat{p}_y(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)]$

$= R_x - i\begin{bmatrix} 0 & 0 & 0 & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & -1+\hat{p}_z^2 & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z) - i\begin{bmatrix} 0 & -\hat{p}_x(1+\hat{p}_z) & -\hat{p}_x\hat{p}_y & 0 \\ 0 & -\hat{p}_y(1+\hat{p}_z) & -\hat{p}_y^2 & 0 \\ 0 & 1-\hat{p}_z^2 & -(1+\hat{p}_z)\hat{p}_y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)$

$+ \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} i \begin{bmatrix} 0 & 0 & 0 & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & -1+\hat{p}_z^2 & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)^2]$

$= R_x - i\begin{bmatrix} 0 & -\hat{p}_x(1+\hat{p}_z) & -\hat{p}_x\hat{p}_y & 0 \\ (1+\hat{p}_z)\hat{p}_x & 0 & -\hat{p}_x^2-2\hat{p}_y^2 & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_x^2+2\hat{p}_y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z) - i\begin{bmatrix} 0 & 0 & 2\hat{p}_x\hat{p}_y & 0 \\ 0 & 0 & 2\hat{p}_y^2 & 0 \\ -2\hat{p}_x\hat{p}_y & -2\hat{p}_y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)$

$= R_x - i\begin{bmatrix} 0 & -\hat{p}_x(1+\hat{p}_z) & \hat{p}_x\hat{p}_y & 0 \\ (1+\hat{p}_z)\hat{p}_x & 0 & -\hat{p}_x^2 & 0 \\ -\hat{p}_x\hat{p}_y & \hat{p}_x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)$

$= R_x - \frac{\hat{p}_x}{1+\hat{p}_z}(R\cdot\hat{p}+R_z)$ □

Proof: $e^{i\vec{\omega}\cdot R}R_y e^{-i\vec{\omega}\cdot R}$

$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)]\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}[1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)]$

$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)][\begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -i(1+\hat{p}_z)\hat{p}_x & -i(1+\hat{p}_z)\hat{p}_y & i-i\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \\ -i\hat{p}_x^2 & -i\hat{p}_x\hat{p}_y & -i\hat{p}_x(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)]$

$= R_y - i\begin{bmatrix} -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\hat{p}_x^2 & -\hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)] - i\begin{bmatrix} \hat{p}_x(1+\hat{p}_z) & 0 & \hat{p}_x^2 & 0 \\ \hat{p}_y(1+\hat{p}_z) & 0 & \hat{p}_y\hat{p}_x & 0 \\ -1+\hat{p}_z^2 & 0 & (1+\hat{p}_z)\hat{p}_x & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}/(1+\hat{p}_z)$

$$- i \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & -1+\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)^2$$

$$= R_y - i \begin{bmatrix} 0 & -(1+\hat{p}_z)\hat{p}_y & 2\hat{p}_x^2+\hat{p}_y^2 & 0 \\ (1+\hat{p}_z)\hat{p}_y & 0 & \hat{p}_x\hat{p}_y & 0 \\ -2\hat{p}_x^2-\hat{p}_y^2 & -\hat{p}_x\hat{p}_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)] - i \begin{bmatrix} 0 & 0 & -2\hat{p}_x^2 & 0 \\ 0 & 0 & -2\hat{p}_x\hat{p}_y & 0 \\ 2\hat{p}_x^2 & 2\hat{p}_x\hat{p}_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)$$

$$= R_y - i \begin{bmatrix} 0 & -(1+\hat{p}_z)\hat{p}_y & \hat{p}_y^2 & 0 \\ (1+\hat{p}_z)\hat{p}_y & 0 & -\hat{p}_x\hat{p}_y & 0 \\ -\hat{p}_y^2 & \hat{p}_x\hat{p}_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= R_y - \frac{\hat{p}_y}{1+\hat{p}_z}(R \cdot \hat{p} + R_z)$$ □

Proof: $e^{i\vec{\omega}\cdot R} R_z e^{-i\vec{\omega}\cdot R}$

$$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)] \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)][\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -i\hat{p}_y\hat{p}_x & -i\hat{p}_y^2 & -i\hat{p}_y(1+\hat{p}_z) & 0 \\ i\hat{p}_x^2 & i\hat{p}_x\hat{p}_y & i\hat{p}_x(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= R_z - i \begin{bmatrix} -\hat{p}_y\hat{p}_x & -\hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)$$

$$- i \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ -\hat{p}_x^2 & -\hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)^2$$

$$= R_z - i \begin{bmatrix} 0 & -\hat{p}_x^2-\hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ \hat{p}_x^2+\hat{p}_y^2 & 0 & \hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y(1+\hat{p}_z) & -\hat{p}_x(1+\hat{p}_z) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)$$

$$= R \cdot \hat{p}$$ □

2.4 $e^{i\vec{\omega}\cdot R} L e^{-i\vec{\omega}\cdot R}$ properties

Thm. 2.4.1. $e^{i\vec{\omega}\cdot R} L_x e^{-i\vec{\omega}\cdot R} = L_x - \frac{\hat{p}_x}{1+\hat{p}_z}(L \cdot \hat{p} + L_z), e^{i\vec{\omega}\cdot R} L_y e^{-i\vec{\omega}\cdot R} = L_y - \frac{\hat{p}_y}{1+\hat{p}_z}(L \cdot \hat{p} + L_z), e^{i\vec{\omega}\cdot R} L_z e^{-i\vec{\omega}\cdot R} = L \cdot \hat{p}$

Proof: $e^{i\vec{\omega}\cdot R} L_x e^{-i\vec{\omega}\cdot R}$

$$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)] \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)][\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= L_x + i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \end{bmatrix} /(1+\hat{p}_z)] - i \begin{bmatrix} 0 & 0 & 0 & \hat{p}_x^2 \\ 0 & 0 & 0 & \hat{p}_y\hat{p}_x \\ 0 & 0 & 0 & (1+\hat{p}_z)\hat{p}_x \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= L_x - i \begin{bmatrix} 0 & 0 & 0 & \hat{p}_x^2 \\ 0 & 0 & 0 & \hat{p}_y\hat{p}_x \\ 0 & 0 & 0 & (1+\hat{p}_z)\hat{p}_x \\ -\hat{p}_x^2 & -\hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= L_x - \frac{\hat{p}_x}{1+\hat{p}_z}(L \cdot \hat{p} + L_z)$$ □

Proof: $e^{i\vec{\omega}\cdot R} L_y e^{-i\vec{\omega}\cdot R}$

$$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)][\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \end{bmatrix} /(1+\hat{p}_z)]$$

$$= L_y + i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \end{bmatrix} /(1+\hat{p}_z)] - i \begin{bmatrix} 0 & 0 & 0 & \hat{p}_x\hat{p}_y \\ 0 & 0 & 0 & \hat{p}_y^2 \\ 0 & 0 & 0 & (1+\hat{p}_z)\hat{p}_y \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)$$

$$= L_y - i \begin{bmatrix} 0 & 0 & 0 & \hat{p}_x\hat{p}_y \\ 0 & 0 & 0 & \hat{p}_y^2 \\ 0 & 0 & 0 & (1+\hat{p}_z)\hat{p}_y \\ -\hat{p}_y\hat{p}_x & -\hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \end{bmatrix} /(1+\hat{p}_z)$$

$$= L_y - \frac{\hat{p}_y}{1+\hat{p}_z}(L \cdot \hat{p} + L_z)$$ □

Proof: $e^{i\vec{\omega}\cdot R}L_z e^{-i\vec{\omega}\cdot R}$

$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)] \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&i\\0&0&-i&0 \end{bmatrix} [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & \hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & \hat{p}_y(1+\hat{p}_z) & 0 \\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)]$

$= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x\hat{p}_y & -\hat{p}_x(1+\hat{p}_z) & 0 \\ \hat{p}_y\hat{p}_x & \hat{p}_y^2 & -\hat{p}_y(1+\hat{p}_z) & 0 \\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} /(1+\hat{p}_z)][\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&i\\0&0&-i&0 \end{bmatrix} + i \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \end{bmatrix} /(1+\hat{p}_z)]$

$= L_z + i \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\ -(1+\hat{p}_z)\hat{p}_x & -(1+\hat{p}_z)\hat{p}_y & 1-\hat{p}_z^2 & 0 \end{bmatrix} /(1+\hat{p}_z) - i \begin{bmatrix} 0&0&0&-\hat{p}_x(1+\hat{p}_z)\\0&0&0&-\hat{p}_y(1+\hat{p}_z)\\0&0&0&1-\hat{p}_z^2\\0&0&0&0 \end{bmatrix} /(1+\hat{p}_z)$

$= L_z - i \begin{bmatrix} 0&0&0&-\hat{p}_x(1+\hat{p}_z)\\0&0&0&-\hat{p}_y(1+\hat{p}_z)\\0&0&0&1-\hat{p}_z^2\\ (1+\hat{p}_z)\hat{p}_x & (1+\hat{p}_z)\hat{p}_y & -1+\hat{p}_z^2 & 0 \end{bmatrix} /(1+\hat{p}_z)$

$= L\cdot\hat{p}$ □

2.5 Raising and lowering operator of $\varepsilon(\vec{p},h)$

Thm. 2.5.1. $\varepsilon(\vec{p},h) = e^{i\vec{\omega}\cdot R}e^{-ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0},h) = [i\lambda_m(\hat{p},1),0],[\frac{E}{m}i\lambda_m(\hat{p},0),\frac{i|\vec{p}|}{m}],[i\lambda_m(\hat{p},-1),0]$

Cor. 2.5.1.

$$\begin{cases} e^{-ln[\gamma_v(1+v)]L_z}R_x e^{ln[\gamma_v(1+v)]L_z}e^{-i\vec{\omega}\cdot R} = \frac{E}{m}[R_x - \frac{\hat{p}_x}{1+\hat{p}_z}(R\cdot\hat{p}+R_z)] - \frac{i|\vec{p}|}{m}[L_y - \frac{\hat{p}_y}{1+\hat{p}_z}(L\cdot\hat{p}+L_z)] \\ e^{i\vec{\omega}\cdot R}e^{-ln[\gamma_v(1+v)]L_z}R_y e^{ln[\gamma_v(1+v)]L_z}e^{-i\vec{\omega}\cdot R} = \frac{E}{m}[R_y - \frac{\hat{p}_y}{1+\hat{p}_z}(R\cdot\hat{p}+R_z)] + \frac{i|\vec{p}|}{m}[L_x - \frac{\hat{p}_x}{1+\hat{p}_z}(L\cdot\hat{p}+L_z)] \\ e^{i\vec{\omega}\cdot R}e^{-ln[\gamma_v(1+v)]L_z}R_z e^{ln[\gamma_v(1+v)]L_z}e^{-i\vec{\omega}\cdot R} = R\cdot\hat{p} \end{cases}$$

Def. 2.5.1.

$$\begin{cases} \hat{J}_x(\vec{p},1;R,L) := \frac{E}{m}[R_x - \frac{\hat{p}_x}{1+\hat{p}_z}(R\cdot\hat{p}+R_z)] - \frac{i|\vec{p}|}{m}[L_y - \frac{\hat{p}_y}{1+\hat{p}_z}(L\cdot\hat{p}+L_z)] \\ \hat{J}_y(\vec{p},1;R,L) := \frac{E}{m}[R_y - \frac{\hat{p}_y}{1+\hat{p}_z}(R\cdot\hat{p}+R_z)] + \frac{i|\vec{p}|}{m}[L_x - \frac{\hat{p}_x}{1+\hat{p}_z}(L\cdot\hat{p}+L_z)] \\ \hat{J}_z(\vec{p},1;R,L) := R\cdot\hat{p} \end{cases}$$

Cor. 2.5.2. $[\hat{J}_i(\vec{p},1;R,L),\hat{J}_j(\vec{p},1;R,L)] = \varepsilon_{ij}{}^k\hat{J}_k(\vec{p},1;R,L)$

Cor. 2.5.3.

$$\begin{cases} \hat{J}_+(\vec{p},1;R,L) := e^{i\vec{\omega}\cdot R}e^{-ln[\gamma_v(1+v)]L_z}(R_x+iR_y)e^{ln[\gamma_v(1+v)]L_z}e^{-i\vec{\omega}\cdot R} \\ = \frac{E}{m}[(R_x+iR_y) - \frac{\hat{p}_x+i\hat{p}_y}{1+\hat{p}_z}(R\cdot\hat{p}+R_z)] - \frac{|\vec{p}|}{m}[(L_x+iL_y) - \frac{\hat{p}_x+i\hat{p}_y}{1+\hat{p}_z}(L\cdot\hat{p}+L_z)] \\ \hat{J}_-(\vec{p},1;R,L) := e^{i\vec{\omega}\cdot R}e^{-ln[\gamma_v(1+v)]L_z}(R_x-iR_y)e^{ln[\gamma_v(1+v)]L_z}e^{-i\vec{\omega}\cdot R} \\ = \frac{E}{m}[(R_x-iR_y) - \frac{\hat{p}_x-i\hat{p}_y}{1+\hat{p}_z}(R\cdot\hat{p}+R_z)] + \frac{|\vec{p}|}{m}[(L_x-iL_y) - \frac{\hat{p}_x-i\hat{p}_y}{1+\hat{p}_z}(L\cdot\hat{p}+L_z)] \\ \hat{J}_z(\vec{p},1;R,L) := e^{i\vec{\omega}\cdot R}e^{-ln[\gamma_v(1+v)]L_z}R_z e^{ln[\gamma_v(1+v)]L_z}e^{-i\vec{\omega}\cdot R} = R\cdot\hat{p} \end{cases}$$

Cor. 2.5.4.

$$\begin{cases} \hat{J}_+(\vec{p},1;R,L)\varepsilon(\vec{p},h) = \sqrt{2-h(h+1)}\varepsilon(\vec{p},h+1), -1\le h\le 1 \\ \hat{J}_-(\vec{p},1;R,L)\varepsilon_(\vec{p},h) = \sqrt{2-h(h+1)}\varepsilon(\vec{p},h-1), -1\le h\le 1 \\ \hat{J}_z(\vec{p},1;R,L)\varepsilon(\vec{p},h) = h\varepsilon(\vec{p},h), -1\le h\le 1 \end{cases}$$

2.6 Definition-Spin basis decomposition: $n = (n-1)\oplus 1$

Def. 2.6.1. $-n\le h\le n$

$$\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h) := \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1) + \frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)$$

Def. 2.6.2. $-n-\frac{1}{2}\le h\le n+\frac{1}{2}$

$$\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h) = \frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

Cor. 2.6.1.

$$\bar{\varepsilon}_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h) = \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\bar{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\bar{\varepsilon}_c(\vec{p},1) + \frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\bar{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\bar{\varepsilon}_c(\vec{p},0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\bar{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\bar{\varepsilon}_c(\vec{p},-1)$$

Cor. 2.6.2.

$$\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h) = \sum_{h'=1}^{-1}\frac{\sqrt{C_{n+h}^{1+h'}C_{n-h}^{1-h'}}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-h')\varepsilon_c(\vec{p},h'), \bar{\varepsilon}_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h) = \sum_{h'=1}^{-1}\frac{\sqrt{C_{n+h}^{1+h'}C_{n-h}^{1-h'}}}{\sqrt{C_{2n}^2}}\bar{\varepsilon}_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-h')\bar{\varepsilon}_c(\vec{p},h')$$

2.7 Corollary-$\varepsilon_{a\cdot\cdot bc}(\vec{p},h)$ is a spin eigenstate

Def. 2.7.1. $\Omega(n;R) := R\otimes I_{4^{n-1}} + I_4\otimes R\otimes I_{4^{n-2}} + \cdot\cdot + I_{4^{n-1}}\otimes R$

Thm. 2.7.1. $[\Omega(n;R)\cdot\hat{p}]\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h) = h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h), -n\le h\le n$

Proof: $[\Omega(n;R)\cdot\hat{p}]\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h)$

$= [\Omega(n-1;R)\otimes I_4 + I_{4^{n-1}}\otimes R]\cdot\hat{p}$

$[\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)\varepsilon_{\otimes c}(\vec{p},1) + \frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\varepsilon_{\otimes c}(\vec{p},0) + \frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)\varepsilon_{\otimes c}(\vec{p},-1)]$

$= [\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)\varepsilon_{\otimes c}(\vec{p},1) + \frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\varepsilon_{\otimes c}(\vec{p},0) + \frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)\varepsilon_{\otimes c}(\vec{p},-1)]$

$= h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h), -n\le h\le n$ □

Thm. 2.7.2. $\Omega^2(n;R)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h) = n(n+1)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h)$

he above theorem can be easily proved using a fully symmetric representation transformation method. From the above, it can be seen that $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)$ is a spin eigenstate. Therefore, the expansion coefficients are CG coefficients, and the actual calculation results also indicate that they are indeed the corresponding CG coefficients. This also provides a unified, standardized, intuitive and complete new method for calculating CG coefficients.

2.8 Klein-GordonRaising and lowering operator of equation spin basis

Def. 2.8.1.

$$\begin{cases}\hat{J}(\vec{p},n;R,L) := \underbrace{\hat{J}(\vec{p},1;R,L)\otimes I_4\otimes\cdot\cdot\otimes I_4}_{n} + \underbrace{I_4\otimes\hat{J}(\vec{p},1;R,L)\otimes\cdot\cdot\otimes I_4}_{n} + \cdots + \underbrace{I_4\otimes\cdot\cdot\otimes I_4\otimes\hat{J}(\vec{p},1;R,L)}_{n}\\ \hat{Q}(\vec{p},n;R,L) := \underbrace{\hat{Q}(\vec{p},1;R,L)\otimes I_4\otimes\cdot\cdot\otimes I_4}_{n} + \underbrace{I_4\otimes\hat{Q}(\vec{p},1;R,L)\otimes\cdot\cdot\otimes I_4}_{n} + \cdots + \underbrace{I_4\otimes\cdot\cdot\otimes I_4\otimes\hat{Q}(\vec{p},1;R,L)}_{n}\end{cases}$$

Cor. 2.8.1. $[\hat{J}_i(\vec{p},n;R,L), \hat{J}_j(\vec{p},n;R,L)] = \varepsilon_{ij}{}^k\hat{J}_k(\vec{p},n;R,L)$

Thm. 2.8.1. $\hat{J}_+(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h) = \sqrt{n(n+1)-h(h+1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h+1), -n\le h\le n$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $n'=1$, the following is established.
$\hat{J}_+(\vec{p},1;R,L)\varepsilon(\vec{p},h) = \sqrt{2-h(h+1)}\varepsilon(\vec{p},h+1), -1\le h\le 1$
Step 2: Assume when $n'=n-1$, the following is established.
$\hat{J}_+(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h) = \sqrt{(n-1)n-h(h+1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1), -n+1\le h\le n-1$
Step 3: When $n'=n,\ -n\le h\le n, \hat{J}_+(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h)$

$= \frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}[\hat{J}_+(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)]\varepsilon_{\otimes c}(\vec{p},1) + \frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}[\hat{J}_+(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)]\varepsilon_{\otimes c}(\vec{p},0)$

$+ \frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}[\hat{J}_+(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)]\varepsilon_{\otimes c}(\vec{p},-1) + \frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)\hat{J}_+(\vec{p},\frac{1}{2};R,L)\varepsilon_{\otimes c}(\vec{p},1)$

$+ \frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\hat{J}_+(\vec{p},\frac{1}{2};R,L)\varepsilon_{\otimes c}(\vec{p},0) + \frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)\hat{J}_+(\vec{p},\frac{1}{2};R,L)\varepsilon_{\otimes c}(\vec{p},-1)$

$= \frac{\sqrt{C^2_{n+h}}\sqrt{(n-1)n-(h-1)h}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\varepsilon_{\otimes c}(\vec{p},1) + \frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\sqrt{2}\varepsilon_{\otimes c}(\vec{p},1)$

$+ \frac{\sqrt{C^1_{n+h}C^1_{n-h}}\sqrt{(n-1)n-h(h+1)}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)\varepsilon_{\otimes c}(\vec{p},0) + \frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)\sqrt{2}\varepsilon_{\otimes c}(\vec{p},0)$

$+ \frac{\sqrt{C^2_{n-h}}\sqrt{(n-1)n-(h+1)(h+2)}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+2)\varepsilon_{\otimes c}(\vec{p},-1)$

$= \frac{\sqrt{n(n+1)-h(h+1)}\sqrt{C^2_{n+h+1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\varepsilon_{\otimes c}(\vec{p},1) + \frac{\sqrt{n(n+1)-h(h+1)}\sqrt{C^1_{n+h+1}C^1_{n-h-1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)\varepsilon_{\otimes c}(\vec{p},0)$

$+\frac{\sqrt{n(n+1)-h(h+1)}\sqrt{C^2_{n-h-1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+2)\varepsilon_{\otimes c}(\vec{p},-1)$

$=\sqrt{n(n+1)-h(h+1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h+1)$

This step proves that when $n'=n$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Thm. 2.8.2. $\hat{J}_-(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h)=\sqrt{n(n+1)-h(h-1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h-1),-n\le h\le n$

Proof: Using mathematical induction to prove this theorem.

Step 1: When $n'=1$, the following is established.

$\hat{J}_-(\vec{p},1;R,L)\varepsilon(\vec{p},h)=\sqrt{2-h(h-1)}\varepsilon(\vec{p},h-1),-1\le h\le 1$

Step 2: Assume when $n'=n-1$, the following is established.

$\hat{J}_-(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)=\sqrt{(n-1)n-h(h-1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1),-n+1\le h\le n-1$

Step 3: When $n'=n,\ -n\le h\le n$, $\hat{J}_-(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h)$

$=\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}[\hat{J}_-(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)]\varepsilon_{\otimes c}(\vec{p},1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}[\hat{J}_-(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)]\varepsilon_{\otimes c}(\vec{p},0)$

$+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}[\hat{J}_-(\vec{p},n-1;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)]\varepsilon_{\otimes c}(\vec{p},-1)+\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)\hat{J}_-(\vec{p},\tfrac{1}{2};R,L)\varepsilon_{\otimes c}(\vec{p},1)$

$+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\hat{J}_-(\vec{p},\tfrac{1}{2};R,L)\varepsilon_{\otimes c}(\vec{p},0)+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h+1)\hat{J}_-(\vec{p},\tfrac{1}{2};R,L)\varepsilon_{\otimes c}(\vec{p},-1)$

$=\frac{\sqrt{C^2_{n+h}}\sqrt{(n-1)n-(h-1)(h-2)}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-2)\varepsilon_{\otimes c}(\vec{p},1)$

$+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}\sqrt{(n-1)n-h(h-1)}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)\varepsilon_{\otimes c}(\vec{p},0)+\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)\sqrt{2}\varepsilon_{\otimes c}(\vec{p},0)$

$+\frac{\sqrt{C^2_{n-h}}\sqrt{(n-1)n-(h+1)h}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\varepsilon_{\otimes c}(\vec{p},-1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\sqrt{2}\varepsilon_{\otimes c}(\vec{p},-1)$

$=\frac{\sqrt{n(n+1)-h(h-1)}\sqrt{C^2_{n+h-1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-2)\varepsilon_{\otimes c}(\vec{p},1)+\frac{\sqrt{n(n+1)-h(h-1)}\sqrt{C^1_{n+h-1}C^1_{n-h+1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h-1)\varepsilon_{\otimes c}(\vec{p},0)$

$+\frac{\sqrt{n(n+1)-h(h-1)}\sqrt{C^2_{n-h+1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n-1}}(\vec{p},h)\varepsilon_{\otimes c}(\vec{p},-1)$

$=\sqrt{n(n+1)-h(h-1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h-1)$

This step proves that when $n'=n$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Cor. 2.8.2. $\hat{J}^2(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h)=n(n+1)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{p},h),-n\le h\le n$

Cor. 2.8.3.

$$\begin{cases}\hat{J}_+(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h)=\sqrt{n(n+1)-h(h+1)}\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h+1),-n\le h\le n\\ \hat{J}_-(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h)=\sqrt{n(n+1)-h(h-1)}\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h-1),-n\le h\le n\\ \hat{J}_z(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h)=h\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h),-n\le h\le n\end{cases}$$

Cor. 2.8.4.

$$\begin{cases}\hat{J}^2(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h)=n(n+1)\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h),\hat{J}^2(\vec{p},*1;R,L)\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h)=2\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h)\\ \hat{J}_z(\vec{p},n;R,L)\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h)=h\varepsilon_{\underbrace{a\otimes b\otimes\cdot\cdot}_{n}}(\vec{p},h),\hat{J}^2=\hat{J}_z^2+\frac{1}{2}(\hat{J}_+\hat{J}_-+\hat{J}_+\hat{J}_-),-n\le h\le n\\ \delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdot\cdot\}}_{n}}(\vec{p},h)\end{cases}$$

Lem. 2.8.1. $(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(p)=(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}(-p)=4im\delta^b_a$

Thm. 2.8.3. $\hat{J}(\vec{p},s;R,L)=\frac{1}{(i4m)^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\otimes\mu_\varsigma\otimes}(\bar{C}\gamma_b)^{\eta_\varsigma\otimes\xi_\varsigma\otimes}\cdot\cdot}^{n}\hat{J}(\vec{p},s;\gamma_a)\underbrace{\mathbb{X}^{a'}_{\lambda'_\varsigma\otimes\mu'_\varsigma\otimes}(p)\mathbb{X}^{b'}_{\eta'_\varsigma\otimes\xi'_\varsigma\otimes}(p)\cdot\cdot}_{n}$

Proof: $\hat{J}_+(\vec{p},s;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h+1)}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h+1)$

$\Rightarrow\frac{1}{(i\sqrt{2})^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\otimes\mu_\varsigma\otimes}(\bar{C}\gamma_b)^{\eta_\varsigma\otimes\xi_\varsigma\otimes}\cdot\cdot}^{n}\hat{J}_+(\vec{p},s;\gamma_a)\frac{1}{(2\sqrt{2}m)^n}\underbrace{\mathbb{X}^{a'}_{\lambda'_\varsigma\otimes\mu'_\varsigma\otimes}(p)\mathbb{X}^{b'}_{\eta'_\varsigma\otimes\xi'_\varsigma\otimes}(p)\cdot\cdot}_{n}\varepsilon_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p},h)$

$=\sqrt{s(s+1)-h(h+1)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)$

$\Rightarrow\hat{J}_+(\vec{p},s;R,L)=\frac{1}{(i4m)^n}\overbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\otimes\mu_\varsigma\otimes}(\bar{C}\gamma_b)^{\eta_\varsigma\otimes\xi_\varsigma\otimes}\cdot\cdot}^{n}\hat{J}_+(\vec{p},s;\gamma_a)\underbrace{\mathbb{X}^{a'}_{\lambda'_\varsigma\otimes\mu'_\varsigma\otimes}(p)\mathbb{X}^{b'}_{\eta'_\varsigma\otimes\xi'_\varsigma\otimes}(p)\cdot\cdot}_{n}$ □

2.9 Corollary-$\varepsilon_{ab\cdot\cdot c}(\vec{p},h)$ orthogonality

Def. 2.9.1. $\bar{\varepsilon}^a(\vec{p},h)\varepsilon_a(\vec{p},h')=\delta_{hh'},-1\le h',h\le 1$

Def. 2.9.2. $\sum_{h=1}^{-1}\varepsilon_a(\vec{p},h)\bar{\varepsilon}_b(\vec{p},h)=\delta_{ab}+\frac{p_ap_b}{m^2}$

Thm. 2.9.1. $\bar{\varepsilon}^{\overbrace{a\cdot\cdot bc}^{n}}(\vec{p},h')\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\delta_{hh'},-n\le h',h\le n$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $n'=1$, the following is established.
$\bar{\varepsilon}^a(\vec{p},h')\varepsilon_a(\vec{p},h)=\delta_{hh'},-1\le h',h\le 1$
Step 2: Assume when $n'=n-1$, the following is established.
$\bar{\varepsilon}^{\overbrace{a\cdot\cdot b}^{n-1}}(\vec{p},h')\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)=\delta_{hh'},-n+1\le h',h\le n-1$

Step 3: When $n'=n$, $\bar{\varepsilon}^{\overbrace{a\cdot\cdot bc}^{n}}(\vec{p},h')\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h),-n\le h',h\le n$

$=[\frac{\sqrt{C^2_{n+h'}}}{\sqrt{C^2_{2n}}}\bar{\varepsilon}^{\overbrace{a\cdot\cdot b}^{n-1}}(\vec{p},h'-1)\bar{\varepsilon}^c(\vec{p},1)+\frac{\sqrt{C^1_{n+h'}C^1_{n-h'}}}{\sqrt{C^2_{2n}}}\bar{\varepsilon}^{\overbrace{a\cdot\cdot b}^{n-1}}(\vec{p},h')\bar{\varepsilon}^c(\vec{p},0)+\frac{\sqrt{C^2_{n-h'}}}{\sqrt{C^2_{2n}}}\bar{\varepsilon}^{\overbrace{a\cdot\cdot b}^{n-1}}(\vec{p},h'+1)\bar{\varepsilon}^c(\vec{p},-1)]$
$[\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)]$
$=\frac{\sqrt{C^2_{n+h'}}}{\sqrt{C^2_{2n}}}\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\delta_{hh'}+\frac{\sqrt{C^1_{n+h'}C^1_{n-h'}}}{\sqrt{C^2_{2n}}}\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\delta_{hh'}+\frac{\sqrt{C^2_{n-h'}}}{\sqrt{C^2_{2n}}}\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\delta_{hh'}=\delta_{hh'}$
This step proves that when $n'=n$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

2.10 Corollary-$p^a\varepsilon_{ab\cdot\cdot c}(\vec{p},h)$ nullity

Def. 2.10.1. $p^a\varepsilon_a(\vec{p},h)=0,-1\le h',h\le 1$

Thm. 2.10.1. $p^a\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=0,-n\le h\le n$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $n'=1$, the following is established.
$p^a\varepsilon_a(\vec{p},h)=\delta_{hh'},-1\le h\le 1$
Step 2: Assume when $n'=n-1$, the following is established.
$p^a\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)=0,-n+1\le h\le n-1$
Step 3: When $n'=n$, $p^a\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h),-n\le h\le n$

$=p^a[\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)]$
$=0+0+0=0$
This step proves that when $n'=n$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

2.11 Corollary-$\varepsilon_{ab\cdot\cdot c}(\vec{p},h)$ tracelessness

Def. 2.11.1. $\bar{\varepsilon}_a(\vec{p},h)=(-1)^h\varepsilon_a(\vec{p},-h),-1\le h\le 1$

Lem. 2.11.1. $\delta^{ab}\varepsilon_{ab}(\vec{p},h)=0,-2\le h\le 2$

Proof: $\delta^{ab}\varepsilon_{ab}(\vec{p},h),-2\le h\le 2$

$$=\delta^{ab}[\frac{\sqrt{C_{2+h}^2}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},h-1)\varepsilon_b(\vec{p},1)+\frac{\sqrt{C_{2+h}^1C_{2-h}^1}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},h)\varepsilon_b(\vec{p},0)+\frac{\sqrt{C_{2-h}^2}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},h+1)\varepsilon_b(\vec{p},-1)]$$

$$=\begin{cases}\delta^{ab}\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},1)=0;h=2\\ \delta^{ab}\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},-1)=0;h=-2\\ \delta^{ab}[\frac{\sqrt{C_3^2}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},1)+\frac{\sqrt{C_3^1C_1^1}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},0)]=0;h=1\\ \delta^{ab}[\frac{\sqrt{C_3^1C_1^1}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},0)+\frac{\sqrt{C_3^2}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},-1)]=0;h=-1\\ \delta^{ab}[\frac{\sqrt{C_2^2}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},1)+\frac{\sqrt{C_2^1C_2^1}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},0)+\frac{\sqrt{C_2^2}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},-1)]=0;h=0\end{cases}$$ □

Thm. 2.11.1. $\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h)=0,n\ge 2,-n\le h\le n$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $n'=2$, the following is established.
$\delta^{ab}\varepsilon_{ab}(\vec{p},h)=0,-2\le h\le 2$
Step 2: Assume when $2\le n'=n-1$, the following is established.
$\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n-1}}(\vec{p},h)=0,-n+1\le h\le n-1$
Step 3: When $3\le n'=n$, $\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h),-n\le h\le n$

$$=\delta^{ab}[\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)]$$
$$=0+0+0=0$$

This step proves that when $n'=n$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

2.12 Corollary-Spin basis decomposition: $2=1\oplus 1$

Cor. 2.12.1. $\varepsilon_{ab}(\vec{p},h)=\sum_{h'=1}^{-1}\frac{\sqrt{C_{2+h}^{1+h'}C_{2-h}^{1-h'}}}{\sqrt{C_4^2}}\varepsilon_a(\vec{p},h-h')\varepsilon_b(\vec{p},h')$

$$=\begin{cases}\varepsilon_{ab}(\vec{p},2)=\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},1)\\ \varepsilon_{ab}(\vec{p},1)=\frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},1)]\\ \varepsilon_{ab}(\vec{p},0)=\frac{1}{\sqrt{6}}[\varepsilon_a(\vec{p},1)\varepsilon_b(\vec{p},-1)+\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},1)+2\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},0)]\\ \varepsilon_{ab}(\vec{p},-1)=\frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},0)+\varepsilon_a(\vec{p},0)\varepsilon_b(\vec{p},-1)]\\ \varepsilon_{ab}(\vec{p},-2)=\varepsilon_a(\vec{p},-1)\varepsilon_b(\vec{p},-1)\end{cases}$$

Cor. 2.12.2. $\varepsilon_{ab}(\vec{p},h)=\varepsilon_{ba}(\vec{p},h),-2\le h\le 2$

2.13 Corollary-Spin basis decomposition: $n=(n-2)\oplus 2$

Thm. 2.13.1. $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\sum_{h'=2}^{-2}\frac{\sqrt{C_{n+h}^{2+h'}C_{n-h}^{2-h'}}}{\sqrt{C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-h')\varepsilon_{bc}(\vec{p},h')$

Proof: $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)$

$$=\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)$$

$$=\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_c(\vec{p},1)$$
$$[\frac{\sqrt{C_{n+h-2}^2}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-2)\varepsilon_b(\vec{p},1)+\frac{\sqrt{C_{n+h-2}^1C_{n-h}^1}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-1)\varepsilon_b(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_b(\vec{p},-1)]$$
$$+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_c(\vec{p},0)$$
$$[\frac{\sqrt{C_{n+h-1}^2}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-1)\varepsilon_b(\vec{p},1)+\frac{\sqrt{C_{n+h-1}^1C_{n-h-1}^1}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_b(\vec{p},0)+\frac{\sqrt{C_{n-h-1}^2}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+1)\varepsilon_b(\vec{p},-1)]$$

$$+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_c(\vec{p},-1)$$

$$[\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_b(\vec{p},1)+\frac{\sqrt{C_{n+h}^1C_{n-h-2}^1}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+1)\varepsilon_b(\vec{p},0)+\frac{\sqrt{C_{n-h-2}^2}}{\sqrt{C_{2n-2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+2)\varepsilon_b(\vec{p},-1)]$$

$$=[\frac{\sqrt{C_{n+h-2}^2}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-2)\varepsilon_b(\vec{p},1)\varepsilon_c(\vec{p},1)$$

$$+[\frac{\sqrt{C_{n+h-1}^2}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-1)\varepsilon_b(\vec{p},1)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C_{n+h-2}^1C_{n-h}^1}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-1)\varepsilon_b(\vec{p},0)\varepsilon_c(\vec{p},1)]$$

$$+[\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_b(\vec{p},1)\varepsilon_c(\vec{p},-1)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_b(\vec{p},-1)\varepsilon_c(\vec{p},1)]$$

$$+\frac{\sqrt{C_{n+h-1}^1C_{n-h-1}^1}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_b(\vec{p},0)\varepsilon_c(\vec{p},0)$$

$$+[\frac{\sqrt{C_{n+h}^1C_{n-h-2}^1}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+1)\varepsilon_b(\vec{p},0)\varepsilon_c(\vec{p},-1)+\frac{\sqrt{C_{n-h-1}^2}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+1)\varepsilon_b(\vec{p},-1)\varepsilon_c(\vec{p},0)]$$

$$+\frac{\sqrt{C_{n-h-2}^2}}{\sqrt{C_{2n-2}^2}}\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+2)\varepsilon_b(\vec{p},-1)\varepsilon_c(\vec{p},-1)]$$

$$=\frac{1}{2!}\frac{\sqrt{C_{n+h}^4C_4^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-2)\varepsilon_{\{b}(\vec{p},1)\varepsilon_{c\}}(\vec{p},1)+\frac{1}{2!}\frac{\sqrt{C_{n-h}^4C_4^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+2)\varepsilon_{\{b}(\vec{p},-1)\varepsilon_{c\}}(\vec{p},-1)$$

$$+\frac{\sqrt{C_{n+h}^3C_{n-h}^1C_3^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-1)\varepsilon_{\{b}(\vec{p},1)\varepsilon_{c\}}(\vec{p},0)+\frac{\sqrt{C_{n+h}^1C_{n-h}^3C_3^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+1)\varepsilon_{\{b}(\vec{p},0)\varepsilon_{c\}}(\vec{p},-1)$$

$$+\frac{\sqrt{C_{n+h}^2C_{n-h}^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_{\{b}(\vec{p},1)\varepsilon_{c\}}(\vec{p},-1)+\frac{\sqrt{C_{n+h}^2C_{n-h}^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_{\{b}(\vec{p},0)\varepsilon_{c\}}(\vec{p},0)$$

$$=\frac{\sqrt{C_{n+h}^4C_{n-h}^0}}{\sqrt{2^2C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-2)\varepsilon_{\{b}(\vec{p},1)\varepsilon_{c\}}(\vec{p},1)+\frac{\sqrt{C_{n+h}^0C_{n-h}^4}}{\sqrt{2^2C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+2)\varepsilon_{\{b}(\vec{p},-1)\varepsilon_{c\}}(\vec{p},-1)$$

$$+\frac{\sqrt{C_{n+h}^3C_{n-h}^1}}{\sqrt{2^1C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-1)\varepsilon_{\{b}(\vec{p},1)\varepsilon_{c\}}(\vec{p},0)+\frac{\sqrt{C_{n+h}^1C_{n-h}^3}}{\sqrt{2^1C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+1)\varepsilon_{\{b}(\vec{p},0)\varepsilon_{c\}}(\vec{p},-1)$$

$$+\frac{\sqrt{C_{n+h}^2C_{n-h}^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_{\{b}(\vec{p},1)\varepsilon_{c\}}(\vec{p},-1)+\frac{\sqrt{C_{n+h}^2C_{n-h}^2}}{\sqrt{C_{2n}^4C_4^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_{\{b}(\vec{p},0)\varepsilon_{c\}}(\vec{p},0)$$

$$=\frac{\sqrt{C_{n+h}^4C_{n-h}^0}}{\sqrt{C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-2)\varepsilon_{bc}(\vec{p},2)+\frac{\sqrt{C_{n+h}^0C_{n-h}^4}}{\sqrt{C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+2)\varepsilon_{bc}(\vec{p},-2)$$

$$+\frac{\sqrt{C_{n+h}^3C_{n-h}^1}}{\sqrt{C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-1)\varepsilon_{bc}(\vec{p},1)+\frac{\sqrt{C_{n+h}^1C_{n-h}^3}}{\sqrt{C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h+1)\varepsilon_{bc}(\vec{p},-1)$$

$$+\frac{\sqrt{C_{n+h}^2C_{n-h}^2}}{\sqrt{C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h)\varepsilon_{bc}(\vec{p},0)$$

$$=\sum_{h'=2}^{-2}\frac{\sqrt{C_{n+h}^{2+h'}C_{n-h}^{2-h'}}}{\sqrt{C_{2n}^4}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-h')\varepsilon_{bc}(\vec{p},h') \qquad \square$$

Cor. 2.13.1. $\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\varepsilon_{\underbrace{a\cdot\cdot cb}_{n}}(\vec{p},h), n\geq 2, -n\leq h\leq n$

2.14 Corollary-Spin basis decomposition: $n+n'=n\oplus n'$

Thm. 2.14.1. $\varepsilon_{\underbrace{a\cdot\cdot}_{n}\underbrace{b\cdot\cdot c}_{n'}}(\vec{p},h)=\sum_{h'=n'}^{-n'}\frac{\sqrt{C_{n+n'+h}^{n'+h'}C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^{2n'}}}\varepsilon_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdot\cdot c}_{n'}}(\vec{p},h')$

Proof: For n' using mathematical induction to prove this theorem.
Step 1: When $n''=1$, the following is established.

$$\varepsilon_{\underbrace{a\cdot\cdot}_{n}\underbrace{b\cdot\cdot}_{1}}(\vec{p},h)=\sum_{h'=1}^{-1}\frac{\sqrt{C_{n+1+h}^{1+h'}C_{n+1-h}^{1-h'}}}{\sqrt{C_{2n+2}^2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdot\cdot}_{1}}(\vec{p},h'), -n-1\leq h\leq n+1$$

Step 2: Assume when $n''=n'-1$, the following is established.

$$\varepsilon_{\underbrace{a\cdot\cdot}_{n}\underbrace{b\cdot\cdot}_{n'-1}}(\vec{p},h)=\sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C_{n+n'-1+h}^{n'-1+h'}C_{n+n'-1-h}^{n'-1-h'}}}{\sqrt{C_{2n+2n'-2}^{2n'-2}}}\varepsilon_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdot\cdot}_{n'-1}}(\vec{p},h'), -n-n'+1\leq h\leq n+n'+1$$

Step 3: When $n''=n'$, $-n-n'\leq h\leq n+n'$, $\varepsilon_{\underbrace{a\cdot\cdot}_{n}\underbrace{b\cdot\cdot c}_{n'}}(\vec{p},h)$

$$= \frac{\sqrt{C^2_{n+n'+h}}}{\sqrt{C^2_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}\underbrace{b\cdots}_{n'-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+n'+h}C^1_{n+n'-h}}}{\sqrt{C^2_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}\underbrace{b\cdots}_{n'-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^2_{n+n'-h}}}{\sqrt{C^2_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}\underbrace{b\cdots}_{n'-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)$$

$$= \frac{\sqrt{C^2_{n+n'+h}}}{\sqrt{C^2_{2n+2n'}}}\sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C^{n'-1+h'}_{n+n'-2+h}C^{n'-1-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-1-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},1)$$

$$+ \frac{\sqrt{C^1_{n+n'+h}C^1_{n+n'-h}}}{\sqrt{C^2_{2n+2n'}}}\sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C^{n'-1+h'}_{n+n'-1+h}C^{n'-1-h'}_{n+n'-1-h}}}{\sqrt{C^{2n'-2}_{2n+2n'-2}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},0)$$

$$+ \frac{\sqrt{C^2_{n+n'-h}}}{\sqrt{C^2_{2n+2n'}}}\sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C^{n'-1+h'}_{n+n'+h}C^{n'-1-h'}_{n+n'-2-h}}}{\sqrt{C^{2n'-2}_{2n+2n'-2}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h+1-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},-1)$$

$$= \sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C^{n'+1+h'}_{n+n'+h}C^{n'-1-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\frac{\sqrt{C^2_{n'+1+h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-1-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},1)$$

$$+ \sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\frac{\sqrt{C^1_{n'+h'}C^1_{n'-h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},0)$$

$$+ \sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C^{n'-1+h'}_{n+n'+h}C^{n'+1-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\frac{\sqrt{C^2_{n'+1-h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h+1-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},-1)$$

$$= \sum_{h'=n'}^{-n'+2}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\frac{\sqrt{C^2_{n'+h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h'-1)\varepsilon_c(\vec{p},1)$$

$$+ \sum_{h'=n'-1}^{-n'+1}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\frac{\sqrt{C^1_{n'+h'}C^1_{n'-h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},0)$$

$$+ \sum_{h'=n'-2}^{-n'}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\frac{\sqrt{C^2_{n'-h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h'+1)\varepsilon_c(\vec{p},-1)$$

$$= \sum_{h'=n'}^{-n'}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\frac{\sqrt{C^2_{n'+h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h'-1)\varepsilon_c(\vec{p},1)$$

$$+ \sum_{h'=n'}^{-n'}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\frac{\sqrt{C^1_{n'+h'}C^1_{n'-h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h')\varepsilon_c(\vec{p},0)$$

$$+ \sum_{h'=n'}^{-n'}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\frac{\sqrt{C^2_{n'-h'}}}{\sqrt{C^2_{2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdots}_{n'-1}}(\vec{p},h'+1)\varepsilon_c(\vec{p},-1)$$

$$= \sum_{h'=n'}^{-n'}\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdots c}_{n'}}(\vec{p},h')$$

This step proves that when $n''=n'$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Cor. 2.14.1. $-n_1-n_2\le h\le n_1+n_2$

$$\begin{cases}\varepsilon_{\underbrace{a\cdots}_{n_1}\underbrace{b\cdots}_{n_2}}(\vec{p},h)=\sum\limits_{h_1=n_1}^{-n_1}\sum\limits_{h_2=n_2}^{-n_2}\frac{\sqrt{C^{n_2+h_2}_{n_1+n_2+h}C^{n_2-h_2}_{n_1+n_2-h}}}{\sqrt{C^{2n_2}_{2n_1+2n_2}}}\varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p},h_1)\varepsilon_{\underbrace{b\cdots}_{n_2}}(\vec{p},h_2)\delta(h-h_1-h_2)\\ \varepsilon_{\underbrace{a\cdots}_{n_1}\underbrace{b\cdots}_{n_2}}(\vec{p},h)=\sum\limits_{h_1=n_1}^{-n_1}\sum\limits_{h_2=n_2}^{-n_2}\frac{\sqrt{C^{n_1+h_1}_{n_1+n_2+h}C^{n_1-h_1}_{n_1+n_2-h}}}{\sqrt{C^{2n_1}_{2n_1+2n_2}}}\varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p},h_1)\varepsilon_{\underbrace{b\cdots}_{n_2}}(\vec{p},h_2)\delta(h-h_1-h_2)\end{cases}$$

Cor. 2.14.2. $-n_1-n_2\le h\le n_1+n_2, \varepsilon_{\underbrace{a\cdots}_{n_1}\underbrace{b\cdots}_{n_2}}(\vec{p},h)$

$$=\sum_{h_1=n_1}^{-n_1}\sum_{h_2=n_2}^{-n_2}[\tfrac{(2n_1)!(2n_2)!}{(2n_1+2n_2)!}\tfrac{(n_1+h_1+n_2+h_2)!}{(n_1+h_1)!(n_2+h_2)!}\tfrac{(n_1-h_1+n_2-h_2)!}{(n_1-h_1)!(n_2-h_2)!}]^{1/2}\varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p},h_1)\varepsilon_{\underbrace{b\cdots}_{n_2}}(\vec{p},h_2)\delta(h-h_1-h_2)$$

2.15 Corollary-Spin basis reverse synthesis

Cor. 2.15.1. $\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')=\varepsilon_{\underbrace{a\cdots}_{n}\underbrace{b\cdots c}_{n'}}(\vec{p},h)\bar{\varepsilon}^{\overbrace{b\cdots c}^{n'}}(\vec{p},h')$

Cor. 2.15.2. $\frac{\sqrt{C^{n'+h'}_{n+n'+h}C^{n'-h'}_{n+n'-h}}}{\sqrt{C^{2n'}_{2n+2n'}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')=\bar{\varepsilon}^{\overbrace{b\cdots c}^{n'}}(\vec{p},h')\varepsilon_{\underbrace{b\cdots c}_{n'}\underbrace{a\cdots}_{n}}(\vec{p},h)$

2.16 Corollary-Spin basis decomposition: $n_1+n_2\cdot\cdot+n_n=n_1\oplus n_2\cdot\cdot\oplus n_n$

Cor. 2.16.1. $-\sum\limits_{i=1}^{n}n_i\leq h\leq\sum\limits_{i=1}^{n}n_i,\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}\underbrace{b\cdot\cdot}_{n_2}\cdot\cdot\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h)$

$$=\sum_{h_1=n_1}^{-n_1}\sum_{h_2=n_2}^{-n_2}\cdot\cdot\sum_{h_n=n_n}^{-n_n}[\frac{\prod\limits_{i=1}^{n}(2n_i)!}{[\sum\limits_{i=1}^{n}(2n_i)]!}\frac{[\sum\limits_{i=1}^{n}(n_i+h_i)]!}{\prod\limits_{i=1}^{n}(n_i+h_i)!}\frac{[\sum\limits_{i=1}^{n}(n_i-h_i)]!}{\prod\limits_{i=1}^{n}(n_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}}(\vec{p},h_1)\varepsilon_{\underbrace{b\cdot\cdot}_{n_2}}(\vec{p},h_2)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h_n)\delta(h-\sum_{i=1}^{n}h_i)$$

2.17 Corollary-$\varepsilon_{ab\cdot\cdot c}(\vec{p},h)$ full symmetry

Thm. 2.17.1. $\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h)=\frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdot\cdot c\}}_{n}}(\vec{p},h),-n\leq h\leq n$

Proof: Using mathematical induction to prove this theorem.
Step 1: $n'=1,2$
$\varepsilon_a(\vec{p},h)=\frac{1}{1!}\varepsilon_a(\vec{p},h),-1\leq h\leq 1;\varepsilon_{ab}(\vec{p},h)=\frac{1}{2!}\varepsilon_{\{ab\}}(\vec{p},h),-2\leq h\leq 2$
Step 2: Assume when $n'=n-1$, the following is established.
$\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)=\frac{1}{(n-1)!}\varepsilon_{\underbrace{\{a\cdot\cdot b\}}_{n-1}}(\vec{p},h),-n+1\leq h\leq n-1$
Step 3: $2\leq n'=n-n\leq h\leq n,\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)$

$$=\sum_{h'=1}^{-1}\frac{\sqrt{C_{n+h}^{1+h'}C_{n+h}^{1-h'}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-h')\varepsilon_c(\vec{p},h')=\sum_{h'=2}^{-2}\frac{\sqrt{C_{n+h}^{2+h'}C_{n-h}^{2-h'}}}{\sqrt{C_{2n}^{4}}}\varepsilon_{\underbrace{a\cdot\cdot}_{n-2}}(\vec{p},h-h')\varepsilon_{bc}(\vec{p},h')$$

$$\Rightarrow\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\frac{1}{(n-1)!}\varepsilon_{\underbrace{\{a\cdot\cdot b\}c}_{n}}(\vec{p},h),\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\varepsilon_{\underbrace{a\cdot\cdot cb}_{n}}(\vec{p},h),-n\leq h\leq n$$

$$\Leftrightarrow\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\frac{1}{n!}\varepsilon_{\underbrace{\{a\cdot\cdot bc\}}_{n}}(\vec{p},h),-n\leq h\leq n$$

This step proves that when $n'=n$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

2.18 Summary of $\varepsilon_{ab\cdot\cdot c}(\vec{p},h)$ properties

Thm. 2.18.1.

$$\begin{cases}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h)=\sum\limits_{h'=1}^{-1}\frac{\sqrt{C_{n+h}^{1+h'}C_{n+h}^{1-h'}}}{\sqrt{C_{2n}^{2}}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n-1}}(\vec{p},h-h')\varepsilon_c(\vec{p},h')\\ \bar{\varepsilon}_a(\vec{p},h)=(-1)^h\varepsilon_a(\vec{p},-h),p^a\varepsilon_a(\vec{p},h)=0,\bar{\varepsilon}^a(\vec{p},h)\varepsilon_a(\vec{p},h')=\delta_{hh'},-1\leq h',h\leq 1\end{cases}$$

$$\Rightarrow$$

$$\begin{cases}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h)=\frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdot\cdot c\}}_{n}}(\vec{p},h),\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h)=0\\ \bar{\varepsilon}^{\overbrace{ab\cdot\cdot c}^{n}}(\vec{p},h')\varepsilon_{\underbrace{ab\cdot\cdot c}_{n}}(\vec{p},h)=\delta_{hh'},-n\leq h',h\leq n\end{cases}$$

2.19 Operator expression of plane wave solutions for Klein-Gordon equation

Thm. 2.19.1. $(-\partial^c\partial_c+m^2)A_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n}}(x)=\frac{1}{n!}A_{\underbrace{\{ab\cdot\cdot\}}_{n}}(x)$

$$A_{\underbrace{ab\cdot\cdot}_{n}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2^nE}}\sum_{h=n}^{-n}\frac{\hat{J}_-^{n-h}(\vec{p},n;R,L)}{(n-h)!\sqrt{C_{2n}^{n-h}}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)[a(\vec{p},h)e^{ip\cdot x}+(-1)^nb^+(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$A_{\underbrace{ab\cdot\cdot}_{n}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2^nE}}\sum_{h=n}^{-n}\frac{\hat{J}_+^{n+h}(\vec{p},n;R,L)}{(n+h)!\sqrt{C_{2n}^{n+h}}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)[a(\vec{p},h)e^{ip\cdot x}+(-1)^nb^+(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

3 Reorganization and analysis of Rarita-Schwinger equation spin basis

3.1 Definition-Spin basis decomposition: $n+\frac{1}{2}=n\oplus\frac{1}{2}$

Def. 3.1.1. $-n-\frac{1}{2}\leq h\leq n+\frac{1}{2}$

$$\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

Cor. 3.1.1. $\bar{\varepsilon}^{\overbrace{ab\cdot\cdot}^{n}\tau_\varsigma}(\vec{p},h')\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=\delta_{hh'},-n-\frac{1}{2}\leq h\leq n+\frac{1}{2}$

Cor. 3.1.2. $p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=0,-n-\frac{1}{2}\leq h\leq n+\frac{1}{2}$

Cor. 3.1.3. $\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)=0,-\frac{5}{2}\le h\le\frac{5}{2}$

3.2 Corollary-$\varepsilon_{ab\cdot\cdot c\tau_\varsigma}(\vec{p},h)$ is a spin eigenstate

Thm. 3.2.1. $[\Omega(n;R)\otimes I_4+I_{4^n}\otimes\hat{J}(\vec{p},\frac{1}{2};\gamma_a)]\cdot\hat{p}\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n}\otimes\tau_\varsigma}(\vec{p},h)=h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n}\otimes\tau_\varsigma}(\vec{p},h),-n-\frac{1}{2}\le h\le n+\frac{1}{2}$

Ass. 3.2.1. $[\Omega(n;R)\otimes I_4+I_{4^n}\otimes\hat{J}(\vec{p},\frac{1}{2};\gamma_a)]^2\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n}\otimes\tau_\varsigma}(\vec{p},h)=(n+\frac{1}{2})(n+\frac{3}{2})\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b}_{n}\otimes\tau_\varsigma}(\vec{p},h)$

3.3 Rarita-SchwingerRaising and lowering operator of equation spin basis

Def. 3.3.1. $\hat{J}(\vec{p},n+\frac{1}{2};R,L,\gamma_a):=[\hat{J}(\vec{p},n;R,L)\otimes I_4+I_{4^n}\otimes\hat{J}(\vec{p},\frac{1}{2};\gamma_a)]$

Cor. 3.3.1. $[\hat{J}_i(\vec{p},n+\frac{1}{2};R,L,\gamma_a),\hat{J}_j(\vec{p},n+\frac{1}{2};R,L,\gamma_a)]=\varepsilon_{ij}{}^k\hat{J}_k(\vec{p},n+\frac{1}{2};R,L,\gamma_a)$

Thm. 3.3.1.

$$\begin{cases}\hat{J}_+(\vec{p},n+\frac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{a\otimes\cdot\cdot}_{n}\otimes\tau_\varsigma}(\vec{p},h)=\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h+1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot}_{n}\otimes\tau_\varsigma}(\vec{p},h+1),-n-\frac{1}{2}\le h\le n+\frac{1}{2}\\ \hat{J}_-(\vec{p},n+\frac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{a\otimes\cdot\cdot}_{n}\otimes\tau_\varsigma}(\vec{p},h)=\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h-1)}\varepsilon_{\underbrace{a\otimes\cdot\cdot}_{n}\otimes\tau_\varsigma}(\vec{p},h-1),-n-\frac{1}{2}\le h\le n+\frac{1}{2}\\ \hat{J}_z(\vec{p},n+\frac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{a\otimes\cdot\cdot}_{n}\otimes\tau_\varsigma}(\vec{p},h)=h\varepsilon_{\underbrace{a\otimes\cdot\cdot}_{n}\otimes\tau_\varsigma}(\vec{p},h-1),-n-\frac{1}{2}\le h\le n+\frac{1}{2}\end{cases}$$

Proof: $-n-\frac{1}{2}\le h\le n+\frac{1}{2}$

$$\begin{aligned}
&\hat{J}_+(\vec{p},n+\tfrac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\\
&=[\hat{J}_+(\vec{p},n;R,L)\otimes I_4+I_{4^n}\otimes\hat{J}_+(\vec{p},\tfrac{1}{2};\gamma_a)][\tfrac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\tfrac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]\\
&=[\tfrac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\hat{J}_+(\vec{p},n;R,L,\gamma_a)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\tfrac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\hat{J}_+(\vec{p},n;R,L,\gamma_a)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]\\
&+\tfrac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})\hat{J}_+(\vec{p},\tfrac{1}{2};R,L,\gamma_a)u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]\\
&=\frac{\sqrt{n+1/2+h}\sqrt{n(n+1)-(h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\tfrac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})\\
&+\frac{\sqrt{n+1/2-h}\sqrt{n(n+1)-(h+\frac{1}{2})(h+\frac{3}{2})}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{3}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})\\
&=\frac{\sqrt{n+1/2+(h+1)}\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h+1)}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})\\
&+\frac{\sqrt{n+1/2-(h+1)}\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h+1)}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{3}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})\\
&=\sqrt{(n+\tfrac{1}{2})(n+\tfrac{3}{2})-h(h+1)}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h+1)
\end{aligned}$$

□

Proof: $-n-\frac{1}{2}\le h\le n+\frac{1}{2}$

$$\begin{aligned}
&\hat{J}_-(\vec{p},n+\tfrac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},h)\\
&=[\hat{J}_-(\vec{p},n;R,L)\otimes I_4+I_{4^n}\otimes\hat{J}_-(\vec{p},\tfrac{1}{2};\gamma_a)][\tfrac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\tfrac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]\\
&=[\tfrac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\hat{J}_-(\vec{p},n;R,L,\gamma_a)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\tfrac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\hat{J}_-(\vec{p},n;R,L,\gamma_a)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]\\
&+\tfrac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+\tfrac{1}{2})\hat{J}_-(\vec{p},\tfrac{1}{2};R,L,\gamma_a)u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]\\
&=\frac{\sqrt{n+1/2+h}\sqrt{n(n+1)-(h-\frac{1}{2})(h-\frac{3}{2})}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{3}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})\\
&+\frac{\sqrt{n+1/2-h}\sqrt{n(n+1)-(h+\frac{1}{2})(h-\frac{1}{2})}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})+\tfrac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})
\end{aligned}$$

$$= \frac{\sqrt{n+1/2+(h-1)}\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h-1)}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{3}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+ \frac{\sqrt{n+1/2-(h-1)}\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h-1)}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$= \sqrt{(n+\tfrac{1}{2})(n+\tfrac{3}{2})-h(h-1)}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h-1) \qquad \square$$

Proof: $-n-\frac{1}{2} \le h \le n+\frac{1}{2}$

$$\hat{J}_z(\vec{p},n+\tfrac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h)$$

$$= [\hat{J}_z(\vec{p},n;R,L)\otimes I_4 + I_{4^n}\otimes \hat{J}_z(\vec{p},\tfrac{1}{2};\gamma_a)][\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$= [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\hat{J}_z(\vec{p},n;R,L)\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\hat{J}_z(\vec{p},n;R,L)\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{1}{2})\hat{J}_z(\vec{p},\tfrac{1}{2};\gamma_a)u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h+\tfrac{1}{2})\hat{J}_z(\vec{p},\tfrac{1}{2};\gamma_a)u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$= [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}(h-\tfrac{1}{2})\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}(h+\tfrac{1}{2})\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+ [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{1}{2})(\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h+\tfrac{1}{2})(-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$= h[\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$= h\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p},h) \qquad \square$$

Cor. 3.3.2. $\hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_+\hat{J}_-)$

$$\begin{cases} \hat{J}^2(\vec{p},n+\frac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{a\otimes b\otimes\cdots}_{n}\tau_\varsigma}(\vec{p},h) = (n+\frac{1}{2})(n+\frac{3}{2})\varepsilon_{\underbrace{a\otimes b\otimes\cdots}_{n}\tau_\varsigma}(\vec{p},h), -n-\frac{1}{2}\le h\le n+\frac{1}{2} \\ \hat{J}_z(\vec{p},n+\frac{1}{2};R,L,\gamma_a)\varepsilon_{\underbrace{a\otimes\cdots}_{n}\otimes\tau_\varsigma}(\vec{p},h) = h\varepsilon_{\underbrace{a\otimes\cdots}_{n}\otimes\tau_\varsigma}(\vec{p},h-1), -n-\frac{1}{2}\le h\le n+\frac{1}{2} \end{cases}$$

3.4 Corollary-$\varepsilon_{ab\cdots c\tau_\varsigma}(\vec{p},h)$ orthogonality

Thm. 3.4.1. $\bar{\varepsilon}^{\overbrace{a\cdots bc}^{n}\tau_\varsigma}(\vec{p},h')\varepsilon_{\underbrace{a\cdots bc}_{n}\tau_\varsigma}(\vec{p},h) = \delta_{hh'}, -n-\frac{1}{2}\le h\le n+\frac{1}{2}$

3.5 Corollary-$p^a\varepsilon_{ab\cdots c\tau_\varsigma}(\vec{p},h)$ nullity

Thm. 3.5.1. $p^a\varepsilon_{\underbrace{a\cdots bc}_{n}\tau_\varsigma}(\vec{p},h) = 0, \gamma^a\varepsilon_{\underbrace{a\cdots bc}_{n}[\tau_\varsigma]}(\vec{p},h) = 0, -n-\frac{1}{2}\le h\le n+\frac{1}{2}$

3.6 Corollary-$\varepsilon_{ab\cdots c\tau_\varsigma}(\vec{p},h)$ tracelessness

Thm. 3.6.1. $\delta^{ab}\varepsilon_{\underbrace{ab\cdots c}_{n}\tau_\varsigma}(\vec{p},h) = 0, n\ge 2, -n-\frac{1}{2}\le h\le n+\frac{1}{2}$

3.7 Corollary-$\varepsilon_{ab\cdots c\tau_\varsigma}(\vec{p},h)$ full symmetry

Thm. 3.7.1. $\varepsilon_{\underbrace{ab\cdots c}_{n}\tau_\varsigma}(\vec{p},h) = \frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdots c\}}_{n}\tau_\varsigma}(\vec{p},h), -n-\frac{1}{2}\le h\le n+\frac{1}{2}$

3.8 Corollary-Spin basis decomposition: $n+n'+\frac{1}{2} = n\oplus n'+\frac{1}{2}$

Thm. 3.8.1. $\varepsilon_{\underbrace{a\cdots}_{n}\underbrace{b\cdots}_{n'}\tau_\varsigma}(\vec{p},h) = \sum_{h'=n'+1/2}^{-n'-1/2}\frac{\sqrt{C_{n+n'+1/2+h}^{n'+1/2+h'}C_{n+n'+1/2-h}^{n'+1/2-h'}}}{\sqrt{C_{2n+2n'+1}^{2n'+1}}}\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p},h-h')\varepsilon_{\underbrace{b\cdots}_{n'}\tau_\varsigma}(\vec{p},h')$

Cor. 3.8.1. $-n_1-n_2-\frac{1}{2}\le h\le n_1+n_2+\frac{1}{2}$

$$\begin{cases} \varepsilon_{\underbrace{a\cdots}_{n_1}\underbrace{b\cdots}_{n_2}\tau_\varsigma}(\vec{p},h) = \sum_{h_1=n_1}^{-n_1}\sum_{h_2=n_2+1/2}^{-n_2-1/2}\frac{\sqrt{C_{n_1+n_2+1/2+h}^{n_2+1/2+h_2}C_{n_1+n_2+1/2-h}^{n_2+1/2-h_2}}}{\sqrt{C_{2n_1+2n_2+1}^{2n_2+1}}}\varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p},h_1)\varepsilon_{\underbrace{b\cdots}_{n_2}\tau_\varsigma}(\vec{p},h_2)\delta(h-h_1-h_2) \\ \varepsilon_{\underbrace{a\cdots}_{n_1}\underbrace{b\cdots}_{n_2}\tau_\varsigma}(\vec{p},h) = \sum_{h_1=n_1}^{-n_1}\sum_{h_2=n_2+1/2}^{-n_2-1/2}\frac{\sqrt{C_{n_1+n_2+1/2+h}^{n_1+h_1}C_{n_1+n_2+1/2-h}^{n_1-h_1}}}{\sqrt{C_{2n_1+2n_2+1}^{2n_1}}}\varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p},h_1)\varepsilon_{\underbrace{b\cdots}_{n_2}\tau_\varsigma}(\vec{p},h_2)\delta(h-h_1-h_2) \end{cases}$$

Cor. 3.8.2. $-n_1-n_2-\frac{1}{2}\le h\le n_1+n_2+\frac{1}{2},\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}\underbrace{b\cdot\cdot}_{n_2}\tau_\varsigma}(\vec{p},h)$

$$=\sum_{h_1=n_1}^{-n_1}\sum_{h_2=n_2+1/2}^{-n_2-1/2}[\frac{(2n_1)!(2n_2+1)!}{(2n_1+2n_2+1)!}\frac{(n_1+h_1+n_2+1/2+h_2)!}{(n_1+h_1)!(n_2+1/2+h_2)!}\frac{(n_1-h_1+n_2+1/2-h_2)!}{(n_1-h_1)!(n_2+1/2-h_2)!}]^{1/2}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}}(\vec{p},h_1)\varepsilon_{\underbrace{b\cdot\cdot}_{n_2}\tau_\varsigma}(\vec{p},h_2)\delta(h-h_1-h_2)$$

3.9 Corollary-Spin basis reverse synthesis

Cor. 3.9.1. $\varepsilon_{\underbrace{a\cdot\cdot}_{n}}(\vec{p},h-h')=\frac{\sqrt{C_{2n+2n'+1}^{2n'+1}}}{\sqrt{C_{n+n'+1/2+h}^{n'+1/2+h'}C_{n+n'+1/2-h}^{n'+1/2-h'}}}\varepsilon_{\underbrace{a\cdot\cdot}_{n}\underbrace{b\cdot\cdot}_{n'}\tau_\varsigma}(\vec{p},h)\bar{\varepsilon}^{\overbrace{b\cdot\cdot}^{n'}\tau_\varsigma}(\vec{p},h')$

3.10 Corollary-Spin basis decomposition: $n_1+n_2\cdot\cdot+n_n+\frac{1}{2}=n_1\oplus n_2\cdot\cdot\oplus n_n\oplus\frac{1}{2}$

Cor. 3.10.1. $-n-\frac{1}{2}\le h\le n+\frac{1}{2},n_0=\frac{1}{2};\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}\underbrace{b\cdot\cdot}_{n_2}\cdot\cdot\underbrace{c\cdot\cdot}_{n_n}\tau_\varsigma}(\vec{p},h)$

$$=\sum_{h_0=n_0}^{-n_0}\sum_{h_1=n_1}^{-n_1}\cdot\cdot\sum_{h_n=n_n}^{-n_n}[\frac{\prod_{i=0}^{n}(2n_i)!}{[\sum_{i=0}^{n}(2n_i)]!}\frac{[\sum_{i=0}^{n}(n_i+h_i)]!}{\prod_{i=0}^{n}(n_i+h_i)!}\frac{[\sum_{i=0}^{n}(n_i-h_i)]!}{\prod_{i=0}^{n}(n_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}}(\vec{p},h_1)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h_n)\delta(h-\sum_{i=0}^{n}h_i)u_{\tau_\varsigma}(\vec{p},h_0)$$

Proof: $-n-\frac{1}{2}\le h\le n+\frac{1}{2},n_0=\frac{1}{2}$

$$\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}\underbrace{b\cdot\cdot}_{n_2}\cdot\cdot\underbrace{c\cdot\cdot}_{n_n}\tau_\varsigma}(\vec{p},h)=\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}\underbrace{b\cdot\cdot}_{n_2}\cdot\cdot\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h-\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}\underbrace{b\cdot\cdot}_{n_2}\cdot\cdot\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h+\tfrac{1}{2})u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}}\sum_{h_1=n_1}^{-n_1}\cdot\cdot\sum_{h_n=n_n}^{-n_n}[\frac{\prod_{i=1}^{n}(2n_i)!}{[\sum_{i=1}^{n}(2n_i)]!}\frac{[\sum_{i=1}^{n}(n_i+h_i)]!}{\prod_{i=1}^{n}(n_i+h_i)!}\frac{[\sum_{i=1}^{n}(n_i-h_i)]!}{\prod_{i=1}^{n}(n_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}}(\vec{p},h_1)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h_n)\delta(h-\tfrac{1}{2}-\sum_{i=1}^{n}h_i)u_{\tau_\varsigma}(\vec{p},\tfrac{1}{2})$$

$$+\frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}}\sum_{h_1=n_1}^{-n_1}\cdot\cdot\sum_{h_n=n_n}^{-n_n}[\frac{\prod_{i=1}^{n}(2n_i)!}{[\sum_{i=1}^{n}(2n_i)]!}\frac{[\sum_{i=1}^{n}(n_i+h_i)]!}{\prod_{i=1}^{n}(n_i+h_i)!}\frac{[\sum_{i=1}^{n}(n_i-h_i)]!}{\prod_{i=1}^{n}(n_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}}(\vec{p},h_1)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h_n)\delta(h+\tfrac{1}{2}-\sum_{i=1}^{n}h_i)u_{\tau_\varsigma}(\vec{p},-\tfrac{1}{2})$$

$$=\sum_{h_0=n_0}^{-n_0}\sum_{h_1=n_1}^{-n_1}\cdot\cdot\sum_{h_n=n_n}^{-n_n}[\frac{\prod_{i=0}^{n}(2n_i)!}{[\sum_{i=0}^{n}(2n_i)]!}\frac{[\sum_{i=0}^{n}(n_i+h_i)]!}{\prod_{i=0}^{n}(n_i+h_i)!}\frac{[\sum_{i=0}^{n}(n_i-h_i)]!}{\prod_{i=0}^{n}(n_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{n_1}}(\vec{p},h_1)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{n_n}}(\vec{p},h_n)\delta(h-\sum_{i=0}^{n}h_i)u_{\tau_\varsigma}(\vec{p},h_0)\qquad\square$$

3.11 Operator expression of plane wave solutions for Rarita-Schwinger equation

Thm. 3.11.1. $s:=n+\frac{1}{2}$

$(\gamma^c\partial_c+m)A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0,\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0,\gamma^aA_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x)=\frac{1}{n!}A_{\underbrace{\{ab\cdot\cdot\}\tau_\varsigma}_{n}}(x)$

$$A_{\underbrace{ab\cdot\cdot}_{n}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{\sqrt{m}}{\sqrt{2^nE}}\sum_{h=s}^{-s}\frac{\hat{J}_-^{s-h}(\vec{p},s;R,L,\gamma_a)}{(s-h)!\sqrt{C_{2s}^{s-h}}}[a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},s)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},s)e^{-ip\cdot x}]d^3\vec{p}$$

$$A_{\underbrace{ab\cdot\cdot}_{n}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{\sqrt{m}}{\sqrt{2^nE}}\sum_{h=s}^{-s}\frac{\hat{J}_+^{s+h}(\vec{p},s;R,L,\gamma_a)}{(s+h)!\sqrt{C_{2s}^{s+h}}}[a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},-s)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p},-s)e^{-ip\cdot x}]d^3\vec{p}$$

4 PenroseReorganization and analysis of equation spin basis

4.1 Neutrino spin bases are common eigenstate of spin and helicity

Pro. 4.1.1. $\begin{cases}\sigma^2(\frac{1}{2})\lambda(\hat{p},\frac{\varsigma}{2})=\frac{1}{2}(\frac{1}{2}+1)\lambda(\hat{p},\frac{\varsigma}{2})\\ \sigma(\frac{1}{2})\cdot\hat{p}\lambda(\hat{p},\frac{\varsigma}{2})=\frac{\varsigma}{2}\lambda(\hat{p},\frac{\varsigma}{2})\end{cases}$

4.2 Definition-Spin basis decomposition: $s=(s-\frac{1}{2})\oplus\frac{1}{2}$

Def. 4.2.1. $\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p},h-\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p},h+\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{1}{2}),-s\le h\le s$

Cor. 4.2.1. $\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^{1}}}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p},h-h')\lambda_{C_\varsigma}(\hat{p},h'),-s\le h\le s$

4.3 Corollary-$\lambda_{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}(\hat{p},h)$ is a spin eigenstate

Thm. 4.3.1. $[\Omega(s)\cdot\hat{p}]\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\hat{p},h)=h\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\hat{p},h),-s\le h\le s$

Proof: $[\Omega(s)\cdot\hat{p}]\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\hat{p},h)$

$=[\Omega(s-\frac{1}{2})\otimes I+I_{2^{2s-1}}\otimes\sigma(\frac{1}{2})]\cdot\hat{p}$

$[\frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\hat{p},h-\frac{1}{2})\lambda_{\otimes C_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\hat{p},h+\frac{1}{2})\lambda_{\otimes C_\varsigma}(\hat{p},-\frac{1}{2})],-s\le h\le s$

$=[\frac{\sqrt{s+h}}{\sqrt{2s}}h\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\hat{p},h-\frac{1}{2})\lambda_{\otimes C_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}h\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\hat{p},h+\frac{1}{2})\lambda_{\otimes C_\varsigma}(\hat{p},-\frac{1}{2})],-s\le h\le s$

$=h\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\hat{p},h),-s\le h\le s$ □

Thm. 4.3.2. $\Omega^2(s)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\hat{p},h)=s(s+1)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\hat{p},h),-s\le h\le s$

The above theorem can be easily proved using a fully symmetric representation transformation method. From the above, it can be seen that $\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)$ is a spin eigenstate. Therefore, the expansion coefficients are CG coefficients, and the actual calculation results also indicate that they are indeed the corresponding CG coefficients. This also provides a unified, standardized, intuitive and complete new method for calculating CG coefficients.

4.4 Raising and lowering operator of Penrose equation spin basis

Thm. 4.4.1.

$$\begin{cases}e^{i\vec{\omega}\cdot\Omega(s)}\Omega_x(s)e^{-i\vec{\omega}\cdot\Omega(s)}=\Omega_x(s)-\hat{p}_x\frac{\Omega(s)\cdot\hat{p}+\Omega_z(s)}{(1+\hat{p}_z)}\\ e^{i\vec{\omega}\cdot\Omega(s)}\Omega_y(s)e^{-i\vec{\omega}\cdot\Omega(s)}=\Omega_y(s)-\hat{p}_y\frac{\Omega(s)\cdot\hat{p}+\Omega_z(s)}{(1+\hat{p}_z)}\\ e^{i\vec{\omega}\cdot\Omega(s)}\Omega_z(s)e^{-i\vec{\omega}\cdot\Omega(s)}=\Omega(s)\cdot\hat{p}\end{cases}$$

Def. 4.4.1.

$$\begin{cases}\hat{J}_x(\hat{p},\Omega(s)):=\{\Omega_x(s)-\frac{\hat{p}_x}{(1+\hat{p}_z)}[\Omega(s)\cdot\hat{p}+\Omega_z(s)]\}\\ \hat{J}_y(\hat{p},\Omega(s)):=\{\Omega_y(s)-\frac{\hat{p}_y}{(1+\hat{p}_z)}[\Omega(s)\cdot\hat{p}+\Omega_z(s)]\}\\ \hat{J}_z(\hat{p},\Omega(s)):=\Omega(s)\cdot\hat{p}\end{cases}$$

Cor. 4.4.1.

$$\begin{cases}\hat{J}_x^2(\hat{p},\Omega(\frac{1}{2}))=\frac{1}{4},\hat{J}_y^2(\hat{p},\Omega(\frac{1}{2}))=\frac{1}{4},\hat{J}_z^2(\hat{p},\Omega(\frac{1}{2}))=\frac{1}{4}\\ [\hat{J}_i(\hat{p},\Omega(s)),\hat{J}_j(\hat{p},\Omega(s))]=\varepsilon_{ij}{}^k\hat{J}_k(\hat{p},\Omega(s))\end{cases}$$

Cor. 4.4.2.

$$\begin{cases}\hat{J}_+(\hat{p},\Omega(s)):=\{[\Omega_x(s)+i\Omega_y(s)]-\frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}[\Omega(s)\cdot\hat{p}+\Omega_z(s)]\}\\ \hat{J}_-(\hat{p},\Omega(s)):=\{[\Omega_x(s)-i\Omega_y(s)]-\frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}[\Omega(s)\cdot\hat{p}+\Omega_z(s)]\}\\ \hat{J}_z(\hat{p},\Omega(s)):=\Omega(s)\cdot\hat{p}\end{cases}$$

Cor. 4.4.3. $\hat{J}(\hat{p},\Omega(s)):=\underbrace{\hat{J}(\vec{p},\sigma(\frac{1}{2}))\otimes I_4\otimes\cdot\cdot\otimes I_4}_{2s}+\underbrace{I_4\otimes\hat{J}(\vec{p},\sigma(\frac{1}{2}))\otimes\cdot\cdot\otimes I_4}_{2s}+\cdots+\underbrace{I_4\otimes\cdot\cdot\otimes I_4\otimes\hat{J}(\vec{p},\sigma(\frac{1}{2}))}_{2s}$

Thm. 4.4.2. $\hat{J}_+(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h)=\sqrt{s(s+1)-h(h+1)}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h+1)$

Proof: Using mathematical induction to prove this theorem.

Step 1: When $s'=\frac{1}{2}$, the following is established.

$\hat{J}_+(\vec{p},\sigma(\frac{1}{2}))\lambda_{\otimes C_\varsigma}(\vec{p},h)=\sqrt{\frac{3}{4}-h(h+1)}\lambda_{\otimes C_\varsigma}(\vec{p},h+1),-\frac{1}{2}\le h\le\frac{1}{2}$

Step 2: Assume when $s'=s-\frac{1}{2}$, the following is established.

$\hat{J}_+(\vec{p},s-\frac{1}{2};\sigma)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h)=\sqrt{(s-\frac{1}{2})(s+\frac{1}{2})-h(h+1)}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h),-s+\frac{1}{2}\le h\le s-\frac{1}{2}$

Step 3: When $s'=s,\ -s\le h\le s,\hat{J}_+(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h)$

$=\frac{\sqrt{s+h}}{\sqrt{2s}}[\hat{J}_+(\vec{p},s-\frac{1}{2};\sigma)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})]\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}[\hat{J}_+(\vec{p},s-\frac{1}{2};\sigma)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})]\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$+\frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})\hat{J}_+(\vec{p},\sigma(\frac{1}{2}))\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})\hat{J}_+(\vec{p},\sigma(\frac{1}{2}))\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$=\frac{\sqrt{s+h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2})$

$+\frac{\sqrt{s-h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h+1-\frac{1}{2})(h+1+\frac{1}{2})}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{3}{2})]\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$= \frac{\sqrt{(s-h)(s+h+1)}\sqrt{s+h+1}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{(s-h)(s+h+1)}\sqrt{s-h-1}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{3}{2})\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$= \sqrt{s(s+1)-h(h+1)}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h+1)$

This step proves that when $s'=s$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Thm. 4.4.3. $\hat{J}_-(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h) = \sqrt{s(s+1)-h(h-1)}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h-1)$

Proof: Using mathematical induction to prove this theorem.

Step 1: When $s'=\frac{1}{2}$, the following is established.

$\hat{J}_-(\vec{p},\sigma(\frac{1}{2}))\lambda_{\otimes C_\varsigma}(\vec{p},h) = \sqrt{\frac{3}{4}-h(h-1)}\lambda_{\otimes C_\varsigma}(\vec{p},h-1), -\frac{1}{2}\le h\le\frac{1}{2}$

Step 2: Assume when $s'=s-\frac{1}{2}$, the following is established.

$\hat{J}_-(\vec{p},s-\frac{1}{2};\sigma)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h) = \sqrt{(s-\frac{1}{2})(s+\frac{1}{2})-h(h-1)}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-1), -s+\frac{1}{2}\le h\le s-\frac{1}{2}$

Step 3: When $s'=s,\ -s\le h\le s, \hat{J}_-(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h)$

$= \frac{\sqrt{s+h}}{\sqrt{2s}}[\hat{J}_-(\vec{p},s-\frac{1}{2};\sigma)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})]\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}}[\hat{J}_-(\vec{p},s-\frac{1}{2};\sigma)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})]\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$+ \frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})\hat{J}_-(\vec{p},\sigma(\frac{1}{2}))\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}}\hat{J}_-(\vec{p},s-\frac{1}{2};\sigma)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})\hat{J}_-(\vec{p},\sigma(\frac{1}{2}))\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$= \frac{\sqrt{s+h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h-1-\frac{1}{2})(h-1+\frac{1}{2})}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{3}{2})\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2})$

$+ \frac{\sqrt{s-h}\sqrt{(s+\frac{1}{2})(s-\frac{1}{2})-(h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})]\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$= \frac{\sqrt{s+h}\sqrt{(s+h-1)(s-h+1)}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{3}{2})\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2})$

$+ \frac{\sqrt{s-h}\sqrt{(s+h)(s-h)}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})]\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$= \frac{\sqrt{(s+h)(s-h+1)}\sqrt{(s+h-1)}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{3}{2})\lambda_{\otimes C_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{(s+h)(s-h+1)}\sqrt{s-h+1}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})\lambda_{\otimes C_\varsigma}(\vec{p},-\frac{1}{2})$

$= \sqrt{s(s+1)-h(h-1)}\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\vec{p},h-1)$

This step proves that when $s'=s$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Cor. 4.4.4.

$$\begin{cases} \hat{J}_+(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) = \sqrt{s(s+1)-h(h+1)}\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h+1), -s\le h\le s \\ \hat{J}_-(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) = \sqrt{s(s+1)-h(h-1)}\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h-1), -s\le h\le s \\ \hat{J}_z(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) = h\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h), -s\le h\le s \end{cases}$$

Cor. 4.4.5. $\hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_+\hat{J}_-)$

$$\begin{cases} \hat{J}^2(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) = s(s+1)\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h), \hat{J}_z(\hat{p},\Omega(s))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) = h\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) \\ \hat{J}^2(\hat{p},*\sigma(\frac{1}{2}))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h) = \frac{3}{4}\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2s}}(\vec{p},h), \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{p},h) = \frac{1}{(2s)!}\lambda_{\underbrace{\{A_\varsigma B_\varsigma\cdot\cdot\}}_{2s}}(\vec{p},h), -s\le h\le s \end{cases}$$

4.5 Corollary-$\lambda_{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}(\hat{p},h)$ orthogonality

Def. 4.5.1. $\lambda^{+A_\varsigma}(\hat{p},h')\lambda_{A_\varsigma}(\hat{p},h) = \delta_{hh'}, -\frac{1}{2}\le h,h'\le\frac{1}{2}$

Thm. 4.5.1. $\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}^{2s}}(\hat{p},h')\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h) = \delta_{hh'}, -s\le h',h\le s$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $s' = \frac{1}{2}$, the following is established.
$\lambda^{+A_\varsigma}(\hat{p}, h')\lambda_{A_\varsigma}(\hat{p}, h) = \delta_{hh'}, -\frac{1}{2} \le h, h' \le \frac{1}{2}$
Step 2: Assume when $s' = s - \frac{1}{2}$, the following is established.

$$\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma}^{2s-1}}(\hat{p}, h')\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p}, h) = \delta_{hh'}, -s + \frac{1}{2} \le h', h \le s - \frac{1}{2}$$

Step 3: When $s' = s$, $\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}^{2s}}(\hat{p}, h')\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p}, h), -s \le h', h \le s$

$$= [\sum_{\bar{h}'=1/2}^{-1/2}\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'}C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^{1}}}\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma}^{2s-1}}(\hat{p}, h' - \bar{h}')\lambda^{+C_\varsigma}(\hat{p}, \bar{h}')][\sum_{\bar{h}=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p}, h - \bar{h})\lambda_{C_\varsigma}(\hat{p}, \bar{h})]$$

$$= \sum_{\bar{h}',\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'}C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^{1}}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma}^{2s-1}}(\hat{p}, h' - \bar{h}')\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p}, h - \bar{h})\delta_{\bar{h}\bar{h}'}]$$

$$= \sum_{\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}}C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma}^{2s-1}}(\hat{p}, h' - \bar{h})\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p}, h - \bar{h})]$$

$$= \sum_{\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h'}^{1/2+\bar{h}}C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}\delta_{hh'}]$$

$$= \sum_{\bar{h}=1/2}^{-1/2}[\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}\frac{\sqrt{C_{s+h}^{1/2+\bar{h}}C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^{1}}}]\delta_{hh'}$$

$= \delta_{hh'}$
This step proves that when $s' = s$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

4.6 Corollary-Spin basis decomposition: $1 = \frac{1}{2} \oplus \frac{1}{2}$

Cor. 4.6.1. $\lambda_{A_\varsigma B_\varsigma}(\hat{p}, h) = \frac{\sqrt{1+h}}{\sqrt{2}}\lambda_{A_\varsigma}(\hat{p}, h - \frac{1}{2})\lambda_{B_\varsigma}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{1-h}}{\sqrt{2}}\lambda_{A_\varsigma}(\hat{p}, h + \frac{1}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{1}{2})$

$$= \begin{cases} \lambda_{A_\varsigma}(\hat{p}, \frac{1}{2})\lambda_{B_\varsigma}(\hat{p}, \frac{1}{2}), h = 1 \\ \frac{1}{\sqrt{2}}\lambda_{\{A_\varsigma}(\hat{p}, \frac{1}{2})\lambda_{B_\varsigma\}}(\hat{p}, -\frac{1}{2}), h = 0 \\ \lambda_{A_\varsigma}(\hat{p}, -\frac{1}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{1}{2}), h = -1 \end{cases}$$

Cor. 4.6.2. $\lambda_{A_\varsigma B_\varsigma}(\hat{p}, h) = \lambda_{B_\varsigma A_\varsigma}(\hat{p}, h), -1 \le h \le 1$

Pro. 4.6.1. $\lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{\varsigma}{2}) = \frac{i}{2}(\sigma, i\varsigma)^a\hat{p}_a\sigma_y$

Cor. 4.6.3. $[(\sigma \otimes I) \cdot (I \otimes \sigma)][\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})] = [\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})]$

Proof: $\sigma \cdot [\lambda(\hat{p}, \frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{\varsigma}{2})]\sigma^T$
$= \frac{i}{2}\sigma \cdot [(\sigma, -i\varsigma)^a\hat{p}_a\sigma_y + (\sigma, i\varsigma)^a\hat{p}_a\sigma_y]\sigma^T$
$= \sigma \cdot [i(\sigma \cdot \hat{p})\sigma_y]\sigma^T$
$= [\sigma_x i(\sigma \cdot \hat{p})\sigma_y\sigma_x^T + \sigma_y i(\sigma \cdot \hat{p})\sigma_y\sigma_y^T + \sigma_z i(\sigma \cdot \hat{p})\sigma_y\sigma_z^T]$
$= [\sigma_x i(\sigma \cdot \hat{p})\sigma_y\sigma_x^T + \sigma_y i(\sigma \cdot \hat{p})\sigma_y\sigma_y^T + \sigma_z i(\sigma \cdot \hat{p})\sigma_y\sigma_z^T]$
$= i(\sigma \cdot \hat{p})\sigma_y = [\lambda(\hat{p}, \frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{\varsigma}{2})]$ □

Cor. 4.6.4.

$$\begin{cases} [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2[\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})] = 2[\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})] \\ [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})] \cdot \hat{p}[\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})] = 0[\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) + \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})] \end{cases}$$

4.7 Corollary-Spin basis decomposition: $0 = \frac{1}{2} \ominus \frac{1}{2}$

Cor. 4.7.1. $F_{A_\varsigma B_\varsigma}(\hat{p}, h) = \frac{1}{\sqrt{2}}\lambda_{[A_\varsigma}(\hat{p}, \frac{1}{2})\lambda_{B_\varsigma]}(\hat{p}, -\frac{1}{2}), h = 0$

Cor. 4.7.2. $[(\sigma \otimes I) \cdot (I \otimes \sigma)][\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) - \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})] = -3[\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{-\varsigma}{2}) - \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, \frac{\varsigma}{2})]$

Proof: $\sigma \cdot [\lambda(\hat{p}, \frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{-\varsigma}{2}) - \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{\varsigma}{2})]\sigma^T$
$= \frac{i}{2}\sigma \cdot [(\sigma, -i\varsigma)^a\hat{p}_a\sigma_y - (\sigma, i\varsigma)^a\hat{p}_a\sigma_y]\sigma^T$
$= \sigma \cdot (i\varsigma\sigma_y)\sigma^T$
$= \sigma_x(i\varsigma\sigma_y)\sigma_x^T + \sigma_y(i\varsigma\sigma_y)\sigma_y^T + \sigma_z(i\varsigma\sigma_y)\sigma_z^T$
$= -3(i\varsigma\sigma_y) = -3[\lambda(\hat{p}, \frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{-\varsigma}{2}) - \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{\varsigma}{2})]$ □

Cor. 4.7.3.
$$\begin{cases}[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]^2[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2})-\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})]=0[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2})-\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})]\\ [\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]\cdot\hat{p}[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2})-\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})]=0[\lambda(\hat{p},\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{-\varsigma}{2})-\lambda(\hat{p},-\frac{\varsigma}{2})\otimes\lambda(\hat{p},\frac{\varsigma}{2})]\end{cases}$$

Cor. 4.7.4. $u(\hat{p},\frac{\kappa}{2})=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix},v(\hat{p},\frac{\kappa}{2})=\frac{\lambda(\hat{p},\frac{\kappa}{2})}{\sqrt{2m(E+\kappa\varsigma|\vec{p}|)}}\otimes\begin{bmatrix}-m\\ \varsigma E+\kappa|\vec{p}|\end{bmatrix}$

4.8 Corollary-Spin basis decomposition: $s=(s-1)\oplus 1$

Thm. 4.8.1. $\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\sum_{h'=1}^{-1}\frac{\sqrt{C_{s+h}^{1+h'}C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2(s-1)}}(\hat{p},h-h')\lambda_{B_\varsigma C_\varsigma}(\hat{p},h'),s\geq 1,-s\leq h\leq s$

Proof: $\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)$

$=\frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p},h-\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p},h+\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{1}{2})$

$=\frac{\sqrt{s+h}}{\sqrt{2s}}[\frac{\sqrt{s+h-1}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h-1)\lambda_{B_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-1}}(\hat{p},h)\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2})]\lambda_{C_\varsigma}(\hat{p},\frac{1}{2})$

$+\frac{\sqrt{s-h}}{\sqrt{2s}}[\frac{\sqrt{s+h}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h)\lambda_{B_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s-h-1}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-1}}(\hat{p},h+1)\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2})]\lambda_{C_\varsigma}(\hat{p},-\frac{1}{2})$

$=[\frac{\sqrt{s+h}}{\sqrt{2s}}\frac{\sqrt{s+h-1}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h-1)\lambda_{B_\varsigma}(\hat{p},\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s+h}}{\sqrt{2s}}\frac{\sqrt{s-h}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-1}}(\hat{p},h)\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},\frac{1}{2})]$

$+[\frac{\sqrt{s-h}}{\sqrt{2s}}\frac{\sqrt{s+h}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h)\lambda_{B_\varsigma}(\hat{p},\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\frac{\sqrt{s-h-1}}{\sqrt{2s-1}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-1}}(\hat{p},h+1)\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{1}{2})]$

$=\frac{\sqrt{C_{s+h}^2C_{s-h}^0}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h-1)\lambda_{B_\varsigma}(\hat{p},\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{C_{s+h}^1C_{s-h}^1}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h)\frac{1}{\sqrt{2}}\lambda_{\{B_\varsigma}(\hat{p},\frac{1}{2})\lambda_{C_\varsigma\}}(\hat{p},-\frac{1}{2})$

$+\frac{\sqrt{C_{s+h}^0C_{s-h}^2}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-1}}(\hat{p},h+1)\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{1}{2})$

$=\frac{\sqrt{C_{s+h}^2C_{s-h}^0}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h-1)\lambda_{B_\varsigma C_\varsigma}(\hat{p},1)+\frac{\sqrt{C_{s+h}^1C_{s-h}^1}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-2}}(\hat{p},h)\lambda_{B_\varsigma C_\varsigma}(\hat{p},0)$

$+\frac{\sqrt{C_{s+h}^0C_{s-h}^2}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s-1}}(\hat{p},h+1)\lambda_{B_\varsigma C_\varsigma}(\hat{p},-1)$

$=\sum_{h'=1}^{-1}\frac{\sqrt{C_{s+h}^{1+h'}C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2(s-1)}}(\hat{p},h-h')\lambda_{B_\varsigma C_\varsigma}(\hat{p},h')$ □

Cor. 4.8.1. $\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\lambda_{\underbrace{A_\varsigma\cdot\cdot C_\varsigma B_\varsigma}_{2s}}(\hat{p},h),s\geq 1,-s\leq h\leq s$

4.9 Corollary-Spin basis decomposition: $s+s'=s\oplus s'$

Thm. 4.9.1. $\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}\underbrace{B_\varsigma\cdot\cdot C_\varsigma}_{2s'}}(\hat{p},h)=\sum_{h'=s'}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}}(\hat{p},h-h')\lambda_{\underbrace{B_\varsigma\cdot\cdot C_\varsigma}_{2s'}}(\hat{p},h'),-s-s'\leq h\leq s+s'$

Proof: For s' using mathematical induction to prove this theorem.
Step 1: When $s''=\frac{1}{2}$, the following is established.

$\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}\underbrace{C_\varsigma}_{1}}(\hat{p},h)=\sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+1/2+h}^{1/2+h'}C_{s+1/2-h}^{1/2-h'}}}{\sqrt{C_{2(s+1/2)}^{1}}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}}(\hat{p},h-h')\lambda_{\underbrace{C_\varsigma}_{1}}(\hat{p},h'),-s-\frac{1}{2}\leq h\leq s+\frac{1}{2}$

Step 2: Assume when $s''=s'-\frac{1}{2}$, the following is established.

$\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}\underbrace{B_\varsigma\cdot\cdot}_{2s'-1}}(\hat{p},h)=\sum_{h'=s'-1/2}^{-s'+1/2}\frac{\sqrt{C_{s+s'-1/2+h}^{s'-1/2+h'}C_{s+s'-1/2-h}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2s'-1}}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}}(\hat{p},h-h')\lambda_{\underbrace{B_\varsigma\cdot\cdot}_{2s'-1}}(\hat{p},h')$

$-s-s'+\frac{1}{2}\leq h\leq s+s'-\frac{1}{2}$

Step 3: When $s''=s'$, $-s-s'\leq h\leq s+s'$, $\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}\underbrace{B_\varsigma\cdot\cdot C_\varsigma}_{2s'}}(\hat{p},h)$

$=\frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}\underbrace{B_\varsigma\cdot\cdot}_{2s'-1}}(\hat{p},h-\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s+s'-h}}{\sqrt{2(s+s')}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s}\underbrace{B_\varsigma\cdot\cdot}_{2s'-1}}(\hat{p},h+\frac{1}{2})\lambda_{C_\varsigma}(\hat{p},-\frac{1}{2})$

$$= \frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}}[\sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'-1+h}^{s'-1/2+h'} C_{s+s'-h}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2s'-1}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-\tfrac{1}{2}-h')\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h')]\lambda_{C_\varsigma}(\hat{p},\tfrac{1}{2})$$

$$+ [\sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'+h}^{s'-1/2+h'} C_{s+s'-1-h}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2s'-1}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h+\tfrac{1}{2}-h')\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h')]\lambda_{C_\varsigma}(\hat{p},-\tfrac{1}{2})$$

$$= [\sum_{h'=s'}^{-s'+1} \frac{\sqrt{C_{s+s'-1+h}^{s'-1+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-h')\frac{\sqrt{s+s'+h}}{\sqrt{2s'}}\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h'-\tfrac{1}{2})]\lambda_{C_\varsigma}(\hat{p},\tfrac{1}{2})$$

$$+ [\sum_{h'=s'-1}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-1-h}^{s'-1-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-h')\frac{\sqrt{s+s'-h}}{\sqrt{2s'}}\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h'+\tfrac{1}{2})]\lambda_{C_\varsigma}(\hat{p},-\tfrac{1}{2})$$

$$= [\sum_{h'=s'}^{-s'+1} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-h')\frac{\sqrt{s'+h'}}{\sqrt{2s'}}\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h'-\tfrac{1}{2})]\lambda_{C_\varsigma}(\hat{p},\tfrac{1}{2})$$

$$+ [\sum_{h'=s'-1}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-h')\frac{\sqrt{s'-h'}}{\sqrt{2s'}}\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h'+\tfrac{1}{2})]\lambda_{C_\varsigma}(\hat{p},-\tfrac{1}{2})$$

$$= [\sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-h')\frac{\sqrt{s'+h'}}{\sqrt{2s'}}\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h'-\tfrac{1}{2})]\lambda_{C_\varsigma}(\hat{p},\tfrac{1}{2})$$

$$+ [\sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-h')\frac{\sqrt{s'-h'}}{\sqrt{2s'}}\lambda_{\underbrace{B_\varsigma\cdots}_{2s'-1}}(\hat{p},h'+\tfrac{1}{2})]\lambda_{C_\varsigma}(\hat{p},-\tfrac{1}{2})$$

$$= \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p}, h-h')\lambda_{\underbrace{B_\varsigma\cdots C_\varsigma}_{2s'}}(\hat{p},h'), -s-s'\le h\le s+s'$$

This step proves that when $s''=s'$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

Cor. 4.9.1. $-s_1-s_2\le h\le s_1+s_2$

$$\begin{cases} \lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}\underbrace{B_\varsigma\cdots}_{2s_2}}(\hat{p},h)=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\frac{\sqrt{C_{s_1+s_2+h}^{s_2+h_2}C_{s_1+s_2-h}^{s_2-h_2}}}{\sqrt{C_{2(s_1+s_2)}^{2s_2}}}\lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}}(\hat{p},h_1)\lambda_{\underbrace{B_\varsigma\cdots}_{2s_2}}(\hat{p},h_2)\delta(h-h_1-h_2) \\ \lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}\underbrace{B_\varsigma\cdots}_{2s_2}}(\hat{p},h)=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\frac{\sqrt{C_{s_1+s_2+h}^{s_1+h_1}C_{s_1+s_2-h}^{s_1-h_1}}}{\sqrt{C_{2(s_1+s_2)}^{2s_1}}}\lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}}(\hat{p},h_1)\lambda_{\underbrace{B_\varsigma\cdots}_{2s_2}}(\hat{p},h_2)\delta(h-h_1-h_2) \end{cases}$$

Cor. 4.9.2. $-s_1-s_2\le h\le s_1+s_2, \lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}\underbrace{B_\varsigma\cdots}_{2s_2}}(\hat{p},h)$

$$= \sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}[\tfrac{(2s_1)!(2s_2)!}{(2s_1+2s_2)!}\tfrac{(s_1+h_1+s_2+h_2)!}{(s_1+h_1)!(s_2+h_2)!}\tfrac{(s_1-h_1+s_2-h_2)!}{(s_1-h_1)!(s_2-h_2)!}]^{1/2}\lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}}(\hat{p},h_1)\lambda_{\underbrace{B_\varsigma\cdots}_{2s_2}}(\hat{p},h_2)\delta(h-h_1-h_2)$$

4.10 Corollary-Spin basis reverse synthesis

Cor. 4.10.1. $\lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p},h-h')=\frac{\sqrt{C_{2(s+s')}^{2s'}}}{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}\lambda_{\underbrace{A_\varsigma\cdots}_{2s}\underbrace{B_\varsigma\cdots C_\varsigma}_{2s'}}(\hat{p},h)\lambda^{+\overbrace{B_\varsigma\cdots C_\varsigma}^{2s'}}(\hat{p},h'), -s-s'\le h\le s+s'$

Cor. 4.10.2. $\lambda_{\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p},h-h')=\frac{\sqrt{C_{2(s+s')}^{2s'}}}{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}\lambda^{+\overbrace{B_\varsigma\cdots C_\varsigma}^{2s'}}(\hat{p},h')\lambda_{\underbrace{B_\varsigma\cdots C_\varsigma}_{2s'}\underbrace{A_\varsigma\cdots}_{2s}}(\hat{p},h), -s-s'\le h\le s+s'$

4.11 Corollary-Spin basis decomposition: $s_1+s_2+s_3=s_1\oplus s_2\oplus s_3$

Cor. 4.11.1. $-s_1-s_2-s_3\le h\le s_1+s_2+s_3, \lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}\underbrace{B_\varsigma\cdots}_{2s_2}\underbrace{C_\varsigma\cdots}_{2s_3}}(\hat{p},h)$

$$= \sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\sum_{h_3=s_3}^{-s_3}[\tfrac{(2s_1)!(2s_2)!(2s_3)!}{(2s_1+2s_2+2s_3)!}\tfrac{(s_1+h_1+s_2+h_2+s_3+h_3)!}{(s_1+h_1)!(s_2+h_2)!(s_3+h_3)!}\tfrac{(s_1-h_1+s_2-h_2+s_3-h_3)!}{(s_1-h_1)!(s_2-h_2)!(s_3-h_3)!}]^{1/2}$$

$$\lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}}(\hat{p},h_1)\lambda_{\underbrace{B_\varsigma\cdots}_{2s_2}}(\hat{p},h_2)\lambda_{\underbrace{C_\varsigma\cdots}_{2s_3}}(\hat{p},h_3)\delta(h-h_1-h_2-h_3)$$

Proof: $-s_1-s_2-s_3\le h\le s_1+s_2+s_3, \lambda_{\underbrace{A_\varsigma\cdots}_{2s_1}\underbrace{B_\varsigma\cdots}_{2s_2}\underbrace{C_\varsigma\cdots}_{2s_3}}(\hat{p},h)$

$$= \sum_{h_1=s_1}^{-s_1}\sum_{h_{23}=s_2+s_3}^{-s_2-s_3}[\tfrac{(2s_1)!(2s_2+2s_3)!}{(2s_1+2s_2+2s_3)!}\tfrac{(s_1+h_1+s_2+s_3+h_{23})!}{(s_1+h_1)!(s_2+s_3+h_{23})!}\tfrac{(s_1-h_1+s_2+s_3-h_{23})!}{(s_1-h_1)!(s_2+s_3-h_{23})!}]^{1/2}$$

$\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}(\hat{p},h_1)\lambda_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}\underbrace{C_\varsigma\cdot\cdot}_{2s_3}}(\hat{p},h_{23})\delta(h-h_1-h_{23})$

$=\sum_{h_1=s_1}^{-s_1}\sum_{h_{23}=s_2+s_3}^{-s_2-s_3}[\frac{(2s_1)!(2s_2+2s_3)!}{(2s_1+2s_2+2s_3)!}\frac{(s_1+h_1+s_2+s_3+h_{23})!}{(s_1+h_1)!(s_2+s_3+h_{23})!}\frac{(s_1-h_1+s_2+s_3-h_{23})!}{(s_1-h_1)!(s_2+s_3-h_{23})!}]^{1/2}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}(\hat{p},h_1)\delta(h-h_1-h_{23})$

$\sum_{h_2=s_2}^{-s_2}\sum_{h_3=s_3}^{-s_3}[\frac{(2s_2)!(2s_3)!}{(2s_2+2s_3)!}\frac{(s_2+h_2+s_3+h_3)!}{(s_2+h_2)!(s_3+h_3)!}\frac{(s_2-h_2+s_3-h_3)!}{(s_2-h_2)!(s_3-h_3)!}]^{1/2}\lambda_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}}(\hat{p},h_2)\lambda_{\underbrace{C_\varsigma\cdot\cdot}_{2s_3}}(\hat{p},h_3)\delta(h_{23}-h_2-h_3)$

$=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\sum_{h_3=s_3}^{-s_3}[\frac{(2s_1)!(2s_2)!(2s_3)!}{(2s_1+2s_2+2s_3)!}\frac{(s_1+h_1+s_2+h_2+s_3+h_3)!}{(s_1+h_1)!(s_2+h_2)!(s_3+h_3)!}\frac{(s_1-h_1+s_2-h_2+s_3-h_3)!}{(s_1-h_1)!(s_2-h_2)!(s_3-h_3)!}]^{1/2}$

$\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}(\hat{p},h_1)\lambda_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}}(\hat{p},h_2)\lambda_{\underbrace{C_\varsigma\cdot\cdot}_{2s_3}}(\hat{p},h_3)\delta(h-h_1-h_2-h_3)$ □

4.12 Corollary-Spin basis decomposition: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Cor. 4.12.1. $-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i,\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}\underbrace{B_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}(\hat{p},h)$

$=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}(\hat{p},h_1)\lambda_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}}(\hat{p},h_2)\cdot\cdot\lambda_{\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}(\hat{p},h_n)\delta(h-\sum_{i=1}^{n}h_i)$

4.13 An important mathematical corollary

Cor. 4.13.1. $\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}\delta(h-\sum_{i=1}^{n}h_i)=\frac{[\sum_{i=1}^{n}(2s_i)]!}{\prod_{i=1}^{n}(2s_i)!},-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i$

4.14 Corollary-$\lambda_{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}(\hat{p},h)$ full symmetry

Thm. 4.14.1. $\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\frac{1}{(2s)!}\lambda_{\underbrace{\{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma\}}_{2s}}(\hat{p},h),-s\le h\le s$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $s'=\frac{1}{2},1$, the following is established.
$\lambda_{A_\varsigma}(\hat{p},h)=\frac{1}{1!}\lambda_{A_\varsigma}(\hat{p},h),-\frac{1}{2}\le h\le\frac{1}{2};\lambda_{A_\varsigma B_\varsigma}(\hat{p},h)=\frac{1}{2!}\lambda_{\{A_\varsigma B_\varsigma\}}(\hat{p},h),-1\le h\le 1$
Step 2: Assume when $s'=s-\frac{1}{2}$, the following is established.
$\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p},h)=\frac{1}{(2s-1)!}\lambda_{\underbrace{\{A_\varsigma\cdot\cdot B_\varsigma\}}_{2s-1}}(\hat{p},h),-s+\frac{1}{2}\le h\le s-\frac{1}{2}$
Step 3: When $1\le s'=s,\ -s\le h\le s,\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)$

$=\sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^{1}}}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma}_{2s-1}}(\hat{p},h-h')\lambda_{C_\varsigma}(\hat{p},h')=\sum_{h'=1}^{-1}\frac{\sqrt{C_{s+h}^{1+h'}C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^{2}}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2(s-1)}}(\hat{p},h-h')\lambda_{B_\varsigma C_\varsigma}(\hat{p},h')$

$\Rightarrow\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\frac{1}{(2s-1)!}\lambda_{\underbrace{\{A_\varsigma\cdot\cdot B_\varsigma\}C_\varsigma}_{2s}}(\hat{p},h),\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\lambda_{\underbrace{A_\varsigma\cdot\cdot C_\varsigma B_\varsigma}_{2s}}(\hat{p},h),-s\le h\le s$

$\Leftrightarrow\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma C_\varsigma}_{2s}}(\hat{p},h)=\frac{1}{(2s)!}\lambda_{\underbrace{\{A_\varsigma\cdot\cdot C_\varsigma B_\varsigma\}}_{2s}}(\hat{p},h),-s\le h\le s$

This step proves that when $s'=s$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

5 Reorganization and analysis of spin equation spin basis

5.1 Definition-Spin basis decomposition: $s=(s-\frac{1}{2})\oplus\frac{1}{2}$

Def. 5.1.1. $-s\le h\le s$

$\lambda_{k_\varsigma}(\hat{p},h;s)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma\cdot\cdot B_\varsigma}^{2s+1}}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma\cdot\cdot}_{2s}}[\frac{\sqrt{s+h}}{\sqrt{2s}}\lambda_{l_\varsigma}(\hat{p},h-\frac{1}{2};s-\frac{1}{2})\lambda_{B_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}\lambda_{l_\varsigma}(\hat{p},h+\frac{1}{2};s-\frac{1}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2})]$

Cor. 5.1.1. $\lambda(\hat{p},h;s)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma\cdot\cdot B_\varsigma}^{2s+1}}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma\cdot\cdot}_{2s}}\sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^{1}}}\lambda(\hat{p},h-h';s-\frac{1}{2})\lambda_{B_\varsigma}(\hat{p},h'),-s\le h\le s$

Cor. 5.1.2. $\lambda(\hat{p},h;s)=N^{A_\varsigma}(s)\sum_{h'=1/2}^{-1/2}\frac{\sqrt{C_{s+h}^{1/2+h'}C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^{1}}}\lambda(\hat{p},h-h';s-\frac{1}{2})\lambda_{A_\varsigma}(\hat{p},h'),-s\le h\le s$

5.2 Corollary-$\lambda(\hat{p},h;s)$ is a spin eigenstate

Thm. 5.2.1. $[\sigma(s)\cdot\hat{p}]\lambda(\hat{p},h;s)=h\lambda(\hat{p},h;s),-s\le h\le s$

Thm. 5.2.2. $\sigma^2(s)\lambda(\hat{p},h;s)=s(s+1)\lambda(\hat{p},h;s),-s\le h\le s$

So $\lambda(\hat{p},h;s)$ is a spin eigenstate. Therefore, the expansion coefficients are just the CG coefficients.

5.3 Corollary-Spin eigenstate $\lambda(\hat{p},h;s)$ raising and lowering operator

Thm. 5.3.1.

$$\begin{cases}e^{i\vec{\omega}\cdot\sigma(s)}\sigma_x(s)e^{-i\vec{\omega}\cdot\sigma(s)}=\sigma_x(s)-\hat{p}_x\frac{\sigma(s)\cdot\hat{p}+\sigma_z(s)}{(1+\hat{p}_z)}\\ e^{i\vec{\omega}\cdot\sigma(s)}\sigma_y(s)e^{-i\vec{\omega}\cdot\sigma(s)}=\sigma_y(s)-\hat{p}_y\frac{\sigma(s)\cdot\hat{p}+\sigma_z(s)}{(1+\hat{p}_z)}\\ e^{i\vec{\omega}\cdot\sigma(s)}\sigma_z(s)e^{-i\vec{\omega}\cdot\sigma(s)}=\sigma(s)\cdot\hat{p}\end{cases}$$

Proof: $e^{i\vec{\omega}\cdot\sigma(s)}\sigma_i(s)e^{-i\vec{\omega}\cdot\sigma(s)}=(e^{-i\vec{\omega}\cdot\gamma})_i{}^j\sigma_j(s)$

$$=[1-i(\gamma\times\hat{p})_z-(\gamma\times\hat{p})_z^2/(1+\hat{p}_z)]_i{}^j\sigma_j(s)$$

$$=[1-i(\gamma_x\hat{p}_y-\gamma_y\hat{p}_x)-(\gamma_x\hat{p}_y-\gamma_y\hat{p}_x)^2/(1+\hat{p}_z)]_i{}^j\sigma_j(s)$$

$$=[1-i\begin{bmatrix}0&0&-i\hat{p}_x\\0&0&-i\hat{p}_y\\i\hat{p}_x&i\hat{p}_y&0\end{bmatrix}-\begin{bmatrix}\hat{p}_x^2&\hat{p}_x\hat{p}_y&0\\\hat{p}_x\hat{p}_y&\hat{p}_y^2&0\\0&0&\hat{p}_x^2+\hat{p}_y^2\end{bmatrix}/(1+\hat{p}_z)]_i{}^j\sigma_j(s)$$

$$=[1-\begin{bmatrix}\hat{p}_x^2&\hat{p}_x\hat{p}_y&\hat{p}_x(1+\hat{p}_z)\\\hat{p}_x\hat{p}_y&\hat{p}_y^2&\hat{p}_y(1+\hat{p}_z)\\-\hat{p}_x(1+\hat{p}_z)&-\hat{p}_y(1+\hat{p}_z)&\hat{p}_x^2+\hat{p}_y^2\end{bmatrix}/(1+\hat{p}_z)]_i{}^j\sigma_j(s)$$

$$=\sigma_i(s)-[\begin{bmatrix}\hat{p}_x^2&\hat{p}_x\hat{p}_y&\hat{p}_x\hat{p}_z+\hat{p}_x\\\hat{p}_x\hat{p}_y&\hat{p}_y^2&\hat{p}_y\hat{p}_z+\hat{p}_y\\-\hat{p}_x(1+\hat{p}_z)&-\hat{p}_y(1+\hat{p}_z)&(1-\hat{p}_z)(1+\hat{p}_z)\end{bmatrix}/(1+\hat{p}_z)]_i{}^j\sigma_j(s)$$

$$=\begin{bmatrix}\sigma_x(s)-\hat{p}_x[\sigma_z(s)+\sigma(s)\cdot\hat{p}]/(1+\hat{p}_z)\\\sigma_y(s)-\hat{p}_y[\sigma_z(s)+\sigma(s)\cdot\hat{p}]/(1+\hat{p}_z)\\\sigma(s)\cdot\hat{p}\end{bmatrix}_i$$ □

Def. 5.3.1.

$$\begin{cases}\hat{J}_x(\hat{p},\sigma(s)):=\bar{\Gamma}(s)\hat{J}_x(\hat{p},\Omega(s))\Gamma(s)=\{\sigma_x(s)-\frac{\hat{p}_x}{(1+\hat{p}_z)}[\sigma(s)\cdot\hat{p}+\sigma_z(s)]\}\\ \hat{J}_y(\hat{p},\sigma(s)):=\bar{\Gamma}(s)\hat{J}_y(\hat{p},\Omega(s))\Gamma(s)=\{\sigma_y(s)-\frac{\hat{p}_y}{(1+\hat{p}_z)}[\sigma(s)\cdot\hat{p}+\sigma_z(s)]\}\\ \hat{J}_z(\hat{p},\sigma(s)):=\bar{\Gamma}(s)\hat{J}_z(\hat{p},\Omega(s))\Gamma(s)=\sigma(s)\cdot\hat{p}\end{cases}$$

Cor. 5.3.1.

$$\begin{cases}\hat{J}_x^2(\hat{p},\sigma(\frac{1}{2}))=\frac{1}{4},\hat{J}_y^2(\hat{p},\sigma(\frac{1}{2}))=\frac{1}{4},\hat{J}_z^2(\hat{p},\sigma(\frac{1}{2}))=\frac{1}{4}\\ [\hat{J}_i(\hat{p},\sigma(s)),\hat{J}_j(\hat{p},\sigma(s))]=\varepsilon_{ij}{}^k\hat{J}_k(\hat{p},\sigma(s)),\hat{J}^2(\hat{p},\sigma(s))=s(s+1)\end{cases}$$

Cor. 5.3.2.

$$\begin{cases}\hat{J}_+(\hat{p},\sigma(s)):=\bar{\Gamma}(s)\hat{J}_+(\hat{p},\Omega(s))\Gamma(s)=\{[\sigma_x(s)+i\sigma_y(s)]-\frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(s)\cdot\hat{p}+\sigma_z(s)]\}\\ \hat{J}_-(\hat{p},\sigma(s)):=\bar{\Gamma}(s)\hat{J}_-(\hat{p},\Omega(s))\Gamma(s)=\{[\sigma_x(s)-i\sigma_y(s)]-\frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(s)\cdot\hat{p}+\sigma_z(s)]\}\\ \hat{J}_z(\hat{p},\sigma(s)):=\bar{\Gamma}(s)\hat{J}_z(\hat{p},\Omega(s))\Gamma(s)=\sigma(s)\cdot\hat{p}\end{cases}$$

Cor. 5.3.3.

$$\begin{cases}\hat{J}_+(\hat{p},\sigma(s))\lambda(\vec{p},h;s)=\sqrt{s(s+1)-h(h+1)}\lambda(\vec{p},h+1;s),-s\le h\le s\\ \hat{J}_-(\hat{p},\sigma(s))\lambda(\vec{p},h;s)=\sqrt{s(s+1)-h(h-1)}\lambda(\vec{p},h-1;s),-s\le h\le s\\ \hat{J}_z(\hat{p},\sigma(s))\lambda(\vec{p},h;s)=h\lambda(\vec{p},h;s),-s\le h\le s\end{cases}$$

Cor. 5.3.4. $\hat{J}^2=\hat{J}_z^2+\frac{1}{2}(\hat{J}_+\hat{J}_-+\hat{J}_+\hat{J}_-)$

$$\begin{cases}\hat{J}^2(\hat{p},\sigma(s))\lambda(\vec{p},h;s)=s(s+1)\lambda(\vec{p},h;s),-s\le h\le s\\ \hat{J}_z(\hat{p},\sigma(s))\lambda(\vec{p},h;s)=h\lambda(\vec{p},h;s),-s\le h\le s\end{cases}$$

5.4 Corollary-$\lambda_{A_\varsigma\cdots B_\varsigma C_\varsigma}(\hat{p},h)$ orthogonality

Thm. 5.4.1. $\lambda^+(\hat{p},h';s)\lambda(\hat{p},h;s)=\delta_{hh'},-s\le h\le s$

5.5 Corollary-Spin basis decomposition: $1=\frac{1}{2}\oplus\frac{1}{2}$

Cor. 5.5.1.

$$\lambda_{k_\varsigma}(\hat{p},h;1)=\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma}[\frac{\sqrt{1+h}}{\sqrt{2}}\lambda_{A_\varsigma}(\hat{p},h-\tfrac{1}{2})\lambda_{B_\varsigma}(\hat{p},\tfrac{1}{2})+\frac{\sqrt{1-h}}{\sqrt{2}}\lambda_{A_\varsigma}(\hat{p},h+\tfrac{1}{2})\lambda_{B_\varsigma}(\hat{p},-\tfrac{1}{2})],-1\le h\le 1$$

5.6 Corollary-Spin basis decomposition: $s+s'=s\oplus s'$

Thm. 5.6.1.

$$\lambda_{k_\varsigma}(\hat{p},h;s+s')=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma\cdots B_\varsigma\cdots}^{2s+2s'}}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma\cdots}_{2s}}\Gamma^{m_\varsigma}_{\underbrace{B_\varsigma\cdots}_{2s'}}\sum_{h'=s'}^{-s'}\frac{\sqrt{C_{s+s'+h}^{s'+h'}C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}}\lambda_{l_\varsigma}(\hat{p},h-h';s)\lambda_{m_\varsigma}(\hat{p},h';s'),-s-s'\le h\le s+s'$$

5.7 Corollary-Spin basis decomposition: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Cor. 5.7.1. $-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i, \lambda_{k_\varsigma}(\hat{p},h;\sum_{i=1}^{n}s_i)=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma\cdot\cdot}^{2s_1}\overbrace{B_\varsigma\cdot\cdot}^{2s_2}\cdots\overbrace{C_\varsigma\cdot\cdot}^{2s_n}}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}\Gamma^{m_\varsigma}_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}}\cdots\Gamma^{n_\varsigma}_{\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}$

$$\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}\left[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}\right]^{\frac{1}{2}}\lambda_{l_\varsigma}(\hat{p},h_1;s_1)\lambda_{m_\varsigma}(\hat{p},h_2;s_2)\cdot\cdot\lambda_{n_\varsigma}(\hat{p},h_n;s_n)\delta(h-\sum_{i=1}^{n}h_i)$$

5.8 Introducing a new constant invariant tensor

Def. 5.8.1. $\Gamma_{k_\varsigma}^{l_\varsigma m_\varsigma\cdot\cdot n_\varsigma}:=\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma\cdot\cdot}^{2s_1}\overbrace{B_\varsigma\cdot\cdot}^{2s_2}\cdots\overbrace{C_\varsigma\cdot\cdot}^{2s_n}}\Gamma^{l_\varsigma}_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}\Gamma^{m_\varsigma}_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}}\cdots\Gamma^{n_\varsigma}_{\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}$

Cor. 5.8.1. $-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i, \lambda_{k_\varsigma}(\hat{p},h;\sum_{i=1}^{n}s_i)=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}$

$$\left[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}\right]^{\frac{1}{2}}\Gamma_{k_\varsigma}^{l_\varsigma m_\varsigma\cdot\cdot n_\varsigma}\lambda_{l_\varsigma}(\hat{p},h_1;s_1)\lambda_{m_\varsigma}(\hat{p},h_2;s_2)\cdot\cdot\lambda_{n_\varsigma}(\hat{p},h_n;s_n)\delta(h-\sum_{i=1}^{n}h_i)$$

6 On unitary transformation of spin bases

6.1 Momentum transformation is equivalent to unitary transformation

Def. 6.1.1. $\hat{p}'=A(\hat{p}\to\hat{p}')\hat{p}, A(\hat{p}\to\hat{p}'):=exp\{i\frac{(\gamma\times\hat{p}')_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}exp\{-i\frac{(\gamma\times\hat{p})_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}$
$=[1+i(\gamma\times\hat{p}')_z-(\gamma\times\hat{p}')_z^2/(1+\hat{p}_z')][1-i(\gamma\times\hat{p})_z-(\gamma\times\hat{p})_z^2/(1+\hat{p}_z)]$

Cor. 6.1.1.

$$\begin{cases}[\sigma(s)\cdot\hat{p}']=[\sigma(s)\cdot A(\hat{p}\to\hat{p}')\hat{p}]=[A(\hat{p}'\to\hat{p})\sigma(s)\cdot\hat{p}]\\ =exp\{i\frac{[\sigma(s)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}[\sigma(s)\cdot\hat{p}]exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}exp\{i\frac{[\sigma(s)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}\\ [\Omega(s)\cdot\hat{p}']=[\Omega(s)\cdot A(\hat{p}\to\hat{p}')\hat{p}]=[A(\hat{p}'\to\hat{p})\Omega(s)\cdot\hat{p}]\\ =exp\{i\frac{[\Omega(s)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}exp\{-i\frac{[\Omega(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}[\Omega(s)\cdot\hat{p}]exp\{i\frac{[\Omega(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}exp\{i\frac{[\Omega(s)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}\end{cases}$$

Cor. 6.1.2.

$$\begin{cases}[\Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot\hat{p}']=[\Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot A(\hat{p}\to\hat{p}')\hat{p}]=[A(\hat{p}'\to\hat{p})\Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot\hat{p}]\\ =exp\{i\frac{[\Omega(s;\sigma(\frac{1}{2})\otimes I)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}exp\{-i\frac{[\Omega(s;\sigma(\frac{1}{2})\otimes I)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}[\Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot\hat{p}]\\ exp\{i\frac{[\Omega(s;\sigma(\frac{1}{2})\otimes I)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}exp\{i\frac{[\Omega(s;\sigma(\frac{1}{2})\otimes I)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}\\ [\Omega(s;R)\cdot\hat{p}']=[\Omega(s;R)\cdot A(\hat{p}\to\hat{p}')\hat{p}]=[A(\hat{p}'\to\hat{p})\Omega(s;R)\cdot\hat{p}]\\ =exp\{i\frac{[\Omega(s;R)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}exp\{-i\frac{[\Omega(s;R)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}[\Omega(s;R)\cdot\hat{p}]\\ exp\{i\frac{[\Omega(s;R)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}exp\{i\frac{[\Omega(s;R)\times\hat{p}']_z}{\sqrt{1-\hat{p}_z'^2}}arccos\hat{p}_z'\}\end{cases}$$

Cor. 6.1.3.

$$\begin{cases}\sigma(s)\cdot\hat{p}=e^{i\vec{\omega}\cdot\sigma(s)}\sigma_z e^{-i\vec{\omega}\cdot\sigma(s)},\Omega(s)\cdot\hat{p}=e^{i\vec{\omega}\cdot\Omega(s)}\Omega_z(s)e^{-i\vec{\omega}\cdot\Omega(s)}\\ \Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot\hat{p}=e^{i\vec{\omega}\cdot\Omega(s;\sigma(\frac{1}{2})\otimes I)}\Omega_z(s;\sigma(\frac{1}{2})\otimes I)e^{-i\vec{\omega}\cdot\Omega(s;\sigma(\frac{1}{2})\otimes I)}\\ \Omega(s;R)\cdot\hat{p}=e^{i\vec{\omega}\cdot\Omega(s;R)}\Omega_z(s;R)e^{-i\vec{\omega}\cdot\Omega(s;R)}\end{cases}$$

Cor. 6.1.4.

$$\begin{cases}\sigma(s)\cdot\hat{p}=exp\{i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\sigma_z exp\{-i\frac{[\sigma(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\\ \Omega(s)\cdot\hat{p}=exp\{i\frac{[\Omega(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\Omega_z(s)exp\{-i\frac{[\Omega(s)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\\ \Omega(s;\sigma(\frac{1}{2})\otimes I)\cdot\hat{p}=exp\{i\frac{[\Omega(s;\sigma(\frac{1}{2})\otimes I)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\Omega_z(s;\sigma(\frac{1}{2})\otimes I)exp\{-i\frac{[\Omega(s;\sigma(\frac{1}{2})\otimes I)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\\ \Omega(s;R)\cdot\hat{p}=exp\{i\frac{[\Omega(s;R)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\Omega_z(s;R)exp\{-i\frac{[\Omega(s;R)\times\hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}arccos\hat{p}_z\}\end{cases}$$

6.2 Spin basis physical decomposition: $1=\frac{1}{2}\oplus\frac{1}{2}$

Cor. 6.2.1. $\lambda_{A_\varsigma B_\varsigma}(\hat{p}',\hat{p},h)=\frac{\sqrt{1+h}}{\sqrt{2}}\lambda_{A_\varsigma}(\hat{p}',h-\frac{1}{2})\lambda_{B_\varsigma}(\hat{p},\frac{1}{2})+\frac{\sqrt{1-h}}{\sqrt{2}}\lambda_{A_\varsigma}(\hat{p}',h+\frac{1}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2})$

Cor. 6.2.2. $\lambda_{A_\varsigma B_\varsigma}(\hat{p}',\hat{p},h)=\begin{cases}\lambda_{A_\varsigma}(\hat{p}',\frac{1}{2})\lambda_{B_\varsigma}(\hat{p},\frac{1}{2}),h=1\\ \frac{1}{\sqrt{2}}\lambda_{\{A_\varsigma}(\hat{p}',\frac{1}{2})\lambda_{B_\varsigma\}}(\hat{p},-\frac{1}{2}),h=0\\ \lambda_{A_\varsigma}(\hat{p}',-\frac{1}{2})\lambda_{B_\varsigma}(\hat{p},-\frac{1}{2}),h=-1\end{cases}$

Cor. 6.2.3. $\lambda(\hat{p}',\hat{p},h)=\begin{cases}\lambda(\hat{p}',\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2}),h=1\\ \frac{1}{\sqrt{2}}[\lambda(\hat{p}',\frac{1}{2})\otimes\lambda(\hat{p},-\frac{1}{2})+\lambda(\hat{p}',-\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})],h=0\\ \lambda(\hat{p}',-\frac{1}{2})\otimes\lambda(\hat{p},-\frac{1}{2}),h=-1\end{cases}$

Cor. 6.2.4. $\lambda^+(\hat{p}',\hat{p},h')\lambda(\hat{p}',\hat{p},h)=\delta_{hh'}$

Cor. 6.2.5. $\tilde{\lambda}(\hat{p}',\hat{p},h)=\begin{cases}\lambda(\hat{p}',\frac{1}{2})\otimes\lambda(\hat{p},\frac{1}{2})+\lambda(\hat{p},\frac{1}{2})\otimes\lambda(\hat{p}',\frac{1}{2}),h=1\\ \lambda(\hat{p}',\frac{1}{2})\otimes\lambda(\hat{p},-\frac{1}{2})+\lambda(\hat{p},-\frac{1}{2})\otimes\lambda(\hat{p}',\frac{1}{2}),h=0\\ \lambda(\hat{p}',-\frac{1}{2})\otimes\lambda(\hat{p},-\frac{1}{2})+\lambda(\hat{p},-\frac{1}{2})\otimes\lambda(\hat{p}',-\frac{1}{2}),h=-1\end{cases}$

Cor. 6.2.6.
$$\begin{cases}\{[\sigma(\frac{1}{2})\cdot\hat{p}']\otimes I+I\otimes[\sigma(\frac{1}{2})\cdot\hat{p}]\}\lambda(\hat{p}',\hat{p},h)=h\lambda(\hat{p}',\hat{p},h)\\ \{[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})\cdot\hat{p}]\otimes I+I\otimes[\sigma(\frac{1}{2})\cdot\hat{p}]\}\lambda(\hat{p}',\hat{p},h)=h\lambda(\hat{p}',\hat{p},h)\end{cases}$$

Cor. 6.2.7.
$$\begin{cases}[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]\cdot\hat{p}\lambda(\hat{p}',h)=h\lambda(\hat{p}',h),-\frac{1}{2}\le h\le\frac{1}{2}\\ [A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]^2\lambda(\hat{p}',h)=\frac{1}{2}(\frac{1}{2}+1)\lambda(\hat{p}',h)\end{cases}\quad\begin{cases}[\sigma(\frac{1}{2})\cdot\hat{p}]\lambda(\hat{p},h)=h\lambda(\hat{p},h),-\frac{1}{2}\le h\le\frac{1}{2}\\ [\sigma(\frac{1}{2})]^2\lambda(\hat{p},h)=\frac{1}{2}(\frac{1}{2}+1)\lambda(\hat{p},h)\end{cases}$$
$$\begin{cases}\{[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]\otimes I\}\cdot\hat{p}\lambda(\hat{p}',\hat{p},h)=\frac{h}{2}\lambda(\hat{p}',\hat{p},h)\\ \{[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]\otimes I\}^2\lambda(\hat{p}',\hat{p},h)=\frac{1}{2}(\frac{1}{2}+1)\lambda(\hat{p}',\hat{p},h)\end{cases}\quad\begin{cases}\{I\otimes\sigma(\frac{1}{2})\}\cdot\hat{p}\lambda(\hat{p}',\hat{p},h)=\frac{h}{2}\lambda(\hat{p}',\hat{p},h)\\ \{I\otimes\sigma(\frac{1}{2})\}^2\lambda(\hat{p}',\hat{p},h)=\frac{1}{2}(\frac{1}{2}+1)\lambda(\hat{p}',\hat{p},h)\end{cases}$$
$$\begin{cases}\{[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]\otimes I+I\otimes\sigma(\frac{1}{2})\}\cdot\hat{p}\lambda(\hat{p}',\hat{p},h)=h\lambda(\hat{p}',\hat{p},h),-1\le h\le 1\\ \{[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]\otimes I+I\otimes\sigma(\frac{1}{2})\}^2\lambda(\hat{p}',\hat{p},h)=1(1+1)\lambda(\hat{p}',\hat{p},h)\end{cases}$$

Cor. 6.2.8. $\lambda^+(\hat{p}',\hat{p},h)\{[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]\otimes I\}\cdot\{I\otimes\sigma(\frac{1}{2})\}\lambda(\hat{p}',\hat{p},h)=\frac{1}{4}\lambda(\hat{p}',\hat{p},h)$

Cor. 6.2.9. $\{[A(\hat{p}'\to\hat{p})\sigma(\frac{1}{2})]\otimes I\}\cdot\{I\otimes\sigma(\frac{1}{2})\}\lambda(\hat{p}',\hat{p},h)=?$

Cor. 6.2.10. $H=-\sum_{i,j}k_{ij}[A(\hat{p}_i)\sigma_i(\frac{1}{2})]\cdot[A(\hat{p}_j)\sigma_j(\frac{1}{2})]$

6.3 Unitary transformation → Z-direction representation of spin basis decomposition for B-W equation: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Def. 6.3.1. $-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i, U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h):=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}$

$$[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_1)U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_2)\cdot\cdot U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_n)\delta(h-\sum_{i=1}^{n}h_i)$$

Cor. 6.3.1.
$$\begin{cases}\Omega^2(s;\sigma(\frac{1}{2})\otimes I)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h)=s(s+1)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h),-s\le h\le s\\ \Omega_z(s;\sigma(\frac{1}{2})\otimes I)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h)=hU_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h),-s\le h\le s\end{cases}$$

Def. 6.3.2. $-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i, V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h):=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}$

$$[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_1)V_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_2)\cdot\cdot V_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_n)\delta(h-\sum_{i=1}^{n}h_i)$$

Cor. 6.3.2.
$$\begin{cases}\Omega^2(s;\sigma(\frac{1}{2})\otimes I)V_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h)=s(s+1)V_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h),-s\le h\le s\\ \Omega_z(s;\sigma(\frac{1}{2})\otimes I)V_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h)=hV_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h),-s\le h\le s\end{cases}$$

6.4 Unitary transformation → Stationary representation of spin basis decomposition for B-W equation: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Def. 6.4.1. $-\sum\limits_{i=1}^{n}s_i\le h\le\sum\limits_{i=1}^{n}s_i, U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec{0},h):=\sum\limits_{h_1=s_1}^{-s_1}\sum\limits_{h_2=s_2}^{-s_2}\cdot\cdot\sum\limits_{h_n=s_n}^{-s_n}$

$$[\frac{\prod\limits_{i=1}^{n}(2s_i)!}{[\sum\limits_{i=1}^{n}(2s_i)]!}\frac{[\sum\limits_{i=1}^{n}(s_i+h_i)]!}{\prod\limits_{i=1}^{n}(s_i+h_i)!}\frac{[\sum\limits_{i=1}^{n}(s_i-h_i)]!}{\prod\limits_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{0},h_1)U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec{0},h_2)\cdot\cdot U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec{0},h_n)\delta(h-\sum\limits_{i=1}^{n}h_i)$$

Cor. 6.4.1.
$$\begin{cases}\Omega^2(s;\sigma(\frac{1}{2})\otimes I)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h)=s(s+1)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h),-s\le h\le s\\ \Omega_z(s;\sigma(\frac{1}{2})\otimes I)U_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h)=hU_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h),-s\le h\le s\end{cases}$$

Def. 6.4.2. $-\sum\limits_{i=1}^{n}s_i\le h\le\sum\limits_{i=1}^{n}s_i, V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec{0},h):=\sum\limits_{h_1=s_1}^{-s_1}\sum\limits_{h_2=s_2}^{-s_2}\cdot\cdot\sum\limits_{h_n=s_n}^{-s_n}$

$$[\frac{\prod\limits_{i=1}^{n}(2s_i)!}{[\sum\limits_{i=1}^{n}(2s_i)]!}\frac{[\sum\limits_{i=1}^{n}(s_i+h_i)]!}{\prod\limits_{i=1}^{n}(s_i+h_i)!}\frac{[\sum\limits_{i=1}^{n}(s_i-h_i)]!}{\prod\limits_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{0},h_1)V_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec{0},h_2)\cdot\cdot V_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec{0},h_n)\delta(h-\sum\limits_{i=1}^{n}h_i)$$

Cor. 6.4.2.
$$\begin{cases}\Omega^2(s;\sigma(\frac{1}{2})\otimes I)V_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h)=s(s+1)V_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h),-s\le h\le s\\ \Omega_z(s;\sigma(\frac{1}{2})\otimes I)V_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h)=hV_{\underbrace{\lambda_\varsigma\otimes\cdot\cdot\otimes\sigma_\varsigma\otimes\tau_\varsigma}_{2s}}(\vec{0},h),-s\le h\le s\end{cases}$$

6.5 Unitary transformation → Z-direction representation of spin basis decomposition for K-G equation: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Def. 6.5.1. $-\sum\limits_{i=1}^{n}n_i\le h\le\sum\limits_{i=1}^{n}n_i,\varepsilon_{\underbrace{a\cdot\cdot}_{2n_1}\underbrace{b\cdot\cdot}_{2n_2}\cdot\cdot\underbrace{c\cdot\cdot}_{2n_n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h)$

$$:=\sum\limits_{h_1=n_1}^{-n_1}\sum\limits_{h_2=n_2}^{-n_2}\cdot\cdot\sum\limits_{h_n=n_n}^{-n_n}[\frac{\prod\limits_{i=1}^{n}(2n_i)!}{[\sum\limits_{i=1}^{n}(2n_i)]!}\frac{[\sum\limits_{i=1}^{n}(n_i+h_i)]!}{\prod\limits_{i=1}^{n}(n_i+h_i)!}\frac{[\sum\limits_{i=1}^{n}(n_i-h_i)]!}{\prod\limits_{i=1}^{n}(n_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{2n_1}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_1)\varepsilon_{\underbrace{b\cdot\cdot}_{2n_2}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_2)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{2n_n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h_n)\delta(h-\sum\limits_{i=1}^{n}h_i)$$

Cor. 6.5.1.
$$\begin{cases}\Omega^2(n;R)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h)=n(n+1)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h),-n\le h\le n\\ \Omega_z(n;R)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h)=h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\begin{bmatrix}0\\0\\p\end{bmatrix},h),-n\le h\le n\end{cases}$$

6.6 Unitary transformation → Stationary representation of spin basis decomposition for K-G equation: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Def. 6.6.1. $-\sum\limits_{i=1}^{n}n_i\le h\le\sum\limits_{i=1}^{n}n_i,\varepsilon_{\underbrace{a\cdot\cdot}_{2n_1}\underbrace{b\cdot\cdot}_{2n_2}\cdot\cdot\underbrace{c\cdot\cdot}_{2n_n}}(\vec{0},h)$

$$:=\sum\limits_{h_1=n_1}^{-n_1}\sum\limits_{h_2=n_2}^{-n_2}\cdot\cdot\sum\limits_{h_n=n_n}^{-n_n}[\frac{\prod\limits_{i=1}^{n}(2n_i)!}{[\sum\limits_{i=1}^{n}(2n_i)]!}\frac{[\sum\limits_{i=1}^{n}(n_i+h_i)]!}{\prod\limits_{i=1}^{n}(n_i+h_i)!}\frac{[\sum\limits_{i=1}^{n}(n_i-h_i)]!}{\prod\limits_{i=1}^{n}(n_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{2n_1}}(\vec{0},h_1)\varepsilon_{\underbrace{b\cdot\cdot}_{2n_2}}(\vec{0},h_2)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{2n_n}}(\vec{0},h_n)\delta(h-\sum\limits_{i=1}^{n}h_i)$$

Cor. 6.6.1.
$$\begin{cases}\Omega^2(n;R)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{0},h)=n(n+1)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{0},h),-n\le h\le n\\ \Omega_z(n;R)\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{0},h)=h\varepsilon_{\underbrace{a\otimes\cdot\cdot\otimes b\otimes c}_{n}}(\vec{0},h),-n\le h\le n\end{cases}$$

6.7 Unitary transformation → Z-direction representation of spin basis decomposition for Penrose equation: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Def. 6.7.1. $-\sum\limits_{i=1}^{n}s_i\le h\le\sum\limits_{i=1}^{n}s_i,\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}\underbrace{B_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h):=\sum\limits_{h_1=s_1}^{-s_1}\sum\limits_{h_2=s_2}^{-s_2}\cdot\cdot\sum\limits_{h_n=s_n}^{-s_n}$

$$[\frac{\prod\limits_{i=1}^{n}(2s_i)!}{[\sum\limits_{i=1}^{n}(2s_i)]!}\frac{[\sum\limits_{i=1}^{n}(s_i+h_i)]!}{\prod\limits_{i=1}^{n}(s_i+h_i)!}\frac{[\sum\limits_{i=1}^{n}(s_i-h_i)]!}{\prod\limits_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h_1)\lambda_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h_2)\cdot\cdot\lambda_{\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h_n)\delta(h-\sum\limits_{i=1}^{n}h_i)$$

Cor. 6.7.1. $\begin{cases} \Omega^2(s)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h)=s(s+1)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h),-s\le h\le s \\ \Omega_z(s)\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h)=h\lambda_{\underbrace{A_\varsigma\otimes\cdot\cdot\otimes B_\varsigma\otimes C_\varsigma}_{2s}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h),-s\le h\le s \end{cases}$

6.8 Unitary transformation → Z-direction representation of spin basis decomposition for spin equation: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

Def. 6.8.1. $-\sum_{i=1}^n s_i\le h\le\sum_{i=1}^n s_i,\lambda_{k_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},h;\sum_{i=1}^n s_i):=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}$

$[\frac{\prod_{i=1}^n(2s_i)!}{[\sum_{i=1}^n(2s_i)]!}\frac{[\sum_{i=1}^n(s_i+h_i)]!}{\prod_{i=1}^n(s_i+h_i)!}\frac{[\sum_{i=1}^n(s_i-h_i)]!}{\prod_{i=1}^n(s_i-h_i)!}]^{\frac12}\Gamma_{k_\varsigma}^{l_\varsigma m_\varsigma\cdot\cdot n_\varsigma}\lambda_{l_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},h_1;s_1)\lambda_{m_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},h_2;s_2)\cdot\cdot\lambda_{n_\varsigma}(\begin{bmatrix}0\\0\\1\end{bmatrix},h_n;s_n)\delta(h-\sum_{i=1}^n h_i)$

Cor. 6.8.1. $\begin{cases}\sigma^2(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=s(s+1)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s),-s\le h\le s\\ \sigma_z(s)\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s)=h\lambda(\begin{bmatrix}0\\0\\1\end{bmatrix},h;s),-s\le h\le s\end{cases}$

7 Unitary transformation → Real physical representation of spin basis decomposition: $s_1+s_2\cdot\cdot+s_n=s_1\oplus s_2\cdot\cdot\oplus s_n$

7.1 The case of arbitrary spin particles with mass

Def. 7.1.1.

$$\begin{cases} U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h):=\prod_{i=1}^n[\otimes exp\{i\frac{[(\sigma(\frac12)\otimes I)\times\hat p_i]_z}{\sqrt{1-\hat p_{iz}^2}}arccos\hat p_{iz}\}\frac{E+m-i|\vec p_i|\gamma_z\gamma_4}{\sqrt{2m(E+m)}}]^{2s_i}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\vec 0,\sum_{i=1}^n s_i;h)\\ \hat J(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a):=\prod_{i=1}^n[\otimes exp\{i\frac{[(\sigma(\frac12)\otimes I)\times\hat p_i]_z}{\sqrt{1-\hat p_{iz}^2}}arccos\hat p_{iz}\}\frac{E+m-i|\vec p_i|\gamma_z\gamma_4}{\sqrt{2m(E+m)}}]^{2s_i}\\ \Omega(\sum_{i=1}^n s_i;\sigma(\frac12)\otimes I)\prod_{i=1}^n[\otimes\frac{E+m+i|\vec p_i|\gamma_z\gamma_4}{\sqrt{2m(E+m)}}exp\{-i\frac{[(\sigma(\frac12)\otimes I)\times\hat p_i]_z}{\sqrt{1-\hat p_{iz}^2}}arccos\hat p_{iz}\}]^{2s_i}\end{cases}$$

Cor. 7.1.1. $-\sum_{i=1}^n s_i\le h\le\sum_{i=1}^n s_i,U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\prod_{i=1}^n(\vec p_i,s_i);h):=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}$

$[\frac{\prod_{i=1}^n(2s_i)!}{[\sum_{i=1}^n(2s_i)]!}\frac{[\sum_{i=1}^n(s_i+h_i)]!}{\prod_{i=1}^n(s_i+h_i)!}\frac{[\sum_{i=1}^n(s_i-h_i)]!}{\prod_{i=1}^n(s_i-h_i)!}]^{\frac12}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec p_1,h_1)U_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec p_2,h_2)\cdot\cdot U_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec p_n,h_n)\delta(h-\sum_{i=1}^n h_i)$

Cor. 7.1.2.

$$\begin{cases}\hat J_+(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=\sqrt{(\sum_{i=1}^n s_i)(\sum_{i=1}^n s_i+1)-h(h+1)}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h+1)\\ \hat J_-(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=\sqrt{(\sum_{i=1}^n s_i)(\sum_{i=1}^n s_i+1)-h(h-1)}U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h-1)\\ \hat J_z(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=hU_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h),-\sum_{i=1}^n s_i\le h\le\sum_{i=1}^n s_i\\ \hat Q(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=-2sU_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)\end{cases}$$

Cor. 7.1.3.

$$\begin{cases}\hat J^2(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=(\sum_{i=1}^n s_i)(\sum_{i=1}^n s_i+1)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)\\ \hat J^2(\vec p_i,*\frac12;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=\frac12(\frac12+1)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)\\ \hat J_z(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=hU_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)\\ \hat Q(\vec p_i,*\frac12;\gamma_a)U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)=-U_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^n(\vec p_i,s_i);h)\\ [\hat J_\alpha(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a),\hat J_\beta(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a)]=\varepsilon_{\alpha\beta}{}^\gamma\hat J_\gamma(\prod_{i=1}^n(\vec p_i,s_i);\gamma_a),-\sum_{i=1}^n s_i\le h\le\sum_{i=1}^n s_i\end{cases}$$

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Cor. 7.1.4. $-\sum_{i=1}^{n} s_i \leq h \leq \sum_{i=1}^{n} s_i, V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) := \sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}$

$$[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s_1}}(\vec{p}_1,h_1)V_{\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s_2}}(\vec{p}_2,h_2)\cdot\cdot V_{\underbrace{\rho_\varsigma\sigma_\varsigma\cdot\cdot}_{2s_n}}(\vec{p}_n,h_n)\delta(h-\sum_{i=1}^{n}h_i)$$

Cor. 7.1.5.

$$\begin{cases}\hat{J}_+(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = \sqrt{(\sum_{i=1}^{n}s_i)(\sum_{i=1}^{n}s_i+1)-h(h+1)}V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h+1)\\ \hat{J}_-(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = \sqrt{(\sum_{i=1}^{n}s_i)(\sum_{i=1}^{n}s_i+1)-h(h-1)}V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h-1)\\ \hat{J}_z(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = hV_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h), -\sum_{i=1}^{n}s_i\leq h\leq\sum_{i=1}^{n}s_i\\ \hat{Q}(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = 2sV_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h)\end{cases}$$

Cor. 7.1.6.

$$\begin{cases}\hat{J}^2(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = (\sum_{i=1}^{n}s_i)(\sum_{i=1}^{n}s_i+1)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h)\\ \hat{J}^2(\vec{p}_i,*\frac{1}{2};\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = \frac{1}{2}(\frac{1}{2}+1)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h)\\ \hat{J}_z(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = hV_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h)\\ \hat{Q}(\vec{p}_i,*\frac{1}{2};\gamma_a)V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h) = V_{\underbrace{\lambda_\varsigma\otimes\mu_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\vec{p}_i,s_i);h)\\ [\hat{J}_\alpha(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a),\hat{J}_\beta(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a)] = \varepsilon_{\alpha\beta}{}^{\gamma}\hat{J}_\gamma(\prod_{i=1}^{n}(\vec{p}_i,s_i);\gamma_a), -\sum_{i=1}^{n}s_i\leq h\leq\sum_{i=1}^{n}s_i\end{cases}$$

7.2 The case of integral spin particles with mass

Def. 7.2.1.

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h) := \prod_{i=1}^{n}[\otimes exp\{i\frac{[R\times\hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}}arccos\hat{p}_{iz}\}\frac{m-|\vec{p}_i|L_z+(E_i-m)L_z^2}{m}]^{l_i}\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\vec{0},h)\\ \hat{J}(\prod_{i=1}^{n}(\vec{p}_i,l_i);R) := \prod_{i=1}^{n}[\otimes exp\{i\frac{[R\times\hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}}arccos\hat{p}_{iz}\}\frac{m-|\vec{p}_i|L_z+(E_i-m)L_z^2}{m}]^{l_i}\\ \Omega(\sum_{i=1}^{n}l_i;R)\prod_{i=1}^{n}[\otimes\frac{m+|\vec{p}_i|L_z+(E_i-m)L_z^2}{m}exp\{-i\frac{[R\times\hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}}arccos\hat{p}_{iz}\}]^{l_i}\end{cases}$$

Cor. 7.2.1. $-\sum_{i=1}^{n}l_i\leq h\leq\sum_{i=1}^{n}l_i, \varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h)$

$$:= \sum_{h_1=l_1}^{-l_1}\sum_{h_2=l_2}^{-l_2}\cdot\cdot\sum_{h_n=l_n}^{-l_n}[\frac{\prod_{i=1}^{n}(2l_i)!}{[\sum_{i=1}^{n}(2l_i)]!}\frac{[\sum_{i=1}^{n}(l_i+h_i)]!}{\prod_{i=1}^{n}(l_i+h_i)!}\frac{[\sum_{i=1}^{n}(l_i-h_i)]!}{\prod_{i=1}^{n}(l_i-h_i)!}]^{\frac{1}{2}}\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}}(\vec{p}_1,h_1)\varepsilon_{\underbrace{b\cdot\cdot}_{l_2}}(\vec{p}_2,h_2)\cdot\cdot\varepsilon_{\underbrace{c\cdot\cdot}_{l_n}}(\vec{p}_n,h_n)\delta(h-\sum_{i=1}^{n}h_i)$$

Cor. 7.2.2.

$$\begin{cases}\hat{J}_+(\prod_{i=1}^{n}(\vec{p}_i,l_i);R)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h) = \sqrt{(\sum_{i=1}^{n}l_i)(\sum_{i=1}^{n}l_i+1)-h(h+1)}\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h+1)\\ \hat{J}_-(\prod_{i=1}^{n}(\vec{p}_i,l_i);R)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h) = \sqrt{(\sum_{i=1}^{n}l_i)(\sum_{i=1}^{n}l_i+1)-h(h-1)}\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h-1)\\ \hat{J}_z(\prod_{i=1}^{n}(\vec{p}_i,l_i);R)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h) = h\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h), -\sum_{i=1}^{n}l_i\leq h\leq\sum_{i=1}^{n}l_i\end{cases}$$

Cor. 7.2.3.

$$\begin{cases}\hat{J}^2(\prod_{i=1}^{n}(\vec{p}_i,l_i);R)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h)=(\sum_{i=1}^{n}l_i)(\sum_{i=1}^{n}l_i+1)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h)\\ \hat{J}^2(\vec{p}_i,*1;R,L)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h)=1(1+1)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h)\\ \hat{J}_z(\prod_{i=1}^{n}(\vec{p}_i,l_i);R)\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h)=h\varepsilon_{\underbrace{a\cdot\cdot}_{l_1}\underbrace{b\cdot\cdot}_{l_2}\cdot\cdot\underbrace{c\cdot\cdot}_{l_n}}(\prod_{i=1}^{n}(\vec{p}_i,l_i);h)\\ [\hat{J}_\alpha(\prod_{i=1}^{n}(\vec{p}_i,l_i);R),\hat{J}_\beta(\prod_{i=1}^{n}(\vec{p}_i,l_i);R)]=\varepsilon_{\alpha\beta}{}^{\gamma}\hat{J}_\gamma(\prod_{i=1}^{n}(\vec{p}_i,l_i);R),-\sum_{i=1}^{n}l_i\le h\le\sum_{i=1}^{n}l_i\end{cases}$$

7.3 The case of arbitrary spin particles without mass

Def. 7.3.1.

$$\begin{cases}\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h):=\prod_{i=1}^{n}\otimes exp\{i\frac{[\sigma(\frac{1}{2})\times\hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}}arccos\hat{p}_{iz}\}\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\begin{bmatrix}0\\0\\1\end{bmatrix},h)\\ \hat{J}(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2})):=\prod_{i=1}^{n}\otimes exp\{i\frac{[\sigma(\frac{1}{2})\times\hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}}arccos\hat{p}_{iz}\}\Omega(s)\prod_{i=1}^{n}\otimes exp\{-i\frac{[\sigma(\frac{1}{2})\times\hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}}arccos\hat{p}_{iz}\}\end{cases}$$

Cor. 7.3.1. $-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i,\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}\underbrace{B_\varsigma\cdot\cdot}_{2s_2}\cdot\cdot\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)$

$$:=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}\lambda_{\underbrace{A_\varsigma\cdot\cdot}_{2s_1}}(\hat{p}_1,h_1)\lambda_{\underbrace{B_\varsigma\cdot\cdot}_{2s_2}}(\hat{p}_2,h_2)\cdot\cdot\lambda_{\underbrace{C_\varsigma\cdot\cdot}_{2s_n}}(\hat{p}_n,h_n)\delta(h-\sum_{i=1}^{n}h_i)$$

Cor. 7.3.2.

$$\begin{cases}\hat{J}_+(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2}))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)=\sqrt{(\sum_{i=1}^{n}s_i)(\sum_{i=1}^{n}s_i+1)-h(h+1)}\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h+1)\\ \hat{J}_-(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2}))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)=\sqrt{(\sum_{i=1}^{n}s_i)(\sum_{i=1}^{n}s_i+1)-h(h-1)}\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h-1)\\ \hat{J}_z(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2}))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)=h\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h),-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i\end{cases}$$

Cor. 7.3.3.

$$\begin{cases}\hat{J}^2(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2}))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)=(\sum_{i=1}^{n}s_i)(\sum_{i=1}^{n}s_i+1)\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)\\ \hat{J}^2(\hat{p}_i,*\sigma(\frac{1}{2}))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)=\frac{1}{2}(\frac{1}{2}+1)\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)\\ \hat{J}_z(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2}))\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h)=h\lambda_{\underbrace{A_\varsigma\otimes B_\varsigma\otimes\cdot\cdot}_{2(s_1+\cdot\cdot+s_n)}}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h),-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i\\ [\hat{J}_\alpha(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2})),\hat{J}_\beta(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2}))]=\varepsilon_{\alpha\beta}{}^{\gamma}\hat{J}_\gamma(\prod_{i=1}^{n}(\hat{p}_i,s_i);\sigma(\frac{1}{2}))\end{cases}$$

— —

Cor. 7.3.4. $-\sum_{i=1}^{n}s_i\le h\le\sum_{i=1}^{n}s_i,\lambda_{k_\varsigma}(\prod_{i=1}^{n}(\hat{p}_i,s_i);h;\sum_{i=1}^{n}s_i):=\sum_{h_1=s_1}^{-s_1}\sum_{h_2=s_2}^{-s_2}\cdot\cdot\sum_{h_n=s_n}^{-s_n}$

$$[\frac{\prod_{i=1}^{n}(2s_i)!}{[\sum_{i=1}^{n}(2s_i)]!}\frac{[\sum_{i=1}^{n}(s_i+h_i)]!}{\prod_{i=1}^{n}(s_i+h_i)!}\frac{[\sum_{i=1}^{n}(s_i-h_i)]!}{\prod_{i=1}^{n}(s_i-h_i)!}]^{\frac{1}{2}}\Gamma_{k_\varsigma}^{l_\varsigma m_\varsigma\cdot\cdot n_\varsigma}\lambda_{l_\varsigma}(\hat{p}_1,h_1;s_1)\lambda_{m_\varsigma}(\hat{p}_2,h_2;s_2)\cdot\cdot\lambda_{n_\varsigma}(\hat{p}_n,h_n;s_n)\delta(h-\sum_{i=1}^{n}h_i)$$

7.4 Self review

The above spin bases are obtained by unitary transformation for the corresponding isomomentum case. Except for the full symmetry breaking, all other properties are still satisfied. In fact its full symmetry still exists, but it is only hidden after unitary transformation. Essentially it still satisfies full symmetry, but it is only visually broken. The above spin bases can also be inverted to return to the isomomentum situation (especially in the z-axis direction). So this also indicates that the total spin of a multi-particle system is independent of the velocity of each particle and is entirely determined by the internal freedom degrees of the particle. After unitary transformation, only the probability is constant and real. But the physical image is not necessarily true, such as the momentum is changed. It can also be argued in another way that the momentum remains unchanged and the representation changes. But this is

not intuitive and natural. I prefer the former, which is more natural and intuitive in physics. It can be considered that the momentum in the expression at this time is the physical and real momentum. Therefore, the spin base at this time is also a physical and real spin base. These physical spin bases are a powerful mathematical tool for latter entanglement analysis. he spin transformation relationship in the above chapter represents a physical system as follows: Multiple different spin particles are combined to form the maximum total spin in a physical system. If the localization is small enough, then they are completely equivalent to a high spin particle. On the contrary, it is an entangled multi particles system. herefore, an entangled system is a state between a completely free system and a completely bound system. Perhaps various elementary particles are synthesized through quantum entanglement. his spin base no longer satisfies the original equation, but satisfies the new equation. But what does the new equation look like? It can be further studied. It perhaps implies new physical content.

8 General theory of spin coupling and CG coefficients [33, 39–41]

8.1 Single spin eigenstate

Axi. 8.1.1. $\begin{cases}\sigma(s)\times\sigma(s)=i\sigma(s),\sigma^2(s)=s(s+1)\\ \sigma^2(s)|s,m=s,\cdots,-s\rangle=s(s+1)|s,m=s,\cdots,-s\rangle\\ \sigma_z(s)|s,m=s,\cdots,-s\rangle=m|s,m=s,\cdots,-s\rangle\end{cases}$

Axi. 8.1.2. $\begin{cases}\hat{J}_k\times\hat{J}_k=i\hat{J}_k\\ \hat{J}_k^2|(j_k,m_k)\rangle=j_k(j_k+1)|(j_k,m_k)\rangle\\ J_{kz}|(j_k,m_k)\rangle=m_k|(j_k,m_k)\rangle\end{cases},\begin{cases}\hat{J}_k=\sigma(j_k)\\ |(j_k,m_k)\rangle\sim e^{(i\omega+\varsigma\varepsilon)\cdot\sigma(j_k)}\end{cases}$

Axi. 8.1.3. $\langle(j_k,m_k')|(j_k,m_k)\rangle=\delta_{m_k'm_k},\sum\limits_{m_k}|(j_k,m_k)\rangle\langle(j_k,m_k)|=1$

8.2 Multi spin coupling eigenstates

Def. 8.2.1. $|(j_1,m_1);\cdots;(j_n,m_n)\rangle:=|(j_1,m_1)\rangle\otimes\cdots\otimes|(j_n,m_n)\rangle$

Def. 8.2.2. $\hat{J}_k:=I_{2j_1+1}\otimes\cdots\otimes I_{2j_{k-1}+1}\otimes\sigma(j_k)\otimes I_{2j_{k+1}+1}\otimes\cdots\otimes I_{2j_n+1},\hat{J}=\sum\limits_{k=1}^{n}\hat{J}_k$

Def. 8.2.3. $\begin{cases}\hat{J}_k^2|(j_1,m_1);\cdots;(j_n,m_n)\rangle=j_k(j_k+1)|(j_1,m_1);\cdots;(j_n,m_n)\rangle\\ J_{kz}|(j_1,m_1);\cdots;(j_n,m_n)\rangle=m_k|(j_1,m_1);\cdots;(j_n,m_n)\rangle\end{cases}$

Def. 8.2.4. $\begin{cases}\hat{J}_k^2|j_1,j_2\cdots j_n;(j,m)\rangle=j_k(j_k+1)|j_1,j_2\cdots j_n;(j,m)\rangle\\ \hat{J}^2|j_1,j_2\cdots j_n;(j,m)\rangle=j(j+1)|j_1,j_2\cdots j_n;(j,m)\rangle\\ J_z|j_1,j_2\cdots j_n;(j,m)\rangle=m|j_1,j_2\cdots j_n;(j,m)\rangle\end{cases}$

8.3 Spin eigenstate expansion

Cor. 8.3.1. $\begin{cases}|j_1,j_2\cdots j_n;(j,m)\rangle=\sum\limits_{m_k}|(j_1,m_1);\cdots;(j_n,m_n)\rangle\langle(j_1,m_1);\cdots;(j_n,m_n)|j_1,j_2\cdots j_n;(j,m)\rangle\\ |j_1,j_2;(j,m)\rangle=\sum\limits_{m_k}|(j_1,m_1);(j_2,m_2)\rangle\langle(j_1,m_1);(j_2,m_2)|j_1,j_2;(j,m)\rangle\end{cases}$

$$[\Updownarrow]$$

Cor. 8.3.2. $\begin{cases}|(j_1,m_1);\cdots;(j_n,m_n)\rangle=\sum\limits_{m_k}|j_1,j_2\cdots j_n;(j,m)\rangle\langle j_1,j_2\cdots j_n;(j,m)|(j_1,m_1);\cdots;(j_n,m_n)\rangle\\ |(j_1,m_1);(j_2,m_2)\rangle=\sum\limits_{m_k}|j_1,j_2;(j,m)\rangle\langle j_1,j_2;(j,m)|(j_1,m_1);(j_2,m_2)\rangle\end{cases}$

8.4 Matrix transformation of spin eigenstate

Cor. 8.4.1. $\begin{cases}|j_1,j_2\cdots j_n;(j,m)\rangle=S_{1\cdots n}|(j_1,m_1);\cdots;(j_n,m_n)\rangle\\ |j_1,j_2;(j,m)\rangle=S_{12}|(j_1,m_1);(j_2,m_2)\rangle\end{cases}$

$$[\Updownarrow]$$

Cor. 8.4.2. $\begin{cases}|(j_1,m_1);\cdots;(j_n,m_n)\rangle=S_{1\cdots n}^{-1}|j_1,j_2\cdots j_n;(j,m)\rangle\\ |(j_1,m_1);(j_2,m_2)\rangle=S_{12}^{-1}|j_1,j_2;(j,m)\rangle\end{cases}$

8.5 Spin eigenstate orthogonality

Cor. 8.5.1. $\begin{cases}\langle (j_1,m_1');\cdots;(j_n,m_n')|(j_1,m_1);\cdots;(j_n,m_n)\rangle=\delta_{m_1'm_1}\cdot\cdot\delta_{m_n'm_n}\\ \langle (j_1,m_1');(j_2,m_2')|(j_1,m_1);(j_2,m_2)\rangle=\delta_{m_1'm_1}\delta_{m_2'm_2}\end{cases}$

$[\Updownarrow]$

Cor. 8.5.2. $\begin{cases}\langle j_1,j_2\cdots j_n;(j,m')|j_1,j_2\cdots j_n;(j,m)\rangle=\delta_{m'm}\\ \langle j_1,j_2;(j,m')|j_1,j_2;(j,m)\rangle=\delta_{m'm}\end{cases}$

8.6 Spin eigenstate completeness

Cor. 8.6.1. $\begin{cases}\sum\limits_{m_k}|(j_1,m_1);\cdots;(j_n,m_n)\rangle\langle (j_1,m_1);\cdots;(j_n,m_n)|=1\\ \sum\limits_{m_k}|(j_1,m_1);(j_2,m_2)\rangle\langle (j_1,m_1);(j_2,m_2)|=1\end{cases}$

$[\Updownarrow]$

Cor. 8.6.2. $\begin{cases}\sum\limits_{m_k}|j_1,j_2\cdots j_n;(j,m)\rangle\langle j_1,j_2\cdots j_n;(j,m)|=1\\ \sum\limits_{m_k}|j_1,j_2;(j,m)\rangle\langle j_1,j_2;(j,m)|=1\end{cases}$

8.7 General Racah formula for coupling CG coefficients of two angular momentum

Thm. 8.7.1. $|j_1,j_2;(j_3,m_3)\rangle=\sum\limits_{m_1=j_1}^{-j_1}\sum\limits_{m_2=j_2}^{-j_2}|(j_1,m_1);(j_2,m_2)\rangle\langle (j_1,m_1);(j_2,m_2)|j_1,j_2;(j_3,m_3)\rangle$

$CG_{Racah}=\langle (j_1,m_1);(j_2,m_2)|j_1,j_2;(j_3,m_3)\rangle=\delta(m_1+m_2-m_3)$
$\{(2j_3+1)\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!\}^{1/2}$
$[\sum\limits_r(-1)^r r!(j_1+j_2-j_3-r)!(j_1-m_1-r)!(j_3-j_1-m_2+r)!(j_2+m_2-r)!(j_3-j_2+m_1+r)!]^{-1}$

8.8 Racah formula of CG coefficients for synthesizing two particles into one particle

Thm. 8.8.1. $|j_1,j_2;(j_1+j_2,m_3)\rangle=\sum\limits_{m_1=j_1}^{-j_1}\sum\limits_{m_2=j_2}^{-j_2}|(j_1,m_1);(j_2,m_2)\rangle\langle (j_1,m_1);(j_2,m_2)|j_1,j_2;(j_1+j_2,m_3)\rangle$

$\langle (j_1,m_1);(j_2,m_2)|j_1,j_2;(j_1+j_2,m_3)\rangle=\delta(m_1+m_2-m_3)\{\frac{(2j_1)!(2j_2)!(j_1+j_2+m_3)!(j_1+j_2-m_3)!}{(2j_1+2j_2)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\}^{1/2}$

Proof: $\langle (j_1,m_1);(j_2,m_2)|j_1,j_2;(j_1+j_2,m_3)\rangle$
$=\delta(m_1+m_2-m_3)\{\frac{(2j_1)!(2j_2)!}{(2j_1+2j_2)!}(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!\}^{1/2}$
$[(j_1-m_1)!(j_2-m_2)!(j_2+m_2)!(j_1+m_1)!]^{-1}$
$=\delta(m_1+m_2-m_3)\{\frac{(2j_1)!(2j_2)!(j_1+j_2+m_3)!(j_1+j_2-m_3)!}{(2j_1+2j_2)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\}^{1/2}$ □

Cor. 8.8.1. $|n,1;(n+1,m_3)\rangle=\sum\limits_{m_1=n}^{-n}\sum\limits_{m_2=1}^{-1}|(n,m_1);(1,m_2)\rangle\langle (n,m_1);(1,m_2)|n,1;(n+1,m_3)\rangle$

$\langle (n,m_1);(1,m_2)|n,1;(n+1,m_3)\rangle$
$=\delta(m_1+m_2-m_3)\{\frac{(2n)!2!(n+1+m_3)!(n+1-m_3)!}{(2n+2)!(n+m_1)!(n-m_1)!(1+m_2)!(1-m_2)!}\}^{1/2}=\delta(m_1+m_2-m_3)\{\frac{2!C_{2n}^{n-m_1}}{(1+m_2)!(1-m_2)!C_{2n+2}^{n+1-m_3}}\}^{1/2}$

Cor. 8.8.2.

$$\begin{cases}|n,1;(n+1,n+1)\rangle=\frac{\sqrt{C_{2n}^0}}{\sqrt{C_{2n+2}^0}}|(n,n);(1,1)\rangle\\ |n,1;(n+1,n)\rangle=\frac{\sqrt{C_{2n}^1}}{\sqrt{C_{2n+2}^1}}|(n,n-1);(1,1)\rangle+\frac{\sqrt{2C_{2n}^0}}{\sqrt{C_{2n+2}^1}}|(n,n);(1,0)\rangle\\ |n,1;(n+1,n-1)\rangle=\frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n+2}^2}}|(n,n-2);(1,1)\rangle+\frac{\sqrt{2C_{2n}^1}}{\sqrt{C_{2n+2}^2}}|(n,n-1);(1,0)\rangle+\frac{\sqrt{C_{2n}^0}}{\sqrt{C_{2n+2}^2}}|(n,n);(1,-1)\rangle\\ |n,1;(n+1,n-2)\rangle=\frac{\sqrt{C_{2n}^3}}{\sqrt{C_{2n+2}^3}}|(n,n-3);(1,1)\rangle+\frac{\sqrt{2C_{2n}^2}}{\sqrt{C_{2n+2}^3}}|(n,n-2);(1,0)\rangle+\frac{\sqrt{C_{2n}^1}}{\sqrt{C_{2n+2}^3}}|(n,n-1);(1,-1)\rangle\\ |n,1;(n+1,n+1-l)\rangle=\frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+2}^l}}|(n,n-l);(1,1)\rangle+\frac{\sqrt{2C_{2n}^{l-1}}}{\sqrt{C_{2n+2}^l}}|(n,n+1-l);(1,0)\rangle+\frac{\sqrt{C_{2n}^{l-2}}}{\sqrt{C_{2n+2}^l}}|(n,n+2-l);(1,-1)\rangle\end{cases}$$

8.9 Concrete expression of CG coefficients for synthesizing two particles into one Particle

Thm. 8.9.1. $\langle (j_1,m_1);(j_2,m_2)|j_1,j_2;(j_1+j_2,m_3)\rangle=\delta(m_1+m_2-m_3)\{\frac{(2j_1)!(2j_2)!(j_1+j_2+m_3)!(j_1+j_2-m_3)!}{(2j_1+2j_2)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\}^{1/2}$
$=\delta(m_1+m_2-m_3)\{\frac{C_{2j_1}^{2j_1}}{C_{2j_1+2j_2}^{2j_2}}\}^{1/2}$

Cor. 8.9.1.

$$\begin{cases}\langle(j_1,j_1);(j_2,j_2)|j_1,j_2;(j_1+j_2,j_1+j_2)\rangle=1\\ \langle(j_1,j_1);(j_2,j_2-1)|j_1,j_2;(j_1+j_2,j_1+j_2-1)\rangle=\frac{\sqrt{2j_2}}{\sqrt{2j_1+2j_2}}\\ \langle(j_1,j_1-1);(j_2,j_2)|j_1,j_2;(j_1+j_2,j_1+j_2-1)\rangle=\frac{\sqrt{2j_1}}{\sqrt{2j_1+2j_2}}\\ \langle(j_1,j_1);(j_2,j_2-2)|j_1,j_2;(j_1+j_2,j_1+j_2-2)\rangle=\frac{\sqrt{2j_2(2j_2-1)}}{\sqrt{(2j_1+2j_2)(2j_1+2j_2-1)}}\\ \langle(j_1,j_1-1);(j_2,j_2-1)|j_1,j_2;(j_1+j_2,j_1+j_2-2)\rangle=\frac{2\sqrt{2j_1j_2}}{\sqrt{(2j_1+2j_2)(2j_1+2j_2-1)}}\\ \langle(j_1,j_1-2);(j_2,j_2)|j_1,j_2;(j_1+j_2,j_1+j_2-2)\rangle=\frac{\sqrt{2j_1(2j_1-1)}}{\sqrt{(2j_1+2j_2)(2j_1+2j_2-1)}}\\ \cdots\cdots\cdots\\ \langle(j_1,-j_1);(j_2,-j_2+2)|j_1,j_2;(j_1+j_2,-j_1-j_2+2)\rangle=\frac{\sqrt{2j_2(2j_2-1)}}{\sqrt{(2j_1+2j_2)(2j_1+2j_2-1)}}\\ \langle(j_1,-j_1+1);(j_2,-j_2+1)|j_1,j_2;(j_1+j_2,-j_1-j_2+2)\rangle=\frac{2\sqrt{2j_1j_2}}{\sqrt{(2j_1+2j_2)(2j_1+2j_2-1)}}\\ \langle(j_1,-j_1+2);(j_2,-j_2)|j_1,j_2;(j_1+j_2,-j_1-j_2+2)\rangle=\frac{\sqrt{2j_1(2j_1-1)}}{\sqrt{(2j_1+2j_2)(2j_1+2j_2-1)}}\\ \langle(j_1,-j_1);(j_2,-j_2+1)|j_1,j_2;(j_1+j_2,-j_1-j_2+1)\rangle=\frac{\sqrt{2j_2}}{\sqrt{2j_1+2j_2}}\\ \langle(j_1,-j_1+1);(j_2,-j_2)|j_1,j_2;(j_1+j_2,-j_1-j_2+1)\rangle=\frac{\sqrt{2j_1}}{\sqrt{2j_1+2j_2}}\\ \langle(j_1,-j_1);(j_2,-j_2)|j_1,j_2;(j_1+j_2,-j_1-j_2)\rangle=1\end{cases}$$

Cor. 8.9.2. $\langle(1,m_1);(1,m_2)|1,1;(2,m_3)\rangle=\delta(m_1+m_2-m_3)\{\frac{2!2!(2+m_3)!(2-m_3)!}{4!(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!}\}^{1/2}$

Cor. 8.9.3.

$$\begin{cases}\langle(1,1);(1,1)|1,1;(2,2)\rangle=1\\ \langle(1,1);(1,0)|1,1;(2,1)\rangle=\frac{1}{\sqrt2},\langle(1,0);(1,1)|1,1;(2,1)\rangle=\frac{1}{\sqrt2}\\ \langle(1,1);(1,-1)|1,1;(2,0)\rangle=\frac{1}{\sqrt6},\langle(1,0);(1,0)|1,1;(2,0)\rangle=\frac{2}{\sqrt6},\langle(1,-1);(1,1)|1,1;(2,0)\rangle=\frac{1}{\sqrt6}\\ \langle(1,-1);(1,0)|1,1;(2,-1)\rangle=\frac{1}{\sqrt2},\langle(1,0);(1,-1)|1,1;(2,-1)\rangle=\frac{1}{\sqrt2}\\ \langle(1,-1);(1,-1)|1,1;(2,-2)\rangle=1\end{cases}$$

Cor. 8.9.4. $\langle(2,m_1);(1,m_2)|2,1;(3,m_3)\rangle=\delta(m_1+m_2-m_3)\{\frac{4!2!(3+m_3)!(3-m_3)!}{6!(2+m_1)!(2-m_1)!(1+m_2)!(1-m_2)!}\}^{1/2}$

Cor. 8.9.5.

$$\begin{cases}\langle(2,2);(1,1)|2,1;(3,3)\rangle=1\\ \langle(2,2);(1,0)|2,1;(3,2)\rangle=\frac{1}{\sqrt3},\langle(2,1);(1,1)|2,1;(3,2)\rangle=\frac{\sqrt2}{\sqrt3}\\ \langle(2,2);(1,-1)|2,1;(3,1)\rangle=\frac{1}{\sqrt{15}},\langle(2,1);(1,0)|2,1;(3,1)\rangle=\frac{\sqrt8}{\sqrt{15}},\langle(2,0);(1,1)|2,1;(3,1)\rangle=\frac{\sqrt6}{\sqrt{15}}\\ \langle(2,1);(1,-1)|2,1;(3,0)\rangle=\frac{1}{\sqrt5},\langle(2,0);(1,0)|2,1;(3,0)\rangle=\frac{\sqrt3}{\sqrt{15}},\langle(2,-1);(1,1)|2,1;(3,0)\rangle=\frac{1}{\sqrt5}\\ \langle(2,-2);(1,1)|2,1;(3,-1)\rangle=\frac{1}{\sqrt{15}},\langle(2,-1);(1,0)|2,1;(3,-1)\rangle=\frac{\sqrt8}{\sqrt{15}},\langle(2,0);(1,-1)|2,1;(3,-1)\rangle=\frac{\sqrt6}{\sqrt{15}}\\ \langle(2,-2);(1,0)|2,1;(3,-2)\rangle=\frac{1}{\sqrt3},\langle(2,-1);(1,-1)|2,1;(3,-2)\rangle=\frac{\sqrt2}{\sqrt3}\\ \langle(2,-2);(1,-1)|2,1;(3,-3)\rangle=1\end{cases}$$

9 CG coefficients formula for synthesizing multiple particles into one particle [33, 39–41]

9.1 CG coefficients formula for synthesizing multiple photons into a particle

Lem. 9.1.1. $|(n+1,m_3)\rangle=|n,1;(n+1,m_3)\rangle=\sum_{m_1=n}^{-n}\sum_{m_2=1}^{-1}|(n,m_1);(1,m_2)\rangle\langle(n,m_1);(1,m_2)|n,1;(n+1,m_3)\rangle$

$=\sum_{m_1=n}^{-n}\sum_{m_2=1}^{-1}\delta(m_1+m_2-m_3)\{\frac{(2n)!2!(n+1+m_3)!(n+1-m_3)!}{(2n+2)!(n+m_1)!(n-m_1)!(1+m_2)!(1-m_2)!}\}^{1/2}|(n,m_1);(1,m_2)\rangle$

Thm. 9.1.1. $|\overbrace{1,\cdot\cdot,1}^{n+1};(n+1,m_{n+1})\rangle=\sum_{m_n=n}^{-n}\cdot\cdot\sum_{m_1=1}^{-1}\sum_{l_1,\cdot\cdot,l_n=1}^{-1}|(1,m_1);(1,l_n);\cdot\cdot;(1,l_1)\rangle$

$\delta(m_n+l_1-m_{n+1})\cdots\delta(m_1+l_n-m_2)\{\frac{2!^{n+1}}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)!\cdot\cdot(1+l_n)!(1-l_n)!}\}^{1/2}$

Proof: $|\overbrace{1,\cdot\cdot,1}^{n+1};(n+1,m_{n+1})\rangle=\sum_{m_n=n}^{-n}\sum_{l_1=1}^{-1}|(n,m_n);(1,l_1)\rangle\langle(n,m_n);(1,l_1)|n,1;(n+1,m_{n+1})\rangle$

$=\sum_{m_n=n}^{-n}\sum_{l_1=1}^{-1}\delta(m_n+l_1-m_{n+1})\{\frac{(2n)!2!(n+1+m_{n+1})!(n+1-m_{n+1})!}{(2n+2)!(n+m_n)!(n-m_n)!(1+l_1)!(1-l_1)!}\}^{1/2}|(n,m_n);(1,l_1)\rangle$

$= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \sum_{l_1,l_2=1}^{-1} |(n-1, m_{n-1}); (1, l_2); (1, l_1)\rangle$
$\delta(m_n + l_1 - m_{n+1})\delta(m_{n-1} + l_2 - m_n)$
$\{\frac{(2n)!2!(n+1+m_{n+1})!(n+1-m_{n+1})!}{(2n+2)!(n+m_n)!(n-m_n)!(1+l_1)!(1-l_1)!}\}^{1/2}\{\frac{(2n-2)!2!(n+m_n)!(n-m_n)!}{(2n)!(n-1+m_{n-1})!(n-1-m_{n-1})!(1+l_2)!(1-l_2)!}\}^{1/2}$
$= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \sum_{l_1,l_2=1}^{-1} |(n-1, m_{n-1}); (1, l_2); (1, l_1)\rangle\delta(m_n + l_1 - m_{n+1})\delta(m_{n-1} + l_2 - m_n)\delta(m_{n-2} + l_3 - m_{n-1})$
$\{\frac{(2n)!2!(n+1+m_{n+1})!(n+1-m_{n+1})!}{(2n+2)!(n+m_n)!(n-m_n)!(1+l_1)!(1-l_1)!}\}^{1/2}\{\frac{(2n-2)!2!(n+m_n)!(n-m_n)!}{(2n)!(n-1+m_{n-1})!(n-1-m_{n-1})!(1+l_2)!(1-l_2)!}\}^{1/2}$
$\{\frac{(2n-4)!2!(n-1+m_{n-1})!(n-1-m_{n-1})!}{(2n-2)!(n-2+m_{n-2})!(n-2-m_{n-2})!(1+l_3)!(1-l_3)!}\}^{1/2}$
$= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \sum_{m_{n-2}=n-2}^{-(n-2)} \sum_{l_1,l_2=1}^{-1} |(n-2, m_{n-2}); (1, l_3); (1, l_2); (1, l_1)\rangle$
$\delta(m_n + l_1 - m_{n+1})\delta(m_{n-1} + l_2 - m_n)\delta(m_{n-2} + l_3 - m_{n-1})$
$\{\frac{(2n-4)!2!^3}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(n-2+m_{n-2})!(n-2-m_{n-2})!}\}^{1/2}\{\frac{1}{(1+l_1)!(1-l_1)!(1+l_2)!(1-l_2)!(1+l_3)!(1-l_3)!}\}^{1/2}$
$= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \cdot\cdot \sum_{m_1=1}^{-1} \sum_{l_1,\cdot\cdot,l_n=1}^{-1} |(1, m_1); (1, l_n); \cdot\cdot; (1, l_1)\rangle$
$\delta(m_n + l_1 - m_{n+1})\delta(m_{n-1} + l_2 - m_n)\cdots\delta(m_1 + l_n - m_2)\{\frac{2!^{n+1}}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)!\cdot\cdot(1+l_n)!(1-l_n)!}\}^{1/2}$
$= \sum_{m_1=1}^{-1} \sum_{l_1,\cdot\cdot,l_n=1}^{-1} |(1, m_1); (1, l_n); \cdot\cdot; (1, l_1)\rangle$
$\delta(m_{n+1} - m_1 - l_1 \cdots - l_n)\{\frac{2!^{n+1}}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)!\cdot\cdot(1+l_n)!(1-l_n)!}\}^{1/2}$ □

Cor. 9.1.1. $|\overbrace{1, \cdot\cdot, 1}^{n+1}; (n+1, m_{n+1})\rangle = \sum_{m_n=n}^{-n} \cdot\cdot \sum_{m_1=1}^{-1} \sum_{l_1,\cdot\cdot,l_n=1}^{-1} |(1, m_1); (1, l_n); \cdot\cdot; (1, l_1)\rangle$
$\delta(m_n + l_1 - m_{n+1})\cdots\delta(m_1 + l_n - m_2)\{\frac{2!^{n+1}}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)!\cdot\cdot(1+l_n)!(1-l_n)!}\}^{1/2}$
$= \sum_{l_1,\cdot\cdot,l_n=1}^{-1} |(1, m_1); (1, l_1); \cdot\cdot; (1, l_n)\rangle\{\frac{2!^{n+1}}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)!\cdot\cdot(1+l_n)!(1-l_n)!}\}^{1/2}; m_1 = m_{n+1} - \sum_{i=1}^{n} l_i$
$= \frac{\sqrt{(2!)^{n+1}}}{\sqrt{(2n+2)!}} \sum_{h_2=1}^{-1} \cdot\cdot \sum_{h_{n+1}=1}^{-1} |(1, h_1); \cdot\cdot; (1, h_{n+1})\rangle \frac{\sqrt{(n+1+h)!}}{\sqrt{(1+h_1)!\cdot\cdot(1+h_{n+1})!}} \frac{\sqrt{(n+1-h)!}}{\sqrt{(1-h_1)!\cdot\cdot(1-h_{n+1})!}}; h = m_{n+1}, h_1 = h - \sum_{i=2}^{n+1} h_i$

Proof: $|\overbrace{1, \cdot\cdot, 1}^{n+1}; (n+1, m_{n+1})\rangle = \sum_{m_n=n}^{-n} \cdot\cdot \sum_{m_1=1}^{-1} \sum_{l_1,\cdot\cdot,l_n=1}^{-1} |(1, m_1); (1, l_n); \cdot\cdot; (1, l_1)\rangle$
$\delta(m_n + l_1 - m_{n+1})\cdots\delta(m_1 + l_n - m_2)\{\frac{2!^{n+1}}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)!\cdot\cdot(1+l_n)!(1-l_n)!}\}^{1/2}$
$= \sum_{l_1,\cdot\cdot,l_n=1}^{-1} |(1, m_1); (1, l_1); \cdot\cdot; (1, l_n)\rangle\{\frac{2!^{n+1}}{(2n+2)!}\}^{1/2}\{\frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)!\cdot\cdot(1+l_n)!(1-l_n)!}\}^{1/2}; m_1 = m_{n+1} - \sum_{i=1}^{n} l_i$
$= \frac{\sqrt{(2!)^{n+1}}}{\sqrt{(2n+2)!}} \sum_{h_2=1}^{-1} \cdot\cdot \sum_{h_{n+1}=1}^{-1} |(1, h_1); \cdot\cdot; (1, h_{n+1})\rangle \frac{\sqrt{(n+1+h)!}}{\sqrt{(1+h_1)!\cdot\cdot(1+h_{n+1})!}} \frac{\sqrt{(n+1-h)!}}{\sqrt{(1-h_1)!\cdot\cdot(1-h_{n+1})!}}; h = m_{n+1}, h_1 = h - \sum_{i=2}^{n+1} h_i$

$$\begin{cases} m_n + l_1 - m_{n+1} = 0 \\ m_{n-1} + l_2 - m_n = 0 \\ \cdots \\ m_2 + l_{n-1} - m_3 = 0 \\ m_1 + l_n - m_2 = 0 \end{cases} \Leftrightarrow \begin{cases} m_n = m_{n+1} - \sum_{i=1}^{1} l_i \\ \cdots \\ m_3 = m_{n+1} - \sum_{i=1}^{n-2} l_i \\ m_2 = m_{n+1} - \sum_{i=1}^{n-1} l_i \\ m_1 = m_{n+1} - \sum_{i=1}^{n} l_i \end{cases}$$ □

The above proofs are initially mainly solved through mental thinking and concrete verification. Sometimes, although it is possible to think clearly through the arrangement and combination, but it is difficult to write them down clearly and clearly. Later, I have found a more rigorous method to prove it.

9.2 CG coefficients formula for multiple angular momentum coupling in special cases

Can the CG coefficients of multiple angular momentum couplings be obtained in principle by repeatly using the Racah formula? It seems only feasible in special single particle cases. The most common case has already been solved by Racah and Wigner. The formula is complex and physically inconvenient to use. Therefore, in some cases, it is still necessary to regain convenient expressions.

Def. 9.2.1. $j_{k+1} = n_1 + \cdot\cdot + n_k + n_{k+1}, j_k = n_2 + \cdot\cdot + n_k + n_{k+1}, \cdots, j_2 = n_k + n_{k+1}, j_1 = n_{k+1},$

Lem. 9.2.1.
$\langle (j_{k-1}, m_{k-1}); (n_2, l_2) | j_{k-1}, n_2; (j_k, m_k) \rangle = \delta(m_{k-1} + l_k - m_k) \{ \frac{(2j_{k-1})!(2n_2)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!(n_2+l_2)!(n_2-l_2)!} \}^{1/2}$

Thm. 9.2.1. $|j_1, n_k, \cdot\cdot, n_2, n_1; (j,m)\rangle = |j_k, n_1; (j,m)\rangle = |(j,m)\rangle$
$= \sum_{m_k=j_k}^{-j_k} \cdot\cdot \sum_{m_1=j_1}^{-j_1} \sum_{l_1=n_1}^{-n_1} \cdot\cdot \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \cdot\cdot; (n_2, l_2); (n_1, l_1)\rangle$
$\delta(m_k + l_1 - m)\delta(m_{k-1} + l_2 - m_k) \cdot \cdot \delta(m_1 + l_k - m_2)$
$\{(2j+1)\frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!}(j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)!\}^{1/2}$
$[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1}$
$\{\frac{(2j_1)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_1+m_1)!(j_1-m_1)!}\frac{(2n_2)!\cdot\cdot(2n_k)!}{(n_2+l_2)!(n_2-l_2)!\cdot\cdot(n_k+l_k)!(n_k-l_k)!}\}^{1/2}$

Proof: $|j_k, n_1; (j,m)\rangle = \sum_{m_k=j_k}^{-j_k} \sum_{l_1=n_1}^{-n_1} |(j_k, m_k); (n_1, l_1)\rangle \langle (j_k, m_k); (n_1, l_1) | j_k, n_1; (j,m)\rangle$
$= \sum_{m_k=j_k}^{-j_k} \sum_{l_1=n_1}^{-n_1} |(j_k, m_k); (n_1, l_1)\rangle \delta(m_k + l_1 - m)$
$\{(2j+1)\frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!}(j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)!\}^{1/2}$
$[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1}$
$= \sum_{m_k=j_k}^{-j_k} \sum_{m_{k-1}=j_{k-1}}^{-j_{k-1}} \sum_{l_1=n_1}^{-n_1} \sum_{l_2=n_2}^{-n_2} |(j_{k-1}, m_{k-1}); (n_2, l_2); (n_1, l_1)\rangle \delta(m_k + l_1 - m)\delta(m_{k-1} + l_2 - m_k)$
$\{(2j+1)\frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!}(j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)!\}^{1/2}$
$[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1}$
$\{\frac{(2j_{k-1})!(2n_2)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!(n_2+l_2)!(n_2-l_2)!}\}^{1/2}$
$= \sum_{m_k=j_k}^{-j_k} \sum_{m_{k-1}=j_{k-1}}^{-j_{k-1}} \sum_{m_{k-2}=j_{k-2}}^{-j_{k-2}} \sum_{l_1=n_1}^{-n_1} \sum_{l_2=n_2}^{-n_2} \sum_{l_3=n_3}^{-n_3} |j_{k-2}, m_{k-2}; (n_3, l_3); (n_2, l_2); (n_1, l_1)\rangle$
$\delta(m_k + l_1 - m)\delta(m_{k-1} + l_2 - m_k)\delta(m_{k-2} + l_3 - m_{k-1})$
$\{(2j+1)\frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!}(j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)!\}^{1/2}$
$[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1}$
$\{\frac{(2j_{k-1})!(2n_2)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!(n_2+l_2)!(n_2-l_2)!}\}^{1/2}\{\frac{(2j_{k-2})!(2n_3)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!}{(2j_{k-1})!(j_{k-2}+m_{k-2})!(j_{k-2}-m_{k-2})!(n_3+l_3)!(n_3-l_3)!}\}^{1/2}$
$= \sum_{m_k=j_k}^{-j_k} \sum_{m_{k-1}=j_{k-1}}^{-j_{k-1}} \sum_{m_{k-2}=j_{k-2}}^{-j_{k-2}} \sum_{l_1=n_1}^{-n_1} \sum_{l_2=n_2}^{-n_2} \sum_{l_3=n_3}^{-n_3} |j_{k-2}, m_{k-2}; (n_3, l_3); (n_2, l_2); (n_1, l_1)\rangle$
$\delta(m_k + l_1 - m)\delta(m_{k-1} + l_2 - m_k)\delta(m_{k-2} + l_3 - m_{k-1})$
$\{(2j+1)\frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!}(j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)!\}^{1/2}$
$[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1}$
$\{\frac{(2j_{k-2})!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-2}+m_{k-2})!(j_{k-2}-m_{k-2})!}\frac{(2n_2)!(2n_3)!}{(n_2+l_2)!(n_2-l_2)!(n_3+l_3)!(n_3-l_3)!}\}^{1/2}$
$= \sum_{m_k=j_k}^{-j_k} \cdot\cdot \sum_{m_1=j_1}^{-j_1} \sum_{l_1=n_1}^{-n_1} \cdot\cdot \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \cdot\cdot; (n_2, l_2); (n_1, l_1)\rangle$
$\delta(m_k + l_1 - m)\delta(m_{k-1} + l_2 - m_k) \cdot \cdot \delta(m_1 + l_k - m_2)$
$\{(2j+1)\frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!}(j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)!\}^{1/2}$
$[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1}$
$\{\frac{(2j_1)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_1+m_1)!(j_1-m_1)!}\frac{(2n_2)!\cdot\cdot(2n_k)!}{(n_2+l_2)!(n_2-l_2)!\cdot\cdot(n_k+l_k)!(n_k-l_k)!}\}^{1/2}$ □

9.3 CG coefficients formula for synthesizing multiple particles into one particle

Thm. 9.3.1. $|j_1, n_k, \cdot\cdot, n_1; (j_{k+1}, m_{k+1})\rangle ? = |(j_{k+1}, m_{k+1})\rangle$
$= \sum_{m_k=j_k}^{-j_k} \cdot\cdot \sum_{m_1=j_1}^{-j_1} \sum_{l_1=n_1}^{-n_1} \cdot\cdot \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \cdot\cdot; (n_1, l_1)\rangle$
$\delta(m_k + l_1 - m_{k+1})\delta(m_{k-1} + l_2 - m_k) \cdot \cdot \delta(m_1 + l_k - m_2)\{\frac{(2j_1)!(j_{k+1}+m_{k+1})!(j_{k+1}-m_{k+1})!}{(2j_{k+1})!(j_1+m_1)!(j_1-m_1)!}\frac{(2n_1)!\cdot\cdot(2n_k)!}{(n_1+l_1)!(n_1-l_1)!\cdot\cdot(n_k+l_k)!(n_k-l_k)!}\}^{1/2}$
$= \sum_{l_1=n_1}^{-n_1} \cdot\cdot \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \cdot\cdot; (n_1, l_1)\rangle \{\frac{(2j_1)!(2n_1)!\cdot\cdot(2n_k)!}{(2j_{k+1})!(j_1+m_1)!(j_1-m_1)!}\frac{(j_{k+1}+m_{k+1})!(j_{k+1}-m_{k+1})!}{(n_1+l_1)!(n_1-l_1)!\cdot\cdot(n_k+l_k)!(n_k-l_k)!}\}^{1/2}$
$; m_1 = m_{k+1} - \sum_{i=1}^{k} l_i$

$$\begin{cases} m_k + l_1 - m_{k+1} = 0 \\ m_{k-1} + l_2 - m_k = 0 \\ \cdots \\ m_2 + l_{k-1} - m_3 = 0 \\ m_1 + l_k - m_2 = 0 \end{cases} \Leftrightarrow \begin{cases} m_k = m_{k+1} - \sum_{i=1}^{1} l_i \\ \cdots \\ m_3 = m_{k+1} - \sum_{i=1}^{k-2} l_i \\ m_2 = m_{k+1} - \sum_{i=1}^{k-1} l_i \\ m_1 = m_{k+1} - \sum_{i=1}^{k} l_i \end{cases}$$

The CG coefficients formula for synthesizing multiple photons into a particle is only a special case of it.

10 Some special cases for CG coefficients of two angular momentum coupling

10.1 Special case: $\frac{1}{2} \oplus \frac{1}{2} = (0,0)$

Thm. 10.1.1. $|j_1, j_2; (j_3, m_3)\rangle = \sum_{m_1=j_1}^{-j_1} \sum_{m_2=j_2}^{-j_2} |(j_1, m_1); (j_2, m_2)\rangle\langle(j_1, m_1); (j_2, m_2)|j_1, j_2; (j_3, m_3)\rangle$

$CG_{Racah} = \langle(j_1, m_1); (j_2, m_2)|j_1, j_2; (j_3, m_3)\rangle = \delta(m_1 + m_2 - m_3)$

$\{(2j_3+1)\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)!\}^{1/2}$

$[\sum_r (-1)^r r!(j_1+j_2-j_3-r)!(j_1-m_1-r)!(j_3-j_1-m_2+r)!(j_2+m_2-r)!(j_3-j_2+m_1+r)!]^{-1}$

Cor. 10.1.1. $|j_1, \frac{1}{2}; (j_3, m_3)\rangle = \sum_{m_1=j_1}^{-j_1} \sum_{m_2=\frac{1}{2}}^{-\frac{1}{2}} |(j_1, m_1); (\frac{1}{2}, m_2)\rangle\langle(j_1, m_1); (\frac{1}{2}, m_2)|j_1, \frac{1}{2}; (j_3, m_3)\rangle$

$CG_{Racah} = \langle(j_1, m_1); (\frac{1}{2}, m_2)|j_1, \frac{1}{2}; (j_3, m_3)\rangle = \delta(m_1 + m_2 - m_3)$

$\{(2j_3+1)\frac{(j_1+\frac{1}{2}-j_3)!(j_1-\frac{1}{2}+j_3)!(-j_1+\frac{1}{2}+j_3)!}{(j_1+\frac{1}{2}+j_3+1)!}(j_1+m_1)!(j_1-m_1)!(j_3+m_3)!(j_3-m_3)!\}^{1/2}$

$[\sum_r (-1)^r r!(j_1+\frac{1}{2}-j_3-r)!(j_1-m_1-r)!(j_3-j_1-m_2+r)!(\frac{1}{2}+m_2-r)!(j_3-\frac{1}{2}+m_1+r)!]^{-1}$

Cor. 10.1.2.

$$\begin{cases} CG_{Racah} = \langle(j_1, m_1); (\frac{1}{2}, \frac{1}{2})|j_1, \frac{1}{2}; (j_1+\frac{1}{2}, m_3)\rangle = \delta(m_1+\frac{1}{2}-m_3)\{\frac{j_1+\frac{1}{2}+m_3}{2j_1+1}\}^{1/2} \\ CG_{Racah} = \langle(j_1, m_1); (\frac{1}{2}, -\frac{1}{2})|j_1, \frac{1}{2}; (j_1+\frac{1}{2}, m_3)\rangle = \delta(m_1-\frac{1}{2}-m_3)\{\frac{j_1+\frac{1}{2}-m_3}{2j_1+1}\}^{1/2} \end{cases}$$

$$\begin{cases} CG_{Racah} = \langle(j_1, m_1); (\frac{1}{2}, \frac{1}{2})|j_1, \frac{1}{2}; (j_1-\frac{1}{2}, m_3)\rangle = -\delta(m_1+\frac{1}{2}-m_3)\{\frac{j_1+\frac{1}{2}-m_3}{2j_1+1}\}^{1/2} \\ CG_{Racah} = \langle(j_1, m_1); (\frac{1}{2}, -\frac{1}{2})|j_1, \frac{1}{2}; (j_1-\frac{1}{2}, m_3)\rangle = \delta(m_1-\frac{1}{2}-m_3)\{\frac{j_1+\frac{1}{2}+m_3}{2j_1+1}\}^{1/2} \end{cases}$$

Thm. 10.1.2. $|\frac{1}{2}, \frac{1}{2}; (0,0)\rangle = \sum_{m_1=1/2}^{-1/2} \sum_{m_2=1/2}^{-1/2} |(\frac{1}{2}, m_1); (\frac{1}{2}, m_2)\rangle\langle(\frac{1}{2}, m_1); (\frac{1}{2}, m_2)|\frac{1}{2}, \frac{1}{2}; (0,0)\rangle$

$CG_{Racah} = \langle(\frac{1}{2}, m_1); (\frac{1}{2}, m_2)|\frac{1}{2}, \frac{1}{2}; (0,0)\rangle = \delta(m_1 + m_2 - 0)$

$\{(2\cdot 0+1)\frac{(\frac{1}{2}+\frac{1}{2}-0)!(\frac{1}{2}-\frac{1}{2}+0)!(-\frac{1}{2}+\frac{1}{2}+0)!}{(\frac{1}{2}+\frac{1}{2}+0+1)!}(\frac{1}{2}+m_1)!(\frac{1}{2}-m_1)!(\frac{1}{2}+m_2)!(\frac{1}{2}-m_2)!(0+0)!(0-0)!\}^{1/2}$

$[\sum_r (-1)^r r!(\frac{1}{2}+\frac{1}{2}-0-r)!(\frac{1}{2}-m_1-r)!(0-\frac{1}{2}-m_2+r)!(\frac{1}{2}+m_2-r)!(0-\frac{1}{2}+m_1+r)!]^{-1}$

$= \delta(m_1+m_2)\{\frac{1}{2!}(\frac{1}{2}+m_1)!(\frac{1}{2}-m_1)!(\frac{1}{2}+m_2)!(\frac{1}{2}-m_2)!\}^{1/2}$

$[\sum_r (-1)^r r!(1-r)!(\frac{1}{2}-m_1-r)!(-\frac{1}{2}-m_2+r)!(\frac{1}{2}+m_2-r)!(-\frac{1}{2}+m_1+r)!]^{-1}$

$= \delta(m_1+m_2)\frac{1}{\sqrt{2!}}(\frac{1}{2}+m_1)!(\frac{1}{2}-m_1)![\sum_r (-1)^r r!(1-r)![(\frac{1}{2}-m_1-r)!]^2[(-\frac{1}{2}+m_1+r)!]^2]^{-1}$

Cor. 10.1.3.

$$\begin{cases} \langle(\frac{1}{2}, \frac{1}{2}); (\frac{1}{2}, -\frac{1}{2})|\frac{1}{2}, \frac{1}{2}; (0,0)\rangle = \frac{1}{\sqrt{2}} \\ \langle(\frac{1}{2}, -\frac{1}{2}); (\frac{1}{2}, \frac{1}{2})|\frac{1}{2}, \frac{1}{2}; (0,0)\rangle = -\frac{1}{\sqrt{2}} \end{cases} \quad |\frac{1}{2}, \frac{1}{2}; (0,0)\rangle = \frac{1}{\sqrt{2}}|(\frac{1}{2}, \frac{1}{2}); (\frac{1}{2}, -\frac{1}{2})\rangle - \frac{1}{\sqrt{2}}|(\frac{1}{2}, -\frac{1}{2}); (\frac{1}{2}, \frac{1}{2})\rangle$$

10.2 Special case: $1 \oplus 1 = (0,0)$

Thm. 10.2.1. $|1, 1; (0,0)\rangle = \sum_{m_1=1}^{-1} \sum_{m_2=1}^{-1} |(1, m_1); (1, m_2)\rangle\langle(1, m_1); (1, m_2)|1, 1; (0,0)\rangle$

$CG_{Racah} = \langle(1, m_1); (1, m_2)|1, 1; (0,0)\rangle = \delta(m_1 + m_2 - 0)$

$\{(2\cdot 0+1)\frac{(1+1-0)!(1-1+0)!(-1+1+0)!}{(1+1+0+1)!}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!(0+0)!(0-0)!\}^{1/2}$

$[\sum_r(-1)^r r!(1+1-0-r)!(1-m_1-r)!(0-1-m_2+r)!(1+m_2-r)!(0-1+m_1+r)!]^{-1}$
$=\delta(m_1+m_2)\frac{1}{\sqrt{3}}(1+m_1)!(1-m_1)![\sum_r(-1)^r r!(2-r)![(1-m_1-r)!]^2[(-1+m_1+r)!]^2]^{-1}$

Cor. 10.2.1.
$\begin{cases}\langle(1,1);(1,-1)|1,1;(0,0)\rangle=\frac{1}{\sqrt{3}}\\ \langle(1,0);(1,0)|1,1;(0,0)\rangle=-\frac{1}{\sqrt{3}}\\ \langle(1,-1);(1,1)|1,1;(0,0)\rangle=\frac{1}{\sqrt{3}}\end{cases}$ $|1,1;(0,0)\rangle=\frac{1}{\sqrt{3}}|(1,1);(1,-1)\rangle-\frac{1}{\sqrt{3}}|(1,0);(1,0)\rangle+\frac{1}{\sqrt{3}}|(1,-1);(1,1)\rangle$

10.3 Special case: $1\oplus1=(1,0)$

Thm. 10.3.1. $|1,1;(1,m_3)\rangle=\sum_{m_1=1}^{-1}\sum_{m_2=1}^{-1}|(1,m_1);(1,m_2)\rangle\langle(1,m_1);(1,m_2)|1,1;(1,m_3)\rangle$
$CG_{Racah}=\langle(1,m_1);(1,m_2)|1,1;(1,m_3)\rangle=\delta(m_1+m_2-m_3)$
$\{(2\cdot1+1)\frac{(1+1-1)!(1-1+1)!(-1+1+1)!}{(1+1+1+1)!}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!(1+m_3)!(1-m_3)!\}^{1/2}$
$[\sum_r(-1)^r r!(1+1-1-r)!(1-m_1-r)!(1-1-m_2+r)!(1+m_2-r)!(1-1+m_1+r)!]^{-1}$
$=\delta(m_1+m_2-m_3)\{\frac{3}{4!}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!(1+m_3)!(1-m_3)!\}^{1/2}$
$[\sum_r(-1)^r r!(1-r)!(1-m_1-r)!(-m_2+r)!(1+m_2-r)!(m_1+r)!]^{-1}$

Cor. 10.3.1. $\langle(1,m_1);(1,m_2)|1,1;(1,m_3)\rangle$
$=\delta(m_1+m_2-m_3)\{\frac{3}{4!}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!(1+m_3)!(1-m_3)!\}^{1/2}$
$[\sum_r(-1)^r r!(1-r)!(1-m_1-r)!(-m_2+r)!(1+m_2-r)!(m_1+r)!]^{-1}$

Cor. 10.3.2. $\langle(1,m_1);(1,m_2)|1,1;(1,0)\rangle$
$=\delta(m_1+m_2)\frac{1}{\sqrt{8}}(1+m_1)!(1-m_1)![\sum_r(-1)^r r!(1-r)![(1-m_1-r)!(m_1+r)!]^2]^{-1}$

Cor. 10.3.3.
$\begin{cases}\langle(1,1);(1,-1)|1,1;(1,0)\rangle=\frac{1}{\sqrt{2}}\\ \langle(1,0);(1,0)|1,1;(1,0)\rangle=0\\ \langle(1,-1);(1,1)|1,1;(1,0)\rangle=-\frac{1}{\sqrt{2}}\end{cases}$ $|1,1;(1,0)\rangle=\frac{1}{\sqrt{2}}|(1,1);(1,-1)\rangle-\frac{1}{\sqrt{2}}|(1,-1);(1,1)\rangle$

10.4 Special case: $1\oplus1=(1,1)$

Cor. 10.4.1. $\langle(1,m_1);(1,m_2)|1,1;(1,m_3)\rangle$
$=\delta(m_1+m_2-m_3)\{\frac{3}{4!}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!(1+m_3)!(1-m_3)!\}^{1/2}$
$[\sum_r(-1)^r r!(1-r)!(1-m_1-r)!(-m_2+r)!(1+m_2-r)!(m_1+r)!]^{-1}$

Cor. 10.4.2. $\langle(1,m_1);(1,m_2)|1,1;(1,1)\rangle$
$=\delta(m_1+m_2-1)\{\frac{1}{4}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!\}^{1/2}$
$[\sum_r(-1)^r r!(1-r)!(1-m_1-r)!(-m_2+r)!(1+m_2-r)!(m_1+r)!]^{-1}$

Cor. 10.4.3.
$\begin{cases}\langle(1,1);(1,0)|1,1;(1,1)\rangle=\frac{1}{\sqrt{2}}\\ \langle(1,0);(1,1)|1,1;(1,1)\rangle=-\frac{1}{\sqrt{2}}\end{cases}$ $|1,1;(1,1)\rangle=\frac{1}{\sqrt{2}}|(1,1);(1,0)\rangle-\frac{1}{\sqrt{2}}|(1,0);(1,1)\rangle$

10.5 Special case: $1\oplus1=(1,-1)$

Cor. 10.5.1. $\langle(1,m_1);(1,m_2)|1,1;(1,m_3)\rangle$
$=\delta(m_1+m_2-m_3)\{\frac{3}{4!}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!(1+m_3)!(1-m_3)!\}^{1/2}$
$[\sum_r(-1)^r r!(1-r)!(1-m_1-r)!(-m_2+r)!(1+m_2-r)!(m_1+r)!]^{-1}$

Cor. 10.5.2. $\langle(1,m_1);(1,m_2)|1,1;(1,-1)\rangle$
$=\delta(m_1+m_2+1)\{\frac{1}{4}(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!\}^{1/2}$
$[\sum_r(-1)^r r!(1-r)!(1-m_1-r)!(-m_2+r)!(1+m_2-r)!(m_1+r)!]^{-1}$

Cor. 10.5.3.
$\begin{cases}\langle(1,-1);(1,0)|1,1;(1,-1)\rangle=-\frac{1}{\sqrt{2}}\\ \langle(1,0);(1,-1)|1,1;(1,-1)\rangle=\frac{1}{\sqrt{2}}\end{cases}$ $|1,1;(1,-1)\rangle=\frac{1}{\sqrt{2}}|(1,0);(1,-1)\rangle-\frac{1}{\sqrt{2}}|(1,-1);(1,0)\rangle$

10.6 Special case: $1 \oplus 1 = (1, -1)$

Cor. 10.6.1.

$$\begin{cases} \hat{J}_+|s,h\rangle = \sqrt{s(s+1)-h(h+1)}|s,h+1\rangle = \sqrt{(s-h)(s+h+1)}|s,h+1\rangle, -s \leq h \leq s \\ \hat{J}_-|s,h\rangle = \sqrt{s(s+1)-h(h-1)}|s,h-1\rangle = \sqrt{(s-h+1)(s+h)}|s,h-1\rangle, -s \leq h \leq s \\ \hat{J}_z|s,h\rangle = h|s,h-1\rangle, -s \leq h \leq s \end{cases}$$

11 Invariant tensor operator

11.1 Invariant tensor operator

Cor. 11.1.1.

$$\begin{cases} \hat{J}^2 = U(\omega)\hat{J}^2U^+(\omega) \Leftrightarrow U(-\omega)\hat{J}^2U^+(-\omega) = \hat{J}^2 \\ \hat{J}_i = e^{i\omega R}|_i{}^j U(\omega)\hat{J}_jU^+(\omega) \Leftrightarrow U(-\omega)\hat{J}_iU^+(-\omega) = e^{i\omega R}|_i{}^j\hat{J}_j \\ T_{ij} = e^{i\omega R}|_i{}^k e^{i\omega R}|_j{}^l U(\omega)T_{kl}U^+(\omega) \Leftrightarrow U(-\omega)T_{ij}U^+(-\omega) = e^{i\omega R}|_i{}^k e^{i\omega R}|_j{}^l T_{kl} \\ T_{i\cdot\cdot j} = e^{i\omega R}|_i{}^k \cdot\cdot e^{i\omega R}|_j{}^l U(\omega)T_{k\cdot\cdot l}U^+(\omega) \Leftrightarrow U(-\omega)T_{i\cdot\cdot j}U^+(-\omega) = e^{i\omega R}|_i{}^k e^{i\omega R}|_j{}^l T_{k\cdot\cdot l} \end{cases}$$

Chapter31 B-F Formula and Projection Operator Conjecture

1 Polynomial theorem and its generalization with fully symmetric indices

1.1 Binomial expansion of zero order fully symmetric indices

Pro. 1.1.1.

$$\begin{cases}(A+B)^2 = A^2 + 2AB + B^2 \\ (A-B)^2 = A^2 - 2AB + B^2 \\ (A+B)(A-B) = A^2 - B^2 \\ (A-B)(A+B) = A^2 - B^2\end{cases}$$

Thm. 1.1.1. $(A+B)^n = \sum\limits_{i=0}^{n} C_n^i A^i B^{n-i}$

1.2 Zero order fully symmetric indices polynomial expansion

Thm. 1.2.1. $(A_1 + \cdot\cdot + A_l)^n = \sum\limits_{n_1 n_2 \cdot\cdot n_l} \frac{n!}{n_1! n_2! \cdot\cdot n_l!} A_1^{n_1} A_2^{n_2} \cdot\cdot A_l^{n_1}, n_1 + n_2 + \cdot\cdot + n_l = n$

1.3 Binomial expansion of first order fully symmetric indices

Pro. 1.3.1.

$$\begin{cases}[A_{\{a_1} + B_{\{a_1}][A_{a_2\}} + B_{a_2\}}] = A_{\{a_1}A_{a_2\}} + 2A_{\{a_1}B_{a_2\}} + B_{\{a_1}B_{a_2\}} \\ [A_{\{a_1} - B_{\{a_1}][A_{a_2\}} - B_{a_2\}}] = A_{\{a_1}A_{a_2\}} - 2A_{\{a_1}B_{a_2\}} + B_{\{a_1}B_{a_2\}} \\ [A_{\{a_1} + B_{\{a_1}][A_{a_2\}} - B_{a_2\}}] = A_{\{a_1}A_{a_2\}} - B_{\{a_1}B_{a_2\}} \\ [A_{\{a_1} - B_{\{a_1}][A_{a_2\}} + B_{a_2\}}] = A_{\{a_1}A_{a_2\}} - B_{\{a_1}B_{a_2\}} \\ A_{\{a_1}B_{a_2\}} = A_{\{a_2}B_{a_1\}}, A_{\{\cdot\cdot a_i \cdot\cdot a_j \cdot\cdot\}} = A_{\{\cdot\cdot a_j \cdot\cdot a_i \cdot\cdot\}}\end{cases}$$

Thm. 1.3.1. $[A_{\{a_1} + B_{\{a_1}] \cdot\cdot [A_{a_n\}} + B_{a_n\}}] = \sum\limits_{i=0}^{n} C_n^i [A_{\{a_1} \cdot\cdot A_{a_i}][B_{a_{i+1}} \cdot\cdot B_{a_n\}}]$

Cor. 1.3.1. $(A_a + B_a)^n = \sum\limits_{i=0}^{n} C_n^i A_a^i B_a^{n-i}$

1.4 First order fully symmetric indices polynomial expansion

Thm. 1.4.1. $[A_{1\{a_1} + \cdot\cdot + A_{l\{a_1}] \cdot\cdot [A_{1a_n\}} + \cdot\cdot + A_{la_n\}}], n_1 + n_2 + \cdot\cdot + n_l = n$
$= \sum\limits_{n_1 n_2 \cdot\cdot n_l} \frac{n!}{n_1! n_2! \cdot\cdot n_l!} [A_{1\{a_1} \cdot\cdot A_{1a_{n_1}}][A_{2a_{n_1+1}} \cdot\cdot A_{2a_{n_1+n_2}}] \cdot\cdot [A_{la_{n_1+\cdot\cdot+n_{l-1}+1}} \cdot\cdot A_{la_n\}}]$

Cor. 1.4.1. $(A_{1a} + \cdot\cdot + A_{la})^n = \sum\limits_{n_1 n_2 \cdot\cdot n_l} \frac{n!}{n_1! n_2! \cdot\cdot n_l!} A_{1a}^{n_1} A_2^{n_2} \cdot\cdot A_{la}^{n_1}, n_1 + n_2 + \cdot\cdot + n_l = n$

1.5 Binomial expansion of second order fully symmetric indices

Pro. 1.5.1.

$$\begin{cases}[A_{\{a_1(b_1} + B_{\{a_1(b_1}][A_{a_2\}b_2)} + B_{a_2\}b_2)}] = A_{\{a_1(b_1}A_{a_2\}b_2)} + 2A_{\{a_1(b_1}B_{a_2\}b_2)} + B_{\{a_1(b_1}B_{a_2\}b_2)} \\ [A_{\{a_1(b_1} - B_{\{a_1(b_1}][A_{a_2\}b_2)} - B_{a_2\}b_2)}] = A_{\{a_1(b_1}A_{a_2\}b_2)} - 2A_{\{a_1(b_1}B_{a_2\}b_2)} + B_{\{a_1(b_1}B_{a_2\}b_2)} \\ [A_{\{a_1(b_1} + B_{\{a_1(b_1}][A_{a_2\}b_2)} - B_{a_2\}b_2)}] = A_{\{a_1(b_1}A_{a_2\}b_2)} - B_{\{a_1(b_1}B_{a_2\}b_2)} \\ [A_{\{a_1(b_1} - B_{\{a_1(b_1}][A_{a_2\}b_2)} + B_{a_2\}b_2)}] = A_{\{a_1(b_1}A_{a_2\}b_2)} - B_{\{a_1(b_1}B_{a_2\}b_2)}\end{cases}$$

Thm. 1.5.1. $[A_{\{a_1(b_1} + B_{\{a_1(b_1}] \cdot\cdot [A_{a_n\}b_n)} + B_{a_n\}b_n)}] = \sum\limits_{i=0}^{n} C_n^i [A_{\{a_1(b_1} \cdot\cdot A_{a_i b_i}][B_{a_{i+1}b_{i+1}} \cdot\cdot B_{a_n\}b_n)}]$

Cor. 1.5.1. $(A_{ab} + B_{ab})^n = \sum\limits_{i=0}^{n} C_n^i A_{ab}^i B_{ab}^{n-i}$

1.6 Second order fully symmetric indices polynomial expansion

Thm. 1.6.1. $[A_{1\{a_1(b_1}+\cdot\cdot+A_{l\{a_1(b_1}]\cdot\cdot[A_{1a_n\}b_n)}+\cdot\cdot+A_{la_n\}b_n)}], n_1+n_2+\cdot\cdot+n_l=n$
$=\sum\limits_{n_1n_2\cdot\cdot n_l}\frac{n!}{n_1!n_2!\cdot\cdot n_l!}[A_{1\{a_1(b_1}\cdot\cdot A_{1a_{n_1}b_{n_1}}][A_{2a_{n_1+1}b_{n_1+1}}\cdot\cdot A_{2a_{n_1+n_2}b_{n_1+n_2}}]\cdot\cdot[A_{la_{n_1+\cdot\cdot+n_{l-1}+1}b_{n_1+\cdot\cdot+n_{l-1}+1}}\cdot\cdot A_{la_n\}b_n)}]$

Cor. 1.6.1. $(A_{1ab}+\cdot\cdot+A_{lab})^n=\sum\limits_{n_1n_2\cdot\cdot n_l}\frac{n!}{n_1!n_2!\cdot\cdot n_l!}A_{1ab}^{n_1}A_{2ab}^{n_2}\cdot\cdot A_{lab}^{n_l}, n_1+n_2+\cdot\cdot+n_l=n$

1.7 Binomial expansion of multiple order fully symmetric indices

Thm. 1.7.1. $[A_{1\{a_1\cdot\cdot(b_1}+\cdot\cdot+A_{l\{a_1\cdot\cdot(b_1}]\cdot\cdot[A_{1a_n\}\cdot\cdot b_n)}+\cdot\cdot+A_{la_n\}\cdot\cdot b_n)}], n_1+n_2+\cdot\cdot+n_l=n$
$=\sum\limits_{n_1n_2\cdot\cdot n_l}\frac{n!}{n_1!n_2!\cdot\cdot n_l!}[A_{1\{a_1\cdot\cdot(b_1}\cdot\cdot A_{1a_{n_1}\cdot\cdot b_{n_1}}][A_{2a_{n_1+1}\cdot\cdot b_{n_1+1}}\cdot\cdot A_{2a_{n_1+n_2}\cdot\cdot b_{n_1+n_2}}]\cdot\cdot[A_{la_{n_1+\cdot\cdot+n_{l-1}+1}\cdot\cdot b_{n_1+\cdot\cdot+n_{l-1}+1}}\cdot\cdot A_{la_n\}\cdot\cdot b_n)}]$

Cor. 1.7.1. $(A_{1a\cdot\cdot b}+\cdot\cdot+A_{la\cdot\cdot b})^n=\sum\limits_{n_1n_2\cdot\cdot n_l}\frac{n!}{n_1!n_2!\cdot\cdot n_l!}A_{1ab}^{n_1}A_{2a\cdot\cdot b}^{n_2}\cdot\cdot A_{la\cdot\cdot b}^{n_l}, n_1+n_2+\cdot\cdot+n_l=n$

1.8 Multiple order fully symmetric indices polynomial expansion

Cor. 1.8.1.
$[A_{\{a_1}B_{(b_1}+B_{\{a_1}A_{(b_1}][A_{a_2\}}B_{b_2)}+B_{a_2\}}A_{b_2)}]=A_{\{a_1}A_{a_2\}}B_{(b_1}B_{b_2)}+2A_{\{a_1}B_{a_2\}}A_{(b_1}B_{b_2)}+B_{\{a_1}B_{a_2\}}A_{(b_1}A_{b_2)}$

Cor. 1.8.2.
$[A_{\{a_1}B_{(b_1}-B_{\{a_1}A_{(b_1}][A_{a_2\}}B_{b_2)}-B_{a_2\}}A_{b_2)}]=A_{\{a_1}A_{a_2\}}B_{(b_1}B_{b_2)}-2A_{\{a_1}B_{a_2\}}A_{(b_1}B_{b_2)}+B_{\{a_1}B_{a_2\}}A_{(b_1}A_{b_2)}$

Cor. 1.8.3. $[A_{\{a_1}B_{(b_1}+B_{\{a_1}A_{(b_1}][A_{a_2\}}B_{b_2)}-B_{a_2\}}A_{b_2)}]=A_{\{a_1}A_{a_2\}}B_{(b_1}B_{b_2)}-B_{\{a_1}B_{a_2\}}A_{(b_1}A_{b_2)}$

Cor. 1.8.4. $[A_{\{a_1}B_{(b_1}-B_{\{a_1}A_{(b_1}][A_{a_2\}}B_{b_2)}+B_{a_2\}}A_{b_2)}]=A_{\{a_1}A_{a_2\}}B_{(b_1}B_{b_2)}-B_{\{a_1}B_{a_2\}}A_{(b_1}A_{b_2)}$

Thm. 1.8.1. $[A_{\{a_1}B_{(b_1}+B_{\{a_1}A_{(b_1}]\cdot\cdot[A_{a_n\}}B_{b_n)}+B_{a_n\}}A_{b_n)}]=\sum\limits_{i=0}^{n}C_n^i[A_{\{a_1}\cdot\cdot A_{a_i}B_{a_{i+1}}\cdot\cdot B_{a_n\}}][B_{(b_1}\cdot\cdot B_{b_i}A_{b_{i+1}}\cdot\cdot A_{b_n)}]$

Thm. 1.8.2. $[A_{\{a_1}B_{(b_1}-B_{\{a_1}A_{(b_1}]\cdot\cdot[A_{a_n\}}B_{b_n)}-B_{a_n\}}A_{b_n)}]$
$=\sum\limits_{i=0}^{n}(-1)^{n-i}C_n^i[A_{\{a_1}\cdot\cdot A_{a_i}B_{a_{i+1}}\cdot\cdot B_{a_n\}}][B_{(b_1}\cdot\cdot B_{b_i}A_{b_{i+1}}\cdot\cdot A_{b_n)}]$

Thm. 1.8.3. $[A_{\{a_1}B_{(b_1}+B_{\{a_1}A_{(b_1}+C_{\{a_1}C_{(b_1}]\cdot\cdot[A_{a_n\}}B_{b_n)}+B_{a_n\}}A_{b_n)}+C_{a_n\}}C_{b_n)}]$
$=\sum\limits_{n_1n_2\cdot\cdot n_l}\frac{n!}{n_1!n_2!\cdot\cdot n_l!}[A_{\{a_1}\cdot\cdot A_{a_{n_1}}B_{a_{n_1+1}}\cdot\cdot B_{n_1+n_2}C_{a_{n_1+n_2+1}}\cdot\cdot C_{a_n\}}][B_{(b_1}\cdot\cdot B_{b_{n_1}}A_{b_{n_1+1}}\cdot\cdot A_{n_1+n_2}C_{b_{n_1+n_2+1}}\cdot\cdot C_{b_n)}]$

Cor. 1.8.5. $[A_{\{a_1}B_{(b_1}+B_{\{a_1}A_{(b_1}+C_{\{a_1}C_{(b_1}][A_{a_2\}}B_{b_2)}+B_{a_2\}}A_{b_2)}+C_{a_2\}}C_{b_2)}]$
$=[A_{\{a_1}B_{(b_1}+B_{\{a_1}A_{(b_1}][A_{a_2\}}B_{b_2)}+B_{a_2\}}A_{b_2)}]+2[A_{\{a_1}B_{(b_1}+B_{\{a_1}A_{(b_1}]C_{a_2\}}C_{b_2)}+[C_{\{a_1}C_{a_2\}}][C_{(b_1}C_{b_2)}]$
$=A_{\{a_1}A_{a_2\}}B_{(b_1}B_{b_2)}+B_{\{a_1}B_{a_2\}}A_{(b_1}A_{b_2)}+C_{\{a_1}C_{a_2\}}C_{(b_1}C_{b_2)}$
$+2A_{\{a_1}B_{a_2\}}A_{(b_1}B_{b_2)}+2A_{\{a_1}C_{a_2\}}B_{(b_1}C_{b_2)}+2B_{\{a_1}C_{a_2\}}A_{(b_1}C_{b_2)}$

2 Polynomial theorem and its generalization of antisymmetric indices

2.1 Binomial expansion of first order fully symmetric indices

Pro. 2.1.1.
$$\begin{cases}[A_{[a_1}+B_{[a_1}][A_{a_2]}+B_{a_2]}]=0\\ [A_{[a_1}-B_{[a_1}][A_{a_2]}-B_{a_2]}]=0\\ [A_{[a_1}+B_{[a_1}][A_{a_2]}-B_{a_2]}]=2B_{[a_1}A_{a_2]}\\ [A_{[a_1}-B_{[a_1}][A_{a_2]}+B_{a_2]}]=2A_{[a_1}B_{a_2]}\end{cases}$$

Thm. 2.1.1. $[A_{[a_1}+B_{[a_1}]\cdot\cdot[A_{a_n]}+B_{a_n]}]=0$

2.2 First order fully symmetric indices polynomial expansion

Thm. 2.2.1. $[A_{1[a_1}+\cdot\cdot+A_{l[a_1}]\cdot\cdot[A_{1a_n]}+\cdot\cdot+A_{la_n]}]=0$

2.3 Binomial expansion of antisymmetric indices

Pro. 2.3.1.
$$\begin{cases}[A_{[a_1\langle b_1}+B_{[a_1\langle b_1}][A_{a_2]b_2\rangle}+B_{a_2]b_2\rangle}]=A_{[a_1\langle b_1}A_{a_2]b_2\rangle}+2A_{[a_1\langle b_1}B_{a_2]b_2\rangle}+B_{[a_1\langle b_1}B_{a_2]b_2\rangle}\\ [A_{[a_1\langle b_1}-B_{[a_1\langle b_1}][A_{a_2]b_2\rangle}-B_{a_2]b_2\rangle}]=A_{[a_1\langle b_1}A_{a_2]b_2\rangle}-2A_{[a_1\langle b_1}B_{a_2]b_2\rangle}+B_{[a_1\langle b_1}B_{a_2]b_2\rangle}\\ [A_{[a_1\langle b_1}+B_{[a_1\langle b_1}][A_{a_2]b_2\rangle}-B_{a_2]b_2\rangle}]=A_{[a_1\langle b_1}A_{a_2]b_2\rangle}-B_{[a_1\langle b_1}B_{a_2]b_2\rangle}\\ [A_{[a_1\langle b_1}-B_{[a_1\langle b_1}][A_{a_2]b_2\rangle}+B_{a_2]b_2\rangle}]=A_{[a_1\langle b_1}A_{a_2]b_2\rangle}-B_{[a_1\langle b_1}B_{a_2]b_2\rangle}\end{cases}$$

Thm. 2.3.1. $[A_{[a_1\langle b_1}+B_{[a_1\langle b_1}]\cdot\cdot[A_{a_n]b_n\rangle}+B_{a_n]b_n\rangle}]=\sum\limits_{i=1}^{n}C_n^i[A_{[a_1\langle b_1}\cdot\cdot A_{a_ib_i}][B_{a_{i+1}b_{i+1}}\cdot\cdot B_{a_n]b_n\rangle}]$

Cor. 2.3.1. $[A_{[a_1}B_{\langle b_1}+B_{[a_1}A_{\langle b_1}+C_{[a_1}C_{\langle b_1}][A_{a_2]}B_{b_2\rangle}+B_{a_2]}A_{b_2\rangle}+C_{a_2]}C_{b_2\rangle}]$
$=2A_{[a_1}B_{a_2]}A_{\langle b_1}B_{b_2\rangle}+2A_{[a_1}C_{a_2]}B_{\langle b_1}C_{b_2\rangle}+2B_{[a_1}C_{a_2]}A_{\langle b_1}C_{b_2\rangle}$

2.4 Antisymmetric indices polynomial expansion(Wrong?)

Thm. 2.4.1. $[A_{1[a_1\langle b_1}+\cdot\cdot+A_{l[a_1\langle b_1}]\cdot\cdot[A_{1a_n]b_n\rangle}+\cdot\cdot+A_{la_n]b_n\rangle}], n_1+n_2+\cdot\cdot+n_l=n$
$=\sum\limits_{n_1n_2\cdot\cdot n_l}\frac{n!}{n_1!n_2!\cdot\cdot n_l!}[A_{1[a_1\langle b_1}\cdot\cdot A_{1a_{n_1}b_{n_1}}][A_{2a_{n_1+1}b_{n_1+1}}\cdot\cdot A_{2a_{n_1+n_2}b_{n_1+n_2}}]\cdot\cdot[A_{la_{n_1+\cdot\cdot+n_{l-1}+1}b_{n_1+\cdot\cdot+n_{l-1}+1}}\cdot\cdot A_{la_n]b_n\rangle}]$

Thm. 2.4.2. *Even indices:* $[A_{1[a_1\cdot\cdot\langle b_1}+\cdot\cdot+A_{l[a_1\cdot\cdot\langle b_1}]\cdot\cdot[A_{1a_n]\cdot\cdot b_n\rangle}+\cdot\cdot+A_{la_n]\cdot\cdot b_n\rangle}], n_1+n_2+\cdot\cdot+n_l=n$
$=\sum\limits_{n_1n_2\cdot\cdot n_l}\frac{n!}{n_1!n_2!\cdot\cdot n_l!}[A_{1[a_1\cdot\cdot\langle b_1}\cdot\cdot A_{1a_{n_1}\cdot\cdot b_{n_1}}][A_{2a_{n_1+1}\cdot\cdot b_{n_1+1}}\cdot\cdot A_{2a_{n_1+n_2}\cdot\cdot b_{n_1+n_2}}]\cdot\cdot[A_{la_{n_1+\cdot\cdot+n_{l-1}+1}\cdot\cdot b_{n_1+\cdot\cdot+n_{l-1}+1}}\cdot\cdot A_{la_n]\cdot\cdot b_n\rangle}]$

3 Projection operator for spin-n particle Klein-Gordon equation

3.1 Classic expression of projection operator for spin-n particle Klein-Gordon equation

Cor. 3.1.1. $\sum\limits_{h=2}^{-2}(-1)^h\varepsilon_{\{a_1a_2\}}(\vec{p},h)\varepsilon_{(b_1b_2)}(\vec{p},-h)$
$=[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{(b_1}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_2\}}(\vec{p},h)\varepsilon_{b_2)}(\vec{p},-h)]$
$-\frac{1}{3}[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{a_2\}}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{(b_1}(\vec{p},h)\varepsilon_{b_2)}(\vec{p},-h)]$

Cor. 3.1.2. $\sum\limits_{h=3}^{-3}(-1)^h\varepsilon_{\{a_1a_2a_3\}}(\vec{p},h)\varepsilon_{(b_1b_2b_3)}(\vec{p},-h)$
$=[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{(b_1}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_2}(\vec{p},h)\varepsilon_{b_2}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_3\}}(\vec{p},h)\varepsilon_{b_3)}(\vec{p},-h)]$
$-\frac{3}{5}[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{a_2}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{(b_1}(\vec{p},h)\varepsilon_{b_2}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_3\}}(\vec{p},h)\varepsilon_{b_3)}(\vec{p},-h)]$

Cor. 3.1.3. $\sum\limits_{h=4}^{-4}(-1)^h\varepsilon_{\{a_1a_2a_3a_4\}}(\vec{p},h)\varepsilon_{(b_1b_2b_3b_4)}(\vec{p},-h)$
$=[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{(b_1}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_2}(\vec{p},h)\varepsilon_{b_2}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_3}(\vec{p},h)\varepsilon_{b_3}(\vec{p},-h)]$
$[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_4\}}(\vec{p},h)\varepsilon_{b_4)}(\vec{p},-h)]-\frac{6}{7}[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{a_2}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{(b_1}(\vec{p},h)\varepsilon_{b_2}(\vec{p},-h)]$
$[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_3}(\vec{p},h)\varepsilon_{b_3}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_4\}}(\vec{p},h)\varepsilon_{b_4)}(\vec{p},-h)]+\frac{3}{35}[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{a_2}(\vec{p},-h)]$
$[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{(b_1}(\vec{p},h)\varepsilon_{b_2}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_3}(\vec{p},h)\varepsilon_{a_4\}}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{b_3}(\vec{p},h)\varepsilon_{b_4)}(\vec{p},-h)]$

Ass. 3.1.1. $\sum\limits_{h=n}^{-n}(-1)^h\varepsilon_{\{a_1a_2\cdot\cdot a_n\}}(\vec{p},h)\varepsilon_{(b_1b_2\cdot\cdot b_n)}(\vec{p},-h)=\sum\limits_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{\{a_1}(\vec{p},h)\varepsilon_{a_2}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{(b_1}(\vec{p},h)\varepsilon_{b_2}(\vec{p},-h)]\cdot\cdot$
$[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_{2r-1}}(\vec{p},h)\varepsilon_{a_{2r}}(\vec{p},-h)][\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{b_{2r-1}}(\vec{p},h)\varepsilon_{b_{2r}}(\vec{p},-h)]\}$
$[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_{2r+1}}(\vec{p},h)\varepsilon_{b_{2r+1}}(\vec{p},-h)]\cdot\cdot[\sum\limits_{h=1}^{-1}(-1)^h\varepsilon_{a_n\}}(\vec{p},h)\varepsilon_{b_n\}}(\vec{p},-h)]$
$=\sum\limits_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[C_{\{a_1}C_{a_2}-A_{\{a_1}B_{a_2}-B_{\{a_1}A_{a_2}][C_{(b_1}C_{b_2}-A_{(b_1}B_{b_2}-B_{(b_1}A_{b_2}]\cdot\cdot$
$[C_{a_{2r-1}}C_{a_{2r}}-A_{a_{2r-1}}B_{a_{2r}}-B_{a_{2r-1}}A_{a_{2r}}][C_{b_{2r-1}}C_{b_{2r}}-A_{b_{2r-1}}B_{b_{2r}}-B_{b_{2r-1}}A_{b_{2r}}]\}$
$[C_{a_{2r+1}}C_{b_{2r+1}}-A_{a_{2r+1}}B_{b_{2r+1}}-B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[C_{a_n\}}C_{b_n)}-A_{a_n\}}B_{b_n)}-B_{a_n\}}A_{b_n)}]$

The above conjecture is equivalently transformed from the formula constructed by Behrends and Fronsdal [50, 51]. It has not been strictly proven and is essentially a conjecture. It is a prerequisite for many important conclusions to follow.

3.2 Definition expression of projection operator for spin-n particle Klein-Gordon equation

In particular, this section uses the conclusions of the latter chapter in advance, and then uses them to derive important conjectures.

Def. 3.2.1. $A=\varepsilon(\vec{p},1),B=\varepsilon(\vec{p},-1),C=\varepsilon(\vec{p},0)$

Cor. 3.2.1. $A_{r,n}=(-\frac{1}{2})^r\frac{n!(2n-2r-1)!!}{r!(n-2r)!(2n-1)!!}=(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$

Thm. 3.2.1. $\varepsilon_{\{a_1a_2\cdot\cdot a_n\}}(\vec{p},n-2k)$
$= \frac{1}{\sqrt{C_{2n}^{2k}}}\sum_{i=0}^{k|(n-k)}\frac{(\sqrt{2})^{2i}n!}{(n-k-i)!(k-i)!(2i)!}[A_{\{a_1}\cdot\cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}}\cdot\cdot B_{a_{n-2i}}][C_{a_{n+1-2i}}\cdot\cdot C_{a_n\}}]$

Thm. 3.2.2. $\varepsilon_{\{a_1a_2\cdot\cdot a_n\}}(\vec{p},n-2k-1)$
$= \frac{1}{\sqrt{C_{2n}^{2k+1}}}\sum_{i=0}^{k|(n-1-k)}\frac{(\sqrt{2})^{2i+1}n!}{(n-k-i-1)!(k-i)!(2i+1)!}[A_{\{a_1}\cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}}\cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}}\cdot\cdot C_{a_n\}}]$

Thm. 3.2.3. $\varepsilon_{\{a_1a_2\cdot\cdot a_n\}}(\vec{p},n-2k)\varepsilon_{(b_1b_2\cdot\cdot b_n)}(\vec{p},n-2(n-k))$
$= \frac{1}{C_{2n}^{2k}}\sum_{i,j=0}^{k|(n-k)}\frac{2^{i+j}n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}}\cdot\cdot B_{a_{n-2i}}][C_{a_{n+1-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}}\cdot\cdot A_{b_{n-2j}}][C_{b_{n+1-2j}}\cdot\cdot C_{b_n)}]\}$

Thm. 3.2.4. $\varepsilon_{\{a_1a_2\cdot\cdot a_n\}}(\vec{p},n-2k-1)\varepsilon_{(b_1b_2\cdot\cdot b_n)}(\vec{p},n-2(n-1-k)-1)$
$= \frac{1}{C_{2n}^{2k+1}}\sum_{i,j=0}^{k|(n-1-k)}\frac{2^{i+j+1}n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}}\cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}}\cdot\cdot A_{b_{n-2j-1}}][C_{b_{n-2j}}\cdot\cdot C_{b_n)}]\}$

Thm. 3.2.5. $\sum_{h=n}^{-n}(-1)^h\varepsilon_{\{a_1a_2\cdot\cdot a_n\}}(\vec{p},h)\varepsilon_{(b_1b_2\cdot\cdot b_n)}(\vec{p},-h)$
$= \sum_{k=0}^{n}\frac{(-1)^n}{C_{2n}^{2k}}\sum_{i,j=0}^{k|(n-k)}\frac{2^{i+j}n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}}\cdot\cdot B_{a_{n-2i}}][C_{a_{n+1-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}}\cdot\cdot A_{b_{n-2j}}][C_{b_{n+1-2j}}\cdot\cdot C_{b_n)}]\}$
$-\sum_{k=0}^{n-1}\frac{(-1)^n}{C_{2n}^{2k+1}}\sum_{i,j=0}^{k|(n-1-k)}\frac{2^{i+j+1}n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}}\cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}}\cdot\cdot A_{b_{n-2j-1}}][C_{b_{n-2j}}\cdot\cdot C_{b_n)}]\}$

The above theorems were proposed by me through inductive exploration and have been strictly proved in the following chapter.

3.3 Important conjecture of projection operator for spin-n particle Klein-Gordon equation

Ass. 3.3.1. $\sum_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[C_{\{a_1}C_{a_2}-A_{\{a_1}B_{a_2}-B_{\{a_1}A_{a_2}][C_{(b_1}C_{b_2}-A_{(b_1}B_{b_2}-B_{(b_1}A_{b_2}]\cdot\cdot$
$[C_{a_{2r-1}}C_{a_{2r}}-A_{a_{2r-1}}B_{a_{2r}}-B_{a_{2r-1}}A_{a_{2r}}][C_{b_{2r-1}}C_{b_{2r}}-A_{b_{2r-1}}B_{b_{2r}}-B_{b_{2r-1}}A_{b_{2r}}]\}$
$[C_{a_{2r+1}}C_{b_{2r+1}}-A_{a_{2r+1}}B_{b_{2r+1}}-B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[C_{a_n\}}C_{b_n)}-A_{a_n\}}B_{b_n)}-B_{a_n\}}A_{b_n)}]$
$= \sum_{k=0}^{n}\frac{(-1)^n}{C_{2n}^{2k}}\sum_{i,j=0}^{k|(n-k)}\frac{2^{i+j}n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}}\cdot\cdot B_{a_{n-2i}}][C_{a_{n+1-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}}\cdot\cdot A_{b_{n-2j}}][C_{b_{n+1-2j}}\cdot\cdot C_{b_n)}]\}$
$-\sum_{k=0}^{n-1}\frac{(-1)^n}{C_{2n}^{2k+1}}\sum_{i,j=0}^{k|(n-1-k)}\frac{2^{i+j+1}n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}}\cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}}\cdot\cdot A_{b_{n-2j-1}}][C_{b_{n-2j}}\cdot\cdot C_{b_n)}]\}$

The above conjecture is obtained by combining the formula (conjecture) constructed by Behrends and Frontdal [50,51] with the theorem proposed by me in the previous section. It is a prerequisite for many following important conclusions. And in many cases the verification is correct. Currently, no counter examples have been encountered. However, to strictly prove it, it is necessary to first prove the formula constructed by Behrends and Frontdal. At present, I can't finish this proof. Of course, if this conjecture is proved by other methods, the formula constructed by Behrends and Frontdal can be strictly proved.

4 From physics to mathematical abstraction: Projection operator conjecture

In this section, A, B, and C are no longer specifically referred to, but generally refer to commutative variables.

4.1 Mathematical conjecture derived from projection operator for $s = n$ K-G equation

Ass. 4.1.1. $\sum_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[A_{\{a_1}B_{a_2}+B_{\{a_1}A_{a_2}+C_{\{a_1}C_{a_2}][A_{(b_1}B_{b_2}+B_{(b_1}A_{b_2}+C_{(b_1}C_{b_2}]\cdot\cdot$
$[A_{a_{2r-1}}B_{a_{2r}}+B_{a_{2r-1}}A_{a_{2r}}+C_{a_{2r-1}}C_{a_{2r}}][A_{b_{2r-1}}B_{b_{2r}}+B_{b_{2r-1}}A_{b_{2r}}+C_{b_{2r-1}}C_{b_{2r}}]\}$
$[A_{a_{2r+1}}B_{b_{2r+1}}+B_{a_{2r+1}}A_{b_{2r+1}}+C_{a_{2r+1}}C_{b_{2r+1}}]\cdot\cdot[A_{a_n\}}B_{b_n)}+B_{a_n\}}A_{b_n)}+C_{a_n\}}C_{b_n)}]$

$= \sum_{k=0}^{n} \frac{1}{C_{2n}^{2k}} \sum_{i,j=0}^{k|(n-k)} \frac{(-2)^{i+j} n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$
$\{[A_{\{a_1} \cdot \cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}} \cdot \cdot B_{a_{n-2i}}][C_{a_{n+1-2i}} \cdot \cdot C_{a_n\}}]\}\{[B_{(b_1} \cdot \cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}} \cdot \cdot A_{b_{n-2j}}][C_{b_{n+1-2j}} \cdot \cdot C_{b_n)}]\}$
$- \sum_{k=0}^{n-1} \frac{1}{C_{2n}^{2k+1}} \sum_{i,j=0}^{k|(n-1-k)} \frac{(-2)^{i+j+1} n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$
$\{[A_{\{a_1} \cdot \cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}} \cdot \cdot B_{a_{n-2i-1}}][C_{a_{n-2i}} \cdot \cdot C_{a_n\}}]\}\{[B_{(b_1} \cdot \cdot B_{b_{n-k-i-1}}][A_{b_{n-k-i}} \cdot \cdot A_{b_{n-2i-1}}][C_{b_{n-2i}} \cdot \cdot C_{b_n)}]\}$

The above is a more general conjecture. If it holds, the conjecture in the previous section will naturally hold.

Cor. 4.1.1. $C = 0 \Rightarrow$
$\sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[A_{\{a_1} B_{a_2} + B_{\{a_1} A_{a_2}][A_{(b_1} B_{b_2} + B_{(b_1} A_{b_2}] \cdot \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}][A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}]\}$
$[A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [A_{a_n\}} B_{b_n)} + B_{a_n\}} A_{b_n)}]$
$= \sum_{k=0}^{n} \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} [A_{\{a_1} \cdot \cdot A_{a_{n-k}} B_{a_{n+1-k}} \cdot \cdot B_{a_n\}}][B_{(b_1} \cdot \cdot B_{b_{n-k}} A_{b_{n+1-k}} \cdot \cdot A_{b_n)}]$

4.2 New combinatorial identities obtained from projection operator conjecture

Cor. 4.2.1. $A = 0, B = 0, C = \pm 1 \Rightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = 2^n$

Cor. 4.2.2. $A = \pm 1, B = \pm 1, C = 0 \Rightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = \sum_{k=0}^{n} \frac{1}{2^n} \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} = 2^n$

Cor. 4.2.3. $\sum_{k=0}^{n} \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} = \sum_{k=0}^{n} C_{2k}^{k} C_{2n-2k}^{n-k} = 2^{2n}, \sum_{k=0}^{n} \frac{(2k-1)!!}{(2k)!!} \frac{(2n-2k-1)!!}{(2n-2k)!!} = 1$

4.3 Projection operator conjecture C(2i,2j) case

Thm. 4.3.1. $\sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{\leq i,j,n/2-r} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$C_r^{i-l} \{[C_{\{a_1} C_{a_2}] \cdot \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}][A_{a_{2i-2l+1}} B_{a_{2i-2l+2}} + B_{a_{2i-2l+1}} A_{a_{2i-2l+2}}] \cdot \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}]$
$C_r^{j-l} [C_{(b_1} C_{b_2}] \cdot \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}][A_{b_{2j-2l+1}} B_{b_{2j-2l+2}} + B_{b_{2j-2l+1}} A_{b_{2j-2l+2}}] \cdot \cdot [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}]\}$
$C_{n-2r}^{2l} [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [A_{a_{n-2l}} B_{b_{n-2l}} + B_{a_{n-2l}} A_{b_{n-2l}}][C_{a_{n-2l+1}} C_{b_{n-2l+1}}] \cdot \cdot [C_{a_n\}} C_{b_n)}]$
$= \frac{n!n!}{(2n)!} \sum_{k \geq i|j}^{\leq n-i|j} \frac{(-2)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$
$\{[A_{\{a_1} \cdot \cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}} \cdot \cdot B_{a_{n-2i}}][C_{a_{n+1-2i}} \cdot \cdot C_{a_n\}}]\}\{[B_{(b_1} \cdot \cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}} \cdot \cdot A_{b_{n-2j}}][C_{b_{n+1-2j}} \cdot \cdot C_{b_n)}]\}$

The above is an identity which contains denominator term $(2i)!(2j)!$ after $(AB + BA), CC$ type binomial expansion.

Cor. 4.3.1. $\sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{\leq i,j,n/2-r} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} 2^{r+l-i} C_r^{j-l} 2^{r+l-j} C_{n-2r}^{2l}$
$[C_{\{a_1} C_{a_2}] \cdot \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}][A_{a_{2i-2l+1}} B_{a_{2i-2l+2}}] \cdot \cdot [A_{a_{2r-1}} B_{a_{2r}}]$
$[C_{(b_1} C_{b_2}] \cdot \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}][A_{b_{2j-2l+1}} B_{b_{2j-2l+2}}] \cdot \cdot [A_{b_{2r-1}} B_{b_{2r}}]$
$[A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [A_{a_{n-2l}} B_{b_{n-2l}} + B_{a_{n-2l}} A_{b_{n-2l}}][C_{a_{n-2l+1}} C_{b_{n-2l+1}}] \cdot \cdot [C_{a_n\}} C_{b_n)}]$
$= \frac{n!n!}{(2n)!} \sum_{k \geq i|j}^{\leq n-i|j} \frac{(-2)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$
$\{[A_{\{a_1} \cdot \cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}} \cdot \cdot B_{a_{n-2i}}][C_{a_{n+1-2i}} \cdot \cdot C_{a_n\}}]\}\{[B_{(b_1} \cdot \cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}} \cdot \cdot A_{b_{n-2j}}][C_{b_{n+1-2j}} \cdot \cdot C_{b_n)}]\}$

Cor. 4.3.2. $A = 1, B = 1, C = 1 \Rightarrow$
$\sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{\leq i,j,n/2-r} (-1)^{r+i+j} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l} = \sum_{k \geq i|j}^{\leq n-i|j} \frac{2^{2i+2j-n}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$

Cor. 4.3.3. $\sum_{r=0}^{[n/2]} [\sum_{l \geq 0, i|j-r}^{\leq i,j,k-r,n-r-k}](-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} 2^{r+l-i} C_r^{j-l} 2^{r+l-j} C_{n-2r}^{2l} C_{n-2r-2l}^{k-r-l}$
$[C_{\{a_1} C_{a_2}] \cdot \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}][A_{a_{2i-2l+1}} B_{a_{2i-2l+2}}] \cdot \cdot [A_{a_{2r-1}} B_{a_{2r}}]$
$[C_{(b_1} C_{b_2}] \cdot \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}][A_{b_{2j-2l+1}} B_{b_{2j-2l+2}}] \cdot \cdot [A_{b_{2r-1}} B_{b_{2r}}]$
$[A_{a_{2r+1}} B_{b_{2r+1}}] \cdot \cdot [A_{a_{n+r-l-k}} B_{b_{n+r-l-k}}][B_{a_{n+r-l-k+1}} A_{b_{n+r-l-k+1}}] \cdot \cdot [B_{a_{n-2l}} A_{b_{n-2l}}][C_{a_{n-2l+1}} C_{b_{n-2l+1}}] \cdot \cdot [C_{a_n\}} C_{b_n)}]$

$= \frac{n!n!}{(2n)!}\sum_{k=0}^{n}\frac{(-2)^{i+j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}}\cdot\cdot B_{a_{n-2i}}][C_{a_{n+1-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}}\cdot\cdot A_{b_{n-2j}}][C_{b_{n+1-2j}}\cdot\cdot C_{b_n)}]\}$
$\Leftrightarrow \sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-r-k}](-1)^{r+i+j}2^{2r+2l}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}C_r^{j-l}C_{n-2r}^{2l}C_{n-2r-2l}^{k-r-l} = \frac{(-2)^{2i+2j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$

The above final form is the k-order item identity.
4.4 Projection operator conjecture C(2i+1,2j+1) case

Thm. 4.4.1. $\sum_{r=0}^{[n/2]}\sum_{l\geq 0,i|j-r}^{\leq i,j,(n-1)/2-r}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$C_r^{i-l}\{[C_{\{a_1}C_{a_2}]\cdot\cdot[C_{a_{2i-2l-1}}C_{a_{2i-2l}}][A_{a_{2i-2l+1}}B_{a_{2i-2l+2}}+B_{a_{2i-2l+1}}A_{a_{2i-2l+2}}]\cdot\cdot[A_{a_{2r-1}}B_{a_{2r}}+B_{a_{2r-1}}A_{a_{2r}}]$
$C_r^{j-l}[C_{(b_1}C_{b_2}]\cdot\cdot[C_{b_{2j-2l-1}}C_{b_{2j-2l}}][A_{b_{2j-2l+1}}B_{b_{2j-2l+2}}+B_{b_{2j-2l+1}}A_{b_{2j-2l+2}}]\cdot\cdot[A_{b_{2r-1}}B_{b_{2r}}+B_{b_{2r-1}}A_{b_{2r}}]\}$
$C_{n-2r}^{2l+1}[A_{a_{2r+1}}B_{b_{2r+1}}+B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_{n-2l-1}}B_{b_{n-2l-1}}+B_{a_{n-2l-1}}A_{b_{n-2l-1}}][C_{a_{n-2l}}C_{b_{n-2l}}]\cdot\cdot[C_{a_n\}}C_{b_n)}]$
$= -\sum_{k\geq i|j}^{\leq n-1-i|j}\frac{n!n!}{(2n)!}\frac{(-2)^{i+j+1}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}}\cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}}\cdot\cdot A_{b_{n-2j-1}}][C_{b_{n-2j}}\cdot\cdot C_{b_n)}]\}$

The above is an identity which contains denominator term $(2i+1)!(2j+1)!$ **after** $(AB+BA), CC$ **type binomial expansion.**

Cor. 4.4.1. $\sum_{r=0}^{[n/2]}\sum_{l\geq 0,i|j-r}^{\leq i,j,(n-1)/2-r}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}2^{r+l-i}C_r^{j-l}2^{r+l-j}C_{n-2r}^{2l+1}$
$[C_{\{a_1}C_{a_2}]\cdot\cdot[C_{a_{2i-2l-1}}C_{a_{2i-2l}}][A_{a_{2i-2l+1}}B_{a_{2i-2l+2}}]\cdot\cdot[A_{a_{2r-1}}B_{a_{2r}}]$
$[C_{(b_1}C_{b_2}]\cdot\cdot[C_{b_{2j-2l-1}}C_{b_{2j-2l}}][A_{b_{2j-2l+1}}B_{b_{2j-2l+2}}]\cdot\cdot[A_{b_{2r-1}}B_{b_{2r}}]$
$[A_{a_{2r+1}}B_{b_{2r+1}}+B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_{n-2l-1}}B_{b_{n-2l-1}}+B_{a_{n-2l-1}}A_{b_{n-2l-1}}][C_{a_{n-2l}}C_{b_{n-2l}}]\cdot\cdot[C_{a_n\}}C_{b_n)}]$
$= -\sum_{k\geq i|j}^{\leq n-1-i|j}\frac{n!n!}{(2n)!}\frac{(-2)^{i+j+1}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}}\cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}}\cdot\cdot A_{b_{n-2j-1}}][C_{b_{n-2j}}\cdot\cdot C_{b_n)}]\}$

Cor. 4.4.2. $A=1, B=1, C=1 \Rightarrow$
$\sum_{r=0}^{[n/2]}\sum_{l\geq 0,i|j-r}^{\leq i,j,(n-1)/2-r}(-1)^{r+i+j}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}C_r^{j-l}C_{n-2r}^{2l+1} = \sum_{k\geq i|j}^{\leq n-1-i|j}\frac{2^{2i+2j+2-n}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$

Cor. 4.4.3. $\sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-1-r-k}](-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}2^{r+l-i}C_r^{j-l}2^{r+l-j}C_{n-2r}^{2l+1}C_{n-2r-2l-1}^{k-r-l}$
$[C_{\{a_1}C_{a_2}]\cdot\cdot[C_{a_{2i-2l-1}}C_{a_{2i-2l}}][A_{a_{2i-2l+1}}B_{a_{2i-2l+2}}]\cdot\cdot[A_{a_{2r-1}}B_{a_{2r}}]$
$[C_{(b_1}C_{b_2}]\cdot\cdot[C_{b_{2j-2l-1}}C_{b_{2j-2l}}][A_{b_{2j-2l+1}}B_{b_{2j-2l+2}}]\cdot\cdot[A_{b_{2r-1}}B_{b_{2r}}]$
$[A_{a_{2r+1}}B_{b_{2r+1}}]\cdot\cdot[A_{a_{n+r-l-k-1}}B_{b_{n+r-l-k-1}}][B_{a_{n+r-l-k}}A_{b_{n+r-l-k}}]\cdot\cdot[B_{a_{n-2l-1}}A_{b_{n-2l-1}}][C_{a_{n-2l}}C_{b_{n-2l}}]\cdot\cdot[C_{a_n\}}C_{b_n)}]$
$= -\sum_{k=0}^{n-1}\frac{n!n!}{(2n)!}\frac{(-2)^{i+j+1}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}}\cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}}\cdot\cdot A_{b_{n-2j-1}}][C_{b_{n-2j}}\cdot\cdot C_{b_n)}]\}$
$\Leftrightarrow \sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-1-r-k}](-1)^{r+i+j}2^{2r+2l}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}C_r^{j-l}C_{n-2r}^{2l+1}C_{n-2r-2l-1}^{k-r-l}$
$= -\frac{(-2)^{2i+2j+1}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$

The above final form is the identity for k-item.
4.5 Combinatorial identities equivalent to projection operator conjecture

Thm. 4.5.1. $\sum_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[C_{\{a_1}C_{a_2}-A_{\{a_1}B_{a_2}-B_{\{a_1}A_{a_2}][C_{(b_1}C_{b_2}-A_{(b_1}B_{b_2}-B_{(b_1}A_{b_2}]\cdot\cdot$
$[C_{a_{2r-1}}C_{a_{2r}}-A_{a_{2r-1}}B_{a_{2r}}-B_{a_{2r-1}}A_{a_{2r}}][C_{b_{2r-1}}C_{b_{2r}}-A_{b_{2r-1}}B_{b_{2r}}-B_{b_{2r-1}}A_{b_{2r}}]\}$
$[C_{a_{2r+1}}C_{b_{2r+1}}-A_{a_{2r+1}}B_{b_{2r+1}}-B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[C_{a_n\}}C_{b_n)}-A_{a_n\}}B_{b_n)}-B_{a_n\}}A_{b_n)}]$
$= \sum_{k=0}^{n}\frac{(-1)^n}{C_{2n}^{2k}}\sum_{i,j=0}^{k|(n-k)}\frac{2^{i+j}n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}}\cdot\cdot B_{a_{n-2i}}][C_{a_{n+1-2i}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}}\cdot\cdot A_{b_{n-2j}}][C_{b_{n+1-2j}}\cdot\cdot C_{b_n)}]\}$
$-\sum_{k=0}^{n-1}\frac{(-1)^n}{C_{2n}^{2k+1}}\sum_{i,j=0}^{k|(n-1-k)}\frac{2^{i+j+1}n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$

$\{[A_{\{a_1} \cdot\cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}} \cdot\cdot B_{a_{n-2i-1}}][C_{a_{n-2i}} \cdot\cdot C_{a_n\}}]\}\{[B_{(b_1} \cdot\cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}} \cdot\cdot A_{b_{n-2j-1}}][C_{b_{n-2j}} \cdot\cdot C_{b_n)}]\}$

$\Leftrightarrow$

$$\begin{cases}\sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-r-k}](-1)^{r+i+j}2^{2r+2l}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}C_r^{j-l}C_{n-2r}^{2l}C_{n-2r-2l}^{k-r-l} \\ =\frac{2^{2i+2j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\ \sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-1-r-k}](-1)^{r+i+j}2^{2r+2l}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}C_r^{j-l}C_{n-2r}^{2l+1}C_{n-2r-2l-1}^{k-r-l} \\ =\frac{2^{2i+2j+1}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}\end{cases}$$

If the above two combinatorial identities are true, then the projection operator conjecture and the important conjecture in the previous section are naturally true.

4.6 Equivalent Analysis of combinatorial identities for projection operator conjecture

Cor. 4.6.1.

$$\begin{cases}\sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-r-k}](-1)^{r+i+j}\frac{2^{2r+2l}(2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ =\frac{2^{2i+2j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\ \sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-1-r-k}](-1)^{r+i+j}\frac{2^{2r+2l}(2n-2r)!}{(n-r)!(n-k-r-l-1)!(k-r-l)!(2l+1)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ =\frac{2^{2i+2j+1}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{r=0}^{k|(n-k)}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-r-k}](-4)^{r}\frac{4^{l}(2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ =\frac{(-4)^{i+j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\ \sum_{r=0}^{k|(n-k)}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-r-k}](-4)^{r}\frac{4^{l}(2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ =\frac{(-4)^{i+j}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k+1)!}{(n-k-i)!(n-k-j)!}\end{cases}$$

If the above two combinatorial identities are true, then the projection operator conjecture and the important conjecture in the previous section are naturally true. Thus the conjecture has obviously been greatly simplified. The next step is to strive to prove the above two combinatorial identities, and I will do this again when I have spare time.

4.7 Strict proof of combinatorial identities for projection operator conjecture???

Thm. 4.7.1.

$$\sum_{r=0}^{k|(n-k)}[\sum_{l\geq 0,(i|j)-r}^{\leq i,j,k-r,n-r-k}]\frac{(-4)^{r}4^{l}(2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!}=\frac{(-4)^{i+j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$$

Proof:

$$\sum_{r=0}^{k|(n-k)}\sum_{l=0}^{i|j}(-4)^{r}4^{l}\frac{(2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!}=\frac{(-4)^{i+j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$$ □

Thm. 4.7.2.

$$\sum_{r=0}^{k|(n-k)}[\sum_{l\geq 0,(i|j)-r}^{\leq i,j,k-r,n-r-k}]\frac{(-4)^{r}4^{l}(2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!}=\frac{(-4)^{i+j}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k+1)!}{(n-k-i)!(n-k-j)!}$$

Proof:

$$\sum_{r=0}^{k|(n-k)}\sum_{l=0}^{i|j}(-4)^{r}4^{l}\frac{(2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!}\frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!}=\frac{(-4)^{i+j}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k+1)!}{(n-k-i)!(n-k-j)!}$$ □

5 Expression and verification of specific situation for projection operator conjecture

5.1 Value range analysis

Cor. 5.1.1.

$$\sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-r-k}](-1)^{r+i+j}2^{2r+2l}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}C_r^{j-l}C_{n-2r}^{2l}C_{n-2r-2l}^{k-r-l}=\frac{2^{2i+2j}}{(2i)!(2j)!}\frac{(2k)!}{(k-i)!(k-j)!}\frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$$

Cor. 5.1.2. $0,i|j-r\leq l\leq i,j,k-r,n-r-k\Rightarrow|i-j|\leq r\leq k|(n-k),i|j\leq k\leq n-i|j$

Cor. 5.1.3. $\sum_{r=0}^{[n/2]}[\sum_{l\geq 0,i|j-r}^{\leq i,j,k-r,n-1-r-k}](-1)^{r+i+j}2^{2r+2l}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}C_r^{i-l}C_r^{j-l}C_{n-2r}^{2l+1}C_{n-2r-2l-1}^{k-r-l}$

$=\frac{2^{2i+2j+1}}{(2i+1)!(2j+1)!}\frac{(2k+1)!}{(k-i)!(k-j)!}\frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$

Cor. 5.1.4. $0,i|j-r\leq l\leq i,j,k-r,n-1-r-k\Rightarrow|i-j|\leq r\leq k|(n-1-k),i|j\leq k\leq n-1-i|j$

5.2 i=0,j=0 case

5.2.1 C(2i,2j)=(0,0) case

Cor. 5.2.1. $i = 0, j = 0 \Rightarrow l = 0, r \leq k|(n-k), 0 \leq k \leq n$

Cor. 5.2.2. $\sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!}$

Cor. 5.2.3. $\sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[A_{\{a_1}B_{a_2} + B_{\{a_1}A_{a_2}][A_{(b_1}B_{b_2} + B_{(b_1}A_{b_2}] \cdot\cdot [A_{a_{2r-1}}B_{a_{2r}} + B_{a_{2r-1}}A_{a_{2r}}][A_{b_{2r-1}}B_{b_{2r}} + B_{b_{2r-1}}A_{b_{2r}}]\}$
$[A_{a_{2r+1}}B_{b_{2r+1}} + B_{a_{2r+1}}A_{b_{2r+1}}] \cdot\cdot [A_{a_n\}}B_{b_n)} + B_{a_n\}}A_{b_n)}]$
$= \sum_{k=0}^{n} \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} [A_{\{a_1} \cdot\cdot A_{a_{n-k}} B_{a_{n+1-k}} \cdot\cdot B_{a_n\}}][B_{(b_1} \cdot\cdot B_{b_{n-k}} A_{b_{n+1-k}} \cdot\cdot A_{b_n)}]$
$\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!}$

Cor. 5.2.4. $\sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!2^n}{r!(n-r)!(n-2r)!} = \sum_{k=0}^{n} \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} = 2^{2n}$

5.2.2 C(2i+1,2j+1)=(1,1) case

Cor. 5.2.5. $i = 0, j = 0 \Rightarrow l = 0, r \leq k|(n-1-k), 0 \leq k \leq n-1$

Cor. 5.2.6. $\sum_{r=0}^{k|(n-1-k)} (-4)^r \frac{(2n-2r-1)!}{r!(n-1-r)!(k-r)!(n-1-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-1)!}{(n-1-k)!(n-1-k)!}$

Thm. 5.2.1. $\sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{[A_{\{a_1}B_{a_2} + B_{\{a_1}A_{a_2}][A_{(b_1}B_{b_2} + B_{(b_1}A_{b_2}] \cdot\cdot [A_{a_{2r-1}}B_{a_{2r}} + B_{a_{2r-1}}A_{a_{2r}}][A_{b_{2r-1}}B_{b_{2r}} + B_{b_{2r-1}}A_{b_{2r}}]\}$
$C^1_{n-2r}[A_{a_{2r+1}}B_{b_{2r+1}} + B_{a_{2r+1}}A_{b_{2r+1}}] \cdot\cdot [A_{a_{n-1}}B_{b_{n-1}} + B_{a_{n-1}}A_{b_{n-1}}][C_{a_n\}}C_{b_n)}]$
$= \sum_{k=0}^{n-1} \frac{2}{C_{2n}^{2k+1}} \frac{n!n!}{(n-k-1)!(n-k-1)!k!k!1!1!} \{[A_{\{a_1} \cdot\cdot A_{a_{n-k-1}}][B_{a_{n-k}} \cdot\cdot B_{a_{n-1}}][C_{a_n\}}]\}\{[B_{\{b_1} \cdot\cdot B_{b_{n-k-1}}][A_{b_{n-k}} \cdot\cdot A_{b_{n-1}}][C_{b_n\}}]\}$
$\Leftrightarrow \sum_{r=0}^{k|(n-1-k)} (-4)^r \frac{(2n-2r-1)!}{r!(n-1-r)!(k-r)!(n-1-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-1)!}{(n-1-k)!(n-1-k)!}$

Cor. 5.2.7. $\sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!(n-2r)2^{n-2}}{r!(n-r)!(n-2r)!} = \sum_{k=0}^{n-1} \frac{(2k+1)!}{k!k!} \frac{(2n-2k-1)!}{(n-1-k)!(n-1-k)!} = 2^{2n-3}n(n+1)$

5.2.3 (1,1)⇒(0,0)

Thm. 5.2.2.
$\sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r+1)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k+1)!}{(n-k)!(n-k)!} \Rightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!}$
$n \geq 0, 0 \leq k \leq n$

Proof: $\sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r+1)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k+1)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n$
$\Leftrightarrow \sum_{r=0}^{k|(n-k-2)} (-4)^r \frac{(2n-2r-3)!}{r!(n-r-2)!(k-r)!(n-k-r-2)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-3)!}{(n-k-2)!(n-k-2)!}, n \geq 2, 0 \leq k \leq n-2$
$\Leftrightarrow \sum_{r=1}^{(k+1)|(n-k-1)} (-4)^{r-1} \frac{(2n-2r-1)!}{(r-1)!(n-r-1)!(k-r+1)!(n-k-r-1)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-3)!}{(n-k-2)!(n-k-2)!}, n \geq 2, 0 \leq k \leq n-2$
$\Leftrightarrow \sum_{r=1}^{k|(n-k)} (-4)^{r-1} \frac{(2n-2r-1)!}{(r-1)!(n-r-1)!(k-r)!(n-k-r)!} = \frac{(2k-1)!}{(k-1)!(k-1)!} \frac{(2n-2k-1)!}{(n-k-1)!(n-k-1)!}, n \geq 2, 1 \leq k \leq n-1$
$\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2r)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = -\frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 2, 1 \leq k \leq n-1$
$\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2r)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = -\frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 2, 0 \leq k \leq n$
$\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2r)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = -\frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n$
$\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r+1)!-(2n+1)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n$

$\Rightarrow \frac{(2k+1)!}{k!k!}\frac{(2n-2k+1)!}{(n-k)!(n-k)!} - \sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n+1)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2n-2k)(2k)!}{k!k!}\frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n$

$\Leftrightarrow \sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!}\frac{(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n$ □

5.2.4 Summary

Cor. 5.2.8. $\sum_{r=0}^{[n/2]}(-1)^r\frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = 2^n, \sum_{r=0}^{[n/2]}(-1)^r\frac{(2n-2r)!(n-2r)}{r!(n-r)!(n-2r)!} = n(n+1)2^{n-1}$

Cor. 5.2.9. $\sum_{r=0}^{[n/2]}(-1)^r\frac{r(2n-2r)!}{r!(n-r)!(n-2r)!} = -n(n-1)2^{n-2}, \sum_{r=0}^{[n/2]}(-1)^r\frac{(n-r)(2n-2r)!}{r!(n-r)!(n-2r)!} = n(n+3)2^{n-2}$

Cor. 5.2.10. $\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}\frac{(2n-2k-1)!!}{(2n-2k)!!} = 1, \sum_{k=0}^{n}\frac{(2k+1)!!}{(2k)!!}\frac{(2n-2k+1)!!}{(2n-2k)!!} = C_{n+2}^2$

Cor. 5.2.11.
$\sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!}\frac{(2n-2k)!}{(n-k)!(n-k)!}, \sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n-2r+1)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k+1)!}{k!k!}\frac{(2n-2k+1)!}{(n-k)!(n-k)!}$

5.3 i=1,j=0 and i=0,j=1 cases

5.3.1 C(2i,2j)=(2,0) and C(2i,2j)=(0,2) cases

Cor. 5.3.1. $i=1, j=0|i=0, j=1 \Rightarrow l=0, 1 \leq r \leq k|(n-k), 1 \leq k \leq n-1$

Cor. 5.3.2. $\sum_{r=1}^{k|(n-k)}(-1)^{r+1}2^{2r-1}\frac{(2n-2r)!}{(r-1)!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{(k-1)!k!}\frac{(2n-2k)!}{(n-1-k)!(n-k)!}$

Thm. 5.3.1. $\sum_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$C_r^1C_r^0\{[A_{\{a_1}B_{a_2} + B_{\{a_1}A_{a_2}][A_{(b_1}B_{b_2} + B_{(b_1}A_{b_2}]\cdot\cdot$
$[A_{a_{2r-3}}B_{a_{2r-1}} + B_{a_{2r-3}}A_{a_{2r-1}}][A_{b_{2r-3}}B_{b_{2r-1}} + B_{b_{2r-3}}A_{b_{2r-1}}][C_{a_{2r-1}}C_{a_{2r}}][A_{b_{2r-1}}B_{b_{2r}} + B_{b_{2r-1}}A_{b_{2r}}]\}$
$C_{n-2r}^0[A_{a_{2r+1}}B_{b_{2r+1}} + B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_n\}}B_{b_n)} + B_{a_n\}}A_{b_n)}]$
$= -\sum_{k=0}^{n}\frac{n!n!}{(2n)!}\frac{(2n-2k)!(2k)!}{(n-k-1)!(n-k)!(k-1)!k!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-1}}][B_{a_{n-k}}\cdot\cdot B_{a_{n-2}}][C_{a_{n-1}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k}}][A_{b_{n+1-k}}\cdot\cdot A_{b_n)}]\}$

Thm. 5.3.2. $\sum_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$C_r^0C_r^1\{[A_{\{a_1}B_{a_2} + B_{\{a_1}A_{a_2}][A_{(b_1}B_{b_2} + B_{(b_1}A_{b_2}]\cdot\cdot$
$[A_{a_{2r-3}}B_{a_{2r-1}} + B_{a_{2r-3}}A_{a_{2r-1}}][A_{b_{2r-3}}B_{b_{2r-1}} + B_{b_{2r-3}}A_{b_{2r-1}}][A_{a_{2r-1}}B_{a_{2r}} + B_{a_{2r-1}}A_{a_{2r}}][C_{b_{2r-1}}C_{b_{2r}}]\}$
$C_{n-2r}^0[A_{a_{2r+1}}B_{b_{2r+1}} + B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_n\}}B_{b_n)} + B_{a_n\}}A_{b_n)}]$
$= -\sum_{k=0}^{n}\frac{n!n!}{(2n)!}\frac{(2n-2k)!(2k)!}{(n-k)!(n-k-1)!k!(k-1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k}}][B_{a_{n-k+1}}\cdot\cdot B_{a_n}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-1}}][A_{b_{n-k}}\cdot\cdot A_{b_{n-2})}][C_{b_{n-1}}\cdot\cdot C_{b_n\}}]\}$

Cor. 5.3.3. $-\sum_{r=0}^{[n/2]}(-1)^r\frac{(2r)(2n-2r)!2^n}{r!(n-r)!(n-2r)!} = \sum_{k=0}^{n}\frac{(2k)!(2k)}{k!k!}\frac{(2n-2k)!(2n-2k)}{(n-k)!(n-k)!} = n(n-1)2^{2n-1}$

5.3.2 C(2i+1,2j+1)=(3,1) and C(2i+1,2j+1)=(1,3) cases

Cor. 5.3.4.
$\sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!1!}\frac{r!}{1!0!(r-1)!r!} = \frac{(-4)^1}{3!1!}\frac{(2k+1)!}{(k-1)!k!}\frac{(2n-2k+1)!}{(n-k-1)!(n-k)!}$
$\sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n-2r+1)!}{(r-1)!(n-r)!(n-k-r)!(k-r)!} = -\frac{2}{3}\frac{(2k+1)!}{(k-1)!k!}\frac{(2n-2k+1)!}{(n-k-1)!(n-k)!}$

Thm. 5.3.3. $\sum_{r=0}^{[n/2]}(-1)^r\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$C_r^1[C_{\{a_1}C_{a_2}][A_{a_3}B_{a_4} + B_{a_3}A_{a_4}]\cdot\cdot[A_{a_{2r-1}}B_{a_{2r}} + B_{a_{2r-1}}A_{a_{2r}}]C_r^0[A_{b_1}B_{b_2} + B_{b_1}A_{b_2}]\cdot\cdot[A_{b_{2r-1}}B_{b_{2r}} + B_{b_{2r-1}}A_{b_{2r}}]$
$C_{n-2r}^1[A_{a_{2r+1}}B_{b_{2r+1}} + B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_{n-1}}B_{b_{n-1}} + B_{a_{n-1}}A_{b_{n-1}}][C_{a_n\}}C_{b_n)}]$
$= -\sum_{k=1}^{n-2}\frac{(-2)^2}{3!1!}\frac{(2k+1)!}{(k-1)!k!}\frac{(2n-2k-1)!}{(n-k-2)!(n-k-1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-2}}][B_{a_{n-k-1}}\cdot\cdot B_{a_{n-3}}][C_{a_{n-2}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-1}}][A_{b_{n-k}}\cdot\cdot A_{b_{n-1}}][C_{b_n)}]\}$

Thm. 5.3.4. $\sum_{r=0}^{[n/2]}(-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$C_r^0[A_{a_1}B_{a_2}+B_{a_1}A_{a_2}]\cdot\cdot[A_{a_{2r-1}}B_{a_{2r}}+B_{a_{2r-1}}A_{a_{2r}}]C_r^1[C_{\{a_1}C_{a_2}][A_{b_3}B_{b_4}+B_{b_3}A_{b_4}]\cdot\cdot[A_{b_{2r-1}}B_{b_{2r}}+B_{b_{2r-1}}A_{b_{2r}}]$
$C_{n-2r}^1[A_{a_{2r+1}}B_{b_{2r+1}}+B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_{n-1}}B_{b_{n-1}}+B_{a_{n-1}}A_{b_{n-1}}][C_{a_n\}}C_{b_n)}]$
$=-\sum_{k=1}^{n-2}\frac{(-2)^2}{1!3!}\frac{(2k+1)!}{k!(k-1)!}\frac{(2n-2k-1)!}{(n-k-1)!(n-k-2)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-1}}][B_{a_{n-k}}\cdot\cdot B_{a_{n-1}}][C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-2}}][A_{b_{n-k-1}}\cdot\cdot A_{b_{n-3}}][C_{b_{n-2}}\cdot\cdot C_{b_n)}]\}$

Cor. 5.3.5. $\frac{3}{8}\sum_{r=0}^{[n/2]}(-1)^r\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}r(n-2r)2^n=-\sum_{k=1}^{n-2}\frac{(2k+1)!}{(k-1)!k!}\frac{(2n-2k-1)!}{(n-k-2)!(n-k-1)!}$

5.4 i=1,j=1 case

5.4.1 C(2i,2j)=(2,2) case

Cor. 5.4.1. $0,i|j-r\leq l\leq i,j,k-r,n-r-k\Rightarrow|i-j|\leq r\leq k|(n-k),1\leq k\leq n-1$

Thm. 5.4.1. $\sum_{r=0}^{[n/2]}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$\{(C_r^1)^2\{[A_{\{a_1}B_{a_2}+B_{\{a_1}A_{a_2}][A_{(b_1}B_{b_2}+B_{(b_1}A_{b_2}]\cdot\cdot[A_{a_{2r-3}}B_{a_{2r-2}}+B_{a_{2r-3}}A_{a_{2r-2}}][A_{b_{2r-3}}B_{b_{2r-2}}+B_{b_{2r-3}}A_{b_{2r-2}}]$
$[C_{a_{2r-1}}C_{a_{2r}}][C_{b_{2r-1}}C_{b_{2r}}]\}[A_{a_{2r+1}}B_{b_{2r+1}}+B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_n\}}B_{b_n)}+B_{a_n\}}A_{b_n)}]$
$+\{[A_{\{a_1}B_{a_2}+B_{\{a_1}A_{a_2}][A_{(b_1}B_{b_2}+B_{(b_1}A_{b_2}]\cdot\cdot[A_{a_{2r-1}}B_{a_{2r}}+B_{a_{2r-1}}A_{a_{2r}}][A_{b_{2r-1}}B_{b_{2r}}+B_{b_{2r-1}}A_{b_{2r}}]\}$
$C_{n-2r}^2[A_{a_{2r+1}}B_{b_{2r+1}}+B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_{n-2}}B_{b_{n-2}}+B_{a_{n-2}}A_{b_{n-2}}][C_{a_{n-1}}C_{b_{n-1}}C_{a_n\}}C_{b_n)}]\}$
$=\sum_{k=0}^{n}\frac{n!n!}{(2n)!}\frac{(2n-2k)!(2k)!}{(n-k-1)!(n-k-1)!(k-1)!(k-1)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-1}}][B_{a_{n-k}}\cdot\cdot B_{a_{n-2}}][C_{a_{n-1}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-1}}][A_{b_{n-k}}\cdot\cdot A_{b_{n-2}}][C_{b_{n-1}}\cdot\cdot C_{b_n)}]\}$

Cor. 5.4.2. $\sum_{r=0}^{[n/2]}(-1)^r\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}[3r^2-(2n-1)r+\frac{n(n-1)}{2}]2^{n-2}=\sum_{k=0}^{n}\frac{(2n-2k)!(2k)!}{(n-k-1)!(n-k-1)!(k-1)!(k-1)!}$

5.4.2 C(2i+1,2j+1)=(3,3) case

Cor. 5.4.3. $\sum_{r=0}^{k|(n-k)}\sum_{l\geq 0,1-r}^{\leq 1,k-r,n-r-k}(-4)^r\frac{4^l(2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!}\frac{r!}{(1-l)!(1-l)!(r+l-1)!(r+l-1)!}$
$=\frac{(-4)^2}{3!3!}\frac{(2k+1)!}{(k-1)!(k-1)!}\frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!}$

Cor. 5.4.4. $\sum_{r=0}^{k|(n-k)}(-4)^r\frac{4^0(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!1!}\frac{r!}{1!1!(r-1)!(r-1)!}+(-4)^r\frac{4^1(2n-2r+1)!}{(n-r)!(n-k-r-1)!(k-r-1)!3!}\frac{r!}{0!0!r!r!}$
$=\frac{(-4)^2}{3!3!}\frac{(2k+1)!}{(k-1)!(k-1)!}\frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!}$
$\sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!}\frac{r^2}{r!}+(-4)^r\frac{4^1(2n-2r+1)!}{(n-r)!(n-k-r-1)!(k-r-1)!3!}\frac{1}{r!}=\frac{(-4)^2}{3!3!}\frac{(2k+1)!}{(k-1)!(k-1)!}\frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!}$
$\sum_{r=0}^{k|(n-k)}(-4)^r\frac{(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!}\frac{r^2}{r!}+(-4)^r\frac{4^1(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!3!}\frac{(n-k-r)(k-r)}{r!}=\frac{(-4)^2}{3!3!}\frac{(2k+1)!}{(k-1)!(k-1)!}\frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!}$
$\sum_{r=0}^{k|(n-k)}-(-4)^{r-1}\frac{(2n-2r+1)!}{r!(n-r)!(n-k-r)!(k-r)!}[9r^2+6(n-k-r)(k-r)]=\frac{(2k+1)!}{(k-1)!(k-1)!}\frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!}$

Thm. 5.4.2. $\sum_{r=0}^{[n/2]}\sum_{l\geq 0,1-r}^{\leq 1,(n-1)/2-r}(-1)^r\frac{n!n!}{(2n)!}\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$
$C_r^{1-l}[C_{\{a_1}C_{a_2}]\cdot\cdot[C_{a_{1-2l}}C_{a_{2-2l}}]2^{r+l-1}[A_{a_{3-2l}}B_{a_{4-2l}}]\cdot\cdot[A_{a_{2r-1}}B_{a_{2r}}]$
$C_r^{1-l}[C_{(b_1}C_{b_2}]\cdot\cdot[C_{b_{1-2l}}C_{b_{2-2l}}]2^{r+l-1}[A_{b_{3-2l}}B_{b_{4-2l}}]\cdot\cdot[A_{b_{2r-1}}B_{b_{2r}}]$
$C_{n-2r}^{2l+1}[A_{a_{2r+1}}B_{b_{2r+1}}+B_{a_{2r+1}}A_{b_{2r+1}}]\cdot\cdot[A_{a_{n-2l-1}}B_{b_{n-2l-1}}+B_{a_{n-2l-1}}A_{b_{n-2l-1}}][C_{a_{n-2l}}C_{b_{n-2l}}]\cdot\cdot[C_{a_n\}}C_{b_n)}]$
$=-\sum_{k\geq 1}^{\leq n-2}\frac{n!n!}{(2n)!}\frac{(-2)^3}{3!3!}\frac{(2k+1)!}{(k-1)!(k-1)!}\frac{(2n-2k-1)!}{(n-k-2)!(n-k-2)!}$
$\{[A_{\{a_1}\cdot\cdot A_{a_{n-k-2}}][B_{a_{n-k-1}}\cdot\cdot B_{a_{n-3}}][C_{a_{n-2}}\cdot\cdot C_{a_n\}}]\}\{[B_{(b_1}\cdot\cdot B_{b_{n-k-2}}][A_{b_{n-k-1}}\cdot\cdot A_{b_{n-3}}][C_{b_{n-2}}\cdot\cdot C_{b_n)}]\}$

Cor. 5.4.5. $\frac{9}{16}\sum_{r=0}^{[n/2]}(-1)^r\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}2^n[r^2(n-2r)+C_{n-2r}^3]=\sum_{k=1}^{n-2}\frac{(2k+1)!}{(k-1)!(k-1)!}\frac{(2n-2k-1)!}{(n-k-2)!(n-k-2)!}$

6 Analysis of recursive relation for Klein-Gordon equation projection operators

Klein Gordon n+1-projection operator can uniquely expand by 1-projection operator. Then we can assume that it is equal to all possible combinations of n-projection operators multiplied by 1-projection

operators (including undetermined coefficients). Finally, we can calculate out the expansion coefficients of n+1-projection operator=$\sum$ n-projection operator multiplied by 1-projection operator. However the solution is not unique. And there are generally infinite solutions. So there is no clear physical meaning.

Def. 6.0.1. $\begin{cases}\hat{P}_{a_1\cdot\cdot a_n b_1\cdot\cdot b_n}(n)=\frac{1}{(n!)^2}\sum\limits_{P(a)}^{P(b)}\sum\limits_{r=0}^{[n/2]}k_r\hat{P}_{a_1a_2}\hat{P}_{b_1b_2}\cdot\cdot\hat{P}_{a_{2r-1}a_{2r}}\hat{P}_{b_{2r-1}b_{2r}}\prod\limits_{i=2r+1}^{n}\hat{P}_{a_ib_i}\\ \hat{P}_{a_1\cdot\cdot a_n b_1\cdot\cdot b_n}(n):=\eta_{b_1}^{a'_1}\eta_{b_2}^{a'_2}\cdot\cdot\eta_{b_n}^{a'_n}\hat{P}_{a_1\cdot\cdot a_n a'_1\cdot\cdot a'_n}(n)\end{cases}$

6.1 Basic properties of projection operator for spin-1 particle Klein-Gordon equation

Cor. 6.1.1. $P_{a_1b_1}=P_{b_1a_1}, p^{a_1}P_{a_1b_1}=0, P_{a_1c_1}\delta^{c_1d_1}P_{d_1b_1}=P_{a_1b_1}$

Cor. 6.1.2. $\begin{cases}P_{a_1b_1},P_{a_1a_2};P_{b_1b_2},P_{a_2b_2};\\ P_{a_1b_1}P_{a_2b_2},P_{a_1a_2}P_{b_1b_2};\end{cases}$

6.2 Basic properties of projection operator for spin-2 particle Klein-Gordon equation

Cor. 6.2.1. $\begin{cases}P_{a_1b_1;a_2b_2},P_{a_1a_2;b_1b_2};P_{a_1a_2;a_3b_1},P_{a_1b_1;b_2b_3};\\ P_{a_1b_1;a_2b_2}P_{a_3b_3},P_{a_1a_2;b_1b_2}P_{a_3b_3};P_{a_1a_2;a_3b_1}P_{b_2b_3};\end{cases}$

Cor. 6.2.2.
$$\begin{cases}P_{a_1a_2;b_1b_2}(2)=\frac{1}{(2!)^2}\{[P_{\{a_1(b_1}P_{a_2\}b_2)}]-\frac{1}{3}[P_{\{a_1a_2\}}P_{(b_1b_2)}]\}=\frac{2}{(2!)^2}\{P_{a_1b_1}P_{a_2b_2}+P_{a_1b_2}P_{a_2b_1}-\frac{2}{3}P_{a_1a_2}P_{b_1b_2}\}\\ P_{a_1b_1;a_2b_2}(2)=\frac{2}{(2!)^2}\{P_{a_1a_2}P_{b_1b_2}+P_{a_1b_2}P_{a_2b_1}-\frac{2}{3}P_{a_1b_1}P_{a_2b_2}\}\\ P_{a_1a_2;a_3b_1}(2)=\frac{2}{(2!)^2}\{P_{a_1b_1}P_{a_2a_3}+P_{a_1a_3}P_{a_2b_1}-\frac{2}{3}P_{a_1a_2}P_{b_1a_3}\}\\ P_{b_1b_2;b_3a_1}(2)=\frac{2}{(2!)^2}\{P_{b_1a_1}P_{b_2b_3}+P_{b_1b_3}P_{b_2a_1}-\frac{2}{3}P_{b_1b_2}P_{a_1b_3}\}\end{cases}$$

6.3 Basic properties of projection operator for spin-3 particle Klein-Gordon equation

Cor. 6.3.1. $\begin{cases}P_{a_1a_2b_3;b_1b_2a_3},P_{a_1a_2a_3;b_1b_2b_3};P_{a_1a_2b_1;a_3a_4b_2},P_{b_1b_2a_1;b_3b_4a_2};P_{a_1a_2a_3;a_4b_1b_2},P_{a_1a_2b_4;b_1b_2b_3};\\ P_{a_1a_2b_3;b_1b_2a_3}P_{a_4b_4},P_{a_1a_2a_3;b_1b_2b_3}P_{a_4b_4};P_{a_1a_2b_1;a_3a_4b_2}P_{b_3b_4},P_{a_1a_2a_3;a_4b_1b_2}P_{b_3b_4};\end{cases}$

6.4 Basic properties of projection operator for spin-n particle Klein-Gordon equation

Cor. 6.4.1.
$$\begin{cases}P_{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_n;b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n};P_{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_{n-1}a_{n+1};b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n};\\ P_{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_n;b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n}P_{a_{n+1}b_{n+1}}:(n+1)-[(n+1)/2];\\ P_{a_1\cdot\cdot a_lb_l\cdot\cdot b_{n-1};b_1\cdot\cdot b_{l-1}a_{l+1}\cdot\cdot a_{n+1}}P_{b_nb_{n+1}}:n-[n/2];\\ k=n,\cdots,[(n+1)/2],l=n,\cdots,[n/2]+1\\ (n+1)-[(n+1)/2]+n-[n/2]=n+1\end{cases}$$

Cor. 6.4.2. $P_{a_1\cdot\cdot a_{n+1};b_1\cdot\cdot b_{n+1}}$
$$=\frac{1}{[(n+1)!]^2}\{\sum_{k=n}^{[(n+1)/2]}B_kP_{\{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_n;(b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n}P_{a_{n+1}\}b_{n+1})}+\sum_{l=n}^{[n/2]+1}C_lP_{\{a_1\cdot\cdot a_lb_l\cdot\cdot b_{n-1};(b_1\cdot\cdot b_{l-1}a_{l+1}\cdot\cdot a_{n+1}\}}P_{b_nb_{n+1})}\}$$

Cor. 6.4.3.
$$\begin{cases}P_{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_n;b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n};P_{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_{n-1}a_{n+1};b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n};\\ P_{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_n;b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n}P_{a_{n+1}b_{n+1}}:(n+1)-[(n+1)/2];\\ P_{a_1\cdot\cdot a_lb_l\cdot\cdot b_{n-1};b_1\cdot\cdot b_{l-1}a_{l+1}\cdot\cdot a_{n+1}}P_{b_nb_{n+1}}:n-[n/2];\\ k=0,\cdots,[n/2],l=1,\cdots,[(n+1)/2]\\ [n/2]+[(n-1)/2]+2=n+1\end{cases}$$

Cor. 6.4.4. $P_{a_1\cdot\cdot a_{n+1};b_1\cdot\cdot b_{n+1}}$
$$=\frac{1}{[(n+1)!]^2}\{\sum_{k=0}^{[n/2]}B_kP_{\{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_n;(b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n}P_{a_{n+1}\}b_{n+1})}$$
$$+\sum_{j=1}^{[(n+1)/2]}A_jP_{\{a_1\cdot\cdot a_{j-1}b_{j+1}\cdot\cdot b_{n+1};(b_1\cdot\cdot b_ja_j\cdot\cdot a_{n-1}\}}P_{a_na_{n+1})}+\sum_{l=1}^{[(n+1)/2]}C_lP_{\{a_1\cdot\cdot a_lb_l\cdot\cdot b_{n-1};(b_1\cdot\cdot b_{l-1}a_{l+1}\cdot\cdot a_{n+1}\}}P_{b_nb_{n+1})}\}$$

Cor. 6.4.5. $P_{a_1\cdot\cdot a_{n+1};b_1\cdot\cdot b_{n+1}}$
$$=\frac{1}{[(n+1)!]^2}\{\sum_{k=0}^{[n/2]}B_kP_{\{a_1\cdot\cdot a_kb_{k+1}\cdot\cdot b_n;(b_1\cdot\cdot b_ka_{k+1}\cdot\cdot a_n}P_{a_{n+1}\}b_{n+1})}$$
$$+\sum_{l=1}^{[(n+1)/2]}C_lP_{\{a_1\cdot\cdot a_{l-1}b_{l+1}\cdot\cdot b_{n+1};(b_1\cdot\cdot b_la_l\cdot\cdot a_{n-1}\}}P_{a_na_{n+1})}+\sum_{l=1}^{[(n+1)/2]}C_lP_{\{a_1\cdot\cdot a_lb_l\cdot\cdot b_{n-1};(b_1\cdot\cdot b_{l-1}a_{l+1}\cdot\cdot a_{n+1}\}}P_{b_nb_{n+1})}\}$$

Cor. 6.4.6. $P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}}$
$= \frac{1}{[(n+1)!]^2}\{\sum_{k=0}^{0} B_k P_{\{a_1\cdots a_k b_{k+1}\cdots b_n;(b_1\cdots b_k a_{k+1}\cdots a_n} P_{a_{n+1}\}b_{n+1})}$
$+ \sum_{l=1}^{1} C_l P_{\{a_1\cdots a_{l-1}b_{l+1}\cdots b_{n+1};(b_1\cdots b_l a_l\cdots a_{n-1}\}} P_{a_n a_{n+1})} + \sum_{l=1}^{1} C_l P_{\{a_1\cdots a_l b_l\cdots b_{n-1};(b_1\cdots b_{l-1}a_{l+1}\cdots a_{n+1}\}} P_{b_n b_{n+1})}\}$

6.5 Expansion of projection operator for spin-2 particle Klein-Gordon equation(Unique)

Cor. 6.5.1. $P_{a_1a_2b_1b_2}(2) = \frac{1}{(2!)^2}\{[P_{\{a_1(b_1}P_{a_2\}b_2)}] - \frac{1}{3}[P_{\{a_1a_2\}}P_{(b_1b_2)}]\}$

Proof: $P_{a_1a_2;b_1b_2}$
$= \frac{1}{(2!)^2}\{\sum_{k=0}^{0} B_k(2) P_{\{a_1\cdots a_k b_{k+1}\cdots(b_1;b_1\cdots b_k a_{k+1}\cdots a_1}P_{a_2\}b_2)} + \sum_{l=1}^{1} C_l(2) P_{\{a_1\cdots a_l b_l\cdots b_0;(b_1\cdots b_{l-1}a_{l+1}\cdots a_2\}}P_{b_1b_2)}\}$
$= \frac{1}{(2!)^2}\{B_0(2)P_{(b_1\{a_1}P_{a_2\}b_2)} + C_1(2)P_{\{a_1a_2\}}P_{(b_1b_2)}\}$
$\Rightarrow B_0(2) = 1, C_1(2) = -\frac{1}{3}$ □

6.6 Expansion of projection operator for spin-3 particle Klein-Gordon equation (Not unique, so it has no significant physical meaning.)

Cor. 6.6.1. $P_{a_1a_2a_3b_1b_2b_3}(3) = \frac{1}{(3!)^2}\{[P_{\{a_1(b_1}P_{a_2b_2}P_{a_3\}b_3)}] - \frac{3}{5}[P_{\{a_1a_2}P_{(b_1b_2}][P_{a_3\}b_3)}]\}$

Lem. 6.6.1.
$$\begin{cases} P_{\{a_1a_2(b_1b_2}(2)P_{a_3\}b_3)} = \{P_{\{a_1(b_1}P_{a_2b_2} - \frac{1}{3}[P_{\{a_1a_2}P_{(b_1b_2}]\}P_{a_3\}b_3)} \\ P_{\{a_1(b_1;a_2b_2}(2)P_{a_3\}b_3)} = \frac{1}{2}\{P_{\{a_1a_2}P_{(b_1b_2} + \frac{1}{3}[P_{\{a_1(b_1}P_{a_2b_2}]\}P_{a_3\}b_3)} \\ P_{\{a_1a_2;a_3\}(b_1}(2)P_{b_2b_3)} = \frac{2}{3}P_{\{a_1a_2}P_{a_3\}(b_1}P_{b_2b_3)} \end{cases}$$

Proof:
$$\begin{cases} P_{\{a_1a_2;(b_1b_2}(2)P_{a_3\}b_3)} = \frac{2}{(2!)^2}\{P_{\{a_1(b_1}P_{\{a_2(b_2} + P_{\{a_1(b_2}P_{a_2b_1} - \frac{2}{3}P_{a_1a_2}P_{b_1b_2}\}P_{a_3\}b_3)} \\ = \{P_{\{a_1(b_1}P_{a_2b_2} - \frac{1}{3}[P_{\{a_1a_2}P_{(b_1b_2}]\}P_{a_3\}b_3)} \\ P_{\{a_1(b_1;a_2b_2}(2)P_{a_3\}b_3)} = \frac{2}{(2!)^2}\{P_{\{a_1a_2}P_{(b_1b_2} + P_{\{a_1(b_2}P_{a_2b_1} - \frac{2}{3}P_{\{a_1(b_1}P_{a_2b_2}\}P_{a_3\}b_3)} \\ = \frac{1}{2}\{P_{\{a_1a_2}P_{(b_1b_2} + \frac{1}{3}[P_{\{a_1(b_1}P_{a_2b_2}]\}P_{a_3\}b_3)} \\ P_{\{a_1a_2;a_3\}(b_1}(2)P_{b_2b_3)} = \frac{2}{(2!)^2}\{P_{\{a_1(b_1}P_{a_2a_3} + P_{\{a_1a_3}P_{a_2(b_1} - \frac{2}{3}P_{\{a_1a_2}P_{(b_1a_3}\}P_{b_2b_3)} = \frac{2}{3}P_{\{a_1a_2}P_{(b_1b_2}P_{a_3\}b_3)} \\ P_{(b_1b_2;b_3)\{a_1}(2)P_{a_2a_3\}} = \frac{2}{(2!)^2}\{P_{(b_1\{a_1}P_{b_2b_3} + P_{(b_1b_3}P_{b_2\{a_1} - \frac{2}{3}P_{(b_1b_2}P_{\{a_1b_3}\}P_{a_2a_3\}} = \frac{2}{3}P_{\{a_1a_2}P_{(b_1b_2}P_{a_3\}b_3)} \end{cases}$$ □

Thm. 6.6.1. $P_{a_1a_2a_3;b_1b_2b_3}(3)$
$= \frac{1}{(3!)^2}\{P_{\{a_1a_2;(b_1b_2}(2)P_{a_3\}b_3)} - \frac{2}{5}P_{\{a_1a_2;a_3\}(b_1}(2)P_{b_2b_3)}\}$
$= \frac{1}{(3!)^2}\{6P_{\{a_1(b_1;a_2b_2}(2)P_{a_3\}b_3)} - \frac{27}{5}P_{\{a_1a_2;a_3\}(b_1}(2)P_{b_2b_3)}\}$
$= \frac{1}{(3!)^2}\{\frac{27}{25}P_{\{a_1a_2;(b_1b_2}(2)P_{a_3\}b_3)} - \frac{12}{25}P_{\{a_1(b_1;a_2b_2}(2)P_{a_3\}b_3)}\}$
$= \frac{1}{(3!)^2}\{\frac{6}{7}P_{\{a_1a_2;(b_1b_2}(2)P_{a_3\}b_3)} + \frac{6}{7}P_{\{a_1(b_1;a_2b_2}(2)P_{a_3\}b_3)} - \frac{39}{35}P_{\{a_1a_2;a_3\}(b_1}(2)P_{b_2b_3)}\}$
$= \frac{1}{(3!)^2}\{\frac{6}{5}P_{\{a_1a_2;(b_1b_2}(2)P_{a_3\}b_3)} - \frac{6}{5}P_{\{a_1(b_1;a_2b_2}(2)P_{a_3\}b_3)} + \frac{3}{5}P_{\{a_1a_2;a_3\}(b_1}(2)P_{b_2b_3)}\}$

Proof: $P_{a_1a_2a_3;b_1b_2b_3}$
$= \frac{1}{(3!)^2}\{\sum_{k=0}^{1} B_k P_{\{a_1\cdots a_k b_{k+1}\cdots b_2;(b_1\cdots b_k a_{k+1}\cdots a_2}P_{a_3\}b_3)} + \sum_{l=1}^{1} C_l P_{\{a_1\cdots a_l(b_l\cdots b_1;b_1\cdots b_{l-1}a_{l+1}\cdots a_3\}}P_{b_2b_3)}\}$
$= \frac{1}{(3!)^2}\{B_1(3)P_{\{a_1(b_2;b_1a_2}P_{a_3\}b_3)} + B_0(3)P_{(b_1b_2;\{a_1a_2}P_{a_3\}b_3)} + C_1(3)P_{\{a_1(b_1;a_2a_3\}}P_{b_2b_3)}\}$
$= \frac{1}{(3!)^2}\{B_1(3)P_{\{a_1(b_1;a_2b_2}P_{a_3\}b_3)} + B_0(3)P_{\{a_1a_2;(b_1b_2}P_{a_3\}b_3)} + C_1(3)P_{\{a_1a_2;a_3\}(b_1}P_{b_2b_3)}\}$
$= \frac{1}{(3!)^2}\{B_1(3)\frac{1}{2}\{P_{\{a_1a_2}P_{(b_1b_2} + \frac{1}{3}[P_{\{a_1(b_1}P_{a_2b_2}]\}P_{a_3\}b_3)} + B_0(3)\{P_{\{a_1(b_1}P_{a_2b_2} - \frac{1}{3}[P_{\{a_1a_2}P_{(b_1b_2}]\}P_{a_3\}b_3)}$
$+ C_1(3)\frac{2}{3}P_{\{a_1a_2}P_{a_3\}(b_1}P_{b_2b_3)}\}$
$= \frac{1}{(3!)^2}\{[\frac{1}{6}B_1(3) + B_0(3)]P_{\{a_1(b_1}P_{a_2b_2}P_{a_3\}b_3)} + [\frac{1}{2}B_1(3) - \frac{1}{3}B_0(3) + \frac{2}{3}C_1(3)]P_{\{a_1a_2}P_{(b_1b_2}P_{a_3\}b_3)}\}$
$\Leftrightarrow [\frac{1}{6}B_1(3) + B_0(3)] = 1, [\frac{1}{2}B_1(3) - \frac{1}{3}B_0(3) + \frac{2}{3}C_1(3)] = -\frac{3}{5}$
$\Leftarrow B_0(3) = 1, B_1(3) = 0, C_1(3) = -\frac{2}{5}$
$B_0(3) = 0, B_1(3) = 6, C_1(3) = -\frac{27}{5}$
$B_0(3) = \frac{27}{25}, B_1(3) = -\frac{12}{25}, C_1(3) = 0$
$B_0(3) = \frac{6}{7}, B_1(3) = \frac{6}{7}, C_1(3) = -\frac{39}{35}$
$B_0(3) = \frac{6}{5}, B_1(3) = -\frac{6}{5}, C_1(3) = \frac{3}{5}$ □

6.7 Expansion of projection operator for spin-4 particle Klein-Gordon equation (Not unique and complex, temporarily placed.)

Cor. 6.7.1.
$$\begin{cases} P_{a_1a_2b_3;b_1b_2a_3}, P_{a_1a_2a_3;b_1b_2b_3}; P_{a_1a_2b_1;a_3a_4b_2}, P_{b_1b_2a_1;b_3b_4a_2}; P_{a_1a_2a_3;a_4b_1b_2}, P_{a_1a_2b_4;b_1b_2b_3}; \\ P_{a_1a_2b_3;b_1b_2a_3}P_{a_4b_4}, P_{a_1a_2a_3;b_1b_2b_3}P_{a_4b_4}; P_{a_1a_2b_1;a_3a_4b_2}P_{b_3b_4}, P_{a_1a_2a_3;a_4b_1b_2}P_{b_3b_4}; \end{cases}$$

Cor. 6.7.2. $P_{a_1a_2a_3b_1b_2b_3}(3) = \frac{1}{(3!)^2}\{[P_{\{a_1(b_1}P_{a_2b_2}P_{a_3\}b_3)}] - \frac{3}{5}[P_{\{a_1a_2}P_{(b_1b_2}][P_{a_3\}b_3)}]\}$

Cor. 6.7.3. $P_{a_1a_2a_3a_4b_1b_2b_3b_4}(4)$
$= \frac{1}{(4!)^2}\{[P_{\{a_1(b_1}P_{a_2b_2}P_{a_3b_3}P_{a_4\}b_4)}] - \frac{6}{7}[P_{\{a_1a_2}P_{(b_1b_2}][P_{a_3b_3}P_{a_4\}b_4)}] + \frac{3}{35}[P_{\{a_1a_2}P_{(b_1b_2}P_{a_3a_4\}}P_{b_3b_4)}]\}$

6.8 Expansion of projection operator for spin-n particle Klein-Gordon equation (Not unique and complex)

Cor. 6.8.1. $P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = \frac{1}{[(n+1)!]^2}\{\sum_{k=0}^{[n/2]} B_k P_{\{a_1\cdots a_k b_{k+1}\cdots b_n;(b_1\cdots b_k a_{k+1}\cdots a_n}P_{a_{n+1}\}b_{n+1})}$
$+ \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1\cdots a_{l-1}b_{l+1}\cdots b_{n+1};(b_1\cdots b_l a_l\cdots a_{n-1}\}}P_{a_na_{n+1})} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1\cdots a_l b_l\cdots b_{n-1};(b_1\cdots b_{l-1}a_{l+1}\cdots a_{n+1}\}}P_{b_nb_{n+1})}\}$
$P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = \frac{1}{(n+1)!}P_{\{a_1\cdots a_{n+1}\};b_1\cdots b_{n+1}}, P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = \frac{1}{(n+1)!}P_{a_1\cdots a_{n+1};(b_1\cdots b_{n+1})}$
$P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = P_{b_1\cdots b_{n+1};a_1\cdots a_{n+1}}, p^{a_1}P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = 0$
$\delta^{a_1a_2}P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = 0, \delta^{a_{n+1}b_{n+1}}P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = \frac{2n+3}{2n+1}P_{a_1\cdots a_n;b_1\cdots b_n}$

$\Leftrightarrow P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = \frac{1}{[(n+1)!]^2}\{\sum_{k=0}^{[n/2]} B_k P_{\{a_1\cdots a_k b_{k+1}\cdots b_n;(b_1\cdots b_k a_{k+1}\cdots a_n}P_{a_{n+1}\}b_{n+1})}$
$+ \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1\cdots a_{l-1}b_{l+1}\cdots b_{n+1};(b_1\cdots b_l a_l\cdots a_{n-1}\}}P_{a_na_{n+1})} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1\cdots a_l b_l\cdots b_{n-1};(b_1\cdots b_{l-1}a_{l+1}\cdots a_{n+1}\}}P_{b_nb_{n+1})}\}$
$\delta^{a_1a_2}P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = 0, \delta^{a_{n+1}b_{n+1}}P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = \frac{2n+3}{2n+1}P_{a_1\cdots a_n;b_1\cdots b_n}$

$\Leftrightarrow P_{a_1\cdots a_{n+1};b_1\cdots b_{n+1}} = \frac{1}{[(n+1)!]^2}\{\sum_{k=0}^{[n/2]} B_k P_{\{a_1\cdots a_k b_{k+1}\cdots b_n;(b_1\cdots b_k a_{k+1}\cdots a_n}P_{a_{n+1}\}b_{n+1})}$
$+ \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1\cdots a_{l-1}b_{l+1}\cdots b_{n+1};(b_1\cdots b_l a_l\cdots a_{n-1}\}}P_{a_na_{n+1})} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1\cdots a_l b_l\cdots b_{n-1};(b_1\cdots b_{l-1}a_{l+1}\cdots a_{n+1}\}}P_{b_nb_{n+1})}\}$

7 Translational quasi projection operator

7.1 Spin-1 basis completeness

Def. 7.1.1. $\varepsilon_a(\vec{p},\kappa) := [i\lambda_m(\vec{p},\kappa),0]_a, \varepsilon_a(\vec{p},0) := \frac{1}{m}[iE\lambda_m(\vec{p},0), i|\vec{p}|]_a, \varepsilon_a(\vec{p},0;0) := \frac{p_a}{m}$

Cor. 7.1.1.
$$\begin{cases}\sum_{h=1}^{-1}\varepsilon_a(\vec{p},h)\bar{\varepsilon}_b(\vec{p},h) - \varepsilon_a(\vec{p},0;0)\bar{\varepsilon}_b(\vec{p},0;0) = \delta_{ab}, \bar{\varepsilon}_a(\vec{p},h;s) := \varepsilon^+_{a'}(\vec{p},h;s)\eta^{a'}_a \\ \bar{\varepsilon}^a(\vec{p},h')\varepsilon_a(\vec{p},h) = \delta_{h'h}, \bar{\varepsilon}^a(\vec{p},0;0)\varepsilon_a(\vec{p},0;0) = -1, \bar{\varepsilon}^a(\vec{p},0;0)\varepsilon_a(\vec{p},h) = 0, \bar{\varepsilon}^a(\vec{p},h)\varepsilon_a(\vec{p},0;0) = 0\end{cases}$$

Cor. 7.1.2.
$\bar{\varepsilon}_a(\vec{p},h';s')\varepsilon_b(\vec{p},h;s) = \eta_{s's}\delta_{h'h}, \sum_{s=1}^{0}\sum_{h=s}^{-s}\eta_{ss}\varepsilon_a(\vec{p},h;s)\bar{\varepsilon}_b(\vec{p},h;s) = \delta_{ab}; \eta_{11} := 1, \eta_{00} := -1, \eta_{10} := 0, \eta_{01} := 0$

7.2 Conjecture on completeness of general spin bases

Ass. 7.2.1.
$$\begin{cases}\bar{\varepsilon}^{\overbrace{a\cdots b}^{n}}(\vec{p},h';s')\varepsilon_{\underbrace{a\cdots b}_{n}}(\vec{p},h;s) = \eta_{s's}\delta_{h'h}, \sum_{s=n}^{0}\sum_{h=s}^{-s}\eta_{ss}\varepsilon_{\underbrace{a\cdots b}_{n}}(\vec{p},h;s)\bar{\varepsilon}_{\underbrace{a'\cdots b'}_{n}}(\vec{p},h;s) = \frac{1}{(2n)!^2}\overbrace{\delta_{\{a(a'}\cdots\delta_{b\}b')}}^{n} \\ \bar{U}^{\overbrace{\lambda_\varsigma\cdots\mu_\varsigma}^{2s_m}}(\vec{p},h';s')U_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma}_{2s_m}}(\vec{p},h;s) = \eta_{s's}\delta_{h'h}, \sum_{s=s_m}^{-s_m}\sum_{h=s}^{-s}U_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma}_{2s_m}}(\vec{p},h;s)\bar{U}_{\underbrace{\lambda'_\varsigma\cdots\mu'_\varsigma}_{2s_m}}(\vec{p},h;s) = \frac{1}{(2s_m)!^2}\overbrace{\delta_{\{\lambda_\varsigma(\lambda'_\varsigma}\cdots\delta_{\mu_\varsigma\}\mu'_\varsigma)}}^{2s_m} \\ U_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma}_{2s_m}}(\vec{p},h;-s) := V_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma}_{2s_m}}(\vec{p},h;s), 0 \le s \le s_m\end{cases}$$

7.3 Mass conjecture(Motion in high dimensional space-time)

Ass. 7.3.1. $E^2 = \vec{p}^2 + \vec{p}_m^2, (\gamma_a,\gamma_5) := [(\sigma\otimes\sigma_y, \varsigma I\otimes\sigma_x), \varsigma I\otimes\sigma_z]$

7.4 Mathematical preparation (Conclusions from previous chapters)

Cor. 7.4.1.
$$\begin{cases}\lambda(\hat{p},\frac{\kappa}{2})\lambda^+(\hat{p},\frac{\kappa}{2}) = \frac{1}{2}(\kappa\sigma\cdot\hat{p}+I) = \frac{1}{2}(\kappa\sigma,-i)^a\hat{p}_a, \hat{p}_a := (\hat{p},i) \\ \lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2}) = -\frac{1}{2}(\sigma\cdot\hat{p}-I) = -\frac{1}{2}(\sigma,i)^a\hat{p}_a \\ \lambda(\hat{p},\frac{\kappa}{2})\lambda^+(\hat{p},-\frac{\kappa}{2}) = \frac{\kappa}{2}(\sigma\cdot\hat{p}+I)i\sigma_y = \frac{\kappa}{2}(\sigma,i)^a\hat{p}_a i\sigma_y\end{cases}$$

Cor. 7.4.2.
$$\begin{cases}\mu(\vec{p},\frac{\kappa}{2})\mu^+(\vec{p},-\frac{\kappa}{2})=\frac{1}{2}(I+\varsigma\frac{E}{m}\sigma_x-i\kappa\frac{|\vec{p}|}{m}\sigma_y)\\ \mu(\vec{p},\frac{\kappa}{2})\mu^+(\vec{p},\frac{\kappa}{2})=\frac{\varsigma}{2}(I+\varsigma\frac{E}{m}\sigma_x-i\kappa\frac{|\vec{p}|}{m}\sigma_y)\sigma_x\end{cases}$$

Cor. 7.4.3. $u(\vec{p},\frac{\kappa}{2})=\lambda(\hat{p},\frac{\kappa}{2})\otimes\mu(\vec{p},\frac{\kappa}{2}),v(\vec{p},\frac{\kappa}{2})=\lambda(\hat{p},\frac{\kappa}{2})\otimes\nu(\vec{p},\frac{\kappa}{2})$

Cor. 7.4.4. $u(\vec{p},\frac{\kappa}{2})u^+(\vec{p},\frac{\kappa}{2})=\frac{1}{4}[(\kappa\sigma\cdot\hat{p}+I)\otimes(I+\varsigma\frac{E}{m}\sigma_x-i\kappa\frac{|\vec{p}|}{m}\sigma_y)](\varsigma I\otimes\sigma_x)$
$=\frac{1}{4}(i\kappa\vec{\gamma}\cdot\hat{p}\gamma_4\gamma_5+I_4)(I_4+\frac{E}{m}\gamma_4+\kappa\frac{|\vec{p}|}{m}\gamma_4\gamma_5)\gamma_4$
$=\frac{1}{4}(i\kappa\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\kappa\frac{|\vec{p}|}{m})\gamma_4$

Cor. 7.4.5. $u(\vec{p},\frac{\kappa}{2})u^+(\vec{p},-\frac{\kappa}{2})=\frac{1}{4}[\kappa(\sigma\cdot\hat{p}+I)i\sigma_y]\otimes(I+\varsigma\frac{E}{m}\sigma_x-i\kappa\frac{|\vec{p}|}{m}\sigma_y)$
$=\frac{1}{4}\kappa(i\vec{\gamma}\cdot\hat{p}\gamma_2-\gamma_2\gamma_4\gamma_5)(I_4+\frac{E}{m}\gamma_4+\kappa\frac{|\vec{p}|}{m}\gamma_4\gamma_5)$
$=\frac{1}{4}\kappa(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\kappa\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5$

Cor. 7.4.6.
$$\begin{cases}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=\frac{1}{2\sqrt{2m}}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(p)\varepsilon_a(\vec{p},h),V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)=-\frac{1}{2\sqrt{2m}}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(-p)\varepsilon_a(\vec{p},h)\\ \varepsilon_a(\vec{p},h)=-\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\hat{p},h)=\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\hat{p},h)\\ \varepsilon^+_{a'}(\vec{p},h)=\frac{i}{\sqrt{2}}(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}U^+_{\lambda'_\varsigma\mu'_\varsigma}(\hat{p},h)=-\frac{i}{\sqrt{2}}(\gamma_{a'}C)^{\lambda'_\varsigma\mu'_\varsigma}V^+_{\lambda'_\varsigma\mu'_\varsigma}(\hat{p},h)\end{cases}$$

Cor. 7.4.7. $\lambda_m(\hat{p},-1)=\lambda^*_m(\hat{p},1),\lambda_m(\hat{p},0)=-\lambda^*_m(\hat{p},0),\lambda_m(\hat{p},1)=\lambda^*_m(\hat{p},-1)$

Def. 7.4.1. $\varepsilon_a(\vec{p},\kappa):=[i\lambda_m(\vec{p},\kappa),0]_a,\varepsilon_a(\vec{p},0):=\frac{1}{m}[iE\lambda_m(\vec{p},0),i|\vec{p}|]_a,\bar{\varepsilon}_a(\vec{p},h):=\varepsilon^+_{a'}(\vec{p},h)\eta^{a'}_a$

Cor. 7.4.8.
$$\begin{cases}\lambda_m(\hat{p},1;1)=\frac{1}{2\hat{p}_-}\begin{bmatrix}i(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\-1(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\-2i(\hat{p}_+\hat{p}_-)\end{bmatrix},\lambda_m(-\hat{p},1;1)=\frac{\hat{p}_+}{\hat{p}_-}\lambda_m(\hat{p},-1;1)\\ \lambda_m(\hat{p},0;1)=-i\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\\hat{p}_z\end{bmatrix}=-i\hat{p},\lambda_m(-\hat{p},0;1)=-\lambda_m(\hat{p},0;1)\\ \lambda_m(\hat{p},-1;1)=\frac{1}{2\hat{p}_+}\begin{bmatrix}-i(\hat{p}_x\hat{p}_z+i\hat{p}_y)\\-1(\hat{p}_x+i\hat{p}_y\hat{p}_z)\\2i(\hat{p}_+\hat{p}_-)\end{bmatrix},\lambda_m(-\hat{p},-1;1)=\frac{\hat{p}_-}{\hat{p}_+}\lambda_m(\hat{p},1;1)\end{cases}$$

Proof: $\lambda_m(\hat{p},1)\lambda^+_m(\hat{p},-1)=\frac{1}{2\hat{p}_-}\begin{bmatrix}i(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\-1(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\-2i(\hat{p}_+\hat{p}_-)\end{bmatrix}\frac{1}{2\hat{p}_-}\begin{bmatrix}i(\hat{p}_x\hat{p}_z-i\hat{p}_y)\\-1(\hat{p}_x-i\hat{p}_y\hat{p}_z)\\-2i(\hat{p}_+\hat{p}_-)\end{bmatrix}^T$ □

7.5 Second-order translational quasi projection operator

Def. 7.5.1. $\sum\limits_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h-h')$
$$\begin{cases}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)=u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})\\ U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)=\frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})]\\ U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},-1)=u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})\end{cases}$$

Proof: $\sum\limits_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h-2)=U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},-1)$
$=u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})$
$=[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma}[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\mu_\varsigma\mu'_\varsigma}$
$=\frac{1}{16}\{[(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]\otimes[(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$
$=\frac{1}{16}\{[(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)\otimes(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)][(I_4-\frac{E}{m}\gamma_4+\frac{|\vec{p}|}{m}\gamma_4\gamma_5)\otimes(I_4-\frac{E}{m}\gamma_4+\frac{|\vec{p}|}{m}\gamma_4\gamma_5)][\gamma_2\otimes\gamma_2]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$ □

Proof: $\sum\limits_{h=1}^{-1}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h-1)=U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},0)+U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},-1)$
$=\frac{1}{\sqrt{2}}u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})[u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})+u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+\frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})]u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})$
$=\frac{1}{\sqrt{2}}u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+\frac{1}{\sqrt{2}}u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$=\frac{1}{\sqrt{2}}[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma}\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})$
$+\frac{1}{\sqrt{2}}\Lambda_{+\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\frac{1}{2})[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p}-\gamma_4\gamma_5)(\gamma_4\gamma_5-\frac{E}{m}\gamma_5-\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\mu_\varsigma\mu'_\varsigma}$ □

Proof: $\sum_{h=1}^{-1} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h) = U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1) + U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},0) + U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},-1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},-1)$
$= u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})$
$+ \frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2}) + u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})][u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2}) + u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+ u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})$
$= \frac{1}{(2!)^2}\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\}\mu'_\varsigma)}(\vec{p},\frac{1}{2})$ □

Proof: $\sum_{h=1}^{-1} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h+1) = U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},-1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},0) + U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1)$
$= \frac{1}{\sqrt{2}}u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})[u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2}) + u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+ \frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2}) + u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})]u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})$
$= \frac{1}{\sqrt{2}}u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})[u_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2}) + u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+ \frac{1}{\sqrt{2}}u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2}) + u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$= \frac{1}{\sqrt{2}}[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 + \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma}\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})$
$+ \frac{1}{\sqrt{2}}\Lambda_{+\lambda_\varsigma\lambda'_\varsigma}(\vec{p},\frac{1}{2})[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 + \frac{E}{m}\gamma_5 - \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\mu_\varsigma\mu'_\varsigma}$ □

Proof: $\sum_{h=1}^{-1} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},h+2) = U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},-1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1)$
$= u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})u^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})$
$= [\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 + \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma}[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 + \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\mu_\varsigma\mu'_\varsigma}$
$= \frac{1}{16}\{[(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 + \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5] \otimes [(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 + \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$
$= \frac{1}{16}\{[(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5) \otimes (i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)][(I_4 - \frac{E}{m}\gamma_4 - \frac{|\vec{p}|}{m}\gamma_4\gamma_5) \otimes (I_4 - \frac{E}{m}\gamma_4 - \frac{|\vec{p}|}{m}\gamma_4\gamma_5)][\gamma_2 \otimes \gamma_2]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$ □

Proof: $U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},-1) + U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},-1)U^+_{\lambda'_\varsigma\mu'_\varsigma}(\vec{p},1)$
$= 2[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5)\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma}[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5)\gamma_4\gamma_2\gamma_5]_{\mu_\varsigma\mu'_\varsigma}$
$+ 2[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)\frac{|\vec{p}|}{m}\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma}[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)\frac{|\vec{p}|}{m}\gamma_4\gamma_2\gamma_5]_{\mu_\varsigma\mu'_\varsigma}$
$= 2[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5)\gamma_4\gamma_2\gamma_5] \otimes [\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5)\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$
$+ 2[\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)\frac{|\vec{p}|}{m}\gamma_4\gamma_2\gamma_5] \otimes [\frac{1}{4}(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)\frac{|\vec{p}|}{m}\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$
$= \frac{1}{8}\{[(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(I_4 - \frac{E}{m}\gamma_4)\gamma_2] \otimes [(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)(I_4 - \frac{E}{m}\gamma_4)\gamma_2]$
$+ [(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)\frac{|\vec{p}|}{m}\gamma_4\gamma_5\gamma_2] \otimes [(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)\frac{|\vec{p}|}{m}\gamma_4\gamma_5\gamma_2]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$
$= \frac{1}{8}\{[(i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5) \otimes (i\vec{\gamma}\cdot\hat{p} - \gamma_4\gamma_5)][(I_4 - \frac{E}{m}\gamma_4) \otimes (I_4 - \frac{E}{m}\gamma_4) + \frac{\vec{p}^2}{m^2}(\gamma_4\gamma_5) \otimes (\gamma_4\gamma_5)][\gamma_2 \otimes \gamma_2]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$ □

Chapter32 Preliminary Analysis of Spin Entangled States

1 Single-mode plane wave solutions of electron equation

1.1 Double electron synthesis: $\frac{1}{2}\oplus\frac{1}{2}=(1,1)$

Cor. 1.1.1. $\begin{cases}(\gamma^a\partial_{1a}+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1,\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,\frac{1}{2})e^{ip_2x_2}]=0\\(\gamma^a\partial_{2a}+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1,\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,\frac{1}{2})e^{ip_2x_2}]=0\end{cases}$

Cor. 1.1.2. $\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_2x}]=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_2x}]=0\end{cases}$

Cor. 1.1.3. $\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}]=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}[U_{\lambda_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}]=0\end{cases}$

Cor. 1.1.4. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(\vec{p},\frac{1}{2})U_{\mu_\varsigma}(\vec{p},\frac{1}{2})e^{ipx}]=0$

1.2 Double electron synthesis: $\frac{1}{2}\oplus\frac{1}{2}=(1,-1)$

Cor. 1.2.1. $\begin{cases}(\gamma^a\partial_{1a}+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1,-\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,-\frac{1}{2})e^{ip_2x_2}]=0\\(\gamma^a\partial_{2a}+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1,-\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,-\frac{1}{2})e^{ip_2x_2}]=0\end{cases}$

Cor. 1.2.2. $\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_2x}]=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_2x}]=0\end{cases}$

Cor. 1.2.3. $\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}]=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}[U_{\lambda_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}]=0\end{cases}$

Cor. 1.2.4. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}[U_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})U_{\mu_\varsigma}(\vec{p},-\frac{1}{2})e^{ipx}]=0$

1.3 Double electron synthesis: $\frac{1}{2}\oplus\frac{1}{2}=(1,0)$

Cor. 1.3.1.
$$\begin{cases}(\gamma^a\partial_{1a}+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1,\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,-\frac{1}{2})e^{ip_2x_2}]+[U_{\lambda_\varsigma}(\vec{p}_1,-\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,\frac{1}{2})e^{ip_2x_2}]\}=0\\(\gamma^a\partial_{2a}+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1,\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,-\frac{1}{2})e^{ip_2x_2}]+[U_{\lambda_\varsigma}(\vec{p}_1,-\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,\frac{1}{2})e^{ip_2x_2}]\}=0\end{cases}$$

Cor. 1.3.2.
$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_2x}]+[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_2x}]\}=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_2x}]+[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_2x}]\}=0\end{cases}$$

Cor. 1.3.3.
$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}]+[U_{\lambda_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}]\}=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}\{[U_{\lambda_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}]+[U_{\lambda_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}]\}=0\end{cases}$$

Cor. 1.3.4. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p},\frac{1}{2})U_{\mu_\varsigma}(\vec{p},-\frac{1}{2})e^{ipx}]+[U_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})U_{\mu_\varsigma}(\vec{p},\frac{1}{2})e^{ipx}]\}=0$

1.4 Double electron synthesis: $\frac{1}{2}\oplus\frac{1}{2}=(0,0)$

Cor. 1.4.1.
$$\begin{cases}(\gamma^a\partial_{1a}+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1,\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,-\frac{1}{2})e^{ip_2x_2}]-[U_{\lambda_\varsigma}(\vec{p}_1,-\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,\frac{1}{2})e^{ip_2x_2}]\}=0\\(\gamma^a\partial_{2a}+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1,\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,-\frac{1}{2})e^{ip_2x_2}]-[U_{\lambda_\varsigma}(\vec{p}_1,-\frac{1}{2})e^{ip_1x_1}][U_{\mu_\varsigma}(\vec{p}_2,\frac{1}{2})e^{ip_2x_2}]\}=0\end{cases}$$

Cor. 1.4.2.
$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_2x}]-[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_2x}]\}=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_2x}]-[U_{\lambda_\varsigma}(\vec{p}_1+\vec{p}_2,-\frac{1}{2})e^{ip_1x}][U_{\mu_\varsigma}(\vec{p}_1+\vec{p}_2,\frac{1}{2})e^{ip_2x}]\}=0\end{cases}$$

Cor. 1.4.3.
$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}]-[U_{\lambda_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}]\}=0\\(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\mu_\varsigma}\{[U_{\lambda_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}]-[U_{\lambda_\varsigma}(2\vec{p},-\frac{1}{2})e^{ipx}][U_{\mu_\varsigma}(2\vec{p},\frac{1}{2})e^{ipx}]\}=0\end{cases}$$

Cor. 1.4.4. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\{[U_{\lambda_\varsigma}(\vec{p},\frac{1}{2})U_{\mu_\varsigma}(\vec{p},-\frac{1}{2})e^{ipx}]-[U_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})U_{\mu_\varsigma}(\vec{p},\frac{1}{2})e^{ipx}]\}=0$

2 Single-mode plane wave solutions of photon equation

2.1 Double photon synthesis: $1 \oplus 1 = (2, 2)$

Thm. 2.1.1. $[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\psi_{\alpha_\varsigma\beta_\varsigma} = 0, S_{ab}(\gamma, \varsigma) = i\sigma^{\alpha_\varsigma}_{\varsigma ab}\gamma_{\alpha_\varsigma}, \lambda_{\alpha_\varsigma}(\hat{p}, h)$

Cor. 2.1.1.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, 1)e^{ip_2x_2}] = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}[\lambda_{\alpha_\varsigma}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, 1)e^{ip_2x_2}] = 0\end{cases}$$

Cor. 2.1.2.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}] = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}] = 0\end{cases}$$

Cor. 2.1.3.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, 1)e^{ipx}] = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}[\lambda_{\alpha_\varsigma}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, 1)e^{ipx}] = 0\end{cases}$$

Cor. 2.1.4. $[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(\hat{p}, 1)\lambda_{\beta_\varsigma}(\hat{p}, 1)e^{ipx}] = 0$

2.2 Double photon synthesis: $1 \oplus 1 = (2, -2)$

Cor. 2.2.1.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, -1)e^{ip_2x_2}] = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}[\lambda_{\alpha_\varsigma}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, -1)e^{ip_2x_2}] = 0\end{cases}$$

Cor. 2.2.2.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] = 0\end{cases}$$

Cor. 2.2.3.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, -1)e^{ipx}] = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}[\lambda_{\alpha_\varsigma}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, -1)e^{ipx}] = 0\end{cases}$$

Cor. 2.2.4. $[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}[\lambda_{\alpha_\varsigma}(\hat{p}, -1)\lambda_{\beta_\varsigma}(\hat{p}, -1)e^{ipx}] = 0$

2.3 Double photon synthesis: $1 \oplus 1 = (2, 0)$

Cor. 2.3.1.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, -1)e^{ip_2x_2}] + [\lambda_{\alpha_\varsigma}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}\{[\lambda_{\alpha_\varsigma}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, -1)e^{ip_2x_2}] + [\lambda_{\alpha_\varsigma}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0\end{cases}$$

Cor. 2.3.2.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] + [\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}\{[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] + [\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0\end{cases}$$

Cor. 2.3.3.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, -1)e^{ipx}] + [\lambda_{\alpha_\varsigma}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, 1)e^{ipx}]\} = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}\{[\lambda_{\alpha_\varsigma}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, -1)e^{ipx}] + [\lambda_{\alpha_\varsigma}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, 1)e^{ipx}]\} = 0\end{cases}$$

Cor. 2.3.4. $[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(\hat{p}, 1)\lambda_{\beta_\varsigma}(\hat{p}, -1)e^{ipx}] + [\lambda_{\alpha_\varsigma}(\hat{p}, -1)\lambda_{\beta_\varsigma}(\hat{p}, 1)e^{ipx}]\} = 0$

2.4 Double photon synthesis: $1 \oplus 1 = (0, 0)$

Cor. 2.4.1.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, -1)e^{ip_2x_2}] - [\lambda_{\alpha_\varsigma}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}\{[\lambda_{\alpha_\varsigma}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, -1)e^{ip_2x_2}] - [\lambda_{\alpha_\varsigma}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\varsigma}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0\end{cases}$$

Cor. 2.4.2.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] - [\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}\{[\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] - [\lambda_{\alpha_\varsigma}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\varsigma}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0\end{cases}$$

Cor. 2.4.3.
$$\begin{cases}[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, -1)e^{ipx}] - [\lambda_{\alpha_\varsigma}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, 1)e^{ipx}]\} = 0\\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\beta_\varsigma}\{[\lambda_{\alpha_\varsigma}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, -1)e^{ipx}] - [\lambda_{\alpha_\varsigma}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\varsigma}(2\vec{p}, 1)e^{ipx}]\} = 0\end{cases}$$

Cor. 2.4.4. $[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\varsigma}{}^{\alpha_\varsigma}\{[\lambda_{\alpha_\varsigma}(\hat{p}, 1)\lambda_{\beta_\varsigma}(\hat{p}, -1)e^{ipx}] - [\lambda_{\alpha_\varsigma}(\hat{p}, -1)\lambda_{\beta_\varsigma}(\hat{p}, 1)e^{ipx}]\} = 0$

3 Single-mode plane wave solutions of vector field equations

3.1 Double photon synthesis: $1 \oplus 1 = (2, 2)$

Cor. 3.1.1.

$$\begin{cases}(-\partial^{1c}\partial_{1c} + m^2)[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0, \partial^{1a}[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0\\(-\partial^{2c}\partial_{2c} + m^2)[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0, \partial^{2b}[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0\end{cases}$$

Cor. 3.1.2.

$$\begin{cases}(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}] = 0, \partial^{a}[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}] = 0\\(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}] = 0, \partial^{b}[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}] = 0\end{cases}$$

Cor. 3.1.3.

$$\begin{cases}(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0, \partial^{a}[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0\\(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0, \partial^{b}[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0\end{cases}$$

Cor. 3.1.4. $(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)e^{ipx}] = 0, \partial^{a}[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)e^{ipx}] = 0$

3.2 Double photon synthesis: $1 \oplus 1 = (2, -2)$

Cor. 3.2.1.

$$\begin{cases}(-\partial^{1c}\partial_{1c} + m^2)[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0, \partial^{1a}[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0\\(-\partial^{2c}\partial_{2c} + m^2)[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0, \partial^{2b}[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0\end{cases}$$

Cor. 3.2.2.

$$\begin{cases}(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] = 0, \partial^{a}[\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] = 0\\(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] = 0, \partial^{b}[\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] = 0\end{cases}$$

Cor. 3.2.3.

$$\begin{cases}(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] = 0, \partial^{a}[\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] = 0\\(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] = 0, \partial^{b}[\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] = 0\end{cases}$$

Cor. 3.2.4. $(-\partial^{c}\partial_{c} + m^2)[\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, -1)e^{ipx}] = 0, \partial^{a}[\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, -1)e^{ipx}] = 0$

3.3 Double photon synthesis: $1 \oplus 1 \to (2, 0)$

Cor. 3.3.1.

$$\begin{cases}(-\partial^{1c}\partial_{1c} + m^2)\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] + [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\\\partial^{1a}\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] + [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\\(-\partial^{2c}\partial_{2c} + m^2)\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] + [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\\\partial^{2b}\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] + [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\end{cases}$$

Cor. 3.3.2.

$$\begin{cases}(-\partial^{c}\partial_{c} + m^2)\{[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] + [\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}]\} = 0\\\partial^{a}\{[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] + [\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}]\} = 0\\(-\partial^{c}\partial_{c} + m^2)\{[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] + [\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}]\} = 0\\\partial^{b}\{[\varepsilon_a(\vec{p}_1+\vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, -1)e^{ip_2x}] + [\varepsilon_a(\vec{p}_1+\vec{p}_2, -1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2, 1)e^{ip_2x}]\} = 0\end{cases}$$

Cor. 3.3.3.

$$\begin{cases}(-\partial^{c}\partial_{c} + m^2)\{[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] + [\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}]\} = 0\\\partial^{a}\{[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] + [\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}]\} = 0\\(-\partial^{c}\partial_{c} + m^2)\{[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] + [\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}]\} = 0\\\partial^{b}\{[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, -1)e^{ipx}] + [\varepsilon_a(2\vec{p}, -1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}]\} = 0\end{cases}$$

Cor. 3.3.4.

$$\begin{cases}(-\partial^{c}\partial_{c} + m^2)\{[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, -1)e^{ipx}] + [\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 1)e^{ipx}]\} = 0\\\partial^{a}\{[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, -1)e^{ipx}] + [\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 1)e^{ipx}]\} = 0\end{cases}$$

3.4 Double photon synthesis: $1 \oplus 1 \to (1, 0)$

Cor. 3.4.1.

$$\begin{cases}(-\partial^{1c}\partial_{1c} + m^2)\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] - [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\\\partial^{1a}\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] - [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\\(-\partial^{2c}\partial_{2c} + m^2)\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] - [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\\\partial^{2b}\{[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] - [\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}]\} = 0\end{cases}$$

Cor. 3.4.2.

$$\begin{cases}(-\partial^c\partial_c+m^2)\{[\varepsilon_a(\vec{p}_1+\vec{p}_2,1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,-1)e^{ip_2x}]-[\varepsilon_a(\vec{p}_1+\vec{p}_2,-1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,1)e^{ip_2x}]\}=0\\ \partial^a\{[\varepsilon_a(\vec{p}_1+\vec{p}_2,1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,-1)e^{ip_2x}]-[\varepsilon_a(\vec{p}_1+\vec{p}_2,-1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,1)e^{ip_2x}]\}=0\\ (-\partial^c\partial_c+m^2)\{[\varepsilon_a(\vec{p}_1+\vec{p}_2,1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,-1)e^{ip_2x}]-[\varepsilon_a(\vec{p}_1+\vec{p}_2,-1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,1)e^{ip_2x}]\}=0\\ \partial^b\{[\varepsilon_a(\vec{p}_1+\vec{p}_2,1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,-1)e^{ip_2x}]-[\varepsilon_a(\vec{p}_1+\vec{p}_2,-1)e^{ip_1x}][\varepsilon_b(\vec{p}_1+\vec{p}_2,1)e^{ip_2x}]\}=0\end{cases}$$

Cor. 3.4.3.

$$\begin{cases}(-\partial^c\partial_c+m^2)\{[\varepsilon_a(2\vec{p},1)e^{ipx}][\varepsilon_b(2\vec{p},-1)e^{ipx}]-[\varepsilon_a(2\vec{p},-1)e^{ipx}][\varepsilon_b(2\vec{p},1)e^{ipx}]\}=0\\ \partial^a\{[\varepsilon_a(2\vec{p},1)e^{ipx}][\varepsilon_b(2\vec{p},-1)e^{ipx}]-[\varepsilon_a(2\vec{p},-1)e^{ipx}][\varepsilon_b(2\vec{p},1)e^{ipx}]\}=0\\ (-\partial^c\partial_c+m^2)\{[\varepsilon_a(2\vec{p},1)e^{ipx}][\varepsilon_b(2\vec{p},-1)e^{ipx}]-[\varepsilon_a(2\vec{p},-1)e^{ipx}][\varepsilon_b(2\vec{p},1)e^{ipx}]\}=0\\ \partial^b\{[\varepsilon_a(2\vec{p},1)e^{ipx}][\varepsilon_b(2\vec{p},-1)e^{ipx}]-[\varepsilon_a(2\vec{p},-1)e^{ipx}][\varepsilon_b(2\vec{p},1)e^{ipx}]\}=0\end{cases}$$

Cor. 3.4.4.

$$\begin{cases}(-\partial^c\partial_c+m^2)\{[\varepsilon_a(\vec{p},1)e^{ipx}][\varepsilon_b(\vec{p},-1)e^{ipx}]-[\varepsilon_a(\vec{p},-1)e^{ipx}][\varepsilon_b(\vec{p},1)e^{ipx}]\}=0\\ \partial^a\{[\varepsilon_a(\vec{p},1)e^{ipx}][\varepsilon_b(\vec{p},-1)e^{ipx}]-[\varepsilon_a(\vec{p},-1)e^{ipx}][\varepsilon_b(\vec{p},1)e^{ipx}]\}=0\end{cases}$$

4 Single-mode plane wave solutions of particle spin equation

4.1 s-spin equation and its single-mode plane wave solutions

Thm. 4.1.1. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(x)=0$

Cor. 4.1.1.
$$\begin{cases}\psi(\vec{r},t):=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma)e^{-ip\cdot x}]d^3\vec{p}\\ |\vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\frac{i}{|\vec{p}|}\lambda^+(\hat{p},-s\varsigma)\dot{\psi}(\vec{r},t)e^{-ip\cdot x}d^3\vec{r}\\ |\vec{p}|^{(s-\frac{1}{2})}a_2^+(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\lambda^+(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{ip\cdot x}d^3\vec{r}=\frac{1}{(2\pi)^{3/2}}\int\frac{-i}{|\vec{p}|}\lambda^+(\hat{p},-s\varsigma)\dot{\psi}(\vec{r},t)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Cor. 4.1.2.
$$\begin{cases}\psi_+(\vec{p},-s\varsigma;x)=\frac{1}{(2\pi)^{3/2}}|\vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{ip\cdot x}=A_+(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{ip\cdot x}\\ \psi_-(\vec{p},-s\varsigma;x)=\frac{1}{(2\pi)^{3/2}}|\vec{p}|^{(s-\frac{1}{2})}a_2^+(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-ip\cdot x}=A_-(\vec{p},-s\varsigma)\lambda(\hat{p},-s\varsigma)e^{-ip\cdot x}\end{cases}$$

Cor. 4.1.3. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi_\pm(\vec{p},-s\varsigma;x)=0[\Leftrightarrow][sp_a+iS_{ab}(s,\varsigma)p^b]\psi_\pm(\vec{p},-s\varsigma;x)=0$

4.2 Several equivalent forms of single-mode spin equation

Thm. 4.2.1.

$$\begin{array}{lcl}[s\hat{p}_a+iS_{ab}(s,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-s\varsigma;x)=0 & [\Leftrightarrow] & \sigma(s)\cdot\hat{p}\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x),O(s)\cdot\hat{p}\psi_\pm(\vec{p},-s\varsigma;x)=0\\ [\Updownarrow] & & [\Updownarrow]\\ [s\hat{p}-i\sigma(s)\times\hat{p}\pm\varsigma\sigma(s)]\psi_\pm(\vec{p},-s\varsigma;x)=0 & [\Leftrightarrow] & \begin{cases}\sigma(s)\cdot\hat{p}\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ \{s^2\hat{p}-is\sigma(s)\times\hat{p}-\sigma(s)[\sigma(s)\cdot\hat{p}]\}\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}\\ [\Updownarrow] & & [\Updownarrow]\\ \{s\hat{p}-[\sigma(s)\cdot\hat{p},\sigma(s)]\pm\varsigma\sigma(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0 & [\Leftrightarrow] & \begin{cases}\sigma(s)\cdot\hat{p}\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ \{s^2\hat{p}+(s-1)\sigma(s)[\sigma(s)\cdot\hat{p}]-s[\sigma(s)\cdot\hat{p}]\sigma(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}\end{array}$$

Cor. 4.2.1.
$$\begin{cases}\sigma(1)\cdot\hat{p}\psi_\pm(\vec{p},-\varsigma;x)=\mp\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ [\sigma(1)\cdot\hat{p}]\sigma(1)\psi_\pm(\vec{p},-\varsigma;x)=\hat{p}\psi_\pm(\vec{p},-\varsigma;x)\end{cases}$$

4.3 Single-mode plane wave solutions of spin equation moving along z-axis

Single-mode plane wave solutions of spin equation moving in forward direction along z-axis $\hat{p}_a=(0,0,1,1)$:

Thm. 4.3.1.

$$\begin{array}{lcl}[s\hat{p}_a+iS_{ab}(s,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-s\varsigma;x)=0 & [\Leftrightarrow] & \sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x),O_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=0\\ [\Updownarrow] & & [\Updownarrow]\\ [s\hat{p}-i\sigma(s)\times\hat{p}\pm\varsigma\sigma(s)]\psi_\pm(\vec{p},-s\varsigma;x)=0 & [\Leftrightarrow] & \begin{cases}\sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ \{s^2\hat{p}-is\sigma(s)\times\hat{p}-\sigma(s)\sigma_z(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}\\ [\Updownarrow] & & [\Updownarrow]\\ \{s\hat{p}-[\sigma_z(s),\sigma(s)]\pm\varsigma\sigma(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0 & [\Leftrightarrow] & \begin{cases}\sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ \{s^2\hat{p}+(s-1)\sigma(s)\sigma_z(s)-s\sigma_z(s)\sigma(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}\end{array}$$

Cor. 4.3.1. $[s\hat{p}_a+iS_{ab}(s,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-s\varsigma;x)=0[\Leftrightarrow]\begin{cases}\sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ [\sigma_x(s)\mp i\varsigma\sigma_y(s)]\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}$

Cor. 4.3.2. $\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\mp\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \sigma_z(1)\sigma(1)\psi_\pm(\vec{p},-\varsigma;x)=\hat{p}\psi_\pm(\vec{p},-\varsigma;x)\end{cases}$

Single-mode plane wave solutions of spin equation moving in backward direction along z-axis $\hat{p}_a=(0,0,-1,1)$**:**

Thm. 4.3.2.

$$[s\hat{p}_a+iS_{ab}(s,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-s\varsigma;x)=0\ [\Leftrightarrow]\ \sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\pm s\varsigma\psi_\pm(\vec{p},-s\varsigma;x),O_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=0$$

$$[\Updownarrow] \qquad [\Updownarrow]$$

$$[s\hat{p}-i\sigma(s)\times\hat{p}\pm\varsigma\sigma(s)]\psi_\pm(\vec{p},-s\varsigma;x)=0\ [\Leftrightarrow]\ \begin{cases}\sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\pm s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ \{s^2\hat{p}-is\sigma(s)\times\hat{p}+\sigma(s)\sigma_z(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}$$

$$[\Updownarrow] \qquad [\Updownarrow]$$

$$\{s\hat{p}+[\sigma_z(s),\sigma(s)]\pm\varsigma\sigma(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0[\Leftrightarrow]\begin{cases}\sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\pm s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ \{s^2\hat{p}-(s-1)\sigma(s)\sigma_z(s)+s\sigma_z(s)\sigma(s)\}\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}$$

Cor. 4.3.3. $[s\hat{p}_a+iS_{ab}(s,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-s\varsigma;x)=0[\Leftrightarrow]\begin{cases}\sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x)=\pm s\varsigma\psi_\pm(\vec{p},-s\varsigma;x)\\ [\sigma_x(s)\pm i\varsigma\sigma_y(s)]\psi_\pm(\vec{p},-s\varsigma;x)=0\end{cases}$

Cor. 4.3.4. $\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\pm\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \sigma_z(1)\sigma(1)\psi_\pm(\vec{p},-\varsigma;x)=-\hat{p}\psi_\pm(\vec{p},-\varsigma;x)\end{cases}$

4.4 Single-mode plane wave solutions of 1-spin equation moving along z-axis

Single-mode plane wave solutions of spin equation moving in forward direction along z-axis $\hat{p}_a=(0,0,1,1)$**:**

Thm. 4.4.1.

$$[\hat{p}_a+iS_{ab}(1,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-\varsigma;x)=0\ [\Leftrightarrow]\ \sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\mp\varsigma\psi_\pm(\vec{p},-\varsigma;x),O_z(1)\psi_\pm(\vec{p},-\varsigma;x)=0$$

$$[\Updownarrow] \qquad [\Updownarrow]$$

$$[\hat{p}-i\sigma(1)\times\hat{p}\pm\varsigma\sigma(1)]\psi_\pm(\vec{p},-\varsigma;x)=0\ [\Leftrightarrow]\ \begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\mp\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \{\hat{p}-i\sigma(1)\times\hat{p}-\sigma(1)\sigma_z(1)\}\psi_\pm(\vec{p},-\varsigma;x)=0\end{cases}$$

$$[\Updownarrow] \qquad [\Updownarrow]$$

$$\{\hat{p}-[\sigma_z(1),\sigma(1)]\pm\varsigma\sigma(1)\}\psi_\pm(\vec{p},-\varsigma;x)=0[\Leftrightarrow]\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\mp\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \{\hat{p}-\sigma_z(1)\sigma(1)\}\psi_\pm(\vec{p},-\varsigma;x)=0\end{cases}$$

Cor. 4.4.1. $[\hat{p}_a+iS_{ab}(1,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-\varsigma;x)=0[\Leftrightarrow]\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\mp\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ [\sigma_x(1)\mp i\varsigma\sigma_y(1)]\psi_\pm(\vec{p},-\varsigma;x)=0\end{cases}$

Cor. 4.4.2. $\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\mp\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \sigma_z(1)\sigma(1)\psi_\pm(\vec{p},-\varsigma;x)=\hat{p}\psi_\pm(\vec{p},-\varsigma;x)\end{cases}$

Single-mode plane wave solutions of spin equation moving in backward direction along z-axis $\hat{p}_a=(0,0,-1,1)$**:**

Thm. 4.4.2.

$$[\hat{p}_a+iS_{ab}(1,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-\varsigma;x)=0\ [\Leftrightarrow]\ \sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\pm\varsigma\psi_\pm(\vec{p},-\varsigma;x),O_z(1)\psi_\pm(\vec{p},-\varsigma;x)=0$$

$$[\Updownarrow] \qquad [\Updownarrow]$$

$$[\hat{p}-i\sigma(1)\times\hat{p}\pm\varsigma\sigma(1)]\psi_\pm(\vec{p},-\varsigma;x)=0\ [\Leftrightarrow]\ \begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\pm\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \{\hat{p}-i\sigma(1)\times\hat{p}+\sigma(1)\sigma_z(1)\}\psi_\pm(\vec{p},-\varsigma;x)=0\end{cases}$$

$$[\Updownarrow] \qquad [\Updownarrow]$$

$$\{\hat{p}+[\sigma_z(1),\sigma(1)]\pm\varsigma\sigma(1)\}\psi_\pm(\vec{p},-\varsigma;x)=0[\Leftrightarrow]\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\pm\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \{s^2\hat{p}+\sigma_z(1)\sigma(1)\}\psi_\pm(\vec{p},-\varsigma;x)=0\end{cases}$$

Cor. 4.4.3. $[\hat{p}_a+iS_{ab}(1,\varsigma)\hat{p}^b]\psi_\pm(\vec{p},-\varsigma;x)=0[\Leftrightarrow]\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\pm\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ [\sigma_x(1)\pm i\varsigma\sigma_y(1)]\psi_\pm(\vec{p},-\varsigma;x)=0\end{cases}$

Cor. 4.4.4. $\begin{cases}\sigma_z(1)\psi_\pm(\vec{p},-\varsigma;x)=\pm\varsigma\psi_\pm(\vec{p},-\varsigma;x)\\ \sigma_z(1)\sigma(1)\psi_\pm(\vec{p},-\varsigma;x)=-\hat{p}\psi_\pm(\vec{p},-\varsigma;x)\end{cases}$

4.5 Single-mode plane wave solutions of spin vector

Cor. 4.5.1. $\begin{cases}[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi_\pm(\vec{p},-s\varsigma;x)=0\\ \hat{P}_a := -i\partial_a, \hat{W}_a(s,\varsigma) := \varsigma S_{ab}(s,\varsigma)\partial^b\end{cases} \Rightarrow \begin{cases}\hat{P}_a\psi_\pm(\vec{p},-s\varsigma;x)=\pm p_a\psi_\pm(\vec{p},-s\varsigma;x)\\ \hat{W}_a(s,\varsigma)\psi_\pm(\vec{p},-s\varsigma;x)=\mp s\varsigma p_a\psi_\pm(\vec{p},-s\varsigma;x)\end{cases}$

Cor. 4.5.2. $W_a(s,\varsigma)\psi_\pm(\vec{p},-s\varsigma;x) = -s\varsigma\vec{p}_a\psi_\pm(\vec{p},-s\varsigma;x), W_a(s,\varsigma) = -i * S_{ab}(s,\varsigma)p^b = i\varsigma S_{ab}(s,\varsigma)p^b$

4.6 Complete quantum states of single-mode plane wave solutions along positive z-axis

Cor. 4.6.1. $\begin{cases}\sigma^2(s)\psi_\pm(\vec{p},-s\varsigma;x) = s(s+1)\psi_\pm(\vec{p},-s\varsigma;x), \sigma(s)\times\sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1)\\ \sigma_z(s)\psi_\pm(\vec{p},-s\varsigma;x) = \mp s\varsigma\psi_\pm(\vec{p},-s\varsigma;x), [\sigma_x(s)\mp i\varsigma\sigma_y(s)]\psi_\pm(\vec{p},-s\varsigma;x)=0\\ \hat{P}_a\psi_\pm(\vec{p},-s\varsigma;x) = \pm p_a\psi_\pm(\vec{p},-s\varsigma;x), \hat{W}_a(s,\varsigma)\psi_\pm(\vec{p},-s\varsigma;x) = \mp s\varsigma p_a\psi_\pm(\vec{p},-s\varsigma;x)\end{cases}$

4.7 Complete quantum states of single-mode photon along positive z-axis

Cor. 4.7.1. $\begin{cases}\gamma^2\Psi_\pm(\vec{p},-\varsigma;x) = 2\Psi_\pm(\vec{p},-\varsigma;x), \gamma\times\gamma = i\gamma, \gamma^2 = 2\\ \gamma_z\Psi_\pm(\vec{p},-\varsigma;x) = \mp\varsigma\Psi_\pm(\vec{p},-\varsigma;x), [\gamma_x \mp i\varsigma\gamma_y]\Psi_\pm(\vec{p},-\varsigma;x) = 0\\ \hat{P}_a\Psi_\pm(\vec{p},-\varsigma;x) = \pm p_a\Psi_\pm(\vec{p},-\varsigma;x), \hat{W}_a(1,\varsigma)\Psi_\pm(\vec{p},-\varsigma;x) = \mp\varsigma p_a\Psi_\pm(\vec{p},-\varsigma;x)\end{cases}$

4.8 Dual-mode syntropic and synchronous plane wave solutions of s-spin equation

Def. 4.8.1. $\begin{cases}[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(x) = 0, \vec{p}\neq 0\\ \psi(x) := \frac{1}{(2\pi)^{3/2}}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x} + a_2^+(\vec{p},-s\varsigma)e^{-ip\cdot x}]\end{cases}$

4.9 Dual-mode heterotropic and synchronous positive plane wave solutions of s-spin equation

Def. 4.9.1. $\begin{cases}[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(x) = 0, \vec{p}\neq 0\\ \psi(x) := \frac{1}{(2\pi)^{3/2}}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{i\vec{p}\cdot\vec{r}} + a_2(-\vec{p},-s\varsigma)e^{-i\vec{p}\cdot\vec{r}}]e^{-iEt}\end{cases}$

4.10 Dual-mode heterotropic and synchronous negative plane wave solutions of s-spin equation

Def. 4.10.1. $\begin{cases}[s\partial_a + iS_{ab}(s,\varsigma)\partial^b]\psi(x) = 0, \vec{p}\neq 0\\ \psi(x) := \frac{1}{(2\pi)^{3/2}}|\vec{p}|^{(s-\frac{1}{2})}\lambda(\hat{p},-s\varsigma)[a_1^+(\vec{p},-s\varsigma)e^{-i\vec{p}\cdot\vec{r}} + a_2^+(-\vec{p},-s\varsigma)e^{i\vec{p}\cdot\vec{r}}]e^{iEt}\end{cases}$

4.11 Guess on dual-mode entangled plane wave solutions of s-spin equation

Ass. 4.11.1.

$$\begin{cases}[\sigma(s)\otimes I + I\otimes\sigma(s)]\partial_t\psi(\vec{r},\vec{r}';t) = s\varsigma(\nabla-\nabla')\psi(\vec{r},\vec{r}';t) - i\varsigma\{[\sigma(s)\otimes I]\times\nabla - [I\otimes\sigma(s)]\times\nabla'\}\psi(\vec{r},\vec{r}';t)\\ \psi(\vec{r},\vec{r}';t) = a(\vec{p})\lambda(\hat{p},-s\varsigma)\otimes Ie^{i(\vec{p}\cdot\vec{r}-Et)} + b(\vec{p})I\otimes\lambda(\hat{p},s\varsigma)e^{i(-\vec{p}\cdot\vec{r}'-Et)} + c(\vec{p})\lambda(\hat{p},-s\varsigma)\otimes\lambda(\hat{p},s\varsigma)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')-iEt}\end{cases}$$

5 Several examples of particles spin coupling

5.1 Diagonally coupling representation of two electron spins

Cor. 5.1.1. *Diagonal coupling representation*

$$\begin{cases}[\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, [\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\ [\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, [\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\\ [\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = 2\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, [\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = -\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\\ [\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = 0\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, [\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = 0\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\end{cases}$$

5.2 Separated uncoupling representation of two electron spins

Cor. 5.2.1. *Separated uncoupling representation*

$$\begin{cases}[\sigma(\frac{1}{2})\otimes I]^2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{3}{4}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{3}{4}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, [\sigma(\frac{1}{2})\otimes I]_z\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\ [\sigma(\frac{1}{2})\otimes I]^2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{3}{4}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{3}{4}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, [\sigma(\frac{1}{2})\otimes I]_z\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\\ [\sigma(\frac{1}{2})\otimes I]^2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \frac{3}{4}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \frac{3}{4}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, [\sigma(\frac{1}{2})\otimes I]_z\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\\ [\sigma(\frac{1}{2})\otimes I]^2\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \frac{3}{4}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \frac{3}{4}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, [\sigma(\frac{1}{2})\otimes I]_z\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, [I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\end{cases}$$

5.3 Separated coupling representation of two electron spins

Cor. 5.3.1. $\hat{S}(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&\sqrt{1}&0\\0&0&0&\sqrt{2}\\0&-\sqrt{1}&\sqrt{1}&0\end{bmatrix},\hat{S}^{+}(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&-\sqrt{1}\\0&\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\end{bmatrix}$

Cor. 5.3.2. $\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\\0&\sqrt{1}&0&-\sqrt{1}\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&\sqrt{1}&0\\0&0&0&\sqrt{2}\\0&-\sqrt{1}&\sqrt{1}&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$

Cor. 5.3.3. $\left\{|\tfrac{1}{2}\rangle=\begin{bmatrix}1\\0\end{bmatrix},|\tfrac{-1}{2}\rangle=\begin{bmatrix}0\\1\end{bmatrix}\right.\quad\left\{|1\rangle=\begin{bmatrix}1\\0\\0\end{bmatrix},|0\rangle=\begin{bmatrix}0\\1\\0\end{bmatrix},|-1\rangle=\begin{bmatrix}0\\0\\1\end{bmatrix}\right.$

Thm. 5.3.1. $S(1)[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]S^{+}(1)=\sigma(1)\oplus\sigma(0)$

Cor. 5.3.4. *Separated coupling representation*

$$\begin{cases}\frac{[\sigma(\frac{1}{2})\otimes I]^2}{[I\otimes\sigma(\frac{1}{2})]^2}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\frac{3}{4}\begin{bmatrix}1\\0\\0\\0\end{bmatrix},[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=2\begin{bmatrix}1\\0\\0\\0\end{bmatrix},[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\ \frac{[\sigma(\frac{1}{2})\otimes I]^2}{[I\otimes\sigma(\frac{1}{2})]^2}\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}=\frac{3}{4}\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix},[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}=2\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix},[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}=0\\ \frac{[\sigma(\frac{1}{2})\otimes I]^2}{[I\otimes\sigma(\frac{1}{2})]^2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\frac{3}{4}\begin{bmatrix}0\\0\\0\\1\end{bmatrix},[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=2\begin{bmatrix}0\\0\\0\\1\end{bmatrix},[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=-\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\\ \frac{[\sigma(\frac{1}{2})\otimes I]^2}{[I\otimes\sigma(\frac{1}{2})]^2}\begin{bmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}=\frac{3}{4}\begin{bmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix},[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]^2\begin{bmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}=0,[\sigma(\frac{1}{2})\otimes I+I\otimes\sigma(\frac{1}{2})]_z\begin{bmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}=0\end{cases}$$

Cor. 5.3.5. $\hat{S}^{+}(1)=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&-\sqrt{1}\\0&\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\end{bmatrix}\Rightarrow|s,m\rangle=\begin{cases}\hat{S}^{+}(1)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=|1,1\rangle=|\frac{1}{2}\rangle\otimes|\frac{1}{2}\rangle;\\\hat{S}^{+}(1)\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=|1,0\rangle=\frac{1}{\sqrt{2}}(|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle+|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle);\\\hat{S}^{+}(1)\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=|1,-1\rangle=|\frac{-1}{2}\rangle\otimes|\frac{-1}{2}\rangle;\\\hat{S}^{+}(1)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=|0,0\rangle=\frac{1}{\sqrt{2}}(-|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle+|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle);\end{cases}$

Cor. 5.3.6. $\begin{bmatrix}|1,1\rangle\\|1,0\rangle\\|1,-1\rangle\\|0,0\rangle\end{bmatrix}=[\hat{S}^{+}(1)\otimes I_4]\begin{bmatrix}|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\end{bmatrix}\otimes\begin{bmatrix}|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\end{bmatrix}=[I_4\otimes\hat{S}(1)]\begin{bmatrix}|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\end{bmatrix}\otimes\begin{bmatrix}|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\end{bmatrix}$

Cor. 5.3.7. $\begin{bmatrix}|\frac{1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\\0&\sqrt{1}&0&-\sqrt{1}\end{bmatrix}\begin{bmatrix}|1,1\rangle\\|1,0\rangle\\|1,-1\rangle\\|0,0\rangle\end{bmatrix},\begin{bmatrix}|1,1\rangle\\|1,0\rangle\\|1,-1\rangle\\|0,0\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\\0&\sqrt{1}&0&-\sqrt{1}\end{bmatrix}\begin{bmatrix}|\frac{1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle\end{bmatrix}$

Cor. 5.3.8. $\begin{bmatrix}|\frac{1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&-\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\\0&\sqrt{1}&0&\sqrt{1}\end{bmatrix}\begin{bmatrix}|1,1\rangle\\|0,0\rangle\\|1,-1\rangle\\|1,0\rangle\end{bmatrix},\begin{bmatrix}|1,1\rangle\\|0,0\rangle\\|1,-1\rangle\\|1,0\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&-\sqrt{1}&0&\sqrt{1}\\0&0&\sqrt{2}&0\\0&\sqrt{1}&0&\sqrt{1}\end{bmatrix}\begin{bmatrix}|\frac{1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle\end{bmatrix}$

Cor. 5.3.9. $\begin{bmatrix}|\frac{1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{2}&0&0\\0&0&\sqrt{1}&\sqrt{1}\\0&0&\sqrt{1}&-\sqrt{1}\end{bmatrix}\begin{bmatrix}|1,1\rangle\\|1,-1\rangle\\|1,0\rangle\\|0,0\rangle\end{bmatrix},\begin{bmatrix}|1,1\rangle\\|1,-1\rangle\\|1,0\rangle\\|0,0\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}&0&0&0\\0&\sqrt{2}&0&0\\0&0&\sqrt{1}&\sqrt{1}\\0&0&\sqrt{1}&-\sqrt{1}\end{bmatrix}\begin{bmatrix}|\frac{1}{2}\rangle\otimes|\frac{1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{1}{2}\rangle\otimes|\frac{-1}{2}\rangle\\|\frac{-1}{2}\rangle\otimes|\frac{1}{2}\rangle\end{bmatrix}$

5.4 Spin coupling representation of two massive photons

Thm. 5.4.1. $S(1\otimes 1)[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]S^{+}(1\otimes 1)=\sigma(2)\oplus\sigma(1)\oplus\sigma(0)$

Cor. 5.4.1.

$$S(1\otimes 1)=\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{6}&0&0&0&0&0&0&0&0\\0&\sqrt{3}&0&\sqrt{3}&0&0&0&0&0\\0&0&\sqrt{1}&0&\sqrt{4}&0&\sqrt{1}&0&0\\0&0&0&0&0&\sqrt{3}&0&\sqrt{3}&0\\0&0&0&0&0&0&0&0&\sqrt{6}\\0&-\sqrt{3}&0&\sqrt{3}&0&0&0&0&0\\0&0&-\sqrt{3}&0&0&0&\sqrt{3}&0&0\\0&0&0&0&0&-\sqrt{3}&0&\sqrt{3}&0\\0&0&\sqrt{2}&0&-\sqrt{2}&0&\sqrt{2}&0&0\end{bmatrix},S^{+}(1\otimes 1)=\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{6}&0&0&0&0&0&0&0&0\\0&\sqrt{3}&0&0&0&-\sqrt{3}&0&0&0\\0&0&\sqrt{1}&0&0&0&-\sqrt{3}&0&\sqrt{2}\\0&\sqrt{3}&0&0&0&\sqrt{3}&0&0&0\\0&0&\sqrt{4}&0&0&0&0&0&-\sqrt{2}\\0&0&0&\sqrt{3}&0&0&0&-\sqrt{3}&0\\0&0&\sqrt{1}&0&0&0&\sqrt{3}&0&\sqrt{2}\\0&0&0&\sqrt{3}&0&0&0&\sqrt{3}&0\\0&0&0&0&\sqrt{6}&0&0&0&0\end{bmatrix}$$

Cor. 5.4.2.

$$\begin{cases}\begin{cases}|2,2\rangle=|1\rangle\otimes|1\rangle;\\ |2,1\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|0\rangle)+|0\rangle\otimes|1\rangle;\\ |2,0\rangle=\frac{1}{\sqrt{6}}(|1\rangle\otimes|-1\rangle+2|0\rangle\otimes|0\rangle+|-1\rangle\otimes|1\rangle);\\ |2,-1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|-1\rangle)+|-1\rangle\otimes|0\rangle;\\ |2,-2\rangle=|-1\rangle\otimes|-1\rangle;\end{cases}\\ \begin{cases}|2;1,1\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|0\rangle)-|0\rangle\otimes|1\rangle;\\ |2;1,0\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|-1\rangle)-|-1\rangle\otimes|1\rangle;\\ |2;1,-1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|-1\rangle)-|-1\rangle\otimes|0\rangle;\end{cases}\\ \begin{cases}|2;0,0\rangle=\frac{1}{\sqrt{3}}(|1\rangle\otimes|-1\rangle)-|0\rangle\otimes|0\rangle+|-1\rangle\otimes|1\rangle;\end{cases}\end{cases}$$

$$\begin{cases}\begin{cases}|1\rangle\otimes|1\rangle=|2,2\rangle;\\ |0\rangle\otimes|1\rangle=\frac{1}{\sqrt{2}}(|2,1\rangle-|1,1\rangle);\\ |-1\rangle\otimes|1\rangle=\frac{1}{\sqrt{6}}(|2,0\rangle-\sqrt{3}|1,0\rangle+\sqrt{2}|0,0\rangle);\\ |1\rangle\otimes|0\rangle=\frac{1}{\sqrt{2}}(|2,1\rangle+|1,1\rangle);\\ |0\rangle\otimes|0\rangle=\frac{1}{\sqrt{3}}(\sqrt{2}|2,0\rangle-|0,0\rangle);\end{cases}\\ \begin{cases}|-1\rangle\otimes|0\rangle=\frac{1}{\sqrt{2}}(|2,-1\rangle-|1,-1\rangle);\\ |1\rangle\otimes|-1\rangle=\frac{1}{\sqrt{6}}(|2,0\rangle+\sqrt{3}|1,0\rangle+\sqrt{2}|0,0\rangle);\\ |0\rangle\otimes|-1\rangle=\frac{1}{\sqrt{2}}(|2,-1\rangle+|1,-1\rangle);\end{cases}\\ \begin{cases}|-1\rangle\otimes|-1\rangle=|2,-2\rangle;\end{cases}\end{cases}$$

Cor. 5.4.3.

$$\begin{cases}\begin{cases}|2,2\rangle=|1\rangle\otimes|1\rangle;\\ |2,1\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|0\rangle)+|0\rangle\otimes|1\rangle;\\ |2,0\rangle=\frac{1}{\sqrt{6}}(|1\rangle\otimes|-1\rangle+2|0\rangle\otimes|0\rangle+|-1\rangle\otimes|1\rangle);\\ |2,-1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|-1\rangle)+|-1\rangle\otimes|0\rangle;\\ |2,-2\rangle=|-1\rangle\otimes|-1\rangle;\end{cases}\\ \begin{cases}|2;1,1\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|0\rangle)-|0\rangle\otimes|1\rangle;\\ |2;1,0\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|-1\rangle)-|-1\rangle\otimes|1\rangle;\\ |2;1,-1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|-1\rangle)-|-1\rangle\otimes|0\rangle;\end{cases}\\ \begin{cases}|2;0,0\rangle=\frac{1}{\sqrt{3}}(|1\rangle\otimes|-1\rangle)-|0\rangle\otimes|0\rangle+|-1\rangle\otimes|1\rangle;\end{cases}\end{cases}$$

$$\begin{cases}\begin{cases}|1\rangle\otimes|1\rangle=|2,2\rangle;\\ |1\rangle\otimes|0\rangle=\frac{1}{\sqrt{2}}(|2,1\rangle+|1,1\rangle);\\ |1\rangle\otimes|-1\rangle=\frac{1}{\sqrt{6}}(|2,0\rangle+\sqrt{3}|1,0\rangle+\sqrt{2}|0,0\rangle);\\ |0\rangle\otimes|-1\rangle=\frac{1}{\sqrt{2}}(|2,-1\rangle+|1,-1\rangle);\\ |-1\rangle\otimes|-1\rangle=|2,-2\rangle;\end{cases}\\ \begin{cases}|0\rangle\otimes|1\rangle=\frac{1}{\sqrt{2}}(|2,1\rangle-|1,1\rangle);\\ |0\rangle\otimes|0\rangle=\frac{1}{\sqrt{3}}(\sqrt{2}|2,0\rangle-|0,0\rangle);\\ |-1\rangle\otimes|0\rangle=\frac{1}{\sqrt{2}}(|2,-1\rangle-|1,-1\rangle);\end{cases}\\ \begin{cases}|-1\rangle\otimes|1\rangle=\frac{1}{\sqrt{6}}(|2,0\rangle-\sqrt{3}|1,0\rangle+\sqrt{2}|0,0\rangle);\end{cases}\end{cases}$$

Cor. 5.4.4.

$$\begin{cases}\begin{cases}|2,2\rangle=|1\rangle\otimes|1\rangle;\\ |2,-2\rangle=|-1\rangle\otimes|-1\rangle;\end{cases}\\ \begin{cases}|2,1\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|0\rangle)+|0\rangle\otimes|1\rangle;\\ |1,1\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|0\rangle)-|0\rangle\otimes|1\rangle;\\ |1,-1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|-1\rangle)-|-1\rangle\otimes|0\rangle;\\ |2,-1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|-1\rangle)+|-1\rangle\otimes|0\rangle;\end{cases}\\ \begin{cases}|2,0\rangle=\frac{1}{\sqrt{6}}(|1\rangle\otimes|-1\rangle+2|0\rangle\otimes|0\rangle+|-1\rangle\otimes|1\rangle);\\ |1,0\rangle=\frac{1}{\sqrt{2}}(|1\rangle\otimes|-1\rangle)-|-1\rangle\otimes|1\rangle;\\ |0,0\rangle=\frac{1}{\sqrt{3}}(|1\rangle\otimes|-1\rangle)-|0\rangle\otimes|0\rangle+|-1\rangle\otimes|1\rangle;\end{cases}\end{cases}$$

$$\begin{cases}\begin{cases}|1\rangle\otimes|1\rangle=|2,2\rangle;\\ |-1\rangle\otimes|-1\rangle=|2,-2\rangle;\end{cases}\\ \begin{cases}|1\rangle\otimes|0\rangle=\frac{1}{\sqrt{2}}(|2,1\rangle+|1,1\rangle);\\ |0\rangle\otimes|1\rangle=\frac{1}{\sqrt{2}}(|2,1\rangle-|1,1\rangle);\\ |-1\rangle\otimes|0\rangle=\frac{1}{\sqrt{2}}(|2,-1\rangle-|1,-1\rangle);\\ |0\rangle\otimes|-1\rangle=\frac{1}{\sqrt{2}}(|2,-1\rangle+|1,-1\rangle);\end{cases}\\ \begin{cases}|1\rangle\otimes|-1\rangle=\frac{1}{\sqrt{6}}(|2,0\rangle+\sqrt{3}|1,0\rangle+\sqrt{2}|0,0\rangle);\\ |0\rangle\otimes|0\rangle=\frac{1}{\sqrt{3}}(\sqrt{2}|2,0\rangle-|0,0\rangle);\\ |-1\rangle\otimes|1\rangle=\frac{1}{\sqrt{6}}(|2,0\rangle-\sqrt{3}|1,0\rangle+\sqrt{2}|0,0\rangle);\end{cases}\end{cases}$$

Cor. 5.4.5.

$$\begin{cases}\begin{bmatrix}|2,2\rangle\\|2,-2\rangle\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}|1\rangle\otimes|1\rangle\\|-1\rangle\otimes|-1\rangle\end{bmatrix},\begin{bmatrix}|1\rangle\otimes|1\rangle\\|-1\rangle\otimes|-1\rangle\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}|2,2\rangle\\|2,-2\rangle\end{bmatrix}\\ \begin{bmatrix}|2,1\rangle\\|1,1\rangle\\|1,-1\rangle\\|2,-1\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&-1&1\\0&0&1&1\end{bmatrix}\begin{bmatrix}|1\rangle\otimes|0\rangle\\|0\rangle\otimes|1\rangle\\|-1\rangle\otimes|0\rangle\\|0\rangle\otimes|-1\rangle\end{bmatrix},\begin{bmatrix}|1\rangle\otimes|0\rangle\\|0\rangle\otimes|1\rangle\\|-1\rangle\otimes|0\rangle\\|0\rangle\otimes|-1\rangle\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&-1&1\\0&0&1&1\end{bmatrix}\begin{bmatrix}|2,1\rangle\\|1,1\rangle\\|1,-1\rangle\\|2,-1\rangle\end{bmatrix}\\ \begin{bmatrix}|2,0\rangle\\|1,0\rangle\\|0,0\rangle\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{6}}&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}&-\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}|1\rangle\otimes|-1\rangle\\|0\rangle\otimes|0\rangle\\|-1\rangle\otimes|1\rangle\end{bmatrix},\begin{bmatrix}|1\rangle\otimes|-1\rangle\\|0\rangle\otimes|0\rangle\\|-1\rangle\otimes|1\rangle\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\\frac{2}{\sqrt{6}}&0&-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}|2,0\rangle\\|1,0\rangle\\|0,0\rangle\end{bmatrix}\end{cases}$$

Cor. 5.4.6.

$$S^+(1\otimes 1)\Rightarrow|s,m\rangle=\begin{cases}\begin{cases}|2,2\rangle=|1\rangle\otimes|1\rangle;\\ |2,1\rangle=\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle+|1\rangle\otimes|0\rangle);\\ |2,0\rangle=\frac{1}{\sqrt{6}}(|-1\rangle\otimes|1\rangle+2|0\rangle\otimes|0\rangle+|1\rangle\otimes|-1\rangle);\\ |2,-1\rangle=\frac{1}{\sqrt{2}}(|-1\rangle\otimes|0\rangle+|0\rangle\otimes|-1\rangle);\\ |2,-2\rangle=|-1\rangle\otimes|-1\rangle;\end{cases}\\ \begin{cases}|2;1,1\rangle=\frac{1}{\sqrt{2}}(-|0\rangle\otimes|1\rangle+|1\rangle\otimes|0\rangle);\\ |2;1,0\rangle=\frac{1}{\sqrt{2}}(-|-1\rangle\otimes|1\rangle+|1\rangle\otimes|-1\rangle);\\ |2;1,-1\rangle=\frac{1}{\sqrt{2}}(-|-1\rangle\otimes|0\rangle+|0\rangle\otimes|-1\rangle);\end{cases}\\ \begin{cases}|2;0,0\rangle=\frac{1}{\sqrt{3}}(|-1\rangle\otimes|1\rangle-|0\rangle\otimes|0\rangle+|1\rangle\otimes|-1\rangle);\end{cases}\end{cases}$$

Cor. 5.4.7. $\begin{bmatrix} |2,2\rangle \\ |2,1\rangle \\ |2,0\rangle \\ |2,-1\rangle \\ |2,-2\rangle \\ |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{bmatrix} = [S^+(1\otimes 1)\otimes I_9]\begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix}\otimes\begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix} = [I_9\otimes S(1\otimes 1)]\begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix}\otimes\begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix}$

5.5 Spin coupling of two massless photons

Cor. 5.5.1. *Diagonal coupling representation*

$$[\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} = 6\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, [\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$[\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix} = 6\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}, [\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix} = -2\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}$$

$$[\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix} = 6\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, [\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix} = 0$$

$$[\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix} = 2\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}, [\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix} = 0$$

$$[\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]^2\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix} = 0, [\sigma(2)\oplus\sigma(1)\oplus\sigma(0)]_z\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix} = 0$$

Cor. 5.5.2. *Non diagonal coupling representation*

$$[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]^2\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} = 6\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, [\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]_z\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]^2\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix} = 6\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}, [\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]_z\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix} = -2\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}$$

$$[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]^2\begin{bmatrix}0\\0\\1\\0\\2\\0\\1\\0\\0\end{bmatrix} = 6\begin{bmatrix}0\\0\\1\\0\\2\\0\\1\\0\\0\end{bmatrix}, [\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]_z\begin{bmatrix}0\\0\\1\\0\\2\\0\\1\\0\\0\end{bmatrix} = 0$$

$$[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]^2\begin{bmatrix}0\\0\\-1\\0\\0\\0\\1\\0\\0\end{bmatrix} = 2\begin{bmatrix}0\\0\\-1\\0\\0\\0\\1\\0\\0\end{bmatrix}, [\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]_z\begin{bmatrix}0\\0\\-1\\0\\0\\0\\1\\0\\0\end{bmatrix} = 0$$

$$[\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]^2\begin{bmatrix}0\\0\\1\\0\\-1\\0\\1\\0\\0\end{bmatrix} = 0, [\sigma(1)\otimes I_3+I_3\otimes\sigma(1)]_z\begin{bmatrix}0\\0\\1\\0\\-1\\0\\1\\0\\0\end{bmatrix} = 0$$

Cor. 5.5.3. $S^+(2)\Rightarrow|s,m\rangle = \begin{cases}\begin{cases}|2,2\rangle = |1\rangle\otimes|1\rangle;\\ \frac{\sqrt{1}}{\sqrt{3}}|2,0\rangle+\frac{\sqrt{2}}{\sqrt{3}}|2;0,0\rangle = \frac{1}{\sqrt{2}}(|-1\rangle\otimes|1\rangle+|1\rangle\otimes|-1\rangle);\\ |2,-2\rangle = |-1\rangle\otimes|-1\rangle;\end{cases}\\ \begin{cases}|2;1,0\rangle = \frac{1}{\sqrt{2}}(-|-1\rangle\otimes|1\rangle+|1\rangle\otimes|-1\rangle);\end{cases}\end{cases}$

Cor. 5.5.4. $\begin{cases} |1\rangle \otimes |1\rangle = |2,2\rangle; \\ |-1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{6}}|2,0\rangle - \frac{1}{\sqrt{2}}|2;1,0\rangle + \frac{1}{\sqrt{3}}|2;0,0\rangle; \\ |1\rangle \otimes |-1\rangle) = \frac{1}{\sqrt{6}}|2,0\rangle + \frac{1}{\sqrt{2}}|2;1,0\rangle + \frac{1}{\sqrt{3}}|2;0,0\rangle; \\ |-1\rangle \otimes |-1\rangle = |2,-2\rangle; \end{cases}$

Cor. 5.5.5. $\begin{cases} \begin{cases} |2,2\rangle = |1\rangle \otimes |1\rangle e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)}; \\ \frac{\sqrt{1}}{\sqrt{3}}|2,0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2;0,0\rangle = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |1\rangle e^{-2i\varsigma pt} + |1\rangle \otimes |-1\rangle e^{2i\varsigma pt})e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)}; \\ |2,-2\rangle = |-1\rangle \otimes |-1\rangle e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)}; \end{cases} \\ \begin{cases} |2;1,0\rangle = \frac{1}{\sqrt{2}}(-|-1\rangle \otimes |1\rangle e^{-2i\varsigma pt} + |1\rangle \otimes |-1\rangle e^{2i\varsigma pt})e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)}; \end{cases} \end{cases}$

$= [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle e^{i(\vec{p}\cdot\vec{r}_1+\varsigma pt)} \\ |-1\rangle e^{i(\vec{p}\cdot\vec{r}_1-\varsigma pt)} \end{bmatrix} \otimes \begin{bmatrix} |1\rangle e^{-i(\vec{p}\cdot\vec{r}_2+\varsigma pt)} \\ |-1\rangle e^{-i(\vec{p}\cdot\vec{r}_2-\varsigma pt)} \end{bmatrix}$

Cor. 5.5.6.

$\begin{cases} \begin{cases} |2,2\rangle = |1\rangle \otimes |1\rangle; \\ i|2;1,0\rangle sin(2\varsigma pt) + (\frac{\sqrt{1}}{\sqrt{3}}|2,0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2;0,0\rangle)cos(2\varsigma pt) = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |1\rangle e^{-2i\varsigma pt} + |1\rangle \otimes |-1\rangle e^{2i\varsigma pt}); \\ |2,-2\rangle = |-1\rangle \otimes |-1\rangle; \end{cases} \\ \begin{cases} |2;1,0\rangle cos(2\varsigma pt) + i(\frac{\sqrt{1}}{\sqrt{3}}|2,0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2;0,0\rangle)sin(2\varsigma pt) = \frac{1}{\sqrt{2}}(-|-1\rangle \otimes |1\rangle e^{-2i\varsigma pt} + |1\rangle \otimes |-1\rangle e^{2i\varsigma pt}); \end{cases} \end{cases}$

$= [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle e^{i\varsigma pt} \\ |-1\rangle e^{-i\varsigma pt} \end{bmatrix} \otimes \begin{bmatrix} |1\rangle e^{-i\varsigma pt} \\ |-1\rangle e^{i\varsigma pt} \end{bmatrix}$

Cor. 5.5.7. $\begin{bmatrix} |2,2\rangle \\ \frac{\sqrt{1}}{\sqrt{3}}|2,0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0,0\rangle \\ |2,-2\rangle \\ |1,0\rangle \end{bmatrix} = [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix} \otimes \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix}, \begin{bmatrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{bmatrix} = [I_4 \otimes \hat{S}(1)] \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix}$

Cor. 5.5.8. $\begin{bmatrix} |2,2\rangle \\ \frac{\sqrt{1}}{\sqrt{3}}|2,0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0,0\rangle \\ |2,-2\rangle \\ |1,0\rangle \end{bmatrix} = [\hat{S}^+(1) \otimes I_9] \begin{bmatrix} |1\rangle \otimes |1\rangle \\ \frac{\sqrt{1}}{\sqrt{3}}|-1\rangle \otimes |1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|-1\rangle \otimes |-1\rangle \\ |0\rangle \otimes |0\rangle \\ |1\rangle \otimes |-1\rangle \end{bmatrix} = [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix} \otimes \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix}$

5.6 Spin coupling of three electrons

Cor. 5.6.1. $\hat{S}(\frac{3}{2}) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & -\sqrt{4} & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{1} & \sqrt{4} & 0 \\ 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 \end{bmatrix}, \hat{S}^+(\frac{3}{2}) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{4} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{4} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \end{bmatrix}$

Cor. 5.6.2. $\hat{S}^+(\frac{3}{2}) \Rightarrow$

$|s,m\rangle = \begin{cases} \begin{cases} |\frac{3}{2},\frac{3}{2}\rangle = |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle; \\ |\frac{3}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(|\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \\ |\frac{3}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(|\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \\ |-\frac{3}{2},-\frac{3}{2}\rangle = |\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle; \end{cases} \\ \begin{cases} |\frac{3}{2};\frac{1}{2},\frac{1}{2}\rangle_1 = \frac{1}{\sqrt{6}}(-2|\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \\ |\frac{3}{2};\frac{1}{2},-\frac{1}{2}\rangle_1 = \frac{1}{\sqrt{6}}(-|\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle - |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle + 2|\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \end{cases} \\ \begin{cases} |\frac{3}{2};\frac{1}{2},\frac{1}{2}\rangle_2 = \frac{1}{\sqrt{2}}(-|\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \\ |\frac{3}{2};\frac{1}{2},-\frac{1}{2}\rangle_2 = \frac{1}{\sqrt{2}}(-|\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \end{cases} \end{cases}$

Cor. 5.6.3. $\begin{bmatrix} |\frac{3}{2},\frac{3}{2}\rangle \\ |\frac{3}{2},\frac{1}{2}\rangle \\ |\frac{3}{2},\frac{-1}{2}\rangle \\ |\frac{3}{2},\frac{-3}{2}\rangle \\ |\frac{1}{2},\frac{1}{2}\rangle_1 \\ |\frac{1}{2},\frac{-1}{2}\rangle_1 \\ |\frac{1}{2},\frac{1}{2}\rangle_2 \\ |\frac{1}{2},\frac{-1}{2}\rangle_2 \end{bmatrix} = [\hat{S}^+(\frac{3}{2}) \otimes I_8] \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix} = [I_8 \otimes \hat{S}(\frac{3}{2})] \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |\frac{-1}{2}\rangle \end{bmatrix}$

6 Explain photon entanglement by classical physics and probability theory

(Just exploratory.)

6.1 Description of spin polarization in classical electromagnetism

Def. 6.1.1. $|\circlearrowleft\rangle := \vec{E}_{\circlearrowleft} = A[cos(p\cdot x+\theta)\hat{e}_x + sin(p\cdot x+\theta)\hat{e}_y], |\circlearrowright\rangle := \vec{E}_{\circlearrowright} = A[cos(p\cdot x+\theta)\hat{e}_x - sin(p\cdot x+\theta)\hat{e}_y]$

Def. 6.1.2. $|\rightarrow\rangle := \vec{E}_{\rightarrow} = \sqrt{2}Acos(p\cdot x+\theta)\hat{e}_x, |\uparrow\rangle := \vec{E}_{\uparrow} = \sqrt{2}Asin(p\cdot x+\theta)\hat{e}_y$

Proof: $I_{\circlearrowleft} = \frac{1}{2}\int_V(\vec{E}^2_{\circlearrowleft} + \vec{B}^2_{\circlearrowleft})d^3\vec{r} = \int_V \vec{E}^2_{\circlearrowleft}d^3\vec{r} = \frac{1}{T}\int_0^T \vec{E}^2_{\circlearrowleft}dt = A^2 = h\nu$ □

Proof: $I_{\circlearrowright} = \frac{1}{2}\int_V(\vec{E}^2_{\circlearrowright} + \vec{B}^2_{\circlearrowright})d^3\vec{r} = \int_V \vec{E}^2_{\circlearrowright}d^3\vec{r} = \frac{1}{T}\int_0^T \vec{E}^2_{\circlearrowright}dt = A^2 = h\nu$ □

Proof: $I_{\rightarrow} = \frac{1}{2}\int_V(\vec{E}^2_{\rightarrow} + \vec{B}^2_{\rightarrow})d^3\vec{r} = \int_V \vec{E}^2_{\rightarrow}d^3\vec{r} = \frac{1}{T}\int_0^T \vec{E}^2_{\rightarrow}dt = A^2 = h\nu$ □

Proof: $I_{\uparrow} = \frac{1}{2}\int_V(\vec{E}^2_{\uparrow} + \vec{B}^2_{\uparrow})d^3\vec{r} = \int_V \vec{E}^2_{\uparrow}d^3\vec{r} = \frac{1}{T}\int_0^T \vec{E}^2_{\uparrow}dt = A^2 = h\nu$ □

Cor. 6.1.1. $\begin{cases}|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle + |\uparrow\rangle\}, |\circlearrowright\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle - |\uparrow\rangle\}\\ |\rightarrow\rangle = \frac{1}{\sqrt{2}}\{|\circlearrowleft\rangle + |\circlearrowright\rangle\}, |\uparrow\rangle = \frac{1}{\sqrt{2}}\{|\circlearrowleft\rangle - |\circlearrowright\rangle\}\end{cases}$

6.2 Description of linear polarization in classical electromagnetics

Def. 6.2.1. $|\rightarrow\rangle := \vec{E}_{\rightarrow} = \sqrt{2}Acos(p\cdot x+\theta)\hat{e}_x, |\uparrow\rangle := \vec{E}_{\uparrow} = \sqrt{2}Acos(p\cdot x+\theta)\hat{e}_y$

Cor. 6.2.1. $\begin{cases}|\nearrow\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle + |\uparrow\rangle\}, |\searrow\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle - |\uparrow\rangle\}\\ |\rightarrow\rangle = \frac{1}{\sqrt{2}}\{|\nearrow\rangle + |\searrow\rangle\}, |\uparrow\rangle = \frac{1}{\sqrt{2}}\{|\nearrow\rangle - |\searrow\rangle\}\end{cases}$

Cor. 6.2.2. $\begin{cases}P(\rightarrow) = 1\end{cases} \rightarrow \oslash \rightarrow \begin{cases}P(\nearrow) = \frac{1}{2}\\ P(\searrow) = 0\end{cases} \rightarrow \ominus \rightarrow \begin{cases}P(\rightarrow) = \frac{1}{4}\\ P(\uparrow) = 0\end{cases} \rightarrow \ominus \rightarrow \begin{cases}P(\rightarrow) = \frac{1}{4}\\ P(\uparrow) = 0\end{cases}$

Cor. 6.2.3. $\begin{cases}P(\rightarrow) = 1\end{cases} \rightarrow \otimes \rightarrow \begin{cases}P(\nearrow) = \frac{1}{2}\\ P(\searrow) = \frac{1}{2}\end{cases} \rightarrow \oplus \rightarrow \begin{cases}P(\uparrow) = \frac{1}{2}\\ P(\rightarrow) = \frac{1}{2}\end{cases} \rightarrow \otimes \rightarrow \begin{cases}P(\nearrow) = \frac{1}{2}\\ P(\searrow) = \frac{1}{2}\end{cases}$

7 Multi particles spin system

7.1 Total spin of two electrons tightly coupled system

Cor. 7.1.1. $\vec{s} = \sigma(\frac{1}{2})\otimes I + I\otimes\sigma(\frac{1}{2})$

7.2 Total spin of two electrons loosely coupled system

Cor. 7.2.1. $s_{ij} = \sigma_i(\frac{1}{2})\otimes\sigma_j(\frac{1}{2})$

7.3 Total spin of multi electrons tightly coupled system

Cor. 7.3.1. $\vec{s} = \Omega(s)$

7.4 Total spin of multi electrons loosely coupled system

Cor. 7.4.1. $s_{i_1i_2\cdots i_{2s}} = \sigma_{i_1}(\frac{1}{2})\otimes\sigma_{i_2}(\frac{1}{2})\otimes\cdot\cdot\otimes\sigma_{i_{2s}}(\frac{1}{2})$

8 Conjecture on multi particles entanglement equation

8.1 Physical meaning 1 of fourth order matrix

Thm. 8.1.1. $X_{ab}(\vec{r},t) = \overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r},t) - \frac{1}{2}\delta_{ab}trX(\vec{r},t)]}^{spin\text{-}2\ particle} + \overbrace{\frac{1}{2!}X_{[ab]}(\vec{r},t)}^{spin\text{-}1\ particle} + \overbrace{\frac{1}{4}\delta_{ab}trX(\vec{r},t)}^{spin\text{-}0\ particle}$

Thm. 8.1.2. $(\partial^c\partial_c - m^2)\overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r},t) - \frac{1}{2}\delta_{ab}trX(\vec{r},t)]}^{spin\text{-}2\ particle} = 0, \partial^a\overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r},t) - \frac{1}{2}\delta_{ab}trX(\vec{r},t)]}^{spin\text{-}2\ particle} = 0$

$$\overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r},t) - \frac{1}{2}\delta_{ab}trX(\vec{r},t)]}^{spin\text{-}2\ particle} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\varepsilon_{ab}(\vec{p},h;2)[a(\vec{p},h;2)e^{ip\cdot x} + b^+(\vec{p},h;2)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.1.3. $(\partial^c\partial_c - m^2)\overbrace{\frac{1}{2!}X_{[ab]}(\vec{r},t)}^{spin\text{-}1\ particle} = 0, \partial^a\overbrace{\frac{1}{2!}X_{[ab]}(\vec{r},t)}^{spin\text{-}1\ particle} = 0$

$$\overbrace{\frac{1}{2!}X_{[ab]}(\vec{r},t)}^{spin\text{-}1\ particle} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\varepsilon_{ab}(\vec{p},h;1)[a(\vec{p},h;1)e^{ip\cdot x} + b^+(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.1.4. $(\partial^c\partial_c - m^2)\overbrace{\frac{1}{4}\delta_{ab}trX(\vec{r},t)}^{spin\text{-}0\ particle} = 0$

$$\overbrace{\frac{1}{4}\delta_{ab}trX(\vec{r},t)}^{spin\text{-}0\ particle} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}[\varepsilon_{ab}(\vec{p},0;0) + \frac{p_ap_b}{m^2}][a(\vec{p},0;0)e^{ip\cdot x} + b^+(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}$$

8.2 Physical meaning 2 of fourth order matrix

Thm. 8.2.1. $X_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=\overbrace{\frac{1}{2!}X_{\{\lambda_\varsigma\mu_\varsigma\}}(\vec{r},t)}^{spin\text{-}1\ particle}+\overbrace{\frac{1}{2!}X_{[\lambda_\varsigma\mu_\varsigma]}(\vec{r},t)}^{spin\text{-}0\ particle}$

Thm. 8.2.2. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\overbrace{\frac{1}{2!}X_{\{\lambda_\varsigma\mu_\varsigma\}}(\vec{r},t)}^{spin\text{-}1\ particle}=0$

$$\overbrace{\frac{1}{2!}X_{\{\lambda_\varsigma\mu_\varsigma\}}(\vec{r},t)}^{spin\text{-}1\ particle}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{\frac{m^2}{E}}[a(\vec{p},h;1)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h;1)e^{ip\cdot x}+b^+(\vec{p},h;1)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.2.3. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\overbrace{\frac{1}{2!}X_{[\lambda_\varsigma\mu_\varsigma]}(\vec{r},t)}^{spin\text{-}0\ particle}=0$

$$\overbrace{\frac{1}{2!}X_{[\lambda_\varsigma\mu_\varsigma]}(\vec{r},t)}^{spin\text{-}0\ particle}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}[a(\vec{p},0;0)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0;0)e^{ip\cdot x}+b^+(\vec{p},0;0)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}$$

8.3 Massive dual-photon entanglement equation

Thm. 8.3.1. $(\partial^c\partial_c-m^2)X_{ab}(\vec{r},t)=0,\partial^aX_{ab}(\vec{r},t)=0,\partial^bX_{ab}(\vec{r},t)=0$

$$\begin{cases}X_{ab}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\varepsilon_{ab}(\vec{p},h;2)[a(\vec{p},h;2)e^{ip\cdot x}+b^+(\vec{p},h;2)e^{-ip\cdot x}]d^3\vec{p}\\+\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}\varepsilon_{ab}(\vec{p},h;1)[a(\vec{p},h;1)e^{ip\cdot x}+b^+(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}\\+\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}\varepsilon_{ab}(\vec{p},0;0)[a(\vec{p},0;0)e^{ip\cdot x}+b^+(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}\end{cases}$$

Thm. 8.3.2.

$$\begin{cases}\hat{J}^2(\vec{p},2;R,L)\varepsilon_{a\otimes b}(\vec{p},h;2)=2(2+1)\varepsilon_{a\otimes b}(\vec{p},h;2),\hat{J}_z(\vec{p},2;R,L)\varepsilon_{a\otimes b}(\vec{p},h;2)=h\varepsilon_{a\otimes b}(\vec{p},h;2),2\le h\le 2\\\hat{J}^2(\vec{p},2;R,L)\varepsilon_{a\otimes b}(\vec{p},h;1)=1(1+1)\varepsilon_{a\otimes b}(\vec{p},h;1),\hat{J}_z(\vec{p},2;R,L)\varepsilon_{a\otimes b}(\vec{p},h;1)=h\varepsilon_{a\otimes b}(\vec{p},h;1),1\le h\le 1\\\hat{J}^2(\vec{p},2;R,L)\delta_{a\otimes b}(\vec{p},0;0)=0(0+1)\delta_{a\otimes b}(\vec{p},0;0),\hat{J}_z(\vec{p},2;R,L)\delta_{a\otimes b}(\vec{p},0;0)=0\delta_{a\otimes b}(\vec{p},0;0)\\\hat{J}^2(\vec{p},2;R,L)\varepsilon_{a\otimes b}(\vec{p},0;0)=0(0+1)\varepsilon_{a\otimes b}(\vec{p},0;0),\hat{J}_z(\vec{p},2;R,L)\varepsilon_{a\otimes b}(\vec{p},0;0)=0\varepsilon_{a\otimes b}(\vec{p},0;0)\\\hat{J}^2(\vec{p},2;R,L)\frac{p_a\otimes p_b}{m^2}=0(0+1)\frac{p_a\otimes p_b}{m^2},\hat{J}_z(\vec{p},2;R,L)\frac{p_a\otimes p_b}{m^2}=0\frac{p_a\otimes p_b}{m^2},\varepsilon_{ab}(\vec{p},0;0)+\frac{p_ap_b}{m^2}=-\delta_{ab}\end{cases}$$

8.4 Dual-electron entanglement equation

Thm. 8.4.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}X_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)=0,X_{\lambda_\varsigma\mu_\varsigma}(\vec{r},t)$

$$=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{\frac{m^2}{E}}[a(\vec{p},h;1)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h;1)e^{ip\cdot x}+b^+(\vec{p},h;1)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}$$
$$+\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}[a(\vec{p},0;0)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0;0)e^{ip\cdot x}+b^+(\vec{p},0;0)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.4.2.

$$\begin{cases}\hat{J}^2(\vec{p},1;\gamma_a)U_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1)=1(1+1)U_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1),\hat{J}_z(\vec{p},1;\gamma_a)U_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1)=hU_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1),-1\le h\le 1\\\hat{J}^2(\vec{p},1;\gamma_a)U_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0)=0(0+1)U_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0),\hat{J}_z(\vec{p},1;\gamma_a)U_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0)=0U_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0)\end{cases}$$
$$\begin{cases}\hat{J}^2(\vec{p},1;\gamma_a)V_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1)=1(1+1)V_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1),\hat{J}_z(\vec{p},1;\gamma_a)V_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1)=hV_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},h;1),-1\le h\le 1\\\hat{J}^2(\vec{p},1;\gamma_a)V_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0)=0(0+1)V_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0),\hat{J}_z(\vec{p},1;\gamma_a)V_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0)=0V_{\lambda_\varsigma\otimes\mu_\varsigma}(\vec{p},0;0)\end{cases}$$

8.5 Physical meaning of second order matrix

Thm. 8.5.1. $X_{AB}(\vec{r},t)=\overbrace{\frac{1}{2!}X_{\{AB\}}(\vec{r},t)}^{spin\text{-}1\ particle}+\overbrace{\frac{1}{2!}X_{[AB]}(\vec{r},t)}^{spin\text{-}0\ particle}$

Thm. 8.5.2. $(\partial^c\partial_c-m^2)\overbrace{\frac{1}{2!}X_{\{AB\}}(\vec{r},t)}^{spin\text{-}1\ particle}=0$

$$\overbrace{\frac{1}{2!}X_{\{AB\}}(\vec{r},t)}^{spin\text{-}1\ particle}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{|\vec{p}|}\lambda_{AB}(\hat{p},h;1)[a(\vec{p},h;1)e^{ip\cdot x}+b^+(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.5.3. $(\partial^c\partial_c-m^2)\overbrace{\frac{1}{2!}X_{[AB]}(\vec{r},t)}^{spin\text{-}0\ particle}=0$

$$\overbrace{\frac{1}{2!}X_{[AB]}(\vec{r},t)}^{spin\text{-}0\ particle}=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}\lambda_{AB}(\hat{p},0;0)[a(\vec{p},0;0)e^{ip\cdot x}+b^+(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}$$

8.6 Dual-neutrinos entanglement equation

Thm. 8.6.1. $(\partial^c\partial_c - m^2)X_{AB}(\vec{r},t) = 0, X_{AB}(\vec{r},t)$

$$= \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\sqrt{|\vec{p}|}\lambda_{AB}(\hat{p},h;1)[a(\vec{p},h;1)e^{ip\cdot x}+b^{+}(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}$$

$$+\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}\lambda_{AB}(\hat{p},0;0)[a(\vec{p},0;0)e^{ip\cdot x}+b^{+}(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.6.2.

$$\begin{cases}\hat{J}^2(\vec{p},1;\Omega(1))\lambda_{A\otimes B}(\vec{p},h;1) = 1(1+1)\lambda_{A\otimes B}(\vec{p},h;1), \hat{J}_z(\vec{p},1;\Omega(1))\lambda_{A\otimes B}(\vec{p},h;1) = h\lambda_{A\otimes B}(\vec{p},h;1), -1\le h\le 1\\ \hat{J}^2(\vec{p},1;\Omega(1))\lambda_{A\otimes B}(\vec{p},0;0) = 0(0+1)\lambda_{A\otimes B}(\vec{p},0;0), \hat{J}_z(\vec{p},1;\Omega(1))\lambda_{A\otimes B}(\vec{p},0;0) = 0\lambda_{A\otimes B}(\vec{p},0;0)\end{cases}$$

8.7 Physical meaning of third order matrix

Thm. 8.7.1. $X_{\alpha\beta}(\vec{r},t) = \overbrace{\frac{1}{2!}[X_{\{\alpha\beta\}}(\vec{r},t)-\frac{1}{2}\delta_{\alpha\beta}trX(\vec{r},t)]}^{spin\text{-}2\ particle} + \overbrace{\frac{1}{2!}X_{[\alpha\beta]}(\vec{r},t)}^{spin\text{-}1\ particle} + \overbrace{\frac{1}{4}\delta_{\alpha\beta}trX(\vec{r},t)}^{spin\text{-}0\ particle}$

Thm. 8.7.2. $(\partial^c\partial_c - m^2)\overbrace{\frac{1}{2!}[X_{\{\alpha\beta\}}(\vec{r},t)-\frac{1}{2}\delta_{\alpha\beta}trX(\vec{r},t)]}^{spin\text{-}2\ particle} = 0$

$$\overbrace{\frac{1}{2!}[X_{\{\alpha\beta\}}(\vec{r},t)-\frac{1}{2}\delta_{\alpha\beta}trX(\vec{r},t)]}^{spin\text{-}2\ particle} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\varepsilon_{\alpha\beta}(\vec{p},h;2)[a(\vec{p},h;2)e^{ip\cdot x}+b^{+}(\vec{p},h;2)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.7.3. $(\partial^c\partial_c - m^2)\overbrace{\frac{1}{2!}X_{[\alpha\beta]}(\vec{r},t)}^{spin\text{-}1\ particle} = 0$

$$\overbrace{\frac{1}{2!}X_{[\alpha\beta]}(\vec{r},t)}^{spin\text{-}1\ particle} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\varepsilon_{\alpha\beta}(\vec{p},h;1)[a(\vec{p},h;1)e^{ip\cdot x}+b^{+}(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}$$

Thm. 8.7.4. $(\partial^c\partial_c - m^2)\overbrace{\frac{1}{4}\delta_{\alpha\beta}trX(\vec{r},t)}^{spin\text{-}0\ particle} = 0$

$$\overbrace{\frac{1}{4}\delta_{\alpha\beta}trX(\vec{r},t)}^{spin\text{-}0\ particle} = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}\varepsilon_{\alpha\beta}(\vec{p},0;0)[a(\vec{p},0;0)e^{ip\cdot x}+b^{+}(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}$$

8.8 Massive dual entangled photons equation of third order matrix

Thm. 8.8.1. $(\partial^c\partial_c - m^2)X_{\alpha\beta}(\vec{r},t) = 0$

$$\begin{cases}X_{\alpha\beta}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\varepsilon_{\alpha\beta}(\vec{p},h;2)[a(\vec{p},h;2)e^{ip\cdot x}+b^{+}(\vec{p},h;2)e^{-ip\cdot x}]d^3\vec{p}\\ +\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1}^{-1}\frac{1}{\sqrt{2E}}\varepsilon_{\alpha\beta}(\vec{p},h;1)[a(\vec{p},h;1)e^{ip\cdot x}+b^{+}(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}\\ +\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}\varepsilon_{\alpha\beta}(\vec{p},0;0)[a(\vec{p},0;0)e^{ip\cdot x}+b^{+}(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}\end{cases}$$

Cor. 8.8.1. $\varepsilon_{\alpha\beta}(\vec{p},h;2) = \sum_{h'=1}^{-1}\frac{\sqrt{C_{2+h}^{1+h'}C_{2-h}^{1-h'}}}{\sqrt{C_4^2}}\varepsilon_{\alpha}(\vec{p},h-h')\varepsilon_{\beta}(\vec{p},h') =$

$$\begin{cases}\varepsilon_{\alpha\beta}(\vec{p},2;2) = \varepsilon_{\alpha}(\vec{p},1)\varepsilon_{\beta}(\vec{p},1)\\ \varepsilon_{\alpha\beta}(\vec{p},1;2) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p},1;2)\varepsilon_{\beta}(\vec{p},0)+\varepsilon_{\alpha}(\vec{p},0)\varepsilon_{\beta}(\vec{p},1)]\\ \varepsilon_{\alpha\beta}(\vec{p},0;2) = \frac{1}{\sqrt{6}}[\varepsilon_{\alpha}(\vec{p},1)\varepsilon_{\beta}(\vec{p},-1)+\varepsilon_{\alpha}(\vec{p},-1)\varepsilon_{\beta}(\vec{p},1)+2\varepsilon_{\alpha}(\vec{p},0)\varepsilon_{\beta}(\vec{p},0)]\\ \varepsilon_{\alpha\beta}(\vec{p},-1;2) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p},-1)\varepsilon_{\beta}(\vec{p},0)+\varepsilon_{\alpha}(\vec{p},0)\varepsilon_{\beta}(\vec{p},-1)]\\ \varepsilon_{\alpha\beta}(\vec{p},-2;2) = \varepsilon_{\alpha}(\vec{p},-1)\varepsilon_{\beta}(\vec{p},-1)\end{cases}$$

Cor. 8.8.2. $\varepsilon_{\alpha\beta}(\vec{p},h;1) =$

$$\begin{cases}\varepsilon_{\alpha\beta}(\vec{p},1;1) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p},1)\varepsilon_{\beta}(\vec{p},0)-\varepsilon_{\alpha}(\vec{p},0)\varepsilon_{\beta}(\vec{p},1)]\\ \varepsilon_{\alpha\beta}(\vec{p},0;1) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p},1)\varepsilon_{\beta}(\vec{p},-1)-\varepsilon_{\alpha}(\vec{p},-1)\varepsilon_{\beta}(\vec{p},1)]\\ \varepsilon_{\alpha\beta}(\vec{p},-1;1) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p},-1)\varepsilon_{\beta}(\vec{p},0)-\varepsilon_{\alpha}(\vec{p},0)\varepsilon_{\beta}(\vec{p},-1)]\end{cases}$$

Cor. 8.8.3. $\varepsilon_{\alpha\beta}(\vec{p},0;0) = \frac{1}{\sqrt{3}}[\varepsilon_{\alpha}(\vec{p},1)\varepsilon_{\beta}(\vec{p},-1)+\varepsilon_{\alpha}(\vec{p},-1)\varepsilon_{\beta}(\vec{p},1)-\varepsilon_{\alpha}(\vec{p},0)\varepsilon_{\beta}(\vec{p},0)] = \frac{1}{\sqrt{3}}\delta_{\alpha\beta}$

Thm. 8.8.2.

$$\begin{cases}\hat{J}^2(\vec{p},2;\gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p},h;2)=2(2+1)\varepsilon_{\alpha\otimes\beta}(\vec{p},h;2),\hat{J}_z(\vec{p},2;\gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p},h;2)=h\varepsilon_{\alpha\otimes\beta}(\vec{p},h;2),2\le h\le 2\\ \hat{J}^2(\vec{p},2;\gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p},h;1)=1(1+1)\varepsilon_{\alpha\otimes\beta}(\vec{p},h;1),\hat{J}_z(\vec{p},2;\gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p},h;1)=h\varepsilon_{\alpha\otimes\beta}(\vec{p},h;1),1\le h\le 1\\ \hat{J}^2(\vec{p},2;\gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p},0;0)=0(0+1)\varepsilon_{\alpha\otimes\beta}(\vec{p},0;0),\hat{J}_z(\vec{p},2;\gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p},0;0)=0\varepsilon_{\alpha\otimes\beta}(\vec{p},0;0)\end{cases}$$

8.9 Massive dual entangled gravitinos equation of fourth order matrix

Thm. 8.9.1. $(\partial^c\partial_c-m^2)X_{kl}(\vec{r},t)=0$

$$\begin{cases}X_{kl}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=3}^{-3}\frac{1}{\sqrt{2^3E}}\lambda_{kl}(\vec{p},h;3)[a(\vec{p},h;3)e^{ip\cdot x}+b^{+}(\vec{p},h;3)e^{-ip\cdot x}]d^3\vec{p}\\ +\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=2}^{-2}\frac{1}{\sqrt{2^2E}}\lambda_{kl}(\vec{p},h;2)[a(\vec{p},h;2)e^{ip\cdot x}+b^{+}(\vec{p},h;2)e^{-ip\cdot x}]d^3\vec{p}\\ +\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}\lambda_{kl}(\vec{p},h;1)[a(\vec{p},h;1)e^{ip\cdot x}+b^{+}(\vec{p},h;1)e^{-ip\cdot x}]d^3\vec{p}\\ +\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}\lambda_{kl}(\vec{p},0;0)[a(\vec{p},0;0)e^{ip\cdot x}+b^{+}(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}\end{cases}$$

8.10 Massive dual entangled s-spin particles equation of $2s+1$-order matrix

Thm. 8.10.1. $(\partial^c\partial_c-m^2)X_{kk'}(\vec{r},t)=0,(2s+1)^2=(4s+1)+(4s-1)+\cdot\cdot+3+1$

$$\begin{cases}X_{kk'}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\sum\limits_{n=1}^{2s}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=2n+1}^{-(2n+1)}\frac{1}{\sqrt{2^nE}}\lambda_{kk'}(\vec{p},h;n)[a(\vec{p},h;n)e^{ip\cdot x}+b^{+}(\vec{p},h;n)e^{-ip\cdot x}]d^3\vec{p}\\ +\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2E}}\lambda_{kk'}(\vec{p},0;0)[a(\vec{p},0;0)e^{ip\cdot x}+b^{+}(\vec{p},0;0)e^{-ip\cdot x}]d^3\vec{p}\end{cases}$$

8.11 $s_1\otimes s_2$ entanglement equation conjecture of $(2s_1+1)\times(2s_2+1)$ matrix

Thm. 8.11.1. $(\partial^c\partial_c-m^2)X_{kk'}(\vec{r},t)=0,(s_1+s_2+1)^2=[2(s_1+s_2)+1]+[2(s_1+s_2)-1]+\cdot\cdot+[2|s_1-s_2|+1]$

$$X_{kk'}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\sum\limits_{s=2|s_1-s_2|+1}^{2(s_1+s_2)+1}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=2s+1}^{-(2s+1)}\frac{m^{s-[s]}}{\sqrt{2^{[s]}E}}\lambda_{kk'}(\vec{p},h;s)[a(\vec{p},h;s)e^{ip\cdot x}+b^{+}(\vec{p},h;s)e^{-ip\cdot x}]d^3\vec{p}$$

Chapter33 Internal Component Interaction of Elementary Particles

1 Basic particle internal component hypothesis

Basic particle internal component hypothesis: It is assumed that the elementary particle has internal components, each corresponding to a geometric point. Scalar particle has an internal component corresponding to a geometric point. Neutrino [5] has two internal components corresponding to two geometric points. Photon [7,8] has three internal components corresponding to three geometric points. Gravitino [17] has four internal components corresponding to four geometric points. Graviton [11–14] has five internal components corresponding to five geometric points. electron [4] has four internal components corresponding to four geometric points.

The internal component interaction is essentially quantum entanglement. The form of interaction is different from traditional one, not attraction and repulsion, but mutual internal component correlation. It is a new interaction. owever, the transmission speed of the interaction is still the speed of light, not instantaneous transmission. The formation of baryons by several quarks is a complete internal component interaction. The normal plane wave superposition has no internal component action.

2 Particle recombination theory

2.1 Physical mechanism of two neutrinos synthesizing one photon

Initially, there were two independent neutrinos χ and φ. Then a new interaction occurs that is not yet known, namely the interaction between internal quantities. This interaction makes the second component of the first neutrino χ equal to the first component of the second neutrino φ in any reference system. That is $\chi_2 \equiv \varphi_1$. That is, the two geometric points of the internal component coincide. After this effect occurs, the previously independent covariancy cannot be maintaine. Because of their independent Lorentz transforms to other reference systems, $\chi_2 \equiv \varphi_1$ can't establish. The result is that the two particles form a new covariance together. It will create a new spin and that becomes a new particle: a photon. The process is as follows:

$$\begin{cases} (\sigma, -i\varsigma)^a \partial_a \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \\ (\sigma, -i\varsigma)^a \partial_a \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \end{cases} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a \partial_a \begin{bmatrix} \chi_1 \\ \chi_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes I \tag{33.1}$$

The geometric points of the two internal components coincide: $\chi_2 \equiv \varphi_1$. The meaningful equation becomes:

$$\to (\sigma \otimes I, -i\varsigma)^a \partial_a \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_1 \equiv \chi_2 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_1 \equiv \chi_2 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \tag{33.2}$$

$$\Leftrightarrow [\partial_a + S_{ab}(1,\varsigma)\partial^b] \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\tau(1)} \tag{33.3}$$

2.2 Physical mechanism of 2s neutrinos synthesizing one s-spin particle

Initially, there were 2s independent neutrinos ${}^i\varphi, i = 1, 2, \cdots, 2s$. Then a new interaction occurs that is not yet known, namely the interaction between internal quantities. This interaction makes the second component of the i-order neutrino ${}^i\varphi$ equal to the first component of the i+1-order neutrino ${}^{i+1}\varphi$ in any reference system. That is ${}^i\varphi_2 \equiv^{i+1} \varphi_1$. That is, the two geometric points of the internal component coincide. After this effect occurs, the previously independent covariancy cannot be maintaine. Because of their independent Lorentz transforms to other reference systems, ${}^i\varphi_2 \equiv^{i+1} \varphi_1$ can't establish. The result is that the 2s particles form a new covariance together. It will create a new spin and that

becomes a new particle: a s-spin particle. The process is as follows:

$$\begin{cases} (\sigma,-i\varsigma)^a\partial_a \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \\ (\sigma,-i\varsigma)^a\partial_a \begin{bmatrix} {}^2\varphi_1 \\ {}^2\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^2\varphi_1 \\ {}^2\varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \\ \cdots\cdots \\ (\sigma,-i\varsigma)^a\partial_a \begin{bmatrix} {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})} \end{cases} \tag{33.4}$$

$$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \\ {}^2\varphi_1 \\ {}^2\varphi_2 \\ \cdots \\ {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \\ {}^2\varphi_1 \\ {}^2\varphi_2 \\ \cdots \\ {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}\otimes I_{2s} \tag{33.5}$$

The geometric points of internal components of two adjacent particles coincide: ${}^i\varphi_2 \equiv {}^{i+1}\varphi_1$. The meaningful equation becomes:

$$\to(\sigma\otimes I_{2s},-i\varsigma)^a\partial_a \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_1 \equiv {}^1\varphi_2 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \cdots \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_1 \equiv {}^1\varphi_2 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \cdots \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\sigma(\frac{1}{2})}{}_{old}\otimes e^{(i\omega+\varsigma\epsilon)\cdot\tau(s-\frac{1}{2})}{}_{new} \tag{33.6}$$

$$\Leftrightarrow [s\partial_a + S_{ab}(s,\varsigma)\partial^b] \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \cdots \\ {}^{2s-2}\varphi_2 \equiv {}^{2s-1}\varphi_1 \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \cdots \\ {}^{2s-2}\varphi_2 \equiv {}^{2s-1}\varphi_1 \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\varsigma\epsilon)\cdot\tau(s)} \tag{33.7}$$

The above mathematical process can be correctly understood in various ways. That is, several neutrinos are first divided into several groups. Each group is synthesized into a new particle. Then these new particles are grouped again and each group is also synthesized into a new particle. This process can be repeated until it is synthesized into a single spin-s particle. Therefore, this synthesis process can be implemented in many combinations. So two neutrinos can be synthesized into one photon. hree neutrinos can be synthesized into a gravitino. A neutrino and a photon can be synthesized into a gravitino. Four neutrinos can be synthesized into one graviton. wo neutrinos and a photon can be synthesized into a graviton. Two photons can be synthesized into a graviton. A neutrino and a gravitational neutrino can be synthesized into a graviton, and so on.

3 Constant tensors and new interaction

3.1 New interaction

Def. 3.1.1. $S_I = G\int dx^4\psi_{k_\varsigma}(s)\Gamma^{k_\varsigma}_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdots}_{2s}}(s)\overbrace{\psi_1^{A_\varsigma}\psi_2^{B_\varsigma}\psi_3^{C_\varsigma}\cdots}^{2s}+\{\}^*$

3.2 Graviton synthesis interaction

Def. 3.2.1. $S_{I1111} = G\int dx^4\psi_{k_\varsigma}(2)\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi_1^{A_\varsigma}\psi_2^{B_\varsigma}\psi_3^{C_\varsigma}\psi_4^{D_\varsigma}+\{\}^*$

Def. 3.2.2. $S_{I211} = G\int dx^4\psi_{k_\varsigma}(2)\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi_1^{A_\varsigma B_\varsigma}\psi_2^{C_\varsigma}\psi_3^{D_\varsigma}+\{\}^*$

Def. 3.2.3. $S_{I22} = G\int dx^4\psi_{k_\varsigma}(2)\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi_1^{A_\varsigma B_\varsigma}\psi_2^{C_\varsigma D_\varsigma}+\{\}^*$

Def. 3.2.4. $S_{I31} = G\int dx^4\psi_{k_\varsigma}(2)\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(2)\psi_1^{A_\varsigma B_\varsigma C_\varsigma}\psi_2^{D_\varsigma}+\{\}^*$

3.3 Gravitino synthesis interaction

Def. 3.3.1. $S_{I111} = G\int dx^4 \psi_{k_\varsigma}(\frac{3}{2})\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma}(\frac{3}{2})\psi_1^{A_\varsigma}\psi_2^{B_\varsigma}\psi_3^{C_\varsigma} + \{\}^*$

Def. 3.3.2. $S_{I21} = G\int dx^4 \psi_{k_\varsigma}(\frac{3}{2})\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}(\frac{3}{2})\psi_1^{A_\varsigma B_\varsigma}\psi_2^{C_\varsigma} + \{\}^*$

3.4 Photon synthesis interaction

Def. 3.4.1. $S_{I11} = G\int dx^4 \psi_{k_\varsigma}(1)\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma}(1)\psi_1^{A_\varsigma}\psi_2^{B_\varsigma} + \{\}^*$

Def. 3.4.2. $S_{I11} = G\int dx^4 \psi_{\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}\psi_1^{A_\varsigma}\psi_2^{B_\varsigma} + \{\}^*$

Def. 3.4.3. $S_{I11} = G\int dx^4 \psi_{\alpha_\varsigma}\sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s)\psi_1^{k_\varsigma}(s)\psi_2^{l_\varsigma}(s) + \{\}^*$

3.5 New similar electromagnetic interactions

Thm. 3.5.1. $S = \int dx^4\{-\frac{e}{4}F^{ab}F_{ab} - \nu_e^+(\sigma,-i)^a\partial_a\nu_e - \nu_\mu^+(\sigma,-i)^a\partial_a\nu_\mu - \nu_\tau^+(\sigma,-i)^a\partial_a\nu_\tau\}$
$+ \frac{1}{2}G\int dx^4\{F_{ab}\sigma^{ab}_{+\alpha}\sigma^\alpha_{AB}[(m_e-m_\mu)\nu_e^A\nu_\mu^B + (m_\mu-m_\tau)\nu_\mu^A\nu_\tau^B + (m_\tau-m_e)\nu_\tau^A\nu_e^B]\}$

Thm. 3.5.2. $S_I = \frac{1}{2}G\int dx^4\{F_{ab}\sigma^{ab}_{+\alpha}\sigma^\alpha_{AB}[\alpha_{e\mu}\nu_e^A\nu_\mu^B + \alpha_{\mu\tau}\nu_\mu^A\nu_\tau^B + \alpha_{\tau e}\nu_\tau^A\nu_e^B]\}$

Thm. 3.5.3. $S_I = \frac{1}{2}G\int dx^4\{F_{ab}\sigma^{ab}_{+\alpha}\sigma^\alpha_{AB}(\nu_e^A\nu_\mu^B + \nu_\mu^A\nu_\tau^B + \nu_\tau^A\nu_e^B)\}$

Thm. 3.5.4.
$\nu_e \to \gamma + \bar{\nu}_\mu \to \nu_\tau : \alpha_{e\mu}\alpha_{\mu\tau}$
$\nu_e \to \gamma + \bar{\nu}_\tau \to \nu_\mu : \alpha_{\mu\tau}\alpha_{\tau e}$
$\nu_e \to \gamma + \bar{\nu}_\mu \to \nu_e : \alpha_{e\mu}\alpha_{e\mu}$
$\nu_e \to \gamma + \bar{\nu}_\tau \to \nu_e : \alpha_{\tau e}\alpha_{\tau e}$
$\nu_\mu \to \gamma + \bar{\nu}_\tau \to \nu_e$
$\Psi = \vec{E} + i\vec{B} = \vec{E} + i\nabla\times\vec{A}$
$\Psi_i = E_i + i\varepsilon_i{}^{jk}\partial_j A_k$
$[\Psi_i(x), \Psi_j(x')] = i\varepsilon_i{}^{kl}\partial_{x_k}[A_l(x), E_j(x')] + i\varepsilon_j{}^{kl}\partial_{x'_k}[E_i(x), A_l(x')]$
$[\Psi_i(x), \Psi_j(x')] = -\varepsilon_{ij}{}^k(\partial_{x_k} + \partial_{x'_k})\delta^3(x-x') = 0$

Thm. 3.5.5.
$\Psi = \vec{E} + i\vec{B} = \vec{E} + i\nabla\times\vec{A}$
$\Psi_i = E_i + i\varepsilon_i{}^{jk}\partial_j A_k$
$[\Psi_i(x), \Psi_j^+(x')] = i\varepsilon_i{}^{kl}\partial_{x_k}[A_l(x), E_j(x')] - i\varepsilon_j{}^{kl}\partial_{x'_k}[E_i(x), A_l(x')]$
$[\Psi_i(x), \Psi_j^+(x')] = -\varepsilon_{ij}{}^k(\partial_{x_k} - \partial_{x'_k})\delta^3(x-x') = -2\varepsilon_{ij}{}^k\partial_{(x_k - x'_k)}\delta^3(x-x')$

4 Internal component interaction of particles

4.1 Internal component interaction of photon pair

Ass. 4.1.1. $\varepsilon_{ab}(\vec{p}_1, \vec{p}_2; 0; 1) = \frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p}_1, 1)\varepsilon_b(\vec{p}_2, -1) - \varepsilon_a(\vec{p}_1, -1)\varepsilon_b(\vec{p}_2, 1)]$

Ass. 4.1.2. $\varepsilon_{ab}(\vec{p}, 0; 1) = \frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p}, 1)\varepsilon_b(-\vec{p}, 1) - \varepsilon_a(-\vec{p}, 1)\varepsilon_b(\vec{p}, 1)]$

Chapter34 Plane Wave Solutions for Symmetric and Antisymmetric Equations

Self comment: This chapter provides a unified solution for plane wave solutions of various fully symmetric and antisymmetric equations and also provides leading knowledge for latter physical research.

1 plane wave solutions of Bargmann-Wigner equation

1.1 Two corollaries

Cor. 1.1.1.

$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})u_{\tau_\varsigma}(\vec{p},-\frac{1}{2}) \\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h)=\frac{\sqrt{s+1/2+h}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1/2-h}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h+\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}) \end{cases}$$

Cor. 1.1.2.

$$\begin{cases} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{\sqrt{s+h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})v_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})v_{\tau_\varsigma}(\vec{p},-\frac{1}{2}) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h)=\frac{\sqrt{s+1/2+h}}{\sqrt{2s+1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1/2-h}}{\sqrt{2s+1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h+\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}) \end{cases}$$

1.2 Two lemmas on U-spin basis

Lem. 1.2.1. $\sum\limits_{h=s}^{-s}a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum\limits_{h=s}^{-s}a_{\tau_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)$

$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h)u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h-1)u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\leq h\leq s$

Proof: $\sum\limits_{h=s}^{-s}a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum\limits_{h=s}^{-s}a_{\tau_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)$

$\Leftrightarrow \sum\limits_{h=s}^{-s}a_{\eta_\varsigma}(\vec{p},h)[\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})u_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

$=\sum\limits_{h=s}^{-s}a_{\tau_\varsigma}(\vec{p},h)\frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})$

$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}a_{\eta_\varsigma}(\vec{p},h)u_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}a_{\eta_\varsigma}(\vec{p},h-1)u_{\tau_\varsigma}(\vec{p},-\frac{1}{2})]$

$=\frac{\sqrt{s+h}}{\sqrt{2s}}a_{\tau_\varsigma}(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}a_{\tau_\varsigma}(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\leq h\leq s$

$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h)u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h-1)u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\leq h\leq s$ □

Lem. 1.2.2. $\frac{\sqrt{s+h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h)u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h-1)u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\leq h\leq s$

$$\Leftrightarrow\begin{cases} a_{\eta_\varsigma}(\vec{p},h)=c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\leq h\leq s \\ a_{\eta_\varsigma}(\vec{p},-s)=c_+(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),h=-s \end{cases}$$

Proof: $\frac{\sqrt{s+h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h)u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p},h-1)u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\leq h\leq s$

$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}\{c_+(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})+d_+(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+d_-(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})\}u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}$
$\{c_+(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})+d_+(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+d_-(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})\}u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\leq h\leq s$

$\Leftrightarrow c_+(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_-(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})$
$+[c_-(\vec{p},h)-\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)]u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})$
$+d_+(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_-(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})$
$+d_-(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\leq h\leq s$

$\Leftrightarrow [c_-(\vec{p},h)-\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)]u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})$
$+d_+(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_-(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})$
$+d_-(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})u_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le h\le s$
$\Leftrightarrow \begin{cases}\frac{\sqrt{s+h}}{\sqrt{2s}}c_-(\vec{p},h)=\frac{\sqrt{s+1-h}}{\sqrt{2s}}c_+(\vec{p},h-1),-(s-1)\le h\le s\\ d_+(\vec{p},h)=0,d_-(\vec{p},h)=0,-s\le h\le s\end{cases}$
$\Leftrightarrow \begin{cases}a_{\eta_\varsigma}(\vec{p},h)=c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\le h\le s\\ a_{\eta_\varsigma}(\vec{p},-s)=c_+(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),h=-s\end{cases}$ □

1.3 Two lemmas on V-spin basis

Lem. 1.3.1. $\sum_{h=s}^{-s}b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum_{h=s}^{-s}b^+_{\tau_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)$
$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h)v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h-1)v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le h\le s$

Proof: $\sum_{h=s}^{-s}b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum_{h=s}^{-s}b^+_{\tau_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)$

$\Leftrightarrow \sum_{h=s}^{-s}b^+_{\eta_\varsigma}(\vec{p},h)[\frac{\sqrt{s+h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})v_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})$

$=\sum_{h=s}^{-s}b^+_{\tau_\varsigma}(\vec{p},h)\frac{\sqrt{s+h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h-\frac{1}{2})v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p},h+\frac{1}{2})v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})$

$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}b^+_{\eta_\varsigma}(\vec{p},h)v_{\tau_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}b^+_{\eta_\varsigma}(\vec{p},h-1)v_{\tau_\varsigma}(\vec{p},-\frac{1}{2})]$
$=\frac{\sqrt{s+h}}{\sqrt{2s}}b^+_{\tau_\varsigma}(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}b^+_{\tau_\varsigma}(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\le h\le s$

$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h)v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h-1)v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le h\le s$ □

Lem. 1.3.2. $\frac{\sqrt{s+h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h)v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h-1)v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le h\le s$
$\Leftrightarrow \begin{cases}b^+_{\eta_\varsigma}(\vec{p},h)=d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\le h\le s\\ b^+_{\eta_\varsigma}(\vec{p},-s)=d_+(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),h=-s\end{cases}$

Proof: $\frac{\sqrt{s+h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h)v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p},h-1)v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le h\le s$

$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}\{c_+(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})+d_+(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+d_-(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})\}v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{2s}}$
$\{c_+(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})+d_+(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})+d_-(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})\}v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le$
$h\le s$
$\Leftrightarrow d_+(\vec{p},h)v_{[\eta_\varsigma}(\vec{p},\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_-(\vec{p},h-1)v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})$
$+[d_-(\vec{p},h)-\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)]v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})$
$+c_+(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_-(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})$
$+c_-(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le h\le s$
$\Leftrightarrow [d_-(\vec{p},h)-\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)]v_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})$
$+c_+(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_-(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})$
$+c_-(\vec{p},h)u_{[\eta_\varsigma}(\vec{p},-\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{[\eta_\varsigma}(\vec{p},\frac{1}{2})v_{\tau_\varsigma]}(\vec{p},-\frac{1}{2})=0,-(s-1)\le h\le s$
$\Leftrightarrow \begin{cases}\frac{\sqrt{s+h}}{\sqrt{2s}}d_-(\vec{p},h)=\frac{\sqrt{s+1-h}}{\sqrt{2s}}d_+(\vec{p},h-1),-(s-1)\le h\le s\\ c_+(\vec{p},h)=0,c_-(\vec{p},h)=0,-s\le h\le s\end{cases}$
$\Leftrightarrow \begin{cases}b^+_{\eta_\varsigma}(\vec{p},h)=d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\le h\le s\\ b^+_{\eta_\varsigma}(\vec{p},-s)=d_+(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),h=-s\end{cases}$ □

1.3.1 Two important theorems

Thm. 1.3.1.
$\begin{cases}a_{\eta_\varsigma}(\vec{p},h)=c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\le h\le s\\ a_{\eta_\varsigma}(\vec{p},-s)=c_+(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),h=-s\end{cases}$

$$\Leftrightarrow \begin{cases} \sum\limits_{h=s}^{-s} a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \sum\limits_{(h+1/2)=(s+1/2)}^{-(s+1/2)} a(\vec{p},h+\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\frac{1}{2}) \\ a(\vec{p},-s-\frac{1}{2}) := c_-(\vec{p},-s), a(\vec{p},h+\frac{1}{2}) := \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p},h), -s \le h \le s \end{cases}$$

Proof:

$$\begin{cases} a_{\eta_\varsigma}(\vec{p},h) = c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}), -(s-1) \le h \le s \\ a_{\eta_\varsigma}(\vec{p},-s) = c_+(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}), h = -s \end{cases}$$

$$\Leftrightarrow \sum_{h=s}^{-s} a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \sum_{h=s}^{-s+1}[c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)$$

$$+ \sum_{h=-s}[c_+(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)$$

$$= \sum_{h=s}^{-s+1} c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) + \sum_{h=s}^{-s+1} \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)$$

$$+ \sum_{h=-s}[c_+(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)$$

$$= [\sum_{h=s-1}^{-s} c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) + \sum_{h=s-1}^{-s} \frac{\sqrt{s-h}}{\sqrt{s+h+1}}c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h+1)]$$

$$+ [\sum_{h=s} c_+(\vec{p},s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},s) + \sum_{h=-s} c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)]$$

$$= \sum_{h=s-1}^{-s} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p},h)[\frac{\sqrt{s+h+1}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s+1}}c_+(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h+1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]$$

$$+ [\sum_{h=s} c_+(\vec{p},s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},s) + \sum_{h=-s} c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)]$$

$$= \sum_{h=s-1}^{-s} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p},h)$$

$$[\frac{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},(h+\frac{1}{2})-\frac{1}{2})u_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{(s+\frac{1}{2})-(h+\frac{1}{2})}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},(h+\frac{1}{2})+\frac{1}{2})u_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]$$

$$+ [c_+(\vec{p},s)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2}) + c_-(\vec{p},-s)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},-s-\frac{1}{2})]$$

$$= \sum_{(h+1/2)=(s-1/2)}^{-(s-1/2)} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\frac{1}{2}) + c_+(\vec{p},s)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},s+\frac{1}{2})$$

$$+ c_-(\vec{p},-s)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},-s-\frac{1}{2})$$

$$= \sum_{(h+1/2)=(s+1/2)}^{-(s-1/2)} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\frac{1}{2}) + c_-(\vec{p},-s)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},-s-\frac{1}{2})$$

$$= \sum_{(h+1/2)=(s+1/2)}^{-(s+1/2)} a(\vec{p},h+\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\frac{1}{2})$$

$$, a(\vec{p},h+\frac{1}{2}) := \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p},h), -s \le h \le s; a(\vec{p},-s-\frac{1}{2}) := c_-(\vec{p},-s)$$ □

Thm. 1.3.2.

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p},h) = d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}), -(s-1) \le h \le s \\ b^+_{\eta_\varsigma}(\vec{p},-s) = d_+(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}), h = -s \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum\limits_{h=s}^{-s} b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \sum\limits_{(h+1/2)=(s+1/2)}^{-(s+1/2)} b^+(\vec{p},h+\frac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\frac{1}{2}) \\ b^+(\vec{p},-s-\frac{1}{2}) := d_-(\vec{p},-s), b^+(\vec{p},h+\frac{1}{2}) := \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p},h), -s \le h \le s \end{cases}$$

Proof:

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p},h) = d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}), -(s-1) \le h \le s \\ b^+_{\eta_\varsigma}(\vec{p},-s) = d_+(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}), h = -s \end{cases}$$

$$\Leftrightarrow \sum_{h=s}^{-s} b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h) = \sum_{h=s}^{-s+1}[d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2})]V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)$$

$$+\sum_{h=-s}[d_+(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})+d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})]V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)$$

$$=\sum_{h=s}^{-s+1}d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)+\sum_{h=s}^{-s+1}\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)$$

$$+\sum_{h=-s}[d_+(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})+d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})]V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)$$

$$=[\sum_{h=s-1}^{-s}d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)+\sum_{h=s-1}^{-s}\frac{\sqrt{s-h}}{\sqrt{s+h+1}}d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h+1)]$$

$$+[\sum_{h=s}d_+(\vec{p},s)v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},s)+\sum_{h=-s}d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)]$$

$$=\sum_{h=s-1}^{-s}\frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p},h)[\frac{\sqrt{s+h+1}}{\sqrt{2s+1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{s-h}}{\sqrt{2s+1}}d_+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h+1)v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+[\sum_{h=s}d_+(\vec{p},s)v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},s)+\sum_{h=-s}d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},-s)]$$

$$=\sum_{h=s-1}^{-s}\frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p},h)$$

$$[\frac{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}}{\sqrt{2s+1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},(h+\tfrac{1}{2})-\tfrac{1}{2})v_{\eta_\varsigma}(\vec{p},\tfrac{1}{2})+\frac{\sqrt{(s+\frac{1}{2})-(h+\frac{1}{2})}}{\sqrt{2s+1}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},(h+\tfrac{1}{2})+\tfrac{1}{2})v_{\eta_\varsigma}(\vec{p},-\tfrac{1}{2})]$$

$$+[d_+(\vec{p},s)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},s+\tfrac{1}{2})+d_-(\vec{p},-s)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},-s-\tfrac{1}{2})]$$

$$=\sum_{(h+1/2)=(s-1/2)}^{-(s-1/2)}\frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\tfrac{1}{2})+d_+(\vec{p},s)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},s+\tfrac{1}{2})$$

$$+d_-(\vec{p},-s)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},-s-\tfrac{1}{2})$$

$$=\sum_{(h+1/2)=(s+1/2)}^{-(s-1/2)}\frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\tfrac{1}{2})+d_-(\vec{p},-s)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},-s-\tfrac{1}{2})$$

$$=\sum_{(h+1/2)=(s+1/2)}^{-(s+1/2)}b^+(\vec{p},h+\tfrac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\tfrac{1}{2})$$

$$,b^+(\vec{p},h+\tfrac{1}{2}):=\frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p},h),-s\le h\le s;b^+(\vec{p},-s-\tfrac{1}{2}):=d_-(\vec{p},-s)$$ □

1.4 Use mathematical induction to strictly solve plane wave solutions of B-W equation

Thm. 1.4.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\}}_{2s}}(x)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

$$U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{(2s)!}\sqrt{C_{2s}^{s-h}}\,\underbrace{u_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})u_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot u_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})u_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}$$

$$V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{(2s)!}\sqrt{C_{2s}^{s-h}}\,\underbrace{v_{\{\lambda_\varsigma}(\vec{p},\tfrac{1}{2})v_{\mu_\varsigma}(\vec{p},\tfrac{1}{2})\cdot\cdot}_{s+h}\underbrace{\cdot\cdot v_{\sigma_\varsigma}(\vec{p},-\tfrac{1}{2})v_{\tau_\varsigma\}}(\vec{p},-\tfrac{1}{2})}_{s-h}$$

Proof: Use mathematical induction to prove this theorem.
Step 1: When $s'=1/2$, the following is established.
$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma}(x)=0,\psi_{\lambda_\varsigma}(x)=\psi_{\lambda_\varsigma}(x)$
$\Leftrightarrow$

$$\psi_{\lambda_\varsigma}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=1/2}^{-1/2}\frac{m^{1/2}}{\sqrt{E}}[a(\vec{p},h)U_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

Step 1: Assume when $s'=s$, the following is established.

$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\}}_{2s}}(x)$
$\Leftrightarrow$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

Step 3: When $s'=s+1/2$,

$$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2s+1)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma\}}_{2s+1}}(x)$$

$$\Leftrightarrow$$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma\tau_\varsigma}_{2s+1}}(x)\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\ \int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\ =\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a_{\tau_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+_{\tau_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\ \sum\limits_{h=s}^{-s}a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum\limits_{h=s}^{-s}a_{\tau_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)\\ \sum\limits_{h=s}^{-s}b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)=\sum\limits_{h=s}^{-s}b^+_{\tau_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p},h)\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}\frac{m^s}{\sqrt{E}}[a_{\eta_\varsigma}(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\ a_{\eta_\varsigma}(\vec{p},h)=c_+(\vec{p},h)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p},h-1)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\le h\le s\\ a_{\eta_\varsigma}(\vec{p},-s)=c_+(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},\frac{1}{2})+c_-(\vec{p},-s)u_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),h=-s\\ b^+_{\eta_\varsigma}(\vec{p},h)=d_+(\vec{p},h)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+\frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p},h-1)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),-(s-1)\le h\le s\\ b^+_{\eta_\varsigma}(\vec{p},-s)=d_+(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},\frac{1}{2})+d_-(\vec{p},-s)v_{\eta_\varsigma}(\vec{p},-\frac{1}{2}),h=-s\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{(h+1/2)=(s+1/2)}^{-(s+1/2)}\frac{m^s}{\sqrt{E}}\\ [\sqrt{m}a(\vec{p},h+\frac{1}{2})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\frac{1}{2})e^{ip\cdot x}+\sqrt{m}b^+(\vec{p},h+\frac{1}{2})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p},h+\frac{1}{2})e^{-ip\cdot x}]d^3\vec{p}\\ \sqrt{m}a(\vec{p},-s-\frac{1}{2}):=c_-(\vec{p},-s),\sqrt{m}a(\vec{p},h+\frac{1}{2}):=\frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}}c_+(\vec{p},h),-s\le h\le s\\ \sqrt{m}b^+(\vec{p},-s-\frac{1}{2}):=d_-(\vec{p},-s),\sqrt{m}b^+(\vec{p},h+\frac{1}{2}):=\frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}}d_+(\vec{p},h),-s\le h\le s\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s+1/2}^{-(s+1/2)}\frac{m^{s+1/2}}{\sqrt{E}}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma}_{2s+1}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\eta_\varsigma}_{2s+1}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\ a(\vec{p},-s-\frac{1}{2}):=\frac{c_-(\vec{p},-s)}{\sqrt{m}},a(\vec{p},h):=\frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+h}}\frac{c_+(\vec{p},h-\frac{1}{2})}{\sqrt{m}},-s+\frac{1}{2}\le h\le s+\frac{1}{2}\\ b^+(\vec{p},-s-\frac{1}{2}):=\frac{d_-(\vec{p},-s)}{\sqrt{m}},b^+(\vec{p},h):=\frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+h}}\frac{d_+(\vec{p},h-\frac{1}{2})}{\sqrt{m}},-s+\frac{1}{2}\le h\le s+\frac{1}{2}\end{cases}$$

This step proves that when $s'=s+1/2$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

2 Plane wave solutions of B-W equation in N+1 dimensional space-time

2.1 Properties of U-spin basis for B-W equation in N+1 dimensional space-time

2.1.1 U-spin basis lemma on symmetry conditions

Lem. 2.1.1. $\sum\limits_{n_1+\cdot\cdot+n_l}^{=2s}a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum\limits_{n_1+\cdot\cdot+n_l}^{=2s}a_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\ +\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

Proof: $\sum\limits_{n_1+\cdot\cdot+n_l}^{=2s} a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum\limits_{n_1+\cdot\cdot+n_l}^{=2s} a_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$

$$\Leftrightarrow \sum_{n_1+\cdot\cdot+n_l}^{=2s} a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)[\frac{\sqrt{n_1}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};1)$$

$$+\frac{\sqrt{n_2}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};2)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)U_{\tau_\varsigma}(\vec{p};l)]$$

$$=\sum_{n_1+\cdot\cdot+n_l}^{=2s} a_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)[\frac{\sqrt{n_1}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)U_{\eta_\varsigma}(\vec{p};1)$$

$$+\frac{\sqrt{n_2}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)U_{\eta_\varsigma}(\vec{p};2)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)U_{\eta_\varsigma}(\vec{p};l)]$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1+1,n_2-1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\cdots\cdots\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l-1)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$ □

Lem. 2.1.2.

$$\begin{cases}a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum\limits_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)+\sum\limits_{k=1}^{l}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

$\Leftrightarrow$

$$a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k),d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)=0$$

$$\begin{cases}a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)u_{\eta_\varsigma}(\vec{p};l),n_1\geq 1\end{cases}$$

$$\begin{cases}a_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l)=c(\vec{p};0,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)\\+\frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p};0,n_2,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};3)+\cdots\\+\frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)u_{\eta_\varsigma}(\vec{p};l),n_2\geq 1\end{cases}$$

$$\begin{cases}a_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l)=c(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2)\\+\frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};3)+\frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};4)+\cdots\\+\frac{\sqrt{n_l+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)u_{\eta_\varsigma}(\vec{p};l),n_3\geq 1\end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases}a_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l)=c(\vec{p};0,\cdot\cdot,0,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,\cdot\cdot,0,n_l;2)u_{\eta_\varsigma}(\vec{p};2)\\+\cdots+c(\vec{p};0,\cdot\cdot,0,n_l;l-1)u_{\eta_\varsigma}(\vec{p};l-1)+\frac{\sqrt{n_l}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l;l)u_{\eta_\varsigma}(\vec{p};l),n_l=2s\geq 1\end{cases}$$

Proof:

$$\begin{cases}a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum\limits_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)+\sum\limits_{k=1}^{l}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

$\Leftrightarrow$

$$a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k),d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)=0$$

$$\begin{cases}c(\vec{p};n_1+1,n_2,\cdot\cdot,n_l;2)=\frac{\sqrt{n_2+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2+1,\cdot\cdot,n_l;1)\cdots\\c(\vec{p};n_1+1,n_2,\cdot\cdot,n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_l+1;1)\end{cases}$$

$$\begin{cases} c(\vec{p};0,n_2+1,\cdot\cdot,n_l;3)=\frac{\sqrt{n_3+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3+1,\cdot\cdot,n_l;2)\cdots \\ c(\vec{p};0,n_2+1,\cdot\cdot,n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,\cdot\cdot,n_l+1;2) \end{cases}$$

$$\begin{cases} c(\vec{p};0,0,n_3+1,\cdot\cdot,n_l;4)=\frac{\sqrt{n_4+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,n_4+1,\cdot\cdot,n_l;3)\cdots \\ c(\vec{p};0,0,n_3+1,\cdot\cdot,n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l+1;3) \end{cases}$$

$\cdots$

$$\begin{cases} c(\vec{p};0,\cdot,0,n_{l-1}+1,n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}+1}}c(\vec{p};0,\cdot\cdot,0,n_{l-1},n_l+1;l-1) \end{cases}$$

$\Leftrightarrow$

$$a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k),d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)=0$$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};2) \\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)u_{\eta_\varsigma}(\vec{p};l),n_1\geq 1 \end{cases}$$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l)=c(\vec{p};0,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1) \\ +\frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p};0,n_2,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};3)+\cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)u_{\eta_\varsigma}(\vec{p};l),n_2\geq 1 \end{cases}$$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l)=c(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2) \\ +\frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};3)+\frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};4)+\cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)u_{\eta_\varsigma}(\vec{p};l),n_3\geq 1 \end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l)=c(\vec{p};0,\cdot\cdot,0,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,\cdot\cdot,0,n_l;2)u_{\eta_\varsigma}(\vec{p};2) \\ +\cdots+c(\vec{p};0,\cdot\cdot,0,n_l;l-1)u_{\eta_\varsigma}(\vec{p};l-1)+\frac{\sqrt{n_l}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l;l)u_{\eta_\varsigma}(\vec{p};l),n_l=2s\geq 1 \end{cases}$$

□

Cor. 2.1.1. $a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)$

$$\begin{cases} c(\vec{p};n_1,n_2,\cdot\cdot,n_l;2)=\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1),n_1\geq 1\cdots \\ c(\vec{p};n_1,n_2,\cdot\cdot,n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1),n_1\geq 1 \end{cases}$$

$$\begin{cases} c(\vec{p};0,n_2,\cdot\cdot,n_l;3)=\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2),n_2\geq 1\cdots \\ c(\vec{p};0,n_2,\cdot\cdot,n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2),n_2\geq 1 \end{cases}$$

$\cdots$

$$\begin{cases} c(\vec{p};0,\cdot\cdot,0,n_{l-1},n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}}c(\vec{p};0,\cdot\cdot,0,n_{l-1}-1,n_l+1;l),n_{l-1}\geq 1 \end{cases}$$

$\Leftrightarrow$

$$a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)$$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};2) \\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)u_{\eta_\varsigma}(\vec{p};l),n_1\geq 1 \end{cases}$$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l)=c(\vec{p};0,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1) \\ +\frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p};0,n_2,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};3)+\cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)u_{\eta_\varsigma}(\vec{p};l),n_2\geq 1 \end{cases}$$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l)=c(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2) \\ +\frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};3)+\frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};4)+\cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)u_{\eta_\varsigma}(\vec{p};l),n_3\geq 1 \end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases} a_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l)=c(\vec{p};0,\cdot\cdot,0,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,\cdot\cdot,0,n_l;2)u_{\eta_\varsigma}(\vec{p};2) \\ +\cdots+c(\vec{p};0,\cdot\cdot,0,n_l;l-1)u_{\eta_\varsigma}(\vec{p};l-1)+\frac{\sqrt{n_l}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l;l)u_{\eta_\varsigma}(\vec{p};l),n_l=2s\geq 1 \end{cases}$$

Lem. 2.1.3.

$$\begin{cases} c(\vec{p};n_1,n_2,\cdot\cdot,n_l;2)=\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1),n_1\geq 1\cdots \\ c(\vec{p};n_1,n_2,\cdot\cdot,n_l;l)=\frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1),n_1\geq 1 \end{cases}$$

$\Leftrightarrow$

$$\sum_{n_1\cdot\cdot+n_l=2s}\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$$

$$= \sum_{n_1 \cdot\cdot + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdot\cdot, n_l)$$
$$+ \sum_{n_2 \cdot\cdot + n_l = 2s} \sum_{k=2}^{l} c(\vec{p}; 0, n_2, \cdot\cdot, n_l; k) u_{\eta_\varsigma}(\vec{p}; k) U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdot\cdot, n_l)$$

Proof:

$$\begin{cases} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_l; 1), n_1 \geq 1 \cdots \\ c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l+1; 1), n_1 \geq 1 \end{cases}$$

$\Leftrightarrow$

$$\sum_{n_1 \cdot\cdot + n_l = 2s} \sum_{k=1}^{l} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; k) u_{\eta_\varsigma}(\vec{p}; k) U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_l)$$
$$= \sum_{n_1 + \cdot\cdot + n_l = 2s}^{n_1 \neq 0} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l) [\frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 1)$$
$$+ \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 2) + \cdot\cdot + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l+1; 1) u_{\eta_\varsigma}(\vec{p}; l)]$$
$$+ \sum_{n_1 + \cdot\cdot + n_l = 2s}^{n_1 = 0} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdot\cdot, n_l) [c(\vec{p}; 0, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 1)$$
$$+ c(\vec{p}; 0, n_2, \cdot\cdot, n_l; 2) u_{\eta_\varsigma}(\vec{p}; 2) + \cdot\cdot + c(\vec{p}; 0, n_2, \cdot\cdot, n_l; l) u_{\eta_\varsigma}(\vec{p}; l)]$$
$$= \sum_{n_1 \cdot\cdot + n_l = 2s}^{1 \leq n_1 \leq 2s} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l) c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 1)$$
$$+ \sum_{n_1 \cdot\cdot + n_l = 2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_2 \leq 2s} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1+1, n_2-1, \cdot\cdot, n_l) \frac{\sqrt{n_2}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 2) + \cdot\cdot$$
$$+ \sum_{n_1 \cdot\cdot + n_l = 2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_l \leq 2s} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1+1, n_2, \cdot\cdot, n_l-1) \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; l)$$
$$+ \sum_{n_1 \cdot\cdot + n_l = 2s}^{n_1 = 0} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdot\cdot, n_l) [c(\vec{p}; 0, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 1)$$
$$+ c(\vec{p}; 0, n_2, \cdot\cdot, n_l; 2) u_{\eta_\varsigma}(\vec{p}; 2) + \cdot\cdot + c(\vec{p}; 0, n_2, \cdot\cdot, n_l; l) u_{\eta_\varsigma}(\vec{p}; l)]$$
$$= \sum_{n_1 \cdot\cdot + n_l = 2s} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l) c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 1)$$
$$+ \sum_{n_1 \cdot\cdot + n_l = 2s} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1+1, n_2-1, \cdot\cdot, n_l) \frac{\sqrt{n_2}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; 2) + \cdot\cdot$$
$$+ \sum_{n_1 \cdot\cdot + n_l = 2s} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1+1, n_2, \cdot\cdot, n_l-1) \frac{\sqrt{n_l}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) u_{\eta_\varsigma}(\vec{p}; l)$$
$$+ \sum_{n_1 \cdot\cdot + n_l = 2s}^{n_1 = 0} U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdot\cdot, n_l) [c(\vec{p}; 0, n_2, \cdot\cdot, n_l; 2) u_{\eta_\varsigma}(\vec{p}; 2) + \cdot\cdot + c(\vec{p}; 0, n_2, \cdot\cdot, n_l; l) u_{\eta_\varsigma}(\vec{p}; l)]$$
$$= \sum_{n_1 \cdot\cdot + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdot\cdot, n_l)$$
$$+ \sum_{n_2 \cdot\cdot + n_l = 2s} \sum_{k=2}^{l} c(\vec{p}; 0, n_2, \cdot\cdot, n_l; k) u_{\eta_\varsigma}(\vec{p}; k) U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdot\cdot, n_l)$$ □

Cor. 2.1.2.

$$\begin{cases} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_l; 1), n_1 \geq 1 \cdots \\ c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l+1; 1), n_1 \geq 1 \end{cases}$$
$$\begin{cases} c(\vec{p}; 0, n_2, \cdot\cdot, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3+1, \cdot\cdot, n_l; 2), n_2 \geq 1 \cdots \\ c(\vec{p}; 0, n_2, \cdot\cdot, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \cdot\cdot, n_l+1; 2), n_2 \geq 1 \end{cases}$$
$\cdots$
$$\Big\{ c(\vec{p}; 0, \cdot\cdot, 0, n_{l-1}, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}} c(\vec{p}; 0, \cdot\cdot, 0, n_{l-1}-1, n_l+1; l), n_{l-1} \geq 1$$
$\Leftrightarrow$
$$\sum_{n_1 \cdot\cdot + n_l = 2s} \sum_{k=1}^{l} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; k) u_{\eta_\varsigma}(\vec{p}; k) U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_l)$$
$$= \sum_{n_1 + \cdot\cdot + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_l; 1) U_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdot \cdot \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdot\cdot, n_l)$$

$$+\sum_{n_2+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3,\cdot\cdot,n_l;2)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,n_2+1,n_3,\cdot\cdot,n_l)$$
$$+\sum_{n_3+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,\cdot\cdot,n_l;3)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,0,n_3+1,\cdot\cdot,n_l)$$
$$+\cdots+\sum_{n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_l+1}}c(\vec{p};0,\cdot\cdot,0,n_l;l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_l+1)$$

Lem. 2.1.4.

$$\sum_{n_1+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)$$
$$+\sum_{n_2+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3,\cdot\cdot,n_l;2)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,n_2+1,n_3,\cdot\cdot,n_l)$$
$$+\sum_{n_3+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,\cdot\cdot,n_l;3)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,0,n_3+1,\cdot\cdot,n_l)$$
$$+\cdots+\sum_{n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_l+1}}c(\vec{p};0,\cdot\cdot,0,n_l;l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_l+1)$$
$$=\sum_{n_1+\cdot\cdot+n_l=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$$
$$\begin{cases}a(\vec{p};n_1,n_2,\cdot\cdot,n_l):=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l;1),n_1\neq0\\ a(\vec{p};0,n_2,\cdot\cdot,n_l):=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l;2),n_2\neq0\\ a(\vec{p};0,0,n_3,\cdot\cdot,n_l):=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l;3),n_3\neq0\\ \cdots\\ a(\vec{p};0,0,\cdot\cdot,0,n_l):=\frac{\sqrt{2s+1}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l-1;l),n_l\neq0\end{cases}$$

Proof: $\sum_{n_1\cdot\cdot+n_l=2s}\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$

$$=\sum_{n_1+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)$$
$$+\sum_{n_2+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3,\cdot\cdot,n_l;2)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,n_2+1,n_3,\cdot\cdot,n_l)$$
$$+\sum_{n_3+\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,\cdot\cdot,n_l;3)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,0,n_3+1,\cdot\cdot,n_l)$$
$$+\cdots+\sum_{n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_l+1}}c(\vec{p};0,\cdot\cdot,0,n_l;l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_l+1)$$
$$=\sum_{n_1+\cdot\cdot+n_l=2s+1}^{n_1\neq0}\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l;1)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$$
$$+\sum_{n_2+\cdot\cdot+n_l=2s+1}^{n_1=0,n_2\neq0}\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l;2)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,n_2,n_3,\cdot\cdot,n_l)$$
$$+\sum_{n_3+\cdot\cdot+n_l=2s+1}^{n_1=0,n_2=0,n_3\neq0}\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l;3)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,0,n_3,\cdot\cdot,n_l)$$
$$+\cdots+\sum_{n_l=2s+1}^{n_1=0,\cdot\cdot,n_{l-1}=0,n_l\neq0}\frac{\sqrt{2s+1}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l-1;l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_l)$$
$$=\sum_{n_1+\cdot\cdot+n_l=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$$
$$\begin{cases}a(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l;1),n_1\neq0\\ a(\vec{p};0,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l;2),n_2\neq0\\ a(\vec{p};0,0,n_3,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l;3),n_3\neq0\\ \cdots\\ a(\vec{p};0,0,\cdot\cdot,0,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l-1;l),n_l\neq0\end{cases}$$
□

2.1.2 Several corollaries

Cor. 2.1.3. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};1)$
$+\frac{\sqrt{n_2}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};2)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)U_{\tau_\varsigma}(\vec{p};l)$

Cor. 2.1.4. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_2}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};0,n_2-1,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};2)$
$+\frac{\sqrt{n_3}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};0,n_2,n_3-1,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};0,0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};0,n_2,\cdot\cdot,n_l-1)U_{\tau_\varsigma}(\vec{p};l)$

Cor. 2.1.5. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)U_{\eta_\varsigma}(\vec{p};1)$
$+\frac{\sqrt{n_2}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)U_{\eta_\varsigma}(\vec{p};2)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s+1}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)U_{\eta_\varsigma}(\vec{p};l)$

2.1.3 An important theorem

Thm. 2.1.1.

$$\sum_{n_1+\cdot\cdot+n_l=2s}a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum_{n_1+\cdot\cdot+n_l=2s}a_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$$
$\Leftrightarrow$
$$a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)$$
$$\sum_{n_1\cdot\cdot+n_l}^{=2s}a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum_{n_1+\cdot\cdot+n_l}^{=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$$
$$\begin{cases}a(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l;1),n_1\neq0\\a(\vec{p};0,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l;2),n_2\neq0\\a(\vec{p};0,0,n_3,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l;3),n_3\neq0\\\cdots\\a(\vec{p};0,0,\cdot\cdot,0,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l-1;l),n_l\neq0\end{cases}$$

Proof:

$$\sum_{n_1+\cdot\cdot+n_l=2s}a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum_{n_1+\cdot\cdot+n_l=2s}a_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$$
$\Leftrightarrow$
$$\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)u_{\tau_\varsigma]}(\vec{p};2)\\+\cdot\cdot+\frac{\sqrt{n_l+1}}{\sqrt{2s}}a_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)u_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$
$\Leftrightarrow$
$$a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)$$
$$\begin{cases}a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)u_{\eta_\varsigma}(\vec{p};l),n_1\geq1\end{cases}$$
$$\begin{cases}a_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l)=c(\vec{p};0,n_2,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)\\+\frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p};0,n_2,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};3)+\cdots\\+\frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)u_{\eta_\varsigma}(\vec{p};l),n_2\geq1\end{cases}$$
$$\begin{cases}a_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l)=c(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)u_{\eta_\varsigma}(\vec{p};2)\\+\frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};3)+\frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)u_{\eta_\varsigma}(\vec{p};4)+\cdots\\+\frac{\sqrt{n_l+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)u_{\eta_\varsigma}(\vec{p};l),n_3\geq1\end{cases}$$
$\cdots\cdots\cdots$
$$\begin{cases}a_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l)=c(\vec{p};0,\cdot\cdot,0,n_l;1)u_{\eta_\varsigma}(\vec{p};1)+c(\vec{p};0,\cdot\cdot,0,n_l;2)u_{\eta_\varsigma}(\vec{p};2)\\+\cdots+c(\vec{p};0,\cdot\cdot,0,n_l;l-1)u_{\eta_\varsigma}(\vec{p};l-1)+\frac{\sqrt{n_l}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l;l)u_{\eta_\varsigma}(\vec{p};l),n_l=2s\geq1\end{cases}$$

$\Leftrightarrow$
$$a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)$$
$$\sum_{n_1\cdot\cdot+n_l}^{=2s}a_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum_{n_1+\cdot\cdot+n_l}^{=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$$

$$\begin{cases} a(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_l;1),n_1\neq 0\\ a(\vec{p};0,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l;2),n_2\neq 0\\ a(\vec{p};0,0,n_3,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_l;3),n_3\neq 0\\ \cdots\\ a(\vec{p};0,0,\cdot\cdot,0,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_l}}c(\vec{p};0,\cdot\cdot,0,n_l-1;l),n_l\neq 0 \end{cases}$$ □

2.2 Properties of V-spin basis for B-W equation in N+1 dimensional space-time

2.2.1 V-spin basis lemma on symmetry conditions

Lem. 2.2.1. $\sum_{n_1+\cdot\cdot+n_l}^{=2s} b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum_{n_1+\cdot\cdot+n_l}^{=2s} b^+_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)v_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

Proof: $\sum_{n_1+\cdot\cdot+n_l}^{=2s} b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum_{n_1+\cdot\cdot+n_l}^{=2s} b^+_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$

$$\Leftrightarrow\sum_{n_1+\cdot\cdot+n_l}^{=2s} b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)[\frac{\sqrt{n_1}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)V_{\tau_\varsigma}(\vec{p};1)$$

$$+\frac{\sqrt{n_2}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)V_{\tau_\varsigma}(\vec{p};2)+\cdots+\frac{\sqrt{n_l}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)V_{\tau_\varsigma}(\vec{p};l)]$$

$$=\sum_{n_1+\cdot\cdot+n_l}^{=2s} b^+_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)[\frac{\sqrt{n_1}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)V_{\eta_\varsigma}(\vec{p};1)$$

$$+\frac{\sqrt{n_2}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)V_{\eta_\varsigma}(\vec{p};2)+\cdots+\frac{\sqrt{n_l}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)V_{\eta_\varsigma}(\vec{p};l)]$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1)v_{\tau_\varsigma]}(\vec{p};l)=0\\ \frac{\sqrt{n_1+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1+1,n_2-1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l+1)v_{\tau_\varsigma]}(\vec{p};l)=0\cdots\cdots\\ \frac{\sqrt{n_1+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l-1)v_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1)v_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)v_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$ □

Lem. 2.2.2.

$$\begin{cases}b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)+\sum_{k=1}^{l}c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k)\\ \frac{\sqrt{n_1+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)v_{\tau_\varsigma]}(\vec{p};l)=0\end{cases}$$
$\Leftrightarrow$

$$b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum_{k=1}^{l}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k),c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)=0$$

$$\begin{cases}b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{n_1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};2)\\ +\cdots+\frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)v_{\eta_\varsigma}(\vec{p};l),n_1\geq 1\end{cases}$$

$$\begin{cases}b^+_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l)=d(\vec{p};0,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)\\ +\frac{\sqrt{n_2}}{\sqrt{n_2}}d(\vec{p};0,n_2,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};3)+\cdots\\ +\frac{\sqrt{n_l+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)v_{\eta_\varsigma}(\vec{p};l),n_2\geq 1\end{cases}$$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l) = d(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + d(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) \\ +\frac{\sqrt{n_3}}{\sqrt{n_3}}d(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};4) + \cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)v_{\eta_\varsigma}(\vec{p};l), n_3 \geq 1 \end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l) = d(\vec{p};0,\cdot\cdot,0,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + d(\vec{p};0,\cdot\cdot,0,n_l;2)v_{\eta_\varsigma}(\vec{p};2) \\ +\cdots + d(\vec{p};0,\cdot\cdot,0,n_l;l-1)v_{\eta_\varsigma}(\vec{p};l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}}d(\vec{p};0,\cdot\cdot,0,n_l;l)v_{\eta_\varsigma}(\vec{p};l), n_l = 2s \geq 1 \end{cases}$$

Proof:

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l) = \sum_{k=1}^{l} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k) + \sum_{k=1}^{l} c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)u_{\eta_\varsigma}(\vec{p};k) \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_l)v_{\tau_\varsigma]}(\vec{p};2) \\ +\cdots + \frac{\sqrt{n_l+1}}{\sqrt{2s}}b^+_{[\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l+1)v_{\tau_\varsigma]}(\vec{p};l) = 0 \end{cases}$$

$\Leftrightarrow$

$$b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l) = \sum_{k=1}^{l} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k), c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k) = 0$$

$$\begin{cases} d(\vec{p};n_1+1,n_2,\cdot\cdot,n_l;2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1+1}}d(\vec{p};n_1,n_2+1,\cdot\cdot,n_l;1)\cdots \\ d(\vec{p};n_1+1,n_2,\cdot\cdot,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1+1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l+1;1) \end{cases}$$

$$\begin{cases} d(\vec{p};0,n_2+1,\cdot\cdot,n_l;3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2+1}}d(\vec{p};0,n_2,n_3+1,\cdot\cdot,n_l;2)\cdots \\ d(\vec{p};0,n_2+1,\cdot\cdot,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2+1}}d(\vec{p};0,n_2,\cdot\cdot,n_l+1;2) \end{cases}$$

$$\begin{cases} d(\vec{p};0,0,n_3+1,\cdot\cdot,n_l;4) = \frac{\sqrt{n_4+1}}{\sqrt{n_3+1}}d(\vec{p};0,0,n_3,n_4+1,\cdot\cdot,n_l;3)\cdots \\ d(\vec{p};0,0,n_3+1,\cdot\cdot,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_3+1}}d(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l+1;3) \end{cases}$$

$\cdots$

$$\begin{cases} d(\vec{p};0,\cdot,0,n_{l-1}+1,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}+1}}d(\vec{p};0,\cdot\cdot,0,n_{l-1},n_l+1;l-1) \end{cases}$$

$\Leftrightarrow$

$$b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l) = \sum_{k=1}^{l} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k), c(\vec{p};n_1,n_2,\cdot\cdot,n_l;k) = 0$$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};2) \\ +\cdots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)v_{\eta_\varsigma}(\vec{p};l), n_1 \geq 1 \end{cases}$$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l) = d(\vec{p};0,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1) \\ +\frac{\sqrt{n_2}}{\sqrt{n_2}}d(\vec{p};0,n_2,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};3) + \cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)v_{\eta_\varsigma}(\vec{p};l), n_2 \geq 1 \end{cases}$$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l) = d(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + d(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) \\ +\frac{\sqrt{n_3}}{\sqrt{n_3}}d(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};4) + \cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)v_{\eta_\varsigma}(\vec{p};l), n_3 \geq 1 \end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l) = d(\vec{p};0,\cdot\cdot,0,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + d(\vec{p};0,\cdot\cdot,0,n_l;2)v_{\eta_\varsigma}(\vec{p};2) \\ +\cdots + d(\vec{p};0,\cdot\cdot,0,n_l;l-1)v_{\eta_\varsigma}(\vec{p};l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}}d(\vec{p};0,\cdot\cdot,0,n_l;l)v_{\eta_\varsigma}(\vec{p};l), n_l = 2s \geq 1 \end{cases}$$

□

Cor. 2.2.1. $b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l) = \sum_{k=1}^{l} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)$

$$\begin{cases} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1), n_1 \geq 1\cdots \\ d(\vec{p};n_1,n_2,\cdot\cdot,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1), n_1 \geq 1 \end{cases}$$

$$\begin{cases} d(\vec{p};0,n_2,\cdot\cdot,n_l;3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2), n_2 \geq 1\cdots \\ d(\vec{p};0,n_2,\cdot\cdot,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2), n_2 \geq 1 \end{cases}$$

$\cdots$

$$\begin{cases} d(\vec{p};0,\cdot\cdot,0,n_{l-1},n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}}d(\vec{p};0,\cdot\cdot,0,n_{l-1}-1,n_l+1;l), n_{l-1} \geq 1 \end{cases}$$

$\Leftrightarrow$

$$b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l) = \sum_{k=1}^{l} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)$$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};2) \\ +\cdots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)v_{\eta_\varsigma}(\vec{p};l), n_1 \geq 1 \end{cases}$$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l) = d(\vec{p};0,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1) \\ +\frac{\sqrt{n_2}}{\sqrt{n_2}}d(\vec{p};0,n_2,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};3) + \cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)v_{\eta_\varsigma}(\vec{p};l), n_2 \geq 1 \end{cases}$$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l) = d(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + d(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) \\ +\frac{\sqrt{n_3}}{\sqrt{n_3}}d(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};4) + \cdots \\ +\frac{\sqrt{n_l+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)v_{\eta_\varsigma}(\vec{p};l), n_3 \geq 1 \end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases} b^+_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l) = d(\vec{p};0,\cdot\cdot,0,n_l;1)v_{\eta_\varsigma}(\vec{p};1) + d(\vec{p};0,\cdot\cdot,0,n_l;2)v_{\eta_\varsigma}(\vec{p};2) \\ +\cdots + d(\vec{p};0,\cdot\cdot,0,n_l;l-1)v_{\eta_\varsigma}(\vec{p};l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}}d(\vec{p};0,\cdot\cdot,0,n_l;l)v_{\eta_\varsigma}(\vec{p};l), n_l = 2s \geq 1 \end{cases}$$

Lem. 2.2.3.

$$\begin{cases} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1), n_1 \geq 1 \cdots \\ d(\vec{p};n_1,n_2,\cdot\cdot,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1), n_1 \geq 1 \end{cases}$$

$\Leftrightarrow$

$$\sum_{n_1\cdot\cdot+n_l=2s}\sum_{k=1}^{l} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$$
$$= \sum_{n_1\cdot\cdot+n_l=2s}\frac{\sqrt{2s+1}}{\sqrt{n_1+1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l)$$
$$+ \sum_{n_2\cdot\cdot+n_l=2s}\sum_{k=2}^{l} d(\vec{p};0,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_l)$$

Proof:

$$\begin{cases} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1), n_1 \geq 1 \cdots \\ d(\vec{p};n_1,n_2,\cdot\cdot,n_l;l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1), n_1 \geq 1 \end{cases}$$

$\Leftrightarrow$

$$\sum_{n_1\cdot\cdot+n_l=2s}\sum_{k=1}^{l} d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)$$
$$= \sum_{n_1+\cdot\cdot+n_l=2s}^{n_1\neq 0} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)[\frac{\sqrt{n_1}}{\sqrt{n_1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)$$
$$+ \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};2) + \cdot\cdot + \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)v_{\eta_\varsigma}(\vec{p};l)]$$
$$+ \sum_{n_1+\cdot\cdot+n_l=2s}^{n_1=0} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_l)[d(\vec{p};0,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)$$
$$+ d(\vec{p};0,n_2,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) + \cdot\cdot + d(\vec{p};0,n_2,\cdot\cdot,n_l;l)v_{\eta_\varsigma}(\vec{p};l)]$$
$$= \sum_{n_1\cdot\cdot+n_l=2s}^{1\leq n_1\leq 2s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)$$
$$+ \sum_{n_1\cdot\cdot+n_l=2s}^{0\leq n_1\leq 2s-1, 1\leq n_2\leq 2s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1+1,n_2-1,\cdot\cdot,n_l)\frac{\sqrt{n_2}}{\sqrt{n_1+1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};2) + \cdot\cdot$$
$$+ \sum_{n_1\cdot\cdot+n_l=2s}^{0\leq n_1\leq 2s-1, 1\leq n_l\leq 2s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l-1)\frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};l)$$
$$+ \sum_{n_1\cdot\cdot+n_l=2s}^{n_1=0} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_l)[d(\vec{p};0,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)$$
$$+ d(\vec{p};0,n_2,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) + \cdot\cdot + d(\vec{p};0,n_2,\cdot\cdot,n_l;l)v_{\eta_\varsigma}(\vec{p};l)]$$
$$= \sum_{n_1\cdot\cdot+n_l=2s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)$$
$$+ \sum_{n_1\cdot\cdot+n_l=2s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1+1,n_2-1,\cdot\cdot,n_l)\frac{\sqrt{n_2}}{\sqrt{n_1+1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};2) + \cdot\cdot$$
$$+ \sum_{n_1\cdot\cdot+n_l=2s} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_l-1)\frac{\sqrt{n_l}}{\sqrt{n_1+1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};l)$$
$$+ \sum_{n_1\cdot\cdot+n_l=2s}^{n_1=0} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_l)[d(\vec{p};0,n_2,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2) + \cdot\cdot + d(\vec{p};0,n_2,\cdot\cdot,n_l;l)v_{\eta_\varsigma}(\vec{p};l)]$$

$$= \sum_{n_1 \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \cdots, n_l; 1) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdots, n_l)$$

$$+ \sum_{n_2 \cdots + n_l = 2s} \sum_{k=2}^{l} d(\vec{p}; 0, n_2, \cdots, n_l; k) v_{\eta_\varsigma}(\vec{p}; k) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdots, n_l) \qquad \square$$

Cor. 2.2.2.

$$\begin{cases} d(\vec{p}; n_1, n_2, \cdots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2+1, \cdots, n_l; 1), n_1 \geq 1 \cdots \\ d(\vec{p}; n_1, n_2, \cdots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2, \cdots, n_l+1; 1), n_1 \geq 1 \end{cases}$$

$$\begin{cases} d(\vec{p}; 0, n_2, \cdots, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3+1, \cdots, n_l; 2), n_2 \geq 1 \cdots \\ d(\vec{p}; 0, n_2, \cdots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3, \cdots, n_l+1; 2), n_2 \geq 1 \end{cases}$$

$$\cdots$$

$$\begin{cases} d(\vec{p}; 0, \cdots, 0, n_{l-1}, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}} d(\vec{p}; 0, \cdots, 0, n_{l-1}-1, n_l+1; l), n_{l-1} \geq 1 \end{cases}$$

$$\Leftrightarrow$$

$$\sum_{n_1 \cdots + n_l = 2s} \sum_{k=1}^{l} d(\vec{p}; n_1, n_2, \cdots, n_l; k) v_{\eta_\varsigma}(\vec{p}; k) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1, \cdots, n_l)$$

$$= \sum_{n_1 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \cdots, n_l; 1) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdots, n_l)$$

$$+ \sum_{n_2 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, n_3, \cdots, n_l; 2) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, n_2+1, n_3, \cdots, n_l)$$

$$+ \sum_{n_3 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, \cdots, n_l; 3) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, 0, n_3+1, \cdots, n_l)$$

$$+ \cdots + \sum_{n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} d(\vec{p}; 0, \cdots, 0, n_l; l) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, \cdots, 0, n_l+1)$$

Lem. 2.2.4.

$$\sum_{n_1 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \cdots, n_l; 1) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdots, n_l)$$

$$+ \sum_{n_2 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, n_3, \cdots, n_l; 2) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, n_2+1, n_3, \cdots, n_l)$$

$$+ \sum_{n_3 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, \cdots, n_l; 3) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, 0, n_3+1, \cdots, n_l)$$

$$+ \cdots + \sum_{n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} d(\vec{p}; 0, \cdots, 0, n_l; l) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, \cdots, 0, n_l+1)$$

$$= \sum_{n_1 + \cdots + n_l = 2s+1} b^+(\vec{p}; n_1, n_2, \cdots, n_l) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1, n_2, \cdots, n_l)$$

$$\begin{cases} b^+(\vec{p}; n_1, n_2, \cdots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2, \cdots, n_l; 1), n_1 \neq 0 \\ b^+(\vec{p}; 0, n_2, \cdots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3, \cdots, n_l; 2), n_2 \neq 0 \\ b^+(\vec{p}; 0, 0, n_3, \cdots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3-1, \cdots, n_l; 3), n_3 \neq 0 \\ \cdots \\ b^+(\vec{p}; 0, 0, \cdots, 0, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \cdots, 0, n_l-1; l), n_l \neq 0 \end{cases}$$

Proof: $\sum_{n_1 \cdots + n_l = 2s} \sum_{k=1}^{l} d(\vec{p}; n_1, n_2, \cdots, n_l; k) v_{\eta_\varsigma}(\vec{p}; k) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma}_{2s}}(\vec{p}; n_1, \cdots, n_l)$

$$= \sum_{n_1 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \cdots, n_l; 1) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdots, n_l)$$

$$+ \sum_{n_2 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, n_3, \cdots, n_l; 2) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, n_2+1, n_3, \cdots, n_l)$$

$$+ \sum_{n_3 + \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, \cdots, n_l; 3) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, 0, n_3+1, \cdots, n_l)$$

$$+ \cdots + \sum_{n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} d(\vec{p}; 0, \cdots, 0, n_l; l) V_{\underbrace{\lambda_\varsigma \mu_\varsigma \cdots \sigma_\varsigma \tau_\varsigma \eta_\varsigma}_{2s+1}}(\vec{p}; 0, \cdots, 0, n_l+1)$$

$$= \sum_{n_1+\cdot\cdot+n_l=2s+1}^{n_1\neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l; 1) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l)$$
$$+ \sum_{n_2+\cdot\cdot+n_l=2s+1}^{n_1=0, n_2\neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3, \cdot\cdot, n_l; 2) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p}; 0, n_2, n_3, \cdot\cdot, n_l)$$
$$+ \sum_{n_3+\cdot\cdot+n_l=2s+1}^{n_1=0, n_2=0, n_3\neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3-1, \cdot\cdot, n_l; 3) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p}; 0, 0, n_3, \cdot\cdot, n_l)$$
$$+ \cdots + \sum_{n_l=2s+1}^{n_1=0, \cdot\cdot, n_{l-1}=0, n_l\neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \cdot\cdot, 0, n_l-1; l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p}; 0, \cdot\cdot, 0, n_l)$$
$$= \sum_{n_1+\cdot\cdot+n_l=2s+1} b^+(\vec{p}; n_1, n_2, \cdot\cdot, n_l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l)$$

$$\begin{cases} b^+(\vec{p}; n_1, n_2, \cdot\cdot, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l; 1), n_1 \neq 0 \\ b^+(\vec{p}; 0, n_2, \cdot\cdot, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3, \cdot\cdot, n_l; 2), n_2 \neq 0 \\ b^+(\vec{p}; 0, 0, n_3, \cdot\cdot, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3-1, \cdot\cdot, n_l; 3), n_3 \neq 0 \\ \cdots \\ b^+(\vec{p}; 0, 0, \cdot\cdot, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \cdot\cdot, 0, n_l-1; l), n_l \neq 0 \end{cases} \qquad \square$$

2.2.2 Several corollaries

Cor. 2.2.3. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l) = \frac{\sqrt{n_1}}{\sqrt{2s}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l) v_{\tau_\varsigma}(\vec{p}; 1)$
$+ \frac{\sqrt{n_2}}{\sqrt{2s}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p}; n_1, n_2-1, \cdot\cdot, n_l) v_{\tau_\varsigma}(\vec{p}; 2) + \cdots + \frac{\sqrt{n_l}}{\sqrt{2s}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l-1) v_{\tau_\varsigma}(\vec{p}; l)$

Cor. 2.2.4. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdot\cdot, n_l) = \frac{\sqrt{n_2}}{\sqrt{2s}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p}; 0, n_2-1, \cdot\cdot, n_l) v_{\tau_\varsigma}(\vec{p}; 2)$
$+ \frac{\sqrt{n_3}}{\sqrt{2s}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p}; 0, n_2, n_3-1, \cdot\cdot, n_l) v_{\tau_\varsigma}(\vec{p}; 0, 0, 1, \cdot\cdot, 0) + \cdots + \frac{\sqrt{n_l}}{\sqrt{2s}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p}; 0, n_2, \cdot\cdot, n_l-1) v_{\tau_\varsigma}(\vec{p}; l)$

Cor. 2.2.5. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l) = \frac{\sqrt{n_1}}{\sqrt{2s+1}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l) v_{\eta_\varsigma}(\vec{p}; 1)$
$+ \frac{\sqrt{n_2}}{\sqrt{2s+1}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; n_1, n_2-1, \cdot\cdot, n_l) v_{\eta_\varsigma}(\vec{p}; 2) + \cdots + \frac{\sqrt{n_l}}{\sqrt{2s+1}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l-1) v_{\eta_\varsigma}(\vec{p}; l)$

2.2.3 An important theorem

Thm. 2.2.1.

$$\sum_{n_1+\cdot\cdot+n_l=2s} b^+_{\eta_\varsigma}(\vec{p}; n_1, \cdot\cdot, n_l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_l) = \sum_{n_1+\cdot\cdot+n_l=2s} b^+_{\tau_\varsigma}(\vec{p}; n_1, \cdot\cdot, n_l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_l)$$
$\Leftrightarrow$
$$b^+_{\eta_\varsigma}(\vec{p}; n_1, n_2, \cdot\cdot, n_l) = \sum_{k=1}^{l} d(\vec{p}; n_1, n_2, \cdot\cdot, n_l; k) v_{\eta_\varsigma}(\vec{p}; k)$$
$$\sum_{n_1\cdot\cdot+n_l}^{=2s} b^+_{\eta_\varsigma}(\vec{p}; n_1, n_2, \cdot\cdot, n_l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_l) = \sum_{n_1+\cdot\cdot+n_l}^{=2s+1} b^+(\vec{p}; n_1, n_2, \cdot\cdot, n_l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p}; n_1, n_2, \cdot\cdot, n_l)$$
$$\begin{cases} b^+(\vec{p}; n_1, n_2, \cdot\cdot, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2, \cdot\cdot, n_l; 1), n_1 \neq 0 \\ b^+(\vec{p}; 0, n_2, \cdot\cdot, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3, \cdot\cdot, n_l; 2), n_2 \neq 0 \\ b^+(\vec{p}; 0, 0, n_3, \cdot\cdot, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3-1, \cdot\cdot, n_l; 3), n_3 \neq 0 \\ \cdots \\ b^+(\vec{p}; 0, 0, \cdot\cdot, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \cdot\cdot, 0, n_l-1; l), n_l \neq 0 \end{cases}$$

Proof:

$$\sum_{n_1+\cdot\cdot+n_l=2s} b^+_{\eta_\varsigma}(\vec{p}; n_1, \cdot\cdot, n_l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_l) = \sum_{n_1+\cdot\cdot+n_l=2s} b^+_{\tau_\varsigma}(\vec{p}; n_1, \cdot\cdot, n_l) V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_l)$$
$\Leftrightarrow$
$$\begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}} b^+_{[\eta_\varsigma}(\vec{p}; n_1+1, n_2, \cdot\cdot, n_l) v_{\tau_\varsigma]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b^+_{[\eta_\varsigma}(\vec{p}; n_1, n_2+1, \cdot\cdot, n_l) v_{\tau_\varsigma]}(\vec{p}; 2) \\ + \cdots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b^+_{[\eta_\varsigma}(\vec{p}; n_1, n_2, \cdot\cdot, n_l+1) v_{\tau_\varsigma]}(\vec{p}; l) = 0 \end{cases}$$
$\Leftrightarrow$

$b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum\limits_{k=1}^{l}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)$

$\begin{cases}b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{n_1}}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l+1;1)v_{\eta_\varsigma}(\vec{p};l),n_1\geq 1\end{cases}$

$\begin{cases}b^+_{\eta_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_l)=d(\vec{p};0,n_2,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)\\+\frac{\sqrt{n_2}}{\sqrt{n_2}}d(\vec{p};0,n_2,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};3)+\cdots\\+\frac{\sqrt{n_l+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l+1;2)v_{\eta_\varsigma}(\vec{p};l),n_2\geq 1\end{cases}$

$\begin{cases}b^+_{\eta_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_l)=d(\vec{p};0,0,n_3,\cdot\cdot,n_l;1)v_{\eta_\varsigma}(\vec{p};1)+d(\vec{p};0,0,n_3,\cdot\cdot,n_l;2)v_{\eta_\varsigma}(\vec{p};2)\\+\frac{\sqrt{n_3}}{\sqrt{n_3}}d(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};3)+\frac{\sqrt{n_4+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_l;3)v_{\eta_\varsigma}(\vec{p};4)+\cdots\\+\frac{\sqrt{n_l+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,\cdot\cdot,n_l+1;3)v_{\eta_\varsigma}(\vec{p};l),n_3\geq 1\end{cases}$

$\cdots\cdots\cdots$

$\begin{cases}b^+_{\eta_\varsigma}(\vec{p};0,\cdot\cdot,0,n_l)=d(\vec{p};0,\cdot\cdot,0,n_l;1)v_{\eta_\varsigma}(\vec{p};1)+d(\vec{p};0,\cdot\cdot,0,n_l;2)v_{\eta_\varsigma}(\vec{p};2)\\+\cdots+d(\vec{p};0,\cdot\cdot,0,n_l;l-1)v_{\eta_\varsigma}(\vec{p};l-1)+\frac{\sqrt{n_l}}{\sqrt{n_l}}d(\vec{p};0,\cdot\cdot,0,n_l;l)v_{\eta_\varsigma}(\vec{p};l),n_l=2s\geq 1\end{cases}$

$\Leftrightarrow$

$b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\sum\limits_{k=1}^{l}d(\vec{p};n_1,n_2,\cdot\cdot,n_l;k)v_{\eta_\varsigma}(\vec{p};k)$

$\sum\limits_{n_1\cdot\cdot+n_l}^{=2s}b^+_{\eta_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum\limits_{n_1+\cdot\cdot+n_l}^{=2s+1}b^+(\vec{p};n_1,n_2,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$

$\begin{cases}b^+(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_1}}d(\vec{p};n_1-1,n_2,\cdot\cdot,n_l;1),n_1\neq 0\\b^+(\vec{p};0,n_2,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_2}}d(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_l;2),n_2\neq 0\\b^+(\vec{p};0,0,n_3,\cdot\cdot,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_3}}d(\vec{p};0,0,n_3-1,\cdot\cdot,n_l;3),n_3\neq 0\\\cdots\\b^+(\vec{p};0,0,\cdot\cdot,0,n_l)=\frac{\sqrt{2s+1}}{\sqrt{n_l}}d(\vec{p};0,\cdot\cdot,0,n_l-1;l),n_l\neq 0\end{cases}$ □

2.3 Use mathematical induction to solve plane wave solutions of B-W equation in N+1-D

Thm. 2.3.1.

$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\}}_{2s}}(x)$

$\Leftrightarrow$

$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdot\cdot+n_l}^{=2s}\frac{m^s}{\sqrt{E}}$

$[a(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}$

Proof: Use mathematical induction to prove this theorem.
Step 1: When $s'=1/2$, the following is established.

$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma}(x)=0,\psi_{\lambda_\varsigma}(x)=\psi_{\lambda_\varsigma}(x)$

$\Leftrightarrow$

$\psi_{\lambda_\varsigma}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdot\cdot+n_l}^{=1}\frac{m^{1/2}}{\sqrt{E}}[a(\vec{p};n_1,\cdot\cdot,n_l)U_{\lambda_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+(\vec{p},h)V_{\lambda_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}$

Step 1: Assume when $s'=s$, the following is established.

$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\}}_{2s}}(x)$

$\Leftrightarrow$

$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdot\cdot+n_l}^{=2s}\frac{m^s}{\sqrt{E}}$

$[a(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}$

Step 3: When $s'=s+1/2$,

$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2s+1)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma\}}_{2s+1}}(x)$

$\Leftrightarrow$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdots+n_l}^{=2s}\frac{m^s}{\sqrt{E}}\\ [a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma\tau_\varsigma}_{2s+1}}(x)\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdots+n_l}^{=2s}\frac{m^s}{\sqrt{E}}\\ [a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}\\ \int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdots+n_l}^{=2s}\frac{m^s}{\sqrt{E}}\\ [a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}\\ =\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdots+n_l}^{=2s}\frac{m^s}{\sqrt{E}}\\ [a_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{n_1+\cdots+n_l}^{=2s}\frac{m^s}{\sqrt{E}}\\ [a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}\\ \sum\limits_{n_1+\cdots+n_l}^{=2s}a_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum\limits_{n_1+\cdots+n_l}^{=2s}a_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)\\ \sum\limits_{n_1+\cdots+n_l}^{=2s}b^+_{\eta_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)=\sum\limits_{n_1+\cdots+n_l}^{=2s}b^+_{\tau_\varsigma}(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\eta_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=s}^{-s}\frac{m^{s+1/2}}{\sqrt{E}}\\ [a(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)e^{ip\cdot x}+b^+(\vec{p};n_1,n_2,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\eta_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)e^{-ip\cdot x}]d^3\vec{p}\end{cases}$$

This step proves that when $s'=s$, the proposition is established.

Step 4: Based on the above inductive reasoning, the theorem has been proved. □

3 An intuitive solution for plane wave solution of K-G equation

3.1 Mathematical basis of intuitive solution

Lem. 3.1.1.

$$\begin{cases}\sum\limits_{h=n}^{-n}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)]=\sum\limits_{h=n}^{-n}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)]\\ (p^cp_c+m^2)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdot\cdot\}}_{n}}(\vec{p},h)\\ a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=a_a(\vec{p},h)\varepsilon_{\underbrace{db\cdot\cdot}_{n}}(\vec{p},h),a_d(\vec{p},h):=a(\vec{p},h;1)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_d(\vec{p},-1)\end{cases}$$

Lem. 3.1.2.

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)\\ \sum\limits_{h=n}^{-n}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)]=\sum\limits_{h=n}^{-n}[a_a(\vec{p},h)\varepsilon_{\underbrace{db\cdot\cdot}_{n}}(\vec{p},h)],-n\le h\le n\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)\\ a(\vec{p},h;1)=\frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h+1;0),a(\vec{p},h;-1)=\frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h-1;0)\\ a(\vec{p},n;-1)=\frac{1}{\sqrt{C^2_{2n}}}a(\vec{p},n-2;1),a(\vec{p},-n;1)=\frac{1}{\sqrt{C^2_{2n}}}a(\vec{p},-n+2;-1)\\ -n+1\le h\le n-1\end{cases}$$

Proof:

$$\begin{cases}\sum_{h=n}^{-n}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)]=\sum_{h=n}^{-n}[a_a(\vec{p},h)\varepsilon_{\underbrace{db\cdot\cdot}_{n}}(\vec{p},h)]\\ a_d(\vec{p},h):=a(\vec{p},h;1)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_d(\vec{p},-1)\end{cases}$$

$$\Leftrightarrow\sum_{h=n}^{-n}[a(\vec{p},h;1)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)$$

$$=\sum_{h=n}^{-n}[a(\vec{p},h;1)\varepsilon_a(\vec{p},1)+a(\vec{p},h;0)\varepsilon_a(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_a(\vec{p},-1)]\varepsilon_{\underbrace{db\cdot\cdot}_{n}}(\vec{p},h)$$

$$\Leftrightarrow\sum_{h=n}^{-n}[a(\vec{p},h;1)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_d(\vec{p},-1)]$$

$$[\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)\varepsilon_a(\vec{p},1)+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\varepsilon_a(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)\varepsilon_a(\vec{p},-1)]$$

$$=\sum_{h=n}^{-n}[a(\vec{p},h;1)\varepsilon_a(\vec{p},1)+a(\vec{p},h;0)\varepsilon_a(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_a(\vec{p},-1)]$$

$$[\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)\varepsilon_d(\vec{p},1)+\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\varepsilon_d(\vec{p},0)+\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)\varepsilon_d(\vec{p},-1)]$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)a(\vec{p},h;1)=\sum_{h=n}^{-n}a(\vec{p},h;1)\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)a(\vec{p},h;0)=\sum_{h=n}^{-n}a(\vec{p},h;1)\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)a(\vec{p},h;-1)=\sum_{h=n}^{-n}a(\vec{p},h;1)\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)\end{cases}$$

$$\begin{cases}\sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h;1)=\sum_{h=n}^{-n}a(\vec{p},h;0)\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h;0)=\sum_{h=n}^{-n}a(\vec{p},h;0)\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h;-1)=\sum_{h=n}^{-n}a(\vec{p},h;0)\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)\end{cases}$$

$$\begin{cases}\sum_{h=n}^{-n}\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)a(\vec{p},h;1)=\sum_{h=n}^{-n}a(\vec{p},h;-1)\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)a(\vec{p},h;0)=\sum_{h=n}^{-n}a(\vec{p},h;-1)\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)a(\vec{p},h;-1)=\sum_{h=n}^{-n}a(\vec{p},h;-1)\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)a(\vec{p},h;0)=\sum_{h=n}^{-n}a(\vec{p},h;1)\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h-1)a(\vec{p},h;-1)=\sum_{h=n}^{-n}a(\vec{p},h;1)\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)\\ \sum_{h=n}^{-n}\frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h+1)a(\vec{p},h;0)=\sum_{h=n}^{-n}a(\vec{p},h;-1)\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h=n-1}^{-n-1}\frac{\sqrt{C_{n+h+1}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h+1;0)=\sum_{h=n}^{-n}a(\vec{p},h;1)\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n-1}^{-n-1}\frac{\sqrt{C_{n+h+1}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h+1;-1)=\sum_{h=n+1}^{-n+1}a(\vec{p},h-1;1)\frac{\sqrt{C_{n-h+1}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n+1}^{-n+1}\frac{\sqrt{C_{n-h+1}^2}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h-1;0)=\sum_{h=n}^{-n}a(\vec{p},h;-1)\frac{\sqrt{C_{n+h}^1C_{n-h}^1}}{\sqrt{C_{2n}^2}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h=n-1}^{-n+1}\frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h+1;0)=\sum_{h=n-1}^{-n+1}a(\vec{p},h;1)\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n-1}^{-n+1}\frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h+1;-1)=\sum_{h=n-1}^{-n+1}a(\vec{p},h-1;1)\frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\\ \sum_{h=n-1}^{-n+1}\frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)a(\vec{p},h-1;0)=\sum_{h=n-1}^{-n+1}a(\vec{p},h;-1)\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{b\cdot\cdot}_{n-1}}(\vec{p},h)\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}\frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^2_{2n}}}a(\vec{p},h+1;0)=\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}a(\vec{p},h;1)\\ \frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^2_{2n}}}a(\vec{p},h+1;-1)=\frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^2_{2n}}}a(\vec{p},h-1;1)\\ \frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^2_{2n}}}a(\vec{p},h-1;0)=\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}a(\vec{p},h;-1)\\ -n+1\le h\le n-1\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}a(\vec{p},h;1)=\frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h+1;0),a(\vec{p},h;-1)=\frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h-1;0)\\ a(\vec{p},n;-1)=\frac{1}{\sqrt{C^2_{2n}}}a(\vec{p},n-2;1),a(\vec{p},-n;1)=\frac{1}{\sqrt{C^2_{2n}}}a(\vec{p},-n+2;-1)\\ -n+1\le h\le n-1\end{cases}$$

□

Cor. 3.1.1.

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)\\ \sum_{h=n}^{-n}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)]=\sum_{h=n}^{-n}[a_a(\vec{p},h)\varepsilon_{\underbrace{db\cdot\cdot}_{n}}(\vec{p},h)],-n\le h\le n\end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases}\varepsilon_{\underbrace{a\cdot\cdot bc}_{n}}(\vec{p},h)=\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h-1)\varepsilon_c(\vec{p},1)+\frac{\sqrt{C^1_{n+h}C^1_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h)\varepsilon_c(\vec{p},0)+\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^2_{2n}}}\varepsilon_{\underbrace{a\cdot\cdot b}_{n-1}}(\vec{p},h+1)\varepsilon_c(\vec{p},-1)\\ a(\vec{p},h;1)=\frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h+1;0),a(\vec{p},h;-1)=\frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h-1;0)\\ a(\vec{p},n;-1)=\frac{1}{\sqrt{C^2_{2n}}}a(\vec{p},n-2;1)=\frac{1}{\sqrt{4n}}a(\vec{p},n-1;0),a(\vec{p},-n;1)=\frac{1}{\sqrt{C^2_{2n}}}a(\vec{p},-n+2;-1)=\frac{1}{\sqrt{4n}}a(\vec{p},-n+1;0)\\ -n+1\le h\le n-1\end{cases}$$

$$\Rightarrow\sum_{h=n}^{-n}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)]$$

$$=\sum_{h=n-1}^{-n+1}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)]+[a_d(\vec{p},n)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)]+[a_d(\vec{p},-n)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)]$$

$$=\sum_{h=n-1}^{-n+1}[a(\vec{p},h;1)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)$$

$$+[a(\vec{p},n;1)\varepsilon_d(\vec{p},1)+a(\vec{p},n;0)\varepsilon_d(\vec{p},0)+a(\vec{p},n;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)$$

$$+[a(\vec{p},-n;1)\varepsilon_d(\vec{p},1)+a(\vec{p},-n;0)\varepsilon_d(\vec{p},0)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$

$$=\sum_{h=n-1}^{-n+1}[\frac{\sqrt{C^2_{n+h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h+1;0)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+\frac{\sqrt{C^2_{n-h+1}}}{\sqrt{C^1_{n+h}C^1_{n-h}}}a(\vec{p},h-1;0)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)$$

$$+[a(\vec{p},n;1)\varepsilon_d(\vec{p},1)+a(\vec{p},n;0)\varepsilon_d(\vec{p},0)+a(\vec{p},n;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)$$

$$+[a(\vec{p},-n;1)\varepsilon_d(\vec{p},1)+a(\vec{p},-n;0)\varepsilon_d(\vec{p},0)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$

$$=[\sum_{h=n}^{-n+2}\frac{\sqrt{C^2_{n+h}}}{\sqrt{C^1_{n+h-1}C^1_{n-h+1}}}a(\vec{p},h;0)\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\sum_{h=n-1}^{-n+1}a(\vec{p},h;0)\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)$$

$$+\sum_{h=n-2}^{-n}\frac{\sqrt{C^2_{n-h}}}{\sqrt{C^1_{n+h+1}C^1_{n-h-1}}}a(\vec{p},h;0)\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)]$$

$$+[a(\vec{p},n;1)\varepsilon_d(\vec{p},1)+a(\vec{p},n;0)\varepsilon_d(\vec{p},0)+\frac{1}{\sqrt{4n}}a(\vec{p},n-1;0)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)$$

$$+[\frac{1}{\sqrt{4n}}a(\vec{p},-n+1;0)\varepsilon_d(\vec{p},1)+a(\vec{p},-n;0)\varepsilon_d(\vec{p},0)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$

$$=[\sum_{h=n}^{-n+2}\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}a(\vec{p},h;0)\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\sum_{h=n-1}^{-n+1}a(\vec{p},h;0)\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)$$

$$+\sum_{h=n-2}^{-n}\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}a(\vec{p},h;0)\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)]$$
$$+[a(\vec{p},n;1)\varepsilon_d(\vec{p},1)+a(\vec{p},n;0)\varepsilon_d(\vec{p},0)+\frac{1}{\sqrt{4n}}a(\vec{p},n-1;0)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)$$
$$+[\frac{1}{\sqrt{4n}}a(\vec{p},-n+1;0)\varepsilon_d(\vec{p},1)+a(\vec{p},-n;0)\varepsilon_d(\vec{p},0)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$
$$=\sum_{h=n-2}^{-n+2}\frac{\sqrt{C^2_{2n+2}}}{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}a(\vec{p},h;0)$$
$$[\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)]$$
$$+\sqrt{n}a(\vec{p},n;0)\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-1)+\frac{\sqrt{2n-1}}{2}a(\vec{p},n-1;0)\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-2)$$
$$+a(\vec{p},n-1;0)\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-1)+a(\vec{p},-n+1;0)\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n+1)$$
$$+\sqrt{n}a(\vec{p},-n;0)\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n+1)+\frac{\sqrt{2n-1}}{2}a(\vec{p},-n+1;0)\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n+2)$$
$$+[a(\vec{p},n;1)\varepsilon_d(\vec{p},1)+a(\vec{p},n;0)\varepsilon_d(\vec{p},0)+\frac{1}{\sqrt{4n}}a(\vec{p},n-1;0)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)$$
$$+[\frac{1}{\sqrt{4n}}a(\vec{p},-n+1;0)\varepsilon_d(\vec{p},1)+a(\vec{p},-n;0)\varepsilon_d(\vec{p},0)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$
$$=\sum_{h=n-2}^{-n+2}\frac{\sqrt{C^2_{2n+2}}}{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}a(\vec{p},h;0)$$
$$[\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)]$$
$$+a(\vec{p},n-1;0)[\frac{\sqrt{2n-1}}{2}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-2)+\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-1)+\frac{1}{\sqrt{4n}}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)]$$
$$+a(\vec{p},-n+1;0)[\frac{1}{\sqrt{4n}}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)+)\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n+1)+\frac{\sqrt{2n-1}}{2}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n+2)]$$
$$+a(\vec{p},n;0)[\sqrt{n}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-1)+\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)]$$
$$+a(\vec{p},-n;0)[\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)+\sqrt{n}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n+1)]$$
$$+a(\vec{p},n;1)\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$
$$=\sum_{h=n-1}^{-n+1}\frac{\sqrt{C^2_{2n+2}}}{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}a(\vec{p},h;0)$$
$$[\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)]$$
$$+a(\vec{p},n;0)[\sqrt{n}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n-1)+\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)]$$
$$+a(\vec{p},-n;0)[\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)+\sqrt{n}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n+1)]$$
$$+a(\vec{p},n;1)\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$
$$=\sum_{h=n}^{-n}\frac{\sqrt{C^2_{2n+2}}}{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}a(\vec{p},h;0)$$
$$[\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)]$$
$$+a(\vec{p},n;1)\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},n)+a(\vec{p},-n;-1)\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},-n)$$
$$=\sum_{h=n+1}^{-n-1}a(\vec{p},h)$$
$$[\frac{\sqrt{C^2_{n+1+h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\frac{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)+\frac{\sqrt{C^2_{n+1-h}}}{\sqrt{C^2_{2n+2}}}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1)]$$
$$a(\vec{p},h):=\frac{\sqrt{C^2_{2n+2}}}{\sqrt{C^1_{n+1+h}C^1_{n+1-h}}}a(\vec{p},h;0),-n\le h\le n;a(\vec{p},n+1):=a(\vec{p},n;1),a(\vec{p},-n-1):=a(\vec{p},-n;-1)$$
$$=\sum_{h=n+1}^{-n-1}a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot d}_{n+1}}(\vec{p},h)$$

$a(\vec{p},h):=\frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{n+1+h}^1C_{n+1-h}^1}}a(\vec{p},h;0),-n\le h\le n;a(\vec{p},n+1):=a(\vec{p},n;1),a(\vec{p},-n-1):=a(\vec{p},-n;-1)$

$\varepsilon_{\underbrace{ab\cdot\cdot d}_{n+1}}(\vec{p},h):=\frac{\sqrt{C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}}\varepsilon_d(\vec{p},1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h-1)+\frac{\sqrt{C_{n+1+h}^1C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}}\varepsilon_d(\vec{p},0)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)$
$+\frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}}\varepsilon_d(\vec{p},-1)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h+1),-n-1\le h\le n+1$

3.2 An intuitive solution to plane wave solution of K-G equation?

Thm. 3.2.1.
$(-\partial^c\partial_c+m^2)A_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n}}(x)\ \textit{fully symmetric}\Leftrightarrow$

$$\begin{cases}A_{\underbrace{ab\cdot\cdot}_{n}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}[a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\(p^cp_c+m^2)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\ \textit{fully symmetric}\\(p^cp_c+m^2)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\ \textit{fully symmetric}\end{cases}$$

Proof: Use mathematical induction to prove this theorem.
Step 1: When $n'=1$, the following is established.
$(-\partial^c\partial_c+m^2)A_a(x)=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n}}(x)=0\Leftrightarrow$

$$\begin{cases}A_a(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}[a(\vec{p},h)\varepsilon_a(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_a(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\(p^cp_c+m^2)\varepsilon_a(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0\\(p^cp_c+m^2)\tilde{\varepsilon}_a(\vec{p},h)=0,p^a\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0\end{cases}$$

Step 2: When $n'=n$, the following is established.
$(-\partial^c\partial_c+m^2)A_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n}}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n}}(x)$ fully symmetric $\Leftrightarrow$

$$\begin{cases}A_{\underbrace{ab\cdot\cdot}_{n}}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}[a(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\(p^cp_c+m^2)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\text{ fully symmetric}\\(p^cp_c+m^2)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)\text{ fully symmetric}\end{cases}$$

Step 3: When $n'=n+1$,
$(-\partial^c\partial_c+m^2)A_{\underbrace{ab\cdot\cdot}_{n+1}}(x)=0,\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n+1}}(x)=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n+1}}(x)=0,A_{\underbrace{ab\cdot\cdot}_{n+1}}(x)$ fully symmetric

$\Leftrightarrow$
$$\begin{cases}(-\partial^c\partial_c+m^2)A_{\underbrace{ab\cdot\cdot}_{n}d}(x)=0,\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}d}(x)=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n}d}(x)=0\\A_{\underbrace{ab\cdot\cdot}_{n}d}(x)=\frac{1}{n!}A_{\underbrace{\{ab\cdot\cdot\}}_{n}d}(x),A_{\underbrace{ab\cdot\cdot}_{n}d}(x)=A_{\underbrace{db\cdot\cdot}_{n}a}(x)\end{cases}$$
$\Leftrightarrow$
$$\begin{cases}A_{\underbrace{ab\cdot\cdot}_{n}d}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{ip\cdot x}+b_d^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}\\(p^cp_c+m^2)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdot\cdot\}}_{n}}(\vec{p},h)\\(p^cp_c+m^2)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{n!}\tilde{\varepsilon}_{\underbrace{\{ab\cdot\cdot\}}_{n}}(\vec{p},h)\\A_{\underbrace{ab\cdot\cdot}_{n}d}(x)=A_{\underbrace{db\cdot\cdot}_{n}a}(x),a_d(\vec{p},h):=a(\vec{p},h;1)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_d(\vec{p},-1)\\b_d^+(\vec{p},h):=b^+(\vec{p},h;1)\tilde{\varepsilon}_d(\vec{p},1)+b^+(\vec{p},h;0)\tilde{\varepsilon}_d(\vec{p},0)+b^+(\vec{p},h;-1)\tilde{\varepsilon}_d(\vec{p},-1)\end{cases}$$
$\Leftrightarrow$

$$\begin{cases} A_{\underbrace{ab\cdot\cdot}_{n}d}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=n}^{-n}\frac{1}{\sqrt{2^nE}}[a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{ip\cdot x}+b_d^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p} \\ (p^cp_c+m^2)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdot\cdot\}}_{n}}(\vec{p},h) \\ (p^cp_c+m^2)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,p^a\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=0,\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=\frac{1}{n!}\tilde{\varepsilon}_{\underbrace{\{ab\cdot\cdot\}}_{n}}(\vec{p},h) \\ a_d(\vec{p},h)\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=a_a(\vec{p},h)\varepsilon_{\underbrace{db\cdot\cdot}_{n}}(\vec{p},h),b_d^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p},h)=b_a^+(\vec{p},h)\tilde{\varepsilon}_{\underbrace{db\cdot\cdot}_{n}}(\vec{p},h) \\ a_d(\vec{p},h):=a(\vec{p},h;1)\varepsilon_d(\vec{p},1)+a(\vec{p},h;0)\varepsilon_d(\vec{p},0)+a(\vec{p},h;-1)\varepsilon_d(\vec{p},-1) \\ b_d^+(\vec{p},h):=b^+(\vec{p},h;1)\tilde{\varepsilon}_d(\vec{p},1)+b^+(\vec{p},h;0)\tilde{\varepsilon}_d(\vec{p},0)+b^+(\vec{p},h;-1)\tilde{\varepsilon}_d(\vec{p},-1) \end{cases}$$

This step proves that when $n'=n+1$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

4 Plane wave solutions of antisymmetric tensor field equations in 4D

4.1 Plane wave solutions of Klein-Gordon equation for spin-1 particles

Thm. 4.1.1. $\partial^bF_{ab}+m^2A_a=0,F_{ab}=\partial_aA_b-\partial_bA_a\Leftrightarrow(\partial^b\partial_b-m^2)A_a=0,\partial^aA_a=0$

$$A_a(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}[a(\vec{p},h)\varepsilon_a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)\tilde{\varepsilon}_a(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

Thm. 4.1.2. $\begin{cases}\frac{1}{2!}\partial^{[a}F^{bc]}+mF^{abc}=0,\partial_aF^{ab}=0 \\ \partial^{[a}F^{bcd]}=0,\partial_aF^{abc}+mF^{bc}=0\end{cases}\Leftrightarrow\begin{cases}\partial_aF^{abc}+mF^{bc}=0,\frac{1}{2!}\partial^{[a}F^{bc]}+mF^{abc}=0 \\ \partial^cF_{cab}-m^2A_{ab}=0,F_{cab}=\frac{1}{2!}\partial_{[c}A_{ab]};A_{ab}:=\frac{-1}{m}F_{ab}\end{cases}$

Thm. 4.1.3. $\partial^cF_{cab}-m^2A_{ab}=0,F_{cab}=\frac{1}{2!}\partial_{[c}A_{ab]}\Leftrightarrow(\partial^c\partial_c-m^2)A_{ab}=0,\partial^aA_{ab}=0,A_{ab}=-A_{ba}$

Thm. 4.1.4. $(\partial^c\partial_c-m^2)A_{ab}=0,\partial^aA_{ab}=0,A_{ab}=-A_{ba}$

$$\Leftrightarrow A_{ab}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}[a(\vec{p},h)\varepsilon_{ab}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_{ab}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$

Proof: $(\partial^c\partial_c-m^2)A_{ab}=0,\partial^aA_{ab}=0,A_{ab}=-A_{ba}$

$$\Leftrightarrow A_{ab}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}[a_b(\vec{p},h)\varepsilon_a(\vec{p},h)e^{ip\cdot x}+b_b^+(\vec{p},h)\tilde{\varepsilon}_a(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p},A_{ab}=-A_{ba}$$

$$\Leftrightarrow\sum\limits_{h=1}^{-1}a_{\{a}(\vec{p},h)\varepsilon_{b\}}(\vec{p},h)=0,a_a(\vec{p},h)=\sum\limits_{h'=1}^{-1}a(\vec{p},h;h')\varepsilon_a(\vec{p},h')+c(\vec{p},h;0)\frac{p_a}{m}$$

$$\Leftrightarrow\sum\limits_{h,h'=1}^{-1}a(\vec{p},h;h')\varepsilon_{\{a}(\vec{p},h')\varepsilon_{b\}}(\vec{p},h)+c(\vec{p},h;0)\frac{1}{m}p_{\{a}\varepsilon_{b\}}(\vec{p},h)=0$$

$$\Leftrightarrow\sum\limits_{h,h'=1}^{-1}a(\vec{p},h;h')\varepsilon_{\{a}(\vec{p},h')\varepsilon_{b\}}(\vec{p},h)=0,c(\vec{p},h;0)=0$$

$$\Leftrightarrow a(\vec{p},-1;1)=-a(\vec{p},1;-1),a(\vec{p},0;1)=-a(\vec{p},1;0),a(\vec{p},-1;0)=-a(\vec{p},0;-1),a(\vec{p},h;h)=0$$

$$\Leftrightarrow\sum\limits_{h=1}^{-1}a_b(\vec{p},h)\varepsilon_a(\vec{p},h)=\sum\limits_{h',h=1}^{-1}a(\vec{p},h;h')\varepsilon_a(\vec{p},h)\varepsilon_b(\vec{p},h')$$

$$=a(\vec{p},1;0)\varepsilon_{[a}(\vec{p},1)\varepsilon_{b]}(\vec{p},0)+a(\vec{p},1;-1)\varepsilon_{[a}(\vec{p},1)\varepsilon_{b]}(\vec{p},-1)+a(\vec{p},0;-1)\varepsilon_{[a}(\vec{p},0)\varepsilon_{b]}(\vec{p},-1)$$

$$=a(1)\varepsilon_{ab}(\vec{p},1)+a(0)\varepsilon_{ab}(\vec{p},0)+a(-1)\varepsilon_{ab}(\vec{p},-1)=\sum\limits_{h=1}^{-1}a(h)\varepsilon_{ab}(\vec{p},h)$$

$a(1)=\sqrt{2}a(\vec{p},1;0),a(0)=\sqrt{2}a(\vec{p},1;-1),a(-1)=\sqrt{2}a(\vec{p},0;-1)$

$\varepsilon_{ab}(\vec{p},1):=\frac{1}{\sqrt{2}}\varepsilon_{[a}(\vec{p},1)\varepsilon_{b]}(\vec{p},0),\varepsilon_{ab}(\vec{p},0):=\frac{1}{\sqrt{2}}\varepsilon_{[a}(\vec{p},1)\varepsilon_{b]}(\vec{p},-1),\varepsilon_{ab}(\vec{p},-1):=\frac{1}{\sqrt{2}}\varepsilon_{[a}(\vec{p},0)\varepsilon_{b]}(\vec{p},-1)$

$$\Leftrightarrow A_{ab}(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{-1}\frac{1}{\sqrt{2E}}[a(\vec{p},h)\varepsilon_{ab}(\vec{p},h)e^{ip\cdot x}+b^+(\vec{p},h)\tilde{\varepsilon}_{ab}(\vec{p},h)e^{-ip\cdot x}]d^3\vec{p}$$ □

Thm. 4.1.5. $\partial^dF_{dabc}-m^2A_{abc}=0,F_{dabc}=\frac{1}{3!}\partial_{[d}A_{abc]}\Leftrightarrow(\partial^d\partial_d-m^2)A_{abc}=0,\partial^aA_{abc}=0,A_{abc}=\frac{1}{3!}A_{[abc]}$

4.2 Plane wave solutions of Klein-Gordon equation for spin-1 particles in n=N+1-D

Thm. 4.2.1. $\partial^bF_{ab}+m^2A_a=0,F_{ab}=\partial_aA_b-\partial_bA_a\Leftrightarrow(\partial^b\partial_b-m^2)A_a=0,\partial^aA_a=0$

$$A_a(\vec{r},t)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{N}\frac{1}{\sqrt{2E}}[a(\vec{p},h)\varepsilon_a(\vec{p},h)e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)\tilde{\varepsilon}_a(\vec{p},h)e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^3\vec{p}$$

Def. 4.2.1. $L=\{\begin{bmatrix}0&0&0&0&0&0&i\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&\cdot\cdot\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\-i&0&0&\cdot\cdot&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&i\\0&0&0&0&0&0&0\\0&0&0&0&0&0&\cdot\cdot\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&-i&0&\cdot\cdot&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&i\\0&0&0&0&0&0&\cdot\cdot\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&-i&\cdot\cdot&0&0&0\end{bmatrix},\cdot\cdot,\begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&\cdot\cdot\\0&0&0&0&0&0&i\\0&0&0&0&0&0&0\\0&0&0&\cdot\cdot&-i&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&\cdot\cdot\\0&0&0&0&0&0&0\\0&0&0&0&0&0&i\\0&0&0&\cdot\cdot&0&-i&0\end{bmatrix}\}$

Cor. 4.2.1. $L_{\vec{p}} := L_{\vec{v}} = e^{-\{ln[\gamma_v(1+v)]\}\hat{v}\cdot L} = 1 - \gamma_v(\vec{v}\cdot L) + \frac{\gamma_v - 1}{v^2}(\vec{v}\cdot L)^2 = 1 - \frac{1}{m}(\vec{p}\cdot L) + \frac{1}{m(E+m)}(\vec{p}\cdot L)^2$

Cor. 4.2.2. $L_{\vec{p}} = \frac{1}{m}\begin{bmatrix} m & 0 & 0 & -ip_x \\ 0 & m & 0 & -ip_y \\ 0 & 0 & m & -ip_z \\ ip_x & ip_y & ip_z & E \end{bmatrix} + \frac{1}{m(E+m)}\begin{bmatrix} p_x p_x & p_x p_y & p_x p_z & 0 \\ p_y p_x & p_y p_y & p_y p_z & 0 \\ p_z p_x & p_z p_y & p_z p_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Cor. 4.2.3. $\varepsilon(\vec{p},h) := L_{\vec{p}}\begin{bmatrix} 0_1 \\ \cdot\cdot \\ 1_h \\ \cdot\cdot \\ 0_n \end{bmatrix} = \begin{bmatrix} 0_1 \\ \cdot\cdot \\ 1_h \\ \cdot\cdot \\ 0_n \end{bmatrix} + \frac{p_h}{m(E+m)}\begin{bmatrix} \vec{p} \\ i(E+m) \end{bmatrix}, \varepsilon_a(\vec{p},n) := L_{\vec{p}}\begin{bmatrix} 0_1 \\ \cdot\cdot \\ 0_0 \\ \cdot\cdot \\ 1_n \end{bmatrix} = -i\frac{p_a}{m}; h = 1, \cdot\cdot, N$

Cor. 4.2.4. $p^a\varepsilon_a(\vec{p},h) = 0, \varepsilon_a(\vec{p},h)\eta^{aa'}\varepsilon^+_{a'}(\vec{p},h') = \delta_{hh'}, \sum_{h=1}^{N}\varepsilon_a(\vec{p},h)\varepsilon^+_{a'}(\vec{p},h) = \eta_{aa'} + \frac{p_a p^+_{a'}}{m^2}$

Cor. 4.2.5. $L_{\vec{v}}\begin{bmatrix} \vec{0} \\ i \end{bmatrix} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L}\begin{bmatrix} \vec{0} \\ i \end{bmatrix} = \begin{bmatrix} \gamma_v\vec{v} \\ i\gamma_v \end{bmatrix}, L_{\vec{v}}\begin{bmatrix} \vec{0} \\ im \end{bmatrix} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L}\begin{bmatrix} \vec{0} \\ im \end{bmatrix} = \begin{bmatrix} \vec{p} \\ iE \end{bmatrix}$

Thm. 4.2.2. $-\frac{i}{4}[\Gamma_i, \Gamma_j] = S_{ij} \Rightarrow \Gamma_i = ?$

Cor. 4.2.6. $\begin{cases} \lambda_m(\hat{p},1;1) = S_m(1)\lambda(\hat{p},1;1) = e^{i\vec{\omega}\cdot\gamma}\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2\hat{p}_-}\begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+\hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p},1;1) = \frac{\hat{p}_+}{\hat{p}_-}\lambda_m(\hat{p},-1;1) \\ \lambda_m(\hat{p},0;1) = S_m(1)\lambda(\hat{p},0;1) = e^{i\vec{\omega}\cdot\gamma}\begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix} = -i\begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} = -i\hat{p}, \lambda_m(-\hat{p},0;1) = -\lambda_m(\hat{p},0;1) \\ \lambda_m(\hat{p},-1;1) = S_m(1)\lambda(\hat{p},-1;1) = e^{i\vec{\omega}\cdot\gamma}\frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2\hat{p}_+}\begin{bmatrix} -i(\hat{p}_x\hat{p}_z + i\hat{p}_y) \\ -1(\hat{p}_x + i\hat{p}_y\hat{p}_z) \\ 2i(\hat{p}_+\hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p},-1;1) = \frac{\hat{p}_-}{\hat{p}_+}\lambda_{em}(\hat{p},1;1) \end{cases}$

Cor. 4.2.7. $\varepsilon(\vec{p},\pm1) = [L_{\vec{p}}S^+_m(1)e^{-i\vec{\omega}\cdot R}]\varepsilon(\vec{p},\pm1;1)$

Cor. 4.2.8. $\frac{1}{2\hat{p}_-}\begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+\hat{p}_-) \end{bmatrix} = [](\begin{bmatrix} 0_1 \\ \cdot\cdot \\ 1_h \\ \cdot\cdot \\ 0_n \end{bmatrix} + \frac{p_x}{m(E+m)}\begin{bmatrix} \vec{p} \\ i(E+m) \end{bmatrix})$

Cor. 4.2.9. $(\gamma^a\otimes\sigma_y\partial_a + I\otimes\sigma_z\otimes\sigma_y\partial_u + I_4\otimes\sigma_x\partial_v + m)\psi = 0$
$\Rightarrow (\gamma^a\otimes\sigma_y\partial_a + I\otimes\sigma_z\otimes\sigma_y iM + I_4\otimes\sigma_x 0 + m)\psi = 0$
$\Rightarrow (\gamma^a\otimes\sigma_z\partial_a + I\otimes\sigma_z\otimes\sigma_z iM + m)\psi' = 0$

4.3 Plane wave solutions of Klein-Gordon equation for spin-2 particles in n=N+1-D

Thm. 4.3.1. $(\partial^c\partial_c - m^2)A_{ab} = 0, \partial^a A_{ab} = 0, A_{ab} = -A_{ba}$

$A_{ab}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{h,h'=1}^{N}\frac{1}{\sqrt{2E}}[a(\vec{p};h,h')\varepsilon_{[a}(\vec{p},h)\varepsilon_{b]}(\vec{p},h')e^{ip\cdot x} + b^+(\vec{p};h,h')\tilde{\varepsilon}_{[a}(\vec{p},h)\tilde{\varepsilon}_{b]}(\vec{p},h')e^{-ip\cdot x}d^3\vec{p}$

Chapter35 Covariant Quantization of Massive Particles in High Dimension

Self comment: In this chapter, the covariant quantization of massive particles is generalized to the general N+1 dimensional space-time. N+1 dimensional space-time case is similar to 4-dimensional space-time case. For particles described by the Bargmann-Wigner equation or Dirac equation, it is generally possible to describe both charged complex particles and uncharged Majorana particles. The principal commutation rule in both cases is consistent, but the rest are generally zero for charged complex particles. For uncharged Majorana particles, the rest of the commutative or anti commutative brackets are naturally derived from the principal commutative rule and Majorana conditions. And they are generally not zero. In this chapter, we only discuss the case of complex particles and generally only give the principal commutation rule. The Majorana particle case is no longer specifically discussed. If we want to obtain the quantum field theory of the Majorana particle case, we only need to add the Majorana condition to the complex particle case. Then we will naturally obtain it.

1 Lorentz boost transformation in N+1 dimensional space-time

1.1 Lorentz boost transformation for vector in N+1 dimensional space-time

Def. 1.1.1. $\Omega(s) := \frac{1}{2}(\Gamma \otimes I_{l^{2s-1}} + I_l \otimes \Gamma \otimes I_{l^{2s-2}} + \cdot\cdot + I_{l^{2s-1}} \otimes \Gamma), l = 2^{[\frac{N-1}{2}]}$

Def. 1.1.2. $L = \{ \begin{bmatrix} 0 & 0&0&0&0&0&i \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\cdot\cdot \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ -i&0&0&\cdot\cdot&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&i \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\cdot\cdot \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&-i&0&\cdot\cdot&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&i \\ 0&0&0&0&0&0&\cdot\cdot \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&-i&\cdot\cdot&0&0&0 \end{bmatrix}, \cdot\cdot, \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\cdot\cdot \\ 0&0&0&0&0&0&i \\ 0&0&0&0&0&0&0 \\ 0&0&0&\cdot\cdot&-i&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\cdot\cdot \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&i \\ 0&0&0&\cdot\cdot&0&-i&0 \end{bmatrix} \}$

Cor. 1.1.1. $L_{\vec{v}} = e^{-\{ln[\gamma_v(1+v)]\}\hat{v}\cdot L} = 1 - \gamma_v(\vec{v}\cdot L) + \frac{\gamma_v - 1}{v^2}(\vec{v}\cdot L)^2, L_{\vec{v}}L_{-\vec{v}} = L_{-\vec{v}}L_{\vec{v}} = I$

Cor. 1.1.2. $L_{\vec{v}} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = \begin{bmatrix} \gamma_v\vec{v} \\ i\gamma_v \end{bmatrix}, L_{\vec{v}} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = \begin{bmatrix} \vec{p} \\ iE_{\vec{p}} \end{bmatrix}$

1.2 Lorentz boost transformation for neutrino spinor in N+1 dimensional space-time

Pro. 1.2.1. $(\vec{v}\cdot\Gamma)^2 = v^2, (\vec{v}\cdot i\vec{\gamma}\gamma_0)^2 = v^2$

Cor. 1.2.1. $\Lambda_{\varsigma\vec{v}} = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\frac{1}{2}\Gamma} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1+\gamma_v - \varsigma\gamma_v\vec{v}\cdot\Gamma), c = \frac{(1+\gamma_v)}{\sqrt{2(\gamma_v+1)}}, s = -\frac{\varsigma\gamma_v}{\sqrt{2(\gamma_v+1)}}$

1.3 Lorentz boost transformation for electron spinor in N+1 dimensional space-time

Cor. 1.3.1. $D_{\varsigma\vec{v}} = e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)} = \frac{1}{\sqrt{2(\gamma_v+1)}}[1+\gamma_v - i\varsigma\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_0]$

Cor. 1.3.2. $D_{\vec{v}} = e^{-ln\frac{E+|\vec{p}|}{m}\hat{p}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)} = \frac{m - i\gamma^a p_a\gamma_0}{\sqrt{2m(E+m)}}$

Proof: $D_{\vec{v}} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)} = \frac{1+\gamma_v - i\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_0}{\sqrt{2(\gamma_v+1)}} = \frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_0}{\sqrt{2m(E+m)}} = \frac{m-i\gamma^a p_a\gamma_0}{\sqrt{2m(E+m)}}$ □

1.4 Polynomial representation of Lorentz boost transformation for photon spinor in N+1-D

Cor. 1.4.1. $e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(1)} = 1 - \varsigma\gamma_v v[\hat{v}\cdot\Omega(1)] + (\gamma_v - 1)[\hat{v}\cdot\Omega(1)]^2$

1.5 Polynomial representation of Lorentz boost transformation for gravitino spinor in N+1-D

Cor. 1.5.1. $e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(\frac{3}{2})} = \frac{(\gamma_v+1)}{\sqrt{2(\gamma_v+1)}}(1 - \frac{\gamma_v-1}{4}) - \frac{2\varsigma\gamma_v v}{\sqrt{2(\gamma_v+1)}}(1 - \frac{\gamma_v - 1}{12})[\hat{v}\cdot\Omega(\frac{3}{2})]$
$+ \frac{\gamma_v^2-1}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^2 - \frac{1}{3}\frac{2\varsigma\gamma_v v(\gamma_v-1)}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^3$

1.6 Polynomial representation of Lorentz boost transformation for graviton spinor in N+1-D

Cor. 1.6.1. $e^{-\varsigma ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(2)} = 1 - \varsigma\gamma_v(1 - \frac{\gamma_v-1}{3})[\vec{v}\cdot\Omega(2)] + \frac{\gamma_v-1}{v^2}(1 - \frac{\gamma_v-1}{6})[\vec{v}\cdot\Omega(2)]^2$
$- \frac{1}{3}\frac{\varsigma\gamma_v(\gamma_v-1)}{v^2}[\vec{v}\cdot\Omega(2)]^3 + \frac{1}{6}\frac{(\gamma_v-1)^2}{v^4}[\vec{v}\cdot\Omega(2)]^4$

2 Electron covariant quantization in N+1 dimensional space-time

2.1 Electron equation in N+1 dimensional space-time [4]

Electronic equations in even dimensional space-time:

Def. 2.1.1. $(\gamma^a\partial_a + m)\psi = 0, \psi = \begin{bmatrix}\varphi\\ \eta\end{bmatrix}, \gamma^a = (\Gamma\otimes\sigma_y, \varsigma I_*\otimes\sigma_z) = (\vec{\gamma}, \varsigma I_*\otimes\sigma_z)$

Electronic equations in odd dimensional space-time:

Def. 2.1.2. $(\gamma^a\partial_a + m)\psi = 0, \psi = \begin{bmatrix}\varphi\\ \eta\end{bmatrix}, \gamma^a = (\Gamma\otimes\sigma_y, I_*\otimes\sigma_x, \varsigma I_*\otimes\sigma_z) = (\vec{\gamma}, \varsigma I_*\otimes\sigma_z)$

2.2 Electron static and kinetic solutions in N+1 dimensional space-time

Unified writing method for electronic equation in N+1 dimensional space-time:

Def. 2.2.1. $(\gamma^a\partial_a + m)\psi = 0, \psi = \begin{bmatrix}\varphi\\ \eta\end{bmatrix}, \gamma^a = (\vec{\gamma}, \varsigma I_*\otimes\sigma_z)$

Cor. 2.2.1. $\partial_{t_0}\psi(\vec{0}) = -im\gamma_0\psi(\vec{0}) \Leftrightarrow \psi(\vec{0}) = e^{-i\gamma_0 m t_0}\psi_0, \forall\psi_0$

Cor. 2.2.2. $\psi(\vec{p}) = \frac{m - i\gamma^a p_a\gamma_0}{\sqrt{2m(E+m)}}e^{i\gamma_0(\vec{p}\cdot\vec{r}-Et)}\psi_p = \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})e^{i\gamma_0(\vec{p}\cdot\vec{r}-Et)}\psi_p$

Cor. 2.2.3. $\psi^{(+\varsigma)}(\vec{p}) = \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}\varphi\\ 0\end{bmatrix}e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}, \psi^{(-\varsigma)}(\vec{p}) = \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0\\ \eta\end{bmatrix}e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}$

2.3 Properties of plane wave solutions for electron in N+1 dimensional space-time

Cor. 2.3.1. $\begin{cases}[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]^+ = \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\\ [\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})][\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})] = -\frac{i\gamma^a p_a\gamma_0}{m}\\ [\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\gamma_0[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})] = \gamma_0\\ [\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})][\sqrt{\frac{E+m}{2m}}(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})] = 1\end{cases}$

Pro. 2.3.1. $\begin{cases}[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})] = \frac{(\varsigma m - i\gamma^a p_a)\gamma_0}{2m}\\ [\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})] = \frac{(-\varsigma m - i\gamma^a p_a)\gamma_0}{2m}\end{cases}$

Pro. 2.3.2. $\begin{bmatrix}\varphi\\ 0\end{bmatrix}^+\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\begin{bmatrix}\eta\\ 0\end{bmatrix} \equiv 0, \begin{bmatrix}0\\ \varphi\end{bmatrix}^+\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\begin{bmatrix}0\\ \eta\end{bmatrix} \equiv 0$

Pro. 2.3.3. $\begin{cases}\begin{bmatrix}\varphi\\ 0\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})][\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}\eta\\ 0\end{bmatrix} = \frac{E}{m}\varphi^+\eta\\ \begin{bmatrix}0\\ \varphi\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})][\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}0\\ \eta\end{bmatrix} = \frac{E}{m}\varphi^+\eta\end{cases}$

Pro. 2.3.4. $\begin{cases}\begin{bmatrix}\varphi\\ 0\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})][\sqrt{\frac{E+m}{2m}}(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}0\\ \eta\end{bmatrix} = 0\\ \begin{bmatrix}0\\ \varphi\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})][\sqrt{\frac{E+m}{2m}}(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}\eta\\ 0\end{bmatrix} = 0\end{cases}$

Pro. 2.3.5. $\begin{cases}\begin{bmatrix}\varphi\\ 0\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\gamma_0[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}\eta\\ 0\end{bmatrix} = \varphi^+\eta\\ \begin{bmatrix}0\\ \varphi\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\gamma_0[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}0\\ \eta\end{bmatrix} = -\varphi^+\eta\end{cases}$

Pro. 2.3.6. $\begin{cases}\begin{bmatrix}\varphi\\ 0\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\gamma_0[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}0\\ \eta\end{bmatrix} = 0\\ \begin{bmatrix}0\\ \varphi\end{bmatrix}^+[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\gamma_0[\sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})]\begin{bmatrix}\eta\\ 0\end{bmatrix} = 0\end{cases}$

2.4 Electron spin basis in N+1 dimensional space-time

Def. 2.4.1. $u_\varsigma(\vec{p}, h) := \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}1\\ 0\\ 0_{l-2}\\ 0_l\end{bmatrix}, \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0\\ 1\\ 0_{l-2}\\ 0_l\end{bmatrix}, \cdot\cdot, \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_{l-2}\\ 0\\ 1\\ 0_l\end{bmatrix}$

Def. 2.4.2. $v_\varsigma(\vec{p}, h) := \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_l\\ 1\\ 0\\ 0_{l-2}\end{bmatrix}, \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_l\\ 0\\ 1\\ 0_{l-2}\end{bmatrix}, \cdot\cdot, \sqrt{\frac{E+m}{2m}}(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_l\\ 0_{l-2}\\ 0\\ 1\end{bmatrix}$

2.5 Dirac basis is a common eigenstate of spin, helicity and charge three operators in N+1-D

Pro. 2.5.1.

$$\begin{cases}\sigma^2(\frac{1}{2})\otimes I_* u(\vec{p},h)=\frac{1}{2}(\frac{1}{2}+1)u(\vec{p},h)\\ \sigma(\frac{1}{2})\cdot\hat{p}\otimes I_* u(\vec{p},h)=(-1)^{h+1}\frac{1}{2}u(\vec{p},h)\\ \hat{Q}(\vec{p})u(\vec{p},h)=-u(\vec{p},h),h=1,2,\cdot\cdot,l\\ \frac{E+m}{2m}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})[(\frac{l+1}{2})-\sigma_z(\frac{l-1}{2})](1+\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})u(\vec{p},h)=hu(\vec{p},h)\\ \textit{Description electronics: } (s,h;Q)=(\frac{1}{2};h,-1)\end{cases}$$

$$\begin{cases}\sigma^2(\frac{1}{2})\otimes I_* v(\vec{p},h)=\frac{1}{2}(\frac{1}{2}+1)v(\vec{p},h)\\ \sigma(\frac{1}{2})\cdot\hat{p}\otimes I_* v(\vec{p},h)=(-1)^{h+1}\frac{1}{2}v(\vec{p},h)\\ \hat{Q}(\vec{p})v(\vec{p},h)=v(\vec{p},h),h=1,2,\cdot\cdot,l\\ \frac{E+m}{2m}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})[(\frac{l+1}{2})-\sigma_z(\frac{l-1}{2})](1+\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})v(\vec{p},h)=hv(\vec{p},h)\\ \textit{Description positron: } (s,h;Q)=(\frac{1}{2};h,1)\end{cases}$$

2.6 Electron spin space in N+1 dimensional space-time

2.7 Properties of electron spin basis in N+1 dimensional space-time (Also true under general representation)

Cor. 2.7.1. $\bar{u}_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h')=\varsigma\delta_{hh'},\bar{v}_\varsigma(\vec{p},h)v_\varsigma(\vec{p},h')=-\varsigma\delta_{hh'},\bar{u}_\varsigma(\vec{p},h)v_\varsigma(\vec{p},h')=0,\bar{v}_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h')=0$

Cor. 2.7.2.
$u^+_\varsigma(\vec{p},h)u_\varsigma(\vec{p},h')=\frac{E}{m}\delta_{hh'},v^+_\varsigma(\vec{p},h)v_\varsigma(\vec{p},h')=\frac{E}{m}\delta_{hh'},u^+_\varsigma(\vec{p},h)v_\varsigma(-\vec{p},h')=0,v^+_\varsigma(\vec{p},h)u_\varsigma(-\vec{p},h')=0$

Cor. 2.7.3. $\sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)=\frac{\varsigma m-i\gamma^a p_a}{2m},\sum\limits_h v_\varsigma(\vec{p},h)\bar{v}_\varsigma(\vec{p},h)=\frac{-\varsigma m-i\gamma^a p_a}{2m}$

Cor. 2.7.4. $\sum\limits_h u_\varsigma(\vec{p},h)u^+_\varsigma(\vec{p},h)=\frac{(\varsigma m-i\gamma^a p_a)\gamma_0}{2m},\sum\limits_h v_\varsigma(\vec{p},h)v^+_\varsigma(\vec{p},h)=\frac{(-\varsigma m-i\gamma^a p_a)\gamma_0}{2m}$

Cor. 2.7.5. $\begin{cases}\sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)-v_\varsigma(\vec{p},h)\bar{v}_\varsigma(\vec{p},h)]=\varsigma\\ \sum\limits_h u_\varsigma(\vec{p},h)\bar{u}_\varsigma(\vec{p},h)+v_\varsigma(\vec{p},h)\bar{v}_\varsigma(\vec{p},h)]=\frac{-i\gamma^a p_a}{m}\\ \sum\limits_h u_\varsigma(\vec{p},h)u^+_\varsigma(\vec{p},h)+v_\varsigma(-\vec{p},h)v^+_\varsigma(-\vec{p},h)]=\frac{E}{m}\end{cases}$

2.8 Plane wave solutions of electron in N+1 dimensional space-time

Cor. 2.8.1.

$$\psi(\vec{r},t)=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sum\limits_{h=1}^{l}[a_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b^+_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\begin{cases}a_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{E}{m}}u^+_\varsigma(\vec{p},h)\psi(\vec{r},t)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\\ b^+_\varsigma(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{E}{m}}v^+_\varsigma(\vec{p},h)\psi(\vec{r},t)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}d^3\vec{r}\end{cases}$$

2.9 Covariant quantization rules for electron in N+1 dimensional space-time

Cor. 2.9.1. $\begin{cases}\{a_\varsigma(\vec{p},h),a^+_\varsigma(\vec{p}',h')\}=\delta_{hh'}\delta^3(\vec{p}-\vec{p}')\\ \{a_\varsigma(\vec{p},h),a_\varsigma(\vec{p}',h')\}=0\\ \{a^+_\varsigma(\vec{p},h),a^+_\varsigma(\vec{p}',h')\}=0\end{cases}\Rightarrow\begin{cases}\{\psi_\alpha(\vec{r},t),\psi^+_\beta(\vec{r}',t)\}=\delta_{\alpha\beta}\delta^3(\vec{r}-\vec{r}')\\ \{\psi_\alpha(\vec{r},t),\psi_\beta(\vec{r}',t)\}=0\\ \{\psi^+_\alpha(\vec{r},t),\psi^+_\beta(\vec{r}',t)\}=0\end{cases}$

Thm. 2.9.1. $\{\psi(x),\psi^+(x')\}=i(m-\gamma^a\partial_a)\gamma^0\Delta(x-x')$

Proof:

$\{\psi(x),\psi^+(x')\}=\frac{1}{(2\pi)^N}\int\sqrt{\frac{m}{E}}\sqrt{\frac{m}{E'}}\sum\limits_{h,h'=1}^{l}u_\varsigma(\vec{p},h)u^+_\varsigma(\vec{p}',h')e^{i\varsigma(p\cdot x-p'\cdot x')}\{a_\varsigma(\vec{p},h),a^+_\varsigma(\vec{p},h)\}$
$+v_\varsigma(\vec{p},h)v^+_\varsigma(\vec{p}',h')e^{-i\varsigma(p\cdot x-p'\cdot x')}\{b^+_\varsigma(\vec{p},h),b_\varsigma(\vec{p}',h')\}d^N\vec{p}d^N\vec{p}'$
$=\frac{1}{(2\pi)^N}\int\sqrt{\frac{m}{E}}\sqrt{\frac{m}{E'}}\sum\limits_{h,h'=1}^{l}\delta_{hh'}\delta^N(\vec{p}-\vec{p}')[u_\varsigma(\vec{p},h)u^+_\varsigma(\vec{p}',h')e^{i\varsigma(p\cdot x-p'\cdot x')}+v_\varsigma(\vec{p},h)v^+_\varsigma(\vec{p}',h')e^{-i\varsigma(p\cdot x-p'\cdot x')}]d^N\vec{p}d^N\vec{p}'$
$=\frac{1}{(2\pi)^N}\int\frac{m}{E}[\sum\limits_{h=1}^{l}u_\varsigma(\vec{p},h)u^+_\varsigma(\vec{p},h)e^{i\varsigma p\cdot(x-x')}+\sum\limits_{h=1}^{l}v_\varsigma(\vec{p},h)v^+_\varsigma(\vec{p},h)e^{-i\varsigma p\cdot(x-x')}]d^N\vec{p}$
$=\frac{1}{(2\pi)^N}\int\frac{m}{E}[\frac{(\varsigma m-i\gamma^a p_a)\gamma^0}{2m}e^{i\varsigma p\cdot(x-x')}+\frac{(-\varsigma m-i\gamma^a p_a)\gamma^0}{2m}e^{-i\varsigma p\cdot(x-x')}]d^N\vec{p}$

$= \frac{1}{(2\pi)^N}\int \frac{1}{2E}\varsigma(m-\gamma^a\partial_a)\gamma^0[e^{i\varsigma p\cdot(x-x')}-e^{-i\varsigma p\cdot(x-x')}]d^N\vec{p}$
$= i(m-\gamma^a\partial_a)\gamma^0\frac{-i\varsigma}{(2\pi)^N}\int \frac{1}{2E}[e^{i\varsigma p\cdot(x-x')}-e^{-i\varsigma p\cdot(x-x')}]d^N\vec{p}$
$= i(m-\gamma^a\partial_a)\gamma^0\Delta(x-x')$ □

2.10 Conserved charge of Dirac equation in N+1 dimensional space-time

Cor. 2.10.1. $Q=\int\psi^+\psi dr^N=\int\sum_h[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)+b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$

Proof: $Q=\int\psi^+\psi dr^N$
$= \frac{1}{(2\pi)^N}\int\sum_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}+b_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'d^N\vec{p}dr^N$
$= \int\sum_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')+b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^N(\vec{p}-\vec{p}')d^N\vec{p}'d^N\vec{p}$
$= \int\sum_{h,h'}\frac{m}{E}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')+b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]d^N\vec{p}$
$= \int\sum_h[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)+b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$ □

Cor. 2.10.2. $H=i\int\psi^+\partial_t\psi dr^N=\varsigma\int\sum_h E(\vec{p})[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$

Proof: $H=i\int\psi^+\partial_t\psi dr^N$
$= i\frac{1}{(2\pi)^N}\int\sum_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$(-i\varsigma E')[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}-b_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'d^N\vec{p}dr^N$
$= -i\int\sum_{h,h'}\frac{m}{E}(-i\varsigma E')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^N(\vec{p}-\vec{p}')d^N\vec{p}'d^N\vec{p}$
$= -i\int\sum_{h,h'}\frac{m}{E}(-i\varsigma E')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]d^N\vec{p}$
$= \varsigma\int\sum_h E(\vec{p})[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$ □

Cor. 2.10.3. $\vec{P}=-i\int\psi^+\nabla\psi dr^N=\varsigma\int\sum_h\vec{p}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-b_\varsigma(\vec{p},s)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$

Proof: $\vec{P}=-i\int\psi^+\nabla\psi dr^N$
$= -i\frac{1}{(2\pi)^N}\int\sum_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$(i\varsigma\vec{p}')[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}-b_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'d^N\vec{p}dr^N$
$= -i\int\sum_{h,h'}\frac{m}{E}(i\varsigma\vec{p}')[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^N(\vec{p}-\vec{p}')d^N\vec{p}'d^N\vec{p}$
$= -i\int\sum_{h,h'}\frac{m}{E}(i\varsigma\vec{p})[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]d^N\vec{p}$
$= \varsigma\int\sum_h\vec{p}[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$ □

Cor. 2.10.4. $P_u=-i\int\psi^+\partial_u\psi dr^N=\varsigma\int\sum_h p_u[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$

Proof: $P_u=-i\int\psi^+\partial_u\psi dr^N$
$= i\frac{1}{(2\pi)^N}\int\sum_{h,h'}[a_\varsigma^+(\vec{p},h)\sqrt{\frac{m}{E}}u_\varsigma^+(\vec{p},h)e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}+b_\varsigma(\vec{p},h)\sqrt{\frac{m}{E}}v_\varsigma^+(\vec{p},h)e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}]$
$(i\varsigma p'_u)[a_\varsigma(\vec{p}',h')\sqrt{\frac{m}{E'}}u_\varsigma(\vec{p}',h')e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}-b_\varsigma^+(\vec{p}',h')\sqrt{\frac{m}{E'}}v_\varsigma(\vec{p}',h')e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'d^N\vec{p}dr^N$
$= -i\int\sum_{h,h'}\frac{m}{E}(i\varsigma p'_u)[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]\delta^N(\vec{p}-\vec{p}')d^N\vec{p}'d^N\vec{p}$
$= -i\int\sum_{h,h'}\frac{m}{E}(i\varsigma p'_u)[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h')u_\varsigma^+(\vec{p},h)u_\varsigma(\vec{p},h')-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h')v_\varsigma^+(\vec{p},h)v_\varsigma(\vec{p},h')]d^N\vec{p}$
$= \varsigma\int\sum_h p_u[a_\varsigma^+(\vec{p},h)a_\varsigma(\vec{p},h)-b_\varsigma(\vec{p},h)b_\varsigma^+(\vec{p},h)]d^N\vec{p}$ □

3 Covariant quantization of B-W equation in N+1 dimensional space-time

In this section the proof can refer to the methods in 4-dimensional space-time. It is completely possible to translate the proof methods in four dimensional space-time into N+1 dimensional space time. Therefore, it is generally not provided in detail.

3.1 Properties of electron normal spin basis in N+1 dimensional space-time (Also true under general representation)

Def. 3.1.1. $u(\vec{p},h) := \begin{cases} u_+(\vec{p},h), \varsigma = 1 \\ v_-(\vec{p},h), \varsigma = -1 \end{cases}, v(\vec{p},h) := \begin{cases} v_+(\vec{p},h), \varsigma = 1 \\ u_-(\vec{p},h), \varsigma = -1 \end{cases}$

Cor. 3.1.1. $\bar{u}(\vec{p},h)u(\vec{p},h') = \delta_{hh'}, \bar{v}(\vec{p},h)v(\vec{p},h') = -\delta_{hh'}, \bar{u}(\vec{p},h)v(\vec{p},h') = 0, \bar{v}(\vec{p},h)u(\vec{p},h') = 0$

Cor. 3.1.2. $u^+(\vec{p},h)u(\vec{p},h') = \frac{E}{m}\delta_{hh'}, v^+(\vec{p},h)v(\vec{p},h') = \frac{E}{m}\delta_{hh'}, u^+(\vec{p},h)v(-\vec{p},h') = 0, v^+(\vec{p},h)u(-\vec{p},h') = 0$

Cor. 3.1.3. $\sum_h u(\vec{p},h)\bar{u}(\vec{p},h) = \frac{m-i\gamma^a p_a}{2m}, \sum_h v(\vec{p},h)\bar{v}(\vec{p},h) = \frac{-m-i\gamma^a p_a}{2m}$

Cor. 3.1.4. $\sum_h u(\vec{p},h)u^+(\vec{p},h) = \frac{(m-i\gamma^a p_a)\gamma_0}{2m}, \sum_h v(\vec{p},h)v^+(\vec{p},h) = \frac{(-m-i\gamma^a p_a)\gamma_0}{2m}$

Cor. 3.1.5. $\begin{cases} \sum_h u(\vec{p},h)\bar{u}(\vec{p},h) - v(\vec{p},h)\bar{v}(\vec{p},h)] = 1 \\ \sum_h u(\vec{p},h)\bar{u}(\vec{p},h) + v(\vec{p},h)\bar{v}(\vec{p},h)] = \frac{-i\gamma^a p_a}{m} \\ \sum_h u(\vec{p},h)u^+(\vec{p},h) + v(-\vec{p},h)v^+(-\vec{p},h)] = \frac{E}{m} \end{cases}$

3.2 Generalized polynomial theorem for spin basis of Dirac equation in N+1-D

Thm. 3.2.1.

$$\sum_{\sum n_i=2s} \frac{(2s)!}{n_1!n_2!\cdots n_l!} \underbrace{u_{\{\lambda_\varsigma}(\vec{p},1)\cdots}_{n_1}\underbrace{u_{\mu_\varsigma}(\vec{p},2)\cdots}_{n_2}\cdots\underbrace{u_{\tau_\varsigma}(\vec{p},l)\cdots_{\}}}_{n_l}\underbrace{u^+_{(\lambda'_\varsigma}(\vec{p},1)\cdots}_{n_1}\underbrace{u^+_{\mu'_\varsigma}(\vec{p},2)\cdots}_{n_2}\cdots\underbrace{u^+_{\tau'_\varsigma}(\vec{p},l)\cdots_{)}}_{n_l}$$

$$= \overbrace{\sum_{h=1}^{l} u_{\{\lambda_\varsigma}(\vec{p},h)u^+_{(\lambda'_\varsigma}(\vec{p},h)\cdots\sum_{h=1}^{l} u_{\mu_\varsigma}(\vec{p},h)u^+_{\mu'_\varsigma}(\vec{p},h)\cdots\sum_{h=1}^{l} u_{\tau_\varsigma}(\vec{p},h)u^+_{\tau'_\varsigma}(\vec{p},h)\cdots_{\})}}^{2s}$$

Cor. 3.2.1.

$$\sum_{n_1\cdots n_l} \frac{(2s)!}{n_1!n_2!\cdots n_l!} \underbrace{u_{\{i_\varsigma}(\vec{p},1)\cdots}_{n_1}\underbrace{u_{i_\varsigma}(\vec{p},2)\cdots}_{n_2}\cdots\underbrace{u_{i_\varsigma}(\vec{p},l)\cdots_{\}}}_{n_l}\underbrace{u^+_{(i'_\varsigma}(\vec{p},1)\cdots}_{n_1}\underbrace{u^+_{i'_\varsigma}(\vec{p},2)\cdots}_{n_2}\cdots\underbrace{u^+_{i'_\varsigma}(\vec{p},l)\cdots_{)}}_{n_l}$$

$$= \overbrace{\sum_{h=1}^{l} u_{\{i_\varsigma}(\vec{p},h)u^+_{(i'_\varsigma}(\vec{p},h)\cdots\sum_{h=1}^{l} u_{i_\varsigma}(\vec{p},h)u^+_{i'_\varsigma}(\vec{p},h)\cdots\sum_{h=1}^{l} u_{i_\varsigma}(\vec{p},h)u^+_{i'_\varsigma}(\vec{p},h)\cdots_{\})}}^{2s}$$

$\Leftrightarrow$

$$\sum_{n_1\cdots n_l} \frac{(2s)!}{n_1!n_2!\cdots n_l!}[u_{i_\varsigma}(\vec{p},1)u^+_{i'_\varsigma}(\vec{p},1)]^{n_1}[u_{i_\varsigma}(\vec{p},2)u^+_{i'_\varsigma}(\vec{p},2)]^{n_2}\cdots[u_{i_\varsigma}(\vec{p},l)u^+_{i'_\varsigma}(\vec{p},l)]^{n_l} = [\sum_{h=1}^{l} u_{i_\varsigma}(\vec{p},h)u^+_{i'_\varsigma}(\vec{p},h)]^{2s}$$

The above corollary happens to be the polynomial expansion theorem.

3.3 Spin basis for B-W equation in N+1 dimensional space-time

Def. 3.3.1.

$$\begin{cases} U_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdots,n_l) := \frac{1}{\sqrt{(2s)!n_1!n_2!\cdots n_l!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},1)\cdots}_{n_1}\underbrace{u_{\mu_\varsigma}(\vec{p},2)\cdots}_{n_2}\cdots\underbrace{u_{\tau_\varsigma}(\vec{p},l)\cdots_{\}}}_{n_l}, n_1+n_2+\cdots n_l = 2s \\ V_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdots,n_l) := \frac{1}{\sqrt{(2s)!n_1!n_2!\cdots n_l!}}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},1)\cdots}_{n_1}\underbrace{v_{\mu_\varsigma}(\vec{p},2)\cdots}_{n_2}\cdots\underbrace{v_{\tau_\varsigma}(\vec{p},l)\cdots_{\}}}_{n_l}, n_1+n_2+\cdots n_l = 2s \end{cases}$$

3.4 Orthogonal properties of spin basis for B-W equation in N+1 dimensional space-time (Can be seen directly)

Cor. 3.4.1.

$$\begin{cases} \bar{U}^{\overbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdots,n_l)U_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p};n'_1,n'_2,\cdots,n'_l) = \delta_{n_1n'_1}\delta_{n_2n'_2}\cdots\delta_{n_ln'_l} \\ \bar{V}^{\overbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdots,n_l)V_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p};n'_1,n'_2,\cdots,n'_l) = \delta_{n_1n'_1}\delta_{n_2n'_2}\cdots\delta_{n_ln'_l} \end{cases}$$

$$\begin{cases} \bar{U}^{\overbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdots,n_l)V_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p};n'_1,n'_2,\cdots,n'_l) = 0 \\ \bar{V}^{\overbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdots,n_l)U_{\underbrace{\lambda_\varsigma\cdots\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p};n'_1,n'_2,\cdots,n'_l) = 0 \end{cases}$$

Cor. 3.4.2.

$$\begin{cases} U^{+\overbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)=(\frac{E}{m})^{2s}\delta_{n_1n'_1}\delta_{n_2n'_2}\cdot\cdot\delta_{n_ln'_l} \\ V^{+\overbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)=(\frac{E}{m})^{2s}\delta_{n_1n'_1}\delta_{n_2n'_2}\cdot\cdot\delta_{n_ln'_l} \end{cases}$$

$$\begin{cases} U^{+\overbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(-\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)=0 \\ V^{+\overbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}^{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(-\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)=0 \end{cases}$$

3.5 Decomposition of U-spin basis for B-W equation in N+1 dimensional space-time

Thm. 3.5.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$

$$=\sum_{n'_1+\cdot\cdot+n'_l}^{=2s'}\frac{\sqrt{C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_l}^{n'_l}}}{\sqrt{C_{2s}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p};n_1-n'_1,n_2-n'_2,\cdot\cdot,n_l-n'_l)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)$$

Proof: $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_l!}}$

$$\sum_{n'_1+\cdot\cdot+n'_l}^{=2s'}C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_l}^{n'_l}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},1)u_{\mu_\varsigma}(\vec{p},1)\cdot\cdot}_{n_1-n'_1}\underbrace{\cdot\cdot u_{\sigma_\varsigma}(\vec{p},l)u_{\tau_\varsigma\}}(\vec{p},l)}_{n_l-n'_l}\underbrace{u_{(\lambda'_\varsigma}(\vec{p},1)u_{\mu'_\varsigma}(\vec{p},1)\cdot\cdot}_{n'_1}\underbrace{\cdot\cdot u_{\sigma'_\varsigma}(\vec{p},l)u_{\tau'_\varsigma)}(\vec{p},l)}_{n'_l}$$

$$=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_l!}}\sum_{n'_1+\cdot\cdot+n'_l}^{=2s'}C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_l}^{n'_l}$$

$$\sqrt{(2s-2s')!(n_1-n'_1)!(n_2-n'_2)!\cdot\cdot(n_l-n'_l)!}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p};n_1-n'_1,n_2-n'_2,\cdot\cdot,n_l-n'_l)$$

$$\sqrt{(2s')!n'_1!n'_2!\cdot\cdot n'_l!}U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)$$

$$=\sum_{n'_1+\cdot\cdot+n'_l}^{=2s'}\frac{\sqrt{C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_l}^{n'_l}}}{\sqrt{C_{2s}^{2s'}}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p};n_1-n'_1,n_2-n'_2,\cdot\cdot,n_l-n'_l)U_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)$$ □

Cor. 3.5.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};1,0,\cdot\cdot,0)$

$+\frac{\sqrt{n_2}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)U_{\tau_\varsigma}(\vec{p};0,0,\cdot\cdot,1)$

Cor. 3.5.2. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_l)$

$=\frac{\sqrt{n_2}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};0,n_2-1,\cdot\cdot,n_l)U_{\tau_\varsigma}(\vec{p};0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};0,n_2,\cdot\cdot,n_l-1)U_{\tau_\varsigma}(\vec{p};0,0,\cdot\cdot,1)$

3.6 Decomposition of V-spin basis for B-W equation in N+1 dimensional space-time

Thm. 3.6.1. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)$

$$=\sum_{n'_1+\cdot\cdot+n'_l}^{=2s'}\frac{\sqrt{C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_l}^{n'_l}}}{\sqrt{C_{2s}^{2s'}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p};n_1-n'_1,n_2-n'_2,\cdot\cdot,n_l-n'_l)V_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s'}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_l)$$

Proof: $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\sigma'_\varsigma\tau'_\varsigma}_{2s}}(\vec{p},h)=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_l!}}$

$$\sum_{n'_1+\cdot\cdot+n'_l}^{=2s'}C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_l}^{n'_l}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},1)v_{\mu_\varsigma}(\vec{p},1)\cdot\cdot}_{n_1-n'_1}\underbrace{\cdot\cdot v_{\sigma_\varsigma}(\vec{p},l)v_{\tau_\varsigma\}}(\vec{p},l)}_{n_l-n'_l}\underbrace{v_{(\lambda'_\varsigma}(\vec{p},1)v_{\mu'_\varsigma}(\vec{p},1)\cdot\cdot}_{n'_1}\underbrace{\cdot\cdot v_{\sigma'_\varsigma}(\vec{p},l)v_{\tau'_\varsigma)}(\vec{p},l)}_{n'_l}$$

$$=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_l!}}\sum_{n'_1+\cdot\cdot+n'_l}^{=2s'}C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_l}^{n'_l}$$

$$\sqrt{(2s-2s')!(n_1-n_1')!(n_2-n_2')!\cdot\cdot(n_l-n_l')!}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p};n_1-n_1',n_2-n_2',\cdot\cdot,n_l-n_l')$$

$$\sqrt{(2s')!n_1'!n_2'!\cdot\cdot n_l'!}V_{\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\sigma_\varsigma'\tau_\varsigma'}_{2s'}}(\vec{p};n_1',n_2',\cdot\cdot,n_l')$$

$$=\sum_{n_1'+\cdot\cdot+n_l'}^{=2s'}\frac{\sqrt{C_{n_1}^{n_1'}C_{n_2}^{n_2'}\cdot\cdot C_{n_l}^{n_l'}}}{\sqrt{C_{2s}^{2s'}}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2(s-s')}}(\vec{p};n_1-n_1',n_2-n_2',\cdot\cdot,n_l-n_l')V_{\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\sigma_\varsigma'\tau_\varsigma'}_{2s'}}(\vec{p};n_1',n_2',\cdot\cdot,n_l') \qquad \Box$$

Cor. 3.6.1. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)=\frac{\sqrt{n_1}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_l)V_{\tau_\varsigma}(\vec{p};1,0,\cdot\cdot,0)$

$+\frac{\sqrt{n_2}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_l)V_{\tau_\varsigma}(\vec{p};0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_l-1)V_{\tau_\varsigma}(\vec{p};0,0,\cdot\cdot,1)$

Cor. 3.6.2. $V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma\tau_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_l)$

$=\frac{\sqrt{n_2}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};0,n_2-1,\cdot\cdot,n_l)V_{\tau_\varsigma}(\vec{p};0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_l}}{\sqrt{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\sigma_\varsigma}_{2s-1}}(\vec{p};0,n_2,\cdot\cdot,n_l-1)V_{\tau_\varsigma}(\vec{p};0,0,\cdot\cdot,1)$

3.7 Quasi projection operators of B-W equation in N+1 dimensional space-time

Def. 3.7.1.

$$\begin{cases}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},s):=\sum_{n_1\cdot\cdot n_l}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)U^+_{\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\tau_\varsigma'}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)\\ \Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},s):=\sum_{n_1\cdot\cdot n_l}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)U^+_{\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\tau_\varsigma'}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l)\end{cases}$$

Cor. 3.7.1.

$$\begin{cases}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},s)=\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{+\{\lambda_\varsigma(\lambda_\varsigma'}(\vec{p},\tfrac{1}{2})\Lambda_{+\mu_\varsigma\mu_\varsigma'}(\vec{p},\tfrac{1}{2})\cdot\cdot\Lambda_{+\tau_\varsigma\}\tau_\varsigma')}(\vec{p},\tfrac{1}{2})}_{2s}\\ \Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}\underbrace{\lambda_\varsigma'\mu_\varsigma'\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p},s)=\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{-\{\lambda_\varsigma(\lambda_\varsigma'}(\vec{p},\tfrac{1}{2})\Lambda_{-\mu_\varsigma\mu_\varsigma'}(\vec{p},\tfrac{1}{2})\cdot\cdot\Lambda_{-\tau_\varsigma\}\tau_\varsigma')}(\vec{p},\tfrac{1}{2})}_{2s}\end{cases}$$

The above inference can be directly obtained from the generalized polynomial theorem with symmetric indices.

3.8 Conjecture on plane wave solutions for B-W equation in N+1 dimensional space-time [16]

(Proof will be provided in the following chapters.)

Thm. 3.8.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma\}}_{2s}}(\vec{r},t)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{r},t)$$

$$=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{n_1+\cdot\cdot+n_l}^{=2s}\frac{m^s}{\sqrt{E}}[a(\vec{p};n_1,\cdot\cdot,n_l)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{ip\cdot x}+b^+(\vec{p};n_1,\cdot\cdot,n_l)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_l)e^{-ip\cdot x}]d^N\vec{p}$$

$$U_{\underbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l):=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_l!}}\underbrace{u_{\{\lambda_\varsigma}(\vec{p},1)\cdot\cdot}_{n_1}\underbrace{u_{\mu_\varsigma}(\vec{p},2)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{u_{\tau_\varsigma}(\vec{p},l)\cdot\cdot\}}_{n_l},n_1+n_2+\cdot\cdot n_l=2s$$

$$V_{\underbrace{\lambda_\varsigma\cdot\cdot\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_l):=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_l!}}\underbrace{v_{\{\lambda_\varsigma}(\vec{p},1)\cdot\cdot}_{n_1}\underbrace{v_{\mu_\varsigma}(\vec{p},2)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{v_{\tau_\varsigma}(\vec{p},l)\cdot\cdot\}}_{n_l},n_1+n_2+\cdot\cdot n_l=2s$$

3.9 Covariant quantization rules for B-W equation in N+1 dimensional space-time

Def. 3.9.1. $\vec{h}:=(n_1,\cdot\cdot,n_l),\delta_{\vec{h}\vec{h}'}:=\delta_{n_1n_1'}\cdot\cdot\delta_{n_1n_1'}$

Thm. 3.9.1.

$$\begin{cases}[a(\vec{p};\vec{h}),a^+(\vec{p}';\vec{h}')]_{-^{2s+1}}=\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')\\ [b(\vec{p};\vec{h}),b^+(\vec{p}';\vec{h}')]_{-^{2s+1}}=\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')\end{cases}\quad \begin{cases}[a(\vec{p};\vec{h}),a(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\\ [a^+(\vec{p};\vec{h}),a^+(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\\ [b(\vec{p};\vec{h}),b(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\\ [b^+(\vec{p};\vec{h}),b^+(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\end{cases}\quad \begin{cases}[a(\vec{p};\vec{h}),b^+(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\\ [a^+(\vec{p};\vec{h}),b(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\\ [a(\vec{p};\vec{h}),b(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\\ [a^+(\vec{p};\vec{h}),b^+(\vec{p}';\vec{h}')]_{-^{2s+1}}=0\end{cases}$$

$\Rightarrow$

$$\begin{cases} [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x') \\ [\psi^{(+)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x), \psi^{(+)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(+)}(x-x') \\ [\psi^{(-)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x), \psi^{(-)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(-)}(x-x') \end{cases}$$

Proof: $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')] = \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}$

$$[[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{ip\cdot x}+b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-ip\cdot x}, [a^+(\vec{p}',\vec{h}')U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h})e^{-ip'\cdot x'}+b(\vec{p}',\vec{h}')V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')e^{ip'\cdot x'}]$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}$$

$$[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')[a(\vec{p},\vec{h}),a^+(\vec{p}',\vec{h}')]e^{i(p\cdot x-p'\cdot x')}+V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')[b^+(\vec{p},\vec{h}),b(\vec{p}',\vec{h}')]e^{-i(p\cdot x-p'\cdot x')}$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}$$

$$[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')e^{i(p\cdot x-p'\cdot x')}+(-1)^{2s+1}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')e^{-i(p\cdot x-p'\cdot x')}]$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}[\sum_{\vec{h}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{ip\cdot(x-x')}+(-1)^{2s+1}\sum_{\vec{h}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-ip\cdot(x-x')}]$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}[\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)e^{ip\cdot(x-x')}+(-1)^{2s+1}\Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)e^{-ip\cdot(x-x')}]$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}[\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdots}_{2s}e^{ip\cdot(x-x')}$$

$$+(-1)^{2s+1}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\tfrac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\tfrac{1}{2})\cdots}_{2s}e^{-ip\cdot(x-x')}]$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip\cdot(x-x')}$$

$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m+\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m+\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip\cdot(x-x')}\}$$

$$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\frac{-i}{(2\pi)^N}\int d^N\vec{p}\frac{1}{2E}[[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]$$

$$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')$$

$$= \frac{i(2m)^{2s}}{2^{2s-1}}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(-i\partial,s)\Delta(x-x')$$

□

Proof: $[\psi^{(+)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x), \psi^{(+)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{ip\cdot x}, a^+(\vec{p}',\vec{h}')U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h})e^{-ip'\cdot x'}]$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}[U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')[a(\vec{p},\vec{h}),a^+(\vec{p}',\vec{h}')]e^{i(p\cdot x-p'\cdot x')}$$

$$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')e^{i(p\cdot x-p'\cdot x')}$$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}\sum_{\vec{h}} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)e^{ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{+\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{2s}e^{ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\frac{-i}{(2\pi)^N}\int d^N\vec{p}\frac{1}{2E}e^{ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(+)}(x-x')$

$= \frac{i(2m)^{2s}}{2^{2s-1}}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(-i\partial,s)\Delta^{(+)}(x-x')$

□

Proof: $[\psi^{(-)}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^{(-)+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}(EE')^{s-\frac{1}{2}}\sqrt{\frac{m^2}{EE'}}^{2s}[b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-ip\cdot x},b(\vec{p}',\vec{h}')V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')e^{ip'\cdot x'}]$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')[b^+(\vec{p},\vec{h}),b(\vec{p}',\vec{h}')]e^{-i(p\cdot x-p'\cdot x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}d^N\vec{p}'\sum_{\vec{h},\vec{h}'}\sqrt{\frac{m^{2s}}{E}}\sqrt{\frac{m^{2s}}{E'}}(-1)^{2s+1}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p}',\vec{h}')\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')e^{-i(p\cdot x-p'\cdot x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\sum_{\vec{h}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\Lambda_{-\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p},s)e^{-ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\frac{1}{[(2s)!]^2}\underbrace{\Lambda_{-\{\lambda_\varsigma(\lambda'_\varsigma}(\vec{p},\frac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma}(\vec{p},\frac{1}{2})\cdots}_{2s}e^{-ip\cdot(x-x')}$

$= \frac{1}{(2\pi)^N}\int d^N\vec{p}\frac{m^{2s}}{E}(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m+\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m+\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\frac{i}{(2\pi)^N}\int d^N\vec{p}\frac{1}{2E}e^{-ip\cdot(x-x')}$

$= \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta^{(-)}(x-x')$

$= \frac{i(2m)^{2s}}{2^{2s-1}}\Lambda_{+\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(-i\partial,s)\Delta^{(-)}(x-x')$ □

3.10 Reverse reasoning for covariant quantization rules for B-W equation in N+1-D

Thm. 3.10.1. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}} = \frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')$

$\Rightarrow [a(\vec{p},\vec{h}),a^+(\vec{p}',\vec{h}')]_{-^{2s+1}} = \delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}'),[b(\vec{p},\vec{h}),b^+(\vec{p}',\vec{h}')]_{-^{2s+1}} = \delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}'),[rest]_{-^{2s+1}} = 0$

The following has given a detailed proof process for several main commutative brackets.

Proof: $[a(\vec{p},\vec{h}),a^+(\vec{p}',\vec{h}')]_{-^{2s+1}}$

$= \frac{1}{(2\pi)^N}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}e^{-i(p\cdot x-p'\cdot x')}d^N\vec{r}d^N\vec{r}'$

$= \frac{1}{(2\pi)^N}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$

$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')e^{-i(p\cdot x-p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$
$$=\frac{1}{(2\pi)^N}\int d^N\vec{r}d^N\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\{\frac{-i}{(2\pi)^N}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^N\vec{p}_0\}e^{-i(p\cdot x-p'\cdot x')}$$
$$=[\frac{1}{(2\pi)^N}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$
$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip_0\cdot(x-x')}e^{-i(p\cdot x-p'\cdot x')}\}d^N\vec{r}d^N\vec{r}'d^N\vec{p}_0$$
$$=[\frac{1}{(2\pi)^N}]^2\int d^N\vec{r}d^N\vec{r}'d^N\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$
$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}$$
$$+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}\}$$
$$=\int d^N\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$
$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)\delta^N(\vec{p}_0-\vec{p})\delta^N(\vec{p}_0-\vec{p}')$$
$$+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{2iE_0(t-t')}\delta^N(\vec{p}_0+\vec{p})\delta^N(\vec{p}_0+\vec{p}')\}$$
$$=\delta^N(\vec{p}-\vec{p}')(\frac{m}{E})^{4s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})U^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')$$
$$\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p},\vec{h}_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p},\vec{h}_0)+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(-\vec{p},\vec{h}_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(-\vec{p},\vec{h}_0)e^{2iE(t-t')}\}$$
$$=\delta^N(\vec{p}-\vec{p}')(\sum_{\vec{h}_0}\delta_{\vec{h}\vec{h}_0}\delta_{\vec{h}'\vec{h}_0}+0)$$
$$=\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')$$

□

Proof: $[b^+(\vec{p},\vec{h}),b(\vec{p}',\vec{h}')]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^N}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}e^{i(p\cdot x-p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$
$$=\frac{1}{(2\pi)^N}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')e^{i(p\cdot x-p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$
$$=\frac{1}{(2\pi)^N}\int d^N\vec{r}d^N\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\{\frac{-i}{(2\pi)^N}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^N\vec{p}_0\}e^{i(p\cdot x-p'\cdot x')}$$
$$=[\frac{1}{(2\pi)^N}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip_0\cdot(x-x')}e^{i(p\cdot x-p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$
$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip_0\cdot(x-x')}e^{i(p\cdot x-p'\cdot x')}\}d^N\vec{r}d^N\vec{r}'d^N\vec{p}_0$$
$$=[\frac{1}{(2\pi)^N}]^2\int d^N\vec{r}d^N\vec{r}'d^N\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$

$$V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{i(p_0+p)\cdot x}e^{-i(p_0+p')\cdot x'}$$
$$+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{-i(p_0-p)\cdot x}e^{i(p_0-p')\cdot x'}\}$$
$$=\int d^N\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$
$$V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{-2iE_0(t-t')}\delta^N(\vec{p}_0+\vec{p})\delta^N(\vec{p}_0+\vec{p}')$$
$$+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)\delta^N(\vec{p}_0-\vec{p})\delta^N(\vec{p}_0-\vec{p}')\}$$

$$=\delta^N(\vec{p}-\vec{p}')(\frac{m}{E})^{4s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')$$
$$\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(-\vec{p},\vec{h}_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(-\vec{p},\vec{h}_0)e^{-2iE(t-t')}+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p},\vec{h}_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p},\vec{h}_0)\}$$
$$=(-1)^{2s+1}\delta^N(\vec{p}-\vec{p}')(0+\sum_{\vec{h}_0}\delta_{\vec{h}\vec{h}_0}\delta_{\vec{h}'\vec{h}_0})$$
$$=(-1)^{2s+1}\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}')$$

□

Proof: $[a(\vec{p},\vec{h}),b(\vec{p}',\vec{h}')]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^N}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}e^{-i(p\cdot x+p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$

$$=\frac{1}{(2\pi)^N}\int\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')e^{-i(p\cdot x+p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$
$$=\frac{1}{(2\pi)^N}\int d^N\vec{r}d^N\vec{r}'\sqrt{EE'}(\frac{m}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\{\frac{-i}{(2\pi)^N}\int\frac{1}{2E_0}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^N\vec{p}_0\}e^{-i(p\cdot x+p'\cdot x')}$$
$$=[\frac{1}{(2\pi)^N}]^2\int\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$
$$\{\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(m-i\gamma^a p_{0a})\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-i\gamma^b p_{0b})\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{ip_0\cdot(x-x')}e^{-i(p\cdot x+p'\cdot x')}d^N\vec{r}d^N\vec{r}'$$
$$+(-1)^{2s+1}\frac{1}{(2m)^{2s}}\frac{1}{[(2s)!]^2}\overbrace{[(-m-i\gamma^a p_{0a})\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(-m-i\gamma^b p_{0b})\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}e^{-ip_0\cdot(x-x')}e^{-i(p\cdot x+p'\cdot x')}\}d^N\vec{r}d^N\vec{r}'d^N\vec{p}_0$$
$$=[\frac{1}{(2\pi)^N}]^2\int d^N\vec{r}d^N\vec{r}'d^N\vec{p}_0\frac{\sqrt{EE'}}{E_0}(\frac{m^2}{EE'})^{2s}$$
$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{i(p_0-p)\cdot x}e^{-i(p_0+p')\cdot x'}$$
$$+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{-i(p_0+p)\cdot x}e^{i(p_0-p')\cdot x'}\}$$
$$=\int d^N\vec{p}_0\frac{\sqrt{EE'}}{|\vec{p}_0|}(\frac{m^2}{EE'})^{2s}$$
$$U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')\{\sum_{\vec{h}_0}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{-iE_0t'}\delta^N(\vec{p}_0-\vec{p})\delta^N(\vec{p}_0+\vec{p}')$$
$$+(-1)^{2s+1}\sum_{\vec{h}_0}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}_0,\vec{h}_0)e^{iE_0t}\delta^N(\vec{p}_0+\vec{p})\delta^N(\vec{p}_0-\vec{p}')\}$$

$$=\delta^N(\vec{p}+\vec{p}')(\frac{m}{E})^{4s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h})V^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')$$

$$\{\sum_{\vec{h}_0} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p},\vec{h}_0)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p},\vec{h}_0)+(-1)^{2s+1}\sum_{\vec{h}_0} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2s}}(\vec{p}',\vec{h}_0)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2s}}(\vec{p}',\vec{h}_0)e^{2iE(t-t')}\}$$
$$=0+0=0 \qquad \square$$

3.11 Summary of covariant quantization rules for B-W equation in N+1-D
Combining the proofs in the above two sections, the following important theorems are obtained.

Thm. 3.11.1.

$$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=\frac{i}{2^{2s-1}}\frac{1}{[(2s)!]^2}\overbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots\})}^{2s}\Delta(x-x')$$

$$\Leftrightarrow [a(\vec{p},\vec{h}),a^+(\vec{p}',\vec{h}')]_{-^{2s+1}}=\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}'),[b(\vec{p},\vec{h}),b^+(\vec{p}',\vec{h}')]_{-^{2s+1}}=\delta_{\vec{h}\vec{h}'}\delta^N(\vec{p}-\vec{p}'),[rest]_{-^{2s+1}}=0$$

3.12 Various physical operators of B-W equation in N+1 dimensional space-time
Thm. 3.12.1.

$$P_u(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}=\int\sum_{\vec{h}}p_u[a^+(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h})b^+(\vec{p},\vec{h})]d^N\vec{p}$$

Proof: $P_u(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}$

$$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{\vec{h}'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'$$
$$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{\vec{h}}E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{p_uE^{2s-1}}{(E^2)^{2s-1}}[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s}b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}d^N\vec{r}$$
$$=\frac{1}{(2\pi)^N}\int\sum_{\vec{h},\vec{h}'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}[a^+(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$$
$$p_u[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s}b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}'d^N\vec{p}d^N\vec{r}$$
$$=\int d^N\vec{p}'d^N\vec{p}\sum_{\vec{h},\vec{h}'}(\frac{m}{E})^{2s}p_u$$
$$\{\delta^N(\vec{p}-\vec{p}')[a^+(\vec{p},\vec{h}')a(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h}')b^+(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]$$
$$+\delta^N(\vec{p}+\vec{p}')[(-1)^{2s}e^{2iEt}a^+(-\vec{p},\vec{h}')b^+(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})$$
$$+e^{-2iEt}b(\vec{p},\vec{h}')a(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]\}$$
$$=\int\sum_{\vec{h}}p_u[a^+(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h})b^+(\vec{p},\vec{h})]d^N\vec{p} \qquad \square$$

Thm. 3.12.2.

$$Q(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}=\int\sum_{\vec{h}}[a^+(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s-1}b(\vec{p},\vec{h})b^+(\vec{p},\vec{h})]d^N\vec{p}$$

Proof: $Q(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}$

$$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{\vec{h}'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'$$
$$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{\vec{h}}E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{E^{2s-1}}{(E^2)^{2s-1}}[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s-1}b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}d^N\vec{r}$$
$$=\frac{1}{(2\pi)^N}\int\sum_{\vec{h},\vec{h}'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}[a^+(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$$

$[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s-1}b^{+}(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}'d^N\vec{p}d^N\vec{r}$

$=\int d^N\vec{p}'d^N\vec{p}\sum_{\vec{h},\vec{h}'}(\frac{m}{E})^{2s}$

$\{\delta^N(\vec{p}-\vec{p}')[a^{+}(\vec{p},\vec{h}')a(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})+(-1)^{2s-1}b(\vec{p},\vec{h}')b^{+}(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]$

$+\delta^N(\vec{p}+\vec{p}')[(-1)^{2s-1}e^{2iEt}a^{+}(-\vec{p},\vec{h}')b^{+}(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})$

$+e^{-2iEt}b(\vec{p},\vec{h}')a(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]\}$

$=\int\sum_{\vec{h}}[a^{+}(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s-1}b(\vec{p},\vec{h})b^{+}(\vec{p},\vec{h})]d^N\vec{p}$ □

Thm. 3.12.3.

$N(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}=\int\sum_{\vec{h}}[a^{+}(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h})b^{+}(\vec{p},\vec{h})]d^N\vec{p}$

Proof: $N(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}$

$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{\vec{h}'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^{+}(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'$

$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{\vec{h}}E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{E^{2s}}{E^{4s-1}}[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s}b^{+}(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}d^N\vec{r}$

$=\frac{1}{(2\pi)^N}\int\sum_{\vec{h},\vec{h}'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}[a^{+}(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$

$[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s}b^{+}(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}'d^N\vec{p}d^N\vec{r}$

$=\int d^N\vec{p}'d^N\vec{p}\sum_{\vec{h},\vec{h}'}(\frac{m}{E})^{2s}$

$\{\delta^N(\vec{p}-\vec{p}')[a^{+}(\vec{p},\vec{h}')a(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h}')b^{+}(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]$

$+\delta^N(\vec{p}+\vec{p}')[(-1)^{2s}e^{2iEt}a^{+}(-\vec{p},\vec{h}')b^{+}(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})$

$+e^{-2iEt}b(\vec{p},\vec{h}')a(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]\}$

$=\int\sum_{\vec{h}}[a^{+}(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h})b^{+}(\vec{p},\vec{h})]d^N\vec{p}$ □

Thm. 3.12.4.

$\vec{S}(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}=\int\sum_{\vec{h}}\hat{p}[a^{+}(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h})b^{+}(\vec{p},\vec{h})]d^N\vec{p}$

Proof: $\vec{S}(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}$

$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{\vec{h}'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^{+}(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'$

$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{\vec{h}}E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{\hat{p}E^{2s-1}}{(E^2)^{2s-1}}[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s}b^{+}(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}d^N\vec{r}$

$=\frac{1}{(2\pi)^N}\int\sum_{\vec{h},\vec{h}'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}\hat{p}[a^{+}(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$

$$[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s}b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}'d^N\vec{p}d^N\vec{r}$$
$$=\int d^N\vec{p}'d^N\vec{p}\sum_{\vec{h},\vec{h}'}(\frac{m}{E})^{2s}\hat{p}$$
$$\{\delta^N(\vec{p}-\vec{p}')[a^+(\vec{p},\vec{h}')a(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h}')b^+(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]$$
$$+\delta^N(\vec{p}+\vec{p}')[(-1)^{2s}e^{2iEt}a^+(-\vec{p},\vec{h}')b^+(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})$$
$$+e^{-2iEt}b(\vec{p},\vec{h}')a(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]\}$$
$$=\int\sum_{\vec{h}}\hat{p}[a^+(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s}b(\vec{p},\vec{h})b^+(\vec{p},\vec{h})]d^N\vec{p}\qquad\square$$

Thm. 3.12.5.

$$\vec{M}(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}=\int\sum_{\vec{h}}\hat{p}[a^+(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s-1}b(\vec{p},\vec{h})b^+(\vec{p},\vec{h})]d^N\vec{p}$$

Proof: $\vec{M}(s)=\int\psi^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2-\nabla^2})^{4s-1}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)d^N\vec{r}$

$$=\int\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}'=-\infty}^{+\infty}\sum_{\vec{h}'}E'^{s-\frac{1}{2}}\sqrt{\frac{m}{E'}}^{2s}[a^+(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]d^N\vec{p}'$$
$$\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\sum_{\vec{h}}E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}\frac{\hat{p}E^{2s}}{E^{4s-1}}[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s-1}b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}d^N\vec{r}$$
$$=\frac{1}{(2\pi)^N}\int\sum_{\vec{h},\vec{h}'}(\frac{E'}{E})^{s-\frac{1}{2}}\sqrt{\frac{m^2}{E'E}}^{2s}\hat{p}[a^+(\vec{p}',\vec{h}')U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-E't)}+b(\vec{p}',\vec{h}')V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p}',\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$$
$$[a(\vec{p},\vec{h})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{i(\vec{p}\cdot\vec{r}-Et)}+(-1)^{2s-1}b^+(\vec{p},\vec{h})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}'d^N\vec{p}d^N\vec{r}$$
$$=\int d^N\vec{p}'d^N\vec{p}\sum_{\vec{h},\vec{h}'}(\frac{m}{E})^{2s}\hat{p}$$
$$\{\delta^N(\vec{p}-\vec{p}')[a^+(\vec{p},\vec{h}')a(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})+(-1)^{2s-1}b(\vec{p},\vec{h}')b^+(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]$$
$$+\delta^N(\vec{p}+\vec{p}')[(-1)^{2s-1}e^{2iEt}a^+(-\vec{p},\vec{h}')b^+(\vec{p},\vec{h})U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})$$
$$+e^{-2iEt}b(\vec{p},\vec{h}')a(\vec{p},\vec{h})V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(-\vec{p},\vec{h}')U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p},\vec{h})]\}$$
$$=\int\sum_{\vec{h}}\hat{p}[a^+(\vec{p},\vec{h})a(\vec{p},\vec{h})+(-1)^{2s-1}b(\vec{p},\vec{h})b^+(\vec{p},\vec{h})]d^N\vec{p}\qquad\square$$

3.13 Conjecture on potential commutation rules for B-W equation in N+1-D (Probably wrong?)

Ass. 3.13.1.

$$\begin{cases}[A_a(x),A^+_{a'}(x')]=i(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')\\ [A_{ab}(x),A^+_{a'b'}(x')]=\frac{i}{8}\{-\frac{1}{N}[\delta_{\{ab\}}-\frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{(a'b')}-\frac{\partial^+_{(a'}\partial^+_{b')}}{m^2}]+[\eta_{\{a(a'}-\frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{b\}b')}-\frac{\partial_{b\}}\partial^+_{b')}}{m^2}]\}\Delta(x-x')\\ [A_{abc}(x),A^+_{a'b'c'}(x')]\\ ?=\frac{i}{144}\{-\frac{1}{N}[\delta_{\{ab}-\frac{\partial_{\{a}\partial_b}{m^2}][\delta_{(a'b'}-\frac{\partial^+_{(a'}\partial^+_{b'}}{m^2}]+[\eta_{\{a(a'}-\frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{bb'}-\frac{\partial_b\partial^+_{b'}}{m^2}]\}[\eta_{c\}c')}-\frac{\partial_{c\}}\partial^+_{c')}}{m^2}]\Delta(x-x')\end{cases}$$

Ass. 3.13.2. $[A_{\underbrace{abc\cdots}_{n}}(x),A^+_{\underbrace{a'b'c'\cdots}_{n}}(x')]$

$$?=\frac{i}{2^{n-1}(n!)^2}\{-\frac{1}{N}[\delta_{\{ab}-\frac{\partial_{\{a}\partial_b}{m^2}][\delta_{(a'b'}-\frac{\partial^+_{(a'}\partial^+_{b'}}{m^2}]+[\eta_{\{a(a'}-\frac{\partial_{\{a}\partial^+_{(a'}}{m^2}][\eta_{bb'}-\frac{\partial_b\partial^+_{b'}}{m^2}]\}\underbrace{[\eta_{cc'}-\frac{\partial_c\partial^+_{c'}}{m^2}]\cdots_{\})}}_{n-2}\Delta(x-x')$$

Chapter36 Covariate Quantization of Massless Particles in High Dimension

Self comment: In this chapter, the covariant quantization of massless particles is generalized to the general N+1 dimensional space-time. Specially, N+1 dimensional space-time case is different from four dimensional space-time case for fully symmetric Penrose equation. It is generally possible to describe both charged complex particles and uncharged Majorana particles. The principal commutation rule in both cases is consistent, but the rest are generally zero for charged complex particles. For uncharged Majorana particles, the rest of the commutative or anti commutative brackets are naturally derived from the principal commutative rule and Majorana conditions. And they are generally not zero. In this chapter, we only discuss the case of complex particles and generally only give the principal commutation rule. The Majorana particle case is no longer specifically discussed. If we want to obtain the quantum field theory of the Majorana particle case, we only need to add the Majorana condition to the complex particle case. Then we will naturally obtain it.

1 Helicity eigenfunction in N+1 dimensional space-time

1.1 Electron equation under separated representation in even N+1=2n-D space-time [4]

Def. 1.1.1. $(\gamma^a\partial_a + m)\psi = 0, \psi = \begin{bmatrix}\varphi \\ \eta\end{bmatrix}, \gamma^a = (\Gamma\otimes\sigma_y, \varsigma I_*\otimes\sigma_x) \Leftrightarrow \begin{cases}(\Gamma, -i\varsigma)^a\partial_a\varphi = im\eta \\ (\Gamma, i\varsigma)^a\partial_a\eta = -im\varphi\end{cases}$

1.2 Neutrino equation in N+1=2n even dimensional space-time [5]

When the mass m=0, it degenerates into two Weyl neutrino equations:

Cor. 1.2.1. $(\Gamma, -i\varsigma)^a\partial_a\varphi = 0, (\Gamma, i\varsigma)^a\partial_a\eta = 0$

1.3 Helicity eigenfunction along motion direction in N+1=2n even dimensional space-time

Def. 1.3.1.

$$\begin{cases}(I_*\otimes\sigma_z)\lambda(\hat{p},\frac{1}{2}) = \frac{1}{2}\lambda(\vec{p},\frac{1}{2}), (I_*\otimes\sigma_z)\lambda(\hat{p},-\frac{1}{2}) = -\frac{1}{2}\lambda(\vec{p},-\frac{1}{2}), l = 2^{[\frac{N-1}{2}]} = 2^{n-1} \\ \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};1) = \begin{bmatrix}1\\ 0\\ 0_{l/2-2}\\ 0_{l/2}\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};2) = \begin{bmatrix}0\\ 1\\ 0_{l/2-2}\\ 0_{l/2}\end{bmatrix}, \cdot\cdot, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};\frac{l}{2}-1) = \begin{bmatrix}0_{l/2-2}\\ 1\\ 0\\ 0_{l/2}\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};\frac{l}{2}) = \begin{bmatrix}0_{l/2-2}\\ 0\\ 1\\ 0_{l/2}\end{bmatrix} \\ \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};1) = \begin{bmatrix}0_{l/2}\\ 0_{l/2-2}\\ 0\\ 1\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};2) = \begin{bmatrix}0_{l/2}\\ 0_{l/2-2}\\ 1\\ 0\end{bmatrix}, \cdot\cdot, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};\frac{l}{2}-1) = \begin{bmatrix}0_{l/2}\\ 0\\ 1\\ 0_{l/2-2}\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};\frac{l}{2}) = \begin{bmatrix}0_{l/2}\\ 1\\ 0\\ 0_{l/2-2}\end{bmatrix}\end{cases}$$

1.4 Helicity eigenfunction along motion direction in N+1=2n-1 odd dimensional space-time

Def. 1.4.1.

$$\begin{cases}(I_*\otimes\sigma_z)\lambda(\hat{p},\frac{1}{2}) = \frac{1}{2}\lambda(\vec{p},\frac{1}{2}), (I_*\otimes\sigma_z)\lambda(\hat{p},-\frac{1}{2}) = -\frac{1}{2}\lambda(\vec{p},-\frac{1}{2}), l = 2^{[\frac{N-1}{2}]} = 2^{n-2} \\ \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};1) = \begin{bmatrix}1\\ 0\\ 0_{l-2}\\ 0_{l}\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};2) = \begin{bmatrix}0\\ 1\\ 0_{l-2}\\ 0_{l}\end{bmatrix}, \cdot\cdot, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};l-1) = \begin{bmatrix}0_{l-2}\\ 1\\ 0\\ 0_{l}\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{1}{2};l) = \begin{bmatrix}0_{l-2}\\ 0\\ 1\\ 0_{l}\end{bmatrix} \\ \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};1) = \begin{bmatrix}0_{l}\\ 0_{l-2}\\ 0\\ 1\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};2) = \begin{bmatrix}0_{l}\\ 0_{l-2}\\ 1\\ 0\end{bmatrix}, \cdot\cdot, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};l-1) = \begin{bmatrix}0_{l}\\ 0\\ 1\\ 0_{l-2}\end{bmatrix}, \lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},\frac{-1}{2};l) = \begin{bmatrix}0_{l}\\ 1\\ 0\\ 0_{l-2}\end{bmatrix}\end{cases}$$

Def. 1.4.2. $(\Gamma\cdot\hat{p})\lambda(\hat{p},\frac{1}{2};h) = \lambda(\hat{p},\frac{1}{2};h), (\Gamma\cdot\hat{p})\lambda(\hat{p},-\frac{1}{2};h) = -\lambda(\hat{p},-\frac{1}{2};h)$

1.5 Helicity $\Gamma\cdot\hat{p}$ eigenfunction in N+1 dimensional space-time

Def. 1.5.1. $r := \begin{cases}l/2, \textit{even dimensional space-time} \\ l, \textit{odd dimensional space-time}\end{cases}, l = 2^{[\frac{N-1}{2}]}$

Def. 1.5.2.

$$\begin{cases}(\Gamma\cdot\hat{p})\lambda(\hat{p},-\frac{\varsigma}{2};h) = -\varsigma\lambda(\hat{p},-\frac{\varsigma}{2};h), \lambda(\hat{p},-\frac{\varsigma}{2};h) := e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}\lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},-\frac{\varsigma}{2};h) \\ e^{-\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}(\Gamma\cdot\hat{p})e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]} = \Gamma_N, \Gamma\cdot\hat{p} = e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}\Gamma_N e^{-\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}\end{cases}$$

Def. 1.5.3. $(\Gamma\cdot\hat{p})\lambda(\hat{p},\frac{1}{2};h) = \lambda(\hat{p},\frac{1}{2};h), (\Gamma\cdot\hat{p})\lambda(\hat{p},-\frac{1}{2};h) = -\lambda(\hat{p},-\frac{1}{2};h)$

Def. 1.5.4.

$$\begin{cases}(\Gamma\cdot\hat{p})\lambda(\hat{p},-\frac{\varsigma}{2};h)=-\varsigma\lambda(\hat{p},-\frac{\varsigma}{2};h),\lambda(\hat{p},-\frac{\varsigma}{2};h):=e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}\lambda(\begin{bmatrix}0\\ \cdot\cdot\\ 0\\ 1\end{bmatrix},-\frac{\varsigma}{2};h)\\ e^{-\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}(\Gamma\cdot\hat{p})e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}=\Gamma_N,\Gamma\cdot\hat{p}=e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}\Gamma_N e^{-\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i,\Gamma_j]}\end{cases}$$

1.6 Orthogonality and completeness of helicity $\Gamma\cdot\hat{p}$ eigenfunction in N+1-D

Def. 1.6.1. $\lambda(\hat{p},\frac{1}{2};h):=\lambda(\hat{p};h),\lambda(\hat{p},-\frac{1}{2};h):=\lambda(\hat{p};-h),\lambda(\hat{p},\frac{\varsigma}{2};h):=\lambda(\hat{p};h\varsigma),\lambda(\hat{p},-\frac{\varsigma}{2};h):=\lambda(\hat{p};-h\varsigma)$

Cor. 1.6.1. $\lambda^+(\hat{p};h)\lambda(\hat{p};h')=\delta_{hh'}$

$$\sum_{h=1}^{r}[\lambda(\hat{p};h)\lambda^+(\hat{p};h)+\lambda(\hat{p};-h)\lambda^+(\hat{p};-h)]=1,\sum_{h=1}^{r}[\lambda(\hat{p};h)\lambda^+(\hat{p};h)-\lambda(\hat{p};-h)\lambda^+(\hat{p};-h)]=\Gamma\cdot\hat{p}$$

Cor. 1.6.2. $\sum_{h=1}^{r}\lambda(\hat{p};h)\lambda^+(\hat{p};h)=\frac{1}{2}(\Gamma,-i)^a\hat{p}_a,\sum_{h=1}^{r}\lambda(\hat{p};-h)\lambda^+(\hat{p};-h)=-\frac{1}{2}(\Gamma,-i)^a\hat{p}_a$

Cor. 1.6.3. $\sum_{h=1}^{r}\lambda(\hat{p};h\varsigma)\lambda^+(\hat{p};h\varsigma)=-\frac{\varsigma}{2}(\Gamma,i\varsigma)^a\hat{p}_a$

1.7 High spin helicity eigenfunction in N+1 dimensional space-time

Def. 1.7.1.

$$\begin{cases}\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma\cdot\cdot C_\varsigma\cdot\cdot}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_{2r}):=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_r!}}\underbrace{\lambda_{\{A_\varsigma}(\vec{p};-1\varsigma)\cdot\cdot}_{n_1}\underbrace{\lambda_{B_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{\lambda_{C_\varsigma}(\vec{p};-2r\varsigma)\cdot\cdot_{\}}}_{n_{2r}}\\ \lambda_{k_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_{2r}):=\frac{\sqrt{(2s)!}}{\sqrt{n_1!n_2!\cdot\cdot n_{2r}!}}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s;w)\underbrace{\lambda_{A_\varsigma}(\vec{p};-1\varsigma)\cdot\cdot}_{n_1}\underbrace{\lambda_{B_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{\lambda_{C_\varsigma}(\vec{p};-2r\varsigma)\cdot\cdot}_{n_{2r}}\\ \lambda_{A_\varsigma}(\vec{p};-(r+k)\varsigma):=\lambda_{A_\varsigma}(\vec{p};k\varsigma),k=1,2,\cdot\cdot,r,\sum_{k=1}^{2r}n_k=2s\end{cases}$$

Cor. 1.7.1.

$$\begin{cases}[\sigma(s,w)\cdot\hat{p}]\lambda_\varsigma(\vec{p};n_1,n_2,\cdot\cdot,n_{2r})=-\frac{\varsigma}{2}[\sum_{k=1}^{r}(n_k-n_{r+k})]\lambda_\varsigma(\vec{p};n_1,n_2,\cdot\cdot,n_{2r})\\ \sigma(s;w)=s\bar{\Gamma}(s;w)(\Gamma\otimes I_*)\Gamma(s;w),\sum_{k=1}^{2r}n_k=2s\end{cases}$$

Cor. 1.7.2.

$$\begin{cases}[\sigma(s,w)\cdot\hat{p}]\lambda_\varsigma(\vec{p};n_1,n_2,\cdot\cdot,n_r)=-\varsigma s\lambda_\varsigma(\vec{p};n_1,n_2,\cdot\cdot,n_r),\sum_{k=1}^{r}n_k=2s\\ \lambda_{k_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r):=\lambda_{k_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r;0_1,0_2,\cdot\cdot,0_r)\end{cases}$$

$$\begin{cases}[\sigma(s,w)\cdot\hat{p}]\lambda_\varsigma(\vec{p};n_{r+1},n_{r+2},\cdot\cdot,n_{2r})=\varsigma s\lambda_\varsigma(\vec{p};n_{r+1},n_{r+2},\cdot\cdot,n_{2r}),\sum_{k=r+1}^{2r}n_k=2s\\ \lambda_{k_\varsigma}(\vec{p};n_{r+1},n_{r+2},\cdot\cdot,n_{2r}):=\lambda_{k_\varsigma}(\vec{p};0_1,0_2,\cdot\cdot,0_r;n_{r+1},n_{r+2},\cdot\cdot,n_{2r})\end{cases}$$

2 Spin basis and basic properties for Penrose fully symmetric equation in N+1-D

2.1 Generalized polynomial theorem of spin basis for Penrose equation in N+1-D

Thm. 2.1.1.

$$\sum_{n_1+\cdot\cdot+n_r}^{=2s}\frac{(2s)!}{n_1!n_2!\cdot\cdot n_r!}\underbrace{\lambda_{\{A_\varsigma}(\vec{p};-1\varsigma)\cdot\cdot}_{n_1}\underbrace{\lambda_{B_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{\lambda_{C_\varsigma}(\vec{p};-r\varsigma)\cdot\cdot_{\}}}_{n_r}\underbrace{\lambda^+_{(A'_\varsigma}(\vec{p};-1\varsigma)\cdot\cdot}_{n_1}\underbrace{\lambda^+_{B'_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{\lambda^+_{C'_\varsigma}(\vec{p};-r\varsigma)\cdot\cdot_{)}}_{n_r}$$

$$=\overbrace{[\sum_{h=1}^{r}\lambda_{\{A_\varsigma}(\vec{p};-h\varsigma)\lambda^+_{(A'_\varsigma}(\vec{p};-h\varsigma)]\cdot\cdot[\sum_{h=1}^{r}\lambda_{B_\varsigma}(\vec{p};-h\varsigma)\lambda^+_{B'_\varsigma}(\vec{p};-h\varsigma)]\cdot\cdot[\sum_{h=1}^{r}\lambda_{C_\varsigma}(\vec{p};-h\varsigma)\lambda^+_{C'_\varsigma}(\vec{p};-h\varsigma)]\cdot\cdot_{\})}}^{2s}$$

Cor. 2.1.1.

$$\sum_{n_1+\cdot\cdot+n_r}^{=2s}\frac{(2s)!}{n_1!n_2!\cdot\cdot n_r!}[\sum_{h=1}^{r}\lambda_{A_\varsigma}(\vec{p};-1\varsigma)\lambda^+_{A'_\varsigma}(\vec{p};-1\varsigma)]^{n_1}[\sum_{h=1}^{r}\lambda_{A_\varsigma}(\vec{p};-2\varsigma)\lambda^+_{A'_\varsigma}(\vec{p};-2\varsigma)]^{n_2}\cdot\cdot[\sum_{h=1}^{r}\lambda_{A_\varsigma}(\vec{p};-r\varsigma)\lambda^+_{A'_\varsigma}(\vec{p};-r\varsigma)]^{n_r}$$

$$=[\sum_{h=1}^{r}\lambda_{A_\varsigma}(\vec{p};-h\varsigma)\lambda^+_{A'_\varsigma}(\vec{p};-h\varsigma)]^{2s}$$

The above corollary is just the polynomial expansion theorem.

2.2 Spin basis for Penrose fully symmetric equationin in N+1 dimensional space-time

Def. 2.2.1.

$$\begin{cases}[\sigma(\tfrac{1}{2},w)\cdot\hat{p}]\lambda(\vec{p};n_1,n_2,\cdot\cdot,n_r)=-\tfrac{\varsigma}{2}\lambda(\vec{p};n_1,n_2,\cdot\cdot,n_r),n_1+n_2+\cdot\cdot n_r=2s\\ \lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma\cdot\cdot C_\varsigma\cdot\cdot}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r):=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_r!}}\underbrace{\lambda_{\{A_\varsigma}(\vec{p};-1\varsigma)\cdot\cdot}_{n_1}\underbrace{\lambda_{B_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{\lambda_{C_\varsigma}(\vec{p};-r\varsigma)\cdot\cdot_{\}}}_{n_r}\end{cases}$$

Def. 2.2.2.

$$\begin{cases}[\sigma(s,w)\cdot\hat{p}]\lambda(\vec{p};-s\varsigma;n_1,n_2,\cdot\cdot,n_r)=-s\varsigma\lambda(\vec{p};-s\varsigma;n_1,n_2,\cdot\cdot,n_r),n_1+n_2+\cdot\cdot n_r=2s\\ \lambda_{k_\varsigma}(\vec{p};-s\varsigma;n_1,n_2,\cdot\cdot,n_r):=\frac{\sqrt{(2s)!}}{\sqrt{n_1!n_2!\cdot\cdot n_r!}}\Gamma_{k_\varsigma}^{\overbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}^{2s}}(s;w)\underbrace{\lambda_{A_\varsigma}(\vec{p};-1\varsigma)\cdot\cdot}_{n_1}\underbrace{\lambda_{B_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n_2}\cdot\cdot\underbrace{\lambda_{C_\varsigma}(\vec{p};-r\varsigma)\cdot\cdot}_{n_r}\end{cases}$$

2.3 Orthogonal properties of spin basis for Penrose fully symmetric equationin in N+1-D

Cor. 2.3.1. $\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma\cdot\cdot C_\varsigma\cdot\cdot}^{2s}}(\vec{p},-s\varsigma;n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma\cdot\cdot C_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s\varsigma;n'_1,n'_2,\cdot\cdot,n'_r)=\delta_{n_1n'_1}\delta_{n_2n'_2}\cdot\cdot\delta_{n_rn'_r}$

Cor. 2.3.2. $\lambda^{+\overbrace{A_\varsigma\cdot\cdot B_\varsigma\cdot\cdot C_\varsigma\cdot\cdot}^{2s}}(-\vec{p},-s\varsigma;n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma\cdot\cdot B_\varsigma\cdot\cdot C_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s\varsigma;n'_1,n'_2,\cdot\cdot,n'_r)=0$

Cor. 2.3.3. $\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(\vec{p},-s\varsigma;\vec{h})\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s\varsigma;\vec{h}')=\delta_{\vec{h}\vec{h}'},\vec{h}:=(n_1,n_2,\cdot\cdot,n_r)$

Cor. 2.3.4. $\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(-\vec{p},-s\varsigma;\vec{h})\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s\varsigma;\vec{h}')=0$

2.4 Decomposition of spin basis for Penrose fully symmetric equation in N+1-D

Thm. 2.4.1. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma A'_\varsigma B'_\varsigma\cdot\cdot C'_\varsigma D'_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)$

$$=\sum_{n'_1+\cdot\cdot+n'_r}^{=2s'}\frac{\sqrt{C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_r}^{n'_r}}}{\sqrt{C_{2s}^{2s'}}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2(s-s')}}(\vec{p};n_1-n'_1,n_2-n'_2,\cdot\cdot,n_r-n'_r)\lambda_{\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot C'_\varsigma D'_\varsigma}_{2s'}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_r)$$

Proof: $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma A'_\varsigma B'_\varsigma\cdot\cdot C'_\varsigma D'_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_r!}}\sum_{n'_1+\cdot\cdot+n'_r}^{=2s'}C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_r}^{n'_r}$

$$\underbrace{\lambda_{\{A_\varsigma}(\vec{p};-1\varsigma)\lambda_{B_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n_1-n'_1}\cdot\underbrace{\lambda_{C_\varsigma}(\vec{p};-r\varsigma)\lambda_{D_\varsigma\}}(\vec{p};-r\varsigma)}_{n_r-n'_r}\underbrace{\lambda_{(A'_\varsigma}(\vec{p};-1\varsigma)\lambda_{B'_\varsigma}(\vec{p};-2\varsigma)\cdot\cdot}_{n'_1}\cdot\underbrace{\lambda_{C'_\varsigma}(\vec{p};-r\varsigma)\lambda_{D'_\varsigma)}(\vec{p};-r\varsigma)}_{n'_r}$$

$$=\frac{1}{\sqrt{(2s)!n_1!n_2!\cdot\cdot n_r!}}\sum_{n'_1+\cdot\cdot+n'_r}^{=2s'}C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_r}^{n'_r}$$

$$\sqrt{(2s-2s')!(n_1-n'_1)!(n_2-n'_2)!\cdot\cdot(n_r-n'_r)!}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2(s-s')}}(\vec{p};n_1-n'_1,n_2-n'_2,\cdot\cdot,n_r-n'_r)$$

$$\sqrt{(2s')!n'_1!n'_2!\cdot\cdot n'_r!}\lambda_{\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot C'_\varsigma D'_\varsigma}_{2s'}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_r)$$

$$=\sum_{n'_1+\cdot\cdot+n'_r}^{=2s'}\frac{\sqrt{C_{n_1}^{n'_1}C_{n_2}^{n'_2}\cdot\cdot C_{n_r}^{n'_r}}}{\sqrt{C_{2s}^{2s'}}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2(s-s')}}(\vec{p};n_1-n'_1,n_2-n'_2,\cdot\cdot,n_r-n'_r)\lambda_{\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot C'_\varsigma D'_\varsigma}_{2s'}}(\vec{p};n'_1,n'_2,\cdot\cdot,n'_r)$$ □

Cor. 2.4.1. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{\sqrt{n_1}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};1,0,\cdot\cdot,0)$

$$+\frac{\sqrt{n_2}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_r}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r-1)\lambda_{D_\varsigma}(\vec{p};0,0,\cdot\cdot,1)$$

Cor. 2.4.2. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_r)$

$$=\frac{\sqrt{n_2}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};0,n_2-1,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_r}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};0,n_2,\cdot\cdot,n_r-1)\lambda_{D_\varsigma}(\vec{p};0,0,\cdot\cdot,1)$$

2.5 Projection operator of Penrose fully symmetric equation in N+1 dimensional space-time

Cor. 2.5.1. $\sum_{\vec{h}} \lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p},-s\varsigma;\vec{h})\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(\vec{p},-s\varsigma;\vec{h}) = (-\frac{\varsigma}{2})^{2s}\frac{1}{[(2s)!]^2}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\hat{p}_a\hat{p}_b\cdots}^{2s}$

Lem. 2.5.1. $\begin{cases}(\Gamma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}p_b \neq (\Gamma,i\varsigma)^a_{B_\varsigma A'_\varsigma}p_a(\Gamma,i\varsigma)^b_{A_\varsigma B'_\varsigma}p_b, p^ap_a=0\\(\Gamma,i\varsigma)^a_{A_\varsigma A'_\varsigma}p_a(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}p_b \neq (\Gamma,i\varsigma)^a_{A_\varsigma B'_\varsigma}p_a(\Gamma,i\varsigma)^b_{B_\varsigma A'_\varsigma}p_b, p^ap_a=0\end{cases}$

Lem. 2.5.2. $\begin{cases}(\Gamma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b \neq (\Gamma,i\varsigma)^a_{B_\varsigma A'_\varsigma}\partial_a(\Gamma,i\varsigma)^b_{A_\varsigma B'_\varsigma}\partial_b, \partial^a\partial_a=0\\(\Gamma,i\varsigma)^a_{A_\varsigma A'_\varsigma}\partial_a(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\partial_b \neq (\Gamma,i\varsigma)^a_{A_\varsigma B'_\varsigma}\partial_a(\Gamma,i\varsigma)^b_{B_\varsigma A'_\varsigma}\partial_b, \partial^a\partial_a=0\end{cases}$

Direct verification can prove the above two lemmas.

Cor. 2.5.2. $\sum_{\vec{h}} \lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p},-s\varsigma;\vec{h})\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(\vec{p},-s\varsigma;\vec{h}) \neq (-\frac{\varsigma}{2})^{2s}\overbrace{(\Gamma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots}^{2s}\overbrace{\hat{p}_a\hat{p}_b\cdots}^{2s}$

3 Covariant quantization of Penrose fully symmetric equation in N+1-D

3.1 Conjecture on commutative rules for Penrose fully symmetric equation [1,2] in N+1-D

Cor. 3.1.1. $(\Gamma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(x)=0$

$$\begin{cases}\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}\neq 0}\sum\limits_{\vec{h}}|\vec{p}|^{(s-\frac{1}{2})}\lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p},-s\varsigma;\vec{h})[a_1(\vec{p},-s\varsigma;\vec{h})e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma;\vec{h})e^{-ip\cdot x}]d^N\vec{p}\\ \vec{p}|^{(s-\frac{1}{2})}a_1(\vec{p},-s\varsigma;\vec{h})=\frac{1}{(2\pi)^{N/2}}\int\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec{p},-s\varsigma;\vec{h})\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(x)e^{-ip\cdot x}d^N\vec{r}\\ |\vec{p}|^{(s-\frac{1}{2})}a_2^+(\vec{p},-s\varsigma;\vec{h})=\frac{1}{(2\pi)^{N/2}}\int\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec{p},-s\varsigma;\vec{h})\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(x)e^{ip\cdot x}d^N\vec{r}\end{cases}$$

Cor. 3.1.2. $(\Gamma,-i\varsigma)^{A'_\varsigma A_\varsigma}_a\partial^a\psi_{A_\varsigma}(x)=0$

$$\begin{cases}\psi_{A_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}\neq 0}\sum\limits_{h=1}^{l/4}\lambda_{A_\varsigma}(\hat{p};-h\varsigma)[a_1(\vec{p},-s\varsigma;\vec{h})e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma;\vec{h})e^{-ip\cdot x}]d^N\vec{p}\\ a_1(\vec{p},-s\varsigma;\vec{h})=\frac{1}{(2\pi)^{N/2}}\int\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2};\vec{h})\psi_{A_\varsigma}(x)e^{-ip\cdot x}d^N\vec{r}\\ a_2^+(\vec{p},-s\varsigma;\vec{h})=\frac{1}{(2\pi)^{N/2}}\int\lambda^{+A_\varsigma}(\hat{p},-\frac{\varsigma}{2};\vec{h})\psi_{A_\varsigma}(x)e^{ip\cdot x}d^N\vec{r}\end{cases}$$

Ass. 3.1.1. $[\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x-x')$

3.2 Recursive relations between spin bases for Penrose fully symmetric equation in N+1-D

3.2.1 Penrose spin basis lemma on symmetry conditions

Lem. 3.2.1.

$$\sum_{n_1+\cdots+n_r}^{=2s}a_{E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)=\sum_{n_1+\cdots+n_r}^{=2s}a_{D_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma E_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\end{cases}$$

Proof:

$$\sum_{n_1+\cdots+n_r}^{=2s}a_{E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)=\sum_{n_1+\cdots+n_r}^{=2s}a_{D_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma E_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)$$

$$\Leftrightarrow\sum_{n_1+\cdots+n_r}^{=2s}a_{E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)[\frac{\sqrt{n_1}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};1)$$

$$+\frac{\sqrt{n_2}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};2)+\cdots+\frac{\sqrt{n_r}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r-1)\lambda_{D_\varsigma}(\vec{p};r)]$$

$$=\sum_{n_1+\cdots+n_r}^{=2s}a_{D_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)[\frac{\sqrt{n_1}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_r)\lambda_{E_\varsigma}(\vec{p};1)$$

$+\frac{\sqrt{n_2}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdots,n_r)\lambda_{E_\varsigma}(\vec{p};2)+\cdots+\frac{\sqrt{n_r}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdots,n_r-1)\lambda_{E_\varsigma}(\vec{p};r)]$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1-1,n_2+1,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1-1,n_2,\cdots,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1+1,n_2-1,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2-1,\cdots,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\cdots\cdots\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1+1,n_2,\cdots,n_r-1)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2+1,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};r)=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1-1,n_2+1,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1-1,n_2,\cdots,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1+1,n_2,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2+1,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\end{cases}\qquad\square$$

Lem. 3.2.2.

$$\begin{cases}a_{E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdots,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)+\sum_{k=1}^{r}d(\vec{p};n_1,n_2,\cdots,n_r;k)\lambda_{E_\varsigma}(\vec{p};-k)\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1+1,n_2,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2+1,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\end{cases}$$

$\Leftrightarrow$

$$a_{E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdots,n_r;k)\lambda_{E_\varsigma}(\vec{p};k),d(\vec{p};n_1,n_2,\cdots,n_r;k)=0$$

$$\begin{cases}a_{E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)=\frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdots,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdots,n_r;1)\lambda_{E_\varsigma}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdots,n_r+1;1)\lambda_{E_\varsigma}(\vec{p};r),n_1\geq1\end{cases}$$

$$\begin{cases}a_{E_\varsigma}(\vec{p};0,n_2,\cdots,n_r)=c(\vec{p};0,n_2,\cdots,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)\\+\frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p};0,n_2,\cdots,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdots,n_r;2)\lambda_{E_\varsigma}(\vec{p};3)+\cdots\\+\frac{\sqrt{n_r+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdots,n_r+1;2)\lambda_{E_\varsigma}(\vec{p};r),n_2\geq1\end{cases}$$

$$\begin{cases}a_{E_\varsigma}(\vec{p};0,0,n_3,\cdots,n_r)=c(\vec{p};0,0,n_3,\cdots,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)+c(\vec{p};0,0,n_3,\cdots,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)\\+\frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p};0,0,n_3,n_4,\cdots,n_r;3)\lambda_{E_\varsigma}(\vec{p};3)+\frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,n_4+1,\cdots,n_r;3)\lambda_{E_\varsigma}(\vec{p};4)+\cdots\\+\frac{\sqrt{n_r+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdots,n_r+1;3)\lambda_{E_\varsigma}(\vec{p};r),n_3\geq1\end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases}a_{E_\varsigma}(\vec{p};0,\cdots,0,n_r)=c(\vec{p};0,\cdots,0,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)+c(\vec{p};0,\cdots,0,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)\\+\cdots+c(\vec{p};0,\cdots,0,n_r;r-1)\lambda_{E_\varsigma}(\vec{p};r-1)+\frac{\sqrt{n_r}}{\sqrt{n_r}}c(\vec{p};0,\cdots,0,n_r;r)\lambda_{E_\varsigma}(\vec{p};r),n_r=2s\geq1\end{cases}$$

Proof:

$$\begin{cases}a_{E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdots,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)+\sum_{k=1}^{r}d(\vec{p};n_1,n_2,\cdots,n_r;k)\lambda_{E_\varsigma}(\vec{p};-k)\\\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1+1,n_2,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2+1,\cdots,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\end{cases}$$

$\Leftrightarrow$

$$a_{E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdots,n_r;k)\lambda_{E_\varsigma}(\vec{p};k),d(\vec{p};n_1,n_2,\cdots,n_r;k)=0$$

$$\begin{cases}c(\vec{p};n_1+1,n_2,\cdots,n_r;2)=\frac{\sqrt{n_2+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2+1,\cdots,n_r;1)\cdots\\c(\vec{p};n_1+1,n_2,\cdots,n_r;r)=\frac{\sqrt{n_r+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdots,n_r+1;1)\end{cases}$$

$$\begin{cases}c(\vec{p};0,n_2+1,\cdots,n_r;3)=\frac{\sqrt{n_3+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3+1,\cdots,n_r;2)\cdots\\c(\vec{p};0,n_2+1,\cdots,n_r;r)=\frac{\sqrt{n_r+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,\cdots,n_r+1;2)\end{cases}$$

$$\begin{cases}c(\vec{p};0,0,n_3+1,\cdots,n_r;4)=\frac{\sqrt{n_4+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,n_4+1,\cdots,n_r;3)\cdots\\c(\vec{p};0,0,n_3+1,\cdots,n_r;r)=\frac{\sqrt{n_r+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,n_4,\cdots,n_r+1;3)\end{cases}$$

$\cdots$

$$\begin{cases}c(\vec{p};0,\cdot,0,n_{r-1}+1,n_r;r)=\frac{\sqrt{n_r+1}}{\sqrt{n_{r-1}+1}}c(\vec{p};0,\cdots,0,n_{r-1},n_r+1;r-1)\end{cases}$$

$\Leftrightarrow$

$$a_{E_\varsigma}(\vec{p};n_1,n_2,\cdots,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdots,n_r;k)\lambda_{E_\varsigma}(\vec{p};k),d(\vec{p};n_1,n_2,\cdots,n_r;k)=0$$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; n_1, n_2, \cdot\cdot, n_r) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 2) \\ + \cdots + \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_r+1; 1)\lambda_{E_\varsigma}(\vec{p}; r), n_1 \geq 1 \end{cases}$$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; 0, n_2, \cdot\cdot, n_r) = c(\vec{p}; 0, n_2, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}} c(\vec{p}; 0, n_2, \cdot\cdot, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3+1, \cdot\cdot, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 3) + \cdots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \cdot\cdot, n_r+1; 2)\lambda_{E_\varsigma}(\vec{p}; r), n_2 \geq 1 \end{cases}$$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; 0, 0, n_3, \cdot\cdot, n_r) = c(\vec{p}; 0, 0, n_3, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \cdot\cdot, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3, n_4, \cdot\cdot, n_r; 3)\lambda_{E_\varsigma}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, n_4+1, \cdot\cdot, n_r; 3)\lambda_{E_\varsigma}(\vec{p}; 4) + \cdots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, \cdot\cdot, n_r+1; 3)\lambda_{E_\varsigma}(\vec{p}; r), n_3 \geq 1 \end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; 0, \cdot\cdot, 0, n_r) = c(\vec{p}; 0, \cdot\cdot, 0, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) + c(\vec{p}; 0, \cdot\cdot, 0, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 2) \\ + \cdots + c(\vec{p}; 0, \cdot\cdot, 0, n_r; r-1)\lambda_{E_\varsigma}(\vec{p}; r-1) + \frac{\sqrt{n_r}}{\sqrt{n_r}} c(\vec{p}; 0, \cdot\cdot, 0, n_r; r)\lambda_{E_\varsigma}(\vec{p}; r), n_r = 2s \geq 1 \end{cases}$$ □

Cor. 3.2.1. $a_{E_\varsigma}(\vec{p}; n_1, n_2, \cdot\cdot, n_r) = \sum_{k=1}^{r} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; k)\lambda_{E_\varsigma}(\vec{p}; k)$

$$\begin{cases} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_r; 1), n_1 \geq 1 \cdots \\ c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_r+1; 1), n_1 \geq 1 \end{cases}$$

$$\begin{cases} c(\vec{p}; 0, n_2, \cdot\cdot, n_r; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3+1, \cdot\cdot, n_r; 2), n_2 \geq 1 \cdots \\ c(\vec{p}; 0, n_2, \cdot\cdot, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \cdot\cdot, n_r+1; 2), n_2 \geq 1 \end{cases}$$

$\cdots$

$$\begin{cases} c(\vec{p}; 0, \cdot\cdot, 0, n_{r-1}, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_{r-1}}} c(\vec{p}; 0, \cdot\cdot, 0, n_{r-1}-1, n_r+1; r), n_{r-1} \geq 1 \end{cases}$$

$\Leftrightarrow$

$$a_{E_\varsigma}(\vec{p}; n_1, n_2, \cdot\cdot, n_r) = \sum_{k=1}^{r} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; k)\lambda_{E_\varsigma}(\vec{p}; k)$$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; n_1, n_2, \cdot\cdot, n_r) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 2) \\ + \cdots + \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_r+1; 1)\lambda_{E_\varsigma}(\vec{p}; r), n_1 \geq 1 \end{cases}$$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; 0, n_2, \cdot\cdot, n_r) = c(\vec{p}; 0, n_2, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}} c(\vec{p}; 0, n_2, \cdot\cdot, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3+1, \cdot\cdot, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 3) + \cdots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \cdot\cdot, n_r+1; 2)\lambda_{E_\varsigma}(\vec{p}; r), n_2 \geq 1 \end{cases}$$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; 0, 0, n_3, \cdot\cdot, n_r) = c(\vec{p}; 0, 0, n_3, \cdot\cdot, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \cdot\cdot, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3, n_4, \cdot\cdot, n_r; 3)\lambda_{E_\varsigma}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, n_4+1, \cdot\cdot, n_r; 3)\lambda_{E_\varsigma}(\vec{p}; 4) + \cdots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, \cdot\cdot, n_r+1; 3)\lambda_{E_\varsigma}(\vec{p}; r), n_3 \geq 1 \end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases} a_{E_\varsigma}(\vec{p}; 0, \cdot\cdot, 0, n_r) = c(\vec{p}; 0, \cdot\cdot, 0, n_r; 1)\lambda_{E_\varsigma}(\vec{p}; 1) + c(\vec{p}; 0, \cdot\cdot, 0, n_r; 2)\lambda_{E_\varsigma}(\vec{p}; 2) \\ + \cdots + c(\vec{p}; 0, \cdot\cdot, 0, n_r; r-1)\lambda_{E_\varsigma}(\vec{p}; r-1) + \frac{\sqrt{n_r}}{\sqrt{n_r}} c(\vec{p}; 0, \cdot\cdot, 0, n_r; r)\lambda_{E_\varsigma}(\vec{p}; r), n_r = 2s \geq 1 \end{cases}$$

Lem. 3.2.3.

$$\begin{cases} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_r; 1), n_1 \geq 1 \cdots \\ c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_r+1; 1), n_1 \geq 1 \end{cases}$$

$\Leftrightarrow$

$$\sum_{n_1 \cdot\cdot + n_r = 2s} \sum_{k=1}^{r} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; k)\lambda_{E_\varsigma}(\vec{p}; k)\lambda_{\underbrace{A_\varsigma B_\varsigma \cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p}; n_1, \cdot\cdot, n_r)$$

$$= \sum_{n_1 \cdot\cdot + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; 1)\lambda_{\underbrace{A_\varsigma B_\varsigma \cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p}; n_1+1, n_2, \cdot\cdot, n_r)$$

$$+ \sum_{n_2 \cdot\cdot + n_r = 2s} \sum_{k=2}^{r} c(\vec{p}; 0, n_2, \cdot\cdot, n_r; k)\lambda_{E_\varsigma}(\vec{p}; k)\lambda_{\underbrace{A_\varsigma B_\varsigma \cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p}; 0, n_2, \cdot\cdot, n_r)$$

Proof:

$$\begin{cases} c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \cdot\cdot, n_r; 1), n_1 \geq 1 \cdots \\ c(\vec{p}; n_1, n_2, \cdot\cdot, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \cdot\cdot, n_r+1; 1), n_1 \geq 1 \end{cases}$$

$\Leftrightarrow$

$$\sum_{n_1\cdot\cdot+n_r=2s}\sum_{k=1}^{r} c(\vec{p};n_1,n_2,\cdot\cdot,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)$$
$$=\sum_{n_1+\cdot\cdot+n_r=2s}^{n_1\neq 0}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)[\tfrac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)$$
$$+\tfrac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};2)+\cdot\cdot+\tfrac{\sqrt{n_r+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r+1;1)\lambda_{E_\varsigma}(\vec{p};r)]$$
$$+\sum_{n_1+\cdot\cdot+n_r=2s}^{n_1=0}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_r)[c(\vec{p};0,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)$$
$$+c(\vec{p};0,n_2,\cdot\cdot,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)+\cdot\cdot+c(\vec{p};0,n_2,\cdot\cdot,n_r;r)\lambda_{E_\varsigma}(\vec{p};r)]$$
$$=\sum_{n_1\cdot\cdot+n_r=2s}^{1\le n_1\le 2s}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)$$
$$+\sum_{n_1\cdot\cdot+n_r=2s}^{0\le n_1\le 2s-1,1\le n_2\le 2s}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1+1,n_2-1,\cdot\cdot,n_r)\tfrac{\sqrt{n_2}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};2)+\cdot\cdot$$
$$+\sum_{n_1\cdot\cdot+n_r=2s}^{0\le n_1\le 2s-1,1\le n_r\le 2s}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r-1)\tfrac{\sqrt{n_r+1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};r)$$
$$+\sum_{n_1\cdot\cdot+n_r=2s}^{n_1=0}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_r)[c(\vec{p};0,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)$$
$$+c(\vec{p};0,n_2,\cdot\cdot,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)+\cdot\cdot+c(\vec{p};0,n_2,\cdot\cdot,n_r;r)\lambda_{E_\varsigma}(\vec{p};r)]$$
$$=\sum_{n_1\cdot\cdot+n_r=2s}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)$$
$$+\sum_{n_1\cdot\cdot+n_r=2s}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1+1,n_2-1,\cdot\cdot,n_r)\tfrac{\sqrt{n_2}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};2)+\cdot\cdot$$
$$+\sum_{n_1\cdot\cdot+n_r=2s}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r-1)\tfrac{\sqrt{n_r}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};r)$$
$$+\sum_{n_1\cdot\cdot+n_r=2s}^{n_1=0}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_r)[c(\vec{p};0,n_2,\cdot\cdot,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)+\cdot\cdot+c(\vec{p};0,n_2,\cdot\cdot,n_r;r)\lambda_{E_\varsigma}(\vec{p};r)]$$
$$=\sum_{n_1\cdot\cdot+n_r=2s}\tfrac{\sqrt{2s+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r)$$
$$+\sum_{n_2\cdot\cdot+n_r=2s}\sum_{k=2}^{r}c(\vec{p};0,n_2,\cdot\cdot,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_r)\qquad\square$$

Cor. 3.2.2.

$$\begin{cases}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;2)=\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_r;1),n_1\ge 1\cdots\\ c(\vec{p};n_1,n_2,\cdot\cdot,n_r;r)=\frac{\sqrt{n_r+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r+1;1),n_1\ge 1\end{cases}$$
$$\begin{cases}c(\vec{p};0,n_2,\cdot\cdot,n_r;3)=\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_r;2),n_2\ge 1\cdots\\ c(\vec{p};0,n_2,\cdot\cdot,n_r;r)=\frac{\sqrt{n_r+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_r+1;2),n_2\ge 1\end{cases}$$
$$\cdots$$
$$\begin{cases}c(\vec{p};0,\cdot\cdot,0,n_{r-1},n_r;r)=\frac{\sqrt{n_r+1}}{\sqrt{n_{r-1}}}c(\vec{p};0,\cdot\cdot,0,n_{r-1}-1,n_r+1;r),n_{r-1}\ge 1\end{cases}$$
$$\Leftrightarrow$$
$$\sum_{n_1\cdot\cdot+n_r=2s}\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)$$
$$=\sum_{n_1+\cdot\cdot+n_r=2s}\tfrac{\sqrt{2s+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r)$$
$$+\sum_{n_2+\cdot\cdot+n_r=2s}\tfrac{\sqrt{2s+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3,\cdot\cdot,n_r;2)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,n_2+1,n_3,\cdot\cdot,n_r)$$
$$+\sum_{n_3+\cdot\cdot+n_r=2s}\tfrac{\sqrt{2s+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,\cdot\cdot,n_r;3)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,0,n_3+1,\cdot\cdot,n_r)$$
$$+\cdots+\sum_{n_r=2s}\tfrac{\sqrt{2s+1}}{\sqrt{n_r+1}}c(\vec{p};0,\cdot\cdot,0,n_r;r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_r+1)$$

Lem. 3.2.4.

$$\sum_{n_1+\cdot\cdot+n_r=2s}\tfrac{\sqrt{2s+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r)$$

$$+\sum_{n_2+\cdot\cdot+n_r=2s}\frac{\sqrt{2s+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3,\cdot\cdot,n_r;2)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,n_2+1,n_3,\cdot\cdot,n_r)$$
$$+\sum_{n_3+\cdot\cdot+n_r=2s}\frac{\sqrt{2s+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,\cdot\cdot,n_r;3)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,0,n_3+1,\cdot\cdot,n_r)$$
$$+\cdots+\sum_{n_r=2s}\frac{\sqrt{2s+1}}{\sqrt{n_r+1}}c(\vec{p};0,\cdot\cdot,0,n_r;r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_r+1)$$
$$=\sum_{n_1+\cdot\cdot+n_r=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)$$
$$\begin{cases}a(\vec{p};n_1,n_2,\cdot\cdot,n_r):=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r;1),n_1\neq 0\\ a(\vec{p};0,n_2,\cdot\cdot,n_r):=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_r;2),n_2\neq 0\\ a(\vec{p};0,0,n_3,\cdot\cdot,n_r):=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_r;3),n_3\neq 0\\ \cdots\\ a(\vec{p};0,0,\cdot\cdot,0,n_r):=\frac{\sqrt{2s+1}}{\sqrt{n_r}}c(\vec{p};0,\cdot\cdot,0,n_r-1;r),n_r\neq 0\end{cases}$$

Proof: $\sum_{n_1\cdot\cdot+n_r=2s}\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)$

$$=\sum_{n_1+\cdot\cdot+n_r=2s}\frac{\sqrt{2s+1}}{\sqrt{n_1+1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r)$$
$$+\sum_{n_2+\cdot\cdot+n_r=2s}\frac{\sqrt{2s+1}}{\sqrt{n_2+1}}c(\vec{p};0,n_2,n_3,\cdot\cdot,n_r;2)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,n_2+1,n_3,\cdot\cdot,n_r)$$
$$+\sum_{n_3+\cdot\cdot+n_r=2s}\frac{\sqrt{2s+1}}{\sqrt{n_3+1}}c(\vec{p};0,0,n_3,\cdot\cdot,n_r;3)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,0,n_3+1,\cdot\cdot,n_r)$$
$$+\cdots+\sum_{n_r=2s}\frac{\sqrt{2s+1}}{\sqrt{n_r+1}}c(\vec{p};0,\cdot\cdot,0,n_r;r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_r+1)$$
$$=\sum_{n_1+\cdot\cdot+n_r=2s+1}^{n_1\neq 0}\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r;1)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)$$
$$+\sum_{n_2+\cdot\cdot+n_r=2s+1}^{n_1=0,n_2\neq 0}\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_r;2)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,n_2,n_3,\cdot\cdot,n_r)$$
$$+\sum_{n_3+\cdot\cdot+n_r=2s+1}^{n_1=0,n_2=0,n_3\neq 0}\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_r;3)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,0,n_3,\cdot\cdot,n_r)$$
$$+\cdots+\sum_{n_r=2s+1}^{n_1=0,\cdot\cdot,n_{r-1}=0,n_r\neq 0}\frac{\sqrt{2s+1}}{\sqrt{n_r}}c(\vec{p};0,\cdot\cdot,0,n_r-1;r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};0,\cdot\cdot,0,n_r)$$
$$=\sum_{n_1+\cdot\cdot+n_r=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)$$
$$\begin{cases}a(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r;1),n_1\neq 0\\ a(\vec{p};0,n_2,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_r;2),n_2\neq 0\\ a(\vec{p};0,0,n_3,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_r;3),n_3\neq 0\\ \cdots\\ a(\vec{p};0,0,\cdot\cdot,0,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_r}}c(\vec{p};0,\cdot\cdot,0,n_r-1;r),n_r\neq 0\end{cases}\qquad\square$$

3.2.2 Several corollaries

Cor. 3.2.3. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{\sqrt{n_1}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};1)$
$+\frac{\sqrt{n_2}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};2)+\cdot\cdot+\frac{\sqrt{n_r}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r-1)\lambda_{D_\varsigma}(\vec{p};r)$

Cor. 3.2.4. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};0,n_2,\cdot\cdot,n_r)=\frac{\sqrt{n_2}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};0,n_2-1,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};2)$
$+\frac{\sqrt{n_3}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};0,n_2,n_3-1,\cdot\cdot,n_r)\lambda_{D_\varsigma}(\vec{p};0,0,1,\cdot\cdot,0)+\cdot\cdot+\frac{\sqrt{n_r}}{\sqrt{2s}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma}_{2s-1}}(\vec{p};0,n_2,\cdot\cdot,n_r-1)\lambda_{D_\varsigma}(\vec{p};r)$

Cor. 3.2.5. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{\sqrt{n_1}}{\sqrt{2s+1}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1-1,n_2,\cdot\cdot,n_r)\lambda_{E_\varsigma}(\vec{p};1)$

$+\frac{\sqrt{n_2}}{\sqrt{2s+1}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,n_2-1,\cdot\cdot,n_r)\lambda_{E_\varsigma}(\vec{p};2)+\cdots+\frac{\sqrt{n_r}}{\sqrt{2s+1}}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,n_2,\cdot\cdot,n_r-1)\lambda_{E_\varsigma}(\vec{p};r)$

3.2.3 An important theorem

Thm. 3.2.1.

$$\sum_{n_1+\cdot\cdot+n_r=2s}a_{E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)=\sum_{n_1+\cdot\cdot+n_r=2s}a_{D_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma E_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)$$

$\Leftrightarrow$

$$a_{E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)$$

$$\sum_{n_1\cdot\cdot+n_r}^{=2s}a_{E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)=\sum_{n_1+\cdot\cdot+n_r}^{=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)$$

$$\begin{cases}a(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r;1),n_1\neq0\\a(\vec{p};0,n_2,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_r;2),n_2\neq0\\a(\vec{p};0,0,n_3,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_r;3),n_3\neq0\\\cdots\\a(\vec{p};0,0,\cdot\cdot,0,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_r}}c(\vec{p};0,\cdot\cdot,0,n_r-1;r),n_r\neq0\end{cases}$$

Proof:

$$\sum_{n_1+\cdot\cdot+n_r=2s}a_{E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)=\sum_{n_1+\cdot\cdot+n_r=2s}a_{D_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma E_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)$$

$\Leftrightarrow$

$$\begin{cases}\frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1+1,n_2,\cdot\cdot,n_r)\lambda_{D_\varsigma]}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2+1,\cdot\cdot,n_r)\lambda_{D_\varsigma]}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r+1)\lambda_{D_\varsigma]}(\vec{p};r)=0\end{cases}$$

$\Leftrightarrow$

$$a_{E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)$$

$$\begin{cases}a_{E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)+\frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2+1,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};2)\\+\cdots+\frac{\sqrt{n_r+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r+1;1)\lambda_{E_\varsigma}(\vec{p};r),n_1\geq1\end{cases}$$

$$\begin{cases}a_{E_\varsigma}(\vec{p};0,n_2,\cdot\cdot,n_r)=c(\vec{p};0,n_2,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)\\+\frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p};0,n_2,\cdot\cdot,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)+\frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3+1,\cdot\cdot,n_r;2)\lambda_{E_\varsigma}(\vec{p};3)+\cdots\\+\frac{\sqrt{n_r+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_r+1;2)\lambda_{E_\varsigma}(\vec{p};r),n_2\geq1\end{cases}$$

$$\begin{cases}a_{E_\varsigma}(\vec{p};0,0,n_3,\cdot\cdot,n_r)=c(\vec{p};0,0,n_3,\cdot\cdot,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)+c(\vec{p};0,0,n_3,\cdot\cdot,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)\\+\frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p};0,0,n_3,n_4,\cdot\cdot,n_r;3)\lambda_{E_\varsigma}(\vec{p};3)+\frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,n_4+1,\cdot\cdot,n_r;3)\lambda_{E_\varsigma}(\vec{p};4)+\cdots\\+\frac{\sqrt{n_r+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_r+1;3)\lambda_{E_\varsigma}(\vec{p};r),n_3\geq1\end{cases}$$

$\cdots\cdots\cdots$

$$\begin{cases}a_{E_\varsigma}(\vec{p};0,\cdot\cdot,0,n_r)=c(\vec{p};0,\cdot\cdot,0,n_r;1)\lambda_{E_\varsigma}(\vec{p};1)+c(\vec{p};0,\cdot\cdot,0,n_r;2)\lambda_{E_\varsigma}(\vec{p};2)\\+\cdots+c(\vec{p};0,\cdot\cdot,0,n_r;r-1)\lambda_{E_\varsigma}(\vec{p};r-1)+\frac{\sqrt{n_r}}{\sqrt{n_r}}c(\vec{p};0,\cdot\cdot,0,n_r;r)\lambda_{E_\varsigma}(\vec{p};r),n_r=2s\geq1\end{cases}$$

$\Leftrightarrow$

$$a_{E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\sum_{k=1}^{r}c(\vec{p};n_1,n_2,\cdot\cdot,n_r;k)\lambda_{E_\varsigma}(\vec{p};k)$$

$$\sum_{n_1\cdot\cdot+n_r}^{=2s}a_{E_\varsigma}(\vec{p};n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)=\sum_{n_1+\cdot\cdot+n_r}^{=2s+1}a(\vec{p};n_1,n_2,\cdot\cdot,n_r)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1,n_2,\cdot\cdot,n_r)$$

$$\begin{cases}a(\vec{p};n_1,n_2,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p};n_1-1,n_2,\cdot\cdot,n_r;1),n_1\neq0\\a(\vec{p};0,n_2,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p};0,n_2-1,n_3,\cdot\cdot,n_r;2),n_2\neq0\\a(\vec{p};0,0,n_3,\cdot\cdot,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p};0,0,n_3-1,\cdot\cdot,n_r;3),n_3\neq0\\\cdots\\a(\vec{p};0,0,\cdot\cdot,0,n_r)=\frac{\sqrt{2s+1}}{\sqrt{n_r}}c(\vec{p};0,\cdot\cdot,0,n_r-1;r),n_r\neq0\end{cases}$$ □

3.3 Use mathematical induction to solve plane wave solutions of Penrose equation in N+1-D

Thm. 3.3.1. $(\Gamma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x)=0,\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x)=\frac{1}{(2s)!}\psi_{\underbrace{\{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma\}}_{2s}}(x)$

$\Leftrightarrow\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x)$

$$= \frac{1}{(2\pi)^{N/2}} \int\limits_{\vec{p}\neq 0} d^N\vec{p}|\vec{p}|^{(s-\frac{1}{2})} \sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)[a_1(\vec{p};n_1,\cdot\cdot,n_r)e^{ip\cdot x} + a_2^+(\vec{p};n_1,\cdot\cdot,n_r)e^{-ip\cdot x}]$$

$$\begin{cases} |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p};n_1,\cdot\cdot,n_r) = \frac{1}{(2\pi)^{N/2}} \int \lambda^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}^{2s}}(\vec{p};n_1,\cdot\cdot,n_r)\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x)e^{-ip\cdot x}d^N\vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p};n_1,\cdot\cdot,n_r) = \frac{1}{(2\pi)^{N/2}} \int \lambda^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}^{2s}}(\vec{p};n_1,\cdot\cdot,n_r)\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x)e^{ip\cdot x}d^N\vec{r} \end{cases}$$

Proof: Using mathematical induction to prove this theorem.
Step 1: When $s' = \frac{1}{2}$, the following is established.

$$(\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a \psi_{A_\varsigma}(x) = 0, \psi_{A_\varsigma}(x) = \psi_{A_\varsigma}(x)$$

$\Leftrightarrow$

$$\psi_{A_\varsigma}(x) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\vec{p}\neq 0} d^N\vec{p} \sum_{n_1+\cdot\cdot+n_r}^{=1} \lambda_{A_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)[a_1(\vec{p};n_1,\cdot\cdot,n_r)e^{ip\cdot x} + a_2^+(\vec{p};n_1,\cdot\cdot,n_r)e^{-ip\cdot x}]$$

Step 2: Assume when $s' = s$, the following is established.

$$(\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x) = 0, \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x) = \frac{1}{(2s)!}\psi_{\underbrace{\{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma\}}_{2s}}(x)$$

$\Leftrightarrow$

$$\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\vec{p}\neq 0} d^N\vec{p}|\vec{p}|^{(s-\frac{1}{2})}$$

$$\sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)[a_1(\vec{p};n_1,\cdot\cdot,n_r)e^{ip\cdot x} + a_2^+(\vec{p};n_1,\cdot\cdot,n_r)e^{-ip\cdot x}]$$

Step 3: When $s' = s + 1/2$,

$$(\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(x) = 0, \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(x) = \frac{1}{(2s+1)!}\psi_{\underbrace{\{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma\}}_{2s+1}}(x)$$

$\Leftrightarrow$

$$\begin{cases} \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma\}}_{2s+1}}(x) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\vec{p}\neq 0} d^N\vec{p}|\vec{p}|^{(s-\frac{1}{2})} \\ \sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)[a_{1E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)e^{ip\cdot x} + a_{2E_\varsigma}^+(\vec{p};n_1,\cdot\cdot,n_r)e^{-ip\cdot x}] \\ \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(x) = \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma E_\varsigma D_\varsigma}_{2s+1}}(x) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma\}}_{2s+1}}(x) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\vec{p}\neq 0} d^N\vec{p}|\vec{p}|^{(s-\frac{1}{2})} \\ \sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)[a_{1E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r)e^{ip\cdot x} + a_{2E_\varsigma}^+(\vec{p};n_1,\cdot\cdot,n_r)e^{-ip\cdot x}] \\ \sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)a_{1E_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r) = \sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma E_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)a_{1D_\varsigma}(\vec{p};n_1,\cdot\cdot,n_r) \\ \sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)a_{2E_\varsigma}^+(\vec{p};n_1,\cdot\cdot,n_r) = \sum_{n_1+\cdot\cdot+n_r}^{=2s} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma E_\varsigma}_{2s}}(\vec{p};n_1,\cdot\cdot,n_r)a_{2D_\varsigma}^+(\vec{p};n_1,\cdot\cdot,n_r) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(x) \\ = \frac{1}{(2\pi)^{N/2}} \int\limits_{\vec{p}\neq 0} d^N\vec{p}|\vec{p}|^{s} \sum_{n_1+\cdot\cdot+n_r}^{=2s+1} \lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot C_\varsigma D_\varsigma E_\varsigma}_{2s+1}}(\vec{p};n_1,\cdot\cdot,n_r)[a_1(\vec{p};n_1,\cdot\cdot,n_r)e^{ip\cdot x} + a_2^+(\vec{p};n_1,\cdot\cdot,n_r)e^{-ip\cdot x}] \end{cases}$$

This step proves that when $s' = s + 1/2$, the proposition is established.
Step 4: Based on the above inductive reasoning, the theorem has been proved. □

3.4 Covariant commutation rules for Penrose fully symmetric equation in N+1-D

Thm. 3.4.1.

$$\begin{cases} [a_\sigma(\vec{p}, -s\varsigma; \vec{h}), a_{\sigma'}^+(\vec{p}', -s\varsigma; \vec{h}')]_{-^{2s+1}} = \delta_{\sigma\sigma'}\delta_{\vec{h}\vec{h}'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}, -s\varsigma; \vec{h}), a_{\sigma'}(\vec{p}', -s\varsigma; \vec{h}')]_{-^{2s+1}} = 0, [a_\sigma^+(\vec{p}, -s\varsigma; \vec{h}), a_{\sigma'}^+(\vec{p}', -s\varsigma; \vec{h}')]_{-^{2s+1}} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases}[\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x-x')\\ [\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(x),\psi_{\underbrace{E_\varsigma F_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=0,[\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{E'_\varsigma F'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}=0,s\geq 0\end{cases}$$

Proof: $[\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(x')]_{-^{2s+1}}$

$$=\frac{1}{(2\pi)^3}\int\sum_{\vec h,\vec h'}d^3\vec p d^3\vec p'(|\vec p||\vec p'|)^{(s-\frac12)}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(\vec p',-s\varsigma;\vec h')$$
$$[[a_1(\vec p,-s\varsigma;\vec h)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma;\vec h)e^{-ip\cdot x}],[a_1^+(\vec p',-s\varsigma;\vec h')e^{-ip'\cdot x'}+a_2(\vec p',-s\varsigma;\vec h')e^{ip'\cdot x'}]]_{-^{2s+1}}$$

$$=\frac{1}{(2\pi)^3}\int\sum_{\vec h,\vec h'}d^3\vec p d^3\vec p'(|\vec p||\vec p'|)^{(s-\frac12)}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(\vec p',-s\varsigma;\vec h')$$
$$\{[a_1(\vec p,-s\varsigma;\vec h),a_1^+(\vec p',-s\varsigma;\vec h')]_{-^{2s+1}}e^{i(p\cdot x-p'\cdot x')}+[a_2^+(\vec p,-s\varsigma;\vec h),a_2(\vec p',-s\varsigma;\vec h')]_{-^{2s+1}}e^{-i(p\cdot x-p'\cdot x')}\}$$

$$=\frac{1}{(2\pi)^3}\int\sum_{\vec h,\vec h'}d^3\vec p d^3\vec p'(|\vec p||\vec p'|)^{(s-\frac12)}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(\vec p',-s\varsigma;\vec h')\delta^3(\vec p-\vec p')\delta_{\vec h\vec h'}$$
$$[e^{i(p\cdot x-p'\cdot x')}+(-1)^{2s+1}e^{-i(p\cdot x-p'\cdot x')}]$$
$$=\frac{1}{(2\pi)^3}\int\sum_{\vec h}d^3\vec p|\vec p|^{2s-1}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)[e^{i(p\cdot x-p\cdot x')}+(-1)^{2s+1}e^{-i(p\cdot x-p\cdot x')}]$$

$$=i\frac{(-\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\frac{1}{(2\pi)^3}\int\frac{-i}{|2\vec p|}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{p_ap_b\cdots}^{2s}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]d^3\vec p$$
$$=i\frac{(i\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\frac{1}{(2\pi)^3}\int\frac{-i}{|2\vec p|}[e^{ip\cdot(x-x')}-e^{-ip\cdot(x-x')}]d^3\vec p$$
$$=i\frac{(i\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x-x')$$ □

Proof: $[a_1(\vec p,-s\varsigma;\vec h),a_1^+(\vec p',-s\varsigma;\vec h')]$

$$=\frac{1}{(2\pi)^3}\int|\vec p|^{-(2s-1)}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p,-s\varsigma;\vec h)\lambda^{\overbrace{A'_\varsigma B'_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')[\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(x')]e^{-i(p\cdot x-p'\cdot x')}d^3\vec r d^3\vec r'$$

$$=\frac{1}{(2\pi)^3}\int|\vec p|^{-(2s-1)}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p,-s\varsigma;\vec h)\lambda^{\overbrace{A'_\varsigma B'_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')$$
$$i\frac{(i\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x-x')e^{-i(p\cdot x-p'\cdot x')}d^3\vec r d^3\vec r'$$
$$=\frac{1}{(2\pi)^3}\int|\vec p|^{-(2s-1)}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p,-s\varsigma;\vec h)\lambda^{\overbrace{A'_\varsigma B'_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')$$
$$i\frac{(i\varsigma)^{2s}}{2^{2s}[(2s)!]^2}\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s-1}\overbrace{\partial_a\partial_b\cdots}^{2s}\frac{-i}{(2\pi)^3}\int\frac{1}{2|\vec p_0|}[e^{ip_0\cdot(x-x')}-e^{-ip_0\cdot(x-x')}]d^3\vec p_0\}e^{-i(p\cdot x-p'\cdot x')}d^3\vec r d^3\vec r'$$
$$=[\frac{1}{(2\pi)^3}]^2\int d^3\vec p_0d^3\vec r d^3\vec r'|\vec p|^{-(2s-1)}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p,-s\varsigma;\vec h)\lambda^{\overbrace{A'_\varsigma B'_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')(-\frac{\varsigma}{2})^{2s}\frac{1}{[(2s)!]^2}|\vec p_0|^{(2s-1)}$$
$$\overbrace{(\Gamma,i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\hat p_{0a}\hat p_{0b}\cdots}^{2s}[e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}+(-1)^{2s+1}e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}]$$
$$=[\frac{1}{(2\pi)^3}]^2\int d^3\vec p_0d^3\vec r d^3\vec r'|\vec p|^{-(2s-1)}|\vec p_0|^{(2s-1)}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p,-s\varsigma;\vec h)\lambda^{\overbrace{A'_\varsigma B'_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')$$
$$\sum_{\vec h_0}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p_0,-s\varsigma;\vec h_0)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(\vec p_0,-s\varsigma;\vec h_0)[e^{i(p_0-p)\cdot x}e^{-i(p_0-p')\cdot x'}+(-1)^{2s+1}e^{-i(p_0+p)\cdot x}e^{i(p_0+p')\cdot x'}]$$
$$=\int\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p,-s\varsigma;\vec h)\lambda^{\overbrace{A'_\varsigma B'_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')\sum_{\vec h_0}\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p_0,-s\varsigma;\vec h_0)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdots}_{2s}}(\vec p_0,-s\varsigma;\vec h_0)$$
$$[\delta^3(\vec p_0-\vec p)\delta^3(\vec p_0-\vec p')+(-1)^{2s+1}e^{2iE_0(t-t')}\delta^3(\vec p_0+\vec p)\delta^3(\vec p_0+\vec p')]d^3\vec p_0$$

$= \int \sum_{\vec{h}_0} \lambda^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{p}, -s\varsigma; \vec{h})\lambda^{\overbrace{A'_\varsigma B'_\varsigma \cdots}^{2s}}(\vec{p}', -s\varsigma; \vec{h}')\lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p}_0, -s\varsigma; \vec{h}_0)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(\vec{p}_0, -s\varsigma; \vec{h}_0)\delta^3(\vec{p}_0 - \vec{p})\delta^3(\vec{p}_0 - \vec{p}')d^3\vec{p}_0$

$= \sum_{\vec{h}_0} \lambda^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{p}, -s\varsigma; \vec{h})\lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p}', -s\varsigma; \vec{h}_0)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(\vec{p}', -s\varsigma; \vec{h}_0)\lambda^{\overbrace{A'_\varsigma B'_\varsigma \cdots}^{2s}}(\vec{p}', -s\varsigma; \vec{h}')\delta^3(\vec{p} - \vec{p}')$

$= \sum_{\vec{h}_0} \lambda^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{p}, -s\varsigma; \vec{h})\lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p}, -s\varsigma; \vec{h}_0)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(\vec{p}, -s\varsigma; \vec{h}_0)\lambda^{\overbrace{A'_\varsigma B'_\varsigma \cdots}^{2s}}(\vec{p}, -s\varsigma; \vec{h}')\delta^3(\vec{p} - \vec{p}')$

$= \sum_{\vec{h}_0} \delta_{\vec{h}\vec{h}_0}\delta_{\vec{h}'\vec{h}_0}\delta^3(\vec{p} - \vec{p}') = \delta_{\vec{h}\vec{h}'}\delta^3(\vec{p} - \vec{p}')$ □

Self comment: The above proof method is no longer based on the isochronous commutation rule, but directly based on the covariant commutation rule. It seems more difficult, but it's actually simpler. Because there is no need to find complex isochronal commutation rules. Even if it is calculated out, it is still difficult to use. The covariant commutation rule itself is known and very regular and can also be decomposed into the product of spin bases. The entire proof process basically depends on the properties of the spin base and hasn't complex calculations. The other commutative brackets can also be calculated out by using the same method and will not be listed.

Thm. 3.4.2.

$$\begin{cases} [\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(x), \psi^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(x')]_{-^{2s+1}} = i\frac{(i\varsigma)^{2s}}{2^{2s-1}[(2s)!]^2}\overbrace{(\Gamma, i\varsigma)^a_{\{A_\varsigma(A'_\varsigma}(\Gamma, i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdots\})}^{2s}\overbrace{\partial_a\partial_b\cdots}^{2s}\Delta(x - x') \\ [\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(x), \psi_{\underbrace{E_\varsigma F_\varsigma \cdots}_{2s}}(x')]_{-^{2s+1}} = 0, [\psi^+_{\underbrace{A'_\varsigma B'_\varsigma \cdots}_{2s}}(x), \psi^+_{\underbrace{E'_\varsigma F'_\varsigma \cdots}_{2s}}(x')]_{-^{2s+1}} = 0, s \geq 0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} [\psi_{k_\varsigma}(x), \psi^+_{k'_\varsigma}(x')]_{-^{2s+1}} = i\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s, w)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x - x'), \Gamma(0) := 1 \\ [\psi_{k_\varsigma}(x), \psi_{l_\varsigma}(x')]_{-^{2s+1}} = 0, [\psi^+_{k'_\varsigma}(x), \psi^+_{l'_\varsigma}(x')]_{-^{2s+1}} = 0, s \geq 0 \end{cases}$$

3.5 Various physical operators of Penrose fully symmetric equation in N+1-D

Thm. 3.5.1. $P_u(s) = \int \psi^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{r}, t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{r}, t)d^3\vec{r}$

$= \int \sum_{\vec{h}} p_u[a^+_1(\vec{p}, -s\varsigma; \vec{h})a_1(\vec{p}, -s\varsigma; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s\varsigma; \vec{h})a^+_2(\vec{p}, -s\varsigma; \vec{h})]d^3\vec{p}$

Proof: $P_u(s) = \int \psi^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{r}, t)\frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{r}, t)d^3\vec{r}$

$= \frac{1}{(2\pi)^3}\int d^3\vec{p}'d^3\vec{p}d^3\vec{r}\sum_{\vec{h},\vec{h}'}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\lambda^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{p}', -s\varsigma; \vec{h}')\lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p}, -s\varsigma; \vec{h})\frac{p_u}{|\vec{p}|^{2s-1}}$

$[a^+_1(\vec{p}', -s\varsigma; \vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}', -s\varsigma; \vec{h}')e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p}, -s\varsigma; \vec{h})e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s}a^+_2(\vec{p}, -s\varsigma; \vec{h})e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$

$= \int \sum_{\vec{h},\vec{h}'} \vec{p}|^{2s-1}\lambda^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{p}', -s\varsigma; \vec{h}')\lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p}, -s\varsigma; \vec{h})\frac{p_u}{|\vec{p}|^{2s-1}}$

$\{[a^+_1(\vec{p}, -s\varsigma; \vec{h})a_1(\vec{p}, -s\varsigma; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s\varsigma; \vec{h})a^+_2(\vec{p}, -s\varsigma; \vec{h})]\delta^3(\vec{p}' - \vec{p})$

$+ [(-1)^{2s}a^+_1(-\vec{p}, -s\varsigma; \vec{h})a^+_2(\vec{p}, -s\varsigma; \vec{h})e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s\varsigma; \vec{h})a_1(\vec{p}, -s\varsigma; \vec{h})e^{2i|\vec{p}|t}]\delta^3(\vec{p}' + \vec{p})\}d^3\vec{p}'d^3\vec{p}$

$= \int \sum_{\vec{h},\vec{h}'} \lambda^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{p}', -s\varsigma; \vec{h}')\lambda_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{p}, -s\varsigma; \vec{h})p_u[a^+_1(\vec{p}, -s\varsigma; \vec{h})a_1(\vec{p}, -s\varsigma; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s\varsigma; \vec{h})a^+_2(\vec{p}, -s\varsigma; \vec{h})]d^3\vec{p}$

$= \int \sum_{\vec{h}} p_u[a^+_1(\vec{p}, -s\varsigma; \vec{h})a_1(\vec{p}, -s\varsigma; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s\varsigma; \vec{h})a^+_2(\vec{p}, -s\varsigma; \vec{h})]d^3\vec{p}$ □

Thm. 3.5.2. $Q(s) = \int \psi^{+\overbrace{A_\varsigma B_\varsigma \cdots}^{2s}}(\vec{r}, t)\frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma \cdots}_{2s}}(\vec{r}, t)d^3\vec{r}$

$= \int \sum_{\vec{h}}[a^+_1(\vec{p}, -s\varsigma; \vec{h})a_1(\vec{p}, -s\varsigma; \vec{h}) + (-1)^{2s-1}a_2(\vec{p}, -s\varsigma; \vec{h})a^+_2(\vec{p}, -s\varsigma; \vec{h})]d^3\vec{p}$

Proof: $Q(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$=\frac{1}{(2\pi)^3}\int d^3\vec p' d^3\vec p d^3\vec r\sum_{\vec h,\vec h'}|\vec p'|^{s-\frac12}|\vec p|^{s-\frac12}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\frac{1}{|\vec p|^{2s-1}}$
$[a_1^+(\vec p',-s\varsigma;\vec h')e^{-i(\vec p'\cdot\vec r-|\vec p'|t)}+a_2(\vec p',-s\varsigma;\vec h')e^{i(\vec p'\cdot\vec r-|\vec p'|t)}][a_1(\vec p,-s\varsigma;\vec h)e^{i(\vec p\cdot\vec r-|\vec p|t)}+(-1)^{2s-1}a_2^+(\vec p,-s\varsigma;\vec h)e^{-i(\vec p\cdot\vec r-|\vec p|t)}]$
$=\int\sum_{\vec h,\vec h'}\vec p|^{2s-1}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\frac{1}{|\vec p|^{2s-1}}$
$\{[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s-1}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]\delta^3(\vec p'-\vec p)$
$+[(-1)^{2s-1}a_1^+(-\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)e^{-2i|\vec p|t}+a_2(-\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)e^{2i|\vec p|t}]\delta^3(\vec p'+\vec p)\}d^3\vec p' d^3\vec p$
$=\int\sum_{\vec h}[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s-1}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]d^3\vec p$ □

Thm. 3.5.3. $N(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$
$=\int\sum_{\vec h}[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]d^3\vec p$

Proof: $N(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$=\frac{1}{(2\pi)^3}\int d^3\vec p' d^3\vec p d^3\vec r\sum_{\vec h,\vec h'}|\vec p'|^{s-\frac12}|\vec p|^{s-\frac12}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\frac{1}{|\vec p|^{2s-1}}$
$[a_1^+(\vec p',-s\varsigma;\vec h')e^{-i(\vec p'\cdot\vec r-|\vec p'|t)}+a_2(\vec p',-s\varsigma;\vec h')e^{i(\vec p'\cdot\vec r-|\vec p'|t)}][a_1(\vec p,-s\varsigma;\vec h)e^{i(\vec p\cdot\vec r-|\vec p|t)}+(-1)^{2s}a_2^+(\vec p,-s\varsigma;\vec h)e^{-i(\vec p\cdot\vec r-|\vec p|t)}]$
$=\int\sum_{\vec h,\vec h'}\vec p|^{2s-1}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\frac{1}{|\vec p|^{2s-1}}$
$\{[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]\delta^3(\vec p'-\vec p)$
$+[(-1)^{2s}a_1^+(-\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)e^{-2i|\vec p|t}+a_2(-\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)e^{2i|\vec p|t}]\delta^3(\vec p'+\vec p)\}d^3\vec p' d^3\vec p$
$=\int\sum_{\vec h}[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]d^3\vec p$ □

Thm. 3.5.4. $\vec S(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$
$=\int\sum_{\vec h}\hat p[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]d^3\vec p$

Proof: $\vec S(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$

$=\frac{1}{(2\pi)^3}\int d^3\vec p' d^3\vec p d^3\vec r\sum_{\vec h,\vec h'}|\vec p'|^{s-\frac12}|\vec p|^{s-\frac12}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\frac{\hat p}{|\vec p|^{2s-1}}$
$[a_1^+(\vec p',-s\varsigma;\vec h')e^{-i(\vec p'\cdot\vec r-|\vec p'|t)}+a_2(\vec p',-s\varsigma;\vec h')e^{i(\vec p'\cdot\vec r-|\vec p'|t)}][a_1(\vec p,-s\varsigma;\vec h)e^{i(\vec p\cdot\vec r-|\vec p|t)}+(-1)^{2s}a_2^+(\vec p,-s\varsigma;\vec h)e^{-i(\vec p\cdot\vec r-|\vec p|t)}]$
$=\int\sum_{\vec h,\vec h'}\vec p|^{2s-1}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec p',-s\varsigma;\vec h')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec p,-s\varsigma;\vec h)\frac{\hat p}{|\vec p|^{2s-1}}$
$\{[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]\delta^3(\vec p'-\vec p)$
$+[(-1)^{2s}a_1^+(-\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)e^{-2i|\vec p|t}+a_2(-\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)e^{2i|\vec p|t}]\delta^3(\vec p'+\vec p)\}d^3\vec p' d^3\vec p$
$=\int\sum_{\vec h}\hat p[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]d^3\vec p$ □

Thm. 3.5.5. $\vec M(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec r,t)\frac{\hat\nabla(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec r,t)d^3\vec r$
$=\int\sum_{\vec h}\hat p[a_1^+(\vec p,-s\varsigma;\vec h)a_1(\vec p,-s\varsigma;\vec h)+(-1)^{2s-1}a_2(\vec p,-s\varsigma;\vec h)a_2^+(\vec p,-s\varsigma;\vec h)]d^3\vec p$

Proof: $\vec{M}(s)=\int\psi^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec{r},t)\frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}}\psi_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec{r},t)d^3\vec{r}$

$=\frac{1}{(2\pi)^3}\int d^3\vec{p}'d^3\vec{p}d^3\vec{r}\sum_{\vec{h},\vec{h}'}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec{p}',-s\varsigma;\vec{h}')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec{p},-s\varsigma;\vec{h})\frac{\hat{p}}{|\vec{p}|^{2s-1}}$
$[a_1^+(\vec{p}',-s\varsigma;\vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}+a_2(\vec{p}',-s\varsigma;\vec{h}')e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p},-s\varsigma;\vec{h})e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}+(-1)^{2s-1}a_2^+(\vec{p},-s\varsigma;\vec{h})e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}]$
$=\int\sum_{\vec{h},\vec{h}'}\vec{p}|^{2s-1}\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdots}^{2s}}(\vec{p}',-s\varsigma;\vec{h}')\lambda_{\underbrace{A_\varsigma B_\varsigma\cdots}_{2s}}(\vec{p},-s\varsigma;\vec{h})\frac{\hat{p}}{|\vec{p}|^{2s-1}}$
$\{[a_1^+(\vec{p},-s\varsigma;\vec{h})a_1(\vec{p},-s\varsigma;\vec{h})+(-1)^{2s-1}a_2(\vec{p},-s\varsigma;\vec{h})a_2^+(\vec{p},-s\varsigma;\vec{h})]\delta^3(\vec{p}'-\vec{p})$
$+[(-1)^{2s-1}a_1^+(-\vec{p},-s\varsigma;\vec{h})a_2^+(\vec{p},-s\varsigma;\vec{h})e^{-2i|\vec{p}|t}+a_2(-\vec{p},-s\varsigma;\vec{h})a_1(\vec{p},-s\varsigma;\vec{h})e^{2i|\vec{p}|t}]\delta^3(\vec{p}'+\vec{p})\}d^3\vec{p}'d^3\vec{p}$
$=\int\sum_{\vec{h}}\hat{p}[a_1^+(\vec{p},-s\varsigma;\vec{h})a_1(\vec{p},-s\varsigma;\vec{h})+(-1)^{2s-1}a_2(\vec{p},-s\varsigma;\vec{h})a_2^+(\vec{p},-s\varsigma;\vec{h})]d^3\vec{p}$ □

3.6 Action of Bargmann-Wigner equation

Thm. 3.6.1. $S?=\int\psi^+_{A'_\varsigma B'_\varsigma\cdots}(x)\gamma_0^{A'_\varsigma Z_\varsigma}\gamma_0^{B'_\varsigma B_\varsigma}\cdots(\gamma^a{}_{Z_\varsigma}{}^{A_\varsigma}\partial_a+m\delta_{Z_\varsigma}{}^{A_\varsigma})\psi_{A_\varsigma B_\varsigma\cdots}(x)d^4x$

Chapter37 Covariate Quantization of Particles in Low Dimensional Space-time

Self comment: For particles described by the Bargmann-Wigner equation or Dirac equation, it is generally possible to describe both charged complex particles and uncharged Majorana particles. The principal commutation rule in both cases is consistent, but the rest are generally zero for charged complex particles. For uncharged Majorana particles, the rest of the commutative or anti commutative brackets are naturally derived from the principal commutative rule and Majorana conditions. And they are generally not zero. In this chapter, we only discuss the case of complex particles and generally only give the principal commutation rule. The Majorana particle case is no longer specifically discussed. If we want to obtain the quantum field theory of the Majorana particle case, we only need to add the Majorana condition to the complex particle case. Then we will naturally obtain it. The two or three dimensional space-time particles described in this chapter can be considered as the result of four dimensional space-time particles being constrained on the y, z, or z axes. Therefore it has practical significance and can be applied to condensed matter physics. In addition, two-dimensional spatiotemporal particles can also be considered as the result of further confinement of three-dimensional spatiotemporal particles on the y-axis. Three dimensional spatiotemporal particles correspond to quantum surfaces. Two dimensional spatiotemporal particles correspond to quantum wire. One dimensional spatiotemporal particles correspond to quantum dot.

1 Covariate quantization for massive particles in 3-dimensional space-time

1.1 B-W equation with mass in 3-dimensional space-time

1.1.1 Dirac equation spin basis and its plane wave solutions in 3-dimensional space-time

Def. 1.1.1. $u(\vec{p}) := \sqrt{\frac{E+m}{2m}}(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}, v(\vec{p}) := \sqrt{\frac{E+m}{2m}}(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$

Cor. 1.1.1. $u(\vec{p}) = \sigma_x v^*(\vec{p}), v(\vec{p}) = \sigma_x u^*(\vec{p})$

Thm. 1.1.1. $(\gamma^a\partial_a + m)\psi = 0, \gamma^a = (-\sigma_y, \sigma_x, \varsigma\sigma_z)$

$$\psi(\vec{r},t) = \frac{1}{(2\pi)^{1/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}[a(\vec{p})\sqrt{\tfrac{m}{E}}u(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})\sqrt{\tfrac{m}{E}}v(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d\vec{p}$$

$$a(\vec{p}) = \frac{1}{(2\pi)^{1/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{E}{m}}u^+(\vec{p})\psi(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d\vec{r}, b^+(\vec{p}) = \frac{1}{(2\pi)^{1/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{E}{m}}v^+(\vec{p})\psi(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d\vec{r}$$

1.1.2 Properties of Dirac spin basis in 3-dimensional space-time

Cor. 1.1.2. $\begin{cases}\bar{u}(\vec{p})u(\vec{p}) = 1, \bar{v}(\vec{p})v(\vec{p}) = -1, \bar{u}(\vec{p})v(\vec{p}) = 0, \bar{v}(\vec{p})u(\vec{p}) = 0\\ u^+(\vec{p})u(\vec{p}) = \frac{E}{m}, v^+(\vec{p})v(\vec{p}) = \frac{E}{m}, u^+(\vec{p})v(-\vec{p}) = 0, v^+(\vec{p})u(-\vec{p}) = 0\end{cases}$

Cor. 1.1.3. $\begin{cases}u(\vec{p})\bar{u}(\vec{p}) = \frac{m-i\gamma^a p_a}{2m}\\ v(\vec{p})\bar{v}(\vec{p}) = \frac{-m-i\gamma^a p_a}{2m}\end{cases}$ $\begin{cases}u(\vec{p})u^+(\vec{p}) = \frac{(m-i\gamma^a p_a)\gamma^0}{2m} = \frac{m\sigma_z-(\sigma,i\varsigma)^a p_a}{\varsigma 2m}\\ v(\vec{p})v^+(\vec{p}) = \frac{(-m-i\gamma^a p_a)\gamma^0}{2m} = \frac{-m\sigma_z-(\sigma,i\varsigma)^a p_a}{\varsigma 2m}\end{cases}$

Cor. 1.1.4. $u(\vec{p})\bar{u}(\vec{p}) - v(\vec{p},h)\bar{v}(\vec{p}) = 1, u(\vec{p})\bar{u}(\vec{p}) + v(\vec{p},h)\bar{v}(\vec{p}) = \frac{-i\gamma^a p_a}{m}, u(\vec{p})u^+(\vec{p}) + v(-\vec{p},h)v^+(-\vec{p}) = \frac{E}{m}$

1.1.3 Covariant quantization rules for Dirac equation in 3-dimensional space-time

Cor. 1.1.5. $\begin{cases}\{a(\vec{p}), a^+(\vec{p}')\} = \delta(\vec{p}-\vec{p}')\\ \{a(\vec{p}), a(\vec{p}')\} = 0, \{a^+(\vec{p}), a^+(\vec{p}')\} = 0\end{cases} \Rightarrow \{\psi_{\lambda_\varsigma}(x), \psi^+_{\lambda'_\varsigma}(x')\} = i[(m-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}\Delta(x-x')$

1.2 B-W equation in 3-dimensional space-time

1.2.1 Spin basis and its plane wave solutions of B-W equation in 3-dimensional space-time [16]

Def. 1.2.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) := \underbrace{u_{\lambda_\varsigma}(\vec{p})u_{\mu_\varsigma}(\vec{p})\cdot\cdot}_{2s}, V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) := \underbrace{v_{\lambda_\varsigma}(\vec{p})v_{\mu_\varsigma}(\vec{p})\cdot\cdot}_{2s}$

Cor. 1.2.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) = \underbrace{\sigma_x\otimes\sigma_x\cdot\cdot}_{2s}V^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}), V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) = \underbrace{\sigma_x\otimes\sigma_x\cdot\cdot}_{2s}U^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p})$

Thm. 1.2.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)=\frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdots\}}_{2s}}(\vec{r},t)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{s-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2s}[a(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\begin{cases}a(\vec{p})=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\frac{m}{E}}^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^N\vec{r}\\ b^+(\vec{p})=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\frac{m}{E}}^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^N\vec{r}\end{cases}$$

1.2.2 Orthogonal properties of B-W equation spin basis in 3-dimensional space-time

Cor. 1.2.2.

$$\begin{cases}\bar{U}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=1,\bar{V}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=1\\ \bar{U}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=0,\bar{V}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=0\end{cases}$$

$$\begin{cases}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=(\frac{E}{m})^{2s},V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=(\frac{E}{m})^{2s}\\ U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(-\vec{p})=0,V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(-\vec{p})=0\end{cases}$$

1.2.3 Quasi projection operator of B-W equation in 3-dimensional space-time

Cor. 1.2.3.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(2m)^{2s}}\underbrace{[(m-i\gamma^bp_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-i\gamma^cp_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(2m)^{2s}}\underbrace{[(-m-i\gamma^bp_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(-m-i\gamma^cp_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\end{cases}$$

Cor. 1.2.4.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(\varsigma 2m)^{2s}}\underbrace{[m\sigma_z-(\sigma,i\varsigma)^ap_a]_{\lambda_\varsigma\lambda'_\varsigma}[m\sigma_z-(\sigma,i\varsigma)^bp_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(\varsigma 2m)^{2s}}\underbrace{[-m\sigma_z-(\sigma,i\varsigma)^ap_a]_{\lambda_\varsigma\lambda'_\varsigma}[-m\sigma_z-(\sigma,i\varsigma)^bp_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\end{cases}$$

Cor. 1.2.5. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(p)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(p)=(-1)^{2s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(-p)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(-p)$

1.2.4 Covariant commutation rules for B-W equation in 3-dimensional space-time

Thm. 1.2.2. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]=\frac{i}{2^{2s-1}}\underbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 1.2.3. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\underbrace{[-im\sigma_z+(\sigma,i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[-im\sigma_z+(\sigma,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\Delta(x-x')$

1.3 Concrete expression of massive particle potential equation in 3-dimensional space-time

Self comment: This section compares wit four dimensional space-time case. Explore whether is there a K-G or R-S equation equivalent to B-W equation in 3-dimensional space time?

1.3.1 Massive B-W equation with $s=1$ is equivalent to similar K-G equation in 3D

Thm. 1.3.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma}=\psi_{\mu_\varsigma\lambda_\varsigma},A_a=\frac{1}{\sqrt{2}im}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma},\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$
$\Leftrightarrow\partial_aA_b-\partial_bA_a=i\varsigma m\varepsilon_{ab}{}^cA_c,\psi=im\gamma^a\varepsilon A_a\Rightarrow(\partial^b\partial_b-m^2)A_a=0,\partial^aA_a=0$

Thm. 1.3.2. $(\gamma^a\partial_a+m)\psi(x)=0,\psi^T(x)=\psi(x),A_a=\frac{1}{\sqrt{2}im}tr(\bar{\varepsilon}\gamma_a\psi),\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$
$\Leftrightarrow\partial_aA_b-\partial_bA_a=i\varsigma m\varepsilon_{ab}{}^cA_c,\psi=\frac{im}{\sqrt{2}}\gamma^a\varepsilon A_a\Rightarrow(\partial^b\partial_b-m^2)A_a=0,\partial^aA_a=0$

Proof: $(\gamma^a\partial_a+m)\psi(x)=0,\psi=\frac{im}{\sqrt{2}}\gamma^a\varepsilon A_a$
$\Leftrightarrow(\gamma^a\partial_a+m)\frac{im}{\sqrt{2}}\gamma^b\varepsilon A_b=0$
$\Leftrightarrow(\gamma^a\partial_a+m)\gamma^bA_b=0$
$\Leftrightarrow\delta^{ab}\partial_aA_b+i\varsigma\varepsilon^{abc}\partial_aA_b\gamma_c+m\gamma_cA^c=0$
$\Leftrightarrow\partial^aA_a+(i\varsigma\varepsilon^{ab}{}_c\partial_aA_b+mA_c)\gamma^c=0$
$\Leftrightarrow\partial^aA_a=0,i\varsigma\varepsilon^{ab}{}_c\partial_aA_b+mA_c=0$
$\Leftrightarrow\varepsilon^{ab}{}_c\partial_aA_b=i\varsigma mA_c\Leftrightarrow\nabla\times\vec{A}=i\varsigma m\vec{A}$
$\Leftrightarrow\varepsilon^{a'b'c}\varepsilon^{ab}{}_c\partial_aA_b=i\varsigma m\varepsilon_{a'b'}{}^cA_c$
$\Leftrightarrow(\delta^{a'a}\delta^{b'b}-\delta^{a'b}\delta^{b'a})\partial_aA_b=i\varsigma m\varepsilon_{a'b'}{}^cA_c$
$\Leftrightarrow\partial_aA_b-\partial_bA_a=i\varsigma m\varepsilon_{ab}{}^cA_c$
$\Rightarrow\partial^a\partial_aA_b-\partial_b\partial^aA_a=i\varsigma m\varepsilon_{ab}{}^c\partial^aA_c$
$\Leftrightarrow(\partial^a\partial_a-m^2)A_b=0$ □

Thm. 1.3.3. $\gamma^a\partial_a\psi(x)=0,\psi^T(x)=\psi(x),A_a=\frac{1}{\sqrt{2}im}tr(\bar{\varepsilon}\gamma_a\psi),\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$
$\Leftrightarrow\partial_aA_b-\partial_bA_a=0,\partial^aA_a=0,\psi=im\gamma^a\varepsilon A_a\Rightarrow\partial^b\partial_bA_a=0,\partial^aA_a=0$

Proof: $\gamma^a\partial_a\psi(x)=0,\psi=\frac{im}{\sqrt{2}}\gamma^a\varepsilon A_a$
$\Leftrightarrow\gamma^a\partial_a\frac{im}{\sqrt{2}}\gamma^b\varepsilon A_b=0$
$\Leftrightarrow\gamma^a\partial_a\gamma^bA_b=0$
$\Leftrightarrow\delta^{ab}\partial_aA_b+i\varsigma\varepsilon^{abc}\partial_aA_b\gamma_c=0$
$\Leftrightarrow\partial^aA_a+i\varsigma\varepsilon^{ab}{}_c\partial_aA_b=0$
$\Leftrightarrow\partial^aA_a=0,i\varsigma\varepsilon^{ab}{}_c\partial_aA_b=0$
$\Leftrightarrow\partial^aA_a=0,\varepsilon^{ab}{}_c\partial_aA_b=0\Leftrightarrow\partial^aA_a=0,\nabla\times\vec{A}=0$
$\Leftrightarrow\partial^aA_a=0,\varepsilon^{a'b'c}\varepsilon^{ab}{}_c\partial_aA_b=0$
$\Leftrightarrow\partial^aA_a=0,(\delta^{a'a}\delta^{b'b}-\delta^{a'b}\delta^{b'a})\partial_aA_b=0$
$\Leftrightarrow\partial^aA_a=0,\partial_aA_b-\partial_bA_a=0$
$\Rightarrow\partial^aA_a=0,\partial^a\partial_aA_b-\partial_b\partial^aA_a=0$
$\Leftrightarrow\partial^a\partial_aA_b=0,\partial^aA_a=0$ □

1.3.2 Massive B-W equation with $s=\frac{3}{2}$ is equivalent to similar R-S equation in 3D

Thm. 1.3.4. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{1}{3!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\}},A_{a\eta_\varsigma}=\frac{1}{\sqrt{2}im}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}$

$$\Leftrightarrow\begin{cases}\partial_aA_{b\eta_\varsigma}-\partial_bA_{a\eta_\varsigma}=i\varsigma m\varepsilon_{ab}{}^cA_{c\eta_\varsigma}\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma},\gamma^aA_{a[\eta_\varsigma]}=0\end{cases}\Rightarrow(\gamma^b\partial_b+m)A_{a[\eta_\varsigma]}=0,\gamma^aA_{a[\eta_\varsigma]}=0$$

Proof: $\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{1}{3!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\}}\\ A_{a\eta_\varsigma}=\frac{1}{\sqrt{2}im}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma},\gamma^a=(\sigma_x,\sigma_y,\varsigma\sigma_z)\end{cases}$

$$\Leftrightarrow\begin{cases}\partial_aA_{b\eta_\varsigma}-\partial_bA_{a\eta_\varsigma}=i\varsigma m\varepsilon_{ab}{}^cA_{c\eta_\varsigma}\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma},\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}\end{cases}$$

$$\Leftrightarrow\begin{cases}\partial_aA_{b\eta_\varsigma}-\partial_bA_{a\eta_\varsigma}=i\varsigma m\varepsilon_{ab}{}^cA_{c\eta_\varsigma}\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma},\varepsilon^{\mu_\varsigma\eta_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=0\end{cases}$$

$\Leftrightarrow\partial_aA_{b\eta_\varsigma}-\partial_bA_{a\eta_\varsigma}=i\varsigma m\varepsilon_{ab}{}^cA_{c\eta_\varsigma},\gamma^aA_{a[\eta_\varsigma]}=0$
$\Rightarrow\gamma^a\partial_aA_{b[\eta_\varsigma]}-\partial_b\gamma^aA_{a[\eta_\varsigma]}=i\varsigma m\varepsilon_{ab}{}^c\gamma^aA_{c[\eta_\varsigma]},\gamma^aA_{a\eta_\varsigma}=0$
$\Leftrightarrow\gamma^a\partial_aA_{b[\eta_\varsigma]}+\frac{1}{2}m[\gamma_c,\gamma_b]A^c_{[\eta_\varsigma]}=0,\gamma^aA_{a\eta_\varsigma}=0$
$\Leftrightarrow\gamma^a\partial_aA_{b\eta_\varsigma}+\frac{1}{2}m\{\gamma_c,\gamma_b\}A^c_{[\eta_\varsigma]}=0,\gamma^aA_{a\eta_\varsigma}=0$
$\Leftrightarrow(\gamma^b\partial_b+m)A_{a[\eta_\varsigma]}=0,\gamma^aA_{a[\eta_\varsigma]}=0$
$\Leftrightarrow(\gamma^b\partial_b+m)A_{a[\eta_\varsigma]}=0,\gamma^aA_{a[\eta_\varsigma]}=0,\partial^aA_{a\eta_\varsigma}=0$ □

1.3.3 Massive B-W equation with $s=2$ is equivalent to similar K-G equation in 3D

Thm. 1.3.5. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}=\frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}},A_{ab}=(\frac{1}{\sqrt{2}im})^2(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}$

$$\Leftrightarrow\begin{cases}\partial_aA_{bd}-\partial_bA_{ad}=i\varsigma m\varepsilon_{ab}{}^cA_{cd},A_{ab}=A_{ba}\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}=(\frac{im}{\sqrt{2}})^2(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma}A_{ab},\delta^{ab}A_{ab}=0\end{cases}\Rightarrow\begin{cases}(\partial^c\partial_c-m^2)A_{ab}=0,A_{ab}=A_{ba}\\ \delta^{ab}A_{ab}=0,\partial^aA_{ab}=0\end{cases}$$

Proof:
$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}=\frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}}\\ A_{ab}:=(\frac{1}{\sqrt{2}im})^2(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma},\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)\end{cases}$$

$$\Leftrightarrow\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}=\frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}}\\ A_{a\eta_\varsigma\xi_\varsigma}:=\frac{1}{\sqrt{2}im}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}\end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\varsigma\xi_\varsigma} - \partial_b A_{a\eta_\varsigma\xi_\varsigma} = i\varsigma m \varepsilon_{ab}{}^c A_{c\eta_\varsigma\xi_\varsigma}, A_{a\eta_\varsigma\xi_\varsigma} = A_{a\xi_\varsigma\eta_\varsigma} \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma\xi_\varsigma}, \gamma^a A_{a[\eta_\varsigma]\xi_\varsigma} = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\varsigma\xi_\varsigma} - \partial_b A_{a\eta_\varsigma\xi_\varsigma} = i\varsigma m \varepsilon_{ab}{}^c A_{c\eta_\varsigma\xi_\varsigma}, A_{a\eta_\varsigma\xi_\varsigma} = A_{a\xi_\varsigma\eta_\varsigma} \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma\xi_\varsigma}, \gamma^a A_{a[\eta_\varsigma]\xi_\varsigma} = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} \partial_a A_{bd} - \partial_b A_{ad} = i\varsigma m \varepsilon_{ab}{}^c A_{cd}, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = (\frac{im}{\sqrt{2}})^2 (\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma} (\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma} A_{ab} \end{cases}$$
$$\Rightarrow (\partial^c\partial_c - m^2) A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0, \partial^a A_{ab} = 0 \qquad \square$$

1.4 General expression of massive boson potential equation in 3-dimensional space-time

1.4.1 Mathematical preparation

Pro. 1.4.1. $(\gamma^a\varepsilon)_{\lambda'_\varsigma\mu'_\varsigma}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma} = \delta^{\{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma\}}_{\mu'_\varsigma}$

Pro. 1.4.2. $(\gamma^a\varepsilon)_{\lambda'_\varsigma\mu'_\varsigma}\eta_{aa'}(\bar{\varepsilon}\gamma^{a'})^{\lambda_\varsigma\mu_\varsigma} = \delta^{\{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma\}}_{\mu'_\varsigma} - 2|\varepsilon_{\lambda'_\varsigma\mu'_\varsigma}||\varepsilon^{\lambda_\varsigma\mu_\varsigma}|$

1.4.2 Massive B-W equation with $s = n$ is equivalent to similar K-G equation in 3D

Thm. 1.4.1.

$$\begin{cases} (\gamma^a\partial_a + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x) = 0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}} = \frac{1}{(2n)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\}}_{2n}} \\ A_{\underbrace{ab\cdots}_{n}} = (\frac{1}{\sqrt{2}im})^n \underbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}_{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}} \end{cases} \Leftrightarrow \begin{cases} \partial_a A_{\underbrace{bd\cdots}_{n}}(x) - \partial_b A_{\underbrace{ad\cdots}_{n}}(x) = i\varsigma m\varepsilon_{ab}{}^c A_{\underbrace{cd\cdots}_{n}}(x) \\ A_{\underbrace{ab\cdots}_{n}} = \frac{1}{n!}A_{\underbrace{\{ab\cdots\}}_{n}}, \delta^{ab}A_{\underbrace{ab\cdots}_{n}} = 0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}} = (\frac{im}{\sqrt{2}})^n \underbrace{(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}_{n} A_{\underbrace{ab\cdots}_{n}} \end{cases}$$

$$\Rightarrow\Rightarrow \begin{cases} (\partial^c\partial_c - m^2)A_{\underbrace{ab\cdots}_{n}} = 0 \\ A_{\underbrace{ab\cdots}_{n}} = \frac{1}{n!}A_{\underbrace{\{ab\cdots\}}_{n}}, \delta^{ab}A_{\underbrace{ab\cdots}_{n}} = 0, \partial^a A_{\underbrace{ab\cdots}_{n}} = 0 \end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{r},t) = \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty} E^{n-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2n}[a(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$A_{\underbrace{ab\cdots}_{n}}(\vec{r},t) = \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2^nE}}[a(\vec{p})\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}), \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p})$$

1.4.3 Spin bases relations for massive $s = n$ B-W equation and similar K-G equation in 3D

Cor. 1.4.1.

$$\begin{cases} (i\gamma^a p_a + m)U_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}) = 0 \\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p})\ \text{fully symmetric} \\ \varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}) \end{cases} \Leftrightarrow \begin{cases} (p^cp_c + m^2)\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p}) = 0 \\ p_a\varepsilon_{\underbrace{bd\cdots}_{n}}(x) - p_b\varepsilon_{\underbrace{ad\cdots}_{n}} = \varsigma m\varepsilon_{ab}{}^c\varepsilon_{\underbrace{cd\cdots}_{n}} \\ \delta^{ab}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p}) = 0, p^a\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p}) = 0, \varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p})\ \text{fully symmetric} \\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}) = (\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p}) \end{cases}$$

Cor. 1.4.2.

$$\begin{cases} (-i\gamma^a p_a + m)V_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}) = 0 \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p})\ \text{fully symmetric} \\ \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}) \end{cases} \Leftrightarrow \begin{cases} (p^cp_c + m^2)\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p}) = 0 \\ p_a\tilde{\varepsilon}_{\underbrace{bd\cdots}_{n}}(x) - p_b\tilde{\varepsilon}_{\underbrace{ad\cdots}_{n}} = -\varsigma m\varepsilon_{ab}{}^c\tilde{\varepsilon}_{\underbrace{cd\cdots}_{n}} \\ \delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p}) = 0, p^a\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p}) = 0, \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p})\ \text{fully symmetric} \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}) = (\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p}) \end{cases}$$

Cor. 1.4.3.

$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p}) = (\frac{i}{2})^n \overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})[\Leftrightarrow]\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p}) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p}) = (\frac{i}{2})^n \overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})[\Leftrightarrow]\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p}) \end{cases}$$

Cor. 1.4.4. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}) = \underbrace{\varepsilon_a(\vec{p})\varepsilon_b(\vec{p})\cdot\cdot}_{n}, \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}) = \underbrace{\tilde{\varepsilon}_a(\vec{p})\tilde{\varepsilon}_b(\vec{p})\cdot\cdot}_{n}$

1.4.4 Spin basis $\varepsilon_a(\vec{p})$ and its properties of similar Klein-Gordon equation in 3D

Cor. 1.4.5. $u(\vec{p}) := \sqrt{\frac{E+m}{2m}}(1 - \frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$

Thm. 1.4.2. $\varepsilon_a(\vec{p}) = (i\varsigma + \frac{i\varsigma p_x(p_x+i\varsigma p_y)}{m(E+m)}, -1 + \frac{i\varsigma p_y(p_x+i\varsigma p_y)}{m(E+m)}, -\varsigma\frac{p_x+i\varsigma p_y}{m})$

Proof: $u^T(\vec{p})u(\vec{p})$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 - \frac{\varsigma\sigma^*\cdot\vec{p}}{E+m})(1 - \frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 + \frac{\sigma^*\cdot\vec{p}}{E+m}\frac{\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 + \frac{p_x^2-p_y^2+2ip_xp_y\sigma_z}{(E+m)^2})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{1}{2m(E+m)}[(E+m)^2 + p_x^2 - p_y^2 + 2i\varsigma p_xp_y]$
$= 1 + \frac{p_x(p_x+i\varsigma p_y)}{m(E+m)}$ □

Proof: $u^T(\vec{p})\sigma_z u(\vec{p})$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 - \frac{\varsigma\sigma^*\cdot\vec{p}}{E+m})\sigma_z(1 - \frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 - \frac{\sigma^*\cdot\vec{p}}{E+m}\frac{\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\varsigma\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 - \frac{p_x^2-p_y^2+2ip_xp_y\sigma_z}{(E+m)^2})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\varsigma\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{\varsigma}{2m(E+m)}[(E+m)^2 - p_x^2 + p_y^2 - 2i\varsigma p_xp_y]$
$= \varsigma - \frac{ip_y(p_x+i\varsigma p_y)}{m(E+m)}$ □

Proof: $u^T(\vec{p})\sigma_x u(\vec{p})$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 - \frac{\varsigma\sigma^*\cdot\vec{p}}{E+m})\sigma_x(1 - \frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T\sigma_x(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)(1 - \frac{\varsigma\sigma\cdot\vec{p}}{E+m})(1 - \frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}1\\0\end{bmatrix}^T\sigma_x(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\frac{-2\varsigma\sigma\cdot\vec{p}}{E+m}(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}$
$= -\varsigma\frac{(p_x+i\varsigma p_y)}{m}$ □

Proof: $\varepsilon_a(\vec{p}) = -i(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p})$
$= -iu^T(\vec{p})(\bar{\varepsilon}\gamma_a)u(\vec{p})$
$= u^T(\vec{p})(1, i\sigma_z, -i\varsigma\sigma_x)u(\vec{p})$
$= (1 + \frac{p_x(p_x+i\varsigma p_y)}{m(E+m)}, i\varsigma + \frac{p_y(p_x+i\varsigma p_y)}{m(E+m)}, i\frac{p_x+i\varsigma p_y}{m})$
$= (1 + \frac{p_x(p_x-i\varsigma p_y)}{m(E+m)}, -i\varsigma + \frac{p_y(p_x-i\varsigma p_y)}{m(E+m)}, i\frac{(p_x-i\varsigma p_y)}{m})$ □

Cor. 1.4.6. $\varepsilon_a(\vec{p})\varepsilon^+_{a'}(\vec{p}) = \begin{bmatrix} 1+\frac{p_x^2}{m^2} & \frac{p_xp_y}{m^2}-\frac{\varsigma p_\pi}{m} & -\frac{p_xp_\pi}{m^2}-\frac{\varsigma p_y}{m} \\ \frac{p_xp_y}{m^2}+\frac{\varsigma p_\pi}{m} & 1+\frac{p_y^2}{m^2} & -\frac{p_yp_\pi}{m^2}+\frac{\varsigma p_x}{m} \\ \frac{p_xp_\pi}{m^2}-\frac{\varsigma p_y}{m} & \frac{p_yp_\pi}{m^2}+\frac{\varsigma p_x}{m} & -1-\frac{p_\pi^2}{m^2} \end{bmatrix} = \frac{1}{m}\begin{bmatrix} m & -\varsigma p_\pi & -\varsigma p_y \\ \varsigma p_\pi & m & \varsigma p_x \\ -\varsigma p_y & \varsigma p_x & -m \end{bmatrix} + \frac{1}{m^2}\begin{bmatrix} p_xp_x^+ & p_xp_y^+ & p_xp_\pi^+ \\ p_yp_x^+ & p_yp_y^+ & p_yp_\pi^+ \\ p_\pi p_x^+ & p_\pi p_y^+ & p_\pi p_\pi^+ \end{bmatrix}$

Cor. 1.4.7. $\varepsilon_a(\vec{p})\varepsilon^+_{a'}(\vec{p}) = \eta_{aa'} + \frac{p_ap^+_{a'}}{m^2} - \frac{\varsigma\varepsilon_{acd}\eta^c_{a'}p^d}{m}$

Cor. 1.4.8. $\varepsilon_a(\vec{p})\delta^{ab}\varepsilon_b(\vec{p}) = 0, \varepsilon_a(\vec{p})p^a = 0, \varepsilon^+_a(\vec{p})\eta^{aa'}\varepsilon_{a'}(\vec{p}) = 2, \varepsilon^+_a(\vec{p})\delta^{aa'}\varepsilon_{a'}(\vec{p}) = 2(\frac{E}{m})^2$

Cor. 1.4.9. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) = \underbrace{(\eta_{aa'} + \frac{p_ap^+_{a'}}{m^2} - \frac{\varsigma\varepsilon_{acd}\eta^c_{a'}p^d}{m})(\eta_{bb'} + \frac{p_bp^+_{b'}}{m^2} - \frac{\varsigma\varepsilon_{bcd}\eta^c_{b'}p^d}{m})\cdots}_{n}$

1.4.5 Spin basis $\tilde{\varepsilon}_a(\vec{p})$ and its properties of similar Klein-Gordon equation in 3D

Cor. 1.4.10. $v(\vec{p}) := \sqrt{\frac{E+m}{2m}}(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$

Thm. 1.4.3. $\varepsilon_a(\vec{p}) = (i\varsigma+\frac{i\varsigma p_x(p_x+i\varsigma p_y)}{m(E+m)}, -1+\frac{i\varsigma p_y(p_x+i\varsigma p_y)}{m(E+m)}, -\varsigma\frac{p_x+i\varsigma p_y}{m})$

Proof: $v^T(\vec{p})v(\vec{p})$
$= \frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1-\frac{\varsigma\sigma^*\cdot\vec{p}}{E+m})(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1+\frac{\sigma^*\cdot\vec{p}}{E+m}\frac{\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1+\frac{p_x^2-p_y^2+2ip_xp_y\sigma_z}{(E+m)^2})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{1}{2m(E+m)}[(E+m)^2+p_x^2-p_y^2-2i\varsigma p_xp_y]$
$= 1+\frac{p_x(p_x-i\varsigma p_y)}{m(E+m)}$ □

Proof: $v^T(\vec{p})\sigma_z v(\vec{p})$
$= \frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1-\frac{\varsigma\sigma^*\cdot\vec{p}}{E+m})\sigma_z(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$
$= -\frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1-\frac{\sigma^*\cdot\vec{p}}{E+m}\frac{\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\varsigma\begin{bmatrix}0\\1\end{bmatrix}$
$= -\frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1-\frac{p_x^2-p_y^2+2ip_xp_y\sigma_z}{(E+m)^2})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\varsigma\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{-\varsigma}{2m(E+m)}[(E+m)^2-p_x^2+p_y^2+2i\varsigma p_xp_y]$
$= -\varsigma-\frac{ip_y(p_x-i\varsigma p_y)}{m(E+m)}$ □

Proof: $v^T(\vec{p})\sigma_x v(\vec{p})$
$= \frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1-\frac{\varsigma\sigma^*\cdot\vec{p}}{E+m})\sigma_x(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T\sigma_x(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(1-\frac{\varsigma\sigma\cdot\vec{p}}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$
$= \frac{E+m}{2m}\begin{bmatrix}0\\1\end{bmatrix}^T\sigma_x(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\frac{-2\varsigma\sigma\cdot\vec{p}}{E+m}(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$
$= -\varsigma\frac{(p_x-i\varsigma p_y)}{m}$ □

Proof: $\tilde{\varepsilon}_a(\vec{p}) = -i(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p})$
$= -iv^T(\vec{p})(\bar{\varepsilon}\gamma_a)v(\vec{p})$
$= v^T(\vec{p})(1, i\sigma_z, -i\varsigma\sigma_x)v(\vec{p})$
$= (1+\frac{p_x(p_x-i\varsigma p_y)}{m(E+m)}, -i\varsigma+\frac{p_y(p_x-i\varsigma p_y)}{m(E+m)}, i\frac{(p_x-i\varsigma p_y)}{m})$ □

Cor. 1.4.11. $\tilde{\varepsilon}_a(\vec{p}) = \varepsilon^+_{a'}(\vec{p})\eta^{a'}_a, \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}) = \varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p})\underbrace{\eta^{a'}_a\eta^{b'}_b\cdot\cdot}_{n}$

Cor. 1.4.12. $\tilde{\varepsilon}_a(\vec{p})\tilde{\varepsilon}^+_{a'}(\vec{p}) = \eta_{aa'}+\frac{p_ap^+_{a'}}{m^2}+\frac{\varsigma\varepsilon_{acd}\eta^c_{a'}p^d}{m}$

Cor. 1.4.13. $\tilde{\varepsilon}_a(\vec{p})\delta^{ab}\tilde{\varepsilon}_b(\vec{p}) = 0, \tilde{\varepsilon}_a(\vec{p})p^a = 0, \tilde{\varepsilon}^+_a(\vec{p})\eta^{aa'}\tilde{\varepsilon}_{a'}(\vec{p}) = 2, \tilde{\varepsilon}^+_a(\vec{p})\delta^{aa'}\tilde{\varepsilon}_{a'}(\vec{p}) = 2(\frac{E}{m})^2$

Cor. 1.4.14. $\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) = \underbrace{(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2}+\frac{\varsigma\varepsilon_{acd}\eta^c_{a'}p^d}{m})(\eta_{bb'}+\frac{p_bp^+_{b'}}{m^2}+\frac{\varsigma\varepsilon_{bcd}\eta^c_{b'}p^d}{m})\cdots}_{n}$

1.4.6 Relations between various quasi projection operators for massive bosons in 3D

Cor. 1.4.15.
$$\begin{cases}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) = \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}\varepsilon)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(\vec{p})\\ \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) = \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}\varepsilon)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(\vec{p})\end{cases}$$

Cor. 1.4.16.
$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{p}) = \frac{1}{2^{2n}}\overbrace{(\gamma_a\varepsilon)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p})\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{p}) = \frac{1}{2^{2n}}\overbrace{(\gamma_a\varepsilon)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p})\end{cases}$$

Cor. 1.4.17.
$$\begin{cases}[A_{\underbrace{ab\cdots}_{n}}(x),A^{+}_{\underbrace{a'b'\cdots}_{n}}(x')]=\frac{1}{m^{2n}2^{n}}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=\frac{m^{2n}}{2^{n}}\overbrace{(\gamma_a\varepsilon)^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}[A_{\underbrace{ab\cdots}_{n}}(x),A^{+}_{\underbrace{a'b'\cdots}_{n}}(x')]\end{cases}$$

1.4.7 Equivalent expression of quasi projection operators for massive bosons in 3D

Lem. 1.4.1.
$$\begin{cases}u(\vec{p})u^{+}(\vec{p})=\frac{(m-i\gamma^a p_a)\gamma^0}{2m},u_{\lambda_\varsigma}(\vec{p})u^{+}_{\lambda'_\varsigma}(\vec{p})u_{\mu_\varsigma}(\vec{p})u^{+}_{\mu'_\varsigma}(\vec{p})=u_{\lambda_\varsigma}(\vec{p})u^{+}_{\mu'_\varsigma}(\vec{p})u_{\mu_\varsigma}(\vec{p})u^{+}_{\lambda'_\varsigma}(\vec{p})\\ \varepsilon_a(\vec{p})\varepsilon^{+}_{a'}(\vec{p})=\eta_{aa'}+\frac{p_a p^{+}_{a'}}{m^2}-\frac{\varsigma\varepsilon_{acd}\eta^{c}_{a'}p^d}{m},\varepsilon_a(\vec{p})\varepsilon^{+}_{a'}(\vec{p})\varepsilon_b(\vec{p})\varepsilon^{+}_{b'}(\vec{p})=\varepsilon_a(\vec{p})\varepsilon^{+}_{b'}(\vec{p})\varepsilon_b(\vec{p})\varepsilon^{+}_{a'}(\vec{p})\end{cases}$$
$\Leftrightarrow$
$$\begin{cases}[(m-i\gamma^a p_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}=[(m-i\gamma^a p_a)\gamma^0]_{\mu_\varsigma\lambda'_\varsigma}[(m-i\gamma^b p_b)\gamma^0]_{\lambda_\varsigma\mu'_\varsigma}\\ (\eta_{aa'}+\frac{p_a p^{+}_{a'}}{m^2}-\frac{\varsigma\varepsilon_{acd}\eta^{c}_{a'}p^d}{m})(\eta_{bb'}+\frac{p_b p^{+}_{b'}}{m^2}-\frac{\varsigma\varepsilon_{bcd}\eta^{c}_{b'}p^d}{m})=(\eta_{ab'}+\frac{p_a p^{+}_{b'}}{m^2}-\frac{\varsigma\varepsilon_{acd}\eta^{c}_{b'}p^d}{m})(\eta_{ba'}+\frac{p_b p^{+}_{a'}}{m^2}-\frac{\varsigma\varepsilon_{bcd}\eta^{c}_{a'}p^d}{m})\\ [(m-i\gamma^b p_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-i\gamma^c p_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}=m^2(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma^{a'})_{\lambda'_\varsigma\mu'_\varsigma}(\eta_{aa'}+\frac{p_a p^{+}_{a'}}{m^2}-\frac{\varsigma\varepsilon_{acd}\eta^{c}_{a'}p^d}{m})\end{cases}$$

Cor. 1.4.18.
$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})U^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p})=\frac{1}{2^{2n}}\underbrace{[(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma^{a'})_{\lambda'_\varsigma\mu'_\varsigma}(\eta_{aa'}+\frac{p_a p^{+}_{a'}}{m^2}-\frac{\varsigma\varepsilon_{acd}\eta^{c}_{a'}p^d}{m})]\cdots}_{n}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})V^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p})=\frac{1}{2^{2n}}\underbrace{[(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma^{a'})_{\lambda'_\varsigma\mu'_\varsigma}(\eta_{aa'}+\frac{p_a p^{+}_{a'}}{m^2}+\frac{\varsigma\varepsilon_{acd}\eta^{c}_{a'}p^d}{m})]\cdots}_{n}\end{cases}$$

1.4.8 Covariant commutation rules for massive bosons in 3-dimensional space-time

Thm. 1.4.4. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=\frac{i}{2^{2n-1}}\underbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 1.4.5. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=i\frac{i^{2n}}{2^{2n-1}}\underbrace{[-im\sigma_z+(\sigma,i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[-im\sigma_z+(\sigma,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 1.4.6. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^{+}_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')]=\frac{i}{2^{2n-1}}\underbrace{\mathbb{X}^{a}_{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}\underbrace{(\eta_{aa'}-\frac{\partial_a\partial^{+}_{a'}}{m^2}+\frac{i\varsigma\varepsilon_{acd}\eta^{c}_{a'}\partial^d}{m})\cdots}_{n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 1.4.7. $[A_{\underbrace{ab\cdots}_{n}}(x),A^{+}_{\underbrace{a'b'\cdots}_{n}}(x')]=\frac{i}{2^{n-1}}\underbrace{(\eta_{aa'}-\frac{\partial_a\partial^{+}_{a'}}{m^2}+\frac{i\varsigma\varepsilon_{acd}\eta^{c}_{a'}\partial^d}{m})\cdots}_{n}\Delta(x-x')$

1.5 General expression of fermion potential equation in 3-dimensional space-time

1.5.1 Massive B-W equation with $s=n+\frac{1}{2}$ is equivalent to similar R-S equationin 3D

Thm. 1.5.1.
$$\begin{cases}(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x)=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}=\frac{1}{(2n+1)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma\}}_{2n+1}}\\ A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}\\ =(\frac{1}{\sqrt{2}im})^n\underbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}_{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}\end{cases}\Leftrightarrow\begin{cases}\partial_a A_{\underbrace{bd\cdots}_{n}\tau_\varsigma}-\partial_b A_{\underbrace{ad\cdots}_{n}\tau_\varsigma}=i\varsigma m\varepsilon_{ab}{}^{c}A_{\underbrace{cd\cdots}_{n}\tau_\varsigma}\\ A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}=\frac{1}{n!}A_{\underbrace{\{ab\cdots\}}_{n}\tau_\varsigma},\delta^{ab}A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}=0,\gamma^a A_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}=(\frac{im}{\sqrt{2}})^n\underbrace{(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}_{n}A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}\end{cases}$$
$\Rightarrow\Rightarrow$
$$\begin{cases}(\gamma^c\partial_c+m)A_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}=0,\gamma^a A_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}=0\\ A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}=\frac{1}{n!}A_{\underbrace{\{ab\cdots\}}_{n}\tau_\varsigma},\delta^{ab}A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}=0,\partial^a A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}=0\end{cases}$$
$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{r},t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^n\sqrt{\frac{m}{E}}^{2n+1}[a(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}+b^{+}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$A_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{r},t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{\sqrt{m}}{\sqrt{2^n E}}[a(\vec{p})\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p})\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p}),\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})$$

Cor. 1.5.1. $\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=\underbrace{\varepsilon_a(\vec{p})\varepsilon_b(\vec{p})\cdots}_{n}u_{\tau_\varsigma}(\vec{p}),\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=\underbrace{\tilde{\varepsilon}_a(\vec{p})\tilde{\varepsilon}_b(\vec{p})\cdots}_{n}v_{\tau_\varsigma}(\vec{p})$

1.5.2 Spin bases relations for massive $s=n+\frac{1}{2}$ B-W and similar R-S equation in 3D

Cor. 1.5.2.

$$\begin{cases}(i\gamma^a p_a+m)U_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})=0\\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p})\ \textit{fully symmetric}\\ \varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\\ =\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})\end{cases}\Leftrightarrow\begin{cases}(p^c p_c+m^2)\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=0,\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\ \textit{fully symmetric}\\ p_a\varepsilon_{\underbrace{bd\cdots}_{n}\tau_\varsigma}(x)-p_b\varepsilon_{\underbrace{ad\cdots}_{n}\tau_\varsigma}=\varsigma m\varepsilon_{ab}{}^{c}\varepsilon_{\underbrace{cd\cdots}_{n}\tau_\varsigma}\\ \delta^{ab}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=0,p^a\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=0,\gamma^a\varepsilon_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}=0\\ U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\end{cases}$$

Cor. 1.5.3.

$$\begin{cases}(-i\gamma^a p_a+m)V_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})=0\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})\ \textit{fully symmetric}\\ \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\\ =\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})\end{cases}\Leftrightarrow\begin{cases}(p^c p_c+m^2)\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=0,\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\ \textit{fully symmetric}\\ p_a\tilde{\varepsilon}_{\underbrace{bd\cdots}_{n}\tau_\varsigma}(x)-p_b\tilde{\varepsilon}_{\underbrace{ad\cdots}_{n}\tau_\varsigma}=-\varsigma m\varepsilon_{ab}{}^{c}\tilde{\varepsilon}_{\underbrace{cd\cdots}_{n}\tau_\varsigma}\\ \delta^{ab}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=0,p^a\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=0,\gamma^a\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}=0\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\end{cases}$$

Cor. 1.5.4.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})[\Leftrightarrow]\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})[\Leftrightarrow]\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})\end{cases}$$

Cor. 1.5.5. $\tilde{\varepsilon}_{a[\tau_\varsigma]}(\vec{p})=\sigma_x\varepsilon^+_{a'[\tau'_\varsigma]}(\vec{p})\eta_a^{a'},\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}[\tau_\varsigma]}(\vec{p})=\sigma_x\varepsilon^+_{\underbrace{a'b'\cdots}_{n}[\tau'_\varsigma]}(\vec{p})\underbrace{\eta_a^{a'}\eta_b^{b'}\cdots}_{n}$

1.5.3 Relations between various quasi projection operators for massive fermions in 3D

Cor. 1.5.6.

$$\begin{cases}\varepsilon_{\underbrace{ab\cdots\tau_\varsigma}_{n}}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdots\tau'_\varsigma}_{n}}(\vec{p})=\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}\varepsilon)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p})\\ \tilde{\varepsilon}_{\underbrace{ab\cdots\tau_\varsigma}_{n}}(\vec{p})\tilde{\varepsilon}^+_{\underbrace{a'b'\cdots\tau'_\varsigma}_{n}}(\vec{p})=\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}\varepsilon)^{\eta'_\varsigma\xi'_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p})\end{cases}$$

Cor. 1.5.7.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p})=\frac{1}{2^{2n}}\overbrace{(\gamma_a\varepsilon)^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\varepsilon_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdots}_{n}\tau'_\varsigma}(\vec{p})\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p})=\frac{1}{2^{2n}}\overbrace{(\gamma_a\varepsilon)^{\lambda_\varsigma\mu_\varsigma}\cdots}^{n}\overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'_\varsigma\mu'_\varsigma}\cdots}^{n}\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\varsigma}(\vec{p})\tilde{\varepsilon}^+_{\underbrace{a'b'\cdots}_{n}\tau'_\varsigma}(\vec{p})\end{cases}$$

Cor. 1.5.8.

$$\begin{cases}\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\}=\frac{1}{m^{2n}2^n}\overbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}\\ \{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}=\frac{m^{2n}}{2^n}\overbrace{(\gamma_a\varepsilon)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\bar\varepsilon\gamma_{a'})^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\}\end{cases}$$

1.5.4 Equivalent expression of quasi projection operators for massive fermions in 3D

Cor. 1.5.9.

$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec p)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec p)=\frac{1}{2^{2n+1}m}\underbrace{[(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma^{a'})_{\lambda'_\varsigma\mu'_\varsigma}(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2}-\frac{\varsigma\varepsilon_{acd}\eta^c_{a'}p^d}{m})]\cdots}_{n}[(m-i\gamma^cp_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec p)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec p)=\frac{1}{2^{2n+1}m}\underbrace{[(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma^{a'})_{\lambda'_\varsigma\mu'_\varsigma}(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2}+\frac{\varsigma\varepsilon_{acd}\eta^c_{a'}p^d}{m})]\cdots}_{n}[(-m-i\gamma^cp_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}\end{cases}$$

1.5.5 Covariant commutation rules for massive fermions in 3-dimensional space-time

Thm. 1.5.2. $\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n+1}}(x')\}=\frac{i}{2^{2n}}\underbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}_{2n+1}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 1.5.3. $\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n+1}}(x')\}=i\frac{(i\varsigma)^{2n+1}}{2^{2n}}\underbrace{[-im\sigma_z+(\sigma,i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[-im\sigma_z+(\sigma,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}_{2n+1}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 1.5.4. $\{\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')\}$

$$=\frac{i}{2^{2n}m}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}\underbrace{(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}+\frac{i\varsigma\varepsilon_{acd}\eta^c_{a'}\partial^d}{m})\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$$

$$[\Updownarrow]$$

Thm. 1.5.5. $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\}=\frac{i}{2^n}\underbrace{(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}+\frac{i\varsigma\varepsilon_{acd}\eta^c_{a'}\partial^d}{m})\cdot\cdot}_{n}[(m-\gamma^c\partial_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

Self comment: In 3-dimensional space-time there are indeed K-G or R-S equation that is equivalent to B-W equations, and the form is simpler and clearer than four-dimensional ones.

1.6 s-spin equation in 3-dimensional space-time

Thm. 1.6.1. $[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)$ *fully symmetric*, $\gamma_a:=[-\sigma_y,\sigma_x,\varsigma\sigma_z]$

$\Leftrightarrow[s\partial_a+m\gamma_a(s)+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0,S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)],\gamma_a(s):=[-\sigma_y(s),\sigma_x(s),\varsigma\sigma_z(s)]$

Proof: $[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)$ fully symmetric

$\Leftrightarrow[\gamma^a\partial_a+m]\hat\psi(s)=0$
$\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_aD^a\hat\psi(s)=im\sigma_z\otimes I_{2^{2s-1}}\hat\psi(s),D^a=(\partial^x,\partial^y,0,\partial^\pi)$
$\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_aD^a[I\otimes\Gamma(s)]N(s)\psi(s)=im\sigma_z\otimes I_{2^{2s-1}}\hat\psi(s)$
$\Leftrightarrow[I\otimes\Gamma(s)](\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im\sigma_z\otimes I_{2^{2s-1}}\hat\psi(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im[I\otimes\bar\Gamma(s)](\sigma_z\otimes I_{2^{2s-1}})\hat\psi(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im(\sigma_z\otimes I_{2s})[I\otimes\bar\Gamma(s)]\hat\psi(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow Z_bD^b\psi(s)=im\frac{i\varsigma}{\sqrt2}(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow\bar Z_aZ_bD^b\psi(s)=im\frac{i\varsigma}{\sqrt2}\bar Z_a(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow\bar Z_aZ_bD^b\psi(s)=im\frac{i\varsigma}{\sqrt2}\frac{-i\varsigma}{\sqrt2}\bar N(s)(\sigma\otimes I_{2s},i\varsigma)_a(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow\bar Z_aZ_bD^b\psi(s)=-\frac{m}{2s}\bar N(s)[(-\sigma_y,\sigma_x,-i)\otimes I_{2s},\varsigma\sigma_z\otimes I_{2s}]_aN(s)\psi(s)$
$\Leftrightarrow[sD_a+iS_{ab}(s,\varsigma;4)D^b]\psi(s)=-m[(-\sigma_y(s),\sigma_x(s),-is),\varsigma\sigma_z(s)]_a\psi(s)$

$$S_{ab}(s,\varsigma;4)\succ\begin{bmatrix}0&\sigma_z(s)&-\sigma_y(s)&-\varsigma\sigma_x(s)\\-\sigma_z(s)&0&\sigma_x(s)&-\varsigma\sigma_y(s)\\\sigma_y(s)&-\sigma_x(s)&0&-\varsigma\sigma_z(s)\\\varsigma\sigma_x(s)&\varsigma\sigma_y(s)&\varsigma\sigma_z(s)&0\end{bmatrix}$$

$$\Leftrightarrow[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=-m\gamma_a(s)\psi(s),S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)]\succ\begin{bmatrix}0&\sigma_z(s)&-\varsigma\sigma_x(s)\\-\sigma_z(s)&0&-\varsigma\sigma_y(s)\\\varsigma\sigma_x(s)&\varsigma\sigma_y(s)&0\end{bmatrix}$$

$$\Leftrightarrow[s\partial_a+m\gamma_a(s)+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0\qquad\square$$

Lem. 1.6.1. $\gamma_a(s)=[e^{\vartheta}]_a{}^b e^{\frac12\vartheta^{ab}[\gamma_a(s),\gamma_b(s)]}\gamma_b(s)e^{-\frac12\vartheta^{ab}[\gamma_a(s),\gamma_b(s)]}=[e^{i\omega R_z+\epsilon\cdot L}]_a{}^b e^{i\omega\sigma_z(s)+\varsigma\epsilon\cdot\sigma(s)}\gamma_b(s)e^{-i\omega\sigma_z(s)-\varsigma\epsilon\cdot\sigma(s)}$

Thm. 1.6.2.

$$\begin{cases}[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)\ \textit{fully symmetric}\\ \psi_{k_\varsigma}(x):=\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}^{2s}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)\end{cases}\Rightarrow\begin{cases}[\gamma^a(s)\partial_a+sm]\psi_{[k_\varsigma]}(x)=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)=\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}\psi_{k_\varsigma}(x)\end{cases}$$

1.7 B-W equation [16] is equivalent to Penrose equation [1,2] in 3-dimensional space-time

Thm. 1.7.1.

$$\begin{cases}[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)\ \textit{fully symmetric}\\ \psi_{k_\varsigma}(x):=\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}^{2s}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)\end{cases}\Leftrightarrow\begin{cases}[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi_{[k_\varsigma]}(x)=-m\gamma_a(s)\psi_{[k_\varsigma]}(x)\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}(x)=\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2s}}\psi_{k_\varsigma}(x)\end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec r,t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec p=-\infty}^{+\infty}E^{n-\frac12}\sqrt{\frac{m}{E}}^{2s}[a(\vec p,h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p)e^{i(\vec p\cdot\vec r-Et)}+b^+(\vec p,h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p)e^{-i(\vec p\cdot\vec r-Et)}]d^N\vec p$$

$$\psi_{k_\varsigma}(\vec r,t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec p=-\infty}^{+\infty}E^{n-\frac12}\sqrt{\frac{m}{E}}^{2n}[a(\vec p,h)U_{k_\varsigma}(\vec p;s)e^{i(\vec p\cdot\vec r-Et)}+b^+(\vec p,h)V_{k_\varsigma}(\vec p;s)e^{-i(\vec p\cdot\vec r-Et)}]d^N\vec p$$

$$\begin{cases}U_{k_\varsigma}(\vec p;s):=\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p)\Leftrightarrow U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p)=\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}U_{k_\varsigma}(\vec p;s)\\ V_{k_\varsigma}(\vec p;s):=\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p)\Leftrightarrow V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec p)=\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}V_{k_\varsigma}(\vec p;s)\end{cases}$$

Thm. 1.7.2. $(\gamma^a\partial_a+m)\psi(x)=0,\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)\Leftrightarrow(\sigma,-i\varsigma)^a\partial_a\psi(x)=im\sigma_z\psi(x),\sigma=(\sigma_x,\sigma_y)$

Thm. 1.7.3. $(\gamma^a\partial_a+m)\psi(x)=M\sigma_x\psi^*(x),\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)\Leftrightarrow(\sigma,-i\varsigma)^a\partial_a\psi(x)=im\sigma_z\psi(x)+M\sigma_y\psi^*(x)$

1.8 Majorana equation with z-restricted in 4D is equivalent to Penrose equation [1,2] in 3D

Cor. 1.8.1.

$$\begin{cases}(\sigma,-i\varsigma)_a\partial^a\nu(x)-me^{-2i\theta}\sigma_y\nu^*(x)=0\\ \psi(x)=\frac{1}{\sqrt2}\begin{bmatrix}\nu(x)-ie^{-2i\theta}\sigma_y\nu^*(x)\\-\nu(x)-ie^{-2i\theta}\sigma_y\nu^*(x)\end{bmatrix}\end{cases}\Leftrightarrow\begin{cases}(\gamma^a\partial_a+m)\psi(x)=0,\gamma^a=(\sigma\otimes\sigma_y,\varsigma I\otimes\sigma_z)\\ \psi^*(x)=-e^{2i\theta}\sigma_y\otimes\sigma_y\psi(x)\\ \nu(x)=\frac{1}{\sqrt2}[\psi_1(x)+ie^{-2i\theta}\sigma_y\psi_1^*(x)]\end{cases}$$

$$\begin{cases}\nu(x)=\frac{1}{(2\pi)^{N/2}}\int\frac{E+m-\varsigma\vec p\cdot\sigma}{\sqrt{2m(E+m)}}\frac{1}{\sqrt2}(\xi_0e^{i\varsigma p\cdot x}+ie^{-2i\theta}\sigma_y\xi_0^*e^{-i\varsigma p\cdot x})d^N\vec p\\ \psi(x)=\frac{1}{(2\pi)^{N/2}}\int\frac{E+m+\varsigma\vec p\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}}\begin{bmatrix}\xi_0e^{i\varsigma p\cdot x}\\-ie^{-2i\theta}\sigma_y\xi_0^*e^{-i\varsigma p\cdot x}\end{bmatrix}d^3\vec p=\frac{1}{(2\pi)^{N/2}}\int\begin{bmatrix}\frac{(E+m)\xi_0e^{i\varsigma p\cdot x}-\varsigma\vec p\cdot\sigma(ie^{-2i\theta}\sigma_y\xi_0^*)e^{-i\varsigma p\cdot x}}{\sqrt{2m(E+m)}}\\ \frac{-(E+m)(ie^{-2i\theta}\sigma_y\xi_0^*)e^{-i\varsigma p\cdot x}+\varsigma\vec p\cdot\sigma\xi_0e^{i\varsigma p\cdot x}}{\sqrt{2m(E+m)}}\end{bmatrix}d^N\vec p\end{cases}$$

$\xi_0=a(\vec p,\frac12)\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]+a(\vec p,-\frac12)\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ \xi_0=a(\vec p,-\frac{\varsigma}{2})\lambda(\hat p,-\frac{\varsigma}{2})+a(\vec p,\frac{\varsigma}{2})\lambda(\hat p,\frac{\varsigma}{2})$

2 Generalized B-W equation in 3-dimensional space-time

2.1 Generalized B-W equation in 3-dimensional space-time

Cor. 2.1.1.

$(\sigma,-i\varsigma)_a\partial^a\nu(x)-im\sigma_z\nu(x)-Me^{-2i\theta}\sigma_y\nu^*(x)=0\Leftrightarrow(\gamma^a\partial_a+m)\nu(x)=M\sigma_xe^{-2i\theta}\nu^*(x),\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$

$$\begin{cases}[i\varsigma\gamma^a(\vec p,iE_+)^a+(m+M)]\xi_+(\vec p)=0,[-i\varsigma\gamma^a(\vec p,iE_+)^a+(m+M)]\eta_+(\vec p)=0\\ [i\varsigma\gamma^a(\vec p,iE_-)^a+(m-M)]\xi_-(\vec p)=0,[-i\varsigma\gamma^a(\vec p,iE_-)^a+(m-M)]\eta_-(\vec p)=0\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec p\{\xi_+(\vec p)e^{i\varsigma[\vec p\cdot\vec r-E_+t]}-e^{2i\theta}\sigma_x\xi_+^*(\vec p)e^{-i\varsigma[\vec p\cdot\vec r-E_+t]}+\xi_-(\vec p)e^{i\varsigma[\vec p\cdot\vec r-E_-t]}+e^{2i\theta}\sigma_x\xi_-^*(\vec p)e^{-i\varsigma[\vec p\cdot\vec r-E_-t]}\}\end{cases}$$

Proof:
$(\sigma,-i\varsigma)_a\partial^a\nu(x)-im\sigma_z\nu(x)-Me^{-2i\theta}\sigma_y\nu^*(x)=0\Leftrightarrow(\gamma^a\partial_a+m)\nu(x)=M\sigma_xe^{-2i\theta}\nu^*(x),\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$

$\Rightarrow[\partial^a\partial_a-(m+M)^2][\partial^a\partial_a-(m-M)^2]\nu(x)=0,E_+=\sqrt{\vec{p}^2+(m+M)^2},E_-=\sqrt{\vec{p}^2+(m-M)^2}$
$\nu(x)=\frac{1}{(2\pi)^{3/2}}\int d^3\vec{p}\{\xi_+(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}+\eta_+(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}+\xi_-(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}+\eta_-(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}\}$
$\Leftrightarrow$
$i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a\{\xi_+(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}-\eta_+(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}\}$
$+i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a\{\xi_-(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}-\eta_-(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}\}$
$-im\sigma_z$
$\{\xi_+(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}+\eta_+(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}$
$+\xi_-(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}+\eta_-(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}\}$
$-Me^{-2i\theta}\sigma_y$
$\{\xi_+^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}+\eta_+^*(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}$
$+\xi_-^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}+\eta_-^*(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}\}=0$
$\Leftrightarrow$
$\{[i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a-im\sigma_z]\xi_+(\vec{p})-Me^{-2i\theta}\sigma_y\eta_+^*(\vec{p})\}e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}$
$+\{[-i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a-im\sigma_z]\eta_+(\vec{p})-Me^{-2i\theta}\sigma_y\xi_+^*(\vec{p})\}e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}$
$+\{[i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a-im\sigma_z]\xi_-(\vec{p})-Me^{-2i\theta}\sigma_y\eta_-^*(\vec{p})\}\{e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}$
$+\{[-i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a-im\sigma_z]\eta_-(\vec{p})-Me^{-2i\theta}\sigma_y\xi_-^*(\vec{p})\}e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}=0$
$\Leftrightarrow(M\neq0,m\neq0)$
$$\begin{cases}[i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a-im\sigma_z]\xi_+(\vec{p})-Me^{-2i\theta}\sigma_y\eta_+^*(\vec{p})=0\\ [-i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a-im\sigma_z]\eta_+(\vec{p})-Me^{-2i\theta}\sigma_y\xi_+^*(\vec{p})=0\\ [i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a-im\sigma_z]\xi_-(\vec{p})-Me^{-2i\theta}\sigma_y\eta_-^*(\vec{p})=0\\ [-i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a-im\sigma_z]\eta_-(\vec{p})-Me^{-2i\theta}\sigma_y\xi_-^*(\vec{p})=0\end{cases}$$
$\Leftrightarrow(M\neq0,m\neq0)$
$$\begin{cases}M\eta_+(\vec{p})=-e^{-2i\theta}\sigma_y[-i\varsigma(\sigma^*,-i\varsigma)_a(\vec{p},iE_+)^a+im\sigma_z]\xi_+^*(\vec{p})\\ [-i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a-im\sigma_z]\eta_+(\vec{p})-Me^{-2i\theta}\sigma_y\xi_+^*(\vec{p})=0\\ Me^{-2i\theta}\sigma_y\eta_-^*(\vec{p})=-e^{-2i\theta}\sigma_y[-i\varsigma(\sigma^*,-i\varsigma)_a(\vec{p},iE_-)^a+im\sigma_z]\xi_-^*(\vec{p})\\ [-i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a-im\sigma_z]\eta_-(\vec{p})-Me^{-2i\theta}\sigma_y\xi_-^*(\vec{p})=0\end{cases}$$
$\Leftrightarrow(M\neq0,m\neq0)$
$$\begin{cases}(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a\xi_+(\vec{p})=\varsigma\sigma_z(m+M)\xi_+(\vec{p}),(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a\eta_+(\vec{p})=-\varsigma\sigma_z(m+M)\eta_+(\vec{p})\\ M\eta_+^*(\vec{p})=e^{2i\theta}\sigma_y[i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a-im\sigma_z]\xi_+(\vec{p})\\ (\sigma,-i\varsigma)_a(\vec{p},iE_-)^a\xi_-(\vec{p})=\varsigma\sigma_z(m-M)\xi_-(\vec{p}),(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a\eta_-(\vec{p})=-\varsigma\sigma_z(m-M)\eta_-(\vec{p})\\ M\eta_-^*(\vec{p})=e^{2i\theta}\sigma_y[i\varsigma(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a-im\sigma_z]\xi_-(\vec{p})\end{cases}$$
$\Leftrightarrow(M\neq0,m\neq0)$
$$\begin{cases}(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a\xi_+(\vec{p})=\varsigma\sigma_z(m+M)\xi_+(\vec{p}),(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a\eta_+(\vec{p})=-\varsigma\sigma_z(m+M)\eta_+(\vec{p})\\ \eta_+(\vec{p})=-e^{-2i\theta}\sigma_x\xi_+^*(\vec{p})\\ (\sigma,-i\varsigma)_a(\vec{p},iE_-)^a\xi_-(\vec{p})=\varsigma\sigma_z(m-M)\xi_-(\vec{p}),(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a\eta_-(\vec{p})=-\varsigma\sigma_z(m-M)\eta_-(\vec{p})\\ \eta_-(\vec{p})=e^{-2i\theta}\sigma_x\xi_-^*(\vec{p})\end{cases}$$
$\Leftrightarrow$
$$\begin{cases}(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a\xi_+(\vec{p})=\varsigma\sigma_z(m+M)\xi_+(\vec{p}),(\sigma,-i\varsigma)_a(\vec{p},iE_+)^a\eta_+(\vec{p})=-\varsigma\sigma_z(m+M)\eta_+(\vec{p})\\ (\sigma,-i\varsigma)_a(\vec{p},iE_-)^a\xi_-(\vec{p})=\varsigma\sigma_z(m-M)\xi_-(\vec{p}),(\sigma,-i\varsigma)_a(\vec{p},iE_-)^a\eta_-(\vec{p})=-\varsigma\sigma_z(m-M)\eta_-(\vec{p})\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\{\xi_+(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}-e^{-2i\theta}\sigma_x\xi_+^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}\\ +\xi_-(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}+e^{-2i\theta}\sigma_x\xi_-^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}\}\end{cases}$$
$\Leftrightarrow$
$$\begin{cases}[i\varsigma\gamma^a(\vec{p},iE_+)^a+(m+M)]\xi_+(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE_+)^a+(m+M)]\eta_+(\vec{p})=0\\ [i\varsigma\gamma^a(\vec{p},iE_-)^a+(m-M)]\xi_-(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE_-)^a+(m-M)]\eta_-(\vec{p})=0\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\{\xi_+(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}-e^{-2i\theta}\sigma_x\xi_+^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}\\ +\xi_-(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}+e^{-2i\theta}\sigma_x\xi_-^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}\}\end{cases}$$ □

2.2 Generalized Majorana B-W equation in 3-dimensional space-time

Cor. 2.2.1.
$(\sigma,-i\varsigma)_a\partial^a\nu(x)-i(m\pm M)\sigma_z\nu(x)=0\Leftrightarrow(\gamma^a\partial_a+m\pm M)\nu(x)=0,\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$
$\nu(x)=i\sigma_x\nu^*(x),e^{-2i\theta}=-\pm i$
$$\begin{cases}[i\varsigma\gamma^a(\vec{p},iE_\pm)^a+(m\pm M)]\xi_\pm(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE_\pm)^a+(m\pm M)]\sigma_x\xi_\pm^*(\vec{p})=0\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\{\xi_\pm(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_\pm t]}+i\sigma_x\xi_\pm^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_\pm t]}\}\end{cases}$$

Cor. 2.2.2.
$(\sigma,-i\varsigma)_a\partial^a\nu(x)-im\sigma_z\nu(x)=0\Leftrightarrow(\gamma^a\partial_a+m)\nu(x)=0,\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$
$\nu(x)=i\sigma_x\nu^*(x),e^{-2i\theta}=-i$
$$\begin{cases}[i\varsigma\gamma^a(\vec{p},iE)^a+m]\xi(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE)^a+m]\sigma_x\xi^*(\vec{p})=0\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\{\xi(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-Et]}+i\sigma_x\xi^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-Et]}\}\end{cases}$$

2.3 Generalized B-W equation under real representation in 3-dimensional space-time

Cor. 2.3.1.
$S_{xy}S_c(\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}i&1\\1&i\end{bmatrix},[S_{xy}S_c(\frac{1}{2})]^+=\frac{1}{\sqrt{2}}\begin{bmatrix}-i&1\\1&-i\end{bmatrix}$
$S_{xy}S_c(\frac{1}{2})\sigma_x=\frac{1}{\sqrt{2}}\begin{bmatrix}1&i\\i&1\end{bmatrix},[S_{xy}S_c(\frac{1}{2})\sigma_x]^+=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-i\\-i&1\end{bmatrix}$
$\sigma_x[S_{xy}S_c(\frac{1}{2})]^+(-\sigma_y,\sigma_x,\varsigma\sigma_z)S_{xy}S_c(\frac{1}{2})\sigma_x^+=\sigma_x(\sigma_z,\sigma_x,\varsigma\sigma_y)\sigma_x^+=(-\sigma_z,\sigma_x,-\varsigma\sigma_y)$

Cor. 2.3.2.
$(\sigma,-i\varsigma)_a\partial^a\nu(x)+im\sigma_y\nu(x)-Me^{-2i\theta}\sigma_y\nu^*(x)=0\Leftrightarrow(\gamma^a\partial_a+m)\nu(x)=-iMe^{-2i\theta}\nu^*(x),\gamma^a=(-\sigma_z,\sigma_x,-\varsigma\sigma_y)_{xz\pi}$

$$\begin{cases}[i\varsigma\gamma^a(\vec{p},iE_+)^a+(m+M)]\xi_+(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE_+)^a+(m+M)]\eta_+(\vec{p})=0\\ [i\varsigma\gamma^a(\vec{p},iE_-)^a+(m-M)]\xi_-(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE_-)^a+(m-M)]\eta_-(\vec{p})=0\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\{\xi_+(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}+ie^{-2i\theta}\xi_+^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]}+\xi_-(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}-ie^{-2i\theta}\xi_-^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]}\}\end{cases}$$

2.4 Generalized Majorana B-W equation under real representation in 3D

Cor. 2.4.1.
$(\sigma,-i\varsigma)_a\partial^a\nu(x)+i(m\pm M)\sigma_y\nu(x)=0\Leftrightarrow(\gamma^a\partial_a+m\pm M)\nu(x)=0,\gamma^a=(-\sigma_z,\sigma_x,-\varsigma\sigma_y),\nu^*(x)=\nu(x)$
$$\begin{cases}[i\varsigma\gamma^a(\vec{p},iE_\pm)^a+(m\pm M)]\xi_\pm(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE_\pm)^a+(m\pm M)]\xi_\pm^*(\vec{p})=0\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\{\xi_\pm(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-E_\pm t]}+\xi_\pm^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_\pm t]}\}\end{cases}$$

Cor. 2.4.2.
$(\sigma,-i\varsigma)_a\partial^a\nu(x)+im\sigma_y\nu(x)=0\Leftrightarrow(\gamma^a\partial_a+m)\nu(x)=0,\gamma^a=(-\sigma_z,\sigma_x,-\varsigma\sigma_y),\nu^*(x)=\nu(x)$
$$\begin{cases}[i\varsigma\gamma^a(\vec{p},iE)^a+m]\xi(\vec{p})=0,[-i\varsigma\gamma^a(\vec{p},iE)^a+m]\xi^*(\vec{p})=0\\ \nu(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\{\xi(\vec{p})e^{i\varsigma[\vec{p}\cdot\vec{r}-Et]}+\xi^*(\vec{p})e^{-i\varsigma[\vec{p}\cdot\vec{r}-Et]}\}\end{cases}$$

3 B-W equation under visual representation in 3-dimensional space-time

3.1 Dirac equation under visual representation in 3-dimensional space-time

Proof: $D_{\vec{v}}=e^{-ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)}=\frac{1+\gamma_v-i\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_0}{\sqrt{2(\gamma_v+1)}}=\frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_0}{\sqrt{2m(E+m)}}=\frac{m-i\gamma^a p_a\gamma_0}{\sqrt{2m(E+m)}}$ □

Def. 3.1.1. $(\gamma^a\partial_a+m)\psi=0,\gamma^a=(1\otimes\sigma_x,1\otimes\sigma_y,\varsigma1\otimes\sigma_z)$

Def. 3.1.2. $u(\vec{p}):=\sqrt{\frac{E+m}{2m}}(1-\frac{i\varsigma\sigma\cdot\vec{p}\sigma_z}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],v(\vec{p}):=\sqrt{\frac{E+m}{2m}}(1-\frac{i\varsigma\sigma\cdot\vec{p}\sigma_z}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$

Cor. 3.1.1. $u(\vec{p})=\sigma_x v^*(\vec{p}),v(\vec{p})=\sigma_x u^*(\vec{p})$

Cor. 3.1.2. $S_{xy}(\sigma_x,\sigma_y,\sigma_z)S_{xy}^+=(-\sigma_y,\sigma_x,\sigma_z),S_{xy}=\begin{bmatrix}1&0\\0&-i\end{bmatrix},S_{xy}^+=\begin{bmatrix}1&0\\0&i\end{bmatrix}$

3.2 K-G spin basis $\varepsilon_a(\vec{p}),\tilde{\varepsilon}_a(\vec{p})$ and its properties under visual representation in 3D

Thm. 3.2.1. $u(\vec{p}):=\sqrt{\frac{E+m}{2m}}(1-\frac{i\varsigma\sigma\cdot\vec{p}\sigma_z}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\gamma^a=(\sigma_x,\sigma_y,\varsigma\sigma_z)$
$\Rightarrow\varepsilon_a(\vec{p})=-i(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p})=-iu^T(\vec{p})(\bar{\varepsilon}\gamma_a)u(\vec{p})$
$=(i\varsigma+\frac{i\varsigma p_x(p_x+i\varsigma p_y)}{m(E+m)},-1+\frac{i\varsigma p_y(p_x+i\varsigma p_y)}{m(E+m)},-\varsigma\frac{p_x+i\varsigma p_y}{m})=i\varsigma(1+\frac{p_x(p_x+i\varsigma p_y)}{m(E+m)},i\varsigma+\frac{p_y(p_x+i\varsigma p_y)}{m(E+m)},i\frac{p_x+i\varsigma p_y}{m})$

Thm. 3.2.2. $v(\vec{p}):=\sqrt{\frac{E+m}{2m}}(1-\frac{i\varsigma\sigma\cdot\vec{p}\sigma_z}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],\gamma^a=(\sigma_x,\sigma_y,\varsigma\sigma_z)$
$\Rightarrow\tilde{\varepsilon}_a(\vec{p})=-i(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\vec{p})$
$=(-i\varsigma-\frac{i\varsigma p_x(p_x-i\varsigma p_y)}{m(E+m)},-1-\frac{i\varsigma p_y(p_x-i\varsigma p_y)}{m(E+m)},\varsigma\frac{p_x-i\varsigma p_y}{m})=-i\varsigma(1+\frac{p_x(p_x-i\varsigma p_y)}{m(E+m)},-i\varsigma+\frac{p_y(p_x-i\varsigma p_y)}{m(E+m)},i\frac{(p_x-i\varsigma p_y)}{m})$

Cor. 3.2.1. $\tilde{\varepsilon}_a(\vec{p})=\varepsilon_{a'}^+(\vec{p})\eta_a^{a'},\tilde{\varepsilon}_{\underbrace{ab\cdots}_n}(\vec{p})=\varepsilon^+_{\underbrace{a'b'\cdots}_n}(\vec{p})\underbrace{\eta_a^{a'}\eta_b^{b'}\cdots}_n$

3.3 s-spin equation under visual representation in 3-dimensional space-time

Thm. 3.3.1. $[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdots}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x)$ *fully symmetric,* $\gamma_a:=[\sigma_x,\sigma_y,\varsigma\sigma_z]$

$\Leftrightarrow[s\partial_a+m\gamma_a(s)+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=\frac{i\varsigma m}{s\sqrt{2}}\gamma_a(s)\psi(s),S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)],\gamma_a(s):=[\sigma_x(s),\sigma_y(s),\varsigma\sigma_z(s)]$

Proof: $[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdots}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x)$ fully symmetric

$\Leftrightarrow[\gamma^a\partial_a+m]\hat{\psi}(s)=0$
$\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_aD^a\hat{\psi}(s)=-m\hat{\psi}(s),D^a=(\partial^x,\partial^y,\varsigma\partial^\pi,0)$
$\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_aD^a[I\otimes\Gamma(s)]N(s)\psi(s)=-m\hat{\psi}(s)$
$\Leftrightarrow[I\otimes\Gamma(s)](\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=-m\hat{\psi}(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=-m[I\otimes\bar{\Gamma}(s)]\hat{\psi}(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=-mN(s)\psi(s)$
$\Leftrightarrow Z_bD^b\psi(s)=-m\frac{i\varsigma}{\sqrt{2}}N(s)\psi(s)$
$\Leftrightarrow\bar{Z}_aZ_bD^b\psi(s)=-m\frac{-i\varsigma}{\sqrt{2}}\bar{Z}_aN(s)\psi(s)$
$\Leftrightarrow\bar{Z}_aZ_bD^b\psi(s)=-\frac{m}{2}\bar{N}(s)(\sigma\otimes I_{2s},i\varsigma)_aN(s)\psi(s)$
$\Leftrightarrow\bar{Z}_aZ_bD^b\psi(s)=-\frac{m}{2s}[\sigma(s),is\varsigma]_a\psi(s)$
$\Leftrightarrow[sD_a+iS_{ab}(s,\varsigma;4)D^b]\psi(s)=-m[\sigma(s),is\varsigma]_a\psi(s),S_{ab}(s,\varsigma;4)\succ\begin{bmatrix}0&\sigma_z(s)&-\sigma_y(s)&-\varsigma\sigma_x(s)\\-\sigma_z(s)&0&\sigma_x(s)&-\varsigma\sigma_y(s)\\\sigma_y(s)&-\sigma_x(s)&0&-\varsigma\sigma_z(s)\\\varsigma\sigma_x(s)&\varsigma\sigma_y(s)&\varsigma\sigma_z(s)&0\end{bmatrix}$
$\Leftrightarrow[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=-m\gamma_a(s)\psi(s),S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)]\succ\begin{bmatrix}0&\sigma_z(s)&-\varsigma\sigma_y(s)\\-\sigma_z(s)&0&\varsigma\sigma_x(s)\\\varsigma\sigma_y(s)&-\varsigma\sigma_x(s)&0\end{bmatrix}$
$\Leftrightarrow[s\partial_a+m\gamma_a(s)+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0$ □

4 B-W equation without mass in 3-dimensional space-time(m is only a parameter.)

4.1 Penrose equation for massless particles in 3-dimensional space-time

4.1.1 Helicity function for massless particles in 3-dimensional space-time

Def. 4.1.1. $\sigma(\frac{1}{2})\cdot\hat{p}\lambda(\hat{p},h)=h\lambda(\hat{p},h),h=-\frac{1}{2},\frac{1}{2}$

4.1.2 Penrose equation [1,2] and helicity eigenfunction for massless particles in 3D

Def. 4.1.2. $\gamma^a\partial_a\psi(x)=0,\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)=S_{xy}(\sigma_x,\sigma_y,\varsigma\sigma_z)S^+_{xy}\Leftrightarrow(\sigma,-i\varsigma)^a\partial_a\psi(x)=0,\sigma=(\sigma_x,\sigma_y)$

Proof: $\gamma^a\partial_a\psi(x)=0,\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$
$\Leftrightarrow(-\sigma_y\partial_x+\sigma_x\partial_y)\psi(x)=-\varsigma\sigma_z\partial_\pi\psi(x)$
$\Leftrightarrow(\sigma_x\partial_x+\sigma_y\partial_y)\psi(x)=i\varsigma\partial_\pi\psi(x)$
$\Leftrightarrow(\sigma_y\partial_y+\sigma_x\partial_x)\psi(x)=i\varsigma\partial_\pi\psi(x)$
$\Leftrightarrow(\sigma_y\partial_y+\sigma_x\partial_x)\psi(\vec{p})e^{ip\cdot x}=i\varsigma\partial_\pi\psi(\vec{p})e^{ip\cdot x}$
$\Leftrightarrow(\sigma_xp_x+\sigma_yp_y)\psi(\vec{p})e^{ip\cdot x}=i\varsigma p_\pi\psi(\vec{p})e^{ip\cdot x}$
$\Leftrightarrow(\sigma_x\hat{p}_x+\sigma_y\hat{p}_y)\lambda(\hat{p},-\frac{\varsigma}{2})=-\varsigma\lambda(\hat{p},-\frac{\varsigma}{2})$
$\Leftrightarrow(\sigma_x\hat{p}_x+\sigma_y\hat{p}_y)\lambda(\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\0\end{bmatrix},-\frac{\varsigma}{2})=-\varsigma\lambda(\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\0\end{bmatrix},-\frac{\varsigma}{2})$ □

Cor. 4.1.1. $\lambda(\hat{p},\frac{1}{2})=\lambda(\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\0\end{bmatrix},\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\\hat{p}_x+i\hat{p}_y\end{bmatrix},\lambda(\hat{p},-\frac{1}{2})=\lambda(\begin{bmatrix}\hat{p}_x\\\hat{p}_y\\0\end{bmatrix},-\frac{1}{2})=\frac{1}{\sqrt{2}}\begin{bmatrix}-(\hat{p}_x-i\hat{p}_y)\\1\end{bmatrix}$

Cor. 4.1.2. $\lambda(\hat{p},\frac{1}{2})\lambda^+(\hat{p},\frac{1}{2})=\begin{cases}-\frac{i}{2}\gamma^0\gamma^a\hat{p}_a,\varsigma=1\\-\frac{i}{2}\gamma^a\hat{p}_a\gamma^0,\varsigma=-1\end{cases}\quad\lambda(\hat{p},-\frac{1}{2})\lambda^+(\hat{p},-\frac{1}{2})=\begin{cases}-\frac{i}{2}\gamma^a\hat{p}_a\gamma^0,\varsigma=1\\-\frac{i}{2}\gamma^0\gamma^a\hat{p}_a,\varsigma=-1\end{cases}$

Cor. 4.1.3. $\begin{cases}\lambda(\hat{p},-\frac{\varsigma}{2})\lambda^+(\hat{p},-\frac{\varsigma}{2})=-\frac{i}{2}\gamma^a\hat{p}_a\gamma^0=-\frac{\varsigma}{2}(\sigma,i\varsigma)^a\hat{p}_a\\\lambda(\hat{p},\frac{\varsigma}{2})\lambda^+(\hat{p},\frac{\varsigma}{2})=-\frac{i}{2}\gamma^0\gamma^a\hat{p}_a=\frac{\varsigma}{2}(\sigma,-i\varsigma)^a\hat{p}_a\end{cases}$

Cor. 4.1.4. $\sigma(\frac{1}{2})\cdot\hat{p}\lambda(\hat{p},h)=h\lambda(\hat{p},h),h=-\frac{1}{2},\frac{1}{2}$

Cor. 4.1.5. $\lambda^+(\hat{p},h)\lambda(\hat{p},h')=\delta_{hh'},\sum_{h=1/2}^{-1/2}\lambda(\hat{p},h)\lambda^+(\hat{p},h)=1,\sum_{h=1/2}^{-1/2}h\lambda(\hat{p},h)\lambda^+(\hat{p},h)=\sigma(\frac{1}{2})\cdot\hat{p}$

4.1.3 Plane wave solutions and its spin basis for massless particles in 3D

Thm. 4.1.1. $\gamma^a\partial_a\psi(x)=0\Leftrightarrow(\sigma,-i\varsigma)_a\partial^a\psi(x)=0$

Cor. 4.1.6. $\begin{cases}\psi(\vec r,t):=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec p\neq0}\lambda(\hat p,-\frac{\varsigma}{2})[a_1(\vec p,-\frac{\varsigma}{2})e^{ip\cdot x}+a_2^+(\vec p,-\frac{\varsigma}{2})e^{-ip\cdot x}]d^N\vec p\\ a_1(\vec p,-\frac{\varsigma}{2})=\frac{1}{(2\pi)^{N/2}}\int\lambda^+(\hat p,-\frac{\varsigma}{2})\psi(\vec r,t)e^{-ip\cdot x}d^N\vec r\\ a_2^+(\vec p,-\frac{\varsigma}{2})=\frac{1}{(2\pi)^{N/2}}\int\lambda^+(\hat p,-\frac{\varsigma}{2})\psi(\vec r,t)e^{ip\cdot x}d^N\vec r\end{cases}$

Thm. 4.1.2. $\gamma^a{}_{Z_\varsigma}{}^{A_\varsigma}\partial_a\psi_{\underbrace{A_\varsigma B_\varsigma\ldots}_{2s}}(x)=0\Leftrightarrow(\sigma,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\psi_{\underbrace{A_\varsigma B_\varsigma\ldots}_{2s}}(x)=0$

Cor. 4.1.7.

$$\begin{cases}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec p\neq0}|\vec p|^{(s-\frac12)}\underbrace{\lambda_{A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\frac{\varsigma}{2})\cdot\cdot}_{2s}[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^3\vec p\\ \vec p|^{(s-\frac12)}a_1(\vec p,-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\overbrace{\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p,-\frac{\varsigma}{2})\cdot\cdot}^{2s}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(x)e^{-ip\cdot x}d^3\vec r\\ |\vec p|^{(s-\frac12)}a_2^+(\vec p,-s\varsigma)=\frac{1}{(2\pi)^{3/2}}\int\overbrace{\lambda^{+A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda^{+B_\varsigma}(\hat p,-\frac{\varsigma}{2})\cdot\cdot}^{2s}\psi_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(x)e^{ip\cdot x}d^3\vec r\end{cases}$$

Def. 4.1.3. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\hat p,-s\varsigma):=\underbrace{\lambda_{A_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat p,-\frac{\varsigma}{2})\cdot\cdot}_{2s}$

Cor. 4.1.8. $\lambda^{+\overbrace{A_\varsigma B_\varsigma\cdot\cdot}^{2s}}(\hat p,-s\varsigma)\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\hat p,-s\varsigma)=1$

Cor. 4.1.9. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\hat p,-s\varsigma)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(\hat p,-s\varsigma)=\frac{1}{(2|\vec p|)^{2s}}\underbrace{[(-i\gamma^ap_a)\gamma^0]_{A_\varsigma A'_\varsigma}[(-i\gamma^bp_b)\gamma^0]_{B_\varsigma B'_\varsigma}\cdot\cdot}_{2s}$

Cor. 4.1.10. $\lambda_{\underbrace{A_\varsigma B_\varsigma\cdot\cdot}_{2s}}(\hat p,-s\varsigma)\lambda^+_{\underbrace{A'_\varsigma B'_\varsigma\cdot\cdot}_{2s}}(\hat p,-s\varsigma)=\frac{1}{(-\varsigma2|\vec p|)^{2s}}\underbrace{(\sigma,i\varsigma)^a_{A_\varsigma A'_\varsigma}(\sigma,i\varsigma)^b_{B_\varsigma B'_\varsigma}\cdot\cdot}_{2s}\underbrace{p_ap_b\cdot\cdot}_{2s}$

4.2 Concrete expression of massless particle potential equation in 3-dimensional space-time

4.2.1 Massless B-W equation with $s=1$ is equivalent to similar K-G equation in 3D

Thm. 4.2.1. $\gamma^a\partial_a\psi(x)=0,\psi^T(x)=\psi(x),A_a=\frac{1}{\sqrt2im}tr(\bar\varepsilon\gamma_a\psi),\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)$
$\Leftrightarrow\partial_aA_b-\partial_bA_a=0,\partial^aA_a=0,\psi=\frac{im}{\sqrt2}\gamma^a\varepsilon A_a\Rightarrow\partial^b\partial_bA_a=0,\partial^aA_a=0$

Proof: $\gamma^a\partial_a\psi(x)=0,\psi=\frac{im}{\sqrt2}\gamma^a\varepsilon A_a$
$\Leftrightarrow\gamma^a\partial_a\frac{im}{\sqrt2}\gamma^b\varepsilon A_b=0$
$\Leftrightarrow\gamma^a\partial_a\gamma^bA_b=0$
$\Leftrightarrow\delta^{ab}\partial_aA_b+i\varsigma\varepsilon^{abc}\partial_aA_b\gamma_c=0$
$\Leftrightarrow\partial^aA_a+i\varsigma\varepsilon^{ab}{}_c\partial_aA_b\gamma^c=0$
$\Leftrightarrow\partial^aA_a=0,i\varsigma\varepsilon^{ab}{}_c\partial_aA_b=0$
$\Leftrightarrow\partial^aA_a=0,\varepsilon^{ab}{}_c\partial_aA_b=0\Leftrightarrow\partial^aA_a=0,\nabla\times\vec A=0$
$\Leftrightarrow\partial^aA_a=0,\varepsilon^{a'b'c}\varepsilon^{ab}{}_c\partial_aA_b=0$
$\Leftrightarrow\partial^aA_a=0,(\delta^{a'a}\delta^{b'b}-\delta^{a'b}\delta^{b'a})\partial_aA_b=0$
$\Leftrightarrow\partial^aA_a=0,\partial_aA_b-\partial_bA_a=0$
$\Rightarrow\partial^aA_a=0,\partial^a\partial_aA_b-\partial_b\partial^aA_a=0$
$\Leftrightarrow\partial^a\partial_aA_b=0,\partial^aA_a=0$ □

4.2.2 Massless B-W equation with $s=\frac32$ is equivalent to similar R-S equation in 3D

Thm. 4.2.2. $(\gamma^a\partial_a)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{1}{3!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\}},A_{a\eta_\varsigma}=\frac{1}{\sqrt2im}(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}$
$\Leftrightarrow\begin{cases}\partial_aA_{b\eta_\varsigma}-\partial_bA_{a\eta_\varsigma}=0,\partial^aA_{a\eta_\varsigma}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{im}{\sqrt2}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma},\gamma^aA_{a[\eta_\varsigma]}=0\end{cases}\Rightarrow\gamma^b\partial_bA_{a[\eta_\varsigma]}=0,\gamma^aA_{a[\eta_\varsigma]}=0,\partial^aA_{a\eta_\varsigma}=0$

Proof: $\begin{cases}(\gamma^a\partial_a)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)=0,\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{1}{3!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\}}\\ A_{a\eta_\varsigma}=\frac{1}{\sqrt2im}(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma},\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)\end{cases}$
$\Leftrightarrow\begin{cases}\partial_aA_{b\eta_\varsigma}-\partial_bA_{a\eta_\varsigma}=0,\partial^aA_{a\eta_\varsigma}=0\\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\frac{im}{\sqrt2}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma},\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=\psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}\end{cases}$

$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\varsigma} - \partial_b A_{a\eta_\varsigma} = 0, \partial^a A_{a\eta_\varsigma} = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma} = \frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}, \varepsilon^{\mu_\varsigma\eta_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma} = 0 \end{cases}$
$\Leftrightarrow \partial_a A_{b\eta_\varsigma} - \partial_b A_{a\eta_\varsigma} = 0, \gamma^a A_{a[\eta_\varsigma]} = 0, \partial^a A_{a\eta_\varsigma} = 0$
$\Rightarrow \gamma^a \partial_a A_{b[\eta_\varsigma]} - \partial_b \gamma^a A_{a[\eta_\varsigma]} = 0, \gamma^a A_{a\eta_\varsigma} = 0, \partial^a A_{a\eta_\varsigma} = 0$
$\Leftrightarrow \gamma^a \partial_a A_{b[\eta_\varsigma]} = 0, \gamma^a A_{a\eta_\varsigma} = 0, \partial^a A_{a\eta_\varsigma} = 0$
$\Leftrightarrow \gamma^a \partial_a A_{b\eta_\varsigma} = 0, \gamma^a A_{a\eta_\varsigma} = 0, \partial^a A_{a\eta_\varsigma} = 0$
$\Leftrightarrow \gamma^b \partial_b A_{a[\eta_\varsigma]} = 0, \gamma^a A_{a[\eta_\varsigma]} = 0, \partial^a A_{a\eta_\varsigma} = 0$ □

4.2.3 Massless B-W equation with $s = 2$ is equivalent to similar K-G equation in 3D

Thm. 4.2.3. $(\gamma^a\partial_a)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x) = 0, \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}}, A_{ab} = (\frac{1}{\sqrt{2}im})^2(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}$

$\Leftrightarrow \begin{cases} \partial_a A_{bd} - \partial_b A_{ad} = 0, \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab}A_{ab} = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = (\frac{im}{\sqrt{2}})^2(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma}A_{ab} \end{cases} \Rightarrow \begin{cases} \partial^c\partial_c A_{ab} = 0, \partial^a A_{ab} = 0 \\ A_{ab} = A_{ba}, \delta^{ab}A_{ab} = 0 \end{cases}$

Proof:
$\begin{cases} (\gamma^a\partial_a)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x) = 0, \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}} \\ A_{ab} := (\frac{1}{\sqrt{2}im})^2(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}, \gamma^a = (-\sigma_y, \sigma_x, \varsigma\sigma_z) \end{cases}$
$\Leftrightarrow \begin{cases} (\gamma^a\partial_a)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x) = 0, \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{1}{4!}\psi_{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\}} \\ A_{a\eta_\varsigma\xi_\varsigma} := \frac{1}{\sqrt{2}im}(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} \end{cases}$
$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\varsigma\xi_\varsigma} - \partial_b A_{a\eta_\varsigma\xi_\varsigma} = 0, A_{a\eta_\varsigma\xi_\varsigma} = A_{a\xi_\varsigma\eta_\varsigma} \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma\xi_\varsigma}, \gamma^a A_{a[\eta_\varsigma]\xi_\varsigma} = 0, \partial^a A_{a\eta_\varsigma\xi_\varsigma} = 0 \end{cases}$
$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\varsigma\xi_\varsigma} - \partial_b A_{a\eta_\varsigma\xi_\varsigma} = 0, A_{a\eta_\varsigma\xi_\varsigma} = A_{a\xi_\varsigma\eta_\varsigma} \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = \frac{im}{\sqrt{2}}(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma\xi_\varsigma}, \gamma^a A_{a[\eta_\varsigma]\xi_\varsigma} = 0, \partial^a A_{a\eta_\varsigma\xi_\varsigma} = 0 \end{cases}$
$\Leftrightarrow \begin{cases} \partial_a A_{bd} - \partial_b A_{ad} = 0, \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab}A_{ab} = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma} = (\frac{im}{\sqrt{2}})^2(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma}A_{ab} \end{cases}$
$\Rightarrow \partial^c\partial_c A_{ab} = 0, \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab}A_{ab} = 0$ □

4.3 General expression of massless particle potential equation in 3-dimensional space-time

4.3.1 Massless B-W equation with $s = n$ is equivalent to similar K-G equation in 3D

Thm. 4.3.1.

$$\begin{cases} (\gamma^a\partial_a)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x) = 0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}} = \frac{1}{(2n)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots\}}_{2n}} \\ A_{\underbrace{ab\cdots}_{n}} = (\frac{1}{\sqrt{2}im})^n\underbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}_{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}} \end{cases} \Leftrightarrow \begin{cases} \partial_a A_{\underbrace{bd\cdots}_{n}} - \partial_b A_{\underbrace{ad\cdots}_{n}} = 0, \partial^a A_{\underbrace{ab\cdots}_{n}} = 0 \\ A_{\underbrace{ab\cdots}_{n}} = \frac{1}{n!}A_{\underbrace{\{ab\cdots\}}_{n}}, \delta^{ab}A_{\underbrace{ab\cdots}_{n}} = 0 \\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}} = (\frac{im}{\sqrt{2}})^n\underbrace{(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdots}_{n}A_{\underbrace{ab\cdots}_{n}} \end{cases}$$

$\Rightarrow\Rightarrow$

$$\begin{cases} \partial^c\partial_c A_{\underbrace{ab\cdots}_{n}} = 0, \partial^a A_{\underbrace{ab\cdots}_{n}} = 0 \\ A_{\underbrace{ab\cdots}_{n}} = \frac{1}{n!}A_{\underbrace{\{ab\cdots\}}_{n}}, \delta^{ab}A_{\underbrace{ab\cdots}_{n}} = 0 \end{cases}$$

Cor. 4.3.1.

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x) = \frac{1}{(2\pi)^{N/2}}\int_{\vec p\neq 0}|\vec p|^{(n-\frac{1}{2})}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\hat p, -\tfrac{n\varsigma}{2})[a_1(\vec p, -n\varsigma)e^{ip\cdot x} + a_2^+(\vec p, -n\varsigma)e^{-ip\cdot x}]d^N\vec p$$

$$A_{\underbrace{ab\cdots}_{n}}(x) = \frac{1}{(2\pi)^{N/2}}\int_{\vec p\neq 0}\frac{1}{\sqrt{2^nE}}\lambda_{\underbrace{ab\cdots}_{n}}(\vec p)[a_1(\vec p, -s\varsigma)e^{ip\cdot x} + a_2^+(\vec p, -s\varsigma)e^{-ip\cdot x}]d^N\vec p$$

$$\lambda_{\underbrace{ab\cdots}_{n}}(\vec p) = \frac{1}{i^n}\overbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec p, -\tfrac{n\varsigma}{2})$$

4.3.2 Spin bases relations on massive $s=n$ B-W equation and similar K-G equation in 3D

Cor. 4.3.2.

$$\begin{cases}(i\gamma^a p_a+m)\lambda_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\hat p,-\frac{n\varsigma}{2})=0\\ \lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\hat p,-\frac{n\varsigma}{2})\ fully\ symmetric\\ \lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)\\ =\frac{1}{i^n}\overbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec p,-\frac{n\varsigma}{2})\end{cases}\Leftrightarrow\begin{cases}(p^cp_c+m^2)\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)=0\\ p_a\lambda_{\underbrace{bd\cdot\cdot}_{n}}(x)-p_b\lambda_{\underbrace{ad\cdot\cdot}_{n}}=\varsigma m\lambda_{ab}{}^c\lambda_{\underbrace{cd\cdot\cdot}_{n}}\\ \delta^{ab}\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)=0,p^a\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)=0,\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)\ fully\ symmetric\\ \lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec p,-\frac{n\varsigma}{2})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)\end{cases}$$

Cor. 4.3.3.

$$\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec p,-\frac{n\varsigma}{2})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)[\Leftrightarrow]\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)=\frac{1}{i^n}\overbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(\vec p,-\frac{n\varsigma}{2})$$

Cor. 4.3.4. $\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\hat p,-s\varsigma)=\underbrace{\lambda_{\lambda_\varsigma}(\hat p,-\frac{\varsigma}{2})\lambda_{\mu_\varsigma}(\hat p,-\frac{\varsigma}{2})\cdot\cdot}_{2n},\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec p)=\underbrace{\lambda_a(\vec p)\lambda_b(\vec p)\cdot\cdot}_{n}$

Cor. 4.3.5. $\lambda_a(\vec p,-\varsigma)$
$=\frac{1}{i}(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\lambda_{\lambda_\varsigma\mu_\varsigma}(\vec p,-\varsigma)$
$=\frac{1}{i}\lambda_{\lambda_\varsigma}(\vec p,-\frac{\varsigma}{2})(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\lambda_{\mu_\varsigma}(\vec p,-\frac{\varsigma}{2})$
$=(\hat p_x-i\varsigma\hat p_y)(\hat p_x,\hat p_y,i)_a$
$=(\hat p_x-i\varsigma\hat p_y)\hat p_a$
$\lambda^{+a}(\vec p,-\varsigma)\lambda_a(\vec p,-\varsigma)=2,\lambda_a(\vec p,-\varsigma)\lambda^+_{a'}(\vec p,-\varsigma)=\hat p_a\hat p^+_{a'}$

Cor. 4.3.6. $\lambda(\hat p,\frac12)=\lambda(\begin{bmatrix}\hat p_x\\ \hat p_y\\ 0\end{bmatrix},\frac12)=\frac{1}{\sqrt2}\begin{bmatrix}1\\ \hat p_x+i\hat p_y\end{bmatrix},\lambda(\hat p,-\frac12)=\lambda(\begin{bmatrix}\hat p_x\\ \hat p_y\\ 0\end{bmatrix},-\frac12)=\frac{1}{\sqrt2}\begin{bmatrix}-(\hat p_x-i\hat p_y)\\ 1\end{bmatrix}$

4.3.3 Massless B-W equation with $s=n+\frac12$ is equivalent to similar R-S equationin 3D

Thm. 4.3.2.

$$\begin{cases}(\gamma^a\partial_a)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}=\frac{1}{(2n+1)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma\}}_{2n+1}}\\ A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}\\ =(\frac{1}{\sqrt2 im})^n\underbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}_{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}\end{cases}\Leftrightarrow\begin{cases}\partial_aA_{\underbrace{bd\cdot\cdot}_{n}\tau_\varsigma}-\partial_bA_{\underbrace{ad\cdot\cdot}_{n}\tau_\varsigma}=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}=0\\ A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}=\frac{1}{n!}A_{\underbrace{\{ab\cdot\cdot\}}_{n}\tau_\varsigma},\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}=0,\gamma^aA_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}=0\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}=(\frac{im}{\sqrt2})^n\underbrace{(\gamma^a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma^b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}_{n}A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}\end{cases}$$

$\Rightarrow\Rightarrow$

$$\begin{cases}\gamma^c\partial_cA_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}=0,\partial^aA_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}=0\\ A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}=\frac{1}{n!}A_{\underbrace{\{ab\cdot\cdot\}}_{n}\tau_\varsigma},\delta^{ab}A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}=0,\gamma^aA_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}=0\end{cases}$$

Cor. 4.3.7.

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)=\frac{1}{(2\pi)^{N/2}}\int_{\vec p\neq0}|\vec p|^n\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\hat p,-\frac{s\varsigma}{2})[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^N\vec p$$

$$A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int_{\vec p\neq0}\frac{1}{\sqrt{2^nE}}\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec p)[a_1(\vec p,-s\varsigma)e^{ip\cdot x}+a_2^+(\vec p,-s\varsigma)e^{-ip\cdot x}]d^N\vec p$$

$$\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec p)=\frac{1}{i^n}\overbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar\varepsilon\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec p,-\frac{s\varsigma}{2})$$

4.3.4 Spin bases relations on massive $s=n+\frac{1}{2}$ B-W and similar R-S equation in 3D

Cor. 4.3.8.

$$\begin{cases}(i\gamma^a p_a+m)\lambda_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\hat{p},-\frac{n\varsigma}{2})=0\\ \lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\hat{p},-\frac{n\varsigma}{2})\ \textit{fully symmetric}\\ \lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\\ \lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},-\frac{n\varsigma}{2})\end{cases}\Leftrightarrow\begin{cases}(p^c p_c+m^2)\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})=0,p^a\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})=0\\ p_a\lambda_{\underbrace{bd\cdot\cdot}_{n}\tau_\varsigma}(x)-p_b\lambda_{\underbrace{ad\cdot\cdot}_{n}\tau_\varsigma}=\varsigma m\lambda_{ab}{}^{c}\lambda_{\underbrace{cd\cdot\cdot}_{n}\tau_\varsigma}\\ \delta^{ab}\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})=0,\gamma^a\lambda_{\underbrace{ab\cdot\cdot}_{n}[\tau_\varsigma]}=0,\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})\ \textit{fully symmetric}\\ \lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},-\frac{n\varsigma}{2})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})\end{cases}$$

Cor. 4.3.9. $\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},-\frac{n\varsigma}{2})=(\frac{i}{2})^n\overbrace{(\gamma_a\varepsilon)_{\lambda_\varsigma\mu_\varsigma}(\gamma_b\varepsilon)_{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})$

$[\Leftrightarrow]\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p},-\frac{n\varsigma}{2})$

Cor. 4.3.10.

$\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\hat{p},-s\varsigma)=\underbrace{\lambda_{\lambda_\varsigma}(\hat{p},-\frac{\varsigma}{2})\lambda_{\mu_\varsigma}(\hat{p},-\frac{\varsigma}{2})\cdot\cdot\lambda_{\tau_\varsigma}(\hat{p},-\frac{\varsigma}{2})}_{2n+1},\lambda_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})=\underbrace{\lambda_a(\vec{p})\lambda_b(\vec{p})\cdot\cdot}_{n}\lambda_{\tau_\varsigma}(\hat{p},-\frac{\varsigma}{2})=\lambda_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\lambda_{\tau_\varsigma}(\hat{p},-\frac{\varsigma}{2})$

4.4 s-spin equation without mass in 3-dimensional space-time

Thm. 4.4.1.

$$\begin{cases}\gamma^a\partial_a\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)\ \textit{fully symmetric}\\ \psi_{k_\varsigma}(s,\varsigma):=\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}},\gamma^a=(-\sigma_y,\sigma_x,\varsigma\sigma_z)\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s,\varsigma)=0,S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)]\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}=\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}\psi_{k_\varsigma}(s,\varsigma),S_{ab}(s,\varsigma)=\begin{bmatrix}0&\sigma_z(s)&-\varsigma\sigma_x(s)\\-\sigma_z(s)&0&-\varsigma\sigma_y(s)\\ \varsigma\sigma_x(s)&\varsigma\sigma_y(s)&0\end{bmatrix}\end{cases}$$

$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma)e^{-ip\cdot x}]d^N\vec{p}$

$\psi_{k_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}\neq0}|\vec{p}|^{(s-\frac{1}{2})}\lambda_{k_\varsigma}(\hat{p},-s\varsigma)[a_1(\vec{p},-s\varsigma)e^{ip\cdot x}+a_2^+(\vec{p},-s\varsigma)e^{-ip\cdot x}]d^N\vec{p}$

$\lambda_{k_\varsigma}(\vec{p},-s\varsigma):=\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s\varsigma)\Leftrightarrow\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p},-s\varsigma)=\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}\lambda_{k_\varsigma}(\vec{p},-s\varsigma)$

4.4.1 Covariant commutation rules for massless particles in 3-dimensional space-time

Thm. 4.4.2. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]=\frac{i}{2^{2s-1}}\underbrace{[(-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}_{2s}\Delta(x-x')$

$[\Updownarrow]$

Thm. 4.4.3. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2s}}(x')]=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\underbrace{[(\sigma,i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[(\sigma,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdot\cdot}_{2s}\Delta(x-x')$

$[\Updownarrow]$

Thm. 4.4.4. $[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdot\cdot}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdot\cdot}^{2s}\Delta(x-x')$

$[\Updownarrow]$

Thm. 4.4.5. $[A_{\underbrace{ab\cdot\cdot}_{n}}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')]=\frac{i}{2^{n-1}}\underbrace{\frac{\partial_a\partial^+_{a'}}{\nabla^2}\frac{\partial_b\partial^+_{b'}}{\nabla^2}\cdot\cdot}_{n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 4.4.6. $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\} = \frac{i}{2^n}\underbrace{\frac{\partial_a\partial^+_{a'}}{\nabla^2}\frac{\partial_b\partial^+_{b'}}{\nabla^2}\cdot\cdot}_{n}[(-\gamma^c\partial_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 4.4.7. $\{A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')\} = i\varsigma\frac{i}{2^n}\underbrace{\frac{\partial_a\partial^+_{a'}}{\nabla^2}\frac{\partial_b\partial^+_{b'}}{\nabla^2}\cdot\cdot}_{n}[(\sigma,i\varsigma)^c\partial_c]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

5 Covariate quantization for massive particles in 2-dimensional space-time

5.1 Dirac equation in 2-dimensional space-time

5.1.1 Dirac spin basis in 2-dimensional space-time

Def. 5.1.1. $u(\vec{p}) := \sqrt{\frac{E+m}{2m}}(1-\frac{\varsigma p_x\sigma_x}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}1\\0\end{bmatrix}, v(\vec{p}) := \sqrt{\frac{E+m}{2m}}(1-\frac{\varsigma p_x\sigma_x}{E+m})(\frac{1+\varsigma}{2}+\frac{1-\varsigma}{2}\sigma_x)\begin{bmatrix}0\\1\end{bmatrix}$

Cor. 5.1.1. $u(\vec{p}) = \sigma_x v(\vec{p}), v(\vec{p}) = \sigma_x u(\vec{p}), u^*(\vec{p}) = u(\vec{p}), v^*(\vec{p}) = v(\vec{p})$

5.1.2 Properties of Dirac spin basis in 2-dimensional space-time

Pro. 5.1.1. $\begin{cases}\bar{u}(\vec{p})u(\vec{p}) = 1, \bar{v}(\vec{p})v(\vec{p}) = -1, \bar{u}(\vec{p})v(\vec{p}) = 0, \bar{v}(\vec{p})u(\vec{p}) = 0\\ u^+(\vec{p})u(\vec{p}) = \frac{E}{m}, v^+(\vec{p})v(\vec{p}) = \frac{E}{m}, u^+(\vec{p})v(-\vec{p}) = 0, v^+(\vec{p})u(-\vec{p}) = 0\end{cases}$

Pro. 5.1.2. $\begin{cases}u(\vec{p})\bar{u}(\vec{p}) = \frac{m-i\gamma^a p_a}{2m}\\ v(\vec{p})\bar{v}(\vec{p}) = \frac{-m-i\gamma^a p_a}{2m}\end{cases}\quad \begin{cases}u(\vec{p})u^+(\vec{p}) = \frac{(m-i\gamma^a p_a)\gamma^0}{2m} = \frac{m\sigma_z-(\sigma,i\varsigma)^a p_a}{\varsigma 2m}\\ v(\vec{p})v^+(\vec{p}) = \frac{(-m-i\gamma^a p_a)\gamma^0}{2m} = \frac{-m\sigma_z-(\sigma,i\varsigma)^a p_a}{\varsigma 2m}\end{cases}$

Pro. 5.1.3. $u(\vec{p})\bar{u}(\vec{p}) - v(\vec{p},h)\bar{v}(\vec{p}) = 1, u(\vec{p})\bar{u}(\vec{p}) + v(\vec{p},h)\bar{v}(\vec{p}) = \frac{-i\gamma^a p_a}{m}, u(\vec{p})u^+(\vec{p}) + v(-\vec{p},h)v^+(-\vec{p}) = \frac{E}{m}$

5.1.3 Dirac equation [4] and its plane wave solutions in 2-dimensional space-time

Thm. 5.1.1. $(\gamma^a\partial_a + m)\psi = 0, \gamma^a = (-\sigma_y, \varsigma\sigma_z)$

$$\psi(\vec{r},t) = \frac{1}{(2\pi)^{1/2}}\int_{\vec{p}=-\infty}^{+\infty}[a(\vec{p})\sqrt{\tfrac{m}{E}}u(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})\sqrt{\tfrac{m}{E}}v(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d\vec{p}$$

$$a(\vec{p}) = \frac{1}{(2\pi)^{1/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\tfrac{E}{m}}u^+(\vec{p})\psi(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d\vec{r}, b^+(\vec{p}) = \frac{1}{(2\pi)^{1/2}}\int_{\vec{p}=-\infty}^{+\infty}\sqrt{\tfrac{E}{m}}v^+(\vec{p})\psi(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d\vec{r}$$

5.1.4 Covariant quantization rules for Dirac equation in 2-dimensional space-time

Cor. 5.1.2. $\begin{cases}\{a(\vec{p}), a^+(\vec{p}')\} = \delta(\vec{p}-\vec{p}')\\ \{a(\vec{p}), a(\vec{p}')\} = 0, \{a^+(\vec{p}), a^+(\vec{p}')\} = 0\end{cases} \Rightarrow \{\psi(x), \psi^+(x')\} = i(m-\gamma^a\partial_a)\gamma^0\Delta(x-x')$

5.2 B-W equation in 2-dimensional space-time

5.2.1 Spin basis and plane wave solutions of B-W equation [16] in 2-dimensional space-time

Def. 5.2.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) := \underbrace{u_{\lambda_\varsigma}(\vec{p})u_{\mu_\varsigma}(\vec{p})\cdot\cdot}_{2s}, V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) := \underbrace{v_{\lambda_\varsigma}(\vec{p})v_{\mu_\varsigma}(\vec{p})\cdot\cdot}_{2s}$

Cor. 5.2.1. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) = \underbrace{\sigma_x\otimes\sigma_x\cdot\cdot}_{2s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}), V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p}) = \underbrace{\sigma_x\otimes\sigma_x\cdot\cdot}_{2s}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p})$

Thm. 5.2.1. $(\gamma^a\partial_a + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t) = 0, \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t) = \frac{1}{(2s)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\cdot\cdot\}}_{2s}}(\vec{r},t)$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t) = \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{s-\frac{1}{2}}\sqrt{\tfrac{m}{E}}^{2s}[a(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\begin{cases}a(\vec{p}) = \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\tfrac{m}{E}}^{2s}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p})\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d^N\vec{r}\\ b^+(\vec{p}) = \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{-(s-\frac{1}{2})}\sqrt{\tfrac{m}{E}}^{2s}V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}^{2s}}(\vec{p})\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec{r},t)e^{i(\vec{p}\cdot\vec{r}-Et)}d^N\vec{r}\end{cases}$$

5.2.2 Orthogonal properties of spin basis for B-W equation in 2-dimensional space-time

Cor. 5.2.2.
$$\begin{cases}\bar{U}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=1,\bar{V}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=1\\ \bar{U}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=0,\bar{V}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=0\end{cases}$$

Cor. 5.2.3.
$$\begin{cases}U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=(\frac{E}{m})^{2s},U^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(-\vec{p})=0\\ V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})=(\frac{E}{m})^{2s},V^{+\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}(\vec{p})U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(-\vec{p})=0\end{cases}$$

5.2.3 Quasi projection operator of B-W equation in 2-dimensional space-time

Cor. 5.2.4.
$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(2m)^{2s}}\underbrace{[(m-i\gamma^bp_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-i\gamma^cp_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(2m)^{2s}}\underbrace{[(-m-i\gamma^bp_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(-m-i\gamma^cp_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\end{cases}$$

Cor. 5.2.5.
$$\begin{cases}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(\varsigma 2m)^{2s}}\underbrace{[m\sigma_z-(\sigma,i\varsigma)^ap_a]_{\lambda_\varsigma\lambda'_\varsigma}[m\sigma_z-(\sigma,i\varsigma)^bp_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(\vec{p})=\frac{1}{(\varsigma 2m)^{2s}}\underbrace{[-m\sigma_z-(\sigma,i\varsigma)^ap_a]_{\lambda_\varsigma\lambda'_\varsigma}[-m\sigma_z-(\sigma,i\varsigma)^bp_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\end{cases}$$

Cor. 5.2.6. $U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(p)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(p)=(-1)^{2s}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(-p)V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(-p)$

5.2.4 Covariant commutation rules for B-W equation in 2-dimensional space-time

Thm. 5.2.2. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]=\frac{i}{2^{2s-1}}\underbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 5.2.3. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\underbrace{[-im\sigma_z+(\sigma,i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[-im\sigma_z+(\sigma,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\Delta(x-x')$

5.2.5 Properties of $X^a_{\lambda_\varsigma\mu_\varsigma}(p)$ in 2-dimensional space-time

Def. 5.2.2. $X^a_{\lambda_\varsigma\mu_\varsigma}(x):=[(im\gamma^a+\varsigma\sigma_x\varepsilon^{ab}\partial_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma},X^a_{\lambda_\varsigma\mu_\varsigma}(p):=[(im\gamma^a+i\varsigma\sigma_x\varepsilon^{ab}p_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma}$

Pro. 5.2.1. $(\gamma^a\varepsilon)_{\lambda'_\varsigma\mu'_\varsigma}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}=-\delta^{\{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma\}}_{\mu'_\varsigma}-\sigma_{x\lambda'_\varsigma\mu'_\varsigma}\sigma_x^{\lambda_\varsigma\mu_\varsigma}$

Pro. 5.2.2. $(\bar{\varepsilon}\gamma^{a'})^{\lambda_\varsigma\mu_\varsigma}X^a_{\lambda_\varsigma\mu_\varsigma}(p)=i2m\delta^{a'a}$

Proof: $(\bar{\varepsilon}\gamma^{a'})^{\lambda_\varsigma\mu_\varsigma}X^a_{\lambda_\varsigma\mu_\varsigma}(p)$
$=(\bar{\varepsilon}\gamma^{a'})^{\lambda_\varsigma\mu_\varsigma}[(im\gamma^a+i\varsigma\sigma_x\varepsilon^{ab}p_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma}$
$=tr[\bar{\varepsilon}\gamma^{a'}(im\gamma^a+i\varsigma\sigma_x\varepsilon^{ab}p_b)\varepsilon]$
$=tr[\gamma^{a'}(im\gamma^a+i\varsigma\sigma_x\varepsilon^{ab}p_b)]$
$=imtr(\gamma^{a'}\gamma^a)$
$=i2m\delta^{a'a}$ □

Pro. 5.2.3. $[(im\gamma^a+\varsigma\sigma_x\varepsilon^{ab}\partial_b)\varepsilon]_{\lambda'_\varsigma\mu'_\varsigma}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}=-im\delta^{\{\lambda_\varsigma}_{\lambda'_\varsigma}\delta^{\mu_\varsigma\}}_{\mu'_\varsigma}-im\sigma_{x\lambda'_\varsigma\mu'_\varsigma}\sigma_x^{\lambda_\varsigma\mu_\varsigma}+\cdots$

5.2.6 Equivalent expression of quasi projection operators for B-W equation in 2D

Lem. 5.2.1.
$$\begin{cases}u(\vec{p})u^+(\vec{p})=\frac{(m-i\gamma^ap_a)\gamma^0}{2m},u_{\lambda_\varsigma}(\vec{p})u^+_{\lambda'_\varsigma}(\vec{p})u_{\mu_\varsigma}(\vec{p})u^+_{\mu'_\varsigma}(\vec{p})=u_{\lambda_\varsigma}(\vec{p})u^+_{\mu'_\varsigma}(\vec{p})u_{\mu_\varsigma}(\vec{p})u^+_{\lambda'_\varsigma}(\vec{p})\\ \varepsilon_a(\vec{p})\varepsilon^+_{a'}(\vec{p})=\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2},\varepsilon_a(\vec{p})\varepsilon^+_{a'}(\vec{p})\varepsilon_b(\vec{p})\varepsilon^+_{b'}(\vec{p})=\varepsilon_a(\vec{p})\varepsilon^+_{b'}(\vec{p})\varepsilon_b(\vec{p})\varepsilon^+_{a'}(\vec{p})\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}[(m-i\gamma^bp_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-i\gamma^cp_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}=[(m-i\gamma^bp_b)\gamma^0]_{\mu_\varsigma\lambda'_\varsigma}[(m-i\gamma^cp_c)\gamma^0]_{\lambda_\varsigma\mu'_\varsigma}\\ (\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})(\eta_{bb'}+\frac{p_bp^+_{b'}}{m^2})=(\eta_{ba'}+\frac{p_bp^+_{a'}}{m^2})(\eta_{ab'}+\frac{p_ap^+_{b'}}{m^2})\\ [(m-i\gamma^bp_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-i\gamma^cp_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}=X^a_{\lambda_\varsigma\mu_\varsigma}(p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(-p)(\eta_{aa'}+\frac{p_ap^+_{a'}}{m^2})\end{cases}$$

Cor. 5.2.7. $\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p}) = \frac{1}{(2m)^{2n}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(-p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})]\cdots}_{n} \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(\vec{p}) = \frac{1}{(2m)^{2n}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(-p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})]\cdots}_{n} \end{cases}$

Cor. 5.2.8. $\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p}) = \frac{1}{(2m)^{2n+1}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(-p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})]\cdots}_{n}[(m-i\gamma^c p_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma} \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau'_\varsigma}_{2n+1}}(\vec{p}) = \frac{1}{(2m)^{2n+1}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(-p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)(\eta_{aa'}+\frac{p_a p^+_{a'}}{m^2})]\cdots}_{n}[(-m-i\gamma^c p_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma} \end{cases}$

5.2.7 Equivalent expression of covariant commutation rules for B-W equation in 2D

Thm. 5.2.4. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')] = \frac{i}{2^{2n-1}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')$

Thm. 5.2.5.
$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x')] = \frac{i}{2^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}[(m-\gamma^c\partial_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

$[\Updownarrow]$

Thm. 5.2.6.
$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x')] = i\frac{i\varsigma}{2^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}\underbrace{[\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}[-im\sigma_z+(\sigma,i\varsigma)^b\partial_b]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

5.3 Potential equation in 2-dimensional space-time

Self comment: This section compares with four dimensional space-time case. Explore whether is there a K-G or R-S equation equivalent to B-W equation in 2-dimensional space time?

5.3.1 B-W equation with $s=1$ is equivalent to K-G equationin 2-dimensional space-time

Def. 5.3.1. $(\gamma^a\partial_a+m)^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\zeta_\varsigma}^{2s}} = J^{\overbrace{\kappa_\varsigma\mu_\varsigma\cdots\zeta_\varsigma}^{2s}}, \psi^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots\zeta_\varsigma}^{2s}}$ *fully symmetric*, $J^{\overbrace{\kappa_\varsigma\mu_\varsigma\cdots\zeta_\varsigma}^{2s}}\kappa_\varsigma, \gamma^a = (-\sigma_y, \varsigma\sigma_z)$

Thm. 5.3.1. $(\gamma^a\partial_a+m)\psi(x)=0, \psi^T(x)=\psi(x), A_a=\frac{1}{\sqrt{2}im}tr(\bar{\varepsilon}\gamma_a\psi)$
$\Leftrightarrow (\partial^b\partial_b-m^2)A_a=0, \partial^aA_a=0; F_{ab}:=\partial_aA_b-\partial_bA_a, \psi=(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon\frac{A_a}{\sqrt{2}}, S_{ab}(e)=\frac{1}{2}\varsigma\varepsilon^{ab}\sigma_x$

Proof: $(\gamma^a\partial_a+m)\psi(x)=0, \psi^T(x)=\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow (\gamma^a\partial_a+m)(\gamma^b\varepsilon imA_b-\varsigma\sigma_x\varepsilon F_{xy})=0, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow (\gamma^a\partial_a+m)(\gamma^b imA_b-\varsigma\sigma_x F_{xy})=0, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow (\delta^{ab}-i\varsigma\varepsilon^{ab}\sigma_x)im\partial_aA_b+m(im\gamma_bA^b-\varsigma\sigma_xF_{xy})-i\varepsilon^{ab}\gamma_b\partial_aF_{xy}=0, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow im\partial^aA_a-m(F_{xy}-\varepsilon^{ab}\partial_aA_b)\varsigma\sigma_x+i(m^2A^b-\varepsilon^{ab}\partial_aF_{xy})\gamma_b=0, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow im\partial^aA_a=0, m(F_{xy}-\varepsilon^{ab}\partial_aA_b)=0, i(m^2A^b-\varepsilon^{ab}\partial_aF_{xy})\gamma_b=0, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow \partial^aA_a=0, F_{xy}=\varepsilon^{ab}\partial_aA_b, \varepsilon^{ab}\partial_aF_{xy}-m^2A^b=0, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a+\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow \partial^aA_a=0, F_{xy}=\partial_xA_y-\partial_yA_x, \partial_aF^{ab}-m^2A^b=0$
$F_{ab}:=\partial_aA_b-\partial_bA_a=\varepsilon_{ab}F_{xy}, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow \partial^aF_{ab}-m^2A_b=0; F_{ab}:=\partial_aA_b-\partial_bA_a=\varepsilon_{ab}F_{xy}, \psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow (\partial^b\partial_b-m^2)A_a=0, \partial^aA_a=0, \psi=(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon\frac{A_a}{\sqrt{2}}$ □

Proof: $A_a=\frac{1}{\sqrt{2}im}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}$
$\Rightarrow [A_a(x),A^+_{a'}(x')]=\frac{1}{(\sqrt{2}m)^2}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}[\psi_{\lambda_\varsigma\mu_\varsigma},\psi^+_{\lambda'_\varsigma\mu'_\varsigma}]$
$=\frac{1}{(\sqrt{2}m)^2}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}\frac{i}{8}[(m-\gamma^b\partial_b)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^c\partial_c)\gamma^0]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$
$=\frac{i}{2}\frac{1}{(\sqrt{2}m)^2}(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma^c\partial_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\Delta(x-x')$
$=\frac{i}{2}\frac{1}{(2m)^2}[(\bar{\varepsilon}\gamma_a)(m-\gamma^b\partial_b)\gamma^0(\gamma_{a'}\varepsilon)]^{\mu_\varsigma\mu'_\varsigma}[(m-\gamma^c\partial_c)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\Delta(x-x')$
$=\frac{i}{2}\frac{1}{(\sqrt{2}m)^2}tr\{[(\bar{\varepsilon}\gamma_a)(m-\gamma^b\partial_b)\gamma^0(\gamma_{a'}\varepsilon)][-\gamma^0(m-\gamma^{*c}\partial_c)]\}\Delta(x-x')$
$=\frac{i}{2}\frac{1}{(\sqrt{2}m)^2}tr\{[(\gamma^0\gamma_a)(m-\gamma^b\partial_b)\gamma^0(\gamma_{a'}\gamma^0)][(-m-\gamma^c\partial_c)\gamma^0]\}\Delta(x-x')$
$=\frac{i}{2}\frac{1}{(2m)^2}tr\{\gamma_a(m-\gamma^b\partial_b)\gamma^*_{a'}(m+\gamma^c\partial_c)\}\Delta(x-x')$

$= \frac{i}{2}\frac{1}{(\sqrt{2}m)^2}tr\{\gamma_a(m-\gamma_b\partial^b)\gamma_{b'}(m+\gamma_c\partial^c)\}\eta_{a'}^{b'}\Delta(x-x')$
$= \frac{i}{2}\frac{1}{(2m)^2}tr\{\gamma_a[\gamma_{b'}(m+\gamma_b\partial^b)-2\delta_{bb'}\partial^b](m+\gamma_c\partial^c)\}\eta_{a'}^{b'}\Delta(x-x')$
$= \frac{i}{(2m)^2}tr\{\gamma_a[m\gamma_{b'}-\partial_{b'}](m+\gamma_c\partial^c)\}\eta_{a'}^{b'}\Delta(x-x')$
$= \frac{i}{(2m)^2}tr\{\gamma_a(m^2\gamma_{b'}-\gamma_c\partial_{b'}\partial^c)\}\eta_{a'}^{b'}\Delta(x-x')$
$= \frac{i}{(\sqrt{2}m)^2}tr\{(m^2\delta_{ab'}-\delta_{ac}\partial_{b'}\partial^c)\}\eta_{a'}^{b'}\Delta(x-x')$
$= i(\eta_{aa'}-\frac{\partial_a\partial_{a'}^+}{m^2})\Delta(x-x')$ □

Thm. 5.3.2. $[(m-\gamma^b\partial_b)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^c\partial_c)\gamma^0]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x') = X_{\{\lambda_\varsigma\mu_\varsigma\}}(x)X^+_{(\lambda'_\varsigma\mu'_\varsigma)}(x')(\eta_{aa'}-\frac{\partial_a\partial_{a'}^+}{m^2})\Delta(x-x')$

Proof: $\psi = (im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon\frac{A_a}{\sqrt{2}}$
$\Rightarrow A_a = \frac{1}{\sqrt{2}im}tr(\bar{\varepsilon}\gamma_a\psi)$
$\Rightarrow \psi = [(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]\frac{1}{2im}tr(\bar{\varepsilon}\gamma_a\psi)$
$\Rightarrow \psi_{\lambda_\varsigma\mu_\varsigma} = [(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma}[\frac{1}{2im}(\bar{\varepsilon}\gamma_a)^{\tilde{\lambda}_\varsigma\tilde{\mu}_\varsigma}\psi_{\tilde{\lambda}_\varsigma\tilde{\mu}_\varsigma}]$
$\Rightarrow [\psi_{\lambda_\varsigma\mu_\varsigma}(x),\psi^+_{\lambda'_\varsigma\mu'_\varsigma}(x')]$
$= [(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma}[(im\gamma^{a'}+\varepsilon^{a'b'}\varsigma\sigma_x\partial'_{b'})\varepsilon]^+_{\lambda'_\varsigma\mu'_\varsigma}[\frac{1}{2im}(\bar{\varepsilon}\gamma_a)^{\tilde{\lambda}_\varsigma\tilde{\mu}_\varsigma}][\frac{1}{-2im}(\gamma_a\varepsilon)^{\tilde{\lambda}'_\varsigma\tilde{\mu}'_\varsigma}][\psi_{\tilde{\lambda}'_\varsigma\tilde{\mu}'_\varsigma}(x),\psi_{\tilde{\lambda}'_\varsigma\tilde{\mu}'_\varsigma}(x')]$
$\Rightarrow \frac{i}{8}[(m-\gamma^b\partial_b)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^c\partial_c)\gamma^0]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x')$
$= [(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma}[(im\gamma^{a'}+\varepsilon^{a'b'}\varsigma\sigma_x\partial'_{b'})\varepsilon]^+_{\lambda'_\varsigma\mu'_\varsigma}$
$[\frac{1}{2im}(\bar{\varepsilon}\gamma_a)^{\tilde{\lambda}_\varsigma\tilde{\mu}_\varsigma}][\frac{1}{-2im}(\gamma_a\varepsilon)^{\tilde{\lambda}'_\varsigma\tilde{\mu}'_\varsigma}]\frac{i}{8}[(m-\gamma^b\partial_b)\gamma^0]_{\{\tilde{\lambda}_\varsigma(\tilde{\lambda}'_\varsigma}[(m-\gamma^c\partial_c)\gamma^0]_{\tilde{\mu}_\varsigma\}\tilde{\mu}'_\varsigma)}\Delta(x-x')$
$= \frac{i}{2}[(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma}[\bar{\varepsilon}(-im\gamma^{a'}+\varepsilon^{a'b'}\varsigma\sigma_x\partial^+_{b'})]_{\lambda'_\varsigma\mu'_\varsigma}(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')$
$= \frac{i}{2}X^a_{\lambda_\varsigma\mu_\varsigma}(x)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x'), X^a_{\lambda_\varsigma\mu_\varsigma}(x) := [(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_\varsigma\mu_\varsigma}$
$\Rightarrow [(m-\gamma^b\partial_b)\gamma^0]_{\{\lambda_\varsigma(\lambda'_\varsigma}[(m-\gamma^c\partial_c)\gamma^0]_{\mu_\varsigma\}\mu'_\varsigma)}\Delta(x-x') = X^a_{\{\lambda_\varsigma\mu_\varsigma\}}(x)X^{+a'}_{(\lambda'_\varsigma\mu'_\varsigma)}(x')(\eta_{aa'}-\frac{\partial_a\partial^+_{a'}}{m^2})\Delta(x-x')$ □

Thm. 5.3.3. $[F_{ab}(x),F^+_{a'b'}(x')] = -i\eta_{[a<a'}\partial_{b]}\partial^+_{b'>}\Delta(x-x')$

Thm. 5.3.4. $[F_{xy}(x),F^+_{xy}(x')] = im^2\Delta(x-x')$

5.3.2 Similar Klein-Gordon equation with $s=n$ in 2-dimensional space-time

Thm. 5.3.5.

$$\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x)=0,\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x)\ fully\ symmetric\\ A_{\underbrace{ab\cdots}_{n}}(x):=(\frac{1}{\sqrt{2}im})^n\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(x)\end{cases}\Rightarrow\begin{cases}(-\partial^c\partial_c+m^2)A_{\underbrace{ab\cdots}_{n}}(x)=0\\ \partial^aA_{\underbrace{ab\cdots}_{n}}(x)=0,\ A_{\underbrace{ab\cdots}_{n}}(x)\ fully\ symmetric\end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{r},t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}E^{n-\frac{1}{2}}\sqrt{\frac{m}{E}}^{2n}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$A_{\underbrace{ab\cdots}_{n}}(\vec{r},t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{1}{\sqrt{2^nE}}[a(\vec{p})\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)}+b^+(\vec{p})\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p}),\ \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p})=\frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdots}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdots}_{2n}}(\vec{p})$$

5.3.3 Properties of spin basis for similar Klein-Gordon equation in 2D

Cor. 5.3.1. $\varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p})=\underbrace{\varepsilon_a(\vec{p})\varepsilon_b(\vec{p})\cdots}_{n},\ \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p})=\underbrace{\tilde{\varepsilon}_a(\vec{p})\tilde{\varepsilon}_b(\vec{p})\cdots}_{n}$

Cor. 5.3.2. $\varepsilon_a(\vec{p})=\tilde{\varepsilon}_a(\vec{p})=\frac{1}{m}(E,ip_x),\ \varepsilon_{\underbrace{ab\cdots}_{n}}(\vec{p})=\tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}}(\vec{p})$

Proof: $\varepsilon_a(\vec{p})=-i(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p})$
$=-iu^T(\vec{p})(\bar{\varepsilon}\gamma_a)u(\vec{p})$
$=-iu^+(\vec{p})(i,\varsigma\sigma_x)_au(\vec{p})$
$=(\frac{E}{m},-i\varsigma u^+(\vec{p})v(\vec{p}))_a$
$=(\frac{E}{m},i\frac{p_x}{m})_a$ □

Cor. 5.3.3. $\varepsilon_a(\vec{p})\delta^{ab}\varepsilon_b(\vec{p})=1$

Proof: $X_a^{\lambda_\varsigma\mu_\varsigma}(\vec{p})\varepsilon^a(\vec{p})$
$= X_a^{\lambda_\varsigma\mu_\varsigma}(-\frac{E}{m}, -i\frac{p_x}{m})^a$
$\neq U^{\lambda_\varsigma\mu_\varsigma}(\vec{p})$ □

Cor. 5.3.4. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) = \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}\varepsilon)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2n}}(x)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot}_{2n}}(x)$

Cor. 5.3.5. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) = \underbrace{(\eta_{aa'} + \frac{p_a p^+_{a'}}{m^2})(\eta_{bb'} + \frac{p_b p^+_{b'}}{m^2})\cdots}_{n}$

Cor. 5.3.6. $\varepsilon_a(\vec{p})\varepsilon^+_{a'}(\vec{p}) = \eta_{aa'} + \frac{p_a p^+_{a'}}{m^2}, \varepsilon_a(\vec{p})\varepsilon^+_{a'}(\vec{p})\eta^{a'}_b = \varepsilon_a(\vec{p})\varepsilon_b(\vec{p}) = \delta_{ab} + \frac{p_a p_b}{m^2}$

Cor. 5.3.7.
$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{p}) = \frac{1}{(2m)^{2n}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(-p)]\cdots}_{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(\vec{p}) = \frac{1}{(2m)^{2n}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(-p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)]\cdots}_{n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p})\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}}(\vec{p}) \end{cases}$$

Cor. 5.3.8.
$$\begin{cases} [A_{\underbrace{ab\cdot\cdot}_{n}}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')] = \frac{1}{m^{2n}2^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] \\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot}_{2n}}(x')] = \frac{1}{2^{2n-1}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[A_{\underbrace{ab\cdot\cdot}_{n}}(x), A^+_{\underbrace{a'b'\cdot\cdot}_{n}}(x')] \end{cases}$$

5.3.4 Similar R-S equation with $s = n + \frac{1}{2}$ in 2-dimensional space-time

Thm. 5.3.6.
$$\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x) = 0, \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x) \textit{ fully symmetric} \\ A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x) := (\frac{1}{\sqrt{2im}})^n\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x) \end{cases} \Rightarrow \begin{cases} (-\partial^c\partial_c + m^2)A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x) = 0 \\ \partial^a A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x) = 0, A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x) \textit{ fully symmetric} \end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{r},t) = \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty} E^n\sqrt{\frac{m}{E}}^{2n+1}[a(\vec{p},h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p},h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{r},t) = \frac{1}{(2\pi)^{N/2}}\int_{\vec{p}=-\infty}^{+\infty}\frac{\sqrt{m}}{\sqrt{2^n E}}[a(\vec{p})\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p}), \tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}}(\vec{p}) = \frac{1}{i^n}\overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}V_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p})$$

5.3.5 Properties of similar R-S equation spin basis in 2-dimensional space-time

Cor. 5.3.9. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p}) = \underbrace{I\otimes I\otimes\cdot\cdot I}_{2n}\otimes\sigma_x\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})$

Cor. 5.3.10. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}) = \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta_\varsigma\xi_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}(\gamma_{b'}\varepsilon)^{\eta'_\varsigma\xi'_\varsigma}\cdot\cdot}^{n}U_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x)U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\eta'_\varsigma\xi'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x)$

Cor. 5.3.11. $\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}) = \frac{1}{2m}\underbrace{(\eta_{aa'} + \frac{p_a p^+_{a'}}{m^2})(\eta_{bb'} + \frac{p_b p^+_{b'}}{m^2})\cdots}_{n}[(m - i\gamma^c p_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}$

Cor. 5.3.12.
$$\begin{cases} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p})U^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec{p}) = \frac{1}{(2m)^{2n+1}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(-p)]\cdots}_{n}\varepsilon_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})\varepsilon^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}) \\ V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(\vec{p})V^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(\vec{p}) = \frac{1}{(2m)^{2n+1}}\underbrace{[X^a_{\lambda_\varsigma\mu_\varsigma}(-p)X^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(p)]\cdots}_{n}\tilde{\varepsilon}_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(\vec{p})\tilde{\varepsilon}^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(\vec{p}) \end{cases}$$

Cor. 5.3.13.
$$\begin{cases}[A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')]=\frac{1}{m^{2n}2^n}\overbrace{(\bar\varepsilon\gamma_a)^{\lambda_\varsigma\mu_\varsigma}\cdot\cdot}^{n}\overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\varsigma\mu'_\varsigma}\cdot\cdot}^{n}[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')]\\ [\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot\tau_\varsigma}_{2n+1}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdot\cdot\tau'_\varsigma}_{2n+1}}(x')]=\frac{1}{2^{2n-1}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdot\cdot}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdot\cdot}_{n}[A_{\underbrace{ab\cdot\cdot}_{n}\tau_\varsigma}(x),A^+_{\underbrace{a'b'\cdot\cdot}_{n}\tau'_\varsigma}(x')]\end{cases}$$

Self comment: In 2-dimensional space-time, only a small number of B-W equations have equivalent K-G equations, but there are still similar communication rules.

5.4 B-W [16] equation with $s=\frac{3}{2}$ in 2-dimensional space-time

Proof: $(\gamma^a\partial_a+m)^{\kappa_\varsigma}{}_{\lambda_\varsigma}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=0,\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}$ fully symmetric, $\gamma^a=(-\sigma_y,\varsigma\sigma_z)$
$\Leftrightarrow(\partial^b\partial_b-m^2)A_{a\eta_\varsigma}=0,\partial^aA_{a\eta_\varsigma}=0,\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=[(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]^{\lambda_\varsigma\mu_\varsigma}A^{\eta_\varsigma}_a,\bar\varepsilon_{\mu_\varsigma\eta_\varsigma}\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=0$ □

Proof: $\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=[(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]^{\lambda_\varsigma\mu_\varsigma}A^{\eta_\varsigma}_a,\psi^{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}=0$
$\Rightarrow[(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]^{\lambda_\varsigma\mu_\varsigma}\bar\varepsilon_{\mu_\varsigma\eta_\varsigma}A^{\eta_\varsigma}_a=0$
$\Leftrightarrow[(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)]^{\lambda_\varsigma}{}_{\eta_\varsigma}A^{\eta_\varsigma}_a=0$ □

5.5 Spin equation in 2-dimensional space-time

5.5.1 Spin equation for spin-s particles in 2-dimensional space-time

Thm. 5.5.1. $[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)$ *fully symmetric*, $\gamma_a:=[-\sigma_y,\varsigma\sigma_z]$

$$\Leftrightarrow\begin{cases}[s\partial_a+m\gamma_a(s)+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0\\ [\gamma_x(s)\partial_\pi-\gamma_\pi(s)\partial_x]\psi(s)=i\varsigma m\gamma_y(s)\psi(s)\end{cases},S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)],\gamma_a(s):=[-\sigma_y(s),\varsigma\sigma_z(s)]$$

Proof: $[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(x)$ fully symmetric

$\Leftrightarrow[\gamma^a\partial_a+m]\hat\psi(s)=0$
$\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_aD^a\hat\psi(s)=im\sigma_z\otimes I_{2^{2s-1}}\hat\psi(s),D^a=(\partial^x,0,0,\partial^\pi)$
$\Leftrightarrow(\sigma\otimes I_{2^{2s-1}},-i\varsigma)_aD^a[I\otimes\Gamma(s)]N(s)\psi(s)=im\sigma_z\otimes I_{2^{2s-1}}\hat\psi(s)$
$\Leftrightarrow[I\otimes\Gamma(s)](\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im\sigma_z\otimes I_{2^{2s-1}}\hat\psi(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im[I\otimes\bar\Gamma(s)](\sigma_z\otimes I_{2^{2s-1}})\hat\psi(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im(\sigma_z\otimes I_{2s})[I\otimes\bar\Gamma(s)]\hat\psi(s)$
$\Leftrightarrow(\sigma\otimes I_{2s},-i\varsigma)_aD^aN(s)\psi(s)=im(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow Z_bD^b\psi(s)=im\frac{i\varsigma}{\sqrt2}(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow\bar Z_aZ_bD^b\psi(s)=im\frac{i\varsigma}{\sqrt2}\bar Z_a(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow\bar Z_aZ_bD^b\psi(s)=im\frac{i\varsigma}{\sqrt2}\frac{-i\varsigma}{\sqrt2}\bar N(s)(\sigma\otimes I_{2s},i\varsigma)_a(\sigma_z\otimes I_{2s})N(s)\psi(s)$
$\Leftrightarrow\bar Z_aZ_bD^b\psi(s)=-\frac{m}{2s}\bar N(s)[(-\sigma_y,\sigma_x,-i)\otimes I_{2s},\varsigma\sigma_z\otimes I_{2s}]_aN(s)\psi(s)$
$\Leftrightarrow[sD_a+iS_{ab}(s,\varsigma;4)D^b]\psi(s)=-m[(-\sigma_y(s),\sigma_x(s),-is),\varsigma\sigma_z(s)]_a\psi(s)$

$$S_{ab}(s,\varsigma;4)\succ\begin{bmatrix}0&\sigma_z(s)&-\sigma_y(s)&-\varsigma\sigma_x(s)\\-\sigma_z(s)&0&\sigma_x(s)&-\varsigma\sigma_y(s)\\\sigma_y(s)&-\sigma_x(s)&0&-\varsigma\sigma_z(s)\\\varsigma\sigma_x(s)&\varsigma\sigma_y(s)&\varsigma\sigma_z(s)&0\end{bmatrix}$$

$$\Leftrightarrow\begin{cases}[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=-m\gamma_a(s)\psi(s)\\ [\gamma_x(s)\partial_\pi-\gamma_\pi(s)\partial_x]\psi(s)=i\varsigma m\gamma_y(s)\psi(s)\end{cases},S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)]\succ\begin{bmatrix}0&-\varsigma\sigma_x(s)\\\varsigma\sigma_x(s)0&\end{bmatrix}$$

$$\Leftrightarrow\begin{cases}[s\partial_a+m\gamma_a(s)+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0\\ [\gamma_x(s)\partial_\pi-\gamma_\pi(s)\partial_x]\psi(s)=i\varsigma m\gamma_y(s)\psi(s)\end{cases}$$ □

5.5.2 Plane wave solutions and its spin basis of spin equation in 2-dimensional space-time

Thm. 5.5.2.

$$\begin{cases}[\gamma^a\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s}}(x)\ \textit{fully symmetric}\\ \psi_{k_\varsigma}(x):=\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}^{2s}}\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s}}(x)\end{cases}\Leftrightarrow\begin{cases}[s\partial_a+m\gamma_a(s)+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0\\ [\gamma_x(s)\partial_\pi-\gamma_\pi(s)\partial_x]\psi(s)=i\varsigma m\gamma_y(s)\psi(s)\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s}}(x)=\Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot}_{2s}}\psi_{k_\varsigma}(x)\end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec r,t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec p=-\infty}^{+\infty}E^{n-\frac12}\sqrt{\frac{m}{E}}^{2s}[a(\vec p,h)U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec p)e^{i(\vec p\cdot\vec r-Et)}+b^+(\vec p,h)V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdot\cdot}_{2s}}(\vec p)e^{-i(\vec p\cdot\vec r-Et)}]d^N\vec p$$

$$\psi_{k_\varsigma}(\vec r,t)=\frac{1}{(2\pi)^{N/2}}\int_{\vec p=-\infty}^{+\infty}E^{n-\frac12}\sqrt{\frac{m}{E}}^{2n}[a(\vec p,h)U_{k_\varsigma}(\vec p;s)e^{i(\vec p\cdot\vec r-Et)}+b^+(\vec p,h)V_{k_\varsigma}(\vec p;s)e^{-i(\vec p\cdot\vec r-Et)}]d^N\vec p$$

Thm. 5.5.3.

$$\begin{cases} U_{k_\varsigma}(\vec{p};s) := \Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}} U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p}) \Leftrightarrow U_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p}) = \Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}} U_{k_\varsigma}(\vec{p};s) \\ V_{k_\varsigma}(\vec{p};s) := \Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}} V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p}) \Leftrightarrow V_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\vec{p}) = \Gamma^{k_\varsigma}_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}} V_{k_\varsigma}(\vec{p};s) \end{cases}$$

5.5.3 Orthogonality of spin basis for spin equation in 2-dimensional space-time

Thm. 5.5.4. $U^{+k_\varsigma}(\vec{p};s)U_{k_\varsigma}(\vec{p};s) = (\frac{E}{m})^{2s}, V^{+k_\varsigma}(\vec{p};s)V_{k_\varsigma}(\vec{p};s) = (\frac{E}{m})^{2s}$

Thm. 5.5.5.

$$\begin{cases} U_{k_\varsigma}(\vec{p};s)U^+_{k'_\varsigma}(\vec{p};s) = \frac{1}{(\varsigma 2m)^{2s}}\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}\Gamma_{k'_\varsigma}^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}} \underbrace{[m\sigma_z - (\sigma, i\varsigma)^a p_a]_{\lambda_\varsigma\lambda'_\varsigma}[m\sigma_z - (\sigma, i\varsigma)^b p_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s} \\ V_{k_\varsigma}(\vec{p};s)V^+_{k'_\varsigma}(\vec{p};s) = \frac{1}{(\varsigma 2m)^{2s}}\Gamma_{k_\varsigma}^{\overbrace{\lambda_\varsigma\mu_\varsigma\cdots}^{2s}}\Gamma_{k'_\varsigma}^{\overbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}^{2s}} \underbrace{[-m\sigma_z - (\sigma, i\varsigma)^a p_a]_{\lambda_\varsigma\lambda'_\varsigma}[-m\sigma_z - (\sigma, i\varsigma)^b p_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s} \end{cases}$$

5.6 Covariant commutation rules for in 2-dimensional space-time

5.6.1 Carding of covariant commutation rules for massive bosons in 2D

Thm. 5.6.1. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')] = \frac{i}{2^{2n-1}}\underbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 5.6.2. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')] = i\frac{i^{2n}}{2^{2n-1}}\underbrace{[-im\sigma_z + (\sigma, i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[-im\sigma_z + (\sigma, i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2n}\Delta(x-x')$

$[\Updownarrow]$

Thm. 5.6.3. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n}}(x')] = \frac{i}{2^{2n-1}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}\underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}\Delta(x-x')$

5.6.2 Carding of covariant commutation rules for massive fermions in 2D

Thm. 5.6.4. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')] = \frac{i}{2^{2n}}\underbrace{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(m-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2n+1}\Delta(x-x')$

$[\Updownarrow]$

Thm. 5.6.5. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2n+1}}(x')] = i\frac{i^{2n+1}}{2^{2n}}\underbrace{[-im\sigma_z + (\sigma, i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[-im\sigma_z + (\sigma, i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2n+1}\Delta(x-x')$

$[\Updownarrow]$

Thm. 5.6.6.

$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x')] = \frac{i}{2^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}\underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}[(m-\gamma^c\partial_c)\gamma^0]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

$[\Updownarrow]$

Thm. 5.6.7.

$[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x), \psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots\tau_\varsigma}_{2n+1}}(x')] = i\frac{i\varsigma}{2^{2n}}\underbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}(x)\cdots}_{n}\underbrace{\mathbb{X}^{+a'}_{\lambda'_\varsigma\mu'_\varsigma}(x')\cdots}_{n}\underbrace{[\eta_{aa'} - \frac{\partial_a\partial^+_{a'}}{m^2}]\cdots}_{n}[-im\sigma_z + (\sigma, i\varsigma)^b\partial_b]_{\tau_\varsigma\tau'_\varsigma}\Delta(x-x')$

6 B-W equation without mass in 2-dimensional space-time

6.1 Dirac equation without mass in 2-dimensional space-time

Proof: $\gamma^a\partial_a\psi(x) = 0, \gamma^a = (-\sigma_y, \varsigma\sigma_z)$
$\Leftrightarrow (\sigma_x, -i\varsigma)^a\partial_a\psi(x) = 0$
$\Leftrightarrow (\sigma_x, -i\varsigma)^a p_a\lambda(\hat{p}, -\frac{\varsigma}{2}) = 0$
$\Leftrightarrow \sigma_x p_x\lambda(\hat{p}, -\frac{\varsigma}{2}) = -\varsigma|p_x|\lambda(\hat{p}, -\frac{\varsigma}{2})$
$\Leftrightarrow \sigma_x\hat{p}_x\lambda(\hat{p}, -\frac{\varsigma}{2}) = -\varsigma\lambda(\hat{p}, -\frac{\varsigma}{2})$
$\Leftrightarrow \lambda(\hat{p}, -\frac{\varsigma}{2}) = \frac{1}{2\sqrt{2}}\begin{bmatrix}(1-\varsigma) - (1+\varsigma)\hat{p}_x \\ (1+\varsigma) + (1-\varsigma)\hat{p}_x\end{bmatrix}$ □

Cor. 6.1.1.

$$\begin{cases}\psi(\vec{r},t):=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}\neq 0}\lambda(\hat{p},-\frac{\varsigma}{2})[a_1(\vec{p},-\frac{\varsigma}{2})e^{ip\cdot x}+a_2^+(\vec{p},-\frac{\varsigma}{2})e^{-ip\cdot x}]d^N\vec{p}\\ a_1(\vec{p},-\frac{\varsigma}{2})=\frac{1}{(2\pi)^{N/2}}\int\lambda^+(\hat{p},-\frac{\varsigma}{2})\psi(\vec{r},t)e^{-ip\cdot x}d^N\vec{r}\\ a_2^+(\vec{p},-\frac{\varsigma}{2})=\frac{1}{(2\pi)^{N/2}}\int\lambda^+(\hat{p},-\frac{\varsigma}{2})\psi(\vec{r},t)e^{ip\cdot x}d^N\vec{r}\end{cases}$$

$$\lambda^+(\hat{p},-\tfrac{\varsigma}{2})\lambda(\hat{p},-\tfrac{\varsigma}{2})=1,\lambda(\hat{p},-\tfrac{\varsigma}{2})\lambda^+(\hat{p},-\tfrac{\varsigma}{2})=\tfrac{1}{2}(1-\varsigma\sigma_x\hat{p}_x)=-\tfrac{\varsigma}{2}(\sigma_x,i\varsigma)^a\hat{p}_a=-\tfrac{\varsigma}{2}(\sigma,i\varsigma)^a\hat{p}_a$$

6.2 B-W equation without mass [16] in 2-dimensional space-timePlane wave solutions of

Thm. 6.2.1. $\gamma^a{}_{\kappa_\varsigma}{}^{\lambda_\varsigma}\partial_a\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x)=0,\gamma^a=(-\sigma_y,\varsigma\sigma_z)$

$$\begin{cases}\psi(\vec{r},t):=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}\neq 0}\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\hat{p},-s\varsigma)[a_1(\vec{p},-\frac{\varsigma}{2})e^{ip\cdot x}+a_2^+(\vec{p},-\frac{\varsigma}{2})e^{-ip\cdot x}]d^N\vec{p}\\ a_1(\vec{p},-s\varsigma)=\frac{1}{(2\pi)^{N/2}}\int\lambda^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{-ip\cdot x}d^N\vec{r}\\ a_2^+(\vec{p},-\frac{\varsigma}{2})=\frac{1}{(2\pi)^{N/2}}\int\lambda^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\hat{p},-s\varsigma)\psi(\vec{r},t)e^{ip\cdot x}d^N\vec{r}\end{cases}$$

$$\lambda^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\hat{p},-s\varsigma)\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\hat{p},-s\varsigma)=1,\lambda_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\hat{p},-s\varsigma)\lambda^+_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(\hat{p},-s\varsigma)=(-\tfrac{\varsigma}{2})^{2s}\overbrace{(\sigma,i\varsigma)^a(\sigma,i\varsigma)^b\cdots}^{2s}\overbrace{\hat{p}_a\hat{p}_b\cdots}^{2s}$$

6.3 Potential description of B-W equation without mass in 2-dimensional space-time

Thm. 6.3.1. $\gamma^a\partial_a\psi(x)=0,\psi^T(x)=\psi(x),A_a=\frac{1}{\sqrt{2}im}tr(\bar{\varepsilon}\gamma_a\psi),\gamma^a=(-\sigma_y,\varsigma\sigma_z)$
$\Leftrightarrow(\partial^b\partial_b-m^2)A_a=0,\partial^aA_a=0;F_{ab}:=\partial_aA_b-\partial_bA_a,\psi=(im\gamma^a+\varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon\frac{A_a}{\sqrt{2}},S_{ab}(e)=\frac{1}{2}\varsigma\varepsilon^{ab}\sigma_x$

Proof: $\gamma^a\partial_a\psi(x)=0,\psi^T(x)=\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow\gamma^a\partial_a(\gamma^b\varepsilon imA_b-\varsigma\sigma_x\varepsilon F_{xy})=0,\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow\gamma^a\partial_a(\gamma^b imA_b-\varsigma\sigma_xF_{xy})=0,\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow(\delta^{ab}-i\varsigma\varepsilon^{ab}\sigma_x)im\partial_aA_b-i\varepsilon^{ab}\gamma_b\partial_aF_{xy}=0,\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow im\partial^aA_a+m\varepsilon^{ab}\partial_aA_b\varsigma\sigma_x-i\varepsilon^{ab}\partial_aF_{xy}\gamma_b=0,\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow im\partial^aA_a=0,m\varepsilon^{ab}\partial_aA_b=0,-i\varepsilon^{ab}\partial_aF_{xy}\gamma_b=0,\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow\partial^aA_a=0,\varepsilon^{ab}\partial_aA_b=0,\varepsilon^{ab}\partial_aF_{xy}=0,\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$
$\Leftrightarrow\partial^aA_a=0,\partial_aA_b-\partial_bA_a=0,\partial_aF_{xy}=0,\psi(x)=\frac{1}{\sqrt{2}}(\gamma^a\varepsilon imA_a-\varsigma\sigma_x\varepsilon F_{xy})$ □

6.4 s-spin equation without mass in 2-dimensional space-time

Thm. 6.4.1. $\gamma^a\partial_a\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\cdots}_{2s}}(x)=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x)$ *fully symmetric*, $\gamma_a:=[-\sigma_y,\varsigma\sigma_z]$

$$\Leftrightarrow\begin{cases}[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\psi(s)=0\\ [\gamma_x(s)\partial_\pi-\gamma_\pi(s)\partial_x]\psi(s)=0\end{cases},S_{ab}(s,\varsigma)=-i[\gamma_a(s),\gamma_b(s)],\gamma_a(s):=[-\sigma_y(s),\varsigma\sigma_z(s)]$$

6.5 Covariant commutation rules for massless B-W equation in 2-dimensional space-time

Thm. 6.5.1. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]=\frac{i}{2^{2s-1}}\underbrace{[(-\gamma^a\partial_a)\gamma^0]_{\lambda_\varsigma\lambda'_\varsigma}[(-\gamma^b\partial_b)\gamma^0]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 6.5.2. $[\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\cdots}_{2s}}(x),\psi^+_{\underbrace{\lambda'_\varsigma\mu'_\varsigma\cdots}_{2s}}(x')]=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\underbrace{[(\sigma,i\varsigma)^a\partial_a]_{\lambda_\varsigma\lambda'_\varsigma}[(\sigma,i\varsigma)^b\partial_b]_{\mu_\varsigma\mu'_\varsigma}\cdots}_{2s}\Delta(x-x')$

$$[\Updownarrow]$$

Thm. 6.5.3. $[\psi_{k_\varsigma}(x),\psi^+_{k'_\varsigma}(x')]=i\frac{(-1)^{2s}}{2^{s-1}}\Gamma^{\overbrace{abc\cdots}^{2s}}_{k_\varsigma k'_\varsigma}(s)\overbrace{\partial_a\partial_b\partial_c\cdots}^{2s}\Delta(x-x')$

7 Penrose equation for massless particles in 2-dimensional space-time

7.1 Dirac equation [4,5] under separated representation in 2-dimensional space-time

Def. 7.1.1. $(\gamma^a\partial_a+m)\psi=0,\psi=\begin{bmatrix}\varphi\\ \bar{\varphi}\end{bmatrix},\gamma^a=(1\otimes\sigma_y,\varsigma1\otimes\sigma_x)\Leftrightarrow\begin{cases}(1,-i\varsigma)^a\partial_a\varphi=im\bar{\varphi}\\ (1,i\varsigma)^a\partial_a\bar{\varphi}=-im\varphi\end{cases}$

Def. 7.1.2. $\vartheta=\begin{bmatrix}0 & i\varepsilon\\ -i\varepsilon & 0\end{bmatrix}, S_{ab}=\begin{bmatrix}0 & \sigma_y\\ -\sigma_y & 0\end{bmatrix}, S_{ab}(e,\varsigma)=-\frac{i}{4}[\sigma_a,\sigma_b]=\begin{bmatrix}0 & \frac{-\varsigma}{2}\sigma_z\\ \frac{\varsigma}{2}\sigma_z & 0\end{bmatrix}, \psi:=\begin{bmatrix}\varphi\\ \bar{\varphi}\end{bmatrix}$

Cor. 7.1.1. $\Lambda(\begin{bmatrix}\sigma\\ i\tau\end{bmatrix})=e^{\frac{i}{2}\vartheta^{ab}S_{ab}}=e^{-\varepsilon\sigma_y}, \Lambda(\begin{bmatrix}\varphi\\ \bar{\varphi}\end{bmatrix})=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(e,\varsigma)}=e^{\frac{\varsigma}{2}\varepsilon\sigma_z}$

When the mass m=0, it degenerates into two Weyl neutrino equations:

Cor. 7.1.2. $(1,-i\varsigma)^a\partial_a\varphi=0,(1,i\varsigma)^a\partial_a\bar{\varphi}=0,\Lambda(\varphi)=e^{\frac{\varsigma}{2}\varepsilon},\Lambda(\bar{\varphi})=e^{-\frac{\varsigma}{2}\varepsilon}$

7.2 Helicity eigenfunction of massless Dirac equation under separated representation in 2D

Def. 7.2.1. $\gamma^a\partial_a\psi(x)=0,\psi=\begin{bmatrix}\varphi\\ \bar{\varphi}\end{bmatrix},\gamma^a=(1\otimes\sigma_y,\varsigma 1\otimes\sigma_x)\Leftrightarrow\begin{cases}(1,-i\varsigma)^a\partial_a\varphi=0\\ (1,i\varsigma)^a\partial_a\bar{\varphi}=0\end{cases}$

Proof: $(\sigma_z,-i\varsigma)^a\partial_a\psi(x)=0$
$\Leftrightarrow(\sigma_z,-i\varsigma)^a p_a\lambda(\hat{p},-\frac{\varsigma}{2})=0$
$\Leftrightarrow\sigma_z p_z\lambda(\hat{p},-\frac{\varsigma}{2})=-\varsigma|p_z|\lambda(\hat{p},-\frac{\varsigma}{2})$
$\Leftrightarrow\sigma_z\hat{p}_z\lambda(\hat{p},-\frac{\varsigma}{2})=-\varsigma\lambda(\hat{p},-\frac{\varsigma}{2})$
$\Leftrightarrow\lambda(\hat{p},-\varsigma)=\frac{1}{2}\begin{bmatrix}-1+\varsigma\hat{p}_z\\ 1+\varsigma\hat{p}_z\end{bmatrix}$ □

Cor. 7.2.1. $\lambda^+(\hat{p},-\frac{\varsigma}{2})\lambda(\hat{p},-\frac{\varsigma}{2})=1,\lambda(\hat{p},-\frac{\varsigma}{2})\lambda^+(\hat{p},-\frac{\varsigma}{2})=-\frac{\varsigma}{2}(\sigma_z,i\varsigma)^a\hat{p}_a$

7.3 Vector and spinor in two dimensions [42]

7.3.1 Light cone coordinates and derivatives in two dimensions

Def. 7.3.1. $z\equiv\tau+\sigma,\tilde{z}\equiv\tau-\sigma,\tau=\frac{1}{2}(z+\tilde{z}),\sigma=\frac{1}{2}(z-\tilde{z}),z_\varsigma:=\tau+\varsigma\sigma,\bar{z}_\varsigma:=\tau-\varsigma\sigma$

Def. 7.3.2. $\begin{bmatrix}z\\ \tilde{z}\end{bmatrix}=\begin{bmatrix}1 & -i\\ -1 & -i\end{bmatrix}\begin{bmatrix}\sigma\\ i\tau\end{bmatrix},\begin{bmatrix}\sigma\\ i\tau\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -1\\ i & i\end{bmatrix}\begin{bmatrix}z\\ \tilde{z}\end{bmatrix}$

Cor. 7.3.1. $\begin{cases}dz=d\tau+d\sigma,d\tilde{z}=d\tau-d\sigma\\ \partial_z=\frac{1}{2}(\partial_\tau+\partial_\sigma),\partial_{\tilde{z}}=\frac{1}{2}(\partial_\tau-\partial_\sigma)\end{cases}\quad\begin{cases}d\tau=\frac{1}{2}(dz+d\tilde{z}),d\sigma=\frac{1}{2}(dz-d\tilde{z})\\ \partial_\tau=\partial_z+\partial_{\tilde{z}},\partial_\sigma=\partial_z-\partial_{\tilde{z}}\end{cases}$

Cor. 7.3.2. $dz\wedge d\tilde{z}=2d\sigma\wedge d\tau$

Def. 7.3.3. $P_z\equiv -i\partial_z,P_{\tilde{z}}\equiv -i\partial_{\tilde{z}},P_\tau\equiv i\partial_\tau,P_\sigma\equiv -i\partial_\sigma$

Cor. 7.3.3. $P_z=-\frac{1}{2}(P_\tau-P_\sigma),P_{\tilde{z}}=-\frac{1}{2}(P_\tau+P_\sigma),-P_\tau=P_z+P_{\tilde{z}},P_\sigma=P_z-P_{\tilde{z}}$

Cor. 7.3.4. $e^{i(P_\sigma\sigma-P_\tau\tau)}=e^{i(P_z z+P_{\tilde{z}}\tilde{z})}$

7.3.2 Lorentz transformation law of vector and spinor in two dimensions

Cor. 7.3.5. *Vector:* $\Lambda(\begin{bmatrix}\sigma\\ i\tau\end{bmatrix})=e^{-\varepsilon\sigma_y},\sigma,\tau\in R\Leftrightarrow$ *Light cone vector:* $\Lambda(\begin{bmatrix}z\\ \tilde{z}\end{bmatrix})=e^{-\varepsilon\sigma_z},z,\tilde{z}\in R$

Cor. 7.3.6. *Dirac spinor:* $\Lambda(\psi)=e^{\frac{\varsigma}{2}\varepsilon\sigma_z},\psi\in C,$ *Weyl spinor:* $\Lambda(\varphi)=e^{\frac{\varsigma}{2}\varepsilon},\Lambda(\bar{\varphi})=e^{-\frac{\varsigma}{2}\varepsilon},\varphi,\bar{\varphi}\in C$

Cor. 7.3.7. *Majorana spinor:* $\Lambda(\psi)=e^{\frac{\varsigma}{2}\varepsilon\sigma_z},\psi\in R,$ *Majorana-Weyl spinor:* $\Lambda(\varphi)=e^{\frac{\varsigma}{2}\varepsilon},\Lambda(\bar{\varphi})=e^{-\frac{\varsigma}{2}\varepsilon},\varphi,\bar{\varphi}\in R$

Prop. 7.3.1. *Constant tensor:* $(1,-i\varsigma)^a=e^{-\varepsilon\sigma_y}|^a{}_b e^{-\frac{\varsigma}{2}\varepsilon}(1,-i)^b e^{-\frac{\varsigma}{2}\varepsilon},(1,-i\varsigma)^{a'}\partial_{a'}=e^{-\varsigma\varepsilon}(1,-i\varsigma)^a\partial_a$

Prop. 7.3.2. *Constant tensor:* $(1,i\varsigma)^a=e^{-\varepsilon\sigma_y}|^a{}_b e^{\frac{\varsigma}{2}\varepsilon}(1,i\varsigma)^b e^{\frac{\varsigma}{2}\varepsilon},(1,i\varsigma)^{a'}\partial_{a'}=e^{\varsigma\varepsilon}(1,i\varsigma)^a\partial_a$

Cor. 7.3.8. $\partial_{z_\varsigma}=\frac{1}{2}(1,i\varsigma)^a\partial_a,\partial_{\tilde{z}_\varsigma}=-\frac{1}{2}(1,-i\varsigma)^a\partial_a;\partial_z=\frac{1}{2}(1,i)^a\partial_a,\partial_{\tilde{z}}=-\frac{1}{2}(1,-i)^a\partial_a$

Cor. 7.3.9. $\begin{cases}\partial_{z'_\varsigma}=e^{\varsigma\varepsilon}\partial_{z_\varsigma},dz'_\varsigma=e^{-\varsigma\varepsilon}dz_\varsigma,z'_\varsigma=e^{-\varsigma\varepsilon}z_\varsigma\\ \partial_{\tilde{z}'_\varsigma}=e^{-\varsigma\varepsilon}\partial_{\tilde{z}_\varsigma},d\tilde{z}'_\varsigma=e^{\varsigma\varepsilon}d\tilde{z}_\varsigma,\tilde{z}'_\varsigma=e^{\varsigma\varepsilon}\tilde{z}_\varsigma\end{cases}\quad\begin{cases}\partial_{z'}=e^{\varepsilon}\partial_z,dz'=e^{-\varepsilon}dz,z'=e^{-\varepsilon}z\\ \partial_{\tilde{z}'}=e^{-\varepsilon}\partial_{\tilde{z}},d\tilde{z}'=e^{\varepsilon}d\tilde{z},\tilde{z}'=e^{\varepsilon}\tilde{z}\end{cases}$

Cor. 7.3.10. *Invariant:* $dz_\varsigma\partial_{z_\varsigma},d\tilde{z}_\varsigma\partial_{\tilde{z}_\varsigma},dz_\varsigma d\tilde{z}_\varsigma,\partial_{z_\varsigma}\partial_{\tilde{z}_\varsigma};dz\partial_z,d\tilde{z}\partial_{\tilde{z}},dzd\tilde{z},\partial_z\partial_{\tilde{z}}$

Cor. 7.3.11. *Light cone vector:* $\Lambda(\begin{bmatrix}z_\varsigma\\ \tilde{z}_\varsigma\end{bmatrix})=e^{-\varsigma\varepsilon\sigma_z},\Lambda(\begin{bmatrix}\partial_{z_\varsigma}\\ \partial_{\tilde{z}_\varsigma}\end{bmatrix})=e^{\varsigma\varepsilon\sigma_z},$ *spinor:* $\Lambda(\begin{bmatrix}\varphi\\ \bar{\varphi}\end{bmatrix})=e^{\frac{\varsigma}{2}\varepsilon\sigma_z}$

In two dimensions the light cone vector is very similar to the spin and it is just the spinor representation of the vector.

7.3.3 Wick rotation (not used)

Def. 7.3.4. $z \equiv \sigma + i\tau, \bar{z} \equiv z^* = \sigma - i\tau, \sigma = \frac{1}{2}(z+\bar{z}), i\tau = \frac{1}{2}(z-\bar{z})$

Cor. 7.3.12. $\begin{cases} dz = d\sigma + id\tau, d\bar{z} = d\sigma - id\tau, d\sigma = \frac{1}{2}(dz + d\bar{z}), id\tau = \frac{1}{2}(dz - d\bar{z}) \\ \partial_z = \frac{1}{2}(\partial_\sigma + \partial_{i\tau}), \partial_{\bar{z}} = \frac{1}{2}(\partial_\sigma - \partial_{i\tau}), \partial_\sigma = \partial_z + \partial_{\bar{z}}, \partial_{i\tau} = \partial_z - \partial_{\bar{z}} \end{cases} \quad \begin{cases} dz = d^*\bar{z} \\ \partial_z = \partial^*_{\bar{z}} \end{cases}$

7.4 Plane wave solutions of Penrose equation [1,2] in 2-dimensional space-time

Thm. 7.4.1. $(1, -i\varsigma)_a \partial^a \varphi(x) = 0$

$$\Rightarrow \varphi(x) = \frac{1}{\sqrt{\pi}} \int a(p,\varsigma) e^{ip(\tau+\varsigma\sigma)} dp = \begin{cases} \frac{1}{\sqrt{\pi}} \int\limits_0^{+\infty} [a(p)e^{i(p\sigma - |p|\tau)} + b^+(p)e^{-i(p\sigma-|p|\tau)}]dp, \varsigma = - \\ \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{0} [a(p)e^{i(p\sigma - |p|\tau)} + b^+(p)e^{-i(p\sigma-|p|\tau)}]dp, \varsigma = + \end{cases}$$

Proof: $(1, -i\varsigma)_a \partial^a \varphi(x) = 0$
$\Leftrightarrow \partial^a \partial_a \varphi(x) = 0, (1, -i\varsigma)_a \partial^a \varphi(x) = 0$
$\Rightarrow \varphi(x) = \frac{1}{\sqrt{\pi}} \int [a(p)e^{i(p\sigma-|p|\tau)} + b^+(p)e^{-i(p\sigma-|p|\tau)}]dp$
$\Rightarrow (\partial_\sigma - \varsigma\partial_\tau)\varphi(x) = 0$
$\Leftrightarrow \frac{1}{\sqrt{\pi}} \int [i(p+\varsigma|p|)a(p)e^{i(p\sigma-|p|\tau)} - i(p+\varsigma|p|)b^+(p)e^{-i(p\sigma-|p|\tau)}]dp$
$\Leftrightarrow (p+\varsigma|p|)a(p) = 0, (p+\varsigma|p|)b^+(p) = 0$
$\Leftrightarrow a(\varsigma p > 0) = 0, b^+(\varsigma p > 0) = 0$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_0^{+\infty} [a(p)e^{i(p\sigma-|p|\tau)} + b^+(p)e^{-i(p\sigma-|p|\tau)}]dp, \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{0} [a(p)e^{i(p\sigma-|p|\tau)} + b^+(p)e^{-i(p\sigma-|p|\tau)}]dp, \varsigma = + \end{cases}$$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_0^{+\infty} [a(p)e^{ip(\sigma-\tau)} + b^+(p)e^{-ip(\sigma-\tau)}]dp, \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{0} [a(p)e^{ip(\sigma+\tau)} + b^+(p)e^{-ip(\sigma+\tau)}]dp, \varsigma = + \end{cases}$$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} a(p)e^{ip(\sigma-\tau)}; a(p) := b^+(-p), p < 0; \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} a(p)e^{ip(\sigma+\tau)}dp; a(p) := b^+(-p), p > 0; \varsigma = + \end{cases}$$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} a'(p)e^{ip(\tau-\sigma)}dp, \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} a(p)e^{ip(\tau+\sigma)}dp, \varsigma = + \end{cases}$$

$$\Leftrightarrow \varphi(x) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} a(p,\varsigma)e^{ip(\tau+\varsigma\sigma)}dp$$ □

7.5 Causal function in 2-dimensional space-time

Def. 7.5.1. $\Delta(z_\varsigma) := \frac{i}{\pi} \int\limits_0^{+\infty} \frac{1}{2p}(e^{ipz_\varsigma} - e^{-ipz_\varsigma})dp = \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{2p}(e^{ipz_\varsigma} - e^{-ipz_\varsigma})dp = \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{p} e^{ipz_\varsigma}dp$

Def. 7.5.2. $\begin{cases} \Delta^{(+)}(z_\varsigma) := \frac{i}{\pi} \int\limits_{p=0}^{+\infty} \frac{1}{2p} e^{ipz_\varsigma} d\vec{p}, \Delta^{(-)}(z_\varsigma) := -\frac{i}{\pi} \int\limits_{p=0}^{+\infty} \frac{1}{2p} e^{-ipz_\varsigma} d\vec{p}, \Delta^{(-)}(z_\varsigma) = -\Delta^{(+)}(-z_\varsigma) \\ \Delta(z_\varsigma) := \frac{i}{\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{2p}[e^{ipz_\varsigma} - e^{-ipz_\varsigma}\}d\vec{p}, \Delta(z_\varsigma) = \Delta^{(+)}(z_\varsigma) + \Delta^{(-)}(z_\varsigma) \end{cases}$

Def. 7.5.3. $\begin{cases} \frac{1}{\sqrt{-\nabla^2}}\Delta(z_\varsigma) := \frac{i}{\pi} \int\limits_0^{+\infty} \frac{1}{2p^2}(e^{ipz_\varsigma} - e^{-ipz_\varsigma})dp = \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{2p|p|}(e^{ipz_\varsigma} - e^{-ipz_\varsigma})dp = \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{p|p|} e^{ipz_\varsigma}dp \\ \sqrt{-\nabla^2}\Delta(z_\varsigma) := \frac{i}{\pi} \int\limits_0^{+\infty} \frac{1}{2}(e^{ipz_\varsigma} - e^{-ipz_\varsigma})dp = \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{|p|}{2p}(e^{ipz_\varsigma} - e^{-ipz_\varsigma})dp = \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{|p|}{p} e^{ipz_\varsigma}dp \end{cases}$

Pro. 7.5.1. $\Delta^*(z_\varsigma) = \Delta(z_\varsigma), \Delta(-z_\varsigma) = -\Delta(z_\varsigma), (\nabla^2 - \partial_\tau^2)\Delta(z_\varsigma) = 0, \partial_{z_\varsigma}\Delta(z_\varsigma)|_{\tau=0} = -\delta(\sigma)$

Pro. 7.5.2. $\Delta(z_\varsigma - z'_\varsigma) := \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{1}{2p}[e^{ip\cdot(z_\varsigma - z'_\varsigma)} - e^{-ip\cdot(z_\varsigma - z'_\varsigma)}\}d\vec{p}$

$$\begin{cases}\partial_u\Delta(z_\varsigma-z'_\varsigma)=-\partial'_u\Delta(z_\varsigma-z'_\varsigma)\\ \nabla\Delta(z_\varsigma-z'_\varsigma)=-\nabla'\Delta(z_\varsigma-z'_\varsigma)\\ \partial_\pi\Delta(z_\varsigma-z'_\varsigma)=-\partial'_\pi\Delta(z_\varsigma-z'_\varsigma)\end{cases}\quad \begin{cases}(\sqrt{-\nabla^2})^n\Delta(z_\varsigma-z'_\varsigma)=(\sqrt{-\nabla'^2})^n\Delta(z_\varsigma-z'_\varsigma)\\ \frac{1}{(\sqrt{-\nabla^2})^n}\Delta(z_\varsigma-z'_\varsigma)=\frac{1}{(\sqrt{-\nabla'^2})^n}\Delta(z_\varsigma-z'_\varsigma)\\ \partial_\pi^{2n}\Delta(z_\varsigma-z'_\varsigma)=\partial'^{2n}_\pi\Delta(z_\varsigma-z'_\varsigma)\end{cases}$$

7.6 Commutation rules for s-spin equation in 2-dimensional space-time

Cor. 7.6.1. $[s\partial_a+iS_{ab}(s,\varsigma)\partial^b]\varphi(x)=0,\varphi(x)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}a(p,\varsigma)e^{ip(\tau+\varsigma\sigma)}dp$

Thm. 7.6.1. $[a(p,\varsigma),a^+(p',\varsigma)]_{-^{2s+1}}=\delta(p-p')\Rightarrow[\varphi(z_\varsigma),\varphi^+(z'_\varsigma)]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(1,i\varsigma)^a\partial_a(1,i\varsigma)^b\partial_b\cdots}^{2s}\Delta(z_\varsigma-z'_\varsigma)$

Proof: $[a(p,\varsigma),a^+(p',\varsigma)]_{-^{2s+1}}=\delta(p-p')$

$\Rightarrow[a(p),a^+(p')]_{-^{2s+1}}=[b(p),b^+(p')]_{-^{2s+1}}=\delta(p-p')$

$\Rightarrow[\varphi(z_\varsigma),\varphi^+(z'_\varsigma)]_{-^{2s+1}}$

$$=\begin{cases}\frac{1}{\pi}\int_0^{+\infty}\{e^{i(p\sigma-|p|\tau)}e^{-i(p'\sigma'-|p'|\tau')}[a(p),a^+(p')]_{-^{2s+1}}+e^{-i(p\sigma-|p|\tau)}e^{i(p'\sigma'-|p'|\tau')}[b^+(p),b(p')]_{-^{2s+1}}\}dpdp',\varsigma=-\\ \frac{1}{\pi}\int_{-\infty}^{0}\{e^{i(p\sigma-|p|\tau)}e^{-i(p'\sigma'-|p'|\tau')}[a(p),a^+(p')]_{-^{2s+1}}+e^{-i(p\sigma-|p|\tau)}e^{i(p'\sigma'-|p'|\tau')}[b^+(p),b(p')]_{-^{2s+1}}\}dpdp',\varsigma=+\end{cases}$$

$$=\begin{cases}\frac{1}{\pi}\int_0^{+\infty}\{e^{i(p\sigma-|p|\tau)}e^{-i(p'\sigma'-|p'|\tau')}+(-1)^{2s-1}e^{-i(p\sigma-|p|\tau)}e^{i(p'\sigma'-|p'|\tau')}\}[a(p),a^+(p')]_{-^{2s+1}}dpdp',\varsigma=-\\ \frac{1}{\pi}\int_{-\infty}^{0}\{e^{i(p\sigma-|p|\tau)}e^{-i(p'\sigma'-|p'|\tau')}+(-1)^{2s-1}e^{-i(p\sigma-|p|\tau)}e^{i(p'\sigma'-|p'|\tau')}\}[a(p),a^+(p')]_{-^{2s+1}}dpdp',\varsigma=+\end{cases}$$

$$=\begin{cases}\frac{1}{\pi}\int_0^{+\infty}\{e^{i(p\sigma-|p|\tau)}e^{-i(p'\sigma'-|p'|\tau')}+(-1)^{2s-1}e^{-i(p\sigma-|p|\tau)}e^{i(p'\sigma'-|p'|\tau')}\}|p|^{2s-1}\delta(p-p')dpdp',\varsigma=-\\ \frac{1}{\pi}\int_{-\infty}^{0}\{e^{i(p\sigma-|p|\tau)}e^{-i(p'\sigma'-|p'|\tau')}+(-1)^{2s-1}e^{-i(p\sigma-|p|\tau)}e^{i(p'\sigma'-|p'|\tau')}\}|p|^{2s-1}\delta(p-p')dpdp',\varsigma=+\end{cases}$$

$$=\begin{cases}\frac{1}{\pi}\int_0^{+\infty}\{e^{i(p\sigma-|p|\tau)}e^{-i(p\sigma'-|p|\tau')}+(-1)^{2s-1}e^{-i(p\sigma-|p|\tau)}e^{i(p\sigma'-|p|\tau')}\}|p|^{2s-1}dp,\varsigma=-\\ \frac{1}{\pi}\int_{-\infty}^{0}\{e^{i(p\sigma-|p|\tau)}e^{-i(p\sigma'-|p|\tau')}+(-1)^{2s-1}e^{-i(p\sigma-|p|\tau)}e^{i(p\sigma'-|p|\tau')}\}|p|^{2s-1}dp,\varsigma=+\end{cases}$$

$$=\begin{cases}\frac{1}{\pi}\int_0^{+\infty}\{e^{i(p\sigma-|p|\tau)}e^{-i(p\sigma'-|p|\tau')}+(-1)^{2s-1}e^{-i(p\sigma-|p|\tau)}e^{i(p\sigma'-|p|\tau')}\}|p|^{2s-1}dp,\varsigma=-\\ \frac{1}{\pi}\int_0^{+\infty}\{e^{i(-p\sigma-|p|\tau)}e^{-i(p\sigma'-|p|\tau')}+(-1)^{2s-1}e^{-i(-p\sigma-|p|\tau)}e^{i(-p\sigma'-|p|\tau')}\}|p|^{2s-1}dp,\varsigma=+\end{cases}$$

$$=\begin{cases}\frac{1}{\pi}\int_0^{+\infty}\{e^{ip(\sigma-\tau)}e^{-ip(\sigma'-\tau')}+(-1)^{2s-1}e^{-ip(\sigma-\tau)}e^{ip(\sigma'-\tau')}\}p^{2s-1}dp,\varsigma=-\\ \frac{1}{\pi}\int_0^{+\infty}\{(-1)^{2s-1}e^{ip(\sigma+\tau)}e^{-ip(\sigma'+\tau')}+e^{-ip(\sigma+\tau)}e^{ip(\sigma'+\tau')}\}p^{2s-1}dp,\varsigma=+\end{cases}$$

$$=\begin{cases}[\frac{-i}{2}(1,-i)^a\partial_a]^{2s}\frac{1}{\pi}\int_0^{+\infty}\{e^{ip(\sigma-\tau)}e^{-ip(\sigma'-\tau')}-e^{-ip(\sigma-\tau)}e^{ip(\sigma'-\tau')}\}p^{-1}dp,\varsigma=-\\ [\frac{-i}{2}(1,i)^a\partial_a]^{2s}\frac{1}{\pi}\int_0^{+\infty}\{(-1)^{2s-1}e^{ip(\sigma+\tau)}e^{-ip(\sigma'+\tau')}+(-1)^{2s}e^{-ip(\sigma+\tau)}e^{ip(\sigma'+\tau')}\}p^{-1}dp,\varsigma=+\end{cases}$$

$$=\begin{cases}[\frac{-i}{2}(1,-i)^a\partial_a]^{2s}\frac{1}{\pi}\int_0^{+\infty}\{e^{ip(\sigma-\tau)}e^{-ip(\sigma'-\tau')}-e^{-ip(\sigma-\tau)}e^{ip(\sigma'-\tau')}\}p^{-1}dp,\varsigma=-\\ [\frac{i}{2}(1,i)^a\partial_a]^{2s}\frac{1}{\pi}\int_0^{+\infty}\{-e^{ip(\sigma+\tau)}e^{-ip(\sigma'+\tau')}+e^{-ip(\sigma+\tau)}e^{ip(\sigma'+\tau')}\}p^{-1}dp,\varsigma=+\end{cases}$$

$$=\begin{cases}i[\frac{-i}{2}(1,-i)^a\partial_a]^{2s}\frac{i}{\pi}\int_0^{+\infty}\{e^{ip(\tau-\sigma)}e^{-ip(\tau'-\sigma')}-e^{-ip(\tau-\sigma)}e^{ip(\tau'-\sigma')}\}p^{-1}dp,\varsigma=-\\ +i[\frac{i}{2}(1,i)^a\partial_a]^{2s}\frac{i}{\pi}\int_0^{+\infty}\{e^{ip(\tau+\sigma)}e^{-ip(\tau'+\sigma')}-e^{-ip(\tau+\sigma)}e^{ip(\tau'+\sigma')}\}p^{-1}dp,\varsigma=+\end{cases}$$

$$=2i[\frac{i\varsigma}{2}(1,i\varsigma)^a\partial_a]^{2s}\frac{i}{2\pi}\int_0^{+\infty}\{e^{ip(\tau+\varsigma\sigma)}e^{-ip(\tau'+\varsigma\sigma')}-e^{-ip(\tau+\varsigma\sigma)}e^{ip(\tau'+\varsigma\sigma')}\}p^{-1}dp$$

$$=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(1,i\varsigma)^a\partial_a(1,i\varsigma)^b\partial_b\cdots}^{2s}\Delta(z_\varsigma-z'_\varsigma)$$

$$=2i^{2s+1}\partial_{z_\varsigma}^{2s}\Delta(z_\varsigma-z'_\varsigma)$$

$$=2i^{2s-1}\partial_{z_\varsigma}^{2s-1}\delta(z_\varsigma-z'_\varsigma)$$

□

7.7 Hotchpotch of Penrose equation [1, 2] in 2-dimensional space-time

Thm. 7.7.1. $(1,-i\varsigma)_a^{A'_\varsigma A_\varsigma}\partial^a\varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0,\varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=k(\tau+\varsigma\sigma)+\frac{1}{\sqrt{\pi}}\int a(p,\varsigma)e^{ip(\tau+\varsigma\sigma)}dp$

Thm. 7.7.2. $(1,-i\varsigma)_a\partial^a\varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0,\partial^a\partial_a\varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0$

Thm. 7.7.3. $(1,-i\varsigma)_a\partial^a\varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0\Leftrightarrow[s\partial_a+iS_{ab}(s)\partial^b]\varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma\cdot\cdot}_{2s}}(x)=0,iS_{ab}(s)=\begin{bmatrix}0 & is\\ -is & 0\end{bmatrix}$

Prop. 7.7.1. $[s\partial_a+iS_{ab}(s)\partial^b]\varphi(s)=0,\varphi'(s)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s)}\varphi(s)=e^{-s\varepsilon}\varphi(s),\vartheta^{ab}\succ\left[\begin{smallmatrix}0 & i\epsilon\\ -i\epsilon & 0\end{smallmatrix}\right]$

7.8 Commutation rules for $\frac{1}{3}$-spin equation in 2-dimensional space-time

Cor. 7.8.1. $[\frac{1}{3}\partial_a+iS_{ab}(\frac{1}{3},\varsigma)\partial^b]\varphi(x)=0,\varphi(x)=\frac{1}{\sqrt{\pi}}\int a(p,\varsigma)e^{ip(\tau+\varsigma\sigma)}dp$

Thm. 7.8.1. $[a(p,\varsigma),a^+(p',\varsigma)]_\varphi=\delta(p-p')\Rightarrow[\varphi(z_\varsigma),\varphi^+(z'_\varsigma)]_{-^{2s+1}}=i\frac{(i\varsigma)^{2s}}{2^{2s-1}}\overbrace{(1,i\varsigma)^a\partial_a(1,i\varsigma)^b\partial_b\cdot\cdot}^{2s}\Delta(z_\varsigma-z'_\varsigma)$

Ass. 7.8.1. $[a(p,\varsigma),a^+(p',\varsigma)]_{-\frac{5}{3}}=\delta(p-p')\Rightarrow[\varphi(z_\varsigma),\varphi^+(z'_\varsigma)]_{-\frac{5}{3}}=i\frac{(i\varsigma)^{\frac{2}{3}}}{2^{-\frac{1}{3}}}[(1,i\varsigma)^a\partial_a]^{\frac{2}{3}}\Delta(z_\varsigma-z'_\varsigma)$

Chapter38 Potential Analysis of B-W Equation in N+1 Dimensional Space-time

Self comment: This chapter imitates the four-dimensional case. It conducts potential decomposition and detailed mathematical analysis on B-W equation in N+1 dimensional space time. In the spin 1 case, antisymmetric tensor fields naturally appear. And based on commutation rules for the B-W equation, the commutation rules for antisymmetric tensor field are derived. Compared with the four-dimensional case, the potential analysis of the B-W equation is much more complex in N+1 dimensional space-time. Moreover, the fully symmetric B-W equation no longer describes a single spin state, but rather describes multiple fundamental fields. It is precisely this that leads to complexity. As a result, this promotion has lost some of its aesthetic appeal and made the description ugly, which seems to imply that this promotion has become meaningless. For example, it is sufficient to directly study the basic antisymmetric tensor field in the spin-1 case. Because it is a single fundamental field, it is simpler and more fundamental. This chapter does not provide a detailed discussion on higher spin cases such as spin-$\frac{3}{2}$,2,etc. But it only provides two conjectures for the high spin case. It will be strictly proved it until I have spare time in the future.

Through the research in this chapter, I found that the fully symmetric B-W equation is no longer a good method for describing physical fields in above four-dimensional spacetime. At this point, B-W equation does not describe a basic field similar in four dimensional space time, but rather a mixed field. So at this point, it is more appropriate to directly use antisymmetric tensor field description method.

1 Dirac matrix in N+1 dimensional space-time

1.1 Conventional representation of Dirac matrix in N+1 dimensional space-time

Def. 1.1.1.

$$\begin{cases}\gamma_a(1)=(1)\\ \gamma_1(1)=1\end{cases}$$

Def. 1.1.2.

$$\begin{cases}\gamma_a(2):=(\gamma_a(1)\otimes\sigma_x,1\otimes\sigma_y)=(\sigma_x,\sigma_y),\Gamma^a(2):=[\gamma_a(1),i\varsigma]=(1,i\varsigma)\\ C(2):=\gamma_2(2)=\sigma_y,\bar{C}(2)=C^+(2)=C(2),\gamma_1(2)\gamma_2(2)=i\sigma_z=i\gamma_0(2)\\ C^T(2)=-C(2),\gamma_a(2)C(2)=[\gamma_a(2)C(2)]^T,\gamma_{[a}(2)\gamma_{b]}(2)C(2)=\{\gamma_{[a}(2)\gamma_{b]}(2)C(2)\}^T\end{cases}$$

Def. 1.1.3.

$$\begin{cases}\gamma_a(3)=[\gamma_a(2),1\otimes\sigma_z]=(\sigma_x,\sigma_y,\sigma_z)\\ C(3):=\gamma_2(3)=\sigma_y,\bar{C}(3)=C^+(3)=C(3),\gamma_1(3)\cdots\gamma_3(3)=i=i\gamma_0(3)\\ C^T(3)=-C(3),[\gamma_a(3)C(3)]^T=\gamma_a(3)C(3)\end{cases}$$

Def. 1.1.4.

$$\begin{cases}\gamma_a(4)=[\gamma_a(3)\otimes\sigma_y,I\otimes\sigma_x]=(\sigma\otimes\sigma_y,I\otimes\sigma_x),\Gamma^a(4)=[\gamma_a(3),i\varsigma]\\ C(4):=\gamma_2(4)\gamma_4(4)=-i\sigma_y\otimes\sigma_z,\bar{C}(4)=C^+(4)=-C(4),\gamma_1(4)\cdots\gamma_4(4)=I\otimes\sigma_z=\gamma_0(4)\\ [\gamma_a(4)C(4)]^T=\gamma_a(4)C(4),\{\gamma_{[a}(4)\gamma_{b]}(4)C(4)\}^T=\gamma_{[a}(4)\gamma_{b]}(4)C(4)\\ C^T(4)=-C(4),\{\gamma_{[a}(4)\gamma_b(4)\gamma_{c]}(4)C(4)\}^T=-\gamma_{[a}(4)\gamma_b(4)\gamma_{c]}(4)C(4)\\ \{\gamma_{[a}(4)\gamma_b(4)\gamma_c(4)\gamma_{d]}(4)C(4)\}^T=-\gamma_{[a}(4)\gamma_b(4)\gamma_c(4)\gamma_{d]}(4)C(4)\end{cases}$$

Def. 1.1.5.

$$\begin{cases}\gamma_a(5)=[\gamma_a(4),I\otimes\sigma_z]=(\sigma\otimes\sigma_y,I\otimes\sigma_x,I\otimes\sigma_z)\\ C(5):=\gamma_2(5)\gamma_4(5)\gamma_5(5)=-i\sigma_y\otimes I,\bar{C}(5)=C^+(5)=-C(5),\gamma_1(5)\cdots\gamma_5(5)=1=\gamma_0(5)\\ C^T(5)=-C(5),[\gamma_a(5)C(5)]^T=-\gamma_a(5)C(5),\{\gamma_{[a}(5)\gamma_{b]}(5)C(5)\}^T=\gamma_{[a}(5)\gamma_{b]}(5)C(5)\end{cases}$$

Def. 1.1.6.

$$\gamma_a(10)=[[[((\sigma_x,\sigma_y,\sigma_z)\otimes\sigma_y,I\otimes\sigma_x,I\otimes\sigma_z)\otimes\sigma_y,I_4\otimes\sigma_x,I_4\otimes\sigma_z]\otimes\sigma_y,I_8\otimes\sigma_x,I_8\otimes\sigma_z]\otimes\sigma_y,I_{16}\otimes\sigma_x]$$

Def. 1.1.7.

$$\begin{cases}\gamma_a(6)=[\gamma_a(5)\otimes\sigma_y, I_4\otimes\sigma_x], \Gamma^a(6)=[\gamma_a(5), i\varsigma]\\ C(6):=\gamma_2(6)\gamma_4(6)\gamma_5(6)=-i\sigma_y\otimes I\otimes\sigma_y, \bar{C}(6)=C^+(6)=-C(6), \gamma_1(6)\cdots\gamma_6(6)=-iI_4\otimes\sigma_z=-i\gamma_0(6)\\ [\gamma_a(6)C(6)]^T=-\gamma_a(6)C(6), [\gamma_{[a}(6)\gamma_{b]}(6)C(6)]^T=-\gamma_{[a}(6)\gamma_{b]}(6)C(6)\\ [\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_{e]}(6)C(6)]^T=-\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_{e]}(6)C(6)\\ [\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_e(6)\gamma_{f]}(6)C(6)]^T=-\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_e(6)\gamma_{f]}(6)C(6)\\ C^T(6)=C(6), [\gamma_{[a}(6)\gamma_b(6)\gamma_{c]}(6)C(6)]^T=\gamma_{[a}(6)\gamma_b(6)\gamma_{c]}(6)C(6)\\ [\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_{d]}(6)C(6)]^T=\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_{d]}(6)C(6)\end{cases}$$

Def. 1.1.8.

$$\begin{cases}\gamma_a(7)=[\gamma_a(6), I_4\otimes\sigma_z]\\ C(7):=\gamma_2(7)\gamma_4(7)\gamma_5(7)=-i\sigma_y\otimes I\otimes\sigma_y, C(7)=C(6), \bar{C}(7)=C^+(7)=-C(7), \gamma_1(7)\cdots\gamma_7(7)=-i=-i\gamma_0(7)\\ [\gamma_a(7)C(7)]^T=-\gamma_a(7)C(7), [\gamma_{[a}(7)\gamma_{b]}(7)C(7)]^T=-\gamma_{[a}(7)\gamma_{b]}(7)C(7)\\ C^T(7)=C(7), [\gamma_{[a}(7)\gamma_b(7)\gamma_{c]}(7)C(7)]^T=\gamma_{[a}(7)\gamma_b(7)\gamma_{c]}(7)C(7)\end{cases}$$

Def. 1.1.9.

$$\begin{cases}\gamma_a(8)=[\gamma_a(7)\otimes\sigma_y, I_8\otimes\sigma_x], \Gamma^a(8)=[\gamma_a(7), i\varsigma]\\ C(8):=\gamma_2(8)\gamma_4(8)\gamma_5(8)\gamma_8(8)=-\sigma_y\otimes I\otimes\sigma_y\otimes\sigma_z, \bar{C}(8)=C^+(8)=C(8), \gamma_1(8)\cdots\gamma_8(8)=-I_8\otimes\sigma_z=-\gamma_0(8)\\ [\gamma_a(8)C(8)]^T=-\gamma_a(8)C(8), [\gamma_{[a}(8)\gamma_{b]}(8)C(8)]^T=-\gamma_{[a}(8)\gamma_{b]}(8)C(8)\\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_{e]}(8)C(8)]^T=-\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_{e]}(8)C(8)\\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_{f]}(8)C(8)]^T=-\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_{f]}(8)C(8)\\ C^T(8)=C(8), [\gamma_{[a}(8)\gamma_b(8)\gamma_{c]}(8)C(8)]^T=\gamma_{[a}(8)\gamma_b(8)\gamma_{c]}(8)C(8)\\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_{d]}(8)C(8)]^T=\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_{d]}(8)C(8)\\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_{g]}(8)C(8)]^T=\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_{g]}(8)C(8)\\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_g(8)\gamma_{h]}(8)C(8)]^T=\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_g(8)\gamma_{h]}(8)C(8)\end{cases}$$

Def. 1.1.10.

$$\begin{cases}\gamma_a(9)=[\gamma_a(8), I_8\otimes\sigma_z]=[\gamma_a(7)\otimes\sigma_y, I_8\otimes\sigma_x, I_8\otimes\sigma_z]\\ C(9):=\gamma_2(9)\gamma_4(9)\gamma_5(9)\gamma_8(9)\gamma_9(9)=-\sigma_y\otimes I\otimes\sigma_y\otimes I, \bar{C}(9)=C^+(9)=C(9), \gamma_1(9)\cdots\gamma_9(9)=-1=-\gamma_0(9)\\ [\gamma_{[a}(9)\gamma_{b]}(9)C(9)]^T=-\gamma_{[a}(9)\gamma_{b]}(9)C(9), [\gamma_{[a}(9)\gamma_b(9)\gamma_{c]}(9)C(9)]^T=-\gamma_{[a}(9)\gamma_b(9)\gamma_{c]}(9)C(9)\\ C^T(9)=C(9), [\gamma_a(9)C(9)]^T=\gamma_a(9)C(9), [\gamma_{[a}(9)\gamma_b(9)\gamma_c(9)\gamma_{d]}(9)C(9)]^T=\gamma_{[a}(9)\gamma_b(9)\gamma_c(9)\gamma_{d]}(9)C(9)\end{cases}$$

Def. 1.1.11.

$$\begin{cases}\gamma_a(10)=[\gamma_a(9)\otimes\sigma_y, I_{16}\otimes\sigma_x], \Gamma^a(10)=[\gamma_a(9), i\varsigma], \gamma_1(10)\cdots\gamma_{10}(10)=iI_{16}\otimes\sigma_z=i\gamma_0(10)\\ C(10):=\gamma_2(10)\gamma_4(10)\gamma_5(10)\gamma_8(10)\gamma_9(10)=-\sigma_y\otimes I\otimes\sigma_y\otimes I\otimes\sigma_y, \bar{C}(10)=C^+(10)=C(10)\end{cases}$$

$$\begin{cases}C^T(10)=-C(10), [\gamma_a(10)C(10)]^T=\gamma_a(10)C(10)\\ [\gamma_{[a}(10)\gamma_{b]}(10)C(10)]^T=\gamma_{[a}(10)\gamma_{b]}(10)C(10)\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_{c]}(10)C(10)]^T=-\gamma_{[a}(10)\gamma_b(10)\gamma_{c]}(10)C(10)\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_{d]}(10)C(10)]^T=-\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_{d]}(10)C(10)\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_{e]}(10)C(10)]^T=\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_{e]}(10)C(10)\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_{f]}(10)C(10)]^T=\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_{f]}(10)C(10)\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_{g]}(10)C(10)]^T=-\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_{g]}(10)C(10)\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_{h]}(10)C(10)]^T=-\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_{h]}(10)C(10)\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10)\gamma_{i]}(10)C(10)]^T=\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10)\gamma_{i]}(10\\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10)\gamma_i(10)\gamma_{j]}(10)C(10)]^T=\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10\end{cases}$$

Def. 1.1.12. $\gamma_a(10)=[\sigma_x\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y, \sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y, \sigma_z\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y, I\otimes\sigma_x\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y, I\otimes\sigma_z\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y, I_4\otimes\sigma_x\otimes\sigma_y\otimes\sigma_y, I_4\otimes\sigma_z\otimes\sigma_y\otimes\sigma_y, I_8\otimes\sigma_x\otimes\sigma_y, I_8\otimes\sigma_z\otimes\sigma_y, I_{16}\otimes\sigma_x]$

Def. 1.1.13.

$$\begin{cases}\gamma_a(11)=[\gamma_a(10), I_{16}\otimes\sigma_z], \gamma_1(11)\cdots\gamma_{11}(11)=i=i\gamma_0(11)\\ C(11):=\gamma_2(11)\gamma_4(11)\gamma_5(11)\gamma_8(11)\gamma_9(11)=-\sigma_y\otimes I\otimes\sigma_y\otimes I\otimes\sigma_y, C(11)=C(10), \bar{C}(11)=C^+(11)=C(11)\\ C^T(11)=-C(11), [\gamma_a(11)C(11)]^T=\gamma_a(11)C(11)\\ [\gamma_{[a}(11)\gamma_{b]}(11)C(11)]^T=\gamma_{[a}(11)\gamma_{b]}(11)C(11)\\ [\gamma_{[a}(11)\gamma_b(11)\gamma_{c]}(11)C(11)]^T=-\gamma_{[a}(11)\gamma_b(11)\gamma_{c]}(11)C(11)\\ [\gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_{d]}(11)C(11)]^T=-\gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_{d]}(11)C(11)\\ [\gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_d(11)\gamma_{e]}(11)C(11)]^T=\gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_d(11)\gamma_{e]}(11)C(11)\end{cases}$$

Ass. 1.1.1.

$$\begin{cases}\bar{C}(n)=C^{+}(n),C^{+}(n)=(-1)^{[\frac{n}{4}]}C(n),C^{T}(n)=(-1)^{[\frac{n+2}{4}]}C(n)\\ [\gamma_a(n)C(n)]^T=(-1)^{[\frac{n-1}{4}]}[\gamma_a(n)C(n)],[C^{+}(n)\gamma_a(n)]^T=(-1)^{[\frac{n-1}{4}]}[C^{+}(n)\gamma_a(n)]\end{cases}$$

Self comment: The above selection of Dirac matrix is not unique in N+1 dimensional space time. In principle, there are infinite options and just perform a representation transformation. Then the C matrix will also change and no longer be in its original form.

2 Antisymmetric tensor field expansion of second-order matrix in n=N+1-D [43]

2.1 Antisymmetric tensor field expansion of general matrix in even n=N+1-D

Def. 2.1.1.

$$X=\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}+\cdots+\frac{1}{(n!)^2}F^{a_1a_2a_3\cdot\cdot a_n}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_n]}$$

$$\begin{cases}F=2^{-[\frac{n}{2}]}tr(X),F_{a_1}=2^{-[\frac{n}{2}]}tr(\gamma_{a_1}X)\\ F_{a_1a_2}=-2^{-[\frac{n}{2}]}tr(\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]}X),F_{a_1a_2a_3}=-2^{-[\frac{n}{2}]}tr(\frac{1}{3!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}X)\\ F_{a_1a_2a_3a_4}=2^{-[\frac{n}{2}]}tr(\frac{1}{4!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}X),F_{a_1a_2a_3a_4a_5}=2^{-[\frac{n}{2}]}tr(\frac{1}{5!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}X)\\ \cdots F_{a_1a_2\cdot\cdot a_n}=(-1)^{[(n\%4)/2]}2^{-[\frac{n}{2}]}tr(\frac{1}{n!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_n]}X)\end{cases}$$

Def. 2.1.2.

$$\begin{cases}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)=\sum\limits_{i=0}^{n}\frac{1}{(i!)^2}F^{a_1\cdot\cdot a_i}|_{\eta_\varsigma}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})_{\lambda_\varsigma\mu_\varsigma}\\ F_{a_1\cdot\cdot a_i|\eta_\varsigma}(x)=(-1)^{[(i\%4)/2]}\frac{2^{-[\frac{n}{2}]}}{i!}(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})^{\mu_\varsigma\lambda_\varsigma}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)\end{cases}$$

Def. 2.1.3.

$$\begin{cases}\sum\limits_{i\in}^{even}\frac{1}{(i!)^2}F^{a_1\cdot\cdot a_i}|_{\eta_\varsigma}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)_{\lambda_\varsigma\mu_\varsigma}(C^{+}\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}^{\eta_\varsigma\mu_\varsigma}=0\\ \sum\limits_{i\in}^{even}\frac{1}{(i!)^2}(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)(C^{+}\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}F^{a_1\cdot\cdot a_i}(x)=0\end{cases}$$

Def. 2.1.4.

$$\begin{cases}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x)=\sum\limits_{i,j=0}^{n}\frac{1}{(i!j!)^2}F^{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})_{\lambda_\varsigma\mu_\varsigma}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]})_{\eta_\varsigma\xi_\varsigma}\\ F_{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)=(-1)^{[(i\%4)/2]+[(j\%4)/2]}\frac{4^{-[\frac{n}{2}]}}{i!j!}(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})^{\mu_\varsigma\lambda_\varsigma}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]})^{\xi_\varsigma\eta_\varsigma}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x)\end{cases}$$

Def. 2.1.5.

$$\begin{cases}\sum\limits_{i,j\in}^{even}\frac{1}{(i!j!)^2}F^{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)_{\lambda_\varsigma\mu_\varsigma}(C^{+}\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}^{\eta_\varsigma\mu_\varsigma}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]}C)_{\eta_\varsigma\xi_\varsigma}=0\\ \sum\limits_{i,j\in}^{even}\frac{1}{(i!j!)^2}F^{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)(C^{+}\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]}C)=0\end{cases}$$

2.2 Antisymmetric tensor field expansion of general matrix in odd n=N+1-D

Def. 2.2.1.

$$X=\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}+\cdots+\frac{1}{\{[n/2]!\}^2}F^{a_1a_2\cdot\cdot a_{[n/2]}}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{[n/2]}]}$$

$$\begin{cases}F=2^{-[\frac{n}{2}]}tr(X),F_{a_1}=2^{-[\frac{n}{2}]}tr(\gamma_{a_1}X)\\ F_{a_1a_2}=-2^{-[\frac{n}{2}]}tr(\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]}X),F_{a_1a_2a_3}=-2^{-[\frac{n}{2}]}tr(\frac{1}{3!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}X)\\ F_{a_1a_2a_3a_4}=2^{-[\frac{n}{2}]}tr(\frac{1}{4!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}X),F_{a_1a_2a_3a_4a_5}=2^{-[\frac{n}{2}]}tr(\frac{1}{5!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}X)\\ \cdots F_{a_1a_2\cdot\cdot a_{[n/2]}}=(-1)^{[([\frac{n}{2}]\%4)/2]}2^{-[\frac{n}{2}]}tr(\frac{1}{n!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_{[n/2]}]}X)\end{cases}$$

Def. 2.2.2.

$$\begin{cases}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)=\sum\limits_{i=0}^{[n/2]}\frac{1}{(i!)^2}F^{a_1\cdot\cdot a_i}|_{\eta_\varsigma}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})_{\lambda_\varsigma\mu_\varsigma}\\ F_{a_1\cdot\cdot a_i|\eta_\varsigma}(x)=(-1)^{[(i\%4)/2]}\frac{2^{-[\frac{n}{2}]}}{i!}(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})^{\mu_\varsigma\lambda_\varsigma}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}(x)\end{cases}$$

Def. 2.2.3.

$$\begin{cases}\sum\limits_{i\in}^{even}\frac{1}{(i!)^2}F^{a_1\cdot\cdot a_i}|_{\eta_\varsigma}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)_{\lambda_\varsigma\mu_\varsigma}(C^{+}\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}^{\eta_\varsigma\mu_\varsigma}=0\\ \sum\limits_{i\in}^{even}\frac{1}{(i!)^2}(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)(C^{+}\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}F^{a_1\cdot\cdot a_i}(x)=0\end{cases}$$

Def. 2.2.4.

$$\begin{cases}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x)=\sum\limits_{i,j=0}^{[n/2]}\frac{1}{(i!j!)^2}F^{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})_{\lambda_\varsigma\mu_\varsigma}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]})_{\eta_\varsigma\xi_\varsigma}\\ F_{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)=(-1)^{[(i\%4)/2]+[(j\%4)/2]}\frac{4^{-[\frac{n}{2}]}}{i!j!}(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]})^{\mu_\varsigma\lambda_\varsigma}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]})^{\xi_\varsigma\eta_\varsigma}X_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}(x)\end{cases}$$

Def. 2.2.5.

$$\begin{cases} \sum_{i,j\in}^{even} \frac{1}{(i!j!)^2} F^{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)_{\lambda_\varsigma\mu_\varsigma}(C^+\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}^{\eta_\varsigma\mu_\varsigma}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]}C)_{\eta_\varsigma\xi_\varsigma} = 0 \\ \sum_{i,j\in}^{even} \frac{1}{(i!j!)^2} F^{a_1\cdot\cdot a_i|b_1\cdot\cdot b_j}(x)(\gamma_{[a_1}\cdot\cdot\gamma_{a_i]}C)(C^+\gamma_{[c_1}\cdot\cdot\gamma_{c_k]})|_{odd}(\gamma_{[b_1}\cdot\cdot\gamma_{b_j]}C) = 0 \end{cases}$$

Self comment: The second-order Dirac tensor(spin-1) can be naturally decomposed into a set of antisymmetric tensors, so it concretely demonstrates spin-1 theory must be a gauge theory.

2.3 Symmetric matrix expansion in n=N+1 even dimensional space-time

Pro. 2.3.1. $X(2) = \{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1} + \frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}\}C$

Pro. 2.3.2. $X(4) = \{\frac{1}{1!}F^{a_1}\gamma_{a_1} + \frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}\}C$

Pro. 2.3.3. $X(6) = \{\frac{1}{0!}F + \frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]} + \frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C$

Pro. 2.3.4.
$X(8) = \{\frac{1}{(0!)^2}F + \frac{1}{3!}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]} + \frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]} + \frac{1}{(7!)^2}F^{a_1\cdot\cdot a_7}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]} + \frac{1}{(8!)^2}F^{a_1\cdot\cdot a_8}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\}C$

Pro. 2.3.5.
$X(10) = \{\frac{1}{(0!)^2}F + \frac{1}{2!}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]} + \frac{1}{(9!)^2}F^{a_1\cdot\cdot a_9}\gamma_{[a_1}\cdot\cdot\gamma_{a_9]}$
$+ \frac{1}{(10!)^2}F^{a_1\cdot\cdot a_{10}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{10}]}\}C$

2.4 Antisymmetric matrix expansion in n=N+1 even dimensional space-time

Pro. 2.4.1. $X(2) = \frac{1}{(0!)^2}FC$

Pro. 2.4.2. $X(4) = \{\frac{1}{(0!)^2}F + \frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]} + \frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\}C$

Pro. 2.4.3. $X(6) = \{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1} + \frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

Pro. 2.4.4. $X(8) = \{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1} + \frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

Pro. 2.4.5. $X(10) =$
$\{\frac{1}{1!}F^{a_1}\gamma_{a_1} + \frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]} + \frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]} + \frac{1}{(7!)^2}F^{a_1\cdot\cdot a_7}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]} + \frac{1}{(8!)^2}F^{a_1\cdot\cdot a_8}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\}C$

2.5 Symmetric matrix expansion in n=N+1 odd dimensional space-time

Pro. 2.5.1. $X(3) = \frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}C$

Pro. 2.5.2. $X(5) = \frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}C$

Pro. 2.5.3. $X(7) = \{\frac{1}{(0!)^2}F + \frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}\}C$

Pro. 2.5.4. $X(9) = \{\frac{1}{(0!)^2}F + \frac{1}{1!}F^{a_1}\gamma_{a_1} + \frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C$

Pro. 2.5.5. $X(11) = \{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1} + \frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C$

2.6 Antisymmetric matrix expansion in n=N+1 odd dimensional space-time

Pro. 2.6.1. $X(3) = \frac{1}{(0!)^2}FC$

Pro. 2.6.2. $X(5) = \{\frac{1}{(0!)^2}F + \frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}\}C$

Pro. 2.6.3. $X(7) = \{\frac{1}{(1!)^2}F^{a}\gamma_{a} + \frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C$

Pro. 2.6.4. $X(9) = \{\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C$

Pro. 2.6.5. $X(11) = \{\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]} + \frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\}C$

3 Common properties of basic antisymmetric tensor field in N+1 dimensional space-time

3.1 Antisymmetric tensor field without mass in N+1 dimensional space-time

Def. 3.1.1. $\partial^{[a_0}A^{a_1\cdot\cdot a_l]} = 0, \partial_{a_1}A^{a_1\cdot\cdot a_l} = 0; \partial^{[a_0}F^{a_1\cdot\cdot a_l]} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_l} = 0$

3.2 Antisymmetric tensor field with mass in N+1 dimensional space-time

Def. 3.2.1. $\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]} + mF^{a_0\cdot\cdot a_l} = 0, \partial_{a_0}F^{a_0\cdot\cdot a_l} + mA^{a_1\cdot\cdot a_l} = 0$
$\Leftrightarrow \partial_{a_0}\partial^{a_0}A^{a_1\cdot\cdot a_l} - m^2A^{a_1\cdot\cdot a_l} = 0, \partial_{a_1}A^{a_1\cdot\cdot a_l} = 0, F^{a_0\cdot\cdot a_l} = -\frac{1}{(l+1)!m}\partial^{[a_0}A^{a_1\cdot\cdot a_l]}$

3.3 Relations between dual bases of antisymmetric tensor field in N+1-D

Thm. 3.3.1. $\frac{1}{l!}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]} = -(-1)^{(n-l-1)(n-l)/2} i^{-[n/2]}\varepsilon_{a_1\cdot\cdot a_n}\frac{1}{[(n-l)!]^2}\Gamma_0\gamma^{[a_{l+1}}\cdot\cdot\gamma^{a_n]}, \Gamma_0 := -i^{[n/2]}\gamma_1\cdot\cdot\gamma_n$

Cor. 3.3.1. $\begin{cases}\frac{1}{l!}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]} = -i^{[n/2]+l[l+(-1)^n]}\varepsilon_{a_1\cdot\cdot a_n}\frac{1}{[(n-l)!]^2}\Gamma_0\gamma^{[a_{l+1}}\cdot\cdot\gamma^{a_n]}, \Gamma_0 := -i^{[n/2]}\gamma_1\cdot\cdot\gamma_n, \Gamma_0|_{odd} = 1\\ \frac{1}{l!}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]} = -i^{[n/2]+l(l-1)}\varepsilon_{a_1\cdot\cdot a_n}\frac{1}{[(n-l)!]^2}\gamma^{[a_{l+1}}\cdot\cdot\gamma^{a_n]}\Gamma_0, \Gamma_0 := -i^{[n/2]}\gamma_1\cdot\cdot\gamma_n, \Gamma_0|_{odd} = 1\end{cases}$

3.4 Equivalent dual representation of antisymmetric tensor field in N+1-D

Lem. 3.4.1. $*A^{a_1\cdot\cdot a_l} = \frac{1}{(n-l)!}\varepsilon^{a_1 a_2\cdot\cdot a_n}A_{a_{l+1}\cdot\cdot a_n} \Leftrightarrow A_{a_{l+1}\cdot\cdot a_n} = \frac{1}{l!}\varepsilon_{a_1 a_2\cdot\cdot a_n}*A^{a_1\cdot\cdot a_l} = (-1)^{Nl}**A_{a_{l+1}\cdot\cdot a_n}$

Proof: $*A^{a_1\cdot\cdot a_l} = \frac{1}{(n-l)!}\varepsilon^{a_1 a_2\cdot\cdot a_n}A_{a_{l+1}\cdot\cdot a_n}$
$\Rightarrow \varepsilon_{a_1\cdot\cdot a_l b_{l+1}\cdot\cdot b_n}*A^{a_1\cdot\cdot a_l}$
$= \varepsilon_{a_1\cdot\cdot a_l b_{l+1}\cdot\cdot b_n}\frac{1}{(n-l)!}\varepsilon^{a_1\cdot\cdot a_l a_{l+1}\cdot\cdot a_n}A_{a_{l+1}\cdot\cdot a_n}$
$= \frac{l!}{(n-l)!}\delta^{[a_{l+1}}_{b_{l+1}}\cdot\cdot\delta^{a_n]}_{b_n}A_{a_{l+1}\cdot\cdot a_n}$
$= \frac{l!}{(n-l)!}(n-l)!\delta^{b_{l+1}}_{a_{l+1}}\cdot\cdot\delta^{b_n}_{a_n}A_{a_{l+1}\cdot\cdot a_n}$
$= l!A_{b_{l+1}\cdot\cdot b_n}$
$\Rightarrow A_{a_{l+1}\cdot\cdot a_n} = \frac{1}{l!}\varepsilon_{a_1 a_2\cdot\cdot a_n}*A^{a_1\cdot\cdot a_l} = (-1)^{Nl}**A_{a_{l+1}\cdot\cdot a_n}$ □

Proof: $A_{a_{l+1}\cdot\cdot a_n} = (-1)^{Nl}**A_{a_{l+1}\cdot\cdot a_n} = \frac{1}{l!}\varepsilon_{a_1 a_2\cdot\cdot a_n}*A^{a_1\cdot\cdot a_l}$
$\Rightarrow \varepsilon^{b_1\cdot\cdot b_l a_{l+1}\cdot\cdot a_n}A_{a_{l+1}\cdot\cdot a_n}$
$= \varepsilon^{b_1\cdot\cdot b_l a_{l+1}\cdot\cdot a_n}\frac{1}{l!}\varepsilon_{a_1 a_2\cdot\cdot a_n}*A^{a_1\cdot\cdot a_l}$
$= \frac{(n-l)!}{l!}\delta^{b_1}_{[a_1}\cdot\cdot\delta^{b_l}_{a_l]}*A^{a_1\cdot\cdot a_l}$
$= \frac{(n-l)!}{l!}l!\delta^{b_1}_{a_1}\cdot\cdot\delta^{b_l}_{a_l}*A^{a_1\cdot\cdot a_l}$
$= (n-l)!*A^{b_1\cdot\cdot b_l}$
$\Rightarrow *A^{a_1\cdot\cdot a_l} = \frac{1}{(n-l)!}\varepsilon^{a_1 a_2\cdot\cdot a_n}A_{a_{l+1}\cdot\cdot a_n}$ □

Thm. 3.4.1.
$$\begin{cases}\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]} + mF^{a_0\cdot\cdot a_l} = 0\\ *A_{a_{l+1}\cdot\cdot a_n} = \frac{(-1)^{Nl}}{l!}\varepsilon_{a_1\cdot\cdot a_n}A^{a_1\cdot\cdot a_l}\\ *F_{a_{l+1}\cdot\cdot a_{n-1}} = \frac{(-1)^{N(l+1)}}{(l+1)!}\varepsilon_{a_0\cdot\cdot a_{n-1}}F^{a_0\cdot\cdot a_l}\end{cases} \Leftrightarrow \begin{cases}\partial^{a_0}*A_{a_0 a_1\cdot\cdot a_{n-l-1}} + (-1)^{n-l-1}m*F_{a_1\cdot\cdot a_{n-l-1}} = 0\\ A^{a_1\cdot\cdot a_l} = \frac{(-1)^{Nl}}{(n-l)!}\varepsilon^{a_1\cdot\cdot a_n}*A_{a_{l+1}\cdot\cdot a_n}\\ F^{a_0\cdot\cdot a_l} = \frac{(-1)^{N(l+1)}}{(n-l-1)!}\varepsilon^{a_0\cdot\cdot a_{n-1}}*F_{a_{l+1}\cdot\cdot a_{n-1}}\end{cases}$$

Proof: $\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]} + mF^{a_0\cdot\cdot a_l} = 0 \Leftrightarrow \frac{1}{(l+1)!}\varepsilon_{a_0 a_1 a_2\cdot\cdot a_{n-1}}\{\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]} + mF^{a_0\cdot\cdot a_l}\} = 0$
$\Leftrightarrow \frac{1}{l!}\varepsilon_{a_0 a_1 a_2\cdot\cdot a_{n-1}}\partial^{a_0}A^{a_1\cdot\cdot a_l} + \frac{1}{(l+1)!}\varepsilon_{a_0 a_1 a_2\cdot\cdot a_{n-1}}mF^{a_0\cdot\cdot a_l} = 0$
$\Leftrightarrow (-1)^{Nl-l}\partial^{a_0}*A_{a_0 a_{l+1}\cdot\cdot a_{n-1}} + m(-1)^{N(l+1)}*F_{a_{l+1}\cdot\cdot a_{n-1}} = 0$
$\Leftrightarrow \partial^{a_0}*A_{a_0 a_{l+1}\cdot\cdot a_{n-1}} - (-1)^{n-l}m*F_{a_{l+1}\cdot\cdot a_{n-1}} = 0$
$\Leftrightarrow \partial^{a_0}*A_{a_0 a_1\cdot\cdot a_{n-l-1}} + (-1)^{n-l-1}m*F_{a_1\cdot\cdot a_{n-l-1}} = 0$ □

Thm. 3.4.2.
$$\begin{cases}\partial_{a_0}F^{a_0\cdot\cdot a_l} + mA^{a_1\cdot\cdot a_l} = 0\\ *A_{a_{l+1}\cdot\cdot a_n} = \frac{(-1)^{Nl}}{l!}\varepsilon_{a_1\cdot\cdot a_n}A^{a_1\cdot\cdot a_l}\\ *F_{a_{l+1}\cdot\cdot a_{n-1}} = \frac{(-1)^{N(l+1)}}{(l+1)!}\varepsilon_{a_0\cdot\cdot a_{n-1}}F^{a_0\cdot\cdot a_l}\end{cases} \Leftrightarrow \begin{cases}\frac{1}{(n-l-1)!}\partial_{[a_0}*F_{a_1\cdot\cdot a_{n-l-1}]} + (-1)^{n-l-1}m*A_{a_0 a_1\cdot\cdot a_{n-l-1}} = 0\\ A^{a_1\cdot\cdot a_l} = \frac{(-1)^{Nl}}{(n-l)!}\varepsilon^{a_1\cdot\cdot a_n}*A_{a_{l+1}\cdot\cdot a_n}\\ F^{a_0\cdot\cdot a_l} = \frac{(-1)^{N(l+1)}}{(n-l-1)!}\varepsilon^{a_0\cdot\cdot a_{n-1}}*F_{a_{l+1}\cdot\cdot a_{n-1}}\end{cases}$$

Proof: $\partial_{a_0}F^{a_0\cdot\cdot a_l} + mA^{a_1\cdot\cdot a_l} = 0 \Leftrightarrow \frac{1}{l!}\varepsilon_{a_1 a_2\cdot\cdot a_n}\{\partial_{a_0}F^{a_0 a_1\cdot\cdot a_l} + mA^{a_1\cdot\cdot a_l}\} = 0$
$\Leftrightarrow \frac{1}{l!}\varepsilon_{a_1 a_2\cdot\cdot a_n}\{\partial_{a_0}\frac{(-1)^{N(l+1)}}{(n-l-1)!}\varepsilon^{a_0 a_1\cdot\cdot b_{l+1}b_{n-1}}*F_{b_{l+1}\cdot\cdot b_{n-1}} + mA^{a_1\cdot\cdot a_l}\} = 0$
$\Leftrightarrow \frac{1}{l!}\varepsilon_{a_1 a_2\cdot\cdot a_n}\partial_{a_0}\frac{(-1)^{Nl}}{(n-l-1)!}\varepsilon^{a_1\cdot\cdot b_{l+1}b_{n-1}a_0}*F_{b_{l+1}\cdot\cdot b_{n-1}} + (-1)^{Nl}m*A^{a_{l+1}\cdot\cdot a_n} = 0$
$\Leftrightarrow \frac{1}{(n-l-1)!}\delta^{b_{l+1}}_{[a_{l+1}}\cdot\cdot\delta^{b_{n-1}}_{a_{n-1}}\delta^{a_0}_{a_n]}\partial_{a_0}*F_{b_{l+1}\cdot\cdot b_{n-1}} + m*A_{a_{l+1}\cdot\cdot a_n} = 0$
$\Leftrightarrow \frac{1}{(n-l-1)!}\partial_{[a_n}*F_{a_{l+1}\cdot\cdot a_{n-1}]} + m*A_{a_{l+1}\cdot\cdot a_n} = 0$
$\Leftrightarrow \frac{1}{(n-l-1)!}\partial_{[a_0}*F_{a_1\cdot\cdot a_{n-l-1}]} + (-1)^{n-l-1}m*A_{a_0 a_1\cdot\cdot a_{n-l-1}} = 0$ □

Cor. 3.4.1.
$$\begin{cases}\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]} + mF^{a_0\cdot\cdot a_l} = 0\\ \partial_{a_0}F^{a_0\cdot\cdot a_l} + mA^{a_1\cdot\cdot a_l} = 0\end{cases} \Leftrightarrow \begin{cases}\partial_{a_0}*A^{a_0 a_1\cdot\cdot a_{n-l-1}} + (-1)^{n-l-1}m*F^{a_1\cdot\cdot a_{n-l-1}} = 0\\ \frac{1}{(n-l-1)!}\partial^{[a_0}*F^{a_1\cdot\cdot a_{n-l-1}]} + (-1)^{n-l-1}m*A^{a_0 a_1\cdot\cdot a_{n-l-1}} = 0\end{cases}$$

Cor. 3.4.2.
$$\begin{cases}\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]} + mF^{a_0\cdot\cdot a_l} = 0\\ \partial_{a_0}F^{a_0\cdot\cdot a_l} + mA^{a_1\cdot\cdot a_l} = 0\end{cases} \Leftrightarrow \begin{cases}\partial_{a_0}*A^{a_0 a_1\cdot\cdot a_{n-l-1}} + (-1)^{n-l-1}m*F^{a_1\cdot\cdot a_{n-l-1}} = 0\\ \partial_{a_0}F^{a_0\cdot\cdot a_l} + mA^{a_1\cdot\cdot a_l} = 0\end{cases}$$

Cor. 3.4.3.

$$\begin{cases}\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]}+mF^{a_0\cdot\cdot a_l}=0\\ \partial_{a_0}F^{a_0\cdot\cdot a_l}+mA^{a_1\cdot\cdot a_l}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]}+mF^{a_0\cdot\cdot a_l}=0\\ \frac{1}{(n-l-1)!}\partial^{[a_0}{*}F^{a_1\cdot\cdot a_{n-l-1}]}+(-1)^{n-l-1}m{*}A^{a_0a_1\cdot\cdot a_{n-l-1}}=0\end{cases}$$

Cor. 3.4.4.

$$\begin{cases}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}=0\\ \partial_{a_1}F^{a_1\cdot\cdot a_l}=0\end{cases}\Leftrightarrow\begin{cases}\partial_{a_1}{*}F^{a_1\cdot\cdot a_{n-l}}=0\\ \partial^{[a_0}{*}F^{a_1\cdot\cdot a_{n-l}]}=0\end{cases}\Leftrightarrow\begin{cases}\partial_{a_1}{*}F^{a_1\cdot\cdot a_{n-l}}=0\\ \partial_{a_1}F^{a_1\cdot\cdot a_l}=0\end{cases}\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}=0\\ \partial^{[a_0}{*}F^{a_1\cdot\cdot a_{n-l}]}=0\end{cases}$$

3.5 B-W equation derives basic antisymmetric tensor field in even n=N+1-D

Lem. 3.5.1. $(\gamma^a\partial_a+m)\{\frac{1}{(l!)^2}F^{a_1\cdot\cdot a_l}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}+\frac{1}{[(l+1)!]^2}F^{a_1\cdot\cdot a_{l+1}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l+1}]}\}C=0$

$$\Leftrightarrow\begin{cases}\frac{1}{l!}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}+mF^{a_0a_1\cdot\cdot a_l}=0,\partial_{a_1}F^{a_1\cdot\cdot a_l}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_{l+1}]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_l}+mF^{a_1\cdot\cdot a_l}=0\end{cases};1\le l\le n-1$$

Proof: $(\gamma^a\partial_a+m)\{\frac{1}{(l!)^2}F^{a_1\cdot\cdot a_l}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}+\frac{1}{[(l+1)!]^2}F^{a_1\cdot\cdot a_{l+1}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l+1}]}\}=0;1\le l\le n-1$

$\Leftrightarrow(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{l!}F^{a_1\cdot\cdot a_l}\gamma_{a_1}\cdot\cdot\gamma_{a_l}+\frac{1}{(l+1)!}F^{a_1\cdot\cdot a_{l+1}}\gamma_{a_1}\cdot\cdot\gamma_{a_{l+1}}\}=0$

$\Leftrightarrow\{\frac{1}{(l+1)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_l]}+\frac{1}{(l-1)!}(\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_l]}+\cdot\cdot)+\cdot\cdot\}\partial^{a_0}\frac{1}{l!}F^{a_1\cdot\cdot a_l}+m\frac{1}{(l+1)!}F^{a_1\cdot\cdot a_{l+1}}\gamma_{a_1}\cdot\cdot\gamma_{a_{l+1}}$
$+\{\frac{1}{(l+2)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_{l+1}]}+\frac{1}{l!}(\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_{l+1}]}+\cdot\cdot)+\cdot\cdot\}\partial^{a_0}\frac{1}{(l+1)!}F^{a_1\cdot\cdot a_{l+1}}+m\frac{1}{l!}F^{a_1\cdot\cdot a_l}\gamma_{a_1}\cdot\cdot\gamma_{a_l}=0$

$\Leftrightarrow\{\frac{1}{(l+1)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_l]}+\frac{l}{(l-1)!}\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_l]}\}\partial^{a_0}\frac{1}{l!}F^{a_1\cdot\cdot a_l}+\frac{1}{(l+1)!}m\frac{1}{(l+1)!}F^{a_1\cdot\cdot a_{l+1}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l+1}]}$
$+\{\frac{1}{(l+2)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_{l+1}]}+\frac{l+1}{l!}\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_{l+1}]}\}\partial^{a_0}\frac{1}{(l+1)!}F^{a_1\cdot\cdot a_{l+1}}+\frac{1}{l!}m\frac{1}{l!}F^{a_1\cdot\cdot a_l}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}=0$

$$\Leftrightarrow\begin{cases}\frac{1}{(l+1)!}\partial^{[a_0}\frac{1}{l!}F^{a_1\cdot\cdot a_l]}+m\frac{1}{(l+1)!}F^{a_0a_1\cdot\cdot a_l}=0,\partial_{a_1}\frac{1}{l!}F^{a_1\cdot\cdot a_l}=0\\ \partial^{[a_0}\frac{1}{(l+1)!}F^{a_1\cdot\cdot a_{l+1}]}=0,\partial_{a_0}\frac{1}{(l+1)!}F^{a_0a_1\cdot\cdot a_l}+\frac{1}{l+1}m\frac{1}{l!}F^{a_1\cdot\cdot a_l}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{1}{l!}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}+mF^{a_0a_1\cdot\cdot a_l}=0,\partial_{a_1}F^{a_1\cdot\cdot a_l}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_{l+1}]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_l}+mF^{a_1\cdot\cdot a_l}=0\end{cases};1\le l\le n-1\qquad\square$$

Lem. 3.5.2. $(\gamma^a\partial_a+m)\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}\}C=0\Leftrightarrow\begin{cases}\frac{1}{0!}\partial^{a_0}F+mF^{a_0}=0\\ \partial^{[a_0}F^{a_1]}=0,\partial_{a_0}F^{a_0}+mF=0\end{cases};l=0$

Proof: $(\gamma^a\partial_a+m)\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}\}C$

$\Leftrightarrow(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{0!}F+\frac{1}{1!}F^{a_1}\gamma_{a_1}\}=0$

$\Leftrightarrow\frac{1}{1!}\gamma_{a_0}\partial^{a_0}\frac{1}{0!}F+m\frac{1}{1!}F^{a_1}\gamma_{a_1}+\{\frac{1}{2!}\gamma_{[a_0}\gamma_{a_1]}+\frac{1}{0!}\delta_{a_0a_1}\}\partial^{a_0}\frac{1}{1!}F^{a_1}+m\frac{1}{0!}F=0$

$$\Leftrightarrow\begin{cases}\frac{1}{1!}\partial^{a_0}\frac{1}{0!}F+m\frac{1}{1!}F^{a_0}=0\\ \partial^{[a_0}\frac{1}{1!}F^{a_1]}=0,\partial_{a_0}\frac{1}{1!}F^{a_0}+\frac{1}{1}m\frac{1}{0!}F=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{1}{0!}\partial^{a_0}F+mF^{a_0}=0\\ \partial^{[a_0}F^{a_1]}=0,\partial_{a_0}F^{a_0}+mF=0\end{cases};l=0\qquad\square$$

Lem. 3.5.3. $(\gamma^a\partial_a+m)\{\frac{1}{(n!)^2}F^{a_1\cdot\cdot a_n}\gamma_{[a_1}\cdot\cdot\gamma_{a_n]}\}C=0\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1\cdot\cdot a_n]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_n}=0\\ mF^{a_1\cdot\cdot a_n}=0\end{cases};l=n$

Proof: $(\gamma^a\partial_a+m)\{\frac{1}{(n!)^2}F^{a_1\cdot\cdot a_n}\gamma_{[a_1}\cdot\cdot\gamma_{a_n]}\}C=0;l=n$

$\Leftrightarrow(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{n!}F^{a_1\cdot\cdot a_n}\gamma_{a_1}\cdot\cdot\gamma_{a_n}\}=0$

$\Leftrightarrow\{\frac{1}{(n+1)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_n]}+\frac{1}{(n-1)!}(\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_n]}+\cdot\cdot)+\cdot\cdot\}\partial^{a_0}\frac{1}{n!}F^{a_1\cdot\cdot a_n}+m\frac{1}{n!}F^{a_1\cdot\cdot a_n}\gamma_{a_1}\cdot\cdot\gamma_{a_n}=0$

$\Leftrightarrow\{\frac{1}{(n+1)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_n]}+\frac{n}{(n-1)!}\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_n]}\}\partial^{a_0}\frac{1}{n!}F^{a_1\cdot\cdot a_n}+\frac{1}{n!}m\frac{1}{n!}F^{a_1\cdot\cdot a_n}\gamma_{[a_1}\cdot\cdot\gamma_{a_n]}=0$

$$\Leftrightarrow\begin{cases}\frac{1}{(n+1)!}\partial^{[a_0}\frac{1}{n!}F^{a_1\cdot\cdot a_n]}=0,\partial_{a_1}\frac{1}{n!}F^{a_1\cdot\cdot a_n}=0\\ \frac{1}{n+1}m\frac{1}{n!}F^{a_1\cdot\cdot a_n}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1\cdot\cdot a_n]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_n}=0\\ mF^{a_1\cdot\cdot a_n}=0\end{cases};l=n\qquad\square$$

3.6 Properties of basic antisymmetric tensor field in n=N+1 even dimensional space-time

Cor. 3.6.1. $\frac{1}{l!}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}=-i^{[n/2]+l(l+1)}\varepsilon_{a_1\cdot\cdot a_n}\frac{1}{[(n-l)!]^2}\Gamma_0\gamma^{[a_{l+1}}\cdot\cdot\gamma^{a_n]}$

Thm. 3.6.1. $(\gamma^a\partial_a+m)\{\frac{1}{(l!)^2}F^{a_1\cdot\cdot a_l}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}+\frac{1}{[(l+1)!]^2}F^{a_1\cdot\cdot a_{l+1}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l+1}]}\}C=0$

$$\Leftrightarrow\begin{cases}\frac{1}{l!}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}+mF^{a_0a_1\cdot\cdot a_l}=0,\partial_{a_1}F^{a_1\cdot\cdot a_l}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_{l+1}]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_l}+mF^{a_1\cdot\cdot a_l}=0\end{cases}$$

Cor. 3.6.2. $\gamma^a\partial_a\{\frac{1}{(l!)^2}F^{a_1\cdot\cdot a_l}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}+\frac{1}{[(l+1)!]^2}F^{a_1\cdot\cdot a_{l+1}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l+1}]}\}C=0$

$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_l}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_{l+1}]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_l}=0\end{cases}$$

Cor. 3.6.3. $(\gamma^a\partial_a+m)\{\frac{1}{(l!)^2}F^{a_1\cdot\cdot a_l}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}+\frac{1}{[(l+1)!]^2}F^{a_1\cdot\cdot a_{l+1}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l+1}]}\}C=0,m\neq 0$

$$\Leftrightarrow\begin{cases}\frac{1}{l!}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}+mF^{a_0a_1\cdot\cdot a_l}=0\\ \partial_{a_0}F^{a_0a_1\cdot\cdot a_l}+mF^{a_1\cdot\cdot a_l}=0\end{cases}\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_l}-m^2F^{a_1\cdot\cdot a_l}=0,\partial_{a_1}F^{a_1\cdot\cdot a_l}=0\\ F^{a_0a_1\cdot\cdot a_l}=-\frac{1}{l!m}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}\end{cases}$$

Cor. 3.6.4.

$$\begin{cases}\frac{1}{l!}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}+mF^{a_0a_1\cdot\cdot a_l}=0\\ \partial_{a_0}F^{a_0a_1\cdot\cdot a_l}+mF^{a_1\cdot\cdot a_l}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{(n-l-1)!}\partial^{[a_0}*F^{a_1\cdot\cdot a_{n-l-1}]}+(-1)^{n-l-1}m*F^{a_0a_1\cdot\cdot a_{n-l-1}}=0\\ \partial_{a_0}*F^{a_0a_1\cdot\cdot a_{n-l-1}}+(-1)^{n-l-1}m*F^{a_1\cdot\cdot a_{n-l-1}}=0\end{cases}$$

3.7 Properties of basic antisymmetric tensor field in n=N+1 odd dimensional space-time

Cor. 3.7.1. $\frac{1}{l!}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}=-i^{[n/2]+l(l-1)}\varepsilon_{a_1\cdot\cdot a_n}\frac{1}{[(n-l)!]^2}\gamma^{[a_{l+1}}\cdot\cdot\gamma^{a_n]}$

Cor. 3.7.2. $\frac{1}{([\frac{n}{2}]+1)!}\gamma_{[a_1}\cdot\cdot\gamma_{a_{[n/2]+1}]}=-(-i)^{[\frac{n}{2}]\%2}\varepsilon_{a_1\cdot\cdot a_n}\frac{1}{([\frac{n}{2}]!)^2}\gamma^{[a_{[n/2]+2}}\cdot\cdot\gamma^{a_n]}$

Thm. 3.7.1. $(\gamma^a\partial_a+m)\{\frac{1}{(l!)^2}F^{a_1\cdot\cdot a_l}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}+\frac{1}{[(l+1)!]^2}F^{a_1\cdot\cdot a_{l+1}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{l+1}]}\}C=0$

$$\Leftrightarrow\begin{cases}\frac{1}{l!}\partial^{[a_0}F^{a_1\cdot\cdot a_l]}+mF^{a_0a_1\cdot\cdot a_l}=0,\partial_{a_1}F^{a_1\cdot\cdot a_l}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_{l+1}]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_l}+mF^{a_1\cdot\cdot a_l}=0\end{cases};l\leq[\tfrac{n}{2}]-2$$

Lem. 3.7.1. $(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{([\frac{n}{2}]!)^2}F^{a_1\cdot\cdot a_{[n/2]}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{[n/2]}]}\}C=0$

$\Leftrightarrow\frac{1}{[\frac{n}{2}]!}\varepsilon^{a_1\cdot\cdot a_n}\partial_{a_{[n/2]+1}}F_{a_{[n/2]+2}\cdot\cdot a_n}-i^{[\frac{n}{2}]\%2}mF^{a_1\cdot\cdot a_{[n/2]}}=0,\partial_{a_1}F^{a_1\cdot\cdot a_{[n/2]}}=0$

Proof: $(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{([\frac{n}{2}]!)^2}F^{a_1\cdot\cdot a_{[n/2]}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{[n/2]}]}\}C=0$

$\Leftrightarrow(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{([\frac{n}{2}]!)^2}F^{a_1\cdot\cdot a_{[n/2]}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{[n/2]}]}\}=0$

$\Leftrightarrow(\gamma_{a_0}\partial^{a_0}+m)\{F^{a_1\cdot\cdot a_{[n/2]}}\gamma_{a_1}\cdot\cdot\gamma_{a_{[n/2]}}\}=0$

$\Leftrightarrow\gamma_{a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_{[n/2]}}\partial^{a_0}F^{a_1\cdot\cdot a_{[n/2]}}+mF^{a_1\cdot\cdot a_{[n/2]}}\gamma_{a_1}\cdot\cdot\gamma_{a_{[n/2]}}=0$

$\Leftrightarrow\{\frac{1}{([n/2]+1)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_{[n/2]}]}+\frac{1}{([n/2]-1)!}(\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_{[n/2]}]}+\cdot\cdot)+\cdot\cdot\}\partial^{a_0}F^{a_1\cdot\cdot a_{[n/2]}}+mF^{a_1\cdot\cdot a_{[n/2]}}\gamma_{a_1}\cdot\cdot\gamma_{a_{[n/2]}}=0$

$\Leftrightarrow\frac{1}{([\frac{n}{2}]+1)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_{[n/2]}]}\partial^{a_0}F^{a_1\cdot\cdot a_{[n/2]}}+\frac{[\frac{n}{2}]}{([\frac{n}{2}]-1)!}\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_{[n/2]}]}\partial^{a_0}F^{a_1\cdot\cdot a_{[n/2]}}+mF^{a_1\cdot\cdot a_{[n/2]}}\gamma_{a_1}\cdot\cdot\gamma_{a_{[n/2]}}=0$

$\Leftrightarrow-(-i)^{[\frac{n}{2}]\%2}\varepsilon^{a_0\cdot\cdot a_{n-1}}\frac{1}{([\frac{n}{2}]!)^2}\gamma_{[a_{[n/2]+1}}\cdot\cdot\gamma_{a_{n-1}]}\partial_{a_0}F_{a_1\cdot\cdot a_{[n/2]}}$

$+\frac{[\frac{n}{2}]}{([\frac{n}{2}]-1)!}\delta_{a_0a_1}\gamma_{[a_2}\cdot\cdot\gamma_{a_{[n/2]}]}\partial^{a_0}F^{a_1\cdot\cdot a_{[n/2]}}+\frac{1}{[\frac{n}{2}]!}mF^{a_{[n/2]+1}\cdot\cdot a_{n-1}}\gamma_{[a_{[n/2]+1}}\cdot\cdot\gamma_{a_{n-1}]}=0$

$\Leftrightarrow\frac{1}{[\frac{n}{2}]!}\varepsilon^{a_0\cdot\cdot a_{n-1}}\partial_{a_0}F_{a_1\cdot\cdot a_{[n/2]}}-i^{[\frac{n}{2}]\%2}mF^{a_{[n/2]+1}\cdot\cdot a_{n-1}}=0,\partial_{a_1}F^{a_1\cdot\cdot a_{[n/2]}}=0$

$\Leftrightarrow\frac{1}{[\frac{n}{2}]!}\varepsilon^{a_1\cdot\cdot a_n}\partial_{a_{[n/2]+1}}F_{a_{[n/2]+2}\cdot\cdot a_n}-i^{[\frac{n}{2}]\%2}mF^{a_1\cdot\cdot a_{[n/2]}}=0,\partial_{a_1}F^{a_1\cdot\cdot a_{[n/2]}}=0$ □

Cor. 3.7.3.

$\gamma_{a_0}\partial^{a_0}\{\frac{1}{([\frac{n}{2}]!)^2}F^{a_1\cdot\cdot a_{[n/2]}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{[n/2]}]}\}C=0\Leftrightarrow\varepsilon^{a_1\cdot\cdot a_n}\partial_{a_{[n/2]+1}}F_{a_{[n/2]+2}\cdot\cdot a_n}=0,\partial_{a_1}F^{a_1\cdot\cdot a_{[n/2]}}=0$

Cor. 3.7.4. $(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{([\frac{n}{2}]!)^2}F^{a_1\cdot\cdot a_{[n/2]}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{[n/2]}]}\}C=0,m\neq 0$

$\Leftrightarrow\frac{1}{[\frac{n}{2}]!}\varepsilon^{a_1\cdot\cdot a_n}\partial_{a_{[n/2]+1}}F_{a_{[n/2]+2}\cdot\cdot a_n}-i^{[\frac{n}{2}]\%2}mF^{a_1\cdot\cdot a_{[n/2]}}=0$

Cor. 3.7.5. $\frac{1}{[\frac{n}{2}]!}\varepsilon^{a_1\cdot\cdot a_n}\partial_{a_{[n/2]+1}}F_{a_{[n/2]+2}\cdot\cdot a_n}-i^{[\frac{n}{2}]\%2}mF^{a_1\cdot\cdot a_{[n/2]}}=0$

$\Leftrightarrow\partial_{a_0}*F^{a_0\cdot\cdot a_{[n/2]}}-(-i)^{[\frac{n}{2}]\%2}mF^{a_1\cdot\cdot a_{[n/2]}}=0\Leftrightarrow\frac{1}{[\frac{n}{2}]!}\partial_{[a_0}F_{a_1\cdot\cdot a_{[n/2]}]}-i^{[\frac{n}{2}]\%2}m*F_{a_0\cdot\cdot a_{[n/2]}}=0$

4 Covariant commutation rules for basic antisymmetric tensor field in N+1-D

4.1 Derive commutation rules for basic antisymmetric tensor field from B-W equation

Ass. 4.1.1.

$$\begin{cases}\bar{C}(n)=C^{+}(n), C^{+}(n)=(-1)^{[\frac{n}{4}]}C(n), C^{T}(n)=(-1)^{[\frac{n+2}{4}]}C(n)\\ [\gamma_a(n)C(n)]^T=(-1)^{[\frac{n-1}{4}]}[\gamma_a(n)C(n)], [C^{+}(n)\gamma_a(n)]^T=(-1)^{[\frac{n-1}{4}]}[C^{+}(n)\gamma_a(n)]\\ \gamma_0^T(n)=\gamma_0(n), n\geq 3; \gamma_0^T(2)=-\gamma_0(2)\\ C(n)\gamma_0^T(n)C^{+}(n)=(-1)^{\xi(n)}\gamma_0^T(n), C(n)\gamma_0^T(n)\gamma_a^T(n)C^{+}(n)=(-1)^{\xi(n)+\eta(n)}\gamma_0^T(n)\gamma_a(n)\\ (-1)^{\xi(n)}=(-1)^{\eta(n)}:=(-1)^{[\frac{n-1}{4}]}(-1)^{[\frac{n+2}{4}]}, n\geq 3\\ (-1)^{\xi(n)+1}=(-1)^{\eta(n)}:=(-1)^{[\frac{n-1}{4}]}(-1)^{[\frac{n+2}{4}]}, n\geq 3\end{cases}$$

Thm. 4.1.1. $[F_{a_1a_2\cdot\cdot a_l}(x), F^{+}_{a_1'a_2'\cdot\cdot a_l'}(x')]=-i\frac{(-1)^{\delta_{2,n}}}{2^{[\frac{n}{2}]}}\begin{cases}\frac{1}{(l+1)!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l}_{a_l'}\eta^{a]}_{a']}\partial_a\partial^{+a'}\Delta(x-x'), (-1)^{\eta(n)+l}=1\\ -\frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\partial^{a_l]}\partial_{a_l']}\Delta(x-x'), (-1)^{\eta(n)+l}=-1\end{cases}$

Proof: $[F_{a_1a_2\cdot\cdot a_l}(x), F^{+}_{a_1'a_2'\cdot\cdot a_l'}(x')]$

$$=\frac{4^{-[\frac{n}{2}]}}{(l!)^2}(C^{+}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}(C^{+}\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']})^{*\mu'\lambda'}[\psi_{\lambda\mu}(x),\psi^{+}_{\lambda'\mu'}(x')]$$
$$=\frac{4^{-[\frac{n}{2}]}}{(l!)^2}(C^{+}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}(C^{+}\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']})^{+\lambda'\mu'}[\psi_{\lambda\mu}(x),\psi^{+}_{\lambda'\mu'}(x')]$$
$$=\frac{4^{-[\frac{n}{2}]}}{(l!)^2}(C^{+}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}(\gamma_{[a_l'}\cdot\cdot\gamma_{a_2'}\gamma_{a_1']}C)^{\lambda'\mu'}[\psi_{\lambda\mu}(x),\psi^{+}_{\lambda'\mu'}(x')]$$
$$=\frac{4^{-[\frac{n}{2}]}}{(l!)^2}i^{l(l-1)}(C^{+}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}(\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}C)^{\lambda'\mu'}[\psi_{\lambda\mu}(x),\psi^{+}_{\lambda'\mu'}(x')]$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}(C^{+}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}(\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}C)^{\lambda'\mu'}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}[(m-\gamma^b\partial_b)\gamma^0]_{\mu\mu'}\Delta(x-x')$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}(C^{+}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}(\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}C)^{\lambda'\mu'}[(m-\gamma^b\partial_b)\gamma^0]^T_{\mu'\mu}\Delta(x-x')$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}tr\{C^{+}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}C[(m-\gamma^b\partial_b)\gamma^0]^T\}\Delta(x-x')$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}tr\{\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}C[(m-\gamma^b\partial_b)\gamma^0]^TC^{+}\}\Delta(x-x')$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}(-1)^{\xi(n)}tr\{\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}[\gamma^0(m-(-1)^{\eta(n)}\gamma^b\partial_b)]\}\Delta(x-x')$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}(-1)^{\xi(n)}tr\{\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}[(m+(-1)^{\eta(n)}\gamma^b\partial_b^{+})\gamma^0]\}\Delta(x-x')$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}(-1)^{\xi(n)}$$
$$\{tr(m^2\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]}\gamma^0\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}\gamma^0)-(-1)^{\eta(n)}tr(\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]}\gamma_a\gamma_0\gamma_{[a_1'}\gamma_{a_2'}\cdot\cdot\gamma_{a_l']}\gamma_{a'}\gamma_0)\partial^a\partial^{+a'}\}\Delta(x-x')$$
$$=i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}(-1)^{\xi(n)}\{i^{l(l+1)}2^{[\frac{n}{2}]}(l!)^2m^2\frac{1}{l!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l]}_{a_l']}$$
$$-(-1)^{\eta(n)}i^{(l+1)(l+2)}2^{[\frac{n}{2}]}(l!)^2\{\frac{1}{(l+1)!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l}_{a_l'}\eta^{a]}_{a']}-\frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\delta^{a_l]a}\delta_{a_l']a'}\}\partial_a\partial^{+a'}\}\Delta(x-x')$$
$$=i2^{-[\frac{n}{2}]-1}i^{2l^2}(-1)^{\xi(n)}$$
$$\{\frac{1}{l!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l]}_{a_l']}m^2+(-1)^{\eta(n)+l}\{\frac{1}{(l+1)!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l}_{a_l'}\eta^{a]}_{a']}-\frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\delta^{a_l]a}\delta_{a_l']a'}\}\partial_a\partial^{+a'}\}\Delta(x-x')$$
$$=i2^{-[\frac{n}{2}]}(-1)^{\xi(n)+l}$$
$$\{\frac{1+(-1)^{\eta(n)+l}}{2}\frac{1}{l!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l]}_{a_l']}m^2-(-1)^{\eta(n)+l}\frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\partial^{a_l]}\partial_{a_l']}\}\Delta(x-x')$$
$$=i\frac{(-1)^{\xi(n)+l}}{2^{[\frac{n}{2}]}}\begin{cases}\frac{1}{(l+1)!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l}_{a_l'}\eta^{a]}_{a']}\partial_a\partial^{+a'}\Delta(x-x'), (-1)^{\eta(n)+l}=1\\ \frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\partial^{a_l]}\partial_{a_l']}\Delta(x-x'), (-1)^{\eta(n)+l}=-1\end{cases}$$
$$=-i\frac{(-1)^{\delta_{2,n}}}{2^{[\frac{n}{2}]}}\begin{cases}\frac{1}{(l+1)!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l}_{a_l'}\eta^{a]}_{a']}\partial_a\partial^{+a'}\Delta(x-x'), (-1)^{\eta(n)+l}=1\\ -\frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\partial^{a_l]}\partial_{a_l']}\Delta(x-x'), (-1)^{\eta(n)+l}=-1\end{cases}$$ □

4.2 Conjecture on commutation rules for basic antisymmetric tensor field in n=N+1-D

Def. 4.2.1. $\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]}+mF^{a_0\cdot\cdot a_l}=0, \partial_{a_0}F^{a_0\cdot\cdot a_l}+mA^{a_1\cdot\cdot a_l}=0$
$\Leftrightarrow \partial_{a_0}\partial^{a_0}A^{a_1\cdot\cdot a_l}-m^2A^{a_1\cdot\cdot a_l}=0, \partial_{a_1}A^{a_1\cdot\cdot a_l}=0, F^{a_0\cdot\cdot a_l}=-\frac{1}{(l+1)!m}\partial^{[a_0}A^{a_1\cdot\cdot a_l]}$

Cor. 4.2.1.

$$\begin{cases}\frac{1}{l!}\partial^{[a_0}A^{a_1\cdot\cdot a_l]}+mF^{a_0a_1\cdot\cdot a_l}=0\\ \partial_{a_0}F^{a_0a_1\cdot\cdot a_l}+mA^{a_1\cdot\cdot a_l}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{(n-l-1)!}\partial^{[a_0}*F^{a_1\cdot\cdot a_{n-l-1}]}+(-1)^{n-l-1}m*A^{a_0a_1\cdot\cdot a_{n-l-1}}=0\\ \partial_{a_0}*A^{a_0a_1\cdot\cdot a_{n-l-1}}+(-1)^{n-l-1}m*F^{a_1\cdot\cdot a_{n-l-1}}=0\end{cases}$$

Lem. 4.2.1. $\frac{1}{(l+1)!}\eta_{[a_1\langle a'_1}\eta_{a_2a'_2}\cdot\cdot\eta_{a_{l-1}a'_{l-1}}\eta_{a_la'_l}\eta_{a_{l+1}]a'_{l+1}\rangle}\partial^{a_{l+1}}\partial^{+a'_{l+1}}\Delta(x-x')$
$=\{\frac{1}{l!}\eta_{[a_1\langle a'_1}\eta_{a_2a'_2}\cdot\cdot\eta_{a_{l-1}a'_{l-1}}\eta_{a_l]a'_l\rangle}m^2-\frac{1}{(l-1)!}\eta_{[a_1\langle a'_1}\eta_{a_2a'_2}\cdot\cdot\eta_{a_{l-1}a'_{l-1}}\partial_{a_l]}\partial^+_{a'_l\rangle}\}\Delta(x-x')$

Ass. 4.2.1.

$$\begin{cases}[A_{a_1\cdot\cdot a_l}(x),A^+_{a'_1\cdot\cdot a'_l}(x')]=i\frac{2^{-[\frac{n}{2}]}}{(l+1)!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_l}_{a'_l}\eta^{a]}_{a']}\partial_a\partial^{+a'}\Delta(x-x')\\ [F_{a_0a_1\cdot\cdot a_l}(x),F^+_{a'_0a'_1\cdot\cdot a'_l}(x')]=-i\frac{2^{-[\frac{n}{2}]}}{l!}\eta^{[a_0}_{[a'_0}\cdot\cdot\eta^{a_{l-1}}_{a'_{l-1}}\partial^{a_l]}\partial_{a'_l]}\Delta(x-x')\end{cases}$$

$[\Leftrightarrow]$

$$\begin{cases}[*A_{a_0\cdot\cdot a_{n-l-1}}(x),*A^+_{a'_0\cdot\cdot a'_{n-l-1}}(x')]=-i\frac{2^{-[\frac{n}{2}]}}{(n-l-1)!}\eta^{[a_0}_{[a'_0}\cdot\cdot\eta^{a_{n-l-2}}_{a'_{n-l-2}}\partial^{a_{n-l-1}]}\partial_{a'_{n-l-1}]}\Delta(x-x')\\ [*F_{a_1\cdot\cdot a_{n-l-1}}(x),*F^+_{a'_1\cdot\cdot a'_{n-l-1}}(x')]=i\frac{2^{-[\frac{n}{2}]}}{(n-l)!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_{n-l-1}}_{a'_{n-l-1}}\eta^{a]}_{a']}\partial_a\partial^{+a'}\Delta(x-x')\end{cases}$$

5 Full coupling antisymmetric tensor field set

5.1 B-W general vector field equation in n=N+1 even dimensional space-time

Def. 5.1.1.
$X=\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}+\cdot\cdot+\frac{1}{(n!)^2}F^{a_1a_2a_3\cdot\cdot a_n}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\cdot\cdot\gamma_{a_n]}$

$$\begin{cases}F=2^{-[\frac{n}{2}]}tr(X),F_{a_1}=2^{-[\frac{n}{2}]}tr(\gamma_{a_1}X)\\ F_{a_1a_2}=-2^{-[\frac{n}{2}]}tr(\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]}X),F_{a_1a_2a_3}=-2^{-[\frac{n}{2}]}tr(\frac{1}{3!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}X)\\ F_{a_1a_2a_3a_4}=2^{-[\frac{n}{2}]}tr(\frac{1}{4!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}X),F_{a_1a_2a_3a_4a_5}=2^{-[\frac{n}{2}]}tr(\frac{1}{5!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}X)\\ \cdots F_{a_1a_2\cdot\cdot a_n}=(-1)^{[(n\%4)/2]}2^{-[\frac{n}{2}]}tr(\frac{1}{n!}\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_n]}X)\end{cases}$$

Thm. 5.1.1. $(\gamma_a\partial^a+m)\psi(x)=0\Leftrightarrow$

$$\begin{cases}mF+\partial_{a_0}F^{a_0}=0,\frac{1}{0!}\partial^{a_1}F+mF^{a_1}+\partial_{a_0}F^{a_0a_1}=0,\frac{1}{1!}\partial^{[a_1}F^{a_2]}+mF^{a_1a_2}+\partial_{a_0}F^{a_0a_1a_2}=0\\ \frac{1}{2!}\partial^{[a_1}F^{a_2a_3]}+mF^{a_1a_2a_3}+\partial_{a_0}F^{a_0a_1a_2a_3}=0,\cdots,\frac{1}{(n-2)!}\partial^{[a_1}F^{a_2\cdot\cdot a_{n-1}]}+mF^{a_1\cdot\cdot a_{n-1}}+\partial_{a_0}F^{a_0a_1\cdot\cdot a_{n-1}}=0\\ \frac{1}{(n-1)!}\partial^{[a_1}F^{a_2\cdot\cdot a_n]}+mF^{a_1\cdot\cdot a_n}=0,\frac{1}{n!}\partial^{[a_0}F^{a_1\cdot\cdot a_n]}=0\end{cases}$$

Proof:

$$(\gamma_a\partial^a+m)\psi(x)=0\Leftrightarrow\begin{cases}(\gamma_a\partial^a+m)\psi(x)=0\\ X=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\cdot\cdot+\frac{1}{(n!)^2}F^{a_1\cdot\cdot a_n}\gamma_{[a_1}\cdot\cdot\gamma_{a_n]}\}C\end{cases}$$

$\Leftrightarrow$

$(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\cdot\cdot+\frac{1}{(n!)^2}F^{a_1\cdot\cdot a_n}\gamma_{[a_1}\cdot\cdot\gamma_{a_n]}\}C=0$

$\Leftrightarrow$

$(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{0!}F+\frac{1}{1!}F^{a_1}\gamma_{a_1}+\frac{1}{2!}F^{a_1a_2}\gamma_{a_1}\gamma_{a_2}+\cdot\cdot+\frac{1}{n!}F^{a_1\cdot\cdot a_n}\gamma_{a_1}\cdot\cdot\gamma_{a_n}\}=0$

$\Leftrightarrow$

$\partial^{a_0}\{\frac{1}{0!}\gamma_{a_0}F+\frac{1}{1!}F^{a_1}\gamma_{a_0}\gamma_{a_1}+\frac{1}{2!}F^{a_1a_2}\gamma_{a_0}\gamma_{a_1}\gamma_{a_2}+\cdot\cdot+\frac{1}{n!}F^{a_1\cdot\cdot a_n}\gamma_{a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_n}\}$
$+m\{\frac{1}{0!}F+\frac{1}{1!}F^{a_1}\gamma_{a_1}+\frac{1}{2!}F^{a_1a_2}\gamma_{a_1}\gamma_{a_2}+\cdot\cdot+\frac{1}{n!}F^{a_1\cdot\cdot a_n}\gamma_{a_1}\cdot\cdot\gamma_{a_n}\}=0$

$\Leftrightarrow$

$\{\frac{1}{0!}\gamma_{a_0}\partial^{a_0}F+\frac{1}{1!}\partial^{a_0}F^{a_1}(\frac{1}{2!}\gamma_{[a_0}\gamma_{a_1]}+\frac{1}{0!}\delta_{a_0a_1})+\frac{1}{2!}\partial^{a_0}F^{a_1a_2}(\frac{1}{3!}\gamma_{[a_0}\gamma_{a_1}\gamma_{a_2]}+\frac{1}{1!}\delta_{a_0[a_1}\gamma_{a_2]})$
$+\frac{1}{3!}\partial^{a_0}F^{a_1\cdot\cdot a_3}(\frac{1}{4!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_3]}+\frac{1}{2!}\delta_{a_0[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_3]})$
$+\frac{1}{4!}\partial^{a_0}F^{a_1\cdot\cdot a_4}(\frac{1}{5!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_4]}+\frac{1}{3!}\delta_{a_0[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_4]})$
$+\cdot\cdot+\frac{1}{n!}\partial^{a_0}F^{a_1\cdot\cdot a_n}(\frac{1}{(n+1)!}\gamma_{[a_0}\gamma_{a_1}\cdot\cdot\gamma_{a_n]}+\frac{1}{(n-1)!}\delta_{a_0[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_n]})\}$
$+m\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_0}\gamma_{a_0}+\frac{1}{(2!)^2}F^{a_0a_1}\gamma_{[a_0}\gamma_{a_1]}+\cdot\cdot+\frac{1}{(n!)^2}F^{a_0\cdot\cdot a_{n-1}}\gamma_{[a_0}\cdot\cdot\gamma_{a_{n-1}]}\}=0$

$\Leftrightarrow$

$mF+\partial_{a_0}F^{a_0}=0$
$\frac{1}{0!}\partial^{a_1}F+mF^{a_1}+\partial_{a_0}F^{a_0a_1}=0$
$\frac{1}{1!}\partial^{[a_1}F^{a_2]}+mF^{a_1a_2}+\partial_{a_0}F^{a_0a_1a_2}=0$
$\frac{1}{2!}\partial^{[a_1}F^{a_2a_3]}+mF^{a_1a_2a_3}+\partial_{a_0}F^{a_0a_1a_2a_3}=0$
$\frac{1}{3!}\partial^{[a_1}F^{a_2\cdot\cdot a_4]}+mF^{a_1\cdot\cdot a_4}+\partial_{a_0}F^{a_0a_1\cdot\cdot a_4}=0$
$\cdots$
$\frac{1}{(n-3)!}\partial^{[a_1}F^{a_2\cdot\cdot a_{n-2}]}+mF^{a_1\cdot\cdot a_{n-2}}+\partial_{a_0}F^{a_0a_1\cdot\cdot a_{n-2}}=0$
$\frac{1}{(n-2)!}\partial^{[a_1}F^{a_2\cdot\cdot a_{n-1}]}+mF^{a_1\cdot\cdot a_{n-1}}+\partial_{a_0}F^{a_0a_1\cdot\cdot a_{n-1}}=0$
$\frac{1}{(n-1)!}\partial^{[a_1}F^{a_2\cdot\cdot a_n]}+mF^{a_1\cdot\cdot a_n}=0$
$\frac{1}{n!}\partial^{[a_0}F^{a_1\cdot\cdot a_n]}\equiv 0$

□

Cor. 5.1.1. $\gamma_a\partial^a\psi(x)=0\Leftrightarrow\begin{cases}\partial_{a_0}F^{a_0}=0,\frac{1}{0!}\partial^{a_1}F+\partial_{a_0}F^{a_0a_1}=0,\frac{1}{1!}\partial^{[a_1}F^{a_2]}+\partial_{a_0}F^{a_0a_1a_2}=0\\ \frac{1}{2!}\partial^{[a_1}F^{a_2a_3]}+\partial_{a_0}F^{a_0a_1a_2a_3}=0,\cdots,\frac{1}{(n-2)!}\partial^{[a_1}F^{a_2\cdot\cdot a_{n-1}]}+\partial_{a_0}F^{a_0a_1\cdot\cdot a_{n-1}}=0\\ \frac{1}{(n-1)!}\partial^{[a_1}F^{a_2\cdot\cdot a_n]}=0,\frac{1}{n!}\partial^{[a_0}F^{a_1\cdot\cdot a_n]}=0\end{cases}$

5.2 Commutation rules for full coupling antisymmetric tensor field

Lem. 5.2.1. $\frac{1}{(l+1)!}\eta_{[a_1\langle a_1'}\eta_{a_2a_2'}\cdot\cdot\eta_{a_{l-1}a_{l-1}'}\eta_{a_la_l'}\eta_{a_{l+1}]a_{l+1}'\rangle}\partial^{a_{l+1}}\partial^{+a_{l+1}'}\Delta(x-x')$
$=\{\frac{1}{l!}\eta_{[a_1\langle a_1'}\eta_{a_2a_2'}\cdot\cdot\eta_{a_{l-1}a_{l-1}'}\eta_{a_l]a_l'\rangle}m^2-\frac{1}{(l-1)!}\eta_{[a_1\langle a_1'}\eta_{a_2a_2'}\cdot\cdot\eta_{a_{l-1}a_{l-1}'}\partial_{a_l]}\partial^+_{a_l'\rangle}\}\Delta(x-x')$

Ass. 5.2.1.

$$\begin{cases}[F^{a_1a_2\cdot\cdot a_l}(x),F^+_{a_1'a_2'\cdot\cdot a_l'}(x')]=-i\frac{(-1)^{\delta_{2,n}}}{2^{[\frac{n}{2}]+1}}\{\frac{1}{(l+1)!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l}_{a_l'}\eta^{a]}_{a']}\partial_a\partial^{+a'}-\frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\partial^{a_l]}\partial_{a_l']}\}\Delta(x-x')\\ [F^{a_1a_2\cdot\cdot a_l}(x),F^+_{a_1'a_2'\cdot\cdot a_l'}(x')]=-i\frac{(-1)^{\delta_{2,n}}}{2^{[\frac{n}{2}]+1}}\{\frac{1}{l!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\cdot\cdot\eta^{a_l]}_{a_l']}m^2-\frac{2}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\partial^{a_l]}\partial_{a_l']}\}\Delta(x-x')\end{cases}$$

5.3 Relations between antisymmetric tensor field basis and B-W basis in even n=N+1-D

Def. 5.3.1.

$$\begin{cases}u(\vec p,h)u^T(\vec p,h')=\sum\limits_{l=0}^{n}\frac{1}{(l!)^2}U_{a_1\cdot\cdot a_l}(\vec p;h,h')\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C,v(\vec p,h)v^T(\vec p,h')=\sum\limits_{l=0}^{n}\frac{1}{(l!)^2}V_{a_1\cdot\cdot a_l}(\vec p;h,h')\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C\\ u(\vec p,h)v^T(\vec p,h')=\sum\limits_{l=0}^{n}\frac{1}{(l!)^2}X_{a_1\cdot\cdot a_l}(\vec p;h,h')\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C,v(\vec p,h)u^T(\vec p,h')=\sum\limits_{l=0}^{n}\frac{1}{(l!)^2}Y_{a_1\cdot\cdot a_l}(\vec p;h,h')\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C\\ h,h'=1,\cdot\cdot,2^{[\frac{N-1}{2}]}\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}u(\vec p,h)u^T(\vec p,h')=\sum\limits_{l=0}^{n}\frac{1}{l!}U_{a_1\cdot\cdot a_l}(\vec p;h,h')\frac{1}{l!}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]};h,h'=-2^{[\frac{N-1}{2}]},\cdot\cdot,-1,1,\cdot\cdot,2^{[\frac{N-1}{2}]}\\ U_{a_1a_2\cdot\cdot a_l}(\vec p;h,h')=(-1)^{[(l\%4)/2]}2^{-[\frac{n}{2}]}u^T(\vec p,h')\frac{1}{l!}C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}u(\vec p,h)\end{cases}$$

Def. 5.3.2.

$$\begin{cases}\frac{1}{l!}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}=\frac{m^2}{E^2}\sum\limits_{h,h'}W^{a_1\cdot\cdot a_l}(\vec p;h,h')u(\vec p,h)u^T(\vec p,h');h,h'=-2^{[\frac{N-1}{2}]},\cdot\cdot,-1,1,\cdot\cdot,2^{[\frac{N-1}{2}]}\\ W^{a_1\cdot\cdot a_l}(\vec p;h,h')=u^+(\vec p,h)\frac{1}{l!}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}u^*(\vec p,h')\end{cases}$$

Cor. 5.3.1.

$$\begin{cases}u(\vec p,h)u^T(\vec p,h')=\frac{m^2}{E^2}\sum\limits_{l=0}^{n}\sum\limits_{h'',h'''}\frac{1}{l!}U_{a_1\cdot\cdot a_l}(\vec p;h,h')W^{a_1\cdot\cdot a_l}(\vec p;h'',h''')u(\vec p,h'')u^T(\vec p,h''')\\ \sum\limits_{l=0}^{n}\frac{1}{l!}U_{a_1\cdot\cdot a_l}(\vec p;h,h')W^{a_1\cdot\cdot a_l}(\vec p;h,h')=1\end{cases}$$

Cor. 5.3.2.

$$\begin{cases}\frac{1}{l!}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}=\frac{m^2}{E^2}\sum\limits_{l=0}^{n}\sum\limits_{h,h'}\frac{1}{l!}W^{a_1\cdot\cdot a_l}(\vec p;h,h')U_{a_1'\cdot\cdot a_l'}(\vec p;h,h')\frac{1}{l!}\gamma^{[a_1'}\cdot\cdot\gamma^{a_l']}\\ \frac{m^2}{E^2}\sum\limits_{h,h'}W^{a_1\cdot\cdot a_l}(\vec p;h,h')U_{a_1\cdot\cdot a_l}(\vec p;h,h')=1\end{cases}$$

5.4 Conjecture of commutation rules for antisymmetric tensor field without mass in N+1-D

Ass. 5.4.1. $\partial_{[a_0}F_{a_1\cdot\cdot a_l]}=0,\partial^{a_1}F_{a_1\cdot\cdot a_l}=0,F_{a_1\cdot\cdot a_l}=\frac{1}{l!}F_{[a_1\cdot\cdot a_l]}$
$\Rightarrow[F^{a_1a_2\cdot\cdot a_l}(x),F^+_{a_1'a_2'\cdot\cdot a_l'}(x')]=?-i\frac{1}{2^{[\frac{n}{2}]}}\frac{1}{(l-1)!}\eta^{[a_1}_{[a_1'}\cdot\cdot\eta^{a_{l-1}}_{a_{l-1}'}\partial^{a_l]}\partial_{a_l']}\Delta(x-x')$

6 B-W vector field equation in even dimensional space-time

6.1 Symmetric B-W vector field equation in two dimensional space-time

Lem. 6.1.1. $\begin{cases}(\gamma^a\partial_a+m)X(2)=0\\ X(2)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases}\Leftrightarrow\begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0,\partial_aF^a=0\\ \partial^{[a}F^{bc]}=0,\partial_aF^{ab}+mF^b=0\end{cases}$

Proof: $(\gamma^a\partial_a+m)X(2)=0,X(2)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C$
$\Leftrightarrow(\gamma_a\partial^a+m)\{\frac{1}{(1!)^2}F^b\gamma_b+\frac{1}{(2!)^2}F^{bc}\gamma_{[b}\gamma_{c]}\}C=0$
$\Leftrightarrow(\gamma_a\partial^a+m)\{\frac{1}{1!}F^b\gamma_b+\frac{1}{2!}F^{bc}\gamma_b\gamma_c\}=0$
$\Leftrightarrow\gamma_a\gamma_b\partial^a\frac{1}{1!}F^b+\gamma_a\gamma_b\gamma_c\partial^a\frac{1}{2!}F^{bc}+m\frac{1}{1!}F^b\gamma_b+m\frac{1}{2!}F^{bc}\gamma_b\gamma_c=0$
$\Leftrightarrow\{\frac{1}{2!}\gamma_{[a}\gamma_{b]}+\delta_{ab}\}\partial^a\frac{1}{1!}F^b+m\frac{1}{2!}F^{bc}\gamma_b\gamma_c+\{\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+(\delta_{a[b}\gamma_{c]}+\gamma_a\delta_{bc})\}\partial^a\frac{1}{2!}F^{bc}+m\frac{1}{1!}F^b\gamma_b=0$
$\Leftrightarrow\frac{1}{2!}\gamma_{[a}\gamma_{b]}\partial^a\frac{1}{1!}F^b+\delta_{ab}\partial^a\frac{1}{1!}F^b+m\frac{1}{2!}F^{bc}\gamma_b\gamma_c+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}\partial^a\frac{1}{2!}F^{bc}+\frac{2}{1!}\delta_{ab}\gamma_c\partial^a\frac{1}{2!}F^{bc}+m\frac{1}{1!}F^b\gamma_b=0$
$\Leftrightarrow\frac{1}{2!}\gamma_{[a}\gamma_{b]}\partial^a\frac{1}{1!}F^b+\partial_a\frac{1}{1!}F^a+\frac{1}{2!}m\frac{1}{2!}F^{ab}\gamma_{[a}\gamma_{b]}+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}\partial^a\frac{1}{2!}F^{bc}+\frac{2}{1!}\gamma_b\partial_a\frac{1}{2!}F^{ab}+m\frac{1}{1!}F^b\gamma_b=0$
$\Leftrightarrow\begin{cases}\frac{1}{2!}\partial^{[a}\frac{1}{1!}F^{b]}+m\frac{1}{2!}F^{ab}=0,\partial_a\frac{1}{1!}F^a=0\\ \partial^{[a}\frac{1}{2!}F^{bc]}=0,\partial_a\frac{1}{2!}F^{ab}+\frac{1}{2}m\frac{1}{1!}F^b=0\end{cases}$
$\Leftrightarrow\begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0,\partial_aF^a=0\\ \partial^{[a}F^{bc]}=0,\partial_aF^{ab}+mF^b=0\end{cases}$ □

Cor. 6.1.1. $\begin{cases}\gamma^a\partial_a X(2)=0\\ X(2)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases} \Leftrightarrow \begin{cases}\partial^{[a}F^{b]}=0,\partial_a F^a=0\\ \partial^{[a}F^{bc]}=0,\partial_a F^{ab}=0\end{cases}$

Cor. 6.1.2. $(\gamma^a\partial_a+m)X(2)=0, X(2)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C, m\neq 0$
$\Leftrightarrow \partial_a F^{ab}+mF^b=0, \partial^{[a}F^{b]}+mF^{ab}=0 \Leftrightarrow \partial_b\partial^b F^a-m^2F^a=0, \partial_a F^a=0, F^{ab}=-\frac{1}{m}\partial^{[a}F^{b]}$

Thm. 6.1.1. $\begin{cases}(\gamma^a\partial_a+m)X(2)=0\\ X(2)=X^T(2)\end{cases} \Leftrightarrow \begin{cases}\partial_b\partial^b F^a-m^2F^a=0,\partial_a F^a=0\\ X(2)=\{\frac{1}{1!}\gamma_a+\frac{1}{2!m}\gamma_{[a}\gamma_{b]}\partial^b\}CF^a\end{cases}$

Cor. 6.1.3. $\begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0\\ \partial_a F^{ab}+mF^b=0\end{cases} \Leftrightarrow \begin{cases}\partial^a{*F}+m{*F^a}=0\\ \partial_a{*F^a}+m{*F}=0\end{cases}$

6.2 Antisymmetric B-W vector field equation in two dimensional space-time

Lem. 6.2.1. $(\gamma^a\partial_a+m)X(2)=0, X(2)=\frac{1}{(0!)^2}FC \Leftrightarrow \partial^a F=0, mF=0$

Proof: $(\gamma^a\partial_a+m)X(2)=0, X(2)=\frac{1}{(0!)^2}FC$
$\Leftrightarrow (\gamma_a\partial^a+m)\frac{1}{(0!)^2}FC=0$
$\Leftrightarrow (\gamma_a\partial^a+m)\frac{1}{0!}F=0$
$\Leftrightarrow \partial^a F=0, mF=0$ □

Cor. 6.2.1. $\gamma^a\partial_a X(2)=0, X(2)=\frac{1}{(0!)^2}FC \Leftrightarrow \partial^a F=0$

Cor. 6.2.2. $(\gamma^a\partial_a+m)X(2)=0, X(2)=\frac{1}{(0!)^2}FC, m\neq 0 \Leftrightarrow F=0$

Thm. 6.2.1. $(\gamma^a\partial_a+m)X(2)=0, X(2)=-X^T(2) \Leftrightarrow F=0$

Cor. 6.2.3. $F=0 \Leftrightarrow {*F^{ab}}=0$

6.3 Symmetric B-W vector field equation in four dimensional space-time

Lem. 6.3.1. $\begin{cases}(\gamma^a\partial_a+m)X(4)=0\\ X(4)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases} \Leftrightarrow \begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0,\partial_a F^a=0\\ \partial^{[a}F^{bc]}=0,\partial_a F^{ab}+mF^b=0\end{cases}$

Proof: $(\gamma^a\partial_a+m)X(4)=0, X(4)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C$
$\Leftrightarrow (\gamma_a\partial^a+m)\{\frac{1}{(1!)^2}F^b\gamma_b+\frac{1}{(2!)^2}F^{bc}\gamma_{[b}\gamma_{c]}\}C=0$
$\Leftrightarrow (\gamma_a\partial^a+m)\{\frac{1}{1!}F^b\gamma_b+\frac{1}{2!}F^{bc}\gamma_b\gamma_c\}=0$
$\Leftrightarrow \gamma_a\gamma_b\partial^a\frac{1}{1!}F^b+\gamma_a\gamma_b\gamma_c\partial^a\frac{1}{2!}F^{bc}+m\frac{1}{1!}F^b\gamma_b+m\frac{1}{2!}F^{bc}\gamma_b\gamma_c=0$
$\Leftrightarrow \{\frac{1}{2!}\gamma_{[a}\gamma_{b]}+\delta_{ab}\}\partial^a\frac{1}{1!}F^b+m\frac{1}{2!}F^{bc}\gamma_b\gamma_c+\{\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+(\delta_{a[b}\gamma_{c]}+\gamma_a\delta_{bc})\}\partial^a\frac{1}{2!}F^{bc}+m\frac{1}{1!}F^b\gamma_b=0$
$\Leftrightarrow \frac{1}{2!}\gamma_{[a}\gamma_{b]}\partial^a\frac{1}{1!}F^b+\delta_{ab}\partial^a\frac{1}{1!}F^b+m\frac{1}{2!}F^{bc}\gamma_b\gamma_c+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}\partial^a\frac{1}{2!}F^{bc}+\frac{2}{1!}\delta_{ab}\gamma_c\partial^a\frac{1}{2!}F^{bc}+m\frac{1}{1!}F^b\gamma_b=0$
$\Leftrightarrow \frac{1}{2!}\gamma_{[a}\gamma_{b]}\partial^a\frac{1}{1!}F^b+\partial_a\frac{1}{1!}F^a+\frac{1}{2!}m\frac{1}{2!}F^{ab}\gamma_{[a}\gamma_{b]}+\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}\partial^a\frac{1}{2!}F^{bc}+\frac{2}{1!}\gamma_b\partial_a\frac{1}{2!}F^{ab}+m\frac{1}{1!}F^b\gamma_b=0$
$\Leftrightarrow \begin{cases}\frac{1}{2!}\partial^{[a}\frac{1}{1!}F^{b]}+m\frac{1}{2!}F^{ab}=0,\partial_a\frac{1}{1!}F^a=0\\ \partial^{[a}\frac{1}{2!}F^{bc]}=0,\partial_a\frac{1}{2!}F^{ab}+\frac{1}{2}m\frac{1}{1!}F^b=0\end{cases}$
$\Leftrightarrow \begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0,\partial_a F^a=0\\ \partial^{[a}F^{bc]}=0,\partial_a F^{ab}+mF^b=0\end{cases}$ □

Cor. 6.3.1. $\begin{cases}\gamma^a\partial_a X(4)=0\\ X(4)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases} \Leftrightarrow \begin{cases}\partial^{[a}F^{b]}=0,\partial_a F^a=0\\ \partial^{[a}F^{bc]}=0,\partial_a F^{ab}=0\end{cases}$

Cor. 6.3.2. $(\gamma^a\partial_a+m)X(4)=0, X(4)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C, m\neq 0$
$\Leftrightarrow \partial_a F^{ab}+mF^b=0, \partial^{[a}F^{b]}+mF^{ab}=0 \Leftrightarrow \partial_b\partial^b F^a-m^2F^a=0, \partial_a F^a=0, F^{ab}=-\frac{1}{m}\partial^{[a}F^{b]}$

Thm. 6.3.1. $\begin{cases}(\gamma^a\partial_a+m)X(4)=0\\ X(4)=X^T(4)\end{cases} \Leftrightarrow \begin{cases}\partial_b\partial^b F^a-m^2F^a=0,\partial_a F^a=0\\ X(4)=\{\frac{1}{1!}\gamma_a+\frac{1}{2!m}\gamma_{[a}\gamma_{b]}\partial^b\}CF^a\\ =\{-\frac{1}{2!}\gamma^{[a_1}\gamma^{a_2]}+\frac{1}{3!m}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3]}\partial_{a_3}\}\Gamma_0 C\frac{1}{2!}{*F_{a_1a_2}}\end{cases}$

Cor. 6.3.3. $\begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0\\ \partial_a F^{ab}+mF^b=0\end{cases} \Leftrightarrow \begin{cases}\frac{1}{2!}\partial^{[a}{*F^{bc]}}+m{*F^{abc}}=0\\ \partial_a{*F^{abc}}+m{*F^{bc}}=0\end{cases}$

Lem. 6.3.2.

$\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]} = -\varepsilon_{a_1a_2a_3a_4}\frac{1}{(2!)^2}\Gamma_0\gamma^{[a_3}\gamma^{a_4]} = -\varepsilon_{a_1a_2a_3a_4}\frac{1}{(2!)^2}\gamma^{[a_3}\gamma^{a_4]}\Gamma_0$

$\frac{1}{1!}\gamma_{a_1} = -\varepsilon_{a_1a_2a_3a_4}\frac{1}{(3!)^2}\Gamma_0\gamma^{[a_2}\gamma^{a_3}\gamma^{a_4]} = \varepsilon_{a_1a_2a_3a_4}\frac{1}{(3!)^2}\gamma^{[a_2}\gamma^{a_3}\gamma^{a_4]}\Gamma_0$

Cor. 6.3.4. $X(4) = \frac{1}{2!}\gamma^{[a_1}\gamma^{a_2]}C\frac{1}{2!}F_{a_1a_2} + \frac{1}{3!m}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3]}\partial_{a_3}\Gamma_0 C\frac{1}{2!}{*F}_{a_1a_2}$

Proof:

$X(4) = \{\frac{1}{(1!)^2}F^a\gamma_a + \frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C$

$= \{\frac{1}{1!}F^{a_1}\varepsilon_{a_1a_2a_3a_4}\frac{1}{(3!)^2}\gamma^{[a_2}\gamma^{a_3}\gamma^{a_4]} - \frac{1}{2!}F^{a_1a_2}\varepsilon_{a_1a_2a_3a_4}\frac{1}{(2!)^2}\gamma^{[a_3}\gamma^{a_4]}\}\Gamma_0 C$

$= \{-\frac{1}{(3!)^2}{*F}_{a_2a_3a_4}\gamma^{[a_2}\gamma^{a_3}\gamma^{a_4]} - \frac{1}{(2!)^2}{*F}_{a_3a_4}\gamma^{[a_3}\gamma^{a_4]}\}\Gamma_0 C$

$= \{-\frac{1}{(2!)^2}{*F}_{a_1a_2}\gamma^{[a_1}\gamma^{a_2]} - \frac{1}{(3!)^2}{*F}_{a_1a_2a_3}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3]}\}\Gamma_0 C$

$= \{-\frac{1}{2!}\gamma^{[a_1}\gamma^{a_2]} + \frac{1}{3!m}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3]}\partial_{a_3}\Gamma_0 C\frac{1}{2!}{*F}_{a_1a_2}$

$= \frac{1}{2!}\gamma^{[a_1}\gamma^{a_2]}C\frac{1}{2!}F_{a_1a_2} + \frac{1}{3!m}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3]}\partial_{a_3}\Gamma_0 C\frac{1}{2!}{*F}_{a_1a_2}$

$= \{\frac{1}{(2!)^2}\gamma^{[a_3}\gamma^{a_4]} + \frac{1}{3!m}\gamma^{[a_1}\gamma^{a_2}\gamma^{a_3']}\partial_{a_3'}\Gamma_0 C\frac{1}{(2!)^2}\varepsilon_{a_1a_2a_3a_4}\}F^{a_3a_4}$ □

Def. 6.3.1.

$$\begin{cases}\gamma_a(4) = [\gamma_a(3)\otimes\sigma_y, I\otimes\sigma_x] = (\sigma\otimes\sigma_y, I\otimes\sigma_x), \Gamma^a(4) = [\gamma_a(3), i\varsigma]\\ C(4) := \gamma_2(4)\gamma_4(4) = -i\sigma_y\otimes\sigma_z, \gamma_1(4)\cdots\gamma_4(4) = I\otimes\sigma_z = \gamma_0(4)\\ [\gamma_a(4)C(4)]^T = \gamma_a(4)C(4), \{\gamma_{[a}(4)\gamma_{b]}(4)C(4)\}^T = \gamma_{[a}(4)\gamma_{b]}(4)C(4)\\ C^T(4) = -C(4), \{\gamma_{[a}(4)\gamma_b(4)\gamma_{c]}(4)C(4)\}^T = -\gamma_{[a}(4)\gamma_b(4)\gamma_{c]}(4)C(4)\\ \{\gamma_{[a}(4)\gamma_b(4)\gamma_c(4)\gamma_{d]}(4)C(4)\}^T = -\gamma_{[a}(4)\gamma_b(4)\gamma_c(4)\gamma_{d]}(4)C(4)\end{cases}$$

Proof: $[F_{a_1}(x), F^+_{a_1'}(x')]$

$= \frac{2^{-4}}{(1!)^2}\bar{C}^{\lambda\eta}(\gamma_{a_1})_\eta{}^\mu(\gamma_{a_1'})^{\mu'}{}_{\eta'}\bar{C}^{+\eta'\lambda'}[\psi_{\lambda\mu}(x), \psi^+_{\lambda'\mu'}(x')]$

$= \frac{2^{-4}}{(1!)^2}(\bar{C}\gamma_{a_1})^{\lambda\mu}(\gamma_{a_1'}C)^{\lambda'\mu'}\frac{i}{2^3}[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda(\lambda'}[(m-\gamma^b\partial_b)\gamma^0]_{\mu\}\mu')}\Delta(x-x')$

$= i\frac{2^{-5}}{(1!)^2}(\bar{C}\gamma_{a_1})^{\lambda\mu}(\gamma_{a_1'}C)^{\lambda'\mu'}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}[(m-\gamma^b\partial_b)\gamma^0]_{\mu\mu'}\Delta(x-x')$

$= i\frac{2^{-5}}{(1!)^2}(\bar{C}\gamma_{a_1})^{\mu\lambda}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}(\gamma_{a_1'}C)^{\lambda'\mu'}[(m-\gamma^b\partial_b)\gamma^0]^T_{\mu'\mu}\Delta(x-x')$

$= i\frac{2^{-5}}{(1!)^2}tr\{(\bar{C}\gamma_{a_1})[(m-\gamma^a\partial_a)\gamma^0](\gamma_{a_1'}C)[(m-\gamma^b\partial_b)\gamma^0]^T\}\Delta(x-x')$

$= i\frac{2^{-5}}{(1!)^2}tr\{\gamma_{a_1}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{a_1'}C[(m-\gamma^b\partial_b)\gamma^0]^T\bar{C}\}\Delta(x-x')$

$= -i\frac{2^{-5}}{(1!)^2}tr\{\gamma_{a_1}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{a_1'}[\gamma^0(m+\gamma^b\partial_b)]\}\Delta(x-x')$

$= -i\frac{2^{-5}}{(1!)^2}tr\{\gamma_{a_1}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{a_1'}[(m-\gamma^b\partial_b^+)\gamma^0]\}\Delta(x-x')$

$= -i\frac{2^{-5}}{(1!)^2}\{m^2 tr(\gamma_{a_1}\gamma^0\gamma_{a_1'}\gamma^0) + tr(\gamma_{a_1}\gamma^a\partial_a\gamma^0\gamma_{a_1'}\gamma^b\partial_b^+\gamma^0)\}\Delta(x-x')$

$= \frac{i}{4}(m^2\eta_{a_1a_1'} - \partial_{a_1}\partial^+_{a_1'})\Delta(x-x')$ □

6.4 Antisymmetric B-W vector field equation in four dimensional space-time

Lem. 6.4.1.

$$\begin{cases}(\gamma^a\partial_a + m)X(4) = 0\\ X(4) = \{\frac{1}{(0!)^2}F + \frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]} + \frac{1}{(4!)^2}F^{abcd}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\}C\end{cases} \Leftrightarrow \begin{cases}F = 0\\ \frac{1}{3!}\partial^{[a}F^{bcd]} + mF^{abcd} = 0, \partial_a F^{abc} = 0\\ \partial^{[a}F^{bcde]} = 0, \partial_a F^{abcd} + mF^{bcd} = 0\end{cases}$$

Cor. 6.4.1.

$$\begin{cases}\gamma^a\partial_a X(4) = 0\\ X(4) = \{\frac{1}{(0!)^2}F + \frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]} + \frac{1}{(4!)^2}F^{abcd}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\}C\end{cases} \Leftrightarrow \begin{cases}F = 0\\ \partial^{[a}F^{bcd]} = 0, \partial_a F^{abc} = 0\\ \partial^{[a}F^{bcde]} = 0, \partial_a F^{abcd} = 0\end{cases}$$

Cor. 6.4.2. $(\gamma^a\partial_a + m)X(4) = 0, X(4) = \{\frac{1}{(0!)^2}F + \frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]} + \frac{1}{(4!)^2}F^{abcd}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\}C, m \neq 0$

$$\Leftrightarrow \begin{cases}F = 0\\ \frac{1}{3!}\partial^{[a}F^{bcd]} + mF^{abcd} = 0\\ \partial_a F^{abcd} + mF^{bcd} = 0\end{cases} \Leftrightarrow \begin{cases}F = 0, \partial_a F^{abc} = 0\\ \partial_a\partial^a F^{bcd} - m^2F^{bcd} = 0\\ F^{abcd} = -\frac{1}{3!m}\partial^{[a}F^{bcd]}\end{cases}$$

Thm. 6.4.1. $$\begin{cases}(\gamma^a\partial_a + m)X(4) = 0\\ X(4) = -X^T(4)\end{cases} \Leftrightarrow \begin{cases}\partial_d\partial^d F^{abc} - m^2F^{abc} = 0, \partial_a F^{abc} = 0\\ X(4) = \{\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]} + \frac{1}{4!m}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\partial^d\}C\frac{1}{3!}F^{abc}\end{cases}$$

Cor. 6.4.3. $$\begin{cases}\frac{1}{3!}\partial^{[a}F^{bcd]} + mF^{abcd} = 0\\ \partial_a F^{abcd} + mF^{bcd} = 0\end{cases} \Leftrightarrow \begin{cases}\frac{1}{0!}\partial^a{*F} + m{*F}^a = 0\\ \partial_a{*F}^a + m{*F} = 0\end{cases}$$

6.5 Symmetric B-W vector field equation in six dimensional space-time

Lem. 6.5.1.

$$\begin{cases}(\gamma^a\partial_a+m)X(6)=0\\X(6)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{(4!)^2}F^{abcd}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\}C\end{cases}\Leftrightarrow\begin{cases}F=0\\\frac{1}{3!}\partial^{[a}F^{bcd]}+mF^{abcd}=0,\partial_aF^{abc}=0\\\partial^{[a}F^{bcde]}=0,\partial_aF^{abcd}+mF^{bcd}=0\end{cases}$$

Proof: $(\gamma^a\partial_a+m)X(6)=0,X(6)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{(4!)^2}F^{abcd}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\}C$

$\Leftrightarrow(\gamma_a\partial^a+m)\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{bcd}\gamma_{[b}\gamma_c\gamma_{d]}+\frac{1}{(4!)^2}F^{bcde}\gamma_{[b}\gamma_c\gamma_d\gamma_{e]}\}C=0$

$\Leftrightarrow(\gamma_a\partial^a+m)\{\frac{1}{3!}F^{bcd}\gamma_b\gamma_c\gamma_d+\frac{1}{4!}F^{bcde}\gamma_b\gamma_c\gamma_d\gamma_e\}=0,F=0$

$\Leftrightarrow\gamma_a\gamma_b\gamma_c\gamma_d\partial^a\frac{1}{3!}F^{bcd}+\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\partial^a\frac{1}{4!}F^{bcde}+m\frac{1}{3!}F^{bcd}\gamma_b\gamma_c\gamma_d+m\frac{1}{4!}F^{bcde}\gamma_b\gamma_c\gamma_d\gamma_e=0$

$\Leftrightarrow\{\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}+\frac{1}{2!}(\delta_{ab}\gamma_{[c}\gamma_{d]}+\delta_{ac}\gamma_{[d}\gamma_{b]}+\delta_{ad}\gamma_{[b}\gamma_{c]}+\gamma_{[a}\gamma_{b]}\delta_{cd}+\gamma_{[c}\gamma_{a]}\delta_{bd}+\gamma_{[a}\gamma_{d]}\delta_{bc})$
$+(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\}\partial^a\frac{1}{3!}F^{bcd}+\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\partial^a\frac{1}{4!}F^{bcde}+m\frac{1}{3!}F^{bcd}\gamma_b\gamma_c\gamma_d+m\frac{1}{4!}F^{bcde}\gamma_b\gamma_c\gamma_d\gamma_e=0$

$\Leftrightarrow\{\frac{1}{4!}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}+3\delta_{ab}\gamma_{[c}\gamma_{d]}\}\partial^a\frac{1}{3!}F^{bcd}+\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\partial^a\frac{1}{4!}F^{bcde}+m\frac{1}{3!}F^{bcd}\gamma_b\gamma_c\gamma_d+m\frac{1}{4!}F^{bcde}\gamma_b\gamma_c\gamma_d\gamma_e=0$

$\Leftrightarrow\gamma_a\gamma_b\gamma_c\gamma_d\gamma_e\partial^a\frac{1}{4!}F^{bcde}+m\frac{1}{3!}F^{bcd}\gamma_b\gamma_c\gamma_d=0,\partial_a\frac{1}{3!}F^{abc}=0,\frac{1}{4!}\partial^{[a}\frac{1}{3!}F^{bcd]}+m\frac{1}{4!}F^{abcd}=0$

$\Leftrightarrow\{\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{1}{3!}(\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]}+\cdot\cdot)+\frac{1}{1!}(\delta_{ab}\delta_{cd}\gamma_e+\cdot\cdot)\}\partial^a\frac{1}{4!}F^{bcde}+m\frac{1}{3!}F^{bcd}\gamma_b\gamma_c\gamma_d=0$

$\Leftrightarrow\{\frac{1}{5!}\gamma_{[a}\gamma_b\gamma_c\gamma_d\gamma_{e]}+\frac{4}{3!}\delta_{ab}\gamma_{[c}\gamma_d\gamma_{e]}\}\partial^a\frac{1}{4!}F^{bcde}+m\frac{1}{3!}F^{bcd}\gamma_b\gamma_c\gamma_d=0$

$$\Leftrightarrow\begin{cases}\frac{1}{4!}\partial^{[a}\frac{1}{3!}F^{bcd]}+m\frac{1}{4!}F^{abcd}=0,\partial_a\frac{1}{3!}F^{abc}=0\\\partial^{[a}\frac{1}{4!}F^{bcde]}=0,\partial_a\frac{1}{4!}F^{abcd}+\frac{1}{4}m\frac{1}{3!}F^{bcd}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{1}{3!}\partial^{[a}F^{bcd]}+mF^{abcd}=0,\partial_aF^{abc}=0\\\partial^{[a}F^{bcde]}=0,\partial_aF^{abcd}+mF^{bcd}=0\end{cases}$$ □

Cor. 6.5.1. $$\begin{cases}\gamma^a\partial_aX(6)=0\\X(6)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{(4!)^2}F^{abcd}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\}C\end{cases}\Leftrightarrow\begin{cases}F=0\\\partial^{[a}F^{bcd]}=0,\partial_aF^{abc}=0\\\partial^{[a}F^{bcde]}=0,\partial_aF^{abcd}=0\end{cases}$$

Cor. 6.5.2. $(\gamma^a\partial_a+m)X(6)=0,X(6)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{(4!)^2}F^{abcd}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\}C,m\neq0$

$$\Leftrightarrow\begin{cases}F=0\\\frac{1}{3!}\partial^{[a}F^{bcd]}+mF^{abcd}=0\\\partial_aF^{abcd}+mF^{bcd}=0\end{cases}\Leftrightarrow\begin{cases}F=0,\partial_aF^{abc}=0\\\partial_a\partial^aF^{bcd}-m^2F^{bcd}=0\\F^{abcd}=-\frac{1}{3!m}\partial^{[a}F^{bcd]}\end{cases}$$

Thm. 6.5.1. $$\begin{cases}(\gamma^a\partial_a+m)X(6)=0\\X(6)=X^T(6)\end{cases}\Leftrightarrow\begin{cases}\partial_d\partial^dF^{abc}-m^2F^{abc}=0,\partial_aF^{abc}=0\\X(6)=\{\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+\frac{1}{4!m}\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\partial^d\}C\frac{1}{3!}F^{abc}\end{cases}$$

Cor. 6.5.3. $$\begin{cases}\frac{1}{3!}\partial^{[a}F^{bcd]}+mF^{abcd}=0\\\partial_aF^{abcd}+mF^{bcd}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{2!}\partial^{[a}{*F}^{bc]}+m{*F}^{abc}=0\\\partial_a{*F}^{abc}+m{*F}^{bc}=0\end{cases}$$

6.6 Antisymmetric B-W vector field equation in six dimensional space-time

Lem. 6.6.1. $(\gamma^a\partial_a+m)X(6)=0$
$X(6)=\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}+\frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1]}+mF^{a_0a_1}=0,\partial_{a_1}F^{a_1}=0\\\partial^{[a_0}F^{a_1a_2]}=0,\partial_{a_0}F^{a_0a_1}+mF^{a_1}=0\end{cases}\begin{cases}\frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]}+mF^{a_0a_1\cdot\cdot a_5}=0,\partial_{a_1}F^{a_1\cdot\cdot a_5}=0\\\partial^{[a_0}F^{a_1\cdot\cdot a_6]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_5}+mF^{a_1\cdot\cdot a_5}=0\end{cases}$$

Cor. 6.6.1. $\gamma^a\partial_aX(6)=0,X(6)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}+\frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1]}=0,\partial_{a_1}F^{a_1}=0\\\partial^{[a_0}F^{a_1a_2]}=0,\partial_{a_0}F^{a_0a_1}=0\end{cases}\begin{cases}\partial^{[a_0}F^{a_1\cdot\cdot a_5]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_5}=0\\\partial^{[a_0}F^{a_1\cdot\cdot a_6]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_5}=0\end{cases}$$

Cor. 6.6.2. $(\gamma^a\partial_a+m)X(6)=0,m\neq0$
$X(6)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}+\frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1]}+mF^{a_0a_1}=0\\\partial_{a_0}F^{a_0a_1}+mF^{a_1}=0\\\frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]}+mF^{a_0a_1\cdot\cdot a_5}=0\\\partial_{a_0}F^{a_0a_1\cdot\cdot a_5}+mF^{a_1\cdot\cdot a_5}=0\end{cases}\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1}-m^2F^{a_1}=0,\partial_{a_1}F^{a_1}=0\\F^{a_0a_1}=-\frac{1}{m}\partial^{[a_0}F^{a_1]}\\\partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_5}-m^2F^{a_1\cdot\cdot a_5}=0,\partial_{a_1}F^{a_1\cdot\cdot a_5}=0\\F^{a_0a_1\cdot\cdot a_5}=-\frac{1}{5!m}\partial^{[a_0}F^{a_1\cdot\cdot a_5]}\end{cases}$$

Thm. 6.6.1. $(\gamma^a\partial_a+m)X(6)=0, X(6)=-X^T(6)$

$$\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1}-m^2F^{a_1}=0,\partial_{a_1}F^{a_1}=0;\partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_5}-m^2F^{a_1\cdot\cdot a_5}=0,\partial_{a_1}F^{a_1\cdot\cdot a_5}=0\\ X(6)=\{\frac{1}{1!}\gamma_{a_1}+\frac{1}{2!m}\gamma_{[a_1}\gamma_{a_2]}\partial^{a_2}\}C\frac{1}{1!}F^{a_1}+\{\frac{1}{5!}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}+\frac{1}{6!}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\partial^{a_6}\}C\frac{1}{5!}F^{a_1\cdot\cdot a_5}\end{cases}$$

Cor. 6.6.3.
$$\begin{cases}\frac{1}{1!}\partial^{[a_0}F^{a_1]}+mF^{a_0a_1}=0\\ \partial_{a_0}F^{a_0a_1}+mF^{a_1}=0\\ \frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]}+mF^{a_0a_1\cdot\cdot a_5}=0\\ \partial_{a_0}F^{a_0a_1\cdot\cdot a_5}+mF^{a_1\cdot\cdot a_5}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{4!}\partial^{[a_0}*F^{a_1\cdot\cdot a_4]}+m*F^{a_0a_1\cdot\cdot a_4}=0\\ \partial_{a_0}*F^{a_0a_1\cdot\cdot a_4}+m*F^{a_1\cdot\cdot a_4}=0\\ \frac{1}{0!}\partial^{a_0}*F+m*F^{a_0}=0\\ \partial_{a_0}*F^{a_0}+m*F=0\end{cases}$$

6.7 Symmetric B-W vector field equation in eight dimensional space-time

Lem. 6.7.1. $(\gamma^a\partial_a+m)X(8)=0$

$X(8)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}+\frac{1}{(7!)^2}F^{a_1\cdot\cdot a_7}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdot\cdot a_8}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\}C$

$$\Leftrightarrow\begin{cases}F=0\\ \frac{1}{3!}\partial^{[a_0}F^{a_1\cdot\cdot a_3]}+mF^{a_0\cdot\cdot a_3}=0,\partial_{a_1}F^{a_1\cdot\cdot a_3}=0;\partial^{[a_0}F^{a_1\cdot\cdot a_4]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_3}+mF^{a_1\cdot\cdot a_3}=0\\ \frac{1}{7!}\partial^{[a_0}F^{a_1\cdot\cdot a_7]}+mF^{a_0a_1\cdot\cdot a_7}=0,\partial_{a_1}F^{a_1\cdot\cdot a_7}=0;\partial^{[a_0}F^{a_1\cdot\cdot a_8]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_7}+mF^{a_1\cdot\cdot a_7}=0\end{cases}$$

Proof: $(\gamma^a\partial_a+m)X(8)=0$

$X(8)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}+\frac{1}{(7!)^2}F^{a_1\cdot\cdot a_7}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdot\cdot a_8}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\}C$

$\Leftrightarrow$

$$\begin{cases}F=0\\ (\gamma^a\partial_a+m)\{\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\}=0\\ (\gamma^a\partial_a+m)\{\frac{1}{(7!)^2}F^{a_1\cdot\cdot a_7}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdot\cdot a_8}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\}=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}F=0\\ \frac{1}{3!}\partial^{[a_0}F^{a_1\cdot\cdot a_3]}+mF^{a_0\cdot\cdot a_3}=0,\partial_{a_1}F^{a_1\cdot\cdot a_3}=0;\partial^{[a_0}F^{a_1\cdot\cdot a_4]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_3}+mF^{a_1\cdot\cdot a_3}=0\\ \frac{1}{7!}\partial^{[a_0}F^{a_1\cdot\cdot a_7]}+mF^{a_0a_1\cdot\cdot a_7}=0,\partial_{a_1}F^{a_1\cdot\cdot a_7}=0;\partial^{[a_0}F^{a_1\cdot\cdot a_8]}=0,\partial_{a_0}F^{a_0a_1\cdot\cdot a_7}+mF^{a_1\cdot\cdot a_7}=0\end{cases}$$ □

Cor. 6.7.1. $\gamma^a\partial_aX(8)=0$

$X(8)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}+\frac{1}{(7!)^2}F^{a_1\cdot\cdot a_7}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdot\cdot a_8}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\}C$

$$\Leftrightarrow\begin{cases}F=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_3]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_3}=0;\partial^{[a_0}F^{a_1\cdot\cdot a_4]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_4}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_7]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_7}=0;\partial^{[a_0}F^{a_1\cdot\cdot a_8]}=0,\partial_{a_1}F^{a_1\cdot\cdot a_8}=0\end{cases}$$

Cor. 6.7.2. $(\gamma^a\partial_a+m)X(8)=0, m\neq0$

$X(8)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}+\frac{1}{(7!)^2}F^{a_1\cdot\cdot a_7}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdot\cdot a_8}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\}C$

$$\Leftrightarrow\begin{cases}F=0\\ \frac{1}{3!}\partial^{[a_0}F^{a_1\cdot\cdot a_3]}+mF^{a_0\cdot\cdot a_3}=0,\partial_{a_0}F^{a_0\cdot\cdot a_3}+mF^{a_1\cdot\cdot a_3}=0\\ \frac{1}{7!}\partial^{[a_0}F^{a_1\cdot\cdot a_7]}+mF^{a_0\cdot\cdot a_7}=0,\partial_{a_0}F^{a_0\cdot\cdot a_7}+mF^{a_1\cdot\cdot a_7}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}F=0\\ \partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_3}-m^2F^{a_1\cdot\cdot a_3}=0,\partial_{a_1}F^{a_1\cdot\cdot a_3}=0,F^{a_0\cdot\cdot a_3}=-\frac{1}{3!m}\partial^{[a_0}F^{a_1\cdot\cdot a_3]}\\ \partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_7}-m^2F^{a_1\cdot\cdot a_7}=0,\partial_{a_1}F^{a_1\cdot\cdot a_7}=0,F^{a_0\cdot\cdot a_7}=-\frac{1}{7!m}\partial^{[a_0}F^{a_1\cdot\cdot a_7]}\end{cases}$$

Thm. 6.7.1.

$$\begin{cases}(\gamma^a\partial_a+m)X(8)=0\\ X(8)=X^T(8)\end{cases}\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_3}-m^2F^{a_1\cdot\cdot a_3}=0,\partial_{a_1}F^{a_1\cdot\cdot a_3}=0\\ \partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_7}-m^2F^{a_1\cdot\cdot a_7}=0,\partial_{a_1}F^{a_1\cdot\cdot a_7}=0\\ X(8)=\{\frac{1}{3!}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}+\frac{1}{4!m}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\partial^{a_4}\}C\frac{1}{3!}F^{a_1\cdot\cdot a_3}\\ +\{\frac{1}{7!}\gamma_{[a_1}\cdot\cdot\gamma_{a_7]}+\frac{1}{8!m}\gamma_{[a_1}\cdot\cdot\gamma_{a_8]}\partial^{a_8}\}C\frac{1}{7!}F^{a_1\cdot\cdot a_7}\end{cases}$$

Cor. 6.7.3.
$$\begin{cases}\frac{1}{3!}\partial^{[a_0}F^{a_1\cdot\cdot a_3]}+mF^{a_0\cdot\cdot a_3}=0\\ \partial_{a_0}F^{a_0\cdot\cdot a_3}+mF^{a_1\cdot\cdot a_3}=0\\ \frac{1}{7!}\partial^{[a_0}F^{a_1\cdot\cdot a_7]}+mF^{a_0\cdot\cdot a_7}=0\\ \partial_{a_0}F^{a_0\cdot\cdot a_7}+mF^{a_1\cdot\cdot a_7}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{4!}\partial^{[a_0}*F^{a_1\cdot\cdot a_4]}+m*F^{a_0\cdot\cdot a_4}=0\\ \partial_{a_0}*F^{a_0\cdot\cdot a_4}+m*F^{a_1\cdot\cdot a_4}=0\\ \partial^{a_0}*F+m*F^{a_0}=0\\ \partial_{a_0}*F^{a_0}+m*F=0\end{cases}$$

6.8 Antisymmetric B-W vector field equation in eight dimensional space-time

Lem. 6.8.1. $(\gamma^a\partial_a+m)X(8)=0$

$X(8)=\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}+\frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

$$\Leftrightarrow \begin{cases} \partial^{[a_0}F^{a_1]} + mF^{a_0a_1} = 0, \partial_{a_1}F^{a_1} = 0 \\ \partial^{[a_0}F^{a_1a_2]} = 0, \partial_{a_0}F^{a_0a_1} + mF^{a_1} = 0 \end{cases} \begin{cases} \frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} + mF^{a_0a_1\cdot\cdot a_5} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0 \\ \partial^{[a_0}F^{a_1\cdot\cdot a_6]} = 0, \partial_{a_0}F^{a_0a_1\cdot\cdot a_5} + mF^{a_1\cdot\cdot a_5} = 0 \end{cases}$$

Cor. 6.8.1. $\gamma^a\partial_aX(8) = 0, X(8) = \{\frac{1}{(1!)^2}F^a\gamma_a + \frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

$$\Leftrightarrow \begin{cases} \partial^{[a_0}F^{a_1]} = 0, \partial_{a_1}F^{a_1} = 0 \\ \partial^{[a_0}F^{a_1a_2]} = 0, \partial_{a_0}F^{a_0a_1} = 0 \end{cases} \begin{cases} \partial^{[a_0}F^{a_1\cdot\cdot a_5]} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0 \\ \partial^{[a_0}F^{a_1\cdot\cdot a_6]} = 0, \partial_{a_0}F^{a_0a_1\cdot\cdot a_5} = 0 \end{cases}$$

Cor. 6.8.2. $(\gamma^a\partial_a + m)X(8) = 0, m \neq 0$
$X(8) = \{\frac{1}{(1!)^2}F^a\gamma_a + \frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\}C$

$$\Leftrightarrow \begin{cases} \partial^{[a_0}F^{a_1]} + mF^{a_0a_1} = 0 \\ \partial_{a_0}F^{a_0a_1} + mF^{a_1} = 0 \\ \frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} + mF^{a_0a_1\cdot\cdot a_5} = 0 \\ \partial_{a_0}F^{a_0a_1\cdot\cdot a_5} + mF^{a_1\cdot\cdot a_5} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0}\partial^{a_0}F^{a_1} - m^2F^{a_1} = 0, \partial_{a_1}F^{a_1} = 0 \\ F^{a_0a_1} = -\frac{1}{m}\partial^{[a_0}F^{a_1]} \\ \partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_5} - m^2F^{a_1\cdot\cdot a_5} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0 \\ F^{a_0a_1\cdot\cdot a_5} = -\frac{1}{5!m}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} \end{cases}$$

Thm. 6.8.1. $(\gamma^a\partial_a + m)X(8) = 0, X(8) = -X^T(8)$

$$\Leftrightarrow \begin{cases} \partial_{a_0}\partial^{a_0}F^{a_1} - m^2F^{a_1} = 0, \partial_{a_1}F^{a_1} = 0; \partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_5} - m^2F^{a_1\cdot\cdot a_5} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0 \\ X(6) = \{\frac{1}{1!}\gamma_{a_1} + \frac{1}{2!m}\gamma_{[a_1}\gamma_{a_2]}\partial^{a_2}\}CF^{a_1} + \{\frac{1}{5!}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{6!}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\partial^{a_6}\}C\frac{1}{5!}F^{a_1\cdot\cdot a_5} \end{cases}$$

Cor. 6.8.3.
$$\begin{cases} \frac{1}{1!}\partial^{[a_0}F^{a_1]} + mF^{a_0a_1} = 0 \\ \partial_{a_0}F^{a_0a_1} + mF^{a_1} = 0 \\ \frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} + mF^{a_0a_1\cdot\cdot a_5} = 0 \\ \partial_{a_0}F^{a_0a_1\cdot\cdot a_5} + mF^{a_1\cdot\cdot a_5} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{6!}\partial^{[a_0}*F^{a_1\cdot\cdot a_6]} + m*F^{a_0a_1\cdot\cdot a_6} = 0 \\ \partial_{a_0}*F^{a_0a_1\cdot\cdot a_6} + m*F^{a_1\cdot\cdot a_6} = 0 \\ \frac{1}{2!}\partial^{[a_0}*F^{a_1a_2]} + m*F^{a_0a_1a_2} = 0 \\ \partial_{a_0}*F^{a_0a_1a_2} + m*F^{a_1a_2} = 0 \end{cases}$$

6.9 Symmetric B-W vector field equation in ten dimensional space-time

Lem. 6.9.1. $(\gamma^a\partial_a + m)X(10) = 0, X(10) = \{\frac{1}{(0!)^2}F + \frac{1}{2!}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}$
$+ \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]} + \frac{1}{(9!)^2}F^{a_1\cdot\cdot a_9}\gamma_{[a_1}\cdot\cdot\gamma_{a_9]} + \frac{1}{(10!)^2}F^{a_1\cdot\cdot a_{10}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{10}]}\}C$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1a_2} = 0 \\ \frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} + mF^{a_0a_1\cdot\cdot a_5} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0; \partial^{[a_0}F^{a_1\cdot\cdot a_6]} = 0, \partial_{a_0}F^{a_0a_1\cdot\cdot a_5} + mF^{a_1\cdot\cdot a_5} = 0 \\ \frac{1}{9!}\partial^{[a_0}F^{a_1\cdot\cdot a_9]} + mF^{a_0a_1\cdot\cdot a_9} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_9} = 0; \partial^{[a_0}F^{a_1\cdot\cdot a_{10}]} = 0, \partial_{a_0}F^{a_0a_1\cdot\cdot a_9} + mF^{a_1\cdot\cdot a_9} = 0 \end{cases}$$

Proof: $(\gamma^a\partial_a + m)X(10) = 0, X(10) = \{\frac{1}{(0!)^2}F + \frac{1}{2!}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}$
$+ \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]} + \frac{1}{(9!)^2}F^{a_1\cdot\cdot a_9}\gamma_{[a_1}\cdot\cdot\gamma_{a_9]} + \frac{1}{(10!)^2}F^{a_1\cdot\cdot a_{10}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{10}]}\}C$
$\Leftrightarrow$

$$\begin{cases} F = 0, F^{a_1a_2} = 0 \\ (\gamma^b\partial_b + m)\{\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}\} = 0 \\ (\gamma^b\partial_b + m)\{\frac{1}{(9!)^2}F^{a_1\cdot\cdot a_9}\gamma_{[a_1}\cdot\cdot\gamma_{a_9]} + \frac{1}{(10!)^2}F^{a_1\cdot\cdot a_{10}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{10}]}\} = 0 \end{cases}$$
$\Leftrightarrow$
$$\begin{cases} \frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} + mF^{a_0a_1\cdot\cdot a_5} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0; \partial^{[a_0}F^{a_1\cdot\cdot a_6]} = 0, \partial_{a_0}F^{a_0a_1\cdot\cdot a_5} + mF^{a_1\cdot\cdot a_5} = 0 \\ \frac{1}{9!}\partial^{[a_0}F^{a_1\cdot\cdot a_9]} + mF^{a_0a_1\cdot\cdot a_9} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_9} = 0; \partial^{[a_0}F^{a_1\cdot\cdot a_{10}]} = 0, \partial_{a_0}F^{a_0a_1\cdot\cdot a_9} + mF^{a_1\cdot\cdot a_9} = 0 \end{cases}$$ □

Cor. 6.9.1. $\gamma^a\partial_aX(10) = 0, X(10) = \{\frac{1}{(0!)^2}F + \frac{1}{2!}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}$
$+ \frac{1}{(9!)^2}F^{a_1\cdot\cdot a_9}\gamma_{[a_1}\cdot\cdot\gamma_{a_9]} + \frac{1}{(10!)^2}F^{a_1\cdot\cdot a_{10}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{10}]}\}C$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1a_2} = 0 \\ \partial^{[a_0}F^{a_1\cdot\cdot a_5]} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0; \partial^{[a_0}F^{a_1\cdot\cdot a_6]} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_6} = 0 \\ \partial^{[a_0}F^{a_1\cdot\cdot a_9]} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_9} = 0; \partial^{[a_0}F^{a_1\cdot\cdot a_{10}]} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_{10}} = 0 \end{cases}$$

Cor. 6.9.2.
$(\gamma^b\partial_b + m)X(10) = 0, X(10) = \{\frac{1}{(0!)^2}F + \frac{1}{2!}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]} + \frac{1}{(6!)^2}F^{a_1\cdot\cdot a_6}\gamma_{[a_1}\cdot\cdot\gamma_{a_6]}$
$+ \frac{1}{(9!)^2}F^{a_1\cdot\cdot a_9}\gamma_{[a_1}\cdot\cdot\gamma_{a_9]} + \frac{1}{(10!)^2}F^{a_1\cdot\cdot a_{10}}\gamma_{[a_1}\cdot\cdot\gamma_{a_{10}]}\}C, m \neq 0$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1a_2} = 0 \\ \frac{1}{5!}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} + mF^{a_0\cdot\cdot a_5} = 0, \partial_{a_0}F^{a_0\cdot\cdot a_5} + mF^{a_1\cdot\cdot a_5} = 0 \\ \frac{1}{9!}\partial^{[a_0}F^{a_1\cdot\cdot a_9]} + mF^{a_0\cdot\cdot a_9} = 0, \partial_{a_0}F^{a_0\cdot\cdot a_9} + mF^{a_1\cdot\cdot a_9} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1a_2} = 0 \\ \partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_5} - m^2F^{a_1\cdot\cdot a_5} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_5} = 0, F^{a_0\cdot\cdot a_5} = -\frac{1}{5!m}\partial^{[a_0}F^{a_1\cdot\cdot a_5]} \\ \partial_{a_0}\partial^{a_0}F^{a_1\cdot\cdot a_9} - m^2F^{a_1\cdot\cdot a_9} = 0, \partial_{a_1}F^{a_1\cdot\cdot a_9} = 0, F^{a_0\cdot\cdot a_9} = -\frac{1}{9!m}\partial^{[a_0}F^{a_1\cdot\cdot a_9]} \end{cases}$$

Thm. 6.9.1.

$$\begin{cases}(\gamma^a\partial_a+m)X(10)=0\\ X(10)=X^T(10)\end{cases} \Leftrightarrow \begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1\cdots a_5}-m^2F^{a_1\cdots a_5}=0,\partial_{a_1}F^{a_1\cdots a_5}=0\\ \partial_{a_0}\partial^{a_0}F^{a_1\cdots a_9}-m^2F^{a_1\cdots a_9}=0,\partial_{a_1}F^{a_1\cdots a_9}=0\\ X(10)=\{\frac{1}{5!}\gamma_{[a_1}\cdots\gamma_{a_5]}+\frac{1}{6!m}\gamma_{[a_1}\cdots\gamma_{a_6]}\partial^{a_6}\}C\frac{1}{5!}F^{a_1\cdots a_5}\\ +\{\frac{1}{9!}\gamma_{[a_1}\cdots\gamma_{a_9]}+\frac{1}{10!m}\gamma_{[a_1}\cdots\gamma_{a_{10}]}\partial^{a_{10}}\}C\frac{1}{9!}F^{a_1\cdots a_9}\end{cases}$$

Cor. 6.9.3.

$$\begin{cases}\frac{1}{5!}\partial^{[a_0}F^{a_1\cdots a_5]}+mF^{a_0\cdots a_5}=0,\partial_{a_0}F^{a_0\cdots a_5}+mF^{a_1\cdots a_5}=0\\ \frac{1}{9!}\partial^{[a_0}F^{a_1\cdots a_9]}+mF^{a_0\cdots a_9}=0,\partial_{a_0}F^{a_0\cdots a_9}+mF^{a_1\cdots a_9}=0\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}\frac{1}{4!}\partial^{[a_0}*F^{a_1\cdots a_4]}+m*F^{a_0\cdots a_4}=0,\partial_{a_0}*F^{a_0\cdots a_4}+m*F^{a_1\cdots a_4}=0\\ \partial^{a_0}*F+m*F^{a_0}=0,\partial_{a_0}*F^{a_0}+m*F=0\end{cases}$$

Cor. 6.9.4.

$$\begin{cases}\partial^{[a_0}F^{a_1\cdots a_5]}=0,\partial_{a_1}F^{a_1\cdots a_5}=0;\partial^{[a_0}F^{a_1\cdots a_6]}=0,\partial_{a_1}F^{a_1\cdots a_6}=0\\ \partial^{[a_0}F^{a_1\cdots a_9]}=0,\partial_{a_1}F^{a_1\cdots a_9}=0;\partial^{[a_0}F^{a_1\cdots a_{10}]}=0,\partial_{a_1}F^{a_1\cdots a_{10}}=0\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}\partial^{[a_0}*F^{a_1\cdots a_5]}=0,\partial_{a_1}*F^{a_1\cdots a_5}=0;\partial^{[a_0}*F^{a_1\cdots a_4]}=0,\partial_{a_1}*F^{a_1\cdots a_4}=0\\ \partial^{[a_0}*F^{a_1]}=0,\partial_{a_1}*F^{a_1}=0;\partial^{a_0}*F=0\end{cases}$$

6.10 Antisymmetric B-W vector field equation in ten dimensional space-time

Lem. 6.10.1. $(\gamma^b\partial_b+m)X(10)=0,X(10)=$
$\{\frac{1}{1!}F^{a_1}\gamma_{a_1}+\frac{1}{(3!)^2}F^{a_1\cdots a_3}\gamma_{[a_1}\cdots\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdots a_4}\gamma_{[a_1}\cdots\gamma_{a_4]}+\frac{1}{(7!)^2}F^{a_1\cdots a_7}\gamma_{[a_1}\cdots\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdots a_8}\gamma_{[a_1}\cdots\gamma_{a_8]}\}C$

$$\Leftrightarrow\begin{cases}F^{a_1}=0\\ \frac{1}{3!}\partial^{[a_0}F^{a_1\cdots a_3]}+mF^{a_0a_1\cdots a_3}=0,\partial_{a_1}F^{a_1\cdots a_3}=0;\partial^{[a_0}F^{a_1\cdots a_4]}=0,\partial_{a_0}F^{a_0a_1\cdots a_3}+mF^{a_1\cdots a_3}=0\\ \frac{1}{7!}\partial^{[a_0}F^{a_1\cdots a_7]}+mF^{a_0a_1\cdots a_7}=0,\partial_{a_1}F^{a_1\cdots a_7}=0;\partial^{[a_0}F^{a_1\cdots a_8]}=0,\partial_{a_0}F^{a_0a_1\cdots a_7}+mF^{a_1\cdots a_7}=0\end{cases}$$

Cor. 6.10.1. $\gamma^b\partial_bX(10)=0,X(10)=$
$\{\frac{1}{1!}F^{a_1}\gamma_{a_1}+\frac{1}{(3!)^2}F^{a_1\cdots a_3}\gamma_{[a_1}\cdots\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdots a_4}\gamma_{[a_1}\cdots\gamma_{a_4]}+\frac{1}{(7!)^2}F^{a_1\cdots a_7}\gamma_{[a_1}\cdots\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdots a_8}\gamma_{[a_1}\cdots\gamma_{a_8]}\}C$

$$\Leftrightarrow\begin{cases}F^{a_1}=0\\ \partial^{[a_0}F^{a_1\cdots a_3]}=0,\partial_{a_1}F^{a_1\cdots a_3}=0;\partial^{[a_0}F^{a_1\cdots a_4]}=0,\partial_{a_0}F^{a_0a_1\cdots a_3}=0\\ \partial^{[a_0}F^{a_1\cdots a_7]}=0,\partial_{a_1}F^{a_1\cdots a_7}=0;\partial^{[a_0}F^{a_1\cdots a_8]}=0,\partial_{a_0}F^{a_0a_1\cdots a_7}=0\end{cases}$$

Cor. 6.10.2.
$(\gamma^b\partial_b+m)X(10)=0,m\neq0,X(10)=$
$\{\frac{1}{1!}F^{a_1}\gamma_{a_1}+\frac{1}{(3!)^2}F^{a_1\cdots a_3}\gamma_{[a_1}\cdots\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdots a_4}\gamma_{[a_1}\cdots\gamma_{a_4]}+\frac{1}{(7!)^2}F^{a_1\cdots a_7}\gamma_{[a_1}\cdots\gamma_{a_7]}+\frac{1}{(8!)^2}F^{a_1\cdots a_8}\gamma_{[a_1}\cdots\gamma_{a_8]}\}C$

$$\Leftrightarrow\begin{cases}F^{a_1}=0\\ \frac{1}{3!}\partial^{[a_0}F^{a_1\cdots a_3]}+mF^{a_0a_1\cdots a_3}=0,\partial_{a_0}F^{a_0a_1\cdots a_3}+mF^{a_1\cdots a_3}=0\\ \frac{1}{7!}\partial^{[a_0}F^{a_1\cdots a_7]}+mF^{a_0a_1\cdots a_7}=0,\partial_{a_0}F^{a_0a_1\cdots a_7}+mF^{a_1\cdots a_7}=0\end{cases}$$

$$\Leftrightarrow\begin{cases}F^{a_1}=0\\ \partial_{a_0}\partial^{a_0}F^{a_1\cdots a_3}-m^2F^{a_1\cdots a_3}=0,\partial_{a_1}F^{a_1\cdots a_3}=0,F^{a_0a_1\cdots a_3}=-\frac{1}{3!m}\partial^{[a_0}F^{a_1\cdots a_3]}\\ \partial_{a_0}\partial^{a_0}F^{a_1\cdots a_7}-m^2F^{a_1\cdots a_7}=0,\partial_{a_1}F^{a_1\cdots a_7}=0,F^{a_0a_1\cdots a_7}=-\frac{1}{7!m}\partial^{[a_0}F^{a_1\cdots a_7]}\end{cases}$$

Thm. 6.10.1. $(\gamma^a\partial_a+m)X(10)=0,X(10)=-X^T(10)$

$$\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1\cdots a_3}-m^2F^{a_1\cdots a_3}=0,\partial_{a_1}F^{a_1\cdots a_3}=0;\partial_{a_0}\partial^{a_0}F^{a_1\cdots a_7}-m^2F^{a_1\cdots a_7}=0,\partial_{a_1}F^{a_1\cdots a_7}=0\\ X(10)=\{\frac{1}{3!}\gamma_{[a_1}\cdots\gamma_{a_3]}+\frac{1}{4!m}\gamma_{[a_1}\cdots\gamma_{a_4]}\partial^{a_4}\}C\frac{1}{3!}F^{a_1\cdots a_3}+\{\frac{1}{7!}\gamma_{[a_1}\cdots\gamma_{a_7]}+\frac{1}{8!m}\gamma_{[a_1}\cdots\gamma_{a_8]}\partial^{a_8}\}C\frac{1}{7!}F^{a_1\cdots a_7}\end{cases}$$

Cor. 6.10.3.

$$\begin{cases}\frac{1}{3!}\partial^{[a_0}F^{a_1\cdots a_3]}+mF^{a_0a_1\cdots a_3}=0\\ \partial_{a_0}F^{a_0a_1\cdots a_3}+mF^{a_1\cdots a_3}=0\\ \frac{1}{7!}\partial^{[a_0}F^{a_1\cdots a_7]}+mF^{a_0a_1\cdots a_7}=0\\ \partial_{a_0}F^{a_0a_1\cdots a_7}+mF^{a_1\cdots a_7}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{6!}\partial^{[a_0}*F^{a_1\cdots a_6]}+m*F^{a_0\cdots a_6}=0\\ \partial_{a_0}*F^{a_0\cdots a_6}+m*F^{a_1\cdots a_6}=0\\ \frac{1}{2!}\partial^{[a_0}*F^{a_1a_2]}+m*F^{a_0a_1a_2}=0\\ \partial_{a_0}*F^{a_0a_1a_2}+m*F^{a_1a_2}=0\end{cases}$$

6.11 Commutation rules for B-W vector field equation in ten dimensional space-time

Lem. 6.11.1. $\frac{1}{(l+1)!}\eta_{[a_1\langle a_1'}\eta_{a_2a_2'}\cdots\eta_{a_{l-1}a_{l-1}'}\eta_{a_la_l'}\eta_{a_{l+1}]a_{l+1}'\rangle}\partial^{a_{l+1}}\partial^{+a_{l+1}'}\Delta(x-x')$
$=\{\frac{1}{l!}\eta_{[a_1\langle a_1'}\eta_{a_2a_2'}\cdots\eta_{a_{l-1}a_{l-1}'}\eta_{a_l]a_l'\rangle}m^2-\frac{1}{(l-1)!}\eta_{[a_1\langle a_1'}\eta_{a_2a_2'}\cdots\eta_{a_{l-1}a_{l-1}'}\partial_{a_l]}\partial^+_{a_l'\rangle}\}\Delta(x-x')$

Cor. 6.11.1. $F_{a_1a_2a_3a_4a_5}(x)=2^{-5}tr\{\frac{1}{5!}\bar{C}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}X(x)\}=\frac{2^{-5}}{5!}\bar{C}^{\lambda\eta}\{\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}\}_\eta{}^\mu X_{\lambda\mu}(x)$

Pro. 6.11.1. $tr\{\frac{1}{5!}\gamma^{[b_1}\cdot\cdot\gamma^{b_5]}\frac{1}{5!}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\} = 2^5\delta^{b_1}_{[a_1}\cdot\cdot\delta^{b_5}_{a_5]}$

Cor. 6.11.2.

$$\begin{cases} U_{a_1a_2a_3a_4a_5}(\vec{p},h) := \frac{2^{-5}}{5!}\bar{C}^{\lambda\eta}\{\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}\}_\eta{}^\mu U_{\lambda\mu}(\vec{p},h) = 2^{-5}tr\{\bar{C}\frac{1}{5!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}U(\vec{p},h)\} \\ V_{a_1a_2a_3a_4a_5}(\vec{p},h) := \frac{2^{-5}}{5!}\bar{C}^{\lambda\eta}\{\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}\}_\eta{}^\mu V_{\lambda\mu}(\vec{p},h) = 2^{-5}tr\{\bar{C}\frac{1}{5!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}V(\vec{p},h)\} \\ C(10) := \gamma_2(10)\gamma_4(10)\gamma_5(10)\gamma_8(10)\gamma_9(10) = -\sigma_y\otimes I\otimes\sigma_y\otimes I\otimes\sigma_y \\ \bar{C} = C^+, C^T = -C \end{cases}$$

Ass. 6.11.1.

$$\begin{cases} \bar{C}(n) = C^+(n), C^+(n) = (-1)^{[\frac{n}{4}]}C(n), C^T(n) = (-1)^{[\frac{n+2}{4}]}C(n) \\ [\gamma_a(n)C(n)]^T = (-1)^{[\frac{n-1}{4}]}[\gamma_a(n)C(n)], [C^+(n)\gamma_a(n)]^T = (-1)^{[\frac{n-1}{4}]}[C^+(n)\gamma_a(n)] \end{cases}$$

Thm. 6.11.1. $[F_{a_1a_2a_3a_4a_5}(x), F^+_{a'_1a'_2a'_3a'_4a'_5}(x')] = -\frac{i}{2^5}\frac{1}{6!}\eta_{[a_1\langle a'_1}\eta_{a_2a'_2}\cdot\cdot\eta_{a_5a'_5}\eta_{a_6]a'_6\rangle}\partial^{a_6}\partial^{+a'_6}\Delta(x-x')$

Proof: $[F_{a_1a_2a_3a_4a_5}(x), F^+_{a'_1a'_2a'_3a'_4a'_5}(x')]$
$= \frac{2^{-10}}{(5!)^2}(C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})^{\mu\lambda}(C^+\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]})^{*\mu'\lambda'}[\psi_{\lambda\mu}(x), \psi^+_{\lambda'\mu'}(x')]$
$= \frac{2^{-10}}{(5!)^2}(C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})^{\lambda\mu}(C^+\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]})^{+\lambda'\mu'}[\psi_{\lambda\mu}(x), \psi^+_{\lambda'\mu'}(x')]$
$= \frac{2^{-10}}{(5!)^2}(C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})^{\lambda\mu}(\gamma_{[a'_5}\gamma_{a'_4}\gamma_{a'_3}\gamma_{a'_2}\gamma_{a'_1]}C)^{\lambda'\mu'}[\psi_{\lambda\mu}(x), \psi^+_{\lambda'\mu'}(x')]$
$= \frac{2^{-10}}{(5!)^2}(C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})^{\lambda\mu}(\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}C)^{\lambda'\mu'}[\psi_{\lambda\mu}(x), \psi^+_{\lambda'\mu'}(x')]$
$= \frac{2^{-10}}{(5!)^2}(C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})^{\lambda\mu}(\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}C)^{\lambda'\mu'}\frac{i}{2^3}[(m-\gamma^a\partial_a)\gamma^0]_{\{\lambda(\lambda'}[(m-\gamma^b\partial_b)\gamma^0]_{\mu\}\mu')}\Delta(x-x')$
$= i\frac{2^{-11}}{(5!)^2}(C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})^{\lambda\mu}(\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}C)^{\lambda'\mu'}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}[(m-\gamma^b\partial_b)\gamma^0]_{\mu\mu'}\Delta(x-x')$
$= i\frac{2^{-11}}{(5!)^2}(C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})^{\lambda\mu}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}(\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}C)^{\lambda'\mu'}[(m-\gamma^b\partial_b)\gamma^0]^T_{\mu'\mu}\Delta(x-x')$
$= i\frac{2^{-11}}{(5!)^2}tr\{C^+\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}C[(m-\gamma^b\partial_b)\gamma^0]^T\}\Delta(x-x')$
$= i\frac{2^{-11}}{(5!)^2}tr\{\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}C[(m-\gamma^b\partial_b)\gamma^0]^TC^+\}\Delta(x-x')$
$= -i\frac{2^{-11}}{(5!)^2}tr\{\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}[\gamma^0(m+\gamma^b\partial_b)]\}\Delta(x-x')$
$= -i\frac{2^{-11}}{(5!)^2}tr\{\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]}[(m-\gamma^b\partial^+_b)\gamma^0]\}\Delta(x-x')$
$= -i\frac{2^{-11}}{(5!)^2}tr\{m^2(\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})\gamma^0(\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]})\gamma^0\}\Delta(x-x')$
$- i\frac{2^{-11}}{(5!)^2}tr\{(\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]})\gamma_a\gamma_0(\gamma_{[a'_1}\gamma_{a'_2}\gamma_{a'_3}\gamma_{a'_4}\gamma_{a'_5]})\gamma_b\gamma_0\}\partial^a\partial^b\Delta(x-x')$
$= -i\frac{2^{-11}}{(5!)^2}i^{5*6}2^5(5!)^2m^2\frac{1}{5!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5]}_{a'_5]}\Delta(x-x')$
$- i\frac{2^{-11}}{(5!)^2}i^{6*7}2^5(5!)^2\{\frac{1}{6!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5}_{a'_5}\eta^{a]}_{b]} - \frac{1}{4!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_4}_{a'_4}\delta^{a_5]a}\delta_{a'_5]b}\}\partial_a\partial^{+b}\Delta(x-x')$
$= \frac{i}{2^6}\{\frac{1}{5!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5]}_{a'_5]}m^2 + (\frac{1}{6!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5}_{a'_5}\eta^{a]}_{b]} - \frac{1}{4!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_4}_{a'_4}\delta^{a_5]a}\delta_{a'_5]b})\partial_a\partial^{+b}\}\Delta(x-x')$
$= \frac{i}{2^6}\{\eta^{a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5}_{a'_5]}m^2 + (\eta^{a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5}_{a'_5}\eta^{a}_{b]} - \frac{1}{4!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_4}_{a'_4}\delta^{a_5]a}\delta_{a'_5]b})\partial_a\partial^{+b}\}\Delta(x-x')$
$= \frac{i}{2^6}\{\eta^{a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5}_{a'_5]}m^2 + (\eta^{a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5}_{a'_5}\eta^{a}_{b]}\partial_a\partial^{+b} - \frac{1}{4!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_4}_{a'_4}\partial^{a_5]}\partial^+_{a'_5]})\}\Delta(x-x')$
$= \frac{i}{2^5}\{\frac{1}{5!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5]}_{a'_5]}m^2 - \frac{1}{4!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_4}_{a'_4}\partial^{a_5]}\partial^+_{a'_5]}\}\Delta(x-x')$
$= \frac{i}{2^5}\frac{1}{6!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_5}_{a'_5}\eta^{a_6]}_{a'_6]}\partial_{a_6}\partial^{+a'_6}\Delta(x-x')$
$= \frac{i}{2^5}\frac{1}{6!}\eta_{[a_1\langle a'_1}\eta_{a_2a'_2}\cdot\cdot\eta_{a_5a'_5}\eta_{a_6]a'_6\rangle}\partial^{a_6}\partial^{+a'_6}\Delta(x-x')$ □

Cor. 6.11.3. $[F_{a_1a_2\cdot\cdot a_6}(x), F^+_{a'_1a'_2\cdot\cdot a'_6}(x')] = -i\frac{2^{-5}}{5!}\eta_{[a_1\langle a'_1}\eta_{a_2a'_2}\cdot\cdot\eta_{a_5a'_5}\partial_{a_6]}\partial^+_{a'_6\rangle}\Delta(x-x')$

7 B-W vector field equation in odd dimensional space-time

7.1 Symmetric B-W vector field equation in three dimensional space-time

Lem. 7.1.1. $(\gamma^a\partial_a+m)X(3) = 0, X(3) = \frac{1}{(1!)^2}F^a\gamma_aC \Leftrightarrow \varepsilon^{abc}\partial_bF_c - imF^a = 0, \partial_aF^a = 0$

Proof: $(\gamma^a\partial_a+m)X(3) = 0, X(3) = \frac{1}{(1!)^2}F^a\gamma_aC$
$\Leftrightarrow (\gamma_a\partial^a+m)\frac{1}{(1!)^2}F^b\gamma_bC = 0$
$\Leftrightarrow (\gamma_a\partial^a+m)F^b\gamma_b = 0$
$\Leftrightarrow \gamma_a\gamma_b\partial^aF^b + mF^b\gamma_b = 0$
$\Leftrightarrow \{\frac{1}{2!}\gamma_{[a}\gamma_{b]} + \delta_{ab}\}\partial^aF^b + \frac{1}{1!}mF^b\gamma_b = 0$
$\Leftrightarrow \frac{1}{2!}\gamma_{[a}\gamma_{b]}\partial^aF^b + \delta_{ab}\partial^aF^b + \frac{1}{1!}mF^b\gamma_b = 0$
$\Leftrightarrow i\varepsilon^{abc}\frac{1}{(1!)^2}\gamma_c\partial_aF_b + \delta_{ab}\partial^aF^b + \frac{1}{1!}mF^c\gamma_c = 0$
$\Leftrightarrow \varepsilon^{abc}\partial_aF_b - imF^c = 0, \partial_aF^a = 0$
$\Leftrightarrow \varepsilon^{abc}\partial_bF_c - imF^a = 0, \partial_aF^a = 0$ □

Cor. 7.1.1. $\gamma^a\partial_a X(3)=0, X(3)=\frac{1}{(1!)^2}F^a\gamma_a C \Leftrightarrow \varepsilon^{abc}\partial_b F_c=0, \partial_a F^a=0 \Rightarrow \partial_b\partial^b F^a=0$

Cor. 7.1.2. $(\gamma^a\partial_a+m)X(3)=0, X(3)=\frac{1}{(1!)^2}F^a\gamma_a C, m\neq 0 \Leftrightarrow \varepsilon^{abc}\partial_b F_c-imF^a=0 \Rightarrow \partial_b\partial^b F^a-m^2F^a=0$

Thm. 7.1.1. $(\gamma^a\partial_a+m)X(3)=0, X(3)=X^T(3), m\neq 0 \Leftrightarrow \varepsilon^{abc}\partial_b F_c-imF^a=0, X(3)=\frac{1}{(1!)^2}F^a\gamma_a C$

Cor. 7.1.3. $\varepsilon^{abc}\partial_b F_c-imF^a=0 \Leftrightarrow \partial_a{*}F^{ab}+imF^b=0 \Leftrightarrow \partial_{[a}F_{b]}-im{*}F_{ab}=0$

7.2 Antisymmetric B-W vector field equation in three dimensional space-time

Lem. 7.2.1. $(\gamma^a\partial_a+m)X(3)=0, X(3)=\frac{1}{(0!)^2}FC \Leftrightarrow \partial^a F=0, mF=0$

Proof: $(\gamma^a\partial_a+m)X(3)=0, X(3)=\frac{1}{(0!)^2}FC$
$\Leftrightarrow (\gamma_a\partial^a+m)\frac{1}{(0!)^2}FC=0$
$\Leftrightarrow (\gamma_a\partial^a+m)F=0$
$\Leftrightarrow \partial^a F=0, mF=0$ □

Cor. 7.2.1. $\gamma^a\partial_a\frac{1}{(0!)^2}FC=0 \Leftrightarrow \partial^a F=0$

Cor. 7.2.2. $(\gamma^a\partial_a+m)\frac{1}{(0!)^2}FC=0, m\neq 0 \Leftrightarrow F=0$

Thm. 7.2.1. $(\gamma^a\partial_a+m)X(3)=0, X(3)=-X^T(3), m\neq 0 \Leftrightarrow F=0, X(3)=\frac{1}{(0!)^2}FC=0$

Cor. 7.2.3. $F=0 \Leftrightarrow {*}F_{ab}=0$

7.3 Symmetric B-W vector field equation in five dimensional space-time

Lem. 7.3.1. $(\gamma^a\partial_a+m)X(5)=0, X(5)=\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}C \Leftrightarrow \varepsilon^{abcde}\partial_c F_{de}-mF^{ab}=0, \partial_a F^{ab}=0$

Proof: $(\gamma^a\partial_a+m)X(5)=0, X(5)=\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}C$
$\Leftrightarrow (\gamma_a\partial^a+m)\frac{1}{(2!)^2}F^{bc}\gamma_{[b}\gamma_{c]}C=0$
$\Leftrightarrow (\gamma_a\partial^a+m)F^{bc}\gamma_b\gamma_c=0$
$\Leftrightarrow \gamma_a\gamma_b\gamma_c\partial^a F^{bc}+mF^{bc}\gamma_b\gamma_c=0$
$\Leftrightarrow \{\frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}+(\delta_{a[b}\gamma_{c]}+\gamma_a\delta_{bc})\}\partial^a F^{bc}+\frac{1}{2!}mF^{bc}\gamma_{[b}\gamma_{c]}=0$
$\Leftrightarrow \frac{1}{3!}\gamma_{[a}\gamma_b\gamma_{c]}\partial^a F^{bc}+2\delta_{ab}\gamma_c\partial^a F^{bc}+\frac{1}{2!}mF^{bc}\gamma_{[b}\gamma_{c]}=0$
$\Leftrightarrow -\varepsilon^{abcde}\frac{1}{(2!)^2}\gamma_{[d}\gamma_{e]}\partial_a F_{bc}+2\delta_{ab}\gamma_c\partial^a F^{bc}+\frac{1}{2!}mF^{de}\gamma_{[d}\gamma_{e]}=0$
$\Leftrightarrow \frac{1}{2!}\varepsilon^{abcde}\partial_a F_{bc}-mF^{de}=0, \partial_a F^{ab}=0$
$\Leftrightarrow \frac{1}{2!}\varepsilon^{abcde}\partial_c F_{de}-mF^{ab}=0, \partial_a F^{ab}=0$ □

Cor. 7.3.1. $\gamma^a\partial_a X(5)=0, X(5)=\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}C \Leftrightarrow \varepsilon^{abcde}\partial_c F_{de}=0, \partial_a F^{ab}=0$

Cor. 7.3.2. $(\gamma^a\partial_a+m)X(5)=0, X(5)=\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}C, m\neq 0 \Leftrightarrow \frac{1}{2!}\varepsilon^{abcde}\partial_c F_{de}-mF^{ab}=0$

Thm. 7.3.1. $(\gamma^a\partial_a+m)X(5)=0, X(5)=X^T(5), m\neq 0 \Leftrightarrow \frac{1}{2!}\varepsilon^{abcde}\partial_c F_{de}-mF^{ab}=0, X(5)=\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}C$

Cor. 7.3.3. $\frac{1}{2!}\varepsilon^{abcde}\partial_c F_{de}-mF^{ab}=0 \Leftrightarrow \partial_a{*}F^{abc}-mF^{bc}=0 \Leftrightarrow \frac{1}{2!}\partial_{[a}F_{bc]}-m{*}F_{abc}=0$

7.4 Antisymmetric B-W vector field equation in five dimensional space-time

Lem. 7.4.1. $\begin{cases}(\gamma^a\partial_a+m)X(5)=0\\ X(5)=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}\}C\end{cases} \Leftrightarrow \begin{cases}\frac{1}{0!}\partial^{a_0}F+mF^{a_0}=0\\ \partial^{[a_0}F^{a_1]}=0, \partial_{a_0}F^{a_0}+mF=0\end{cases}$

Cor. 7.4.1. $\begin{cases}\gamma^a\partial_a X(5)=0\\ X(5)=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}\}C\end{cases} \Leftrightarrow \begin{cases}\partial^{a_0}F=0\\ \partial^{[a_0}F^{a_1]}=0, \partial_{a_0}F^{a_0}=0\end{cases}$

Cor. 7.4.2. $\begin{cases}(\gamma^a\partial_a+m)X(5)=0, m\neq 0\\ X(5)=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}\}C\end{cases} \Leftrightarrow \begin{cases}\frac{1}{0!}\partial^{a_0}F+mF^{a_0}=0\\ \partial_{a_0}F^{a_0}+mF=0\end{cases} \Leftrightarrow \begin{cases}\partial_{a_0}\partial^{a_0}F-m^2F=0\\ F^{a_0}=-\frac{1}{0!m}\partial^{a_0}F\end{cases}$

Thm. 7.4.1. $\begin{cases}(\gamma^a\partial_a+m)X(5)=0, m\neq 0\\ X(5)=-X^T(5)\end{cases} \Leftrightarrow \begin{cases}\partial_{a_0}\partial^{a_0}F-m^2F=0\\ X(5)=\{\frac{1}{0!}F-\frac{1}{1!m}\gamma_{a_1}\partial^{a_1}\}C\end{cases}$

Cor. 7.4.3. $\begin{cases}\frac{1}{0!}\partial^{a_0}F+mF^{a_0}=0\\ \partial_{a_0}F^{a_0}+mF=0\end{cases} \Leftrightarrow \begin{cases}\frac{1}{4!}\partial^{[a_0}{*}F^{a_1\cdot\cdot a_4]}+m{*}F^{a_0a_1\cdot\cdot a_4}=0\\ \partial_{a_0}{*}F^{a_0a_1\cdot\cdot a_4}+m{*}F^{a_1\cdot\cdot a_4}=0\end{cases}$

7.5 Symmetric B-W vector field equation in seven dimensional space-time

Lem. 7.5.1. $(\gamma^{a_0}\partial_{a_0}+m)X(7)=0, X(7)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C$
$\Leftrightarrow \varepsilon^{a_1\cdot\cdot a_7}\partial_{a_4}F_{a_5\cdot\cdot a_7}-imF^{a_1\cdot\cdot a_3}=0, \partial_{a_1}F^{a_1\cdot\cdot a_3}=0$

Proof: $(\gamma^{a_0}\partial_{a_0}+m)X(7)=0, X(7)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C$
$\Leftrightarrow (\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C=0$
$\Leftrightarrow F=0, (\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C=0$
$\Leftrightarrow \frac{1}{3!}\varepsilon^{a_1\cdot\cdot a_7}\partial_{a_4}F_{a_5\cdot\cdot a_7}-imF^{a_1\cdot\cdot a_3}=0, \partial_{a_1}F^{a_1\cdot\cdot a_3}=0$ □

Cor. 7.5.1. $\gamma^{a_0}\partial_{a_0}=0, X(7)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C$
$\Leftrightarrow \varepsilon^{a_1\cdot\cdot a_7}\partial_{a_4}F_{a_5\cdot\cdot a_7}=0, \partial_{a_1}F^{a_1\cdot\cdot a_3}=0$

Cor. 7.5.2. $(\gamma^{a_0}\partial_{a_0}+m)X(7)=0, X(7)=\{\frac{1}{(0!)^2}F+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C, m\neq 0$
$\Leftrightarrow \frac{1}{3!}\varepsilon^{a_1\cdot\cdot a_7}\partial_{a_4}F_{a_5\cdot\cdot a_7}-imF^{a_1\cdot\cdot a_3}=0$

Thm. 7.5.1. $(\gamma^a\partial_a+m)X(7)=0, X(7)=X^T(7), m\neq 0$
$\Leftrightarrow \frac{1}{3!}\varepsilon^{a_1\cdot\cdot a_7}\partial_{a_4}F_{a_5\cdot\cdot a_7}-imF^{a_1\cdot\cdot a_3}=0, X(7)=\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}C$

Cor. 7.5.3.
$\frac{1}{3!}\varepsilon^{a_1\cdot\cdot a_7}\partial_{a_4}F_{a_5\cdot\cdot a_7}-imF^{a_1\cdot\cdot a_3}=0 \Leftrightarrow \partial_{a_0}{*F}^{a_0\cdot\cdot a_3}+imF^{a_1\cdot\cdot a_3}=0 \Leftrightarrow \frac{1}{3!}\partial_{[a_0}F_{a_1\cdot\cdot a_3]}-im{*F}_{a_0\cdot\cdot a_3}=0$

7.6 Antisymmetric B-W vector field equation in seven dimensional space-time

Lem. 7.6.1. $\begin{cases}(\gamma^a\partial_a+m)X(7)=0\\ X(7)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases} \Leftrightarrow \begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0, \partial_aF^a=0\\ \partial^{[a}F^{bc]}=0, \partial_aF^{ab}+mF^b=0\end{cases}$

Cor. 7.6.1. $\begin{cases}\gamma^a\partial_aX(7)=0\\ X(7)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases} \Leftrightarrow \begin{cases}\partial^{[a}F^{b]}=0, \partial_aF^a=0\\ \partial^{[a}F^{bc]}=0, \partial_aF^{ab}=0\end{cases}$

Cor. 7.6.2. $(\gamma^a\partial_a+m)X(7)=0, X(7)=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C, m\neq 0$
$\Leftrightarrow \partial_aF^{ab}+mF^b=0, \partial^{[a}F^{b]}+mF^{ab}=0 \Leftrightarrow \partial_b\partial^bF^a-m^2F^a=0, \partial_aF^a=0, F^{ab}=-\frac{1}{m}\partial^{[a}F^{b]}$

Thm. 7.6.1. $\begin{cases}(\gamma^a\partial_a+m)X(7)=0\\ X(7)=-X^T(7)\end{cases} \Leftrightarrow \begin{cases}\partial_b\partial^bF^a-m^2F^a=0, \partial_aF^a=0\\ X(7)=\{\frac{1}{1!}\gamma_a+\frac{1}{2!m}\gamma_{[a}\gamma_{b]}\partial^b\}CF^a\end{cases}$

Cor. 7.6.3. $\begin{cases}\frac{1}{1!}\partial^{[a_0}F^{a_1]}+mF^{a_0a_1}=0\\ \partial_{a_0}F^{a_0a_1}+mF^{a_1}=0\end{cases} \Leftrightarrow \begin{cases}\frac{1}{5!}\partial^{[a_0}{*F}^{a_1\cdot\cdot a_5]}-m{*F}^{a_0a_1\cdot\cdot a_5}=0\\ \partial_{a_0}{*F}^{a_0a_1\cdot\cdot a_5}-m{*F}^{a_1\cdot\cdot a_5}=0\end{cases}$

7.7 Symmetric B-W vector field equation in nine dimensional space-time

Lem. 7.7.1. $(\gamma^{a_0}\partial_{a_0}+m)X(9)=0, X(9)=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C$
$\Leftrightarrow \begin{cases}\partial^{a_1}F+mF^{a_1}=0, \partial^{[a_0}F^{a_1]}=0, \partial_{a_1}F^{a_1}+mF=0\\ \frac{1}{4!}\varepsilon^{a_1\cdot\cdot a_9}\partial_{a_5}F_{a_6\cdot\cdot a_9}-mF^{a_1\cdot\cdot a_4}=0, \partial_{a_1}F^{a_1\cdot\cdot a_4}=0\end{cases}$

Proof: $(\gamma^{a_0}\partial_{a_0}+m)X(9)=0, X(9)=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C$
$\Leftrightarrow (\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C=0$
$\Leftrightarrow (\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}\}C=0, (\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C=0$
$\Leftrightarrow \begin{cases}\partial^{a_1}F+mF^{a_1}=0, \partial^{[a_0}F^{a_1]}=0, \partial_{a_1}F^{a_1}+mF=0\\ \frac{1}{4!}\varepsilon^{a_1\cdot\cdot a_9}\partial_{a_5}F_{a_6\cdot\cdot a_9}-mF^{a_1\cdot\cdot a_4}=0, \partial_{a_1}F^{a_1\cdot\cdot a_4}=0\end{cases}$ □

Cor. 7.7.1. $\gamma^{a_0}\partial_{a_0}X(9)=0, X(9)=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C$
$\Leftrightarrow \begin{cases}\partial^{a_1}F=0, \partial^{[a_0}F^{a_1]}=0, \partial_{a_1}F^{a_1}=0\\ \varepsilon^{a_1\cdot\cdot a_9}\partial_{a_5}F_{a_6\cdot\cdot a_9}=0, \partial_{a_1}F^{a_1\cdot\cdot a_4}=0\end{cases}$

Cor. 7.7.2. $(\gamma^{a_0}\partial_{a_0}+m)X(9)=0, X(9)=\{\frac{1}{(0!)^2}F+\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C, m\neq 0$
$\Leftrightarrow \begin{cases}\partial^{a_1}F+mF^{a_1}=0, \partial_{a_1}F^{a_1}+mF=0 \Leftrightarrow \partial_{a_1}\partial^{a_1}F-m^2F=0, F^{a_1}=-\frac{1}{m}\partial^{a_1}F\\ \frac{1}{4!}\varepsilon^{a_1\cdot\cdot a_9}\partial_{a_5}F_{a_6\cdot\cdot a_9}-mF^{a_1\cdot\cdot a_4}=0\end{cases}$

Thm. 7.7.1. $\begin{cases}(\gamma^a\partial_a+m)X(9)=0\\ X(9)=X^T(9)\end{cases} \Leftrightarrow \begin{cases}\partial_{a_1}\partial^{a_1}F-m^2F=0, \frac{1}{4!}\varepsilon^{a_1\cdot\cdot a_9}\partial_{a_5}F_{a_6\cdot\cdot a_9}-mF^{a_1\cdot\cdot a_4}=0\\ X(9)=\{(1-\frac{1}{m}\gamma_{a_1}\partial^{a_1})F+\frac{1}{(4!)^2}F^{a_1a_2a_3a_4}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}\}C\end{cases}$

Cor. 7.7.3. $\frac{1}{4!}\varepsilon^{a_1\cdot\cdot a_9}\partial_{a_5}F_{a_6\cdot\cdot a_9}-mF^{a_1\cdot\cdot a_4}=0 \Leftrightarrow \partial_{a_0}{*F}^{a_0\cdot\cdot a_4}-mF^{a_1\cdot\cdot a_4}=0 \Leftrightarrow \frac{1}{4!}\partial_{[a_0}F_{a_1\cdot\cdot a_4]}-m{*F}_{a_0\cdot\cdot a_4}=0$

7.8 Antisymmetric B-W vector field equation in nine dimensional space-time

Lem. 7.8.1.
$$\begin{cases}(\gamma^a\partial_a+m)X(9)=0\\ X(9)=\{\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C\end{cases}\Leftrightarrow\begin{cases}\frac{1}{2!}\partial^{[a_0}F^{a_1a_2]}+mF^{a_0a_1a_2}=0,\partial_{a_1}F^{a_1a_2}=0\\ \partial^{[a_0}F^{a_1a_2a_3]}=0,\partial_{a_0}F^{a_0a_1a_2}+mF^{a_1a_2}=0\end{cases}$$

Cor. 7.8.1.
$$\begin{cases}\gamma^a\partial_aX(9)=0\\ X(9)=\{\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C\end{cases}\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1a_2]}=0,\partial_{a_1}F^{a_1a_2}=0\\ \partial^{[a_0}F^{a_1a_2a_3]}=0,\partial_{a_0}F^{a_0a_1a_2}=0\end{cases}$$

Cor. 7.8.2. $(\gamma^a\partial_a+m)X(9)=0,m\neq0,X(9)=\{\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\}C$
$$\Leftrightarrow\begin{cases}\frac{1}{2!}\partial^{[a_0}F^{a_1a_2]}+mF^{a_0a_1a_2}=0\\ \partial_{a_0}F^{a_0a_1a_2}+mF^{a_1a_2}=0\end{cases}\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1a_2}-m^2F^{a_1a_2}=0,\partial_{a_1}F^{a_1a_2}=0\\ F^{a_0a_1a_2}=-\frac{1}{2!m}\partial^{[a_0}F^{a_1a_2]}\end{cases}$$

Thm. 7.8.1.
$$\begin{cases}(\gamma^a\partial_a+m)X(9)=0,m\neq0\\ X(9)=-X^T(9)\end{cases}\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1a_2}-m^2F^{a_1a_2}=0,\partial_{a_1}F^{a_1a_2}=0\\ X(9)=\{\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]}-\frac{1}{3!m}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}\partial^{a_3}\}C\frac{1}{2!}F^{a_1a_2}\end{cases}$$

Cor. 7.8.3.
$$\begin{cases}\frac{1}{2!}\partial^{[a_0}F^{a_1a_2]}+mF^{a_0a_1a_2}=0\\ \partial_{a_0}F^{a_0a_1a_2}+mF^{a_1a_2}=0\end{cases}\Leftrightarrow\begin{cases}\frac{1}{6!}\partial^{[a_0}*F^{a_1\cdot\cdot a_6]}+m*F^{a_0a_1\cdot\cdot a_6}=0\\ \partial_{a_0}*F^{a_0a_1\cdot\cdot a_6}+m*F^{a_1\cdot\cdot a_6}=0\end{cases}$$

7.9 Symmetric B-W vector field equation in eleven dimensional space-time

Lem. 7.9.1. $(\gamma^{a_0}\partial_{a_0}+m)X(11)=0,X(11)=\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C$
$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1]}+mF^{a_0a_1}=0,\partial_{a_1}F^{a_1}=0;\partial^{[a_0}F^{a_1a_2]}=0,\partial_{a_0}F^{a_0a_1}+mF^{a_1}=0\\ \frac{1}{5!}\varepsilon^{a_1\cdot\cdot a_{11}}\partial_{a_6}F_{a_7\cdot\cdot a_{11}}-mF^{a_1\cdot\cdot a_5}=0,\partial_{a_1}F^{a_1\cdot\cdot a_5}=0\end{cases}$$

Proof: $(\gamma^{a_0}\partial_{a_0}+m)X(11)=0,X(11)=\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C$
$\Leftrightarrow(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C=0$
$\Leftrightarrow(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}\}C=0,(\gamma_{a_0}\partial^{a_0}+m)\{\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C=0$
$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1]}+mF^{a_0a_1}=0,\partial_{a_1}F^{a_1}=0;\partial^{[a_0}F^{a_1a_2]}=0,\partial_{a_0}F^{a_0a_1}+mF^{a_1}=0\\ \frac{1}{5!}\varepsilon^{a_1\cdot\cdot a_{11}}\partial_{a_6}F_{a_7\cdot\cdot a_{11}}-mF^{a_1\cdot\cdot a_5}=0,\partial_{a_1}F^{a_1\cdot\cdot a_5}=0\end{cases}$$ □

Cor. 7.9.1. $\gamma^{a_0}\partial_{a_0}X(11)=0,X(11)=\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C$
$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1]}=0,\partial_{a_1}F^{a_1}=0,\partial^{[a_0}F^{a_1a_2]}=0,\partial_{a_0}F^{a_0a_1}=0\\ \varepsilon^{a_1\cdot\cdot a_{11}}\partial_{a_6}F_{a_7\cdot\cdot a_{11}}=0,\partial_{a_1}F^{a_1\cdot\cdot a_5}=0\end{cases}$$

Cor. 7.9.2. $(\gamma^{a_0}\partial_{a_0}+m)X(11)=0,X(11)=\{\frac{1}{(1!)^2}F^{a_1}\gamma_{a_1}+\frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C,m\neq0$
$$\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1]}+mF^{a_0a_1}=0,\partial_{a_0}F^{a_0a_1}+mF^{a_1}=0\\ \varepsilon^{a_1\cdot\cdot a_{11}}\partial_{a_6}F_{a_7\cdot\cdot a_{11}}-mF^{a_1\cdot\cdot a_5}=0\end{cases}$$
$$\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1}-m^2F^{a_1}=0,\partial_{a_1}F^{a_1}=0,F^{a_0a_1}=-\frac{1}{m}\partial^{[a_0}F^{a_1]}\\ \frac{1}{5!}\varepsilon^{a_1\cdot\cdot a_{11}}\partial_{a_6}F_{a_7\cdot\cdot a_{11}}-mF^{a_1\cdot\cdot a_5}=0\end{cases}$$

Thm. 7.9.1. $\begin{cases}(\gamma^a\partial_a+m)X(11)=0\\ X(11)=X^T(11),m\neq0\end{cases}\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1}-m^2F^{a_1}=0,\frac{1}{5!}\varepsilon^{a_1\cdot\cdot a_{11}}\partial_{a_6}F_{a_7\cdot\cdot a_{11}}-mF^{a_1\cdot\cdot a_5}=0\\ X(11)=\{(\frac{1}{1!}\gamma_{a_1}+\frac{1}{2!m}\gamma_{[a_1}\gamma_{a_2]}\partial^{a_2})F^{a_1}+\frac{1}{(5!)^2}F^{a_1\cdot\cdot a_5}\gamma_{[a_1}\cdot\cdot\gamma_{a_5]}\}C\end{cases}$

Cor. 7.9.3. $\frac{1}{5!}\varepsilon^{a_1\cdot\cdot a_{11}}\partial_{a_6}F_{a_7\cdot\cdot a_{11}}-mF^{a_1\cdot\cdot a_5}=0\Leftrightarrow\partial_{a_0}*F^{a_0\cdot\cdot a_5}+mF^{a_1\cdot\cdot a_5}=0\Leftrightarrow\frac{1}{5!}\partial_{[a_0}F_{a_1\cdot\cdot a_5]}-m*F_{a_0\cdot\cdot a_5}=0$

7.10 Antisymmetric B-W vector field equation in eleven dimensional space-time

Lem. 7.10.1.
$$\begin{cases}(\gamma^a\partial_a+m)X(11)=0,X(11)=\\ \{\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\}C\end{cases}\Leftrightarrow\begin{cases}\frac{1}{3!}\partial^{[a_0}F^{a_1a_2a_3]}+mF^{a_0a_1a_2a_3}=0,\partial_{a_1}F^{a_1a_2a_3}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_4]}=0,\partial_{a_0}F^{a_0a_1a_2a_3}+mF^{a_1a_2a_3}=0\end{cases}$$

Cor. 7.10.1.
$$\begin{cases}\gamma^a\partial_aX(11)=0\\ X(11)=\{\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\}C\end{cases}\Leftrightarrow\begin{cases}\partial^{[a_0}F^{a_1a_2a_3]}=0,\partial_{a_1}F^{a_1a_2a_3}=0\\ \partial^{[a_0}F^{a_1\cdot\cdot a_4]}=0,\partial_{a_0}F^{a_0a_1a_2a_3}=0\end{cases}$$

Cor. 7.10.2. $(\gamma^a\partial_a+m)X(11)=0,m\neq0,X(11)=\{\frac{1}{(3!)^2}F^{a_1\cdot\cdot a_3}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}+\frac{1}{(4!)^2}F^{a_1\cdot\cdot a_4}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\}C$
$$\Leftrightarrow\begin{cases}\frac{1}{3!}\partial^{[a_0}F^{a_1a_2a_3]}+mF^{a_0a_1a_2a_3}=0\\ \partial_{a_0}F^{a_0a_1a_2a_3}+mF^{a_1a_2a_3}=0\end{cases}\Leftrightarrow\begin{cases}\partial_{a_0}\partial^{a_0}F^{a_1a_2a_3}-m^2F^{a_1a_2a_3}=0,\partial_{a_1}F^{a_1a_2a_3}=0\\ F^{a_0a_1a_2a_3}=-\frac{1}{m}\partial^{[a_0}F^{a_1a_2a_3]}\end{cases}$$

Thm. 7.10.1.

$$\begin{cases}(\gamma^a\partial_a+m)X(11)=0, m\neq 0\\ X(11)=-X^T(11)\end{cases} \Leftrightarrow \begin{cases}\partial_{a_1}\partial^{a_1}F^{a_1a_2a_3}-m^2F^{a_1a_2a_3}=0, \partial_{a_1}F^{a_1a_2a_3}=0\\ X(11)=\{\frac{1}{3!}\gamma_{[a_1}\cdot\cdot\gamma_{a_3]}+\frac{1}{4!}\gamma_{[a_1}\cdot\cdot\gamma_{a_4]}\partial^{a_4}\}C\frac{1}{3!}F^{a_1a_2a_3}\end{cases}$$

Cor. 7.10.3.

$$\begin{cases}\frac{1}{3!}\partial^{[a_0}F^{a_1a_2a_3]}+mF^{a_0a_1a_2a_3}=0\\ \partial_{a_0}F^{a_0a_1a_2a_3}+mF^{a_1a_2a_3}=0\end{cases} \Leftrightarrow \begin{cases}\frac{1}{7!}\partial^{[a_0}{*}F^{a_1\cdot\cdot a_7]}-m{*}F^{a_0a_1\cdot\cdot a_7}=0\\ \partial_{a_0}{*}F^{a_0a_1\cdot\cdot a_7}-m{*}F^{a_1\cdot\cdot a_7}=0\end{cases}$$

8 Antisymmetric B-W vector field equation in n=N+1 even dimensional space-time

8.1 On antisymmetric relation lemma

Lem. 8.1.1. $\sum\limits_{h=1}^{l}a_{\{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma\}}(\vec{p},h)=0\Leftrightarrow[a(\vec{p};h,h')+a(\vec{p};h',h)]=0, c(\vec{p};h,h')=0$

Proof: $\sum\limits_{h=1}^{l}a_{\{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma\}}(\vec{p},h)=0$

$\Leftrightarrow\sum\limits_{h=1}^{l}u_{\{\lambda_\varsigma}(\vec{p},h)a_{\mu_\varsigma\}}(\vec{p},h)=0$

$\Leftrightarrow\sum\limits_{h,h'=1}^{l}u_{\{\lambda_\varsigma}(\vec{p},h)[a(\vec{p};h,h')u_{\mu_\varsigma\}}(\vec{p},h')+c(\vec{p};h,h')v_{\mu_\varsigma\}}(\vec{p},h')]=0$

$\Leftrightarrow\sum\limits_{h,h'=1}^{l}[a(\vec{p};h,h')u_{\{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma\}}(\vec{p},h')+c(\vec{p};h,h')u_{\{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma\}}(\vec{p},h')]=0$

$\Leftrightarrow\sum\limits_{h,h'=1}^{l}[a(\vec{p};h,h')+a(\vec{p};h',h)]u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')+\sum\limits_{h,h'=1}^{l}c(\vec{p};h,h')u_{\{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma\}}(\vec{p},h')]=0$

$\Leftrightarrow[a(\vec{p};h,h')+a(\vec{p};h',h)]=0, c(\vec{p};h,h')=0$ □

8.2 Plane wave solutions of antisymmetric B-W vector field equation in even n=N+1-D

Thm. 8.2.1. $(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=0, \psi_{\lambda_\varsigma\mu_\varsigma}(x)=-\psi_{\mu_\varsigma\lambda_\varsigma}(x)$

$\Leftrightarrow$

$\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\frac{m}{\sqrt{E}}\sum\limits_{h\langle h'=1}^{l}[a(\vec{p};h\langle h')\frac{1}{\sqrt{2}}u_{[\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma]}(\vec{p},h')e^{ip\cdot x}+b^{+}(\vec{p};h\langle h')\frac{1}{\sqrt{2}}v_{[\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma]}(\vec{p},h')e^{-ip\cdot x}]d^N\vec{p}$

Proof:

$(\gamma^a\partial_a+m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=0, \psi_{\lambda_\varsigma\mu_\varsigma}(x)=-\psi_{\mu_\varsigma\lambda_\varsigma}(x)$

$\Leftrightarrow$

$\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^{+}_{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}, \psi_{\lambda_\varsigma\mu_\varsigma}(x)=-\psi_{\mu_\varsigma\lambda_\varsigma}(x)$

$\Leftrightarrow$

$$\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^{+}_{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\\ \int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^{+}_{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\\ =-\int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^{+}_{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^{+}_{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\\ \int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a_{\{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma\}}(\vec{p},h)e^{ip\cdot x}+b^{+}_{\{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma\}}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^{+}_{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\\ \sum\limits_{h=1}^{l}a_{\{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma\}}(\vec{p},h)=0, \sum\limits_{h=1}^{l}b^{+}_{\{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma\}}(\vec{p},h)=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h,h'=1}^{l}[a(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')e^{ip\cdot x}+b^{+}(\vec{p};h,h')v_{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma}(\vec{p},h')e^{-ip\cdot x}]d^N\vec{p}\\ a(\vec{p};h,h')+a(\vec{p};h',h)=0, b^{+}(\vec{p};h,h')+b^{+}(\vec{p};h',h)=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h,h'=1}^{l}[a(\vec{p};h,h')\frac{1}{2!}u_{[\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma]}(\vec{p},h')e^{ip\cdot x}+b^{+}(\vec{p};h,h')\frac{1}{2!}v_{[\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma]}(\vec{p},h')e^{-ip\cdot x}]d^N\vec{p}\\ a(\vec{p};h,h')+a(\vec{p};h',h)=0, b^{+}(\vec{p};h,h')+b^{+}(\vec{p};h',h)=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{-\infty}^{+\infty}\frac{m}{\sqrt{E}}\sum\limits_{h\langle h'=1}^{l}[a(\vec{p};h\langle h')\frac{1}{\sqrt{2}}u_{[\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma]}(\vec{p},h')e^{ip\cdot x}+b^{+}(\vec{p};h\langle h')\frac{1}{\sqrt{2}}v_{[\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma]}(\vec{p},h')e^{-ip\cdot x}]d^N\vec{p}\\ a(\vec{p};h\langle h')=\frac{\sqrt{2}}{\sqrt{m}}a(\vec{p};h,h'),b^{+}(\vec{p};h\langle h')=\frac{\sqrt{2}}{\sqrt{m}}b^{+}(\vec{p};h,h')\end{cases}$$

□

9 Direct solution of antisymmetric tensor field in N+1 dimensional space-time

9.1 Electronic equation [4] in N+1 dimensional space-time

The electron equation in even dimensional spacetime:

Def. 9.1.1. $(\gamma^a\partial_a+m)\psi=0,\psi=\begin{bmatrix}\varphi\\ \eta\end{bmatrix},\gamma^a=(\Gamma\otimes\sigma_y,\varsigma I_*\otimes\sigma_z)=(\vec{\gamma},\varsigma I_*\otimes\sigma_z)$

The electron equation in odd dimensional spacetime:

Def. 9.1.2. $(\gamma^a\partial_a+m)\psi=0,\psi=\begin{bmatrix}\varphi\\ \eta\end{bmatrix},\gamma^a=(\Gamma\otimes\sigma_y,I_*\otimes\sigma_x,\varsigma I_*\otimes\sigma_z)=(\vec{\gamma},\varsigma I_*\otimes\sigma_z)$

9.2 Electron spin basis in N+1 dimensional space-time

Def. 9.2.1. $u_\varsigma(\vec{p},h):=\sqrt{\frac{E+m}{2m}}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}1\\0\\0_{l-2}\\0_l\end{bmatrix},\sqrt{\frac{E+m}{2m}}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0\\1\\0_{l-2}\\0_l\end{bmatrix},\cdot\cdot,\sqrt{\frac{E+m}{2m}}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_{l-2}\\0\\1\\0_l\end{bmatrix}$

Def. 9.2.2. $v_\varsigma(\vec{p},h):=\sqrt{\frac{E+m}{2m}}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_l\\1\\0\\0_{l-2}\end{bmatrix},\sqrt{\frac{E+m}{2m}}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_l\\0\\1\\0_{l-2}\end{bmatrix},\cdot\cdot,\sqrt{\frac{E+m}{2m}}(1-\frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m})\begin{bmatrix}0_l\\0_{l-2}\\0\\1\end{bmatrix}$

9.3 Plane wave solutions of electron in N+1 dimensional space-time

Cor. 9.3.1.

$$\begin{cases}\psi_{\lambda_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{m}{E}}\sum\limits_{h=1}^{l}[a(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^{+}(\vec{p},h)v_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\\ a(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{E}{m}}u^{+}(\vec{p},h)\psi(x)e^{-ip\cdot x}d^3\vec{r},b^{+}(\vec{p},h)=\frac{1}{(2\pi)^{3/2}}\int\limits_{\vec{p}=-\infty}^{+\infty}\sqrt{\frac{E}{m}}v^{+}(\vec{p},h)\psi(x)e^{ip\cdot x}d^3\vec{r}\end{cases}$$

Thm. 9.3.1.

$$\begin{cases}A_{a_1\cdot\cdot a_l}(x)=2^{-[\frac{n}{2}]}tr\{\frac{1}{l!}\bar{C}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}\psi(x)\}=\frac{1}{(2\pi)^{N/2}}\int\sqrt{\frac{E}{m}}\sum\limits_{h=1}^{l}[a(\vec{p},h)U_{a_1\cdot\cdot a_l}(\vec{p},h)e^{ip\cdot x}+b^{+}(\vec{p},h)V^{+}_{a_1\cdot\cdot a_l}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\\ U_{a_1\cdot\cdot a_l}(\vec{p},h)=2^{-[\frac{n}{2}]}tr\{\frac{1}{l!}\bar{C}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}u(\vec{p},h),V_{a_1\cdot\cdot a_l}(\vec{p},h)=2^{-[\frac{n}{2}]}tr\{\frac{1}{l!}\bar{C}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}v(\vec{p},h)\end{cases}$$

9.4 Spin basis of B-W vector field in N+1 dimensional space-time

Def. 9.4.1.

$$\begin{cases}U_{\lambda_\varsigma\mu_\varsigma}(\vec{p};h,h)=\frac{1}{2}u_{\{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma\}}(\vec{p},h),U_{\lambda_\varsigma\mu_\varsigma}(\vec{p};h<h')=\frac{1}{\sqrt{2}}u_{\{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma\}}(\vec{p},h');h,h'=1,2,\cdot\cdot,2^{[\frac{N-1}{2}]}\\ V_{\lambda_\varsigma\mu_\varsigma}(\vec{p};h,h)=\frac{1}{2}v_{\{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma\}}(\vec{p},h),V_{\lambda_\varsigma\mu_\varsigma}(\vec{p};h<h')=\frac{1}{\sqrt{2}}v_{\{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma\}}(\vec{p},h');h,h'=1,2,\cdot\cdot,2^{[\frac{N-1}{2}]}\end{cases}$$

$$\begin{cases}X_{\lambda_\varsigma\mu_\varsigma}(\vec{p};h<h')=\frac{1}{\sqrt{2}}u_{[\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma]}(\vec{p},h');h,h'=1,2,\cdot\cdot,2^{[\frac{N-1}{2}]}\\ Y_{\lambda_\varsigma\mu_\varsigma}(\vec{p};h<h')=\frac{1}{\sqrt{2}}v_{[\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma]}(\vec{p},h');h,h'=1,2,\cdot\cdot,2^{[\frac{N-1}{2}]}\end{cases}$$

9.5 Spin basis and properties of antisymmetric tensor field in N+1 dimensional space-time

Thm. 9.5.1. $\begin{cases}U_{a_1\cdot\cdot a_l}(\vec{p};h,h)=2^{-[\frac{n}{2}]}\{(\frac{1}{l!}C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h)\}\\ U_{a_1\cdot\cdot a_l}(\vec{p};h<h')=2^{-[\frac{n}{2}]}\sqrt{2}\{(\frac{1}{l!}C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')\}\\ U_{a_1\cdot\cdot a_l}(\vec{p};h,h')=2^{-[\frac{n}{2}]}\sqrt{2}^{1-\delta_{hh'}}\{(\frac{1}{l!}C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')\}\end{cases}$

Proof: ???$U_{a_1\cdot\cdot a_l}(\vec{p};h,h')=2^{-[\frac{n}{2}]}\sqrt{2}^{1-\delta_{hh'}}\{(\frac{1}{l!}C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')\}$
$=2^{-[\frac{n}{2}]}\sqrt{2}^{1-\delta_{hh'}}\{\frac{1}{l!}u_{\mu_\varsigma}(\vec{p},h')(C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}u_{\lambda_\varsigma}(\vec{p},h)\}$
$=2^{-[\frac{n}{2}]}\sqrt{2}^{1-\delta_{hh'}}\frac{1}{l!}u^T(\vec{p},h')C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}u(\vec{p},h)$ □

Proof: $\sum\limits_{h\le h'}\sqrt{2}^{1-\delta_{hh'}}[\frac{1}{l!}u^T(\vec{p},h')C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]}u(\vec{p},h)][\frac{1}{l!}u^T(\vec{p},h')C^{+}\gamma_{[a'_1}\cdot\cdot\gamma_{a'_l]}u(\vec{p},h)]^{+}$
$=\sum\limits_{h\le h'}\sqrt{2}^{1-\delta_{hh'}}[\frac{1}{l!}u^T(\vec{p},h')(C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})u(\vec{p},h)\frac{1}{l!}u^{+}(\vec{p},h)(C^{+}\gamma_{[a'_1}\cdot\cdot\gamma_{a'_l]})^{+}u^{*}(\vec{p},h')$
$=\frac{1}{(l!)^2}\sum\limits_{h\le h'}\sqrt{2}^{1-\delta_{hh'}}[u^T(\vec{p},h')(C^{+}\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})u(\vec{p},h)u^{+}(\vec{p},h)(\gamma_{[a'_l}\cdot\cdot\gamma_{a'_1]}C)u^{*}(\vec{p},h')$

$= \frac{i^{l(l-1)}}{(l!)^2} \sum_{h\leq h'} \sqrt{2}^{1-\delta_{hh'}} [u^T(\vec{p},h')(C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})u(\vec{p},h)u^+(\vec{p},h)(\gamma_{[a'_1}\cdot\cdot\gamma_{a'_l]}C)u^*(\vec{p},h')$

$= \frac{i^{l(l-1)}}{(l!)^2} \sum_{h\leq h'} \sqrt{2}^{1-\delta_{hh'}} [u_\mu(\vec{p},h')(C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}u_\lambda(\vec{p},h)u^+_{\lambda'}(\vec{p},h)(\gamma_{[a'_1}\cdot\cdot\gamma_{a'_l]}C)^{\lambda'\mu'}u^+_{\mu'}(\vec{p},h')$

$= \frac{i^{l(l-1)}}{(l!)^2}[(C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}[\sum_h u_\lambda(\vec{p},h)u^+_{\lambda'}(\vec{p},h)](\gamma_{[a'_1}\cdot\cdot\gamma_{a'_l]}C)^{\lambda'\mu'}[\sum_{h'} u_\mu(\vec{p},h')u^+_{\mu'}(\vec{p},h')]$

$= \frac{i^{l(l-1)}}{4m^2(l!)^2}[(C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu\lambda}[(m-i\gamma^a p_a)\gamma^0]_{\lambda\lambda'}(\gamma_{[a'_1}\cdot\cdot\gamma_{a'_l]}C)^{\lambda'\mu'}[(m-i\gamma^b p_b)\gamma^0]^T_{\mu'\mu}$

$= \frac{i^{l(l-1)}}{4m^2(l!)^2}tr\{[(C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})[(m-i\gamma^a p_a)\gamma^0](\gamma_{[a'_1}\cdot\cdot\gamma_{a'_l]}C)[(m-i\gamma^b p_b)\gamma^0]^T\}$

$= i\frac{4^{-[\frac{n}{2}]}}{2(l!)^2}i^{l(l-1)}tr\{C^+\gamma_{[a_1}\gamma_{a_2}\cdot\cdot\gamma_{a_l]}[(m-i\gamma^a p_a)\gamma^0]\gamma_{[a'_1}\gamma_{a'_2}\cdot\cdot\gamma_{a'_l]}C[(m-i\gamma^b p_b)\gamma^0]^T\}\Delta(x-x')$

$= -\frac{2^{[\frac{n}{2}]}(-1)^{\xi(n)+l}}{2m^2}\begin{cases}\frac{1}{(l+1)!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_l}_{a'_l}\eta^{a]}_{a']}p_a p^{+a'}, (-1)^{\eta(n)+l}=1\\ \frac{1}{(l-1)!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_{l-1}}_{a'_{l-1}}p^{a_l]}p_{a'_l]}, (-1)^{\eta(n)+l}=-1\end{cases}$

$= \frac{2^{[\frac{n}{2}]}(-1)^{\delta_{2,n}}}{2m^2}\begin{cases}\frac{1}{(l+1)!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_l}_{a'_l}\eta^{a]}_{a']}p_a p^{+a'}, (-1)^{\eta(n)+l}=1\\ -\frac{1}{(l-1)!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_{l-1}}_{a'_{l-1}}p^{a_l]}p_{a'_l]}, (-1)^{\eta(n)+l}=-1\end{cases}$ □

Cor. 9.5.1. $\sum_{h\leq h'} U_{a_1\cdot\cdot a_l}(\vec{p};h,h')U^+_{a'_1\cdot\cdot a'_l}(\vec{p};h,h') = \frac{(-1)^{\delta_{2,n}}}{2m^2 2^{[\frac{n}{2}]}}\begin{cases}\frac{1}{(l+1)!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\cdot\cdot\eta^{a_l}_{a'_l}\eta^{a]}_{a']}p_a p^{+a'}, (-1)^{\eta(n)+l}=1\\ -\frac{1}{(l-1)!}\eta^{[a_1}_{[a'_1}\cdot\cdot\eta^{a_{l-1}}_{a'_{l-1}}p^{a_l]}p_{a'_l]}, (-1)^{\eta(n)+l}=-1\end{cases}$

Proof: $U^{+a'_1\cdot\cdot a'_l}(\vec{p};h,h')\eta^{a_1}_{a'_1}\cdot\cdot\eta^{a_l}_{a'_l}U_{a_1\cdot\cdot a_l}(\vec{p};h,h')$

$= 4^{-[\frac{n}{2}]}2^{1-\delta_{hh'}}i^{l(l-1)}\{\frac{1}{l!}u^+_{\mu'_\varsigma}(\vec{p},h')u^+_{\lambda'_\varsigma}(\vec{p},h)(\gamma^{[a'_1}\cdot\cdot\gamma^{a'_l]}C)^{\lambda'_\varsigma\mu'_\varsigma}\}\eta^{a_1}_{a'_1}\cdot\cdot\eta^{a_l}_{a'_l}\{(\frac{1}{l!}C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')\}$

$= 4^{-[\frac{n}{2}]}2^{1-\delta_{hh'}}i^{l(l-1)}[u^+_{\lambda'_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)][u^+_{\mu'_\varsigma}(\vec{p},h')u_{\mu_\varsigma}(\vec{p},h')]\{(\frac{1}{l!}\gamma^{[a'_1}\cdot\cdot\gamma^{a'_l]}C)^{\lambda'_\varsigma\mu'_\varsigma}\eta^{a_1}_{a'_1}\cdot\cdot\eta^{a_l}_{a'_l}(\frac{1}{l!}C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}\}$

$= 4^{-[\frac{n}{2}]}2^{1-\delta_{hh'}}i^{l(l+1)}[u^+_{\lambda'_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)][u^+_{\mu'_\varsigma}(\vec{p},h')u_{\mu_\varsigma}(\vec{p},h')]\{(\frac{1}{l!}\gamma_0\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C\gamma_0)^{\lambda'_\varsigma\mu'_\varsigma}(\frac{1}{l!}C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}\}$

$= 4^{-[\frac{n}{2}]}2^{1-\delta_{hh'}}i^{l(l+1)}[\bar{u}_{\lambda'_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)][\bar{u}_{\mu'_\varsigma}(\vec{p},h')u_{\mu_\varsigma}(\vec{p},h')]\{(\frac{1}{l!}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C)^{\lambda'_\varsigma\mu'_\varsigma}(\frac{1}{l!}C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma}\}$ □

9.6 Orthogonality of Dirac matrix in N+1 dimensional space-time?

Proof: $(\frac{1}{l!}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C)^{\lambda'_\varsigma\mu'_\varsigma}(\frac{1}{l!}C^+\gamma_{[a_1}\cdot\cdot\gamma_{a_l]})^{\mu_\varsigma\lambda_\varsigma} = ???\delta^{\lambda_\varsigma\lambda'_\varsigma}\delta^{\mu_\varsigma\mu'_\varsigma}+\cdots$ □

10 Special antisymmetric tensor field in four dimensional space-time

10.1 Summary of relevant properties

Cor. 10.1.1. $v(\vec{p},h) = -\gamma_5 u(\vec{p},h), u(\vec{p},h) = -\gamma_5 v(\vec{p},h)$

Pro. 10.1.1. $\begin{cases}u^T(\vec{p},h)C^+u(\vec{p},h')=0, u^T(\vec{p},h)C^+v(\vec{p},h)=0\\ u^T(\vec{p},\frac{1}{2})C^+v(\vec{p},-\frac{1}{2})=-\varsigma, u^T(\vec{p},-\frac{1}{2})C^+v(\vec{p},\frac{1}{2})=\varsigma\end{cases}$

Pro. 10.1.2. $\begin{cases}u^T(\vec{p},h)C^+\gamma_5 u(\vec{p},h)=0, u^T(\vec{p},h)C^+\gamma_5 v(\vec{p},h')=0\\ u^T(\vec{p},-\frac{1}{2})C^+\gamma_5 u(\vec{p},\frac{1}{2})=-1, u^T(\vec{p},\frac{1}{2})C^+\gamma_5 u(\vec{p},-\frac{1}{2})=1\end{cases}$

Pro. 10.1.3. $\begin{cases}u^T(\hat{p},\frac{\kappa}{2})\bar{C}\gamma_a u(\hat{p},\frac{\kappa}{2})=i\sqrt{2}\varepsilon_a(\vec{p},\kappa), u^T(\hat{p},\frac{\kappa}{2})\bar{C}\gamma_a u(\hat{p},-\frac{\kappa}{2})=i\varepsilon_a(\vec{p},0)\\ u^T(\vec{p},\frac{\kappa}{2})C^+\gamma_a v(\vec{p},\frac{\kappa}{2})=0, u^T(\vec{p},\frac{1}{2})C^+\gamma_a v(\vec{p},-\frac{1}{2})=\frac{-i\varsigma p_a}{m}, u^T(\vec{p},-\frac{1}{2})C^+\gamma_a v(\vec{p},\frac{1}{2})=\frac{i\varsigma p_a}{m}\end{cases}$

10.2 Plane wave solutions of special antisymmetric tensor field in 4D

Lem. 10.2.1.

$\begin{cases}(\gamma^a\partial_a+m)X(4)=0\\ X(4)=\{\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}+\frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]}\}C\end{cases} \Leftrightarrow \begin{cases}\frac{1}{2!}\partial^{[a}F^{bc]}+mF^{abc}=0, \partial_a F^{ab}=0\\ \partial^{[a}F^{bcd]}=0, \partial_a F^{abc}+mF^{bc}=0\end{cases}$

$\Leftrightarrow \begin{cases}\frac{1}{1!}\partial^{[a}{*F}^{b]}+m{*F}^{ab}=0\\ \partial_a{*F}^{ab}+m{*F}^b=0\end{cases}$

Lem. 10.2.2.

$\begin{cases}(\gamma^a\partial_a+m)\psi(x)=0\\ tr\{C^+\psi(x)\}=0, tr\{C^+\gamma_a\psi(x)\}=0, tr\{C^+\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\psi(x)\}=0\\ \psi(x)=\{\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}+\frac{1}{(3!)^2}F^{abc}\gamma_{[a}\gamma_b\gamma_{c]}\}C\end{cases} \Leftrightarrow \begin{cases}\frac{1}{2!}\partial^{[a}F^{bc]}+mF^{abc}=0, \partial_a F^{ab}=0\\ \partial^{[a}F^{bcd]}=0, \partial_a F^{abc}+mF^{bc}=0\end{cases}$

Proof:

$\begin{cases}(\gamma^a\partial_a+m)\psi(x)=0\\ tr\{C^+\psi(x)\}=0, tr\{C^+\gamma_a\psi(x)\}=0, tr\{C^+\gamma_{[a}\gamma_b\gamma_c\gamma_{d]}\psi(x)\}=0\end{cases}$

$\Leftrightarrow$

$$\begin{cases}\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int\sqrt{\frac{m}{E}}\sum_{h=1/2}^{-1/2}[a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)e^{ip\cdot x}+b^+_{\mu_\varsigma}(\vec{p},h)v_{\lambda_\varsigma}(\vec{p},h)e^{-ip\cdot x}]d^N\vec{p}\\ \sum_{h=1/2}^{-1/2}C^{+\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0,\sum_{h=1/2}^{-1/2}(C^+\gamma_a)^{\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\\ \sum_{h=1/2}^{-1/2}(C^+\gamma_{[a}\gamma_b\gamma_c\gamma_{d]})^{\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\end{cases}$$

□

Proof:

$$\begin{cases}\sum_{h=1/2}^{-1/2}C^{+\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\\ \sum_{h=1/2}^{-1/2}(C^+\gamma_a)^{\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\\ \sum_{h=1/2}^{-1/2}(C^+\gamma_{[a}\gamma_b\gamma_c\gamma_{d]})^{\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\end{cases}\Leftrightarrow\begin{cases}\sum_{h=1/2}^{-1/2}C^{+\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\\ \sum_{h=1/2}^{-1/2}(C^+\gamma_a)^{\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\\ \sum_{h=1/2}^{-1/2}(C^+\gamma_5)^{\mu_\varsigma\lambda_\varsigma}a_{\mu_\varsigma}(\vec{p},h)u_{\lambda_\varsigma}(\vec{p},h)=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h,h'=1/2}^{-1/2}C^{+\mu_\varsigma\lambda_\varsigma}[a(\vec{p};h,h')u_{\mu_\varsigma}(\vec{p},h')+c(\vec{p};h,h')v_{\mu_\varsigma}(\vec{p},h')]u_{\lambda_\varsigma}(\vec{p},h)=0\\ \sum_{h,h'=1/2}^{-1/2}(C^+\gamma_a)^{\mu_\varsigma\lambda_\varsigma}[a(\vec{p};h,h')u_{\mu_\varsigma}(\vec{p},h')+c(\vec{p};h,h')v_{\mu_\varsigma}(\vec{p},h')]u_{\lambda_\varsigma}(\vec{p},h)=0\\ \sum_{h,h'=1/2}^{-1/2}(C^+\gamma_5)^{\mu_\varsigma\lambda_\varsigma}[a(\vec{p};h,h')u_{\mu_\varsigma}(\vec{p},h')+c(\vec{p};h,h')v_{\mu_\varsigma}(\vec{p},h')]u_{\lambda_\varsigma}(\vec{p},h)=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h,h'=1/2}^{-1/2}C^{+\mu_\varsigma\lambda_\varsigma}[a(\vec{p};h,h')u_{[\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma]}(\vec{p},h')+c(\vec{p};h,h')u_{[\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma]}(\vec{p},h')]=0\\ \sum_{h,h'=1/2}^{-1/2}(C^+\gamma_a)^{\mu_\varsigma\lambda_\varsigma}[a(\vec{p};h,h')u_{\{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma\}}(\vec{p},h')+c(\vec{p};h,h')u_{\{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma\}}(\vec{p},h')]=0\\ \sum_{h,h'=1/2}^{-1/2}(C^+\gamma_5)^{\mu_\varsigma\lambda_\varsigma}[a(\vec{p};h,h')u_{[\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma]}(\vec{p},h')+c(\vec{p};h,h')u_{[\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma]}(\vec{p},h')]=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h,h'=1/2}^{-1/2}C^{+\lambda_\varsigma\mu_\varsigma}[a(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')+c(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma}(\vec{p},h')]=0\\ \sum_{h,h'=1/2}^{-1/2}(C^+\gamma_a)^{\lambda_\varsigma\mu_\varsigma}[a(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')+c(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma}(\vec{p},h')]=0\\ \sum_{h,h'=1/2}^{-1/2}(C^+\gamma_5)^{\lambda_\varsigma\mu_\varsigma}[a(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')+c(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma}(\vec{p},h')]=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\sum_{h,h'=1/2}^{-1/2}[a(\vec{p};h,h')u^T(\vec{p},h)C^+u(\vec{p},h')+c(\vec{p};h,h')u^T(\vec{p},h)C^+v(\vec{p},h')]=0\\ \sum_{h,h'=1/2}^{-1/2}[a(\vec{p};h,h')u^T(\vec{p},h)C^+\gamma_a u(\vec{p},h')+c(\vec{p};h,h')u^T(\vec{p},h)C^+\gamma_a v(\vec{p},h')]=0\\ \sum_{h,h'=1/2}^{-1/2}[a(\vec{p};h,h')u^T(\vec{p},h)C^+\gamma_5 u(\vec{p},h')+c(\vec{p};h,h')u^T(\vec{p},h)C^+\gamma_5 v(\vec{p},h')]=0\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}c(\vec{p};\frac{1}{2},-\frac{1}{2})=c(\vec{p};-\frac{1}{2},\frac{1}{2})\\ i\sqrt{2}\varepsilon_a(\vec{p},1)a(\vec{p};\frac{1}{2},\frac{1}{2})+i\sqrt{2}\varepsilon_a(\vec{p},-1)a(\vec{p};-\frac{1}{2},-\frac{1}{2})+i\varepsilon_a(\vec{p},0)[a(\vec{p};\frac{1}{2},-\frac{1}{2})+a(\vec{p};-\frac{1}{2},\frac{1}{2})]=0\\ a(\vec{p};\frac{1}{2},-\frac{1}{2})=a(\vec{p};-\frac{1}{2},\frac{1}{2})\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}c(\vec{p};\frac{1}{2},-\frac{1}{2})=c(\vec{p};-\frac{1}{2},\frac{1}{2})\\ a(\vec{p};h,h')=0\end{cases}$$

$\Rightarrow$

$$\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\sqrt{\frac{m}{E}}\sum_{h,h'=1/2}^{-1/2}[a(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)u_{\mu_\varsigma}(\vec{p},h')+c(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma}(\vec{p},h')]e^{ip\cdot x}+\cdots$$

$\Rightarrow$

$$\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\sqrt{\frac{m}{E}}\sum_{h,h'=1/2}^{-1/2}c(\vec{p};h,h')u_{\lambda_\varsigma}(\vec{p},h)v_{\mu_\varsigma}(\vec{p},h')e^{ip\cdot x}+\cdots$$

$\Rightarrow$

$\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\sqrt{\frac{m}{E}}e^{ip\cdot x}$
$\{c(\vec{p};\frac{1}{2},\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})+c(\vec{p};-\frac{1}{2},-\frac{1}{2})u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})$
$+c(\vec{p};\frac{1}{2},-\frac{1}{2})[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})]\}+\cdots$
$\Rightarrow$
$\psi_{\lambda_\varsigma\mu_\varsigma}(x)=\frac{1}{(2\pi)^{N/2}}\int d^N\vec{p}\sqrt{\frac{m}{E}}e^{ip\cdot x}$
$\{c(\vec{p};1)u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})+c(\vec{p};-1)u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+c(\vec{p};0)\frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})]\}+\cdots$
□

10.3 B-W quasi projection operator of special antisymmetric tensor field in 4D

Proof:
$u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})$
$+\frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})+u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v_{\mu_\varsigma}(\vec{p},\frac{1}{2})]\frac{1}{\sqrt{2}}[u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})+u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})]$
$=[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})]+[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})]$
$+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})]+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$=\frac{1}{2}\sum_{h=1/2}^{-1/2}[u_{\lambda_\varsigma}(\vec{p},h)u^+_{\lambda'_\varsigma}(\vec{p},h)]\sum_{h'=1/2}^{-1/2}[v_{\mu_\varsigma}(\vec{p},h')v^+_{\mu'_\varsigma}(\vec{p},h')]$
$+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})]+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})]+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})]$
$=\frac{1}{2}\sum_{h=1/2}^{-1/2}[u_{\lambda_\varsigma}(\vec{p},h)u^+_{\lambda'_\varsigma}(\vec{p},h)]\sum_{h'=1/2}^{-1/2}[v_{\mu_\varsigma}(\vec{p},h')v^+_{\mu'_\varsigma}(\vec{p},h')]$
$+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})]+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]$
$+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},-\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},-\frac{1}{2})]+\frac{1}{2}[u_{\lambda_\varsigma}(\vec{p},-\frac{1}{2})v^+_{\mu'_\varsigma}(\vec{p},-\frac{1}{2})][v_{\mu_\varsigma}(\vec{p},\frac{1}{2})u^+_{\lambda'_\varsigma}(\vec{p},\frac{1}{2})]$
$=\frac{1}{2}\sum_{h=1/2}^{-1/2}[u_{\lambda_\varsigma}(\vec{p},h)u^+_{\lambda'_\varsigma}(\vec{p},h)]\sum_{h'=1/2}^{-1/2}[v_{\mu_\varsigma}(\vec{p},h')v^+_{\mu'_\varsigma}(\vec{p},h')]+\frac{1}{2}\sum_{h=1/2}^{-1/2}[u_{\lambda_\varsigma}(\vec{p},h)v^+_{\mu'_\varsigma}(\vec{p},h)]\sum_{h'=1/2}^{-1/2}[v_{\mu_\varsigma}(\vec{p},h')u^+_{\lambda'_\varsigma}(\vec{p},h')]$
$=\frac{1}{2}\sum_{h=1/2}^{-1/2}[u_{\lambda_\varsigma}(\vec{p},h)u^+_{\lambda'_\varsigma}(\vec{p},h)]\sum_{h'=1/2}^{-1/2}[v_{\mu_\varsigma}(\vec{p},h')v^+_{\mu'_\varsigma}(\vec{p},h')]$
$+\frac{1}{2}\sum_{h=1/2}^{-1/2}[u_{\lambda_\varsigma}(\vec{p},h)[u^+(\vec{p},h)\gamma_5]_{\mu'_\varsigma}]\sum_{h'=1/2}^{-1/2}[v_{\mu_\varsigma}(\vec{p},h')[v^+(\vec{p},h')\gamma_5]_{\lambda'_\varsigma}]$
$=\frac{1}{2}\frac{[(m-i\gamma^a p_a)\gamma_0]_{\lambda_\varsigma\lambda'_\varsigma}}{2m}\frac{[(-m-i\gamma^a p_a)\gamma_0]_{\mu_\varsigma\mu'_\varsigma}}{2m}+\frac{1}{2}\frac{[(m-i\gamma^a p_a)\gamma_0\gamma_5]_{\lambda_\varsigma\mu'_\varsigma}}{2m}\frac{[(-m-i\gamma^a p_a)\gamma_0\gamma_5]_{\mu_\varsigma\lambda'_\varsigma}}{2m}$ □

10.4 Potential commutation rules for special antisymmetric tensor field in 4D

Thm. 10.4.1. $[F_{a_1a_2}(x),F^+_{a'_1a'_2}(x')]=\frac{i}{2^2}\{\frac{1}{2!}\eta^{[a_1}_{[a'_1}\eta^{a_2]}_{a'_2]}m^2-\frac{1}{1!}\eta^{[a_1}_{[a'_1}\partial^{a_2]}\partial^+_{a'_2]}\}\Delta(x-x')=\frac{i}{2^2}\frac{1}{3!}\eta^{[a_1}_{[a'_1}\eta^{a_2}_{a'_2}\eta^{a]}_{b]}\partial_a\partial^{+b}\Delta(x-x')$

Proof: $[F_{a_1a_2}(x),F^+_{a'_1a'_2}(x')]$
$=\frac{2^{-4}}{(2!)^2}(C^+\gamma_{[a_1}\gamma_{a_2]})^{\mu\lambda}(C^+\gamma_{[a'_1}\gamma_{a'_2]})^{*\mu'\lambda'}[\psi_{\lambda\mu}(x),\psi^+_{\lambda'\mu'}(x')]$
$=\frac{2^{-4}}{(2!)^2}(C^+\gamma_{[a_1}\gamma_{a_2]})^{\lambda\mu}(C^+\gamma_{[a'_1}\gamma_{a'_2]})^{+\lambda'\mu'}[\psi_{\lambda\mu}(x),\psi^+_{\lambda'\mu'}(x')]$
$=\frac{2^{-4}}{(2!)^2}(C^+\gamma_{[a_1}\gamma_{a_2]})^{\lambda\mu}(\gamma_{[a'_2}\gamma_{a'_1]}C)^{\lambda'\mu'}[\psi_{\lambda\mu}(x),\psi^+_{\lambda'\mu'}(x')]$
$=-\frac{2^{-4}}{(2!)^2}(C^+\gamma_{[a_1}\gamma_{a_2]})^{\lambda\mu}(\gamma_{[a'_1}\gamma_{a'_2]}C)^{\lambda'\mu'}[\psi_{\lambda\mu}(x),\psi^+_{\lambda'\mu'}(x')]$
$=-\frac{2^{-4}}{(2!)^2}(C^+\gamma_{[a_1}\gamma_{a_2]})^{\lambda\mu}(\gamma_{[a'_1}\gamma_{a'_2]}C)^{\lambda'\mu'}$
$\frac{i}{2^2}\{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}[(-m-\gamma^b\partial_b)\gamma^0]_{\mu\mu'}+[(m-\gamma^a\partial_a)\gamma^0\gamma^5]_{\mu\lambda'}[(-m-\gamma^b\partial_b)\gamma^0\gamma^5]_{\lambda\mu'}\}\Delta(x-x')$
$=-i\frac{2^{-6}}{(2!)^2}(C^+\gamma_{[a_1}\gamma_{a_2]})^{\lambda\mu}(\gamma_{[a'_1}\gamma_{a'_2]}C)^{\lambda'\mu'}$
$\{[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}[(-m-\gamma^b\partial_b)\gamma^0]_{\mu\mu'}+[(m-\gamma^a\partial_a)\gamma^0\gamma^5]_{\lambda\lambda'}[(-m-\gamma^b\partial_b)\gamma^0\gamma^5]_{\mu\mu'}\}\Delta(x-x')$
$=-i\frac{2^{-6}}{(2!)^2}\{(C^+\gamma_{[a_1}\gamma_{a_2]})^{\lambda\mu}[(m-\gamma^a\partial_a)\gamma^0]_{\lambda\lambda'}(\gamma_{[a'_1}\gamma_{a'_2]}C)^{\lambda'\mu'}[(-m-\gamma^b\partial_b)\gamma^0]^T_{\mu'\mu}\Delta(x-x')$
$+(C^+\gamma_{[a_1}\gamma_{a_2]})^{\lambda\mu}[(m-\gamma^a\partial_a)\gamma^0\gamma^5]_{\lambda\lambda'}(\gamma_{[a'_1}\gamma_{a'_2]}C)^{\lambda'\mu'}[(-m-\gamma^b\partial_b)\gamma^0\gamma^5]^T_{\mu'\mu}\Delta(x-x')\}$
$=-i\frac{2^{-6}}{(2!)^2}tr\{(C^+\gamma_{[a_1}\gamma_{a_2]})[(m-\gamma^a\partial_a)\gamma^0](\gamma_{[a'_1}\gamma_{a'_2]}C)[(-m-\gamma^b\partial_b)\gamma^0]^T$
$+(C^+\gamma_{[a_1}\gamma_{a_2]})[(m-\gamma^a\partial_a)\gamma^0\gamma^5](\gamma_{[a'_1}\gamma_{a'_2]}C)[(-m-\gamma^b\partial_b)\gamma^0\gamma^5]^T\}\Delta(x-x')$
$=-i\frac{2^{-6}}{(2!)^2}tr\{\gamma_{[a_1}\gamma_{a_2]}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{[a'_1}\gamma_{a'_2]}C[(-m-\gamma^b\partial_b)\gamma^0]^TC^+$
$+\gamma_{[a_1}\gamma_{a_2]}[(m-\gamma^a\partial_a)\gamma^0\gamma^5]\gamma_{[a'_1}\gamma_{a'_2]}C[(-m-\gamma^b\partial_b)\gamma^0\gamma^5]^TC^+\}\Delta(x-x')$
$=i\frac{2^{-6}}{(2!)^2}tr\{\gamma_{[a_1}\gamma_{a_2]}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{[a'_1}\gamma_{a'_2]}[\gamma^0(-m+\gamma^b\partial_b)]+\gamma_{[a_1}\gamma_{a_2]}[(m-\gamma^a\partial_a)\gamma^0\gamma^5]\gamma_{[a'_1}\gamma_{a'_2]}[\gamma^5\gamma^0(-m+\gamma^b\partial_b)]\}\Delta(x-x')$
$=i\frac{2^{-5}}{(2!)^2}tr\{\gamma_{[a_1}\gamma_{a_2]}[(m-\gamma^a\partial_a)\gamma^0]\gamma_{[a'_1}\gamma_{a'_2]}[(-m-\gamma^b\partial^+_b)\gamma^0]\}\Delta(x-x')$

$= i\frac{2^{-5}}{(2!)^2}tr\{-m^2(\gamma_{[a_1}\gamma_{a_2]})\gamma^0(\gamma_{[a_1'}\gamma_{a_2']})\gamma^0\}\Delta(x-x') + i\frac{2^{-5}}{(2!)^2}tr\{(\gamma_{[a_1}\gamma_{a_2]})\gamma_a\gamma_0(\gamma_{[a_1'}\gamma_{a_2']})\gamma_b\gamma_0\}\partial^a\partial^b\Delta(x-x')$
$= -i\frac{2^{-5}}{(2!)^2}i^{2*3}2^2(2!)^2m^2\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a_2]}_{a_2']}\Delta(x-x')$
$+ i\frac{2^{-5}}{(2!)^2}i^{3*4}2^2(2!)^2\{\frac{1}{3!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\eta^{a]}_{b]} - \frac{1}{1!}\eta^{[a_1}_{[a_1'}\delta^{a_2]a}\delta_{a_2']b}\}\partial_a\partial^{+b}\Delta(x-x')$
$= \frac{i}{2^3}\{\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a_2]}_{a_2']}m^2 + (\frac{1}{3!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\eta^{a]}_{b]} - \frac{1}{1!}\eta^{[a_1}_{[a_1'}\delta^{a_2]a}\delta_{a_2']b})\partial_a\partial^{+b}\}\Delta(x-x')$
$= \frac{i}{2^2}\{\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a_2]}_{a_2']}m^2 - \frac{1}{1!}\eta^{[a_1}_{[a_1'}\partial^{a_2]}\partial^{+}_{a_2']}\}\Delta(x-x')$
$= \frac{i}{2^2}\frac{1}{3!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\eta^{a]}_{b]}\partial_a\partial^{+b}\Delta(x-x')$ □

Cor. 10.4.1. $[F^{a_1a_2a_3}(x), F^{+}_{a_1'a_2'a_3'}(x')] = -\frac{i}{2^2}\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\partial^{a_3]}\partial^{+}_{a_3']}\Delta(x-x')$

Thm. 10.4.2.
$$\begin{cases}[F^{a_1a_2a_3}(x), F^{+}_{a_1'a_2'a_3'}(x')] = -\frac{i}{2^2}\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\partial^{a_3]}\partial^{+}_{a_3']}\Delta(x-x') \Leftrightarrow [*F_{a_0}, *F^{+}_{a_0'}] = \frac{i}{2^2}\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a]}_{a']}\partial_a\partial^{+a'}\Delta(x-x') \\ [F_{a_1a_2}(x), F^{+}_{a_1'a_2'}(x')] = \frac{i}{2^2}\frac{1}{3!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\eta^{a]}_{b]}\partial_a\partial^{+b}\Delta(x-x') \Leftrightarrow [*F_{a_1a_2}(x), *F^{+}_{a_1'a_2'}(x')] = -\frac{i}{2^2}\eta_{[a_1[a_1'}\partial_{a_2]}\partial^{+}_{a_2']}\Delta(x-x')\end{cases}$$

Proof: $[\frac{1}{3!}\varepsilon_{a_0a_1a_2a_3}F^{a_1a_2a_3}(x), \frac{1}{3!}\varepsilon^{a_0'a_1'a_2'a_3'}F^{+}_{a_1'a_2'a_3'}(x')]$
$= -\frac{1}{(3!)^2}\varepsilon_{a_0a_1a_2a_3}\varepsilon^{a_0'a_1'a_2'a_3'}\frac{i}{2^2}\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a_2}_{a_2'}\partial^{a_3]}\partial^{+}_{a_3']}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}\varepsilon_{a_0a_1a_2a_3}\varepsilon^{a_0'a_1'a_2'a_3'}\eta^{a_1}_{a_1'}\eta^{a_2}_{a_2'}\partial^{a_3}\partial^{+}_{a_3'}$
$= -\frac{i}{2^2}\frac{1}{2!}\delta^{a_0'}_{[a_0}\delta^{a_1'}_{a_1}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_3]}\eta^{a_1}_{a_1'}\eta^{a_2}_{a_2'}\partial^{a_3}\partial^{+}_{a_3'}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}(\delta^{a_0'}_{a_0}\delta^{a_1'}_{[a_1}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_3]} - \delta^{a_0'}_{a_1}\delta^{a_1'}_{[a_0}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_3]} + \delta^{a_0'}_{a_2}\delta^{a_1'}_{[a_0}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_3]} - \delta^{a_0'}_{a_3}\delta^{a_1'}_{[a_0}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_2]})\eta^{a_1}_{a_1'}\eta^{a_2}_{a_2'}\partial^{a_3}\partial^{+}_{a_3'}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}\{\delta^{a_0'}_{a_0}(\delta^{a_1'}_{a_1}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_3} - \delta^{a_1'}_{a_1}\delta^{a_2'}_{a_3}\delta^{a_3'}_{a_2} + \delta^{a_1'}_{a_2}\delta^{a_2'}_{a_3}\delta^{a_3'}_{a_1} - \delta^{a_1'}_{a_2}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_3} + \delta^{a_1'}_{a_3}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_2} - \delta^{a_1'}_{a_3}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_1})\eta^{a_1}_{a_1'}\eta^{a_2}_{a_2'}\partial^{a_3}\partial^{+}_{a_3'}$
$- \delta^{a_0'}_{a_1}(\delta^{a_1'}_{a_0}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_3} - \delta^{a_1'}_{a_0}\delta^{a_2'}_{a_3}\delta^{a_3'}_{a_2} + \delta^{a_1'}_{a_2}\delta^{a_2'}_{a_3}\delta^{a_3'}_{a_0} - \delta^{a_1'}_{a_2}\delta^{a_2'}_{a_0}\delta^{a_3'}_{a_3} + \delta^{a_1'}_{a_3}\delta^{a_2'}_{a_0}\delta^{a_3'}_{a_2} - \delta^{a_1'}_{a_3}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_0})\eta^{a_1}_{a_1'}\eta^{a_2}_{a_2'}\partial^{a_3}\partial^{+}_{a_3'}$
$- \delta^{a_0'}_{a_2}(\delta^{a_1'}_{a_1}\delta^{a_2'}_{a_0}\delta^{a_3'}_{a_3} - \delta^{a_1'}_{a_1}\delta^{a_2'}_{a_3}\delta^{a_3'}_{a_0} + \delta^{a_1'}_{a_0}\delta^{a_2'}_{a_3}\delta^{a_3'}_{a_1} - \delta^{a_1'}_{a_0}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_3} + \delta^{a_1'}_{a_3}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_0} - \delta^{a_1'}_{a_3}\delta^{a_2'}_{a_0}\delta^{a_3'}_{a_1})\eta^{a_1}_{a_1'}\eta^{a_2}_{a_2'}\partial^{a_3}\partial^{+}_{a_3'}$
$- \delta^{a_0'}_{a_3}(\delta^{a_1'}_{a_1}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_0} - \delta^{a_1'}_{a_1}\delta^{a_2'}_{a_0}\delta^{a_3'}_{a_2} + \delta^{a_1'}_{a_2}\delta^{a_2'}_{a_0}\delta^{a_3'}_{a_1} - \delta^{a_1'}_{a_2}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_0} + \delta^{a_1'}_{a_0}\delta^{a_2'}_{a_1}\delta^{a_3'}_{a_2} - \delta^{a_1'}_{a_0}\delta^{a_2'}_{a_2}\delta^{a_3'}_{a_1})\eta^{a_1}_{a_1'}\eta^{a_2}_{a_2'}\partial^{a_3}\partial^{+}_{a_3'}\}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}\{\delta^{a_0'}_{a_0}(\eta^{a_1}_{a_1}\eta^{a_2}_{a_2}\partial^{a_3}\partial^{+}_{a_3} - \eta^{a_1}_{a_1}\eta^{a_2}_{a_3}\partial^{a_3}\partial^{+}_{a_2} + \eta^{a_1}_{a_2}\eta^{a_2}_{a_3}\partial^{a_3}\partial^{+}_{a_1} - \eta^{a_1}_{a_2}\eta^{a_2}_{a_1}\partial^{a_3}\partial^{+}_{a_3} + \eta^{a_1}_{a_3}\eta^{a_2}_{a_1}\partial^{a_3}\partial^{+}_{a_2} - \eta^{a_1}_{a_3}\eta^{a_2}_{a_2}\partial^{a_3}\partial^{+}_{a_1})$
$- \delta^{a_0'}_{a_1}(\eta^{a_1}_{a_0}\eta^{a_2}_{a_2}\partial^{a_3}\partial^{+}_{a_3} - \eta^{a_1}_{a_0}\eta^{a_2}_{a_3}\partial^{a_3}\partial^{+}_{a_2} + \eta^{a_1}_{a_2}\eta^{a_2}_{a_3}\partial^{a_3}\partial^{+}_{a_0} - \eta^{a_1}_{a_2}\eta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_3} + \eta^{a_1}_{a_3}\eta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_2} - \eta^{a_1}_{a_3}\eta^{a_2}_{a_2}\partial^{a_3}\partial^{+}_{a_0})$
$- \delta^{a_0'}_{a_2}(\eta^{a_1}_{a_1}\eta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_3} - \eta^{a_1}_{a_1}\eta^{a_2}_{a_3}\partial^{a_3}\partial^{+}_{a_0} + \eta^{a_1}_{a_0}\eta^{a_2}_{a_3}\partial^{a_3}\partial^{+}_{a_1} - \eta^{a_1}_{a_0}\eta^{a_2}_{a_1}\partial^{a_3}\partial^{+}_{a_3} + \eta^{a_1}_{a_3}\eta^{a_2}_{a_1}\partial^{a_3}\partial^{+}_{a_0} - \eta^{a_1}_{a_3}\eta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_1})$
$- \delta^{a_0'}_{a_3}(\eta^{a_1}_{a_1}\eta^{a_2}_{a_2}\partial^{a_3}\partial^{+}_{a_0} - \eta^{a_1}_{a_1}\eta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_2} + \eta^{a_1}_{a_2}\eta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_1} - \eta^{a_1}_{a_2}\eta^{a_2}_{a_1}\partial^{a_3}\partial^{+}_{a_0} + \eta^{a_1}_{a_0}\eta^{a_2}_{a_1}\partial^{a_3}\partial^{+}_{a_2} - \eta^{a_1}_{a_0}\eta^{a_2}_{a_2}\partial^{a_3}\partial^{+}_{a_1})\}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}\{\delta^{a_0'}_{a_0}(4\partial^{a_3}\partial^{+}_{a_3} - 2\partial^{a_3}\partial_{a_3} + \partial^{a_3}\partial^{+}_{a_3} - 4\partial^{a_3}\partial^{+}_{a_3} + \partial^{a_3}\partial^{+}_{a_3} - 2\partial^{a_3}\partial_{a_3})$
$- \delta^{a_0'}_{a_1}(2\eta^{a_1}_{a_0}\partial^{a_3}\partial^{+}_{a_3} - \eta^{a_1}_{a_0}\partial^{a_3}\partial_{a_3} + \partial^{a_1}\partial^{+}_{a_0} - \delta^{a_1}_{a_0}\partial^{a_3}\partial^{+}_{a_3} + \partial^{+a_1}\partial_{a_0} - 2\partial^{+a_1}\partial^{+}_{a_0})$
$- \delta^{a_0'}_{a_2}(2\eta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_3} - 2\partial^{+a_2}\partial^{+}_{a_0} + \partial^{+a_2}\partial_{a_0} - \delta^{a_2}_{a_0}\partial^{a_3}\partial^{+}_{a_3} + \partial^{a_2}\partial^{+}_{a_0} - \eta^{a_2}_{a_0}\partial^{+a_1}\partial^{+}_{a_1})$
$- \delta^{a_0'}_{a_3}(4\partial^{a_3}\partial^{+}_{a_0} - 2\partial^{a_3}\partial_{a_0} + \partial^{a_3}\partial^{+}_{a_0} - 4\partial^{a_3}\partial^{+}_{a_0} + \partial^{a_3}\partial^{+}_{a_0} - 2\partial^{a_3}\partial_{a_0})\}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}\{\delta^{a_0'}_{a_0}(4\partial^{a_3}\partial^{+}_{a_3} - 2\partial^{a_3}\partial_{a_3} + \partial^{a_3}\partial^{+}_{a_3} - 4\partial^{a_3}\partial^{+}_{a_3} + \partial^{a_3}\partial^{+}_{a_3} - 2\partial^{a_3}\partial_{a_3})$
$- (2\eta^{a_0'}_{a_0}\partial^{a_3}\partial^{+}_{a_3} - \eta^{a_0'}_{a_0}\partial^{a_3}\partial_{a_3} + \partial^{a_0'}\partial^{+}_{a_0} - \delta^{a_0'}_{a_0}\partial^{a_3}\partial^{+}_{a_3} + \partial^{+a_0'}\partial_{a_0} - 2\partial^{+a_0'}\partial^{+}_{a_0})$
$- (2\eta^{a_0'}_{a_0}\partial^{a_3}\partial^{+}_{a_3} - 2\partial^{+a_0'}\partial^{+}_{a_0} + \partial^{+a_0'}\partial_{a_0} - \delta^{a_0'}_{a_0}\partial^{a_3}\partial^{+}_{a_3} + \partial^{a_0'}\partial^{+}_{a_0} - \eta^{a_0'}_{a_0}\partial^{+a_1}\partial^{+}_{a_1})$
$- (4\partial^{a_0'}\partial^{+}_{a_0} - 2\partial^{a_0'}\partial_{a_0} + \partial^{a_0'}\partial^{+}_{a_0} - 4\partial^{a_0'}\partial^{+}_{a_0} + \partial^{a_0'}\partial^{+}_{a_0} - 2\partial^{a_0'}\partial_{a_0})\}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}\{\delta^{a_0'}_{a_0}(4\partial^{a_3}\partial^{+}_{a_3} - 4\partial^{a_3}\partial_{a_3}) - 2(2\eta^{a_0'}_{a_0}\partial^{a_3}\partial^{+}_{a_3} - \eta^{a_0'}_{a_0}\partial^{a_3}\partial_{a_3} + \partial^{a_0'}\partial^{+}_{a_0} + \partial^{+a_0'}\partial_{a_0} - 2\partial^{+a_0'}\partial^{+}_{a_0})$
$- (2\partial^{a_0'}\partial^{+}_{a_0} - 4\partial^{a_0'}\partial_{a_0})\}\Delta(x-x')$
$= -\frac{i}{2^2}\frac{1}{2!}\{-8\delta^{a_0'}_{a_0}\partial^2_\pi - 2[\eta^{a_0'}_{a_0}(\nabla^2 - 3\partial^2_\pi) + \partial^{a_0'}\partial^{+}_{a_0} + \partial^{+a_0'}\partial_{a_0} - 2\partial^{+a_0'}\partial^{+}_{a_0}] - (2\partial^{a_0'}\partial^{+}_{a_0} - 4\partial^{a_0'}\partial_{a_0})\}$
$= \frac{i}{2^2}m^2(\eta^{a_0'}_{a_0} - \frac{\partial_{a_0}\partial^{+a_0'}}{m^2})\Delta(x-x')$
$= \frac{i}{2^2}\frac{1}{2!}\eta^{[a_1}_{[a_1'}\eta^{a]}_{a']}\partial_a\partial^{+a'}\Delta(x-x')$ □

11 Concrete calculations of related properties

11.1 Concrete calculation 1 for special antisymmetric tensor field in 4D

Pro. 11.1.1. $\begin{cases}u^T(\vec{p},h)C^+u(\vec{p},h') = 0, u^T(\vec{p},h)C^+v(\vec{p},h) = 0 \\ u^T(\vec{p},\frac{1}{2})C^+v(\vec{p},-\frac{1}{2}) = -\varsigma, u^T(\vec{p},-\frac{1}{2})C^+v(\vec{p},\frac{1}{2}) = \varsigma\end{cases}$

Proof: $u^T(\vec{p},h)C^+u(\vec{p},h) = 0$
$u^T(\vec{p},\frac{1}{2})C^+u(\vec{p},-\frac{1}{2}) = -u^T(\vec{p},-\frac{1}{2})C^+u(\vec{p},\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix}$

$= \frac{i\varsigma\lambda^T(\hat{p},\frac{1}{2})\sigma_y\lambda(\hat{p},-\frac{1}{2})}{2m^3} \otimes (\begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T \begin{bmatrix} m \\ -\varsigma E+|\vec{p}| \end{bmatrix})$
$= 0$ □

Proof: $u^T(\vec{p},h)C^+v(\vec{p},h)=0$
$u^T(\vec{p},\frac{1}{2})C^+v(\vec{p},-\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y\otimes\sigma_z)\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= \frac{i\varsigma\lambda^T(\hat{p},\frac{1}{2})\sigma_y\lambda(\hat{p},-\frac{1}{2})}{2m^2} \otimes (\begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T \begin{bmatrix} -m \\ -\varsigma E+|\vec{p}| \end{bmatrix})$
$= -i\varsigma\lambda^T(\hat{p},\frac{1}{2})\sigma_y\lambda(\hat{p},-\frac{1}{2})$
$= -\varsigma\lambda^+(\hat{p},-\frac{1}{2})\lambda(\hat{p},-\frac{1}{2})$
$= -\varsigma$ □

Proof:
$u^T(\vec{p},-\frac{1}{2})C^+v(\vec{p},\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E-|\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y\otimes\sigma_z)\frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= \frac{i\varsigma\lambda^T(\hat{p},-\frac{1}{2})\sigma_y\lambda(\hat{p},\frac{1}{2})}{2m^2} \otimes (\begin{bmatrix} m \\ \varsigma E-|\vec{p}| \end{bmatrix}^T \begin{bmatrix} -m \\ -\varsigma E-|\vec{p}| \end{bmatrix})$
$= -i\varsigma\lambda^T(\hat{p},-\frac{1}{2})\sigma_y\lambda(\hat{p},\frac{1}{2})$
$= \varsigma\lambda^+(\hat{p},\frac{1}{2})\sigma_y\lambda(\hat{p},\frac{1}{2})$
$= \varsigma$ □

11.2 Concrete calculation 2 for special antisymmetric tensor field in 4D

Pro. 11.2.1. $\begin{cases} u^T(\vec{p},h)C^+\gamma_5u(\vec{p},h)=0, u^T(\vec{p},h)C^+\gamma_5v(\vec{p},h')=0 \\ u^T(\vec{p},-\frac{1}{2})C^+\gamma_5u(\vec{p},\frac{1}{2})=-1, u^T(\vec{p},\frac{1}{2})C^+\gamma_5u(\vec{p},-\frac{1}{2})=1 \end{cases}$

Proof:
$u^T(\vec{p},\frac{1}{2})C^+\gamma_5v(\vec{p},-\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (i\sigma_y\otimes I)\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= \frac{i\lambda^T(\hat{p},\frac{1}{2})\sigma_y\lambda(\hat{p},-\frac{1}{2})}{2m^2} \otimes (\begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix})$
$= 0$ □

Proof:
$u^T(\vec{p},\frac{1}{2})C^+\gamma_5u(\vec{p},-\frac{1}{2})=-u^T(\vec{p},-\frac{1}{2})C^+\gamma_5u(\vec{p},\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (i\sigma_y\otimes I)\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= \frac{i\lambda^T(\hat{p},\frac{1}{2})\sigma_y\lambda(\hat{p},-\frac{1}{2})}{2m^2} \otimes (\begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T \begin{bmatrix} m \\ \varsigma E-|\vec{p}| \end{bmatrix})$
$= 1$ □

11.3 Concrete calculation 3 for special antisymmetric tensor field in 4D

Pro. 11.3.1. $\begin{cases} u^T(\hat{p},\frac{\kappa}{2})\bar{C}\gamma_a u(\hat{p},\frac{\kappa}{2})=i\sqrt{2}\varepsilon_a(\vec{p},\kappa), u^T(\hat{p},\frac{\kappa}{2})\bar{C}\gamma_a u(\hat{p},-\frac{\kappa}{2})=i\varepsilon_a(\vec{p},0) \\ u^T(\vec{p},\frac{\kappa}{2})C^+\gamma_a v(\vec{p},\frac{\kappa}{2})=0, u^T(\vec{p},\frac{1}{2})C^+\gamma_a v(\vec{p},-\frac{1}{2})=\frac{-i\varsigma p_a}{m}, u^T(\vec{p},-\frac{1}{2})C^+\gamma_a v(\vec{p},\frac{1}{2})=\frac{i\varsigma p_a}{m} \end{cases}$

Proof:
$u^T(\vec{p},\frac{1}{2})C^+\gamma_a v(\vec{p},-\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y\otimes\sigma_z)\gamma_a\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (\varsigma I\otimes\sigma_z)(\sigma_x\otimes\sigma_y,\sigma_y\otimes\sigma_y,\sigma_z\otimes\sigma_y,\varsigma I\otimes\sigma_x)\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (-i\varsigma\sigma\otimes\sigma_x, iI\otimes\sigma_y)\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= (\frac{-i\varsigma\vec{p}}{m},\frac{\varsigma E}{m})$
$= \frac{-i\varsigma p_a}{m}$ □

Proof:
$u^T(\vec{p},-\frac{1}{2})C^{+}\gamma_a v(\vec{p},\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E-|\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_a \frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= -\frac{\lambda^+(\hat{p},\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E-|\vec{p}| \end{bmatrix}^T (\varsigma I \otimes \sigma_z)(\sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_y, \varsigma I \otimes \sigma_x) \frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= -\frac{\lambda^+(\hat{p},\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E-|\vec{p}| \end{bmatrix}^T (-i\varsigma\sigma \otimes \sigma_x, iI \otimes \sigma_y) \frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= (\frac{i\varsigma\vec{p}}{m}, -\frac{\varsigma E}{m})$
$= \frac{i\varsigma p_a}{m}$ □

Proof:
$u^T(\vec{p},\frac{1}{2})C^{+}\gamma_a v(\vec{p},\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_a \frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (\varsigma I \otimes \sigma_z)(\sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_y, \varsigma I \otimes \sigma_x) \frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (-i\varsigma\sigma \otimes \sigma_x, iI \otimes \sigma_y) \frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= 0$ □

11.4 Concrete calculation 4 for special antisymmetric tensor field in 4D

Pro. 11.4.1. $u^T(\hat{p},\frac{\kappa}{2})\bar{C}\gamma_{[a}\gamma_{b]}u(\hat{p},\frac{\kappa}{2}) = i\sqrt{2}p_a\varepsilon_b(\vec{p},\kappa)$

Proof:
$u^T(\vec{p},\frac{1}{2})C^{+}\gamma_{[a}\gamma_{b]}v(\vec{p},-\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_{[a}\gamma_{b]} \frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T$
$(\varsigma I \otimes \sigma_z)(i\sigma_z \otimes I, i\sigma_x \otimes I, i\sigma_y \otimes I, -i\varsigma\sigma_x \otimes \sigma_z, -i\varsigma\sigma_y \otimes \sigma_z, -i\varsigma\sigma_z \otimes \sigma_z)$
$\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T$
$(i\varsigma\sigma_z \otimes \sigma_z, i\varsigma\sigma_x \otimes \sigma_z, i\varsigma\sigma_y \otimes \sigma_z, -i\sigma_x \otimes I, -i\sigma_y \otimes I, -i\sigma_z \otimes I)$
$\frac{\lambda(\hat{p},-\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E-|\vec{p}| \end{bmatrix}$
$= (i\varsigma\hat{p}_z, i\varsigma\hat{p}_x, i\varsigma\hat{p}_y, 0, 0, 0)$ □

Proof:
$u^T(\vec{p},\frac{1}{2})C^{+}\gamma_{[a}\gamma_{b]}v(\vec{p},\frac{1}{2})$
$= \frac{\lambda^T(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_{[a}\gamma_{b]} \frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T$
$(\varsigma I \otimes \sigma_z)(i\sigma_z \otimes I, i\sigma_x \otimes I, i\sigma_y \otimes I, -i\varsigma\sigma_x \otimes \sigma_z, -i\varsigma\sigma_y \otimes \sigma_z, -i\varsigma\sigma_z \otimes \sigma_z)$
$\frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$= \frac{\lambda^+(\hat{p},-\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E+|\vec{p}| \end{bmatrix}^T$
$(i\varsigma\sigma_z \otimes \sigma_z, i\varsigma\sigma_x \otimes \sigma_z, i\varsigma\sigma_y \otimes \sigma_z, -i\sigma_x \otimes I, -i\sigma_y \otimes I, -i\sigma_z \otimes I)$
$\frac{\lambda(\hat{p},\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E+|\vec{p}| \end{bmatrix}$
$=$ □

12 Conjecture on higher order generalization of antisymmetric tensor field in N+1-D

12.1 Conjecture of B-W equation with mass and $s=n$ in N+1 dimensional space-time

Def. 12.1.1. $\mathbb{X}^{a_1\cdot\cdot a_l}:=\{\frac{1}{l!}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}+\frac{(-1)^l}{(l+1)!m}\gamma^{[a_1}\cdot\cdot\gamma^{a_{l+1}]}\partial_{a_{l+1}}\}C\frac{1}{l!}$

Ass. 12.1.1.

$$\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n}}{}^{\sigma}=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n}}{}^{\sigma}=\frac{1}{(2n)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma\}}_{2n}}{}^{\sigma}\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n}}{}^{\sigma}=\frac{1}{(l!)^2}F_{a_1\cdot\cdot a_l\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n-2}}{}^{\sigma}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C+\frac{1}{[(l+1)!]^2}F_{a_1\cdot\cdot a_{l+1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n-2}}{}^{\sigma}\gamma^{[a_1}\cdot\cdot\gamma^{a_{l+1}]}C\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}(-\partial^d\partial_d+m^2)A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}}{}^{\sigma}=0\\ \delta^{a_1b_1}A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}}{}^{\sigma}=0,\partial^{a_1}A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}}{}^{\sigma}=0\\ A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}}{}^{\sigma}\ \textit{is fully symmetric for}\ a_1b_1c_1\cdot\cdot,\ \textit{fully antisymmetric for}\ a_1\cdot\cdot a_l,b_1\cdot\cdot b_l,c_1\cdot\cdot c_l,\cdot\cdot\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\rho_\varsigma\tau_\varsigma\cdot\cdot}_{2n}}{}^{\sigma}=\overbrace{\mathbb{X}^{a_1\cdot\cdot a_l}_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^{b_1\cdot\cdot b_l}_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^{c_1\cdot\cdot c_l}_{\rho_\varsigma\tau_\varsigma}\cdot\cdot}^{n}A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}}{}^{\sigma}\end{cases}$$

12.2 Conjecture of B-W equation with mass and $s=n+\frac{1}{2}$ in N+1 dimensional space-time

Ass. 12.2.1.

$$\begin{cases}[\gamma^a(\varsigma)\partial_a+m]\psi_{\underbrace{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n+1}}{}^{\sigma}=0,\psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n+1}}{}^{\sigma}=\frac{1}{(2n+1)!}\psi_{\underbrace{\{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma\}}_{2n+1}}{}^{\sigma}\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n+1}}{}^{\sigma}=\frac{1}{(l!)^2}F_{a_1\cdot\cdot a_l\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n-1}}{}^{\sigma}\gamma^{[a_1}\cdot\cdot\gamma^{a_l]}C+\frac{1}{[(l+1)!]^2}F_{a_1\cdot\cdot a_{l+1}\underbrace{\eta_\varsigma\xi_\varsigma\cdot\cdot\zeta_\varsigma}_{2n-1}}{}^{\sigma}\gamma^{[a_1}\cdot\cdot\gamma^{a_{l+1}]}C\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}[\gamma^d(\varsigma)\partial_d+m]A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}[\zeta_\varsigma]}{}^{\sigma}=0\\ \delta^{a_1b_1}A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}[\zeta_\varsigma]}{}^{\sigma}=0,\gamma^{a_1}(\varsigma)A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}[\zeta_\varsigma]}{}^{\sigma}=0\\ A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}\zeta_\varsigma}{}^{\sigma}\ \textit{is fully symmetric for}\ a_1b_1c_1\cdot\cdot,\ \textit{fully antisymmetric for}\ a_1\cdot\cdot a_l,b_1\cdot\cdot b_l,c_1\cdot\cdot c_l,\cdot\cdot\\ \psi_{\underbrace{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma\rho_\varsigma\tau_\varsigma\cdot\cdot\zeta_\varsigma}_{2n+1}}{}^{\sigma}=\overbrace{\mathbb{X}^{a_1\cdot\cdot a_l}_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^{b_1\cdot\cdot b_l}_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^{c_1\cdot\cdot c_l}_{\rho_\varsigma\tau_\varsigma}\cdot\cdot}^{n}A_{\underbrace{a_1\cdot\cdot a_l|b_1\cdot\cdot b_l|c_1\cdot\cdot c_l|\cdot\cdot}_{n}\zeta_\varsigma}{}^{\sigma}\end{cases}$$

Self comment: The above two conjectures can be strictly proved by mathematical induction. I will talk about them later when I have time. At the same time, it also reveals that physics is far from complete. Because many interesting physical equations can be constructed, which can be infinite in principle. Therefore the development of physics is endless. And it makes people yearn for it endlessly, but also makes people despair to the extreme!

13 Analysis of electromagnetic field equations in N+1 dimensional space-time

13.1 B-W vector field equation in N+1 dimensional space-time

Def. 13.1.1. $\mathbb{X}_a:=[im\gamma_a-2S_{ab}(e)\partial^b]C,\mathbb{X}^a:=[im\gamma^a-2S^{ab}(e)\partial_b]C$

Lem. 13.1.1. $\begin{cases}(\gamma^a\partial_a+m)X=0\\ X=\{\frac{1}{(1!)^2}F^a\gamma_a+\frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases}\Leftrightarrow\begin{cases}\partial^{[a}F^{b]}+mF^{ab}=0,\partial_aF^a=0\\ \partial^{[a}F^{bc]}=0,\partial_aF^{ab}+mF^b=0\end{cases}$

Cor. 13.1.1. $\begin{cases}(\gamma^a\partial_a+m)\psi=0\\ \psi=\{\frac{1}{(1!)^2}imA^a\gamma_a-\frac{i}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]}\}C\end{cases}\Leftrightarrow\begin{cases}\partial_aF^{ab}-m^2A^b=0\\ F^{ab}=\partial^{[a}A^{b]},\partial_aA^a=0\end{cases}$

Lem. 13.1.2. $\frac{2^{-5}}{im}tr(\bar{C}\gamma_{a'}\mathbb{X}^a)A_a=\frac{2^{-5}}{im}(\bar{C}\gamma_{a'})^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}A_a=A_{a'},(\bar{C}\gamma_{a'})^{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}=im2^5\delta^a_{a'}$

Lem. 13.1.3. $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab}=\mathbb{X}^a_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}A_{ab}\Leftrightarrow A_{ab}=A_{ba}$

Cor. 13.1.2. $(\gamma^b\partial_b+m)\mathbb{X}^aA_a\Leftrightarrow(\partial^b\partial_b-m^2)A_a=0,\partial^aA_a=0$

Proof: $(\gamma^c\partial_c+m)\mathbb{X}^aA_a=0$
$\Leftrightarrow(\gamma^c\partial_c+m)[im\gamma^a-2S^{ab}(e)\partial_b]CA_a=0$
$\Leftrightarrow(\gamma^c\partial_c+m)[im\gamma^a-2S^{ab}(e)\partial_b]A_a=0$

$\Leftrightarrow im\gamma^c\gamma^a\partial_c A_a + im^2\gamma^a A_a + \frac{i}{2}\gamma^c\gamma^{[a}\gamma^{b]}\partial_c\partial_b A_a + \frac{i}{2}m\gamma^{[a}\gamma^{b]}\partial_b A_a = 0$
$\Leftrightarrow -\frac{i}{2}\gamma^a\gamma^{[b}\gamma^{c]}\partial_a\partial_b A_c + im^2\gamma^a A_a + \frac{i}{2}m\gamma^{\{a}\gamma^{b\}}\partial_b A_a = 0$
$\Leftrightarrow -\frac{i}{2}(\frac{1}{3}\gamma_{[a}\gamma_b\gamma_{c]} + 2\delta_{a[b}\gamma_{c]})\partial_a\partial_b A_c + im^2\gamma^a A_a + im\delta^{ab}\partial_b A_a = 0$
$\Leftrightarrow -i\delta_{a[b}\gamma_{c]}\partial_a\partial_b A_c + im^2\gamma^a A_a + im\partial^a A_a = 0$
$\Leftrightarrow -i\gamma^c\partial_a\partial^a A_c + i\gamma^b\partial_b(\partial^a A_a) + im^2\gamma^a A_a + im\partial^a A_a = 0$
$\Leftrightarrow \gamma^a[-i\partial_b\partial^b A_a + i\partial_a(\partial^b A_b) + im^2 A_a] + im\partial^a A_a = 0$
$\Leftrightarrow -i\partial_b\partial^b A_a + i\partial_a(\partial^b A_b) + im^2 A_a = 0, im\partial^a A_a = 0$
$\Leftrightarrow (\partial_b\partial^b - m^2)A_a = 0, \partial^a A_a = 0$ □

Cor. 13.1.3. $(\gamma^c\partial_c + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = 0 \Leftrightarrow (\partial^c\partial_c - m^2)A_{ab} = 0, \partial^a A_{ab} = 0$

Proof: $(\gamma^c\partial_c + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = 0$
$\Leftrightarrow (\gamma^c\partial_c + m)\mathbb{X}^a\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = 0$
$\Leftrightarrow (\partial_c\partial^c - m^2)\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = 0, \partial^a\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = 0$
$\Leftrightarrow (\partial_c\partial^c - m^2)\mathbb{X}^b A_{ab} = 0, \partial^a\mathbb{X}^b A_{ab} = 0$
$\Leftrightarrow (\partial_d\partial^d - m^2)[im\gamma^b - 2S^{bc}(e)\partial_c]CA_{ab} = 0, \partial^a[im\gamma^b - 2S^{bc}(e)\partial_c]CA_{ab} = 0$
$\Leftrightarrow (\partial_d\partial^d - m^2)A_{ab} = 0, (\partial_d\partial^d - m^2)(\partial_c A_{ab} - \partial_b A_{ac}) = 0, \partial^a A_{ab} = 0, \partial^a(\partial_c A_{ab} - \partial_b A_{ac}) = 0$
$\Leftrightarrow (\partial_c\partial^c - m^2)A_{ab} = 0, \partial^a A_{ab} = 0$ □

Cor. 13.1.4. $\begin{cases}(\gamma^c\partial_c + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = 0\\ \mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = \mathbb{X}^a_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}A_{ab}\end{cases} \Leftrightarrow \begin{cases}(\partial^c\partial_c - m^2)A_{ab} = 0\\ \partial^a A_{ab} = 0, A_{ab} = A_{ba}\end{cases}$

Cor. 13.1.5. $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = \mathbb{X}^a_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}A_{ab} \Leftrightarrow A_{ab} = A_{ba}$

Proof: $\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = \mathbb{X}^a_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}A_{ab}$
$\Leftrightarrow [im\gamma^a C - 2S^{ac}(e)C\partial_c]_{\lambda_\varsigma\mu_\varsigma}[im\gamma^b C - 2S^{bd}(e)C\partial_d]_{\eta_\varsigma\xi_\varsigma}A_{ab} = [im\gamma^a C - 2S^{ac}(e)C\partial_c]_{\lambda_\varsigma\mu_\varsigma}[im\gamma^b C - 2S^{bd}(e)C\partial_d]_{\eta_\varsigma\xi_\varsigma}A_{ba}$
$\Leftrightarrow (im)^2 A_{ab}[\gamma^a C]_{\lambda_\varsigma\mu_\varsigma}[\gamma^b C]_{\eta_\varsigma\xi_\varsigma} + 4\partial_c\partial_d A_{ab}[S^{ac}(e)C]_{\lambda_\varsigma\mu_\varsigma}[S^{bd}(e)C]_{\eta_\varsigma\xi_\varsigma}$
$- 2im\partial_d A_{ab}[\gamma^a C]_{\lambda_\varsigma\mu_\varsigma}[S^{bd}(e)C]_{\eta_\varsigma\xi_\varsigma} - 2im\partial_c A_{ab}[S^{ac}(e)C]_{\lambda_\varsigma\mu_\varsigma}[\gamma^b C]_{\eta_\varsigma\xi_\varsigma}$
$= (im)^2 A_{ba}[\gamma^a C]_{\lambda_\varsigma\mu_\varsigma}[\gamma^b C]_{\eta_\varsigma\xi_\varsigma} + 4\partial_c\partial_d A_{ba}[S^{ac}(e)C]_{\lambda_\varsigma\mu_\varsigma}[S^{bd}(e)C]_{\eta_\varsigma\xi_\varsigma}$
$- 2im\partial_d A_{ba}[\gamma^a C]_{\lambda_\varsigma\mu_\varsigma}[S^{bd}(e)C]_{\eta_\varsigma\xi_\varsigma} - 2im\partial_c A_{ba}[S^{ac}(e)C]_{\lambda_\varsigma\mu_\varsigma}[\gamma^b C]_{\eta_\varsigma\xi_\varsigma}$
$\Leftrightarrow A_{ab} = A_{ba}, \partial_c\partial_d A_{ab} + \partial_a\partial_b A_{cd} - \partial_a\partial_d A_{cb} - \partial_c\partial_d A_{ab} = \partial_c\partial_d A_{ba} + \partial_a\partial_b A_{dc} - \partial_a\partial_d A_{bc} - \partial_c\partial_d A_{ba},$
$\partial_d A_{ab} - \partial_b A_{ad} = \partial_d A_{ba} - \partial_b A_{da}, \partial_c A_{ab} - \partial_a A_{cb} = \partial_c A_{ba} - \partial_a A_{bc}$
$\Leftrightarrow A_{ab} = A_{ba}$ □

Cor. 13.1.6. $\begin{cases}(\gamma^c\partial_c + m)_{\kappa_\varsigma}{}^{\lambda_\varsigma}\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = 0\\ \mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}A_{ab} = \mathbb{X}^a_{\eta_\varsigma\xi_\varsigma}\mathbb{X}^b_{\lambda_\varsigma\mu_\varsigma}A_{ab}\end{cases} ?? \Leftrightarrow \begin{cases}(\partial^c\partial_c - m^2)A_{ab} = 0\\ \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab}A_{ab} = 0\end{cases}$

Cor. 13.1.7. $\begin{cases}[\gamma^a(\varsigma)\partial_a + m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}A_{\underbrace{abc\cdots}_{n}}{}^{\sigma} = 0 \Leftrightarrow (-\partial^d\partial_d + m^2)A_{\underbrace{abc\cdots}_{n}}{}^{\sigma} = 0\\ A_{\underbrace{abc\cdots}_{n}}{}^{\sigma} = \frac{1}{n!}A_{\underbrace{\{abc\cdots\}}_{n}}{}^{\sigma}, \delta^{ab}A_{\underbrace{abc\cdots}_{n}}{}^{\sigma} = 0, \partial^a A_{\underbrace{abc\cdots}_{n}}{}^{\sigma} = 0\end{cases}$

Cor. 13.1.8. $\begin{cases}[\gamma^a(\varsigma)\partial_a + m]_{\kappa_\varsigma}{}^{\lambda_\varsigma}\overbrace{\mathbb{X}^a_{\lambda_\varsigma\mu_\varsigma}\mathbb{X}^b_{\eta_\varsigma\xi_\varsigma}\cdots}^{n}A_{\underbrace{abc\cdots}_{n}[\zeta_\varsigma]}{}^{\sigma} = 0 \Leftrightarrow [\gamma^d(\varsigma)\partial_d + m]A_{\underbrace{abc\cdots}_{n}[\zeta_\varsigma]}{}^{\sigma} = 0\\ A_{\underbrace{abc\cdots}_{n}[\zeta_\varsigma]}{}^{\sigma} = \frac{1}{n!}A_{\underbrace{\{abc\cdots\}}_{n}[\zeta_\varsigma]}{}^{\sigma}, \delta^{ab}A_{\underbrace{abc\cdots}_{n}[\zeta_\varsigma]}{}^{\sigma} = 0, \gamma^a(\varsigma)A_{\underbrace{abc\cdots}_{n}[\zeta_\varsigma]}{}^{\sigma} = 0\end{cases}$

Ass. 13.1.1. $(\gamma^c\partial_c + m)\mathbb{X}^a\mathbb{X}^b A_{ab} = 0 \Leftrightarrow ??(\partial^c\partial_c - m^2)A_{ab} = 0, \partial^a A_{ab} = 0, \delta^{ab}A_{ab} = 0$

Proof: $(\gamma^c\partial_c + m)\mathbb{X}^a\mathbb{X}^b A_{ab} = 0$
$\Leftrightarrow (\gamma^c\partial_c + m)[im\gamma^a - 2S^{ac}(e)\partial_c]C[im\gamma^b - 2S^{bd}(e)\partial_d]CA_{ab} = 0$
$\Leftrightarrow (\gamma^c\partial_c + m)[-m^2\gamma^a C\gamma^b C + 4S^{ac}(e)CS^{bd}(e)C\partial_c\partial_d - 2im\gamma^a CS^{bd}(e)C\partial_d - 2imS^{ac}(e)C\gamma^b C\partial_c]A_{ab} = 0$ □

Chapter39 Plane Wave Solutions of Relativistic Bose String

1 Mathematical preparation

1.1 Properties of $\delta(x)$ function

Pro. 1.1.1. $\int\limits_{x=-\infty}^{+\infty} x\delta'(x)dx = -\int\limits_{x=-\infty}^{+\infty} \delta(x)dx$

Pro. 1.1.2. $x\delta'(x) = -\delta(x), x^n\delta^{(n)}(x) = (-1)^n n!\delta(x), x\delta^{(n)}(x) = -n\delta^{(n-1)}(x)$

Pro. 1.1.3. $x\delta(x) = 0, x^2\delta'(x) = 0, x^n\delta^{(n-1)}(x) = 0$

Pro. 1.1.4. $\int\limits_{k=0_-}^{0_+} \delta(k)e^{ikx}dx = 1, \int\limits_{k=0_-}^{0_+} \delta'(k)e^{ikx}dx = -ix, \int\limits_{k=0_-}^{0_+} \delta^{(n)}(k)e^{ikx}dx = (-ix)^n$

1.2 $\delta(x)$ function solution of algebraic equation

Pro. 1.2.1. $xf(x) = 0 \Leftarrow f(x) = c\delta(x)$

Pro. 1.2.2. $\psi(k,E)(E^2-k^2) = 0 \Leftarrow \psi(k,E) = a(k,E)\delta(E^2-k^2)$

Pro. 1.2.3. $\psi(k,E)(E^2-k^2) = 0 \Leftarrow \psi(k,E) = c_{00}\delta(E)\delta(k) + c_{01}\delta(E)\delta'(k) + c_{10}\delta'(E)\delta(k) + c_{11}\delta'(E)\delta'(k)$

Pro. 1.2.4. $\psi(k,E)(E^2-k^2) = 0 \Leftarrow \psi(k,E) = \sum\limits_{n=0}^{\infty}[c_{1n}\delta^{(n)}(E+k)\delta(E-k) + c_{2n}\delta^{(n)}(E-k)\delta(E+k)]$

Pro. 1.2.5. $\psi(0,E)(E^2) = 0 \Leftarrow \psi(0,E) = c\delta(E^2)$

2 Plane wave solutions of bose string with different boundary conditions [42,44]

2.1 Wave function expansion for bose closed string equation(Free closed string)

Thm. 2.1.1. $X^u(\tau,\sigma) = \int\limits_{k=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} X^u(k,E)e^{i(k\sigma-E\tau)}dkdE, X^u(\tau,\sigma) = X^u(\tau,\sigma+2\pi)$
$\Leftrightarrow X^u(\tau,\sigma) = \int\limits_{E=-\infty}^{+\infty}\sum\limits_{n=-\infty}^{\infty}\phi^u(n,E)e^{i(n\sigma-E\tau)}dE$

Proof: $X^u(\tau,\sigma) = \int\limits_{k=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} X^u(k,E)e^{i(k\sigma-E\tau)}dkdE, X^u(\tau,\sigma) = X^u(\tau,\sigma+2\pi)$
$\Leftrightarrow \int\limits_{k=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} X^u(k,E)(e^{i2\pi k}-1)e^{i(k\sigma-E\tau)}dkdE = 0$
$\Leftrightarrow X^u(k,E)(e^{i2\pi k}-1) = 0$
$\Leftrightarrow X^u(k,E) = \phi^u(k,E)\delta(e^{i2\pi k}-1)$
$\Leftrightarrow X^u(k,E) = \sum\limits_{n=-\infty}^{\infty}\phi^u(n,E)\delta(k-n)$
$\Leftrightarrow X^u(\tau,\sigma) = \int\limits_{k=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\sum\limits_{n=-\infty}^{\infty}\phi^u(n,E)\delta(k-n)e^{i(k\sigma-E\tau)}dkdE$
$\Leftrightarrow X^u(\tau,\sigma) = \int\limits_{E=-\infty}^{+\infty}\sum\limits_{n=-\infty}^{\infty}\phi^u(n,E)e^{i(n\sigma-E\tau)}dE$ □

Self comment: Fourier expansion and Fourier transformation can be regarded as identities.

Thm. 2.1.2. $\partial_+\partial_-X^u(\tau,\sigma) = 0, X^u(\tau,\sigma) = X^u(\tau,\sigma+2\pi)$
$\Leftrightarrow X^u(\tau,\sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}]$

Proof: $\partial_+\partial_- X^u(\tau,\sigma)=0, X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi)$

$\Leftrightarrow \int\limits_{E=-\infty}^{+\infty}\sum\limits_{n=-\infty}^{\infty}\phi^u(n,E)(E^2-n^2)e^{i(n\sigma-E\tau)}dE=0$

$\Leftrightarrow \phi^u(n,E)(E^2-n^2)=0$

$\Leftrightarrow \phi^u(n,E)=\begin{cases}a^u(n,E)\delta(E^2-n^2), \delta(E^2-n^2)=\frac{1}{2|n|}[\delta(E-n)+\delta(E-n)], n\neq 0\\ a^u(0,0)c\delta(E)-a^u(0,0)\delta'(E), n=0\end{cases}$

$\Leftrightarrow X^u(\tau,\sigma)=a^u(0,0)c+ia^u(0,0)\tau+\sum\limits_{n=-\infty}^{\infty}\frac{1}{2|n|}[a^u(n,n)e^{in(\sigma-\tau)}+a^u(n,-n)e^{in(\sigma+\tau)}]$

$\Leftrightarrow X^u(\tau,\sigma)=a^u(0,0)c+i\alpha^u(0,0)\tau+\sum\limits_{n=-\infty}^{\infty}\frac{1}{2|n|}[a^u(n,n)e^{-in(\tau-\sigma)}+a^u(-n,n)e^{-in(\sigma+\tau)}]$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}]$

$x^u=a^u(0,0)c, p^u=i2\pi Ta^u(0,0), \alpha^u_n=-i\sqrt{\pi T}\frac{n}{|n|}a^u(n,n), \bar{\alpha}^u_n=-i\sqrt{\pi T}\frac{n}{|n|}a^u(-n,n)$ □

Thm. 2.1.3. $\partial_+\partial_- X^u(\tau,\sigma)=0, X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi), X^u(\tau,\sigma)=X^{*u}(\tau,\sigma)$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], \alpha^u_{-n}=\alpha^{*u}_n, \bar{\alpha}^u_{-n}=\bar{\alpha}^{*u}_n$

Self comment: The wave function expansion of the bose closed string equation is the foundation of the entire bose string theory.

2.2 Wave function expansion for bose N-open string equation(Symmetric closed string)

Thm. 2.2.1. $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], X'^u(\tau,\sigma)|_{\sigma=0,\pi}=0$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], \alpha^u_n=\bar{\alpha}^u_n$

Proof: $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], X'^u(\tau,\sigma)|_{\sigma=0,\pi}=0$

$\Rightarrow \frac{-1}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}[\alpha^u_n e^{-in(\tau-\sigma)}-\bar{\alpha}^u_n e^{-in(\tau+\sigma)}]|_{\sigma=0,\pi}=0$

$\Leftrightarrow \sum\limits_{n\neq 0}[\alpha^u_n e^{-in\tau}-\bar{\alpha}^u_n e^{-in\tau}]=0, \sum\limits_{n\neq 0}[\alpha^u_n e^{in\pi}e^{-in\tau}-\bar{\alpha}^u_n e^{-in\pi}e^{-in\tau}]=0,$

$\Leftrightarrow \alpha^u_n=\bar{\alpha}^u_n$

$\Rightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], \alpha^u_n=\bar{\alpha}^u_n$ □

Thm. 2.2.2. $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], X^u(\tau,\sigma)=X^u(\tau,-\sigma)$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}\alpha^u_n e^{-in\tau}cosn\sigma, \alpha^u_n=\bar{\alpha}^u_n$

Proof: $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], X^u(\tau,\sigma)=X^u(\tau,-\sigma)$

$\Rightarrow x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}]=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau+\sigma)}+\bar{\alpha}^u_n e^{-in(\tau-\sigma)}]$

$\Leftrightarrow \alpha^u_n=\bar{\alpha}^u_n$

$\Rightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], \alpha^u_n=\bar{\alpha}^u_n$ □

Cor. 2.2.1. $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], X'^u(\tau,\sigma)|_{\sigma=0,\pi}=0$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], X^u(\tau,\sigma)=X^u(\tau,-\sigma)$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq 0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], \alpha^u_n=\bar{\alpha}^u_n$

Self comment: The equivalence of the N-condition and the symmetric condition also means that the two branches merge into one, and the three are equivalent to each other.

Cor. 2.2.2. $\partial_+\partial_- X^u(\tau,\sigma)=0, X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi), X'^u(\tau,\sigma)|_{\sigma=0,\pi}=0$

$\Leftrightarrow \partial_+\partial_- X^u(\tau,\sigma)=0, X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi), X^u(\tau,\sigma)=X^u(\tau,-\sigma)$

2.3 Wave function expansion for bose D-open string equation(Antisymmetric closed string)

Thm. 2.3.1. $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],\dot{X}^u(\tau,\sigma)|_{\sigma=0,\pi}=0$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],p^u=0,\alpha_n^u=-\bar{\alpha}_n^u$

Proof: $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],\dot{X}^u(\tau,\sigma)|_{\sigma=0,\pi}=0$

$\Rightarrow \frac{p^u}{2\pi T}+\frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}]|_{\sigma=0,\pi}=0$

$\Leftrightarrow \frac{p^u}{2\pi T}+\frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}[\alpha_n^u e^{-in\tau}+\bar{\alpha}_n^u e^{-in\tau}]=0,\ \frac{p^u}{2\pi T}+\frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}[\alpha_n^u e^{in\pi}e^{-in\tau}+\bar{\alpha}_n^u e^{-in\pi}e^{-in\tau}]=0,$

$\Leftrightarrow p^u=0,\alpha_n^u=-\bar{\alpha}_n^u$

$\Rightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],p^u=0,\alpha_n^u=-\bar{\alpha}_n^u$ □

Thm. 2.3.2. $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],X^u(\tau,\sigma)-x^u=-[X^u(\tau,-\sigma)-x^u]$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],p^u=0,\alpha_n^u=-\bar{\alpha}_n^u$

Proof: $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],X^u(\tau,\sigma)-x^u=-[X^u(\tau,-\sigma)-x^u]$

$\Rightarrow \frac{p^u}{\pi T}+\frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}(\alpha_n^u+\bar{\alpha}_n^u)[e^{-in(\tau-\sigma)}+e^{-in(\tau+\sigma)}]=0$

$\Leftrightarrow p^u=0,\alpha_n^u=-\bar{\alpha}_n^u$

$\Rightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],p^u=0,\alpha_n^u=-\bar{\alpha}_n^u$ □

Cor. 2.3.1. $X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],\dot{X}^u(\tau,\sigma)|_{\sigma=0,\pi}=0$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],X^u(\tau,\sigma)-x^u=-[X^u(\tau,-\sigma)-x^u]$

$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],p^u=0,\alpha_n^u=-\bar{\alpha}_n^u$

Self comment: The equivalence of the D-condition and the antisymmetric condition also means that the two branches merge into one, and the three are equivalent to each other.

Cor. 2.3.2. $\partial_+\partial_-X^u(\tau,\sigma)=0,X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi),\dot{X}^u(\tau,\sigma)|_{\sigma=0,\pi}=0$
$\Leftrightarrow \partial_+\partial_-X^u(\tau,\sigma)=0,X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi),X^u(\tau,\sigma)-x^u=-[X^u(\tau,-\sigma)-x^u]$

2.4 Wave function expansion for bose mixing condition open string equation

Thm. 2.4.1.
$X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],[X'^u(\tau,\sigma)cos\theta+\dot{X}^u(\tau,\sigma)sin\theta]|_{\sigma=0,\pi}=0$

$\Leftrightarrow X^u(\tau,\sigma,\theta)=x^u+\frac{p^u}{2\pi T}\tau\delta_{sin\theta,0}+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^u[e^{-in(\tau-\sigma)}+\frac{1-tg\theta}{1+tg\theta}e^{-in(\tau+\sigma)}]$

Proof:
$X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}],[X'^u(\tau,\sigma)cos\theta+\dot{X}^u(\tau,\sigma)sin\theta]|_{\sigma=0,\pi}=0$

$\Rightarrow \frac{p^u}{2\pi T}sin\theta+\frac{-1}{\sqrt{4\pi T}}cos\theta\sum_{n\neq 0}[\alpha_n^u e^{-in(\tau-\sigma)}-\bar{\alpha}_n^u e^{-in(\tau+\sigma)}]+\frac{1}{\sqrt{4\pi T}}sin\theta\sum_{n\neq 0}[\alpha_n^u e^{-in(\tau-\sigma)}+\bar{\alpha}_n^u e^{-in(\tau+\sigma)}]|_{\sigma=0,\pi}=0$

$\Rightarrow \frac{p^u}{2\pi T}sin\theta+\frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}[(-cos\theta+sin\theta)\alpha_n^u e^{-in\tau}+(cos\theta+sin\theta)\bar{\alpha}_n^u e^{-in\tau}]=0$

$,\ \frac{p^u}{2\pi T}sin\theta+\frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}(-1)^n[(-cos\theta+sin\theta)\alpha_n^u e^{-in\tau}+(cos\theta+sin\theta)\bar{\alpha}_n^u e^{-in\tau}]=0$

$\Leftrightarrow p^u sin\theta=0,(cos\theta-sin\theta)\alpha_n^u=(cos\theta+sin\theta)\bar{\alpha}_n^u$

$\Rightarrow X^u(\tau,\sigma,\theta)=x^u+\frac{p^u}{2\pi T}\tau\delta_{sin\theta,0}+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^u[e^{-in(\tau-\sigma)}+\frac{1-tg\theta}{1+tg\theta}e^{-in(\tau+\sigma)}]$ □

Thm. 2.4.2. $X^u(\tau,\sigma,\theta)=x^u+\frac{p^u}{2\pi T}\tau\delta_{sin\theta,0}+\frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^u[e^{-in(\tau-\sigma)}+\frac{1-tg\theta}{1+tg\theta}e^{-in(\tau+\sigma)}]$

$\Rightarrow [X^u(\tau,\sigma,\theta)-x^u](cos\theta+sin\theta)=[X^u(\tau,-\sigma,-\theta)-x^u](cos\theta-sin\theta)$

Self comment: Pure left moving solution: $X^u(\tau,\sigma,\frac{\pi}{4})$**, Pure right moving solution:** $X^u(\tau,\sigma,-\frac{\pi}{4})$**, N string solution:** $X^u(\tau,\sigma,0)$**, D string solution:** $X^u(\tau,\sigma,\pm\frac{\pi}{2})$

2.5 P-brane

Thm. 2.5.1. $X^u(\tau,\sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], \dot{X}^u(\tau,\sigma)|_{\sigma=0,\pi} = 0$

$\Leftrightarrow X^u(\tau,\sigma) = x^u - \frac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^u e^{-in\tau} sinn\sigma, p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u, u = 1,2,\cdots,p$

$X^l(\tau,\sigma) = x^l + \frac{p^l}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^l e^{-in(\tau-\sigma)} + \bar{\alpha}_n^l e^{-in(\tau+\sigma)}], X'^l(\tau,\sigma)|_{\sigma=0,\pi} = 0$

$\Leftrightarrow X^l(\tau,\sigma) = x^l + \frac{p^l}{2\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^l e^{-in\tau} cosn\sigma, \alpha_n^l = \bar{\alpha}_n^l, l = p+1, p+2,\cdots,D$

Self comment: The essence of p-brane is that some dimensions satisfy the N-condition and some dimensions satisfy the D-condition.

2.6 Solutions for general boundary conditions

Self comment: More generally for p-branes, some dimensions satisfy the N-condition, some dimensions that satisfy the D-condition and some dimensions that satisfy the periodic condition.

Thm. 2.6.1. $\partial_+\partial_- X^u(\tau,\sigma) = 0, \dot{X}^I(\tau,\sigma)|_{\sigma=0,\pi} = 0, X'^J(\tau,\sigma)|_{\sigma=0,\pi} = 0, X^u(\tau,\sigma) = X^u(\tau,\sigma+2\pi)$

$\Leftrightarrow X^I(\tau,\sigma) = x^I - \frac{p^I}{2\pi T}\sigma - \frac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\tau} sinn\sigma, \alpha_n^I = -\bar{\alpha}_n^I, I = 1,2,\cdots,p$

$X^J(\tau,\sigma) = x^J + \frac{p^J}{2\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^J e^{-in\tau} cosn\sigma, \alpha_n^J = \bar{\alpha}_n^J, J = p+1, p+2,\cdots,p+l$

$X^K(\tau,\sigma) = x^K + \frac{p^K}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^K e^{-in(\tau-\sigma)} + \bar{\alpha}_n^K e^{-in(\tau+\sigma)}], K = p+l+1,\cdots,D$

Cor. 2.6.1.

$\dot{X}^I(\tau,\sigma) = \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\alpha_n^I e^{-in\tau} sinn\sigma, p^I = 0, \alpha_n^I = -\bar{\alpha}_n^I, I = 1,2,\cdots,p$

$\dot{X}^J(\tau,\sigma) = \frac{p^J}{2\pi T} + \frac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\alpha_n^J e^{-in\tau} cosn\sigma, \alpha_n^J = \bar{\alpha}_n^J, J = p+1, p+2,\cdots,p+l$

$\dot{X}^K(\tau,\sigma) = \frac{p^K}{2\pi T} + \frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}[\alpha_n^K e^{-in(\tau-\sigma)} + \bar{\alpha}_n^K e^{-in(\tau+\sigma)}], K = p+l+1, p+l+2,\cdots,D$

Cor. 2.6.2.

$X'^I(\tau,\sigma) = -\frac{p^K}{2\pi T} - \frac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\alpha_n^I e^{-in\tau} cosn\sigma, p^I = 0, \alpha_n^I = -\bar{\alpha}_n^I, I = 1,2,\cdots,p$

$X'^J(\tau,\sigma) = -\frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\alpha_n^J e^{-in\tau} sinn\sigma, \alpha_n^J = \bar{\alpha}_n^J, J = p+1, p+2,\cdots,p+l$

$X'^K(\tau,\sigma) = -\frac{1}{\sqrt{4\pi T}}\sum_{n\neq 0}[\alpha_n^K e^{-in(\tau-\sigma)} - \bar{\alpha}_n^K e^{-in(\tau+\sigma)}], K = p+l+1, p+l+2,\cdots,D$

Cor. 2.6.3.

$\dot{X}^I(\tau,\sigma) + X'^I(\tau,\sigma) = \frac{1}{\sqrt{\pi T}}\sum_n \bar{\alpha}_n^I e^{-in(\tau+\sigma)}, \bar{\alpha}_n^I = -\alpha_n^I, I = 1,2,\cdots,p$

$\dot{X}^I(\tau,\sigma) - X'^I(\tau,\sigma) = \frac{1}{\sqrt{\pi T}}\sum_n \alpha_n^I e^{-in(\tau-\sigma)}, \alpha_0^I = \frac{p^I}{2\sqrt{\pi T}}, I = 1,2,\cdots,p$

Cor. 2.6.4.

$\dot{X}^J(\tau,\sigma) + X'^J(\tau,\sigma) = \frac{1}{\sqrt{\pi T}}\sum_n \bar{\alpha}_n^J e^{-in(\tau+\sigma)}, \bar{\alpha}_n^J = \alpha_n^J, J = p+1, p+2,\cdots,p+l$

$\dot{X}^J(\tau,\sigma) - X'^J(\tau,\sigma) = \frac{1}{\sqrt{\pi T}}\sum_n \alpha_n^J e^{-in(\tau-\sigma)}, \alpha_0^J = \frac{p^J}{2\sqrt{\pi T}}, J = p+1, p+2,\cdots,p+l$

Cor. 2.6.5.

$\dot{X}^K(\tau,\sigma) + X'^K(\tau,\sigma) = \frac{1}{\sqrt{\pi T}}\sum_n \bar{\alpha}_n^K e^{-in(\tau+\sigma)}, \bar{\alpha}_0^K = \frac{\bar{p}^K}{2\sqrt{\pi T}}, K = p+l+1,\cdots,D$

$\dot{X}^K(\tau,\sigma) - X'^K(\tau,\sigma) = \frac{1}{\sqrt{\pi T}}\sum_n \alpha_n^K e^{-in(\tau-\sigma)}, \alpha_0^K = \frac{p^K}{2\sqrt{\pi T}}, K = p+l+1,\cdots,D$

Cor. 2.6.6. $[\dot{X}^u(\tau,\sigma) + X'^u(\tau,\sigma)][\dot{X}_u(\tau,\sigma) + X'_u(\tau,\sigma)] = \frac{1}{\pi T}\sum_u\sum_n \bar{\alpha}_n^u e^{-in(\tau+\sigma)}\sum_m \bar{\alpha}_m^u e^{-im(\tau+\sigma)}$

$= \frac{1}{\pi T}\sum_m\sum_u\sum_n \bar{\alpha}_n^u\bar{\alpha}_{m-n}^u e^{-im(\tau+\sigma)} = \frac{1}{\pi T}\sum_m \bar{L}_m e^{-im(\tau+\sigma)}, \bar{L}_m = \sum_u\sum_n \bar{\alpha}_n^u\bar{\alpha}_{m-n}^u$

Cor. 2.6.7. $[\dot{X}^u(\tau,\sigma) - X'^u(\tau,\sigma)][\dot{X}_u(\tau,\sigma) - X'_u(\tau,\sigma)] = \frac{1}{\pi T}\sum_u\sum_n \alpha_n^u e^{-in(\tau-\sigma)}\sum_m \alpha_m^u e^{-im(\tau-\sigma)}$

$= \frac{1}{\pi T}\sum_m\sum_u\sum_n \alpha_n^u\alpha_{m-n}^u e^{-im(\tau-\sigma)} = \frac{1}{\pi T}\sum_m L_m e^{-im(\tau-\sigma)}, L_m = \sum_u\sum_n \alpha_n^u\alpha_{m-n}^u$

2.7 Strict and simple solution of bose open string equation

Thm. 2.7.1. $\partial_+\partial_- X(\tau,\sigma)=0 \Leftrightarrow X(\tau,\sigma)=f(\tau+\sigma)+g(\tau-\sigma)$

Proof: $\partial_+\partial_- X(\tau,\sigma)=0 \Leftrightarrow \partial_- X(\tau,\sigma)=h(\tau-\sigma) \Leftrightarrow X(\tau,\sigma)=\int h(\tau-\sigma)d(\tau-\sigma)+f(\tau+\sigma)$
$\Leftrightarrow X(\tau,\sigma)=f(\tau+\sigma)+g(\tau-\sigma)$ □

Cor. 2.7.1. $\partial_+\partial_- X(\tau,\sigma)=0, \partial_\sigma X(\tau,\sigma)|_{\sigma=0,\pi}=0 \Leftrightarrow X(\tau,\sigma)=f(\tau+\sigma)+f(\tau-\sigma), f'(x-\pi)=f'(x+\pi)$

Cor. 2.7.2. $\partial_+\partial_- X(\tau,\sigma)=0, \partial_\tau X(\tau,\sigma)|_{\sigma=0,\pi}=0 \Leftrightarrow X(\tau,\sigma)=f(\tau+\sigma)-f(\tau-\sigma), f'(x-\pi)=f'(x+\pi)$

3 Doubts about completeness of plane wave solutions for bose string equation

3.1 Comparative study: Second wave function expansion method for bose closed string

Thm. 3.1.1. $\partial_+\partial_- X^u(\tau,\sigma)=0 \Leftrightarrow \phi^u(\tau,\sigma)=\int\limits_{k=-\infty}^{+\infty}\frac{1}{2\omega}[a^u(k,\omega)e^{i(k\sigma-\omega\tau)}+a^u(-k,-\omega)e^{-i(k\sigma-\omega\tau)}]dk$

Proof: $\partial_+\partial_- X^u(\tau,\sigma)=0 \Leftrightarrow \int\limits_{k=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} X^u(k,E)(E^2-k^2)e^{i(k\sigma-E\tau)}dkdE=0$

$\Leftrightarrow X^u(k,E)(E^2-k^2)=0$

$\Leftrightarrow \begin{cases} X^u(k,E)|_{(k,E)=0}=C^u\delta(k)\delta(E)+C^u_E\dot{\delta}(E)\delta(k)+C^u_k\delta(E)\delta'(k)+C^u_{Ek}\dot{\delta}(E)\delta'(k) \\ X^u(k,E)|_{(k,E)\neq0}=a^u(k,E)\delta(E^2-k^2)\end{cases}$

$\Leftrightarrow X^u(\tau,\sigma)=\int\limits_{k=0_-}^{0_+}\int\limits_{E=0_-}^{0_+}[C^u\delta(\tau)\delta(\sigma)+C^u_\tau\dot{\delta}(\tau)\delta(\sigma)+C^u_\sigma\delta(\tau)\delta'(\sigma)]e^{i(k\sigma-E\tau)}dkdE$

$+\int\limits_{k=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}a^u(k,E)\delta(E^2-k^2)e^{i(k\sigma-E\tau)}dkdE|_{(k,E)\neq0}$

$\Leftrightarrow X^u(\tau,\sigma)=(C^u+iC^u_\tau\tau-iC^u_\sigma\sigma+C^u_{\tau\sigma}\tau\sigma)$

$+\int\limits_{k=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\frac{1}{2\omega}a^u(k,E)[\delta(E-\omega)+\delta(E+\omega)]e^{i(k\sigma-E\tau)}dkdE|_{(k,E)\neq0}, \omega=|k|$

$\Leftrightarrow X^u(\tau,\sigma)=(C^u+iC^u_\tau\tau-iC^u_\sigma\sigma+C^u_{\tau\sigma}\tau\sigma)+\int\limits_{k=-\infty}^{+\infty}\frac{1}{2\omega}[a^u(k,\omega)e^{i(k\sigma-\omega\tau)}+a^u(k,-\omega)e^{i(k\sigma+\omega\tau)}]dk|_{k\neq0}$

$\Leftrightarrow X^u(\tau,\sigma)=(C^u+iC^u_\tau\tau-iC^u_\sigma\sigma+C^u_{\tau\sigma}\tau\sigma)+\int\limits_{k=-\infty}^{+\infty}\frac{1}{2\omega}[a^u(k,\omega)e^{i(k\sigma-\omega\tau)}+a^u(-k,-\omega)e^{-i(k\sigma-\omega\tau)}]dk|_{k\neq0}$ □

Thm. 3.1.2. $\partial_+\partial_- X^u(\tau,\sigma)=0, X^u(\tau,\sigma)=X^{*u}(\tau,\sigma)$
$\Leftrightarrow X^u(\tau,\sigma)=(c^u+c^u_\tau\tau+c^u_\sigma\sigma+c^u_{\tau\sigma}\tau\sigma)+\int\limits_{k=-\infty}^{+\infty}\frac{1}{2\omega}[a^u(k,\omega)e^{i(k\sigma-\omega\tau)}+a^{*u}(k,\omega)e^{-i(k\sigma-\omega\tau)}]dk|_{k\neq0}$

Thm. 3.1.3. $\partial_+\partial_- X^u(\tau,\sigma)=0, X^u(\tau,\sigma)=X^{*u}(\tau,\sigma), X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi)$
$\Leftrightarrow X^u(\tau,\sigma)=x^u+\frac{p^u}{2\pi T}\tau+\frac{i}{\sqrt{4\pi T}}\sum\limits_{n\neq0}\frac{1}{n}[\alpha^u_n e^{-in(\tau-\sigma)}+\bar{\alpha}^u_n e^{-in(\tau+\sigma)}], \alpha^u_{-n}=\alpha^{*u}_n, \bar{\alpha}^u_{-n}=\bar{\alpha}^{*u}_n$

Proof: $\partial_+\partial_- X^u(\tau,\sigma)=0, X^u(\tau,\sigma)=X^{*u}(\tau,\sigma), X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi)$

$\Leftrightarrow X^u(\tau,\sigma)=(c^u+c^u_\tau\tau+c^u_\sigma\sigma+c^u_{\tau\sigma}\tau\sigma)+\int\limits_{k=-\infty}^{+\infty}\frac{1}{2\omega}[a^u(k,\omega)e^{i(k\sigma-\omega\tau)}+a^{*u}(k,\omega)e^{-i(k\sigma-\omega\tau)}]dk|_{k\neq0}$

$X^u(\tau,\sigma)=X^u(\tau,\sigma+2\pi)$

$\Rightarrow a^u(k,\omega)(e^{i2\pi k}-1)=0, a^{*u}(k,\omega)(e^{-i2\pi k}-1)=0, c^u_\sigma=0, c^u_{\tau\sigma}=0$

$\Leftrightarrow a^u(k,\omega)=\alpha^u(k)\delta(e^{i2\pi k}-1), c^u_\sigma=0, c^u_{\tau\sigma}=0$

$\Rightarrow X^u(\tau,\sigma)=c^u+c^u_\tau\tau+\int\limits_{k=-\infty}^{+\infty}\frac{1}{2\omega}[\alpha^u(k)\delta(e^{i2\pi k}-1)e^{i(k\sigma-\omega\tau)}+\alpha^{*u}(k)\delta(e^{-i2\pi k}-1)e^{-i(k\sigma-\omega\tau)}]dk|_{k\neq0}$

$\Leftrightarrow X^u(\tau,\sigma)=c^u+c^u_\tau\tau+\int\limits_{k=-\infty}^{+\infty}\frac{1}{2\omega}[\alpha^u(k)\sum\limits_{n=-\infty}^{\infty}\delta(k-n)e^{i(k\sigma-\omega\tau)}+\alpha^{*u}(k)\sum\limits_{n=-\infty}^{\infty}\delta(k-n)e^{-i(k\sigma-\omega\tau)}]dk|_{k\neq0}$

$\Leftrightarrow X^u(\tau,\sigma)=c^u+c^u_\tau\tau+\sum\limits_{n\neq0}\frac{1}{2|n|}[\alpha^u(n)e^{i(n\sigma-|n|\tau)}+\alpha^{*u}(n)e^{-i(n\sigma-|n|\tau)}]$

$\Leftrightarrow X^u(\tau,\sigma)=c^u+c^u_\tau\tau+\sum\limits_{n=1}^{\infty}\frac{1}{2n}[\alpha^u(n)e^{in(\sigma-\tau)}+\alpha^{*u}(n)e^{-in(\sigma-\tau)}]$

$+\sum\limits_{n=-\infty}^{-1}-\frac{1}{2n}[\alpha^u(n)e^{in(\tau+\sigma)}+\alpha^{*u}(n)e^{-in(\tau+\sigma)}]$

$\Leftrightarrow X^u(\tau,\sigma)=c^u+c^u_\tau\tau+\sum\limits_{n\neq0}\frac{1}{2n}\alpha'^u_n e^{-in(\tau-\sigma)}+\sum\limits_{n\neq0}\frac{1}{2n}\bar{\alpha}'^u_n e^{-in(\tau+\sigma)}$

$\alpha'^u_n=-\alpha'^{*u}_{-n}=\begin{cases}\alpha^u(n), n>0\\ -\alpha^{*u}(-n), n<0\end{cases}, \bar{\alpha}'^u_n=-\bar{\alpha}'^{*u}_{-n}=\begin{cases}\alpha^u(-n), n>0\\ -\alpha^{*u}(n), n<0\end{cases}$

$\Leftrightarrow X^u(\tau,\sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum\limits_{n\neq 0} \frac{1}{n}[\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}]$

$c_\tau^u = \frac{p^u}{2\pi T}, \alpha_n^u = -i\sqrt{\pi T}\alpha_n'^u, \bar{\alpha}_n^u = -i\sqrt{\pi T}\bar{\alpha}_n'^u, \alpha_{-n}^u = \alpha_n^{*u}, \bar{\alpha}_{-n}^u = \bar{\alpha}_n^{*u}$ □

Self comment: Changing the order of limiting conditions for wave functions requires different mathematical techniques, but the conclusions are still the same.

3.2 Comparative study: Third wave function expansion method for bose closed string

Thm. 3.2.1. $\partial_+\partial_- X^u(\tau,\sigma) = 0 \Leftrightarrow X^u(\tau,\sigma) = \int\limits_{k=-\infty}^{+\infty} \frac{1}{2\omega}[a^u(k,\omega)e^{i(k\sigma-\omega\tau)} + a^u(-k,-\omega)e^{-i(k\sigma-\omega\tau)}]dk$

Proof: $\partial_+\partial_- X^u(\tau,\sigma) = 0 \Leftrightarrow \int\limits_{E=-\infty}^{+\infty} X^u(E,\sigma)e^{-iE\tau}dE = 0$

$\Leftrightarrow (\partial_\sigma^2 + E^2)X^u(E,\sigma) = 0$

$\Leftrightarrow X^u(E,\sigma) = c_{00}^u\delta(E) + c_{01}^u\delta'(E) + c_{10}^u\sigma\delta(E) + c_{11}^u\sigma\delta'(E) + a^u(E)e^{iE\sigma} + b^u(E)e^{-iE\sigma}$

$\Leftrightarrow X^u(\tau,\sigma) = c_{00} + c_{01}\tau + c_{10}\sigma + c_{11}\sigma\tau + \int\limits_{E=-\infty}^{+\infty} [a^u(E)e^{-iE(\tau-\sigma)} + b^u(E)e^{-iE(\tau+\sigma)}]dE$ □

Self comment: Changing the order of limiting conditions for wave functions requires different mathematical techniques, but the conclusions are still the same.

3.3 Properties

Thm. 3.3.1. $\phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{k}=-\infty}^{+\infty} [a(\vec{k},t)e^{i\vec{k}\cdot\vec{r}} + a^*(\vec{k},t)e^{-i\vec{k}\cdot\vec{r}}]d^3\vec{k}$

$\nabla\phi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{k}=-\infty}^{+\infty} i\vec{k}[a(\vec{k},t)e^{i\vec{k}\cdot\vec{r}} - a^*(\vec{k},t)e^{-i\vec{k}\cdot\vec{r}}]d^3\vec{k}$

$\dot{\phi}(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\vec{k}=-\infty}^{+\infty} [\dot{a}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}} + \dot{a}^*(\vec{k},t)e^{-i\vec{k}\cdot\vec{r}}]d^3\vec{k}$

Thm. 3.3.2. $\phi(\vec{r},t)^2 = \int\limits_{\vec{k}=-\infty}^{+\infty} [a(\vec{k},t)a(-\vec{k},t) + a^*(\vec{k},t)a^*(-\vec{k},t) + 2a(\vec{k},t)a^*(\vec{k},t)]d^3\vec{k}$

$\nabla\phi(\vec{r},t)\cdot\nabla\phi(\vec{r},t) = \int\limits_{\vec{k}=-\infty}^{+\infty} \vec{k}^2[a(\vec{k},t)a(-\vec{k},t) + a^*(\vec{k},t)a^*(-\vec{k},t) + 2a(\vec{k},t)a^*(\vec{k},t)]d^3\vec{k}$

$\dot{\phi}(\vec{r},t)^2 = \int\limits_{\vec{k}=-\infty}^{+\infty} [\dot{a}(\vec{k},t)\dot{a}(-\vec{k},t) + \dot{a}^*(\vec{k},t)\dot{a}^*(-\vec{k},t) + 2\dot{a}(\vec{k},t)\dot{a}^*(\vec{k},t)]d^3\vec{k}$

Thm. 3.3.3.

$\int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \phi(\vec{r},t)^2 d^3\vec{r}dt = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} a(\vec{k},E)a(-\vec{k},-E)d^3\vec{k}dE$

$\int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \nabla\phi(\vec{r},t)\cdot\nabla\phi(\vec{r},t)d^3\vec{r}dt = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} \vec{k}^2 a(\vec{k},E)a(-\vec{k},-E)d^3\vec{k}dE$

$\int\limits_{\vec{r}=-\infty}^{+\infty}\int\limits_{t=-\infty}^{+\infty} \dot{\phi}(\vec{r},t)^2 d^3\vec{r}dt = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} E^2 a(\vec{k},E)a(-\vec{k},-E)d^3\vec{k}dE$

$\phi(\vec{r},t) = \frac{1}{(2\pi)^2}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} a(\vec{k},E)e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$S = \frac{1}{2}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} (E^2 - \vec{k}^2 - m^2)a(\vec{k},E)a(-\vec{k},-E)d^3\vec{k}dE \Rightarrow (E^2 - \vec{k}^2 - m^2)a(\vec{k},E) = 0$

Thm. 3.3.4.

$\int\limits_{\vec{r}=-\infty}^{+\infty} \phi(\vec{r},t)^2 d^3\vec{r} = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\int\limits_{E'=-\infty}^{+\infty} a(\vec{k},E)a(-\vec{k},E')e^{-i(E+E')t}d^3\vec{k}dEdE'$

$\int\limits_{\vec{r}=-\infty}^{+\infty} \nabla\phi(\vec{r},t)\cdot\nabla\phi(\vec{r},t)d^3\vec{r} = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\int\limits_{E'=-\infty}^{+\infty} \vec{k}^2 a(\vec{k},E)a(-\vec{k},E')e^{-i(E+E')t}d^3\vec{k}dEdE'$

$\int\limits_{\vec{r}=-\infty}^{+\infty} \dot{\phi}(\vec{r},t)^2 d^3\vec{r} = \int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty}\int\limits_{E'=-\infty}^{+\infty} -EE'a(\vec{k},E)a(-\vec{k},E')e^{-i(E+E')t}d^3\vec{k}dE$

$\phi(\vec{r},t) = \frac{1}{(2\pi)^2}\int\limits_{\vec{k}=-\infty}^{+\infty}\int\limits_{E=-\infty}^{+\infty} a(\vec{k},E)e^{i(\vec{k}\cdot\vec{r}-Et)}d^3\vec{k}dE$

$$H=\frac{1}{2}\int_{\vec{k}=-\infty}^{+\infty}\int_{E=-\infty}^{+\infty}\int_{E'=-\infty}^{+\infty}(\vec{k}^2+m^2-EE')a(\vec{k},E)a(-\vec{k},E')e^{-i(E+E')t}d^3\vec{k}dEdE'$$
$$=\frac{1}{4}\int_{\vec{k}=-\infty}^{+\infty}[a(\vec{k},\omega_k)a(-\vec{k},-\omega_k)+a(-\vec{k},-\omega_k)a(\vec{k},\omega_k)]d^3\vec{k}$$

Chapter40 Preliminary Study on Simple Supersymmetry Theory

1 Electromagnetic field in two dimensions

1.1 Light cone coordinates and derivatives in two dimensions

Def. 1.1.1. $z \equiv \tau + \sigma, \tilde{z} \equiv \tau - \sigma, \tau = \frac{1}{2}(z + \tilde{z}), \sigma = \frac{1}{2}(z - \tilde{z}), z_\varsigma := \tau + \varsigma\sigma, \bar{z}_\varsigma := \tau - \varsigma\sigma$

Def. 1.1.2. $\begin{bmatrix} z \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} \begin{bmatrix} \sigma \\ i\tau \end{bmatrix}, \begin{bmatrix} \sigma \\ i\tau \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix}$

Cor. 1.1.1. $\begin{cases} dz = d\tau + d\sigma, d\tilde{z} = d\tau - d\sigma \\ \partial_z = \frac{1}{2}(\partial_\tau + \partial_\sigma), \partial_{\tilde{z}} = \frac{1}{2}(\partial_\tau - \partial_\sigma) \end{cases} \quad \begin{cases} d\tau = \frac{1}{2}(dz + d\tilde{z}), d\sigma = \frac{1}{2}(dz - d\tilde{z}) \\ \partial_\tau = \partial_z + \partial_{\tilde{z}}, \partial_\sigma = \partial_z - \partial_{\tilde{z}} \end{cases}$

Cor. 1.1.2. $dz \wedge d\tilde{z} = 2d\sigma \wedge d\tau$

Def. 1.1.3. $P_z \equiv -i\partial_z, P_{\tilde{z}} \equiv -i\partial_{\tilde{z}}, P_\tau \equiv i\partial_\tau, P_\sigma \equiv -i\partial_\sigma$

Cor. 1.1.3. $P_z = -\frac{1}{2}(P_\tau - P_\sigma), P_{\tilde{z}} = -\frac{1}{2}(P_\tau + P_\sigma), -P_\tau = P_z + P_{\tilde{z}}, P_\sigma = P_z - P_{\tilde{z}}$

Cor. 1.1.4. $e^{i(P_\sigma \sigma - P_\tau \tau)} = e^{i(P_z z + P_{\tilde{z}} \tilde{z})}$

1.2 Electromagnetic field in two dimensions

Def. 1.2.1. *Field potential:* $A_a = (A_\sigma, A_\pi) = (A_\sigma, iA_\tau)$, *Field strength:* $F_{ab} = \partial_a A_b - \partial_b A_a, : E = -iF_{\sigma\pi}$

Def. 1.2.2. *Light cone field potential:*$\partial_z ? = (\partial_\tau + \partial_\sigma), \partial_{\tilde{z}} ? = (\partial_\tau - \partial_\sigma), A_z \equiv A_\tau + A_\sigma, A_{\tilde{z}} \equiv A_\tau - A_\sigma, A'_z = e^{-\varepsilon} A_z, A'_{\tilde{z}} = e^{\varepsilon} A_{\tilde{z}}$

Def. 1.2.3. *Light cone field source:* $J_z \equiv J_\tau + J_\sigma, J_{\tilde{z}} \equiv J_\tau - J_\sigma, J'_z = e^{-\varepsilon} J_z, J'_{\tilde{z}} = e^{\varepsilon} J_{\tilde{z}}$

Cor. 1.2.1. $F_{\sigma\pi} = \partial_\sigma A_\pi - \partial_\pi A_\sigma = i(\partial_\sigma A_\tau + \partial_\tau A_\sigma) = i(\partial_z A_z - \partial_{\tilde{z}} A_{\tilde{z}}) = iE$

Cor. 1.2.2. $E = \partial_z A_z - \partial_{\tilde{z}} A_{\tilde{z}}, F_{ab} = i\varepsilon_{ab} E$

So in two dimensions, the electric field E is a scalar, and there is no magnetic field, and the field strength F_{ab} can be seen as both a tensor and a scalar.

Cor. 1.2.3. *Electromagnetic field equation:* $\partial^a F_{ab} = -J_b \Leftrightarrow \begin{cases} \partial_\tau E = J_\sigma \\ \partial_\sigma E = -J_\tau \end{cases} \Leftrightarrow \begin{cases} \partial_z E = -\frac{1}{2} J_{\tilde{z}} \\ \partial_{\tilde{z}} E = \frac{1}{2} J_z \end{cases}$

Cor. 1.2.4. *Lorentz condition:* $\partial^a A_a = 0 \Leftrightarrow \partial_z A_z + \partial_{\tilde{z}} A_{\tilde{z}} = 0$

Cor. 1.2.5. $\begin{cases} \partial^a F_{ab} = -J_b \\ \partial^a A_a = 0 \end{cases} \Rightarrow \begin{cases} \partial_z^2 A_z = -\frac{1}{4} J_{\tilde{z}}, \partial_z \partial_{\tilde{z}} A_{\tilde{z}} = \frac{1}{4} J_{\tilde{z}} \\ \partial_{\tilde{z}}^2 A_{\tilde{z}} = -\frac{1}{4} J_z, \partial_z \partial_{\tilde{z}} A_z = \frac{1}{4} J_z \end{cases}$

Cor. 1.2.6. $\begin{cases} \partial^a F_{ab} = 0 \\ \partial^a A_a = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_z A_z = \frac{1}{2} E_0 \\ \partial_{\tilde{z}} A_{\tilde{z}} = -\frac{1}{2} E_0 \end{cases} \Leftrightarrow \begin{cases} A_z = f(\tilde{z}) + \frac{1}{2} E_0 z \\ A_{\tilde{z}} = g(z) - \frac{1}{2} E_0 \tilde{z} \end{cases}$

Cor. 1.2.7. $S = \int dz d\tilde{z} \{-\frac{1}{4} F_{ab} F^{ab}\} = \int dz d\tilde{z} \{\frac{1}{2} E^2\} = \int dz d\tilde{z} \{\frac{1}{2}(\partial_z A_z - \partial_{\tilde{z}} A_{\tilde{z}})^2\}$

Cor. 1.2.8. *Guage transformation:* $\delta A_a = \partial_a \theta \Leftrightarrow \begin{cases} \delta A_z = 2\partial_{\tilde{z}} \theta \\ \delta A_{\tilde{z}} = 2\partial_z \theta \end{cases} \Rightarrow \delta S = 0$

1.3 Vector spinor supersymmetry in two dimensions [45]

Thm. 1.3.1. $S=\int dzd\tilde{z}\{(\partial_z A_z)^2+\bar{\varphi}\partial_z\bar{\varphi}\},\begin{cases}\delta A_z=\bar{\epsilon}(\tilde{z})\bar{\varphi}\\ \delta\bar{\varphi}=-\bar{\epsilon}(\tilde{z})\partial_z A_z\end{cases}\Rightarrow \delta S=0$

Cor. 1.3.1. $\begin{cases}[\delta_{\bar{\epsilon}_1(\tilde{z})},\delta_{\bar{\epsilon}_2(\tilde{z})}]A_z=2\bar{\epsilon}_1(\tilde{z})\bar{\epsilon}_2(\tilde{z})\partial_z A_z\\ [\delta_{\bar{\epsilon}_1(\tilde{z})},\delta_{\bar{\epsilon}_2(\tilde{z})}]\bar{\varphi}=2\bar{\epsilon}_1(\tilde{z})\bar{\epsilon}_2(\tilde{z})\partial_z\bar{\varphi}\end{cases}$

Thm. 1.3.2. $S=\int dzd\tilde{z}\{(\partial_{\tilde{z}} A_{\tilde{z}})^2+\varphi\partial_{\tilde{z}}\varphi\},\begin{cases}\delta A_{\tilde{z}}=\epsilon(z)\varphi\\ \delta\varphi=-\epsilon(z)\partial_{\tilde{z}} A_{\tilde{z}}\end{cases}\Rightarrow \delta S=0$

Cor. 1.3.2. $\begin{cases}[\delta_{\epsilon_1(z)},\delta_{\epsilon_2(z)}]A_{\tilde{z}}=2\epsilon_1(z)\epsilon_2(z)\partial_{\tilde{z}} A_{\tilde{z}}\\ [\delta_{\epsilon_1(z)},\delta_{\epsilon_2(z)}]\varphi=2\epsilon_1(z)\epsilon_2(z)\partial_{\tilde{z}}\varphi\end{cases}$

Thm. 1.3.3. $S=\int dzd\tilde{z}\{(\partial_z A_z)^2+(\partial_{\tilde{z}} A_{\tilde{z}})^2+\bar{\varphi}\partial_z\bar{\varphi}+\varphi\partial_{\tilde{z}}\varphi\},\begin{cases}\delta A_z=\bar{\epsilon}(\tilde{z})\bar{\varphi},\delta A_{\tilde{z}}=\epsilon(z)\varphi\\ \delta\varphi=-\epsilon(z)\partial_{\tilde{z}} A_{\tilde{z}},\delta\bar{\varphi}=-\bar{\epsilon}(\tilde{z})\partial_z A_z\end{cases}\Rightarrow \delta S=0$

Thm. 1.3.4. $S=\int dzd\tilde{z}\{(\partial_z A_z-\partial_{\tilde{z}} A_{\tilde{z}})^2+\bar{\varphi}\partial_z\bar{\varphi}+\varphi\partial_{\tilde{z}}\varphi\},\begin{cases}\delta A_z=\bar{\epsilon}(\tilde{z})\bar{\varphi},\delta A_{\tilde{z}}=\epsilon(z)\varphi\\ \delta\varphi=-\epsilon(z)(\partial_{\tilde{z}} A_{\tilde{z}}-\partial_z A_z)\\ \delta\bar{\varphi}=-\bar{\epsilon}(\tilde{z})(\partial_z A_z-\partial_{\tilde{z}} A_{\tilde{z}})\end{cases}\Rightarrow \delta S=0$

1.4 Majoran-Weyl anyon equation and its corresponding action in two dimensions

Cor. 1.4.1. $[s\partial_a+iS_{ab}(s)\partial^b]\psi(s)=0,\psi'(s)=e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s)}\psi(s)=e^{-s\varepsilon}\psi(s),iS_{ab}(s)=\begin{bmatrix}0&is\\-is&0\end{bmatrix}$
$\Leftrightarrow \partial_z\psi(s)=0$

Cor. 1.4.2. $S=\int dzd\tilde{z}\psi(1-s)\partial_z\psi(s)$

Cor. 1.4.3. $[-s\partial_a+iS_{ab}(s)\partial^b]\psi(s)=0,\psi'(s)=e^{-s\varepsilon}\psi(s),iS_{ab}(s)=\begin{bmatrix}0&is\\-is&0\end{bmatrix}\Leftrightarrow \partial_{\tilde{z}}\psi(s)=0$

Cor. 1.4.4. $S=\int dzd\tilde{z}\psi(-1-s)\partial_{\tilde{z}}\psi(s)$

Cor. 1.4.5. $[s\partial_a+iS_{ab}(-s)\partial^b]\psi(-s)=0,\psi'(s)=e^{s\varepsilon}\psi(-s),iS_{ab}(-s)=\begin{bmatrix}0&-is\\is&0\end{bmatrix}\Leftrightarrow \partial_{\tilde{z}}\psi(-s)=0$

Cor. 1.4.6. $S=\int dzd\tilde{z}\psi(-1+s)\partial_{\tilde{z}}\psi(-s)$

Cor. 1.4.7. $[-s\partial_a+iS_{ab}(-s)\partial^b]\psi(-s)=0,\psi'(s)=e^{s\varepsilon}\psi(-s),iS_{ab}(-s)=\begin{bmatrix}0&-is\\is&0\end{bmatrix}\Leftrightarrow \partial_z\psi(-s)=0$

Cor. 1.4.8. $S=\int dzd\tilde{z}\psi(1+s)\partial_z\psi(-s)$

1.5 Majoran-Weyl anyon action in two dimensions?

Cor. 1.5.1. $S=\sum_s\int dzd\tilde{z}[\psi(s+1)\partial_z\psi(-s)+\psi(s-1)\partial_{\tilde{z}}\psi(-s)]$

1.6 Classical construction of Dirac equation in two dimensions

Massless Dirac equation in two dimensions:

Cor. 1.6.1. $S=\int i\bar{\psi}\gamma^a\partial_a\psi d\sigma d\tau,(\gamma^a,\gamma^3)=(\sigma_x,\sigma_y,\sigma_z)$
$S=\int i\psi^T\sigma_y(\sigma_x\partial_\sigma+\sigma_y\partial_{i\tau})\psi d\sigma d\tau=\int\psi^T(\sigma_z\partial_\sigma+\partial_\tau)\psi d\sigma d\tau$
$=\int[\psi_1(\partial_\tau+\partial_\sigma)\psi_1+\psi_2(\partial_\tau-\partial_\sigma)\psi_2]d\sigma d\tau=\int(\psi_1\partial_z\psi_1+\psi_2\partial_{\tilde{z}}\psi_2)dzd\tilde{z}$

Neutrino equation in two dimensions:

Cor. 1.6.2. $S=\int i\bar{\psi}\frac{1+\gamma^3}{2}\gamma^a\partial_a\psi d\sigma d\tau,(\gamma^a,\gamma^3)=(\sigma_x,\sigma_y,\sigma_z)$
$S=\int\psi_1(\partial_\tau+\partial_\sigma)\psi_1 d\sigma d\tau=\int\psi_1\partial_z\psi_1 dzd\tilde{z}$

Anti neutrino equation in two dimensions:

Cor. 1.6.3. $S=\int i\bar{\psi}\frac{1-\gamma^3}{2}\gamma^a\partial_a\psi d\sigma d\tau,(\gamma^a,\gamma^3)=(\sigma_x,\sigma_y,\sigma_z)$
$S=\int\psi_2(\partial_\tau-\partial_\sigma)\psi_2 d\sigma d\tau=\int\psi_2\partial_{\tilde{z}}\psi_2 dzd\tilde{z}$

2 Left supersymmetric string [42, 44, 45]

2.1 Action and motional equation of left supersymmetric string

Thm. 2.1.1. $S=\int(\partial_z X^u\partial_{\tilde z}X_u+\varphi^u\partial_{\tilde z}\varphi_u)dzd\tilde z,\begin{cases}\delta X^u=\epsilon(z)\varphi^u\\ \delta\varphi^u=-\epsilon(z)\partial_z X^u\equiv-\frac{1}{2}\epsilon(z)(1,i)^a\partial_a X^u\end{cases}\Rightarrow\delta S=0$

Proof: $\delta S=\delta\int(\partial_z X^u\partial_{\tilde z}X_u+\varphi^u\partial_{\tilde z}\varphi_u)dzd\tilde z$
$\Leftrightarrow\delta S=\int(\partial_z\delta X^u\partial_{\tilde z}X_u+\partial_z X^u\partial_{\tilde z}\delta X_u+\delta\varphi^u\partial_{\tilde z}\varphi_u+\varphi^u\partial_{\tilde z}\delta\varphi_u)dzd\tilde z$
$\Leftrightarrow\delta S=\int\{\partial_z[\epsilon(z)\varphi^u]\partial_{\tilde z}X_u+\partial_z X^u\partial_{\tilde z}[\epsilon(z)\varphi_u]+[-\epsilon(z)\partial_z X^u]\partial_{\tilde z}\varphi_u+\varphi^u\partial_{\tilde z}[-\epsilon(z)\partial_z X_u]\}dzd\tilde z$
$\Leftrightarrow\delta S=\int\{\partial_z[\epsilon(z)\varphi^u]\partial_{\tilde z}X_u+\partial_z X^u\epsilon(z)\partial_{\tilde z}\varphi_u+[-\epsilon(z)\partial_z X^u]\partial_{\tilde z}\varphi_u+\varphi^u[-\epsilon(z)\partial_{\tilde z}\partial_z X_u]\}dzd\tilde z$
$\Leftrightarrow\delta S=\int\{\partial_z[\epsilon(z)\varphi^u]\partial_{\tilde z}X_u+\epsilon(z)\varphi^u\partial_z\partial_{\tilde z}X_u\}dzd\tilde z$
$\Leftrightarrow\delta S=\int\partial_z[\epsilon(z)\varphi^u\partial_{\tilde z}X_u]dzd\tilde z$
$\Rightarrow\delta S=0$ □

Closure of supersymmetric transformation:

Cor. 2.1.1. $S=\int(\partial_z X^u\partial_{\tilde z}X_u+\varphi^u\partial_{\tilde z}\varphi_u)dzd\tilde z\equiv\int(\frac{1}{2}\partial^a X^u\partial_a X_u+\varphi^u(-1,i)^a\partial_a\varphi_u)d\tau d\sigma$

$$\begin{cases}[\delta_{\epsilon_1(z)},\delta_{\epsilon_2(z)}]X^u=2\epsilon_1(z)\epsilon_2(z)\partial_z X^u\equiv\frac{1}{2}[\epsilon_1(z)(1,i)^a\epsilon_2(z)-\epsilon_2(z)(1,i)^a\epsilon_1(z)]\partial_a X^u\\ [\delta_{\epsilon_1(z)},\delta_{\epsilon_2(z)}]\varphi^u=2\epsilon_1(z)\epsilon_2(z)\partial_z\varphi^u+\partial_z[\epsilon_1(z)\epsilon_2(z)]\varphi^u\\ \equiv\frac{1}{2}[\epsilon_1(z)(1,i)^a\epsilon_2(z)-\epsilon_2(z)(1,i)^a\epsilon_1(z)]\partial_a\varphi^u+\partial_z[\epsilon_1(z)\epsilon_2(z)]\varphi^u\end{cases}$$

Cor. 2.1.2. $S=\int(\partial_z X^u\partial_{\tilde z}X_u+\varphi^u\partial_{\tilde z}\varphi_u)dzd\tilde z\equiv\int(\frac{1}{2}\partial^a X^u\partial_a X_u+\varphi^u(-1,i)^a\partial_a\varphi_u)d\tau d\sigma$

$$\begin{cases}[\delta_{\epsilon_1},\delta_{\epsilon_2}]X^u=2\epsilon_1\epsilon_2\partial_z X^u\equiv\frac{1}{2}[\epsilon_1(1,i)^a\epsilon_2-\epsilon_2(1,i)^a\epsilon_1]\partial_a X^u\\ [\delta_{\epsilon_1},\delta_{\epsilon_2}]\varphi^u=2\epsilon_1\epsilon_2\partial_z\varphi^u\equiv\frac{1}{2}[\epsilon_1(1,i)^a\epsilon_2-\epsilon_2(1,i)^a\epsilon_1]\partial_a\varphi^u\end{cases}$$

Cor. 2.1.3. $[\delta_{\epsilon_1},\delta_{\epsilon_2}]=2\epsilon_1\epsilon_2\partial_z\equiv\frac{1}{2}[\epsilon_1(1,i)^a\epsilon_2-\epsilon_2(1,i)^a\epsilon_1]\partial_a$

Def. 2.1.1. $\delta_\epsilon Y=[\epsilon Q,Y],\partial_a Y=i[P_a,Y]$

Cor. 2.1.4. $[\delta_{\epsilon_1},\delta_{\epsilon_2}]Y=[\epsilon_2 Q,\delta_{\epsilon_1}Y]-[\epsilon_1 Q,\delta_{\epsilon_2}Y]=[\epsilon_2 Q+,[\epsilon_1 Q,Y]]-[\epsilon_1 Q,[\epsilon_2 Q,Y]]=[[\epsilon_2 Q,\epsilon_1 Q],Y]$

Cor. 2.1.5. $[[\epsilon_2 Q,\epsilon_1 Q],Y]=2\epsilon_1\epsilon_2 i[P_z,Y]\Rightarrow Q^2=iP_z\Rightarrow[Q,P_z]=0$

2.2 Local bose transformation of left supersymmetric string action

Thm. 2.2.1. $S=\int\partial_z X^u\partial_{\tilde z}X_u dzd\tilde z,\delta X^u=v(z)\partial_z X^u\Rightarrow\delta S=0$

Proof: $\delta S=\delta\int\partial_z X^u\partial_{\tilde z}X_u dzd\tilde z$
$\Leftrightarrow\delta S=\int(\partial_z\delta X^u\partial_{\tilde z}X_u+\partial_z X^u\partial_{\tilde z}\delta X_u)dzd\tilde z$
$\Leftrightarrow\delta S=\int\{\partial_z[v(z)\partial_z X^u]\partial_{\tilde z}X_u+\partial_z X^u\partial_{\tilde z}[v(z)\partial_z X_u]\}dzd\tilde z$
$\Leftrightarrow\delta S=\int\{\partial_z[v(z)\partial_z X^u]\partial_{\tilde z}X_u+v(z)\partial_z X^u\partial_z\partial_{\tilde z}X_u\}dzd\tilde z$
$\Leftrightarrow\delta S=\int\partial_z[v(z)\partial_z X^u\partial_{\tilde z}X_u]dzd\tilde z$
$\Rightarrow\delta S=0$ □

Thm. 2.2.2. $S=\int\varphi^u\partial_{\tilde z}\varphi_u dzd\tilde z,\delta\varphi^u=v(z)\partial_z\varphi^u\Rightarrow\delta S?=0$

Proof: $\delta S=\delta\int\varphi^u\partial_{\tilde z}\varphi_u dzd\tilde z$
$\Leftrightarrow\delta S=\int\delta\varphi^u\partial_{\tilde z}\varphi_u+\varphi^u\partial_{\tilde z}\delta\varphi_u dzd\tilde z$
$\Leftrightarrow\delta S=\int[v(z)\partial_z\varphi^u]\partial_{\tilde z}\varphi_u+\varphi^u\partial_{\tilde z}[v(z)\partial_z\varphi_u]dzd\tilde z$
$\Leftrightarrow\delta S=\int v(z)\partial_z[\varphi^u\partial_{\tilde z}\varphi_u]dzd\tilde z$ □

2.3 Global bose transformation of left supersymmetric string action

Cor. 2.3.1. $S=\int\partial_z X^u\partial_{\tilde z}X_u dzd\tilde z,\delta X^u=v\partial_z X^u\Rightarrow\delta S=0$

Cor. 2.3.2. $S=\int\varphi^u\partial_{\tilde z}\varphi_u dzd\tilde z,\delta\varphi^u=v\partial_z\varphi^u\Rightarrow\delta S=0$

Proof: $\delta S=\delta\int\varphi^u\partial_{\tilde z}\varphi_u dzd\tilde z$
$\Leftrightarrow\delta S=\int\delta\varphi^u\partial_{\tilde z}\varphi_u+\varphi^u\partial_{\tilde z}\delta\varphi_u dzd\tilde z$
$\Leftrightarrow\delta S=\int[v\partial_z\varphi^u]\partial_{\tilde z}\varphi_u+\varphi^u\partial_{\tilde z}[v\partial_z\varphi_u]dzd\tilde z$
$\Leftrightarrow\delta S=\int\partial_z[v\varphi^u\partial_{\tilde z}\varphi_u]dzd\tilde z$
$\Rightarrow\delta S=0$ □

2.4 Global transformation closure of left supersymmetric string

Def. 2.4.1. $\delta_v Y = i[vP_z, Y] = v\partial_z$

Cor. 2.4.1. $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]Y = 2i\epsilon_1\epsilon_2[P_z, Y]$

Cor. 2.4.2. $[\delta_{v_1}, \delta_{v_2}]Y = [[iv_2P_z, iv_1P_z], Y] = 0$

Cor. 2.4.3. $[\delta_{v_1}, \delta_{\epsilon_2}]Y = [[\epsilon_2 Q, iv_1P_z], Y] = 0$

Cor. 2.4.4. $[Q, Q] = 2iP_z, [Q, P_z] = 0, [P_z, P_z] = 0$

3 Right supersymmetric string [42, 44, 45]

3.1 Action and motional equation of right supersymmetric string

Thm. 3.1.1. $S = \int(\partial_z X^u \partial_{\tilde{z}} X_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u)dzd\tilde{z}, \begin{cases}\delta X^u = \bar{\epsilon}(\tilde{z})\bar{\varphi}^u \\ \delta\bar{\varphi}^u = -\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X^u \equiv -\frac{1}{2}\bar{\epsilon}(\tilde{z})(-1,i)^a\partial_a X^u\end{cases} \Rightarrow \delta S = 0$

Closure of supersymmetric transformation:

Cor. 3.1.1. $S = \int(\partial_z X^u \partial_{\tilde{z}} X_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u)dzd\tilde{z} \equiv \int(\frac{1}{2}\partial^a X^u \partial_a X_u + \bar{\varphi}^u(1,i)^a\partial_a\bar{\varphi}_u)d\tau d\sigma$

$$\begin{cases}[\delta_{\bar{\epsilon}_1(\tilde{z})}, \delta_{\bar{\epsilon}_2(\tilde{z})}]X^u = 2\bar{\epsilon}_1(\tilde{z})\bar{\epsilon}_2(\tilde{z})\partial_z X^u \\ [\delta_{\bar{\epsilon}_1(\tilde{z})}, \delta_{\bar{\epsilon}_2(\tilde{z})}]\bar{\varphi}^u = 2\bar{\epsilon}_1(\tilde{z})\bar{\epsilon}_2(\tilde{z})\partial_z\bar{\varphi}^u + \partial_z[\bar{\epsilon}_1(\tilde{z})\bar{\epsilon}_2(\tilde{z})]\bar{\varphi}^u\end{cases}$$

Cor. 3.1.2. $S = \int(\partial_z X^u \partial_{\tilde{z}} X_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u)dzd\tilde{z} \equiv \int(\frac{1}{2}\partial^a X^u \partial_a X_u + \bar{\varphi}^u(1,i)^a\partial_a\bar{\varphi}_u)d\tau d\sigma$

$$\begin{cases}[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}]X^u = 2\bar{\epsilon}_1\bar{\epsilon}_2\partial_{\tilde{z}} X^u \\ [\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}]\bar{\varphi}^u = 2\bar{\epsilon}_1\bar{\epsilon}_2\partial_{\tilde{z}}\bar{\varphi}^u\end{cases}$$

Cor. 3.1.3. $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] = 2\bar{\epsilon}_1\bar{\epsilon}_2\partial_{\tilde{z}} \equiv \frac{1}{2}[\bar{\epsilon}_1(-1,i)^a\bar{\epsilon}_2 - \bar{\epsilon}_2(-1,i)^a\bar{\epsilon}_1]\partial_a$

Def. 3.1.1. $\delta_{\bar{\epsilon}} Y = [\bar{\epsilon}\bar{Q}, Y], \partial_a Y = i[P_a, Y]$

Cor. 3.1.4. $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}]Y = [\bar{\epsilon}_2\bar{Q}, \delta_{\bar{\epsilon}_1}Y] - [\bar{\epsilon}_1\bar{Q}, \delta_{\bar{\epsilon}_2}Y] = [\bar{\epsilon}_2\bar{Q}+, [\bar{\epsilon}_1\bar{Q}, Y]] - [\bar{\epsilon}_1\bar{Q}, [\bar{\epsilon}_2\bar{Q}, Y]] = [[\bar{\epsilon}_2\bar{Q}, \bar{\epsilon}_1\bar{Q}], Y]$

Cor. 3.1.5. $[[\bar{\epsilon}_2\bar{Q}, \bar{\epsilon}_1\bar{Q}], Y] = 2\bar{\epsilon}_1\bar{\epsilon}_2 i[P_{\tilde{z}}, Y] \Rightarrow \bar{Q}^2 = iP_{\tilde{z}} \Rightarrow [\bar{Q}, P_{\tilde{z}}] = 0$

3.2 Local bose transformation of right supersymmetric string action

Cor. 3.2.1. $S = \int \partial_z X^u \partial_{\tilde{z}} X_u dzd\tilde{z}, \delta X^u = \bar{v}(\tilde{z})\partial_{\tilde{z}} X^u \Rightarrow \delta S = 0$

Cor. 3.2.2. $S = \int \bar{\varphi}^u \partial_z \bar{\varphi}_u dzd\tilde{z}, \delta\bar{\varphi}^u = \bar{v}(\tilde{z})\partial_{\tilde{z}}\bar{\varphi}^u \Rightarrow \delta S = \int \bar{v}(\tilde{z})\partial_{\tilde{z}}[\bar{\varphi}^u\partial_z\bar{\varphi}_u]dzd\tilde{z}$

3.3 Global bose transformation of right supersymmetric string action

Cor. 3.3.1. $S = \int \partial_z X^u \partial_{\tilde{z}} X_u dzd\tilde{z}, \delta X^u = \bar{v}\partial_{\tilde{z}} X^u \Rightarrow \delta S = 0$

Cor. 3.3.2. $S = \int \bar{\varphi}^u \partial_{\tilde{z}} \bar{\varphi}_u dzd\tilde{z}, \delta\bar{\varphi}^u = \bar{v}\partial_{\tilde{z}}\bar{\varphi}^u \Rightarrow \delta S = 0$

3.4 Global transformation closure of right supersymmetric string action

Def. 3.4.1. $\delta_v Y = i[\bar{v}P_{\tilde{z}}, Y] = \bar{v}\partial_{\tilde{z}}$

Cor. 3.4.1. $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}]Y = 2i\bar{\epsilon}_1\bar{\epsilon}_2[P_{\tilde{z}}, Y]$

Cor. 3.4.2. $[\delta_{\bar{v}_1}, \delta_{\bar{v}_2}]Y = [[i\bar{v}_2P_{\tilde{z}}, i\bar{v}_1P_{\tilde{z}}], Y] = 0$

Cor. 3.4.3. $[\delta_{\bar{v}_1}, \delta_{\bar{\epsilon}_2}]Y = [[\bar{\epsilon}_2\bar{Q}, i\bar{v}_1P_{\tilde{z}}], Y] = 0$

Cor. 3.4.4. $[\bar{Q}, \bar{Q}] = 2iP_{\tilde{z}}, [\bar{Q}, P_{\tilde{z}}] = 0, [P_{\tilde{z}}, P_{\tilde{z}}] = 0$

4 Left and right supersymmetric string [42, 44, 45]

4.1 Action and motional equation of left and right supersymmetric string in mass shell

Cor. 4.1.1. $S_B = \frac{1}{2\pi\alpha'}\int \partial_z X^u \partial_{\tilde{z}} X_u dzd\tilde{z}, S_F = \frac{1}{4\pi}\int(\bar{\varphi}^u\partial_z\bar{\varphi}_u + \varphi^u\partial_{\tilde{z}}\varphi_u)dzd\tilde{z}$

Thm. 4.1.1. $S = \int dzd\tilde{z}(\partial_z X^u \partial_{\tilde{z}} X_u + \varphi^u\partial_{\tilde{z}}\varphi_u + \bar{\varphi}^u\partial_z\bar{\varphi}_u)$

$$\begin{cases}\delta X^u = \epsilon(z)\varphi^u + \bar{\epsilon}(\tilde{z})\bar{\varphi}^u \\ \delta\varphi^u = -\epsilon(z)\partial_z X^u \\ \delta\bar{\varphi}^u = -\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}} X^u\end{cases} \Leftrightarrow \delta\begin{bmatrix}X^u \\ \varphi^u \\ \bar{\varphi}^u\end{bmatrix} = \begin{bmatrix}0 & \epsilon(z) & \bar{\epsilon}(\tilde{z}) \\ -\epsilon(z)\partial_z & 0 & 0 \\ -\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}} & 0 & 0\end{bmatrix}\begin{bmatrix}X^u \\ \varphi^u \\ \bar{\varphi}^u\end{bmatrix} \Rightarrow \delta S = 0$$

Proof: $\delta S = \delta\int dzd\tilde{z}(\partial_z X^u \partial_{\tilde{z}} X_u + \varphi^u\partial_{\tilde{z}}\varphi_u + \bar{\varphi}^u\partial_z\bar{\varphi}_u)$
$\Leftrightarrow \delta S = \int dzd\tilde{z}(\partial_z\delta X^u\partial_{\tilde{z}}X_u + \partial_z X^u\partial_{\tilde{z}}\delta X_u + \delta\varphi^u\partial_{\tilde{z}}\varphi_u + \varphi^u\partial_{\tilde{z}}\delta\varphi_u + \delta\bar{\varphi}^u\partial_z\bar{\varphi}_u + \bar{\varphi}^u\partial_z\delta\bar{\varphi}_u)$
$\Leftrightarrow \delta S = \int dzd\tilde{z}\{\partial_z[\epsilon(z)\varphi^u]\partial_{\tilde{z}}X_u + \bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X_u\partial_z\bar{\varphi}^u + \epsilon(z)\partial_z X^u\partial_{\tilde{z}}\varphi^u + \partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})\bar{\varphi}^u]\partial_z X^u$
$- \epsilon(z)\partial_z X^u\partial_{\tilde{z}}\varphi_u + \epsilon(z)\varphi^u\partial_z\partial_{\tilde{z}}X^u - \bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X^u\partial_z\bar{\varphi}_u + \bar{\epsilon}(\tilde{z})\bar{\varphi}^u\partial_{\tilde{z}}\partial_z X^u\}$
$\Leftrightarrow \delta S = \int dzd\tilde{z}\{\partial_z[\epsilon(z)\varphi^u\partial_{\tilde{z}}X_u] + \partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})\bar{\varphi}^u\partial_z X^u]\}$
$\Rightarrow \delta S = 0$ □

Although the above is a supersymmetric transformation, it only satisfies the closure of the mass shell.

4.2 Action and motional equation of left and right supersymmetric string in non mass shell

Thm. 4.2.1. $S=\int dzd\tilde{z}(\partial_z X^u\partial_{\tilde{z}}X_u+\varphi^u\partial_{\tilde{z}}\varphi_u+\bar{\varphi}^u\partial_z\bar{\varphi}_u+F^uF_u)$

$$\begin{cases}\delta X^u=\epsilon(z)\varphi^u+\bar{\epsilon}(\tilde{z})\bar{\varphi}^u\\ \delta\varphi^u=-\epsilon(z)\partial_z X^u-\bar{\epsilon}(\tilde{z})F^u\\ \delta\bar{\varphi}^u=-\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X^u-\epsilon(z)F^u\\ \delta F^u=\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}\varphi^u+\epsilon(z)\partial_z\bar{\varphi}^u\end{cases}\Leftrightarrow\delta\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}=\begin{bmatrix}0&\epsilon(z)&\bar{\epsilon}(\tilde{z})&0\\ -\epsilon(z)\partial_z&0&0&-\bar{\epsilon}(\tilde{z})\\ -\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}&0&0&-\epsilon(z)\\ 0&\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}&\epsilon(z)\partial_z&0\end{bmatrix}\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}\Rightarrow\delta S=0$$

Proof: $\delta S=\delta\int dzd\tilde{z}(\partial_z X^u\partial_{\tilde{z}}X_u+\varphi^u\partial_{\tilde{z}}\varphi_u+\bar{\varphi}^u\partial_z\bar{\varphi}_u+F^uF_u)$
$\Leftrightarrow\delta S=\int dzd\tilde{z}(\partial_z\delta X^u\partial_{\tilde{z}}X_u+\partial_z X^u\partial_{\tilde{z}}\delta X_u+\delta\varphi^u\partial_{\tilde{z}}\varphi_u+\varphi^u\partial_{\tilde{z}}\delta\varphi_u+\delta\bar{\varphi}^u\partial_z\bar{\varphi}_u+\bar{\varphi}^u\partial_z\delta\bar{\varphi}_u+\frac{1}{2}\delta F^uF_u+\frac{1}{2}F^u\delta F_u)$
$\Leftrightarrow\delta S=\int dzd\tilde{z}\{\partial_z[\epsilon(z)\varphi^u]\partial_{\tilde{z}}X_u+\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X_u\partial_z\bar{\varphi}^u+\epsilon(z)\partial_z X^u\partial_{\tilde{z}}\varphi^u+\partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})\bar{\varphi}^u]\partial_z X^u$
$-\epsilon(z)\partial_z X^u\partial_{\tilde{z}}\varphi_u-\bar{\epsilon}(\tilde{z})F^u\partial_{\tilde{z}}\varphi_u+\epsilon(z)\varphi^u\partial_z\partial_{\tilde{z}}X^u+\partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})F_u]\varphi^u$
$-\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X^u\partial_z\bar{\varphi}_u-\epsilon(z)F^u\partial_z\bar{\varphi}_u+\bar{\epsilon}(\tilde{z})\bar{\varphi}^u\partial_{\tilde{z}}\partial_z X^u+\partial_z[\epsilon(z)F_u]\bar{\varphi}^u$
$+2[\bar{\epsilon}(\tilde{z})F_u]\partial_{\tilde{z}}\varphi^u+2[\epsilon(z)F_u]\partial_z\bar{\varphi}^u\}$
$\Leftrightarrow\delta S=\int dzd\tilde{z}\{\partial_z[\epsilon(z)(\varphi^u\partial_{\tilde{z}}X_u+\bar{\varphi}^uF_u)]+\partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})(\bar{\varphi}^u\partial_z X^u+\varphi^uF_u)]\}$
$\Rightarrow\delta S=0$ □

Although the above is a supersymmetric transformation, it does not meet the closure requirement. The following is not only a supersymmetric transformation, but also a non shell closure.

Thm. 4.2.2. $S=\int dzd\tilde{z}(\partial_z X^u\partial_{\tilde{z}}X_u+\varphi^u\partial_{\tilde{z}}\varphi_u+\bar{\varphi}^u\partial_z\bar{\varphi}_u+F^uF_u)$

$$\begin{cases}\delta X^u=\epsilon(z)\varphi^u+\bar{\epsilon}(\tilde{z})\bar{\varphi}^u\\ \delta\varphi^u=-\epsilon(z)\partial_z X^u-\bar{\epsilon}(\tilde{z})F^u\\ \delta\bar{\varphi}^u=-\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X^u+\epsilon(z)F^u\\ \delta F^u=\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}\varphi^u-\epsilon(z)\partial_z\bar{\varphi}^u\end{cases}\Leftrightarrow\delta\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}=\begin{bmatrix}0&\epsilon(z)&\bar{\epsilon}(\tilde{z})&0\\ -\epsilon(z)\partial_z&0&0&-\bar{\epsilon}(\tilde{z})\\ -\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}&0&0&\epsilon(z)\\ 0&\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}&-\epsilon(z)\partial_z&0\end{bmatrix}\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}\Rightarrow\delta S=0$$

Proof: $\delta S=\delta\int dzd\tilde{z}(\partial_z X^u\partial_{\tilde{z}}X_u+\varphi^u\partial_{\tilde{z}}\varphi_u+\bar{\varphi}^u\partial_z\bar{\varphi}_u+F^uF_u)$
$\Leftrightarrow\delta S=\int dzd\tilde{z}(\partial_z\delta X^u\partial_{\tilde{z}}X_u+\partial_z X^u\partial_{\tilde{z}}\delta X_u+\delta\varphi^u\partial_{\tilde{z}}\varphi_u+\varphi^u\partial_{\tilde{z}}\delta\varphi_u+\delta\bar{\varphi}^u\partial_z\bar{\varphi}_u+\bar{\varphi}^u\partial_z\delta\bar{\varphi}_u+\frac{1}{2}\delta F^uF_u+\frac{1}{2}F^u\delta F_u)$
$\Leftrightarrow\delta S=\int dzd\tilde{z}\{\partial_z[\epsilon(z)\varphi^u]\partial_{\tilde{z}}X_u+\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X_u\partial_z\bar{\varphi}^u+\epsilon(z)\partial_z X^u\partial_{\tilde{z}}\varphi^u+\partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})\bar{\varphi}^u]\partial_z X^u$
$-\epsilon(z)\partial_z X^u\partial_{\tilde{z}}\varphi_u-\bar{\epsilon}(\tilde{z})F^u\partial_{\tilde{z}}\varphi_u+\epsilon(z)\varphi^u\partial_z\partial_{\tilde{z}}X^u+\partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})F_u]\varphi^u$
$-\bar{\epsilon}(\tilde{z})\partial_{\tilde{z}}X^u\partial_z\bar{\varphi}_u+\epsilon(z)F^u\partial_z\bar{\varphi}_u+\bar{\epsilon}(\tilde{z})\bar{\varphi}^u\partial_{\tilde{z}}\partial_z X^u-\partial_z[\epsilon(z)F_u]\bar{\varphi}^u$
$+2[\bar{\epsilon}(\tilde{z})F_u]\partial_{\tilde{z}}\varphi^u-2[\epsilon(z)F_u]\partial_z\bar{\varphi}^u\}$
$\Leftrightarrow\delta S=\int dzd\tilde{z}\{\partial_z[\epsilon(z)(\varphi^u\partial_{\tilde{z}}X_u-\bar{\varphi}^uF_u)]+\partial_{\tilde{z}}[\bar{\epsilon}(\tilde{z})(\bar{\varphi}^u\partial_z X^u+\varphi^uF_u)]\}$
$\Rightarrow\delta S=0$ □

Def. 4.2.1. $\delta_\epsilon\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}=\begin{bmatrix}0&\epsilon&\bar{\epsilon}&0\\ -\epsilon\partial_z&0&0&-\bar{\epsilon}\\ -\bar{\epsilon}\partial_{\tilde{z}}&0&0&\epsilon\\ 0&\bar{\epsilon}\partial_{\tilde{z}}&-\epsilon\partial_z&0\end{bmatrix}\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}$

Thm. 4.2.3. $[\delta_{\epsilon_1},\delta_{\epsilon_2}]\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}=([\epsilon_1,\epsilon_2]\partial_z+[\bar{\epsilon}_1,\bar{\epsilon}_2]\partial_{\tilde{z}})\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}$

Proof: $[\delta_{\epsilon_1},\delta_{\epsilon_2}]\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}=-[\begin{bmatrix}0&\epsilon_1&\bar{\epsilon}_1&0\\ -\epsilon_1\partial_z&0&0&-\bar{\epsilon}_1\\ -\bar{\epsilon}_1\partial_{\tilde{z}}&0&0&\epsilon_1\\ 0&\bar{\epsilon}_1\partial_{\tilde{z}}&-\epsilon_1\partial_z&0\end{bmatrix},\begin{bmatrix}0&\epsilon_2&\bar{\epsilon}_2&0\\ -\epsilon_2\partial_z&0&0&-\bar{\epsilon}_2\\ -\bar{\epsilon}_2\partial_{\tilde{z}}&0&0&\epsilon_2\\ 0&\bar{\epsilon}_2\partial_{\tilde{z}}&-\epsilon_2\partial_z&0\end{bmatrix}]\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}$

$\Leftrightarrow[\delta_{\epsilon_1},\delta_{\epsilon_2}]\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}=([\epsilon_1,\epsilon_2]\partial_z+[\bar{\epsilon}_1,\bar{\epsilon}_2]\partial_{\tilde{z}})\begin{bmatrix}X^u\\ \varphi^u\\ \bar{\varphi}^u\\ F^u\end{bmatrix}$ □

4.3 Supersymmetric algebra of left right supersymmetric string

Def. 4.3.1. $\delta_\epsilon Y=[\epsilon Q+\bar{\epsilon}\bar{Q},Y],\partial_a Y=i[P_a,Y]$

Cor. 4.3.1. $[\delta_{\epsilon_1},\delta_{\epsilon_2}]Y=[\epsilon_2 Q+\bar{\epsilon}_2\bar{Q},\delta_{\epsilon_1}Y]-[\epsilon_1 Q+\bar{\epsilon}_1\bar{Q},\delta_{\epsilon_2}Y]$
$=[\epsilon_2 Q+\bar{\epsilon}_2\bar{Q},[\epsilon_1 Q+\bar{\epsilon}_1\bar{Q},Y]]-[\epsilon_1 Q+\bar{\epsilon}_1\bar{Q},[\epsilon_2 Q+\bar{\epsilon}_2\bar{Q},Y]]$
$=[[\epsilon_2 Q+\bar{\epsilon}_2\bar{Q},\epsilon_1 Q+\bar{\epsilon}_1\bar{Q}],Y]$

Cor. 4.3.2. $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]Y = \{[\epsilon_1, \epsilon_2]\partial_z + [\bar{\epsilon}_1, \bar{\epsilon}_2]\partial_{\tilde{z}}\}Y$
$\Leftrightarrow [[\epsilon_2 Q + \bar{\epsilon}_2\bar{Q}, \epsilon_1 Q + \bar{\epsilon}_1\bar{Q}], Y] = i[[\epsilon_1, \epsilon_2]P_{\tilde{z}} + [\bar{\epsilon}_1, \bar{\epsilon}_2]P_{\tilde{z}}, Y]$
$\Leftrightarrow [[\epsilon_2 Q + \bar{\epsilon}_2\bar{Q}, \epsilon_1 Q + \bar{\epsilon}_1\bar{Q}], Y] = i[2\epsilon_1\epsilon_2 P_{\tilde{z}} + 2\bar{\epsilon}_1\bar{\epsilon}_2 P_{\tilde{z}}, Y]$
$\Leftrightarrow [\epsilon_2 Q + \bar{\epsilon}_2\bar{Q}, \epsilon_1 Q + \bar{\epsilon}_1\bar{Q}] = i(2\epsilon_1\epsilon_2 P_{\tilde{z}} + 2\bar{\epsilon}_1\bar{\epsilon}_2 P_{\tilde{z}})$
$\Leftrightarrow \{Q, \bar{Q}\} = 0, \{Q, Q\} = 2iP_{\tilde{z}}, \{\bar{Q}, \bar{Q}\} = 2iP_{\tilde{z}}$
$\Leftrightarrow \{Q, \bar{Q}\} = 0, \{Q, Q\} = 2iP_{\tilde{z}}, \{\bar{Q}, \bar{Q}\} = 2iP_{\tilde{z}}, [Q, P_{\tilde{z}}] = 0, [Q, P_{\tilde{z}}] = 0, [\bar{Q}, P_{\tilde{z}}] = 0, [\bar{Q}, P_{\tilde{z}}] = 0, [P_z, P_{\tilde{z}}] = 0$

4.4 Hyperspace representation of left right supersymmetric string

Cor. 4.4.1. $D_\theta = \partial_\theta + \theta\partial_z, D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\tilde{z}}$

Cor. 4.4.2. $\hat{X}(z, \tilde{z}, \theta, \bar{\theta}) = X + \theta\varphi + \bar{\theta}\bar{\varphi} + \theta\bar{\theta}F$

Cor. 4.4.3. $S = \int D_\theta \hat{X} D_{\bar{\theta}} \hat{X} d\theta d\bar{\theta} dz d\tilde{z}$

5 Local supersymmetric string action [42, 44, 45]

5.1 Supersymmetric action on the world surface

Thm. 5.1.1. $S = -\frac{1}{2}\int d\sigma d\tau e\{g^{ab}\partial_a X^u \partial_a X_u + i\bar{\psi}^u\gamma^a\partial_a\psi_u + 2i\bar{\chi}_a\gamma^b\gamma^a\psi^u[\partial_b X_u + \frac{i}{2}\bar{\psi}_u\chi_b]\}$

$$\begin{cases} \delta X^u = i\bar{\eta}\psi^u; \\ \delta e^b_\beta = -2i\bar{\eta}\gamma^b\psi_\beta; \\ \delta\psi^u = \gamma^a(\partial_a X^u + i\bar{\psi}^u\chi_a)\eta; \\ \delta\chi_\beta = -D_\beta\eta = -\partial_\beta\eta + \frac{i}{2}\omega_\beta\gamma^x\gamma^y\eta \end{cases}$$

5.2 Supersymmetric action in space-time

Thm. 5.2.1. $S = -\frac{1}{2\pi}\int d\sigma d\tau e\{g^{ab}(\partial_a X^u - i\sum_A \bar{\theta}^A\gamma^u\partial_a\theta^A)(\partial_b X^u - i\sum_A \bar{\theta}^A\gamma^u\partial_b\theta^A)\}$
$+ \frac{1}{\pi}\int\{-idX^u \wedge (\bar{\theta}^1\gamma_u d\theta^1 - \bar{\theta}^2\gamma_u d\theta^2) + \bar{\theta}^1\gamma^u d\theta^1 \wedge \bar{\theta}^2\gamma_u d\theta^2\}$

$$\begin{cases} \delta\theta^A = \varepsilon^A \\ \delta X^u = \frac{i}{2}\sum_A(\bar{\varepsilon}^A\gamma^u\theta^A - \bar{\theta}^A\gamma^u\varepsilon^A) \equiv i\sum_A \bar{\varepsilon}^A\gamma^u\theta^A \end{cases}$$

5.3 Spin representation of supersymmetric theory

Thm. 5.3.1.

$$\begin{cases} [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\psi = 0 \\ [\partial_a + iS_{ab}(1, \varsigma)\partial^b]\hat{Q}^+\psi = 0 \\ [\frac{3}{2}\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b]\hat{Q}^{+2}\psi = 0 \\ [2\partial_a + iS_{ab}(2, \varsigma)\partial^b]\hat{Q}^{+3}\psi = 0 \\ \cdots \end{cases}$$

Chapter41 Explicit Representation of Ground State Wave Function

1 Vacuum state wave function of harmonic oscillator

1.1 Fock representation of harmonic oscillator

Pro. 1.1.1. $a|0\rangle = 0, |n\rangle = \frac{1}{\sqrt{n!}}a^+|0\rangle$

Pro. 1.1.2. $a = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{n} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, a^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{n} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \hat{N} := a^+a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$

Pro. 1.1.3. $|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdots \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \\ \cdots \end{bmatrix}, |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \cdots \\ 0 \\ \cdots \end{bmatrix}, |n\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \\ \cdots \end{bmatrix}$

1.2 Coordinate representation of harmonic oscillator

Pro. 1.2.1. $a = x + \partial_x, a^+ = x - \partial_x$

Pro. 1.2.2. $a|0\rangle = 0 \Rightarrow |0\rangle = \frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}x^2}, |n\rangle = \frac{1}{\sqrt{\pi}\sqrt{n!2^n}}(x-\partial_x)^n e^{-\frac{1}{2}x^2}$

1.3 Coordinate representation of multiple harmonic oscillators

Pro. 1.3.1. $a_i = x_i + \partial_{x_i}, a_i^+ = x_i - \partial_{x_i}$

Pro. 1.3.2. $a_i|0\rangle = 0, i = 1,\cdot\cdot,l \Rightarrow$

$|0\rangle = (\frac{1}{\sqrt{\pi}})^l e^{-\frac{1}{2}\sum\limits_{i=1}^{l} x_i^2}, |n_1,\cdot\cdot,n_l\rangle = (\frac{1}{\sqrt{\pi}})^l (\frac{1}{\sqrt{2}})^{\sum\limits_{i=1}^{l} n_i} \frac{1}{\sqrt{n_1!\cdot\cdot n_l!}}(x_1-\partial_{x_1})^{n_1}\cdot\cdot(x_l-\partial_{x_l})^{n_l} e^{-\frac{1}{2}\sum\limits_{i=1}^{l} x_i^2}$

1.4 Coordinate representation of infinite harmonic oscillators

Pro. 1.4.1. $a_i = x_i + \partial_{x_i}, a_i^+ = x_i - \partial_{x_i}$

Pro. 1.4.2. $a_i|0\rangle = 0, i = 1,\cdot\cdot,l \Rightarrow$

$|0\rangle = \lim\limits_{l\to\infty}(\frac{1}{\sqrt{\pi}})^l e^{-\frac{1}{2}\sum\limits_{i=1}^{l} x_i^2}, |n_1,\cdot\cdot,n_\infty\rangle = \lim\limits_{l\to\infty}(\frac{1}{\sqrt{\pi}})^l (\frac{1}{\sqrt{2}})^{\sum\limits_{i=1}^{l} n_i} \frac{1}{\sqrt{n_1!\cdot\cdot n_l!}}(x_1-\partial_{x_1})^{n_1}\cdot\cdot(x_l-\partial_{x_l})^{n_l} e^{-\frac{1}{2}\sum\limits_{i=1}^{l} x_i^2}$

1.5 Visual representation of infinite harmonic oscillator in coordinates

Pro. 1.5.1. $a_i = x_i + \partial_{x_i}, a_i^+ = x_i - \partial_{x_i}$

Pro. 1.5.2. $a_i|0\rangle = 0, i = 1,\cdot\cdot,+\infty \Rightarrow$

$|0\rangle = (\frac{1}{\sqrt{\pi}})^{+\infty} e^{-\frac{1}{2}\sum\limits_{i=1}^{+\infty} x_i^2}, |n_1,\cdot\cdot,n_\infty\rangle = (\frac{1}{\sqrt{\pi}})^{+\infty} (\frac{1}{\sqrt{2}})^{\sum\limits_{i=1}^{+\infty} n_i} \frac{1}{\sqrt{n_1!\cdot\cdot n_\infty!}}(x_1-\partial_{x_1})^{n_1}\cdot\cdot(x_\infty-\partial_{x_\infty})^{n_\infty} e^{-\frac{1}{2}\sum\limits_{i=1}^{+\infty} x_i^2}$

1.6 Coordinate representation of infinite harmonic oscillator in quantum field theory

Ass. 1.6.1. $a(x) = \frac{1}{\sqrt{2}}[\phi(x) + \frac{\delta}{\delta\phi(x)}], a^+(x) = \frac{1}{\sqrt{2}}[\phi(x) - \frac{\delta}{\delta\phi(x)}]$

Pro. 1.6.1. $a_i|0\rangle = 0, i = 1,\cdot\cdot,l \Rightarrow$

$|0\rangle = e^{-\frac{1}{2}\int\phi^2(x)dx}, |n_1,\cdot\cdot,n_\infty\rangle = \lim\limits_{l\to\infty}(\frac{1}{\sqrt{\pi}})^l (\frac{1}{\sqrt{2}})^{\sum\limits_{i=1}^{l} n_i} \frac{1}{\sqrt{n_1!\cdot\cdot n_l!}}(x_1-\partial_{x_1})^{n_1}\cdot\cdot(x_l-\partial_{x_l})^{n_l} e^{-\frac{1}{2}\sum\limits_{i=1}^{l} x_i^2}$

2 Sequence representation of DNA

2.1 Mathematical description of DNA sequences

Def. 2.1.1. $R := R_1^{l_1}R_2^{l_2}R_3^{l_3}R_4^{l_4}\cdot\cdot R_n^{l_n}, R_i \in \{A,G,T,C\}, l_i > 0, R_i \neq R_{i+1}; N := l_1 + l_2 + \cdot\cdot + l_n$

Def. 2.1.2. $\bar{R} := \bar{R}_1^{l_1}\bar{R}_2^{l_2}\bar{R}_3^{l_3}\bar{R}_4^{l_4}\cdot\cdot\bar{R}_n^{l_n}, \bar{R}_i \in \{A,G,T,C\}, l_i > 0, \bar{R}_i \neq \bar{R}_{i+1}; N := l_1 + l_2 + \cdot\cdot + l_n$

Def. 2.1.3. $RNA := R = R_1^{l_1}R_2^{l_2}R_3^{l_3}R_4^{l_4}\cdot\cdot R_n^{l_n}, DNA := \frac{R}{\bar{R}} = \frac{R_1^{l_1}R_2^{l_2}R_3^{l_3}R_4^{l_4}\cdot\cdot R_n^{l_n}}{\bar{R}_1^{l_1}\bar{R}_2^{l_2}\bar{R}_3^{l_3}\bar{R}_4^{l_4}\cdot\cdot\bar{R}_n^{l_n}}$

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