Any Formal System That Contains Sets Arithmetic and Rational Numbers is Inconsistent

By Jim Rock

Abstract: Gödel proved that any formal system containing arithmetic is incomplete. We show that any such formal system is inconsistent. We establish a collection of nested sets of rational numbers in a descending hierarchy. The sets higher in the descending hierarchy contain element(s) that are not in the sets below them in the hierarchy. Given such a descending set hierarchy, it is easy to develop two arguments that contradict each other. The conclusion of Argument#2 is false. But, Argument#2 is a valid argument.

For rational numbers *a* in [*0, 1*] let the collection of R*a* sets be { *y* is a rational number | $0 \leq y < a$ }

Argument#1: No R*a* contains a largest element.

1) Suppose there is a largest element a' in some individual R*a*.
2) $a' < (a' + a)/2 < a$.
3) Let $b = (a' + a)/2$.
4) Then b is in R*a* and $a' < b$.
5) a' is in R*b* a proper subset of R*a*.

When a largest element is assumed in Argument#1, it leads to a contradiction; so there is no largest element. Every R*a* set element is in one of the proper subsets below R*a* in the set hierarchy. It is a valid proof by contradiction.

Argument#2: Each R*a* contains a largest element.

1) Below each R*a* for all rationals $x < a$ is a collection of R*x* subsets{ *y* is a rational number | $0 \leq y < x$ }.
2) Each R*a* and its collection of R*x* subsets comprise a descending set hierarchy.
3) Each R*x* is missing its index ***"x"***. R*a* contains all the ***"x"*** indices.
4) Since the union of the collection of R*x* sets does not contain any element greater than the elements in all the individual R*x* sets, the union of the collection of R*x* sets does not equal R*a*.
5) There exists at least one R*a* set element *s* ≥ (all values of) ***x*.**
6) Let ***c*** and ***d*** be two elements of a single R*a* set with ***c* > *d***.
7) ***d*** is an element of R*c*, which is a proper subset of R*a*.
8) For any two elements in R*a* the smaller element is contained in a R*x* subset of R***a*.**
9) By steps 6) 7) and 8), **there is at most one R*a* set element** missing from all the R*x* subsets.
10) By steps 5) 9), **each R*a* set contains a largest element *a'*** not in a R*x* set below in the hierarchy.
11) There is no $b = (a' + a)/2$. It would be a second element not in a R*x* set below **R*a*** in the hierarchy. We know by step 8) that isn't possible.

Some people will claim that Argument#2 step 5 is false. For any single R*x* there is always an element of R*a* that is greater than all the elements in that single R*x* set, but no element of R*a* is greater than all the elements in the entire collection of R*x* sets. As argument#1 shows for any element a' in R*a* there is always a larger element $b = (a' + a)/2$ in R*a* and so a' is in the R*x* set where ***x* = *b*.**

However, Argument#2 is also a valid argument. That's why any formal system that contains sets, arithmetic, and rational numbers is inconsistent.

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