Risk Dynamics Modeling and Credit Valuation Adjustment

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ABSTRACT

A broad range of financial products bear credit risk. This paper presents an integrated approach to model credit risk. We focus on the impact of default dependence and rating migration on derivative security valuation, as correlated default risk is one of the most pervasive threats in financial markets. The numerical study shows that the model-implied credit spreads are very close to the market observed credit spreads. Both have the same patterns and trends. The numerical study also indicates that the calculated default correlation is consistent with the market default correlation observed, implying that the model is accurate for computing the market value of credit risk.

Key Words: credit value adjustment (CVA), credit risk modeling, distance to default, default probability, survival probability, asset pricing involving credit risk.

Credit risk is the danger that you will not receive an amount of money you are owed because the party that owes you the money is unable to pay you and defaults on its obligation. Credit risk exists whenever an institution has a relationship where a counterparty has an obligation to make payments in the future. Participants in derivatives market face significant counterparty credit risks, the counterparty may fail to perform on its contingent obligation.

There are two primary types of models that describe default processes in the literature: structural models and reduced-form (or intensity) models. The structural models regard default as an endogenous event, focusing on the capital structure of a firm. In these models, defaults occur as soon as a firm's asset value falls below a certain threshold. Unlike structural models, reduced-form models do not condition default explicitly on the value of the firm, and parameters related to the firm's value needs not be estimated to implement the model.

The reduced-form approach permits a lot of flexibility to obtain realistic default risk estimates, but the structural approach is useful for understanding the economic drivers of default risk (Nagel and Purnanadam (2020)). Many reduced form models also use distance-to-default as one of the state variables driving default intensity (Duffie, Saita, and Wang (2007), Bharath and Shumway (2008) Campbell, Hilscher, and Szilagyi (2008)).

Credit valuation adjustment (CVA) is an adjustment made by an institution to the market value of an OTC derivative contract to take into account credit risk of the counterparty. CVA allows banks to quantify counterparty credit risk as a single, measurable number and becomes an integral part of all valuation and risk management and has taken on significantly higher importance and profile in recent times.

CVA is an adjustment to the valuation of a portfolio in order to explicitly account for the credit worthiness of counterparties. The CVA of an OTC derivatives portfolio with a given counterparty is the market value of the credit risk due to any failure to perform on agreements with that counterparty. This adjustment can be either positive or negative, depending on which of the two counterparties bears the larger burden to the other of exposure and of counterparty default likelihood.

Banerjee and Feinstein (2022) point out that CVA usually neglects adjustments in default probability of indirect counterparties while CVA captures adjustments in default probability of direct counterparties. Barucca et al., (2020) show that if counterparties A and B are embedded in a network of contracts, then indirect counterparties of B can have a very important impact on B's default probabilities.

Bo and Capponi (2014) derive an analytical framework for calculating the bilateral CVA for a large portfolio of credit default swaps. Brigo and Vrins (2016) propose a semi-analytic approach to address wrong way risk, while Glasserman and Yang (2018) use marginal distributions of credit and market risk to calculate CVA and wrong way risk.

The use of generic algorithms for optimizing the portfolio CVA is explored in Chataigner and Cr´epey (2019). Cr´epey (2015) develops a way to calculate CVA under funding constraints using reduced-form backward stochastic differential equations. Abbas-Turki et al. (2022) use neural networks to reduce the computation time for a path-wise CVA calculation.

This paper presents a new framework for calculating CVA with wrong way risk. Wrong-way risk is defined by the International Swaps and Derivatives Association (ISDA) as the risk that occurs when ‘exposure to a counterparty is adversely correlated with the credit quality of that counterparty.

We consider counterparty risk in presence of correlation between the defaults of the counterparty and investor by assuming distances to default for entities are correlated. Given distance to default, one can computes default and survival probabilities and then prices defaultable financial instruments.

We find the impact of default correlation may be significant. The default rate for a group of credits tends to be higher in a recession and lower in a booming economy. This implies that each credit is subject to the same set of macroeconomic environments, and that there exists some form of dependence among the default time of the credits.

Both unilateral and bilateral CVAs are considered. The conditional independence assumption of the reduced-form models is an interesting and important topic in academic research, although it is rarely mentioned in practitioners’ papers. To correct the weakness of this assumption, we also consider correlated and potentially simultaneous defaults.

We conduct numerical study on the model. The numerical results highlight credit spreads and default correlations. Our results show that the model-calculated credit spreads are very close to the market observed credit spreads. Both have the same patterns and trends. The numerical study also indicates that the calculated default correlation is consistent with the market default correlation observed.

The rest of this paper is organized as follows: Section 1 elaborates the credit risk model. Section 2 discusses risky valuation. Section 3 describes the simulation and CVA. Section 4 presents numerical results. The conclusions are given in Section 5.

1. Credit Risk Modeling

The ability to measure and manage credit risk has never been more important. Let's define the log-solvency ratio as

$$x = \log \frac{V_t}{K_t} . \qquad (1)$$

Here, V_t is the firm value and K_t its debt value: firm is in default when V_t falls below K_t .

Suppose x follows the process:

$$dx = (\theta\beta(t) - kx)dt + \sigma\gamma(t)dz \qquad (2)$$

where θ , κ and σ are constant drift, volatility, and mean reversion speed. *z* is the Wiener process. Then, indicator functions from Eq. (1) can be simulated as follows:

$$1_{\{\tau \le T\}} = \begin{cases} 0 & (x_\tau > 0) \\ 1 & (x_\tau \le 0) \end{cases} \qquad (3)$$

To proceed further, we need to calibrate Eq. (2) in the risk-neutral world. It can be done by matching term structure of the risk-neutral default probabilities extracted from the CDS par rates.

The probability density function $f(x_0, x, t)$ is given by (see appendix),

$$f(y_0, y, \bar{t}) = \frac{\lambda(t) e^{-\frac{(y\lambda(t) - y_0 - \bar{\theta}\bar{t})^2}{2\bar{t}}}}{\sqrt{2\pi\bar{t}}} \left[1 - e^{-\frac{2y\lambda(t)y_0}{\bar{t}}} \right] \quad (4)$$

where

$$y = \frac{x}{\sigma}, \quad \bar{\theta} = \frac{\theta}{\sigma}$$

and

$$\bar{t} = \int_0^t dt' \lambda^2(t') \gamma^2(t')$$
$$\lambda(t) = e^{\kappa t}$$

The default probability, $p_D(t)$, is determined as

$$p_D(t) = 1 - \int_0^\infty dy f(y_0, y, \bar{t})$$

Using Eq. (4) we obtain

$$p_D(t) = N\left(-\frac{y_0 + \bar{\theta}\bar{t}}{\sqrt{\bar{t}}} \right) + e^{-2y_0\bar{\theta}} N\left(-\frac{y_0 - \bar{\theta}\bar{t}}{\sqrt{\bar{t}}} \right) \quad (5)$$

Suppose market risk-neutral probabilities, $p_D^{(M)}(t)$, are available. For example, their values can be extracted from the CDS closing rates (bootstrapping). Then

$$p_D^{(M)}(t) = N\left(-\frac{y_0 + \bar{\theta}\bar{t}}{\sqrt{\bar{t}}} \right) + e^{-2y_0\bar{\theta}} N\left(-\frac{y_0 - \bar{\theta}\bar{t}}{\sqrt{\bar{t}}} \right) \quad (6)$$

To simplify our model, we choose $\overline{\theta} = 0$. Then, Eq. (6) reduces to

$$p_D^{(M)}(t) = 2N\left(-\frac{y_0}{\sqrt{\bar{t}}}\right) \qquad (7)$$

To match whole CDS term structure, we calculate time dependent parameter, $\bar{t}$, in Eq. (7) as

$$\bar{t} = \frac{y_0^2}{\left[N^{-1}\left(p_D^{(M)}(t)/2\right)\right]^2} \qquad (8)$$

The remaining unknown parameter is the initial distance to default y_0. We define this parameter by fitting instantaneous CDS spread volatility, $\sigma_I(t)$, to the market spread returns (or implied) volatility, $\sigma_M(t)$. The expression for $\sigma_I(t)$ is given by,

$$\sigma_I(t) = \left|\frac{1}{S}\frac{\partial S}{\partial y}\right|_{y=y_0} \qquad (9)$$

where *S* is the CDS par rate. The value of *S*, at least for investment grades and higher, can be approximated as

$$S \approx \frac{(1-R)}{t}\, p_D(t) \qquad (10)$$

where *R* is the recovery rate. Then, Eq. (9) can be rewritten as

$$\sigma_I(t) \approx \left|\frac{1}{p_D(t)}\frac{\partial p_D(t)}{\partial y}\right|_{y=y_0} \qquad (11)$$

It follows from Eqs. (7, 8)

$$\frac{\partial p_D(t)}{\partial y} = -\sqrt{\frac{2}{\pi \bar{t}}}\, e^{-\frac{y^2}{2\bar{t}}}$$

and Eq. (11) reduces to

$$\sigma_I(t) = \sqrt{\frac{2}{\pi}} \frac{N^{-1}\left(p_D^{(M)}(t)/2\right)}{y_0 p_D^{(M)}(t)} e^{-\frac{\left[N^{-1}\left(p_D^{(M)}(t)/2\right)\right]^2}{2}} \qquad (12)$$

Since y_0 is constant, we cannot match whole CDS spread returns volatility term structure. Therefore, we choose $t = 5y$ that corresponds to the most liquid CDS. In this case, Eq. (12) can be easily solved [$\sigma_I(t) = \sigma_M(t)$]

$$y_0 = -\left(\sqrt{\frac{2}{\pi}} \frac{N^{-1}\left(p_D^{(M)}(t)/2\right)}{p_D^{(M)}(t)\sigma_M(t)} e^{-\frac{\left[N^{-1}\left(p_D^{(M)}(t)/2\right)\right]^2}{2}} \right)_{t=5y} \qquad (13)$$

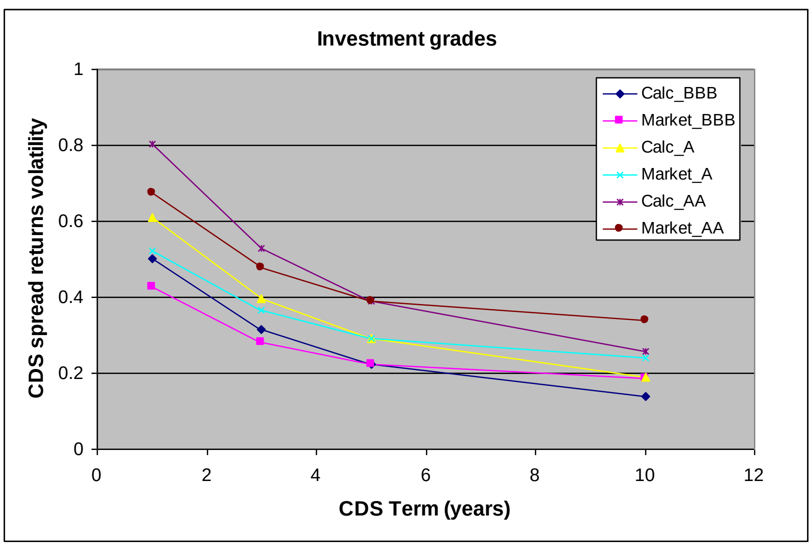

Figure 1. The calculated and market data results for investment grades of the global CDS indices.

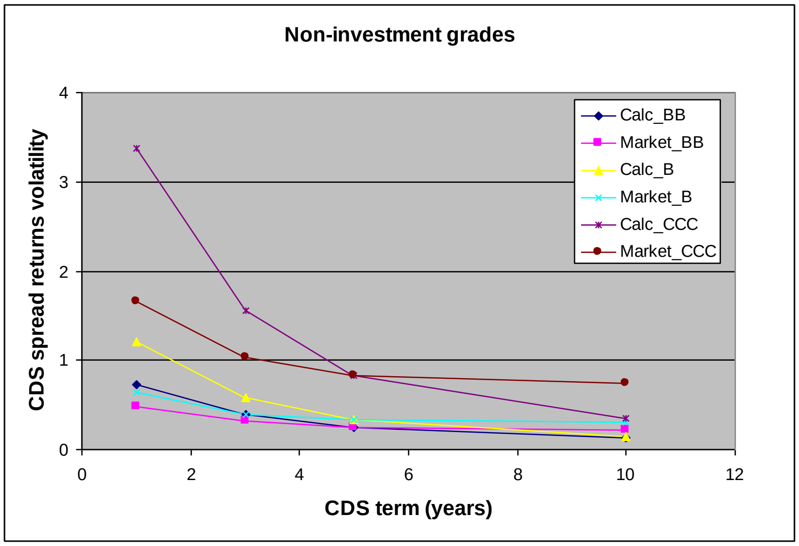

Figure 2. The calculated and market data results for non-investment grades of the global CDS indices.

In Figures 1 and 2, we present the calculated results and market data on the CDS spread returns volatilities. Figures show that the fits are fairly well for investment grades and substantially deviates from market data (see https://finpricing.com/lib/IrBasisCurve.html) for non-investment grades. However, the portfolio has small part of deals with non-investment grades and their contribution to CVA can be regarded as negligible.

We describe correlation structure of the simulated portfolio. It is assumed that counterparties are independent and CVA calculations for these counterparties can be done in parallel. Then, procedure described below is applied for each counterparty.

We assume that distances to default for counterparties, investor and reference names are correlated through Gaussian copula:

$$y_k = a_k^T M + \sqrt{1 - a_k^T a_k}\, \varepsilon_k , \quad k = 1,2,..n$$

Here, $M = (M_1,..,M_s)^T$ are all market indices (credit, IR, commodities etc.) and $\varepsilon_k \sim N(0,1)$ is idiosyncratic component: *k*=1 and *k*=2 correspond to counterparty '1' and investor '2' respectively.

We build correlation matrix, Σ, with the elements:

$$\Sigma = \begin{bmatrix} 1 & a_1^T \Sigma_M a_2 & a_1^T \Sigma_M \\ a_1^T \Sigma_M a_2 & 1 & a_2^T \Sigma_M \\ a_1^T \Sigma_M & a_2^T \Sigma_M & \Sigma_M \end{bmatrix}$$

where Σ_M is the correlation matrix of the market indices M_i.

Finally, we simulate distances to default y_1, y_2 and market risk factors $M = (M_1,..,M_s)^T$ as

$$\begin{aligned} y_1 &= K_1, \\ y_2 &= L_{21} K_1 + L_{22} K_2, \\ M_k &= \sum_{j=1}^{s+2} L_{k+2,j} K_j, \quad k = 1,..s \end{aligned}$$

where $K = (K_1,..,K_{s+2})^T$ are *independent* $N(0,1)$.

2. Valuation Considering Default Risk

For risky bond and credit default swap (CDS), consider cashflow at time *T*. Then, default probability, $p_D(t,T)$, at time bucket *t* is calculated as

$$p_D(t,T) = 2N\left(-\frac{y_t \lambda_t}{\sqrt{\Delta t_{T,t}}} \right)$$

where

$$\Delta t_{T,t} = \frac{y_0^2}{\left[N^{-1}\left(p_D^{(M)}(T)/2 \right) \right]^2} - \frac{y_0^2}{\left[N^{-1}\left(p_D^{(M)}(t)/2 \right) \right]^2}$$

The value of the risky zero-coupon bond is given by

$$Z(t,T) = \left[1-(1-R)\,p_D(t,T)\right]D(t,T)$$

Here, $D(t,T)$ and R are discount factor and recovery rate respectively.

Similarly, par spread, $S_{CDS}(t,T)$, for CDS with maturity T is calculated as

$$S_{CDS}(t,T) = 2(1-R)\frac{\sum_{k=1}^{j} DF(t,T_k)[p_S(t,T_{k-1}) - p_S(t,T_k)]}{\Delta\sum_{k=1}^{j} DF(t,T_k)[p_S(t,T_{k-1}) + p_S(t,T_k)]}, \quad (T_j = T)$$

where day-count fraction $\Delta = 1/4$ for most quoted CDS.

We show that in our model, the price of zero coupon risky bond is a martingale under forward measure.

The price of the risky zero coupon bond is

$$f_0 = D(0,T)E_T\left[R + (1-R)Lim_{t\to T}\, p_S(t,T)\right]$$

where survival probability $p_S(t,T)$ is given by

$$p_S(y_t,t,T) = 1 - p_D(y_t,t,T)$$

By definition we have

$$E_{y_t}[Lim_{t\to T}\, p_S(t,T)] = Lim_{t\to T}\int_0^{\infty} dy_t\, f(y_0,y_t,T-t)\,p_S(y_t,T-t) \qquad (14)$$

It follows from Markov's property of the process Eq. (2)

$$f(y_0, y_T, T) = \int_0^\infty dy_t\, f(y_0, y_t, t)\, f(y_t, y_T, T-t) \qquad (15)$$

By integrating both parts of Eq. (15) on y_T, one can obtain

$$\begin{aligned} p_S(0,T) &= \int_0^\infty dy_T\, f(y_0, y_T, T) \\ p_S(t,T-t) &= \int_0^\infty dy_T\, f(y_t, y_T, T-t) \end{aligned} \qquad (16)$$

Then, Eq. (15) reduces to

$$p_S(0,T) = \int_0^\infty dy_t\, f(y_0, y_t, T-t)\, p_S(y_t, T-t) \qquad (17)$$

Comparison of Eqs. (14, 17) yields

$$E_{y_t}[Lim_{t\to T}\, p_S(t,T)] = p_S(0,T)$$

Then, f_0 is given by,

$$f_0 = D(0,T)\left[R + (1-R)\, p_S(0,T)\right]$$

that confirms the proof.

3. Simulation and CVA

In this section, we describe algorithm for multi-step simulation and CVA calculations. Suppose $y_t^{(i)}(\neq 0)$ is the distance to default of credit name '*i*' at time bucket *t*: at *t*=0, $y_0^{(i)}$ is given by

$$y_0^{(i)} = -\left(\sqrt{\frac{2}{\pi}}\frac{N^{-1}\left(p_D^{(M)}(t)/2\right)}{p_D^{(M)}(t)\sigma_M(t)}e^{-\frac{\left[N^{-1}\left(p_D^{(M)}(t)/2\right)\right]^2}{2}}\right)_{t=5y}^{(i)}$$

Calculate value:

$$q_i = \frac{\lambda_t y_t^{(i)}}{\sqrt{\Delta\bar{t}_i}}$$

where

$$\Delta\bar{t}_i = y_0^{(i)2}\left(\frac{1}{\left[N^{-1}\left(p_D^{(M)}(t+\Delta t)/2\right)\right]^2} - \frac{1}{\left[N^{-1}\left(p_D^{(M)}(t)/2\right)\right]^2}\right)^{(i)}$$

and

$$\lambda_t = e^{\kappa t}$$

Calculate survival probabilities $p_S^{(i)}(t)$

$$p_S^{(i)}(t) = 1 - 2N(-q_i)$$

Simulate two independent uniforms $u_1, u_2 \sim [0,1]$.

<u>Consider a unilateral case where </u>counterparty is in default, but investor is not in default, calculate two correlated normal variables ε_1 and ε_2 as

$$\varepsilon_1^{(I)} = -N^{-1}\left(u_1\left(1 - p_S^{(1)}(t)\right)\right)$$
$$\varepsilon_2^{(I)} = N^{-1}\left(u_2 N\left(\frac{N^{-1}\left(p_S^{(2)}(t)\right) - \varepsilon_1^{(I)}L_{21}}{\sqrt{1-L_{21}^2}}\right)\right)$$

For a unilateral case where counterparty is not in default, investor is in default, we calculate two correlated normal variables ε_1 and ε_2 as

$$\varepsilon_1^{(II)} = N^{-1}\left(u_1 p_S^{(1)}(t)\right)$$
$$\varepsilon_2^{(II)} = -N^{-1}\left(u_2 N\left(-\frac{N^{-1}\left(p_S^{(2)}(t)\right) - \varepsilon_1^{(II)} L_{21}}{\sqrt{1-L_{21}^2}}\right)\right)$$

Simulate independent normal variables $K_j\,(j>2), \varepsilon_k \sim N(0,1)$ and calculate values:

$$y_1 = K_1,$$
$$y_2 = L_{21}K_1 + L_{22}K_2,$$
$$M_k = \sum_{j=1}^{s+2} L_{k+2,j} K_j, \quad k = 1,..s$$

In the unilateral case where counterparty is in default, the distance-to-default is

$$y_k = a_k^T M + \sqrt{1 - a_k^T a_k}\,\varepsilon_k, \quad k = 3,..n$$

with

$$K_1 = \varepsilon_1^{(I)}$$
$$K_2 = \varepsilon_2^{(I)}$$

Simulate independent normal variables $K_j\,(j>2), \varepsilon_k \sim N(0,1)$ and calculate the values.

In the unilateral case where counterparty is not in default, investor is in default, the distance-to-default is given by

$$y_k = a_k^T M + \sqrt{1 - a_k^T a_k}\,\varepsilon_k, \quad k = 3,..n$$

with

$$K_1 = \varepsilon_1^{(II)}$$
$$K_2 = \varepsilon_2^{(II)}$$

For a given scenario ‘*m*’, calculate bilateral *CVA* with netting as

$$CVA_m(t) = (1 - R_{CPN})\,p_{12}^{(I)}(m)\frac{1}{B_m(t)}\max\left[\sum_{i=1}^{k} W(t, y_i^{(I)}),0\right] - (1 - R_{BMO})\,p_{12}^{(II)}(m)\frac{1}{B_m(t)}\max\left[-\sum_{i=1}^{k} W(t, y_i^{(II)}),0\right]$$

The bilateral CVA with no netting is calculated as

$$CVA_m(t) = (1 - R_{CPN})\,p_{12}^{(I)}(m)\frac{1}{B_m(t)}\sum_{i=1}^{k}\max\left[W(t, y_i^{(I)}),0\right] - (1 - R_{BMO})\,p_{12}^{(II)}(m)\frac{1}{B_m(t)}\sum_{i=1}^{k}\max\left[-W(t, y_i^{(I)}),0\right]$$

where

$$p_{12}^{(I)}(m) = N_2\left(-N^{-1}\left(p_S^{(1)}(t)\right), N^{-1}\left(p_S^{(2)}(t)\right); -L_{21}\right)$$
$$p_{12}^{(II)}(m) = N_2\left(N^{-1}\left(p_S^{(1)}(t)\right), -N^{-1}\left(p_S^{(2)}(t)\right); -L_{21}\right)$$

and N_2 is the bivariate standard normal cumulative distribution function. Sum over index ‘*k*’ covers all instruments/deals that should be priced.

Simulate $K_j (j \geq 1), \varepsilon_k \sim N(0,1)$

$$
\begin{aligned}
&y_1 = K_1, \\
&y_2 = L_{21}K_1 + L_{22}K_2, \\
&M_k = \sum_{j=1}^{s+2} L_{k+2,j}K_j, \quad k = 1,..s \\
&y_k = a_k^T M + \sqrt{1 - a_k^T a_k}\,\varepsilon_k, \quad k = 3,..n
\end{aligned}
$$

For counterparty, investor and reference names, calculate variable:

$$u_i = N(y_i)$$

For counterparty, investor and reference name check condition

$$\text{if} \begin{cases} u_i \geq p_S^{(i)}(t) & \text{Default. Stop Simulation for name } 'i' \\ u_i < p_S^{(i)}(t) & \text{Continue} \end{cases}$$

Calculate $y_{t+\Delta t}^{(i)}$ from equation,

$$u_i = N\left(\frac{\lambda_{t+\Delta t} y_{t+\Delta t}^{(i)} - \lambda_t y_t^{(i)}}{\sqrt{\Delta \bar{t}_i}}\right) + N\left(-\frac{\lambda_{t+\Delta t} y_{t+\Delta t}^{(i)} + \lambda_t y_t^{(i)}}{\sqrt{\Delta \bar{t}_i}}\right) - 2N\left(-\frac{\lambda_t y_t^{(i)}}{\sqrt{\Delta \bar{t}_i}}\right)$$

If rating trigger is not applied, repeat step 1. Otherwise, continue procedure.

Up to trigger rating (*trig*), calculate cumulative probability $p_M^{(i)}$ as

$$p_M^{(i)} = p_{AAA,AA}^{(i)} + p_{AA,A}^{(i)} + \ldots + p_{trig-1,trig}^{(i)}$$

where $p_{k-1,k}$ are risk-neutral transition probabilities.

When simulation is completed, calculate total CVA as

$$CVA = \frac{1}{n_m} \sum_{m=1}^{n_m} \sum_{l=1}^{n_{tb}} CVA_m(t_l)$$

where n_m and n_{tb} are numbers of MC scenarios and time buckets respectively.

We describe correlation structure of the simulated portfolio. It is assumed that counterparties are independent and CVA calculations for these counterparties can be done in parallel. Then, procedure described below is applied for each counterparty.

We assume that distances-to-default for counterparties, investor and reference names are correlated through Gaussian copula

$$y_k = a_k^T M + \sqrt{1 - a_k^T a_k}\,\varepsilon_k, \quad k = 1,2,..n$$

Here, $M = (M_1,..,M_s)^T$ are all market indices (credit, IR, commodities etc.) and $\varepsilon_k \sim N(0,1)$ is idiosyncratic component: *k*=1 and *k*=2 correspond to counterparty '1' and investor '2' respectively.

We build correlation matrix, Σ, with the elements,

$$\Sigma = \begin{bmatrix} 1 & a_1^T \Sigma_M a_2 & a_1^T \Sigma_M \\ a_1^T \Sigma_M a_2 & 1 & a_2^T \Sigma_M \\ a_1^T \Sigma_M & a_2^T \Sigma_M & \Sigma_M \end{bmatrix}$$

where Σ_M is the correlation matrix of the market indices M_i.

We apply Cholesky decomposition:

$$\Sigma = L \cdot L^T$$

Finally, we simulate distances-to-default y_1, y_2 and market risk factors $M = (M_1,..,M_s)^T$ as

$$
\begin{aligned}
y_1 &= K_1, \\
y_2 &= L_{21}K_1 + L_{22}K_2, \\
M_k &= \sum_{j=1}^{s+2} L_{k+2,j}K_j, \quad k = 1,..s
\end{aligned}
$$

where $K = (K_1,..,K_{s+2})^T$ are *independent* $N(0,1)$.

4. Numerical Results

In Figure 4, we present CVA(t) results for 10y *CDS* based on unconditional (standard MC) and conditional *MC* simulations: total CVA is presented in Table I.

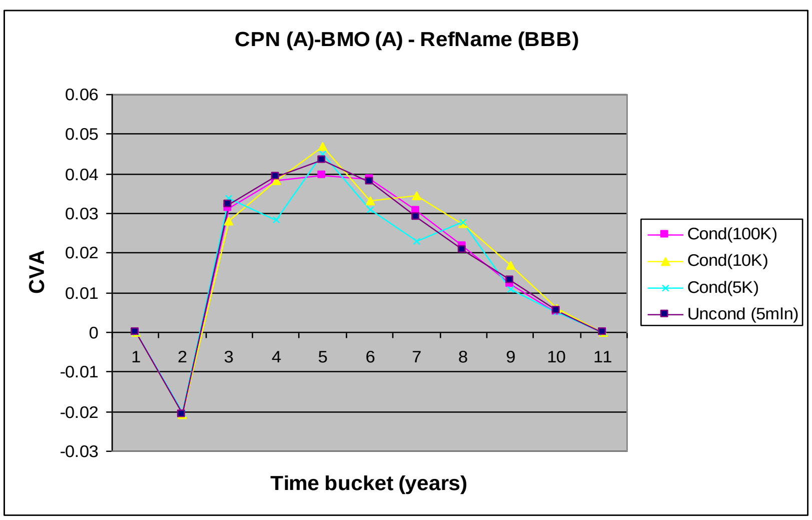

Figure 4. Bilateral *CVA*(t) for BBB 10y CDS: notional is 100$ and recovery rate is 0. Counterparty A and B have rating A while, Reference Name has rating BBB.

# MC scenarios	Bilateral CVA
Uncond (5 mln)	0.199
Cond (100K)	0.197
Cond (10K)	0.211
Cond (5K)	0.184

Table I. Bilateral CVA for BBB 10y CDS: Notional is 100$ and recovery rate is 0.

Using Monte-Carlo simulation we have calculated the default correlation at one year horizon for two issuers with the same rating. Historical and calculated results are presented in Table II. It is seen that model correlations are consistent with those historical observed.

Rating	Asset Correlation	Historical Default Correlation	Model Default Correlation
A	28.74%	0.65%	0.77%
BBB	13.21%	0.59%	0.38%
BB	14.28%	1.68%	1.61%

Table II. One-year historical and model default correlations for different ratings

Time dependence of default correlations is presented in Figures 5 and 6. Obviously, time dependence of model default correlations is in a good agreement with exact results.

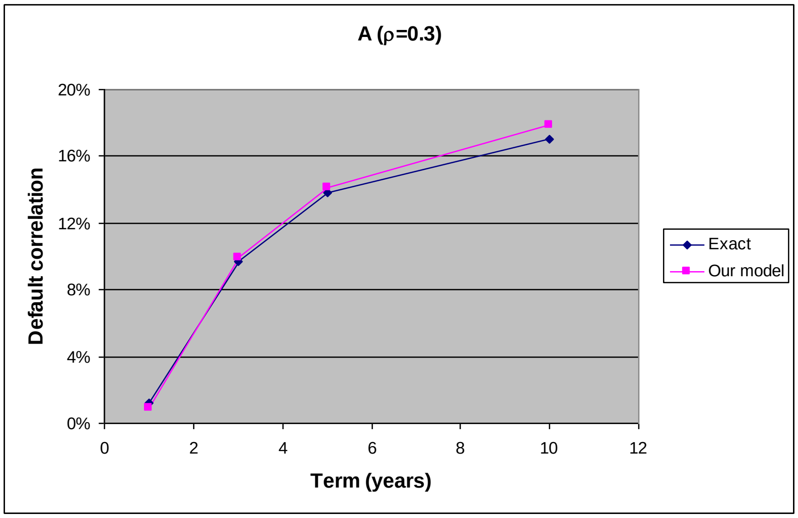

Figure 5. The calculated and exact [6] default correlation for investment grade A: asset correlation is 30%

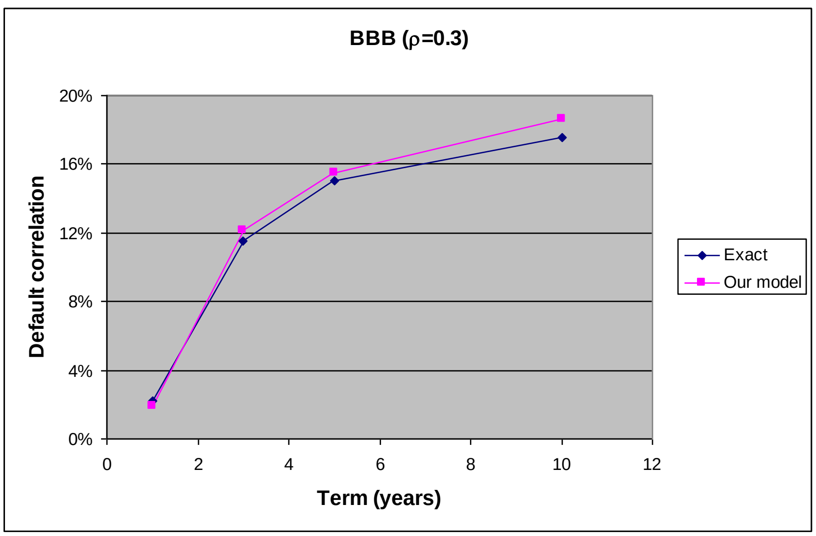

Figure 6. The calculated and exact default correlation for investment grade BBB: asset correlation is 30%

5. Conclusion

As interest in CVA modelling has increased, so too has the attention paid to the role of wrong-way risk in CVA. Wrong-way risk is a correlation between the exposure to a counterparty and the probability of that counterparty defaulting.

In this article, we derive our CVA formulas rigorously from the principals of the fundamental counterparty risk and mitigation arrangements. Both unilateral and bilateral CVAs are considered.

This paper presents a convenient framework that models credit risk based on correlated distances-to-default. Initial distance-to-default can be calibrated by fitting CDS spread volatility to market spread return volatility. Distance-to-default at any future time can be obtained via our model simulation. Given the dynamics of distance-to-default, we derive default probability and survival probability. Furthermore, we can price a risky portfolio and calculate CVA accordingly.

This framework can easily incorporate various credit mitigation techniques, such as netting agreements and margin agreements, and can capture wrong/right way risk. The model gives an integrated view of credit risk including default risk and credit migration. It provides a useful tool for risky valuation. Our theoretic results indicate that the model is a good fit for defaultable portfolio valuation and CVA.

The main goal of this paper is to deepen our understanding of the links between the importance aspect of default, credit migration, and valuation. The numerical study shows that the model can predict credit spread and default correlation very well, implying that the model is accurate for computing the market value of credit risk.

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Appendix. Probability density function for processes with the barrier

According to Eq. (4) we have the standard Brownian motion

$$\begin{aligned} &dx = (\theta\beta(t) - kx)dt + \sigma\gamma(t)dz, \\ &x = \log\frac{V_t}{K_t}. \end{aligned} \tag{1A}$$

Here, $\mu(t)$, $\sigma(t)$ and k are drift, volatility and mean reversion speed: x is the log-solvency ratio. The objective is to find probability density function, $f(x_0, x, t)$, where $f(x_0, x, t) \cdot dx$ is the transition probability to arrive in the vicinity of x at time t starting from x_0 at time zero. It is well known that $f(x_0, x, t)$ should obey forward Kolmogorov equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2 \gamma(t)^2 \cdot \frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left([\theta\beta(t) - kx] \cdot f\right) \qquad (2A)$$

Taking derivatives on the right side of Eq. (2A) we get

$$\frac{\partial f}{\partial t} = \frac{\sigma^2 \gamma(t)^2}{2}\frac{\partial^2 f}{\partial x^2} - \theta\beta(t)\frac{\partial f}{\partial x} + kx\frac{\partial f}{\partial x} + kf \qquad (3A)$$

The function $f(x_0, x, t)$ obeys the following initial and boundary conditions:

$$\begin{aligned} &f(x_0, x, 0) = \delta(x - x_0), \\ &f(x_0, x = 0, t) = 0, \\ &f(x_0, x = \infty, t) = 0, \\ &x_0, x \geq 0 \end{aligned} \qquad (4A)$$

Here, $\delta(x - x_0)$ is the delta-function. To proceed further, we add new function $\tilde{f}$

$$f = \lambda(t)\tilde{f}$$

where

$$\lambda(t) = e^{\kappa t}$$

Then, Eq. (3A) is given by

$$\frac{\partial \tilde{f}}{\partial t} = \frac{\sigma^2 \gamma(t)^2}{2}\frac{\partial^2 \tilde{f}}{\partial x^2} - \theta\beta(t)\frac{\partial \tilde{f}}{\partial x} + kx\frac{\partial \tilde{f}}{\partial x} \qquad (5A)$$

We change variable

$$z = x\lambda(t) \text{ (6A)}$$

and Eq. (5A) reduces to

$$\frac{\partial\tilde{f}}{\partial t} = \lambda(t)^2\gamma(t)^2\left(\frac{\sigma^2}{2}\frac{\partial^2\tilde{f}}{\partial z^2} - \theta\frac{\beta(t)}{\lambda(t)\gamma^2(t)}\frac{\partial\tilde{f}}{\partial z}\right) \qquad (7A)$$

Finally, we change variable

$$\bar{t} = \int_0^t d\tau\lambda(\tau)^2\gamma(\tau)^2$$

and set

$$\beta(t) = \lambda(t)\gamma^2(t)$$

Then, Eq. (7A) reduces to

$$\frac{\partial\tilde{f}}{\partial\bar{t}} = \frac{\sigma^2}{2}\frac{\partial^2\tilde{f}}{\partial z^2} - \theta\frac{\partial\tilde{f}}{\partial z} \qquad (8A)$$

To solve Eq. (8A) we use Laplace transform method. Specifically, we introduce a new function under Laplace transform

$$\overline{f(z_0, z, p)} = \int_0^\infty dt \cdot e^{-pt}\tilde{f}(z_0, z, t) \qquad (9A).$$

Note, the function $\overline{f(z_0, z, p)}$ obeys the same boundary condition (4A). Thus, if transform (9A) is applied to Eq. (8A) we obtain

$$\int_0^\infty dt \cdot e^{-pt}\frac{\partial\tilde{f}}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2\overline{f}}{\partial z^2} - \theta\frac{\partial\overline{f}}{\partial z}. \quad (10A)$$

Integration of the left part of Eq. (10A) yields

$$\int_0^\infty dt \cdot e^{-pt}\frac{\partial\tilde{f}}{\partial t} = e^{-pt}\tilde{f}\Big|_0^\infty + p\cdot\overline{f} = -\delta(z - z_0) + p\cdot\overline{f}. \quad (11A)$$

By substituting Eq. (11A) into Eq. (10A) we finally obtain

$$\frac{\sigma^2}{2}\frac{\partial^2\overline{f}}{\partial z^2} - \theta\frac{\partial\overline{f}}{\partial z} - p\cdot\overline{f} = -\delta(z - z_0) \quad . \qquad (12A)$$

This equation represents the classical problem for building fundamental solution [Green function]. The first step is to solve homogenous equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \overline{f}}{\partial z^2} - \theta\frac{\partial \overline{f}}{\partial z} - p \cdot \overline{f} = 0\,. \qquad (13A)$$

Solution of Eq. (13A) is trivial

$$\begin{aligned} \overline{f} &= Ae^{k_+ z} + Be^{k_- z}, \\ k_\pm &= \frac{\theta \pm \sqrt{\theta^2 + 2p\sigma^2}}{\sigma^2}. \end{aligned} \qquad (14A)$$

To solve non-homogenous Eq. (12A), we write general solution as [it is written in any book on the partial differential equations]

$$\begin{aligned} \overline{f} &= \frac{u_1(z)\cdot u_2(z_0)}{(\sigma^2/2)\cdot W(z_0)}, \quad (z \geq z_0) \\ \overline{f} &= \frac{u_1(z_0)\cdot u_2(z)}{(\sigma^2/2)\cdot W(z_0)}, \quad (z \leq z_0) \end{aligned}\,. \qquad (15A)$$

Here, $u_1(z)$ and $u_2(z)$ are the two independent solutions of the *homogeneous* Eq. (8A) and $W(z_0)$ is the Wronskian of Eq. (12A):

$$W(z) = \begin{vmatrix} u_1(z) & u_2(z) \\ u_1^{'}(z) & u_2^{'}(z) \end{vmatrix}\,. \qquad (16A)$$

To calculate $u_1(z)$ and $u_2(z)$ we use the boundary conditions (4A):

$$\begin{aligned} u_1(z=\infty) &= 0, \\ u_2(z=0) &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} u_1(z) &= e^{k_- z} \\ u_2(z) &= Ae^{k_+ z} + Be^{k_- z} \end{aligned} \qquad (17A)$$

where coefficients *A* and *B* are calculated from the boundary condition

$$A + B = 0 \qquad (18A)$$

By substituting Eqs. (16A, 17A) into Eq. (15A) we obtain

$$\begin{aligned}
\overline{f} &= \frac{e^{\theta(z-z_0)/\sigma^2}}{\sigma^2\Lambda}\left[e^{-\Lambda(z-z_0)} - e^{-\Lambda(z+z_0)}\right], \quad (z \geq z_0) \\
\overline{f} &= \frac{e^{\theta(z-z_0)/\sigma^2}}{\sigma^2\Lambda}\left[e^{\Lambda(z-z_0)} - e^{-\Lambda(z+z_0)}\right], \quad (z \leq z_0) \qquad (19A) \\
\Lambda &= \frac{\sqrt{\theta^2 + 2p\sigma^2}}{\sigma^2}.
\end{aligned}$$

Original function of the Laplace transform (14A) is well known and result is

$$\tilde{f} = \frac{e^{-\frac{(z-z_0-\theta\bar{t})^2}{2\sigma^2\bar{t}}}}{\sqrt{2\pi\sigma^2\bar{t}}}\left[1 - e^{-\frac{2zz_0}{\sigma^2\bar{t}}}\right]. \qquad (20A)$$

Finally, $f(x_0, x, t)$ is given by

$$f(x_0, x, t) = \frac{\lambda(t)e^{-\frac{(x\lambda(t)-x_0-\theta\bar{t})^2}{2\sigma^2\bar{t}}}}{\sqrt{2\pi\sigma^2\bar{t}}}\left[1 - e^{-\frac{2x\lambda(t)x_0}{\sigma^2\bar{t}}}\right] \qquad (21A)$$