Gauge Invariance in Counting and Combinatorics

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There is perhaps no part of mathematics that is more intimately connected with everyday experiences than probability theory and statistics. The element of chance dominates the physical world. Probability is the heart of physics, in particular – quantum physics. At the probability theory lies combinatorics. We make an observation in the combination of n objects taken r objects at a time. We find a sort of combinatorial gauge invariance hidden there in that the combination of n objects taken r at a time or $n - r$ at a time is the same. It has been argued that the Pauli Exclusion Principle is not a principle or cause at all; rather it is an effect of the combinatorics which essentially delivers the Fermi Dirac Statistics. We explore the consequent applications in quantum mechanics and field theory with particle statistics.

Consider n objects and translate it to $n-r$ objects for some $r<n$. Then the combination combinatorial coefficients $C_r^n = C_{n-r}^n$ imply a kind of combinatorial gauge invariance; $n \to n-r$ is a combinatorial gauge transformation. This will be argued below in a more proper context and ramified.

Quantum Mechanics is the theory that successfully describes small systems like fundamental particles, atoms and molecules.

In quantum theory, the fundamental mathematical object describing the quantum systems is the wavefunction Ψ. This Ψ is a function of coordinates and time, and is a complex quantity.

It figures in the Schrödinger equation which describes the evolution of a quantum system.

$$i\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(x,y,z)\Psi \tag{1}$$

Max Born came up with the statistical interpretation of the wave function Ψ. According to the Born rule, the squared absolute value of the wavefunction $|\Psi|^2 dV$ is the probability density of finding the quantum system in some small volume dV.

Now, we know that **the process of measurement entails counting which in turn leads to probabilistic and statistical laws**. Consider the combination of n objects taken r objects at a time. This is given by

$$C_r^n = \frac{n!}{r!(n-r)!} \tag{2}$$

If instead we replace r objects by "$n-r$" objects, we notice that the combination is still the same as above, thus, $C_r^n = C_{n-r}^n$.

Thus, we can interpret this invariance as a combinatorial invariance: a combinatorial redundancy. We propose this as a gauge invariance, inherent in the counting principles.

Now, the expectation value under the Born rule is actually invariant under the Combinatorial Gauge transformation: $C_r^n \rightarrow C_{n-r}^n$, hence the symmetric probability distribution curve. The fact of the matter is that the probability distribution of ψ is gauge invariant under the above gauge transformation, for a symmetric distribution and mirror image of the distribution before the gauge transformation for a asymmetric distribution. Nevertheless, we may assert that *the Born interpretation is invariant under the combinatorial gauge transformation.* We adopt Pascal's identity, viz., $C_r^n + C_{r-1}^n = C_r^{n+1}$ as the fundamental result of the combinatorial gauge theory.

Let us now come to the fundamental statistics in quantum physics, viz.,

1. **The BOSE-EINSTEIN STATISTICS (BES) and**
2. **The FERMI-DIRAC STATISTICS (FDS)**

For the BES, the Bose counting problem is posed thus: *how many ways can r indistinguishable particles be put in n distinguishable boxes?*

This number is given by

$$B_r^n = \frac{(n+r-1)!}{r!(n-1)!} \tag{3}$$

Where B_r^n stands for $n\ Bose\ r$ analogous to C_r^n standing for $n\ Choose\ r$. Particles obeying this rule are called Bosons and carry integer spin values. They can occupy any state simultaneously.

For FDS, the Dirac counting problem is posed thus: *how many ways can r indistinguishable particles be put in n distinguishable boxes, with at most one in each box?*

This number is given by

$$C_r^n = \frac{n!}{r!(n-r)!} \tag{2}$$

Particles obeying this combinatorial rule are called Fermions and carry half odd integer spin values. They obey the Pauli Exclusion Principle which states that

No two Fermi particles with identical quantum numbers can occupy the same quantum state.

Let's reverse this line of reasoning. Let's start with the BES and FDS as given; then notice the combinatorics of the FDS: ***FDS is invariant under the combinatorial gauge. The BES is NOT!!!***

This tells us something: ***Particles obeying the combinatorial gauge invariance carry half odd integer spins and obey the Pauli Exclusion Principle.*** The Pauli exclusion principle is thus a consequence of classical combinatorics. It is an effect. The FDS that springs from classical combinations is the cause. They cannot attain simultaneously the vacuum state. Bosons, on the other hand, can simultaneously occupy the vacuum state. In the atoms of elements such as liquid Helium, the atoms have net whole integer spin and hence at critical low temperature $\sim 0K,$ jump into the lowest energy state and thereby form the BOSE-EINSTEIN CONDENSATE.

Before we go to QUANTUM FIELD THEORY (QFT), we have a look at the origin of combinatorial gauge symmetry.

The BES is given by (3), viz., $B_r^n = \frac{(n+r-1)!}{r!(n-1)!}$ and, the FDS is given by (2) viz., $C_r^n = \frac{n!}{r!(n-r)!}$

The above two statistics are selection of indistinguishable objects. We know that there are arrangement of objects. This is called **PERMUTATIONS**. These are given for the above case by

$$P_r^n = \frac{n!}{(n-r)!} \tag{4}$$

There is an easy relation between **PERMUTATIONS AND COMBINATIONS**, as

$$P_r^n = r!\, C_r^n \tag{5}$$

But for **BES**, we have for the combinatorial gauge transformation $B_r^n \to B_{n-r}^n$, and only for this transformation, a relationship with P_r^n and that too is a monstrosity

$$P_r^n = \frac{(n!)^2}{n(2n-r-1)!} B_{n-r}^n \tag{6}$$

So that,

$$P_r^n = \frac{1}{2}\left[\frac{(n!)^2}{n(2n-r-1)!} B_{n-r}^n + r!\, C_r^n\right] \tag{7}$$

Which can also be rewritten as

$$P_r^n = \frac{1}{2}\left[\frac{(n!)^2}{n(2n-r-1)!} B_{n-r}^n + r!\, C_{n-r}^n\right] \tag{8}$$

Thus, mathematically, the permutation is the super structure from which stem the **BES** and the **FDS**. Finally, the P_r^n and C_r^n are combinatorially gauge related.

This can be seen from $P_r^n = r!\, C_r^n$;

Which is the same as

$$P_{n-r}^n = (n-r)!\, C_{n-r}^n. \tag{9}$$

Now we come to **QFT.** For a world coordinate translation $x^\mu \to x^{\mu'}$, the Poincaré group is given by

$$x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu} \tag{10}$$

Where Λ^{μ}_{ν} is the Lorentz tensor and a^{μ} is an arbitrary translation in Minkowskiian spacetime. Now in **QFT**, for the double cover of the Poincaré group we have irreps as follows:

1. $(0,0)$: $spin\ 0\ representation$ which acts on scalars. The constraint that the Lagrangian is invariant yields free spin 0 Lagrangian and the Euler-Lagrange equations lead to the Klein-Gordon Equation.
2. $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$: $spin\ \frac{1}{2}\ representation$ which acts on spinors. The constraint that the Lagrangian is invariant yields free spin $\frac{1}{2}$ Lagrangian and the Euler-Lagrange equations lead to the Dirac Equation.
3. $\left(\frac{1}{2},\frac{1}{2}\right)$: $spin\ 1\ representation$ which acts on vectors. The constraint that the Lagrangian is invariant yields free spin 1 Lagrangian and the Euler-Lagrange equations lead to the Proca Equation.

Now in the above three irreps, only no. 2 will be combinatorial gauge invariant because it is a Fermionic Field. Fermions are described by spinors. The BES conserves the property of positive definiteness of the time-time component of the energymomentum tensor. Applying, the BES to fermions violates this. The FDS preserves causality. Thus, applying this to the bosons violates causality. Hence the combinatorial gauge invariance lies at the heart of this matter. So, FDS in essence is classical combinatorics. This is very easy to see. The Pauli exclusion puts forth a scheme that only one piece fits into a given square. This a classical combinatoric result. Thus, causality is a classical result. The BES on the other hand does not figure explicitly in the classical combinatorial scheme. Thus, the positive definiteness of the time-time component of the energymomentum tensor is a truly quantum fact. It is deeply rooted in the quantum domain. The future applications of this gauge invariance is still to be foreseen.

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