Natural Units, Pi-Groups and Period Laws

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Abstract

In the context of QFT and Gauge Theory, the introduction of Natural Units, as a quantization in disguise" combined with Buckingham's Pi-Theorem, provides a direct connection with de Rham Periods, as also hinted by Feynman amplitudes, Dessins d'Enfants and Belyi maps models of baryon modes etc.

A program emerges: Physics Laws as Period Laws, and Alpha, as an element of the Pi-groups, a period. Our models of the Physical Reality emerge from the union of Cohomological Physics [1] and Number Theory, helping us understand "the unreasonable effectiveness of Mathematics".

An overview of the Network Model is included, with impacts to Sciences in general. Further prospects for understanding the fine structure constant are presented.

1 Introduction

In this research note we will address a conjectural principle, that Physics Laws are ultimately Period Laws of adequate mathematical models (Algebraic-geometric, with additional data, given by finite groups / structure). The term was suggested in a brief note of Post [1], yet acquiring a wider scope in the context of the author's research aiming to understand the fine structure constant [2][2]. Other related topics are discussed, in the context of the mainstream Mathematical-Physics theories.

It is well documented that Feynman amplitudes are periods. They also correspond to dessins d'enfant, a skeleton" on a Riemann Surface with can be interpreted as 0D and 1D-modes of vibration of the surface" of a baryon [12, 4, 33]:

At a different extreme" the multiplicative laws of Physics ($d = v \cdot t$ etc.), use units that now are replaced by universal constants": Natural Units [34].

A parallel consideration / analogy will be presented: Buckingham's Pi-Theorem, reducing" Physics Units to dimensionless constants and, as an example, cohomology of a Riemann Surface and its de Rham Period Matrix.

1.1 Cohomological Constants vs. Periods

It will be suggested, and much work needed for an actual "implementation", that dimensionless constants in BPT play the role of projective 2-cocycles, when using Natural Units which in fact are cohomological in nature: c and h; or, they are *periods*, e.g. α [2].

Constants of the *1st kind*, result from quantization, *model building* mathematics for the "true reality" [3]. They are "linear (abstract?) algebra" in nature.

Constants of the *2nd kind* are those which are periods, i.e. the result of *pairing two quantities*, as an *algebraic integral*[4] as opposed to evaluate a ratio" [5].

[1] Term introduced by Jim Stasheff; in Gauss's term for Number Theory: The King of Physics rejoins the Queen of Math.

[2] The Ultimate Physics Theory is Number Theory" from a cohomological point of view; the only real number needed, are periods: pairing algebraic functions and domains.

[3] Democrit, Zeno etc.

[4] Duality, Hopf algebras, Hodge structures etc. are by now the norm in Math-Physics.

[5] Newton vs. Leibnitz debate; Cartan's differential forms rules the Theory, but Calculus is education friendly.

The main two are:π (circle or Gauss-Bonnet Th., angle defect in Diff. Geometry etc.) and *alpha* (claim), which (another claim) should be expressible as a series in π [6]

In what follows the above points will be developed, with emphasis on what alpha is, i.e. what is the general framework of ideas and theories associated with it.

2 Natural Units, Buckingham Pi Theorem and de Rham Period Matrix

Natural units were introduced as units for physics quantities corresponding to Nature provided" irreducible systems, instead of man-made [34].

BPT assumes the basis of units being independent (free / vector space). The dimensionless Pi-group constants are reminiscent of 2-cocycles of projective representations. Recall that QM (unitary states) is based on such representations, which geometric correspond to projective spaces (Homogeneous Physics / Conformal transformations). The core of this correspondence is $CP^2 \cong S^3 \cong SU(2)$ (Harmonic oscillator, qubit space, spinors etc.).

The proposed Program aims to bridge Physics Laws and Period Laws of math-models [35, 4], as a next step beyond Natural Units and Pi-groups description of physics phenomena.

2.1 Cohomology Constants"

Quantization lead to Heisenberg group (infinitesimal deformation), revealing the importance of Planck's constant h.

2.1.1 Planck's Constant

It plays a role in several related structures: 1) Heisenberg group (Dirac quantization); 2) Symplectic formalism; 3) Relates the three periods of the Hopf fibration with a canonical gauge field (see [16]):

a) 1-period of magnetic current (fluxon): $\oint_C Adr$; b) 2-period of electric charge; c) 3-period of spin form.

Post [1] showed how (a) and (b) are related (Gauss and Ampere Law). All three should be understood in the context of Hopf bundle, as a basic local" model for Chern-simons Theory [26].

In terms of electric-magnetic charges (periods; see Dirac, Schwinger, Witten etc.):

$$e \cdot g = h.$$

The decomposition depends on the correspondence $C \times C \cong H \cong R \times R^3$; LHS is QC / spinorial formalism and RHS a Lie algebra central extension of the rotation Lie algebra $(R^3, \times)$[7].

This quaternionic Space-Time extension introduces $1/c^2 = \epsilon\mu$. This is the S-T emergent decomposition", corresponding to electric (divergent) and Magnetic (rotational) components in pointwise physics of particle-fields. It also plays the role of a deformation parameter of Galilei group into Lorentzs transformations (corresponding to Mobius transformations $SL_2(C)$ - QC dynamics).

In [3], §2.3 it was explained that the role of alpha is to relate these two sides: classical and quantum[8]:

$$\frac{e^2}{g^2} = \alpha^2 \frac{\epsilon_0}{\mu_0}, \quad \epsilon_0 c = \sqrt{\frac{\epsilon_0}{\mu_0}}\ \textit{Free Space Impedance}.$$

Note: We will drop the subscript from ϵ_0, μ_0, for brevity, as we will only refer to the electric permitivity and magnetic permeability of the vacuum.

[6] Are periods, as algebraic integrals, expressible in terms of the Universal Period" π? Note that e is not a period, rather a Lie base": see Baez's Groupoids and exponentiation. On the other hand almost everything", abstract math structure, has a shadow" on the Real Line"!

[7] ... but finite dimensional Lie algebras have only trivial central extensions ...

[8] Depending on units used, SI with $e/4\pi\epsilon_0$ or cgs form.

The free" vs. bound" states, in the Network Model [57], refers to bonds (mesonic, in Nuclear Physics, or leptonic, as in Chemistry) vs. orbitals (stacks of tori", from a discrete/ finite groups Hopf fibration model; a correspondence is needed, with Schrodinger's Eq. spherical harmonics and Belyi maps).

Thus we have the usual systems of products and ratios (sums and differences):

$$eg = h, e/g = A \quad \epsilon\mu = 1/c^2, \frac{\epsilon}{\mu} = B$$

with *alphs* a proportionality between the two.

2.1.2 ... and String Theory

The open / closed bonds (states) is of course reminiscent of the various String Theories, now unified by M-Theory: role of branes was recognized; except it is difficult to work with unstructured" RS, and tesselations / Hodge data" are/is needed (see Belyi maps [35, 4], Dessins D'enfant and Feynman integrals etc.).

2.1.3 Networks and Manifolds

Modeling Networks as manifolds [57] can be done in 3D/4D as ramified covers of S^3 [31]. The relation with Belyi (algebraic) maps from higher genus RS to S^2 needs investigation.

Now since Lab time" (relativistic) emerges from quantum phase [36], and since the Hopf fibration / GT Klein model $U(1) \rightarrow SU(2)$ is iso with the unitary tangent bundle of the Bloch sphere (QC), it is natural to look at *unitary bundles of RS ramified covers of the sphere*. The Belyi D d'E is the needed Hodge data.

2.1.4 Periods and Fractional Charges

is the $+2/3$ vs. $-1/3$ split between positive and negative elementary charge. These can be viewed as eigenvalues of quark directions as eigenfunctions) associated with the "cuts" of the Bloch-Belyi S^2, i.e. $[0, 1]$ and ∞, with the 2 endpoints and ∞ the 3 marked points? (corresponding to quark directions, if an additional structure is included: $A \in SL_2(Z)$: Mobius transformations are 3-transitive; here algebraic structure and associated integral periods correspond to integral MT.

A congruence group $\Gamma \in SL_2(Z)$ and associated modular curve (see [35, 4]) yield a model for *baryon states* (tie-ing all up: FD, D d'E, tesselated RS / Fuchs groups, Riemann-Hyrwitz Th. etc.).

2.1.5 Monster Group and Prime Graph

It is remarkable that The Monster Group (related to Platonic-Weyl groups, AdS, TOI fermion generations, E6-8 and TOES etc.) correspons to the irreducible Element of Quantum Circuit, the genus zero baryon: Ogg Theorem (why those primes only? For simple finite sporadic groups the associated *prime graph* should be connected ... and genus $g = 0$?).

But Quantum Circuits (QI-processors, made of several components / gates and mesonic-leptonic connection bonds) should have higher genus (Belyi maps of RS, with UT-Bundle; or [31]).

The bridge between "physics code" of SM and an algebraic-geometric model, as above, needs constructed (FD, QLD, framed RS, in the Gauge Theory framework).

2.1.6 ... and Periods

With such a framework (Belyi maps and Ogg's Th. etc.) the Dessin d'Enfants may provide the discrete structure for a *homological basis*. Then a pull back of the canonical gauge field on the Hopf bundle (see Topological Monopole on Hopf fibration), as a model of *baryon quark field*, may yield the *harmonic 1-forms*.

The Hodge-de Rham Period Matrix will yield the Periods. Jacobian varieties (tori) will allow a link with elliptic curves and j-invariant, and beyond (mass from magnetic current model? see H. J. and Schwinger).

2.1.7 Period Quantum Laws

Then the fundamental (Natural) "Laws of Quantum Physics" can be related with such periods. A count of EC at "genus zero primes" (over F_p in Ogg's Th.) could be revealing of more structure and provide insight (not to mention using Weil's Conjecture perspective on Riemann zeros, dual to primes).

2.1.8 Speed of Light is a cohomological constant

The system of Maxwell's eq., in terms of ϵ and μ, yielded the wave eq. and c. Einstein's work on photoelectric effect started from the bold idea that light is quantized *not only in atoms* (bound states), like Planck proposed, but also in "free space".

But there is no "free space", since a photon is transferred from one atom to another, through a fermionic channel (EM-streamer, analog to the tinny chemical bond), as String Theory also proposes (There is no other way to "solve" particle-wave/field duality; the "continuum ether" is in fact a Network [57]; unification boson-fermion, or rather gauge boson (Feynman path / Bosonic String) -fermion source ; branching point / baryon).

So, how to quantize an open EM bond, connecting two H. Oscillators, in an *open version of Bohr-Sommerfeld orbital*? Or rather viewed as a "cobordism" between 2-Riemann spheres.

The *flavor oscillation* of neutrinos and quark flavors must be general: hence a tessalation compatible with the quantum data, should achieve this (frequency, quantum phase / polarization; the photon propagates on a channel, with helicity, but the channel, as a EM-object, will have both e and g aspects: A-connection etc.).

2.2 Constants as Periods: Unification of e, g and spin

The above relation between electric, magnetic and spin of a quantum particle (Hopf GT model) shows that they are deeply related, at the level of the model (e.g. H. J. toroidal loopforms / Wilson loops model).

Intuitively any such quantum channel with associated gauge boson has EM ($U(1)$ connection, E/M fields and spin relating the "3D-loxodromic character" (analogy with MT in 2D), of the lines of EM-field forming the flux-tube (streamer; this may branch, as in lighting streamers "searching" for a "minimal action path"; see lightning streamers); the helicity, spin, momentum should be related in view of the local model $U(1) \to SU(2)$.

2.3 Synchronization and Interference

The process of "path-searching" is related to constructive interference building, which is due to quantum phase "clock" (see feynman's explanation: $exp(i\omega t)$ and [36]). It is similar (Adaptative Network as a universal math model) to how brain learns: pseudovortices (compare with turbulent / pp'-loops forming) help synchronization [38].

Weil mentioned the role of analogy (it conveys a lot of insight and visualization, and using what one knows to understand the new). Here the Brain is the well tested Nature's Real Model for learning / adaptation; the author claims that Physics neglects transient stages of an experiment and phenomena in general, looking for cont. / smooth propagation and steady states mainly. But equilibrium is the exception: the norm is the "Constant Change" (decays are just change of phase transitions, alpha is the Quantum Reynolds Number etc.). The decay is triggered by an interaction, not "spontaneous" / probabilistic; IF a quantum systems if isolated (impossible: that would belong to a different "Universe"; Plato: all is One[9], then it will not "get old"

We return to alpha, to better understand h and c, with e, g as periods of the Hopf-Qubit Fibration / $U(1) \to SU(2)$. Recall that this is the "pixel" of a QC portion of the Universe (3D-RGB frame and 1D-Periodic Clock; the "calibration" / sync means its state: geometry (flavors) and 3D-vibration modes (Fourier frequencies associated to its "faces"? Alternative to investigate: flavor oscillation as dual Platonic geometries: A,B/C and G; dual root data and Langlands Duals)

[9] But there may be such components, isolated for large periods of time, on human scale ...

3 A Framework for Alpha

The main point is to have a correct model for EM (i.e. $U(1)$-Theory) quark field; at this point we will claim that there is a need to unify the electric charge and magnetic "charge" (source), away from the Pointwise QFT framework, where as points, they are inevitably separated[10]

3.1 Schwinger's Dyons

A tentative such unification was done by Schwinger 1969, as *dyons*. By now we know quarks are eigenvectors of the Hopf-qubit model (QM projective formalism meets QC), as more structure for the SM Gauge Theory approach. Experimentally each baryon (Network Node / source) or meson/fermion (Network Channel/ bond) have electric and magnetic structure: neutrons, protons, electrons etc. *not pointwise*, i.e. not isotropic ($SO(3)$-symmetric fields).

Therefore the "dyon" idea is mandatory, and has to be revisited in hindsight of quark model and EPP.

Remark 3.1 *An important methodological point is to consider the historical development and "emergence" of one theory from another, e.g.* $U(1) - GT \rightarrow Maxwel - Heaviside\ EM\ Theory$*; Space-Time-Matter emerges from GT and Network Model [39, 57] etc.*

3.2 H. Jehle's Toroidal Model

In principle H.J. model has all the ingredients needed to proceed to the computation of α by counting Elliptic Curves associated by the Hopf fibration, as a model of dyon".

3.3 The Period Laws for Alpha

Consider Hopf fibration as a $U(1)$-Chern-Simons Theory (see also Topological Monopole article for relating gauge field and Hopf index). The Kiehn 1,2,3-forms and periods are related as $h = eg$, with h unit of 1/2 spin ($SU(2)$-related).

The EM Maxwell formalism ($SL_2(C)$ MT/Lorentz) as a compact form of su_2 has a Hodge duality on d-forms, leading to a relation with ϵ, μ at the level of curvature $F = dA$, corresponding to (E, H) and dual $F*$ corresponding to (D, B).

3.3.1 3-Periods and Self-dual Equations

Can the periods $g = \oint_{S^1} A, e = \oiint_{S^2} F$ and $h = \iiint_{S^3} A \wedge F$ (see CST Lagrangian) be related with the $F*$ dual (Seiberg-witten self-dual and Hitchin Eq.) to yield a relation between e/g and ϵ/μ?

$$deRham\ (Cohomological)\ Periods : e \cdot g = h, \quad Hodge\ duality : det(*) = \epsilon \cdot \mu,$$

$$Quantum\ Reynolds\ Number : \frac{e}{g} = \alpha \sqrt{\frac{\epsilon}{\mu}} \qquad [3,\ 2].$$

Alpha, as a *Quantum Reynolds Number*, is a ratio between the turbulent regime (bound states, when stable)

and laminar regime (free space propagation, without generation of vortex loops" of particle-antiparticle" pairs.

Alternatively:

$$\frac{e}{\sqrt{\epsilon}} = \alpha \frac{g}{\sqrt{\mu}},$$

which suggests a possible redefinition of electric and magnetic charge:

$$e' = e/\sqrt{\epsilon}, g' = g/\sqrt{\mu}, \quad e' \cdot g' = hc.$$

[10] This leads to Maxwel Eq. "symmetry", monopoles, duality: Olive-Montonen.

Note that $hc = h/(1/c)$, a ratio of the two central charges", if we think of $1/c$ as a central charge/ infinitesiomal deformation parameter (Minkowski space / Lie algebra H), complementary somehow to h (symplectic space)[11].

This should yield alpha, when a discrete data, e.g. lattice, RS tesselation/ congruence group [35, 4], is considered, as for a *Hodge structure*? (see [40]).

3.3.2 e and epsilon; g and mu

The ratio $e/4\pi\epsilon$ is an alternative for the unit of electric charge. Similarly, $g/2\pi\mu$ is also a candidate. This suggests a kind of Olive-Montonen duality of Hodge type, which maybe mixes e and g as suggested by Schwinger (compare with $\tau = e + i/g$ in S-duality formalism, weak-strong coupling constant mirror symmetry).

3.3.3 Alpha and Mirror Symmetry

With $A = e/4\pi\epsilon$ and $B = g/2\pi\mu$, the alpha-relation (Pi-group of fundamental natural units) becomes:

$$A/B = \alpha.$$

An analysis of the Calabi-Yau Hodge diamond (due to Hodge structure) and Mirror Symmetry, with duality between the A and B models, may provide additional insight. Again, a finite "Hodge data" may be needed to actually compute it (count structures; see H.J. number of fibrations N; and also F. Potter's conjecture regarding mass of fermions and Klein's j-invariant [35, 4]).

3.3.4 Relaxing Calabi-Yau Condition

Using the CY-manifold as a fiber (instead of extra dimensions) is not appropriate, yet the linear algebra structure (Hodge etc.) may be of interest [40]. The quaternion structure invites to a hyper-Kahler structure to be considered in connection with $UT(S^2) \cong S^3$; to have a 3D-frame (baryons), one would have to consider the frame bundle (6D), and the symplectic vs. complex structure linking $SU(2)$-GT frames and $SU(3)$-gauge group of QCD.

3.4 Alpha and Mass

Returning to the "simplest" occurrence of Alpha, in the Bohr's model of the H-atom [52],

$$E_{Ext}(n) = \frac{1}{2n^2}\alpha^2 E_{Int}, \quad E_{int} = m_e c^2.$$

let us reverse engineering" the *internal energy* E_{Int}, associated with mass. Here $E_{Ext}(n)$ is the potential energy level of the principal number n orbital in the first approximation of Bohr, before Sommerfeld's relativistic model included a *fine structure*, and coined the term *fine structure constant* for Alpha.

3.4.1 Partition of the Energy

In analysing the two terms we use a "brainstorming mode" (association of ideas).

The partition function for the *irreducible H-atom system* has two parts, "external", associated to the electronic orbital, and "internal" associated with the proton shells and orbitals:

$$Z = Z_{Ext} + Z_{Int}, \quad Z_{Ext} = \sum_n \sum_{l,m} e^{-E(n,l,m,s))/kT}, \quad Z_{Int} = \sum_{\Gamma\ ?} e^{-E(\Gamma;N(\Gamma))/kT}, \quad N(\Gamma) =?. \tag{1}$$

By now it is clear that the states of baryons and nuclei are described by finite subgroups of symmetry (Klein geometries $\Gamma \to SU(2), SU(3), SL_2(Z)$) [9, 54, 55, 53][12].

Here T is the associated Lagrange multiplier, which is interpreted as temperature, when k_B is the Boltxmann constant.

[11] ... which corresponds to a complex structure $R^6 = C \times R^3$, complexification of $g = (R^3, \times)$ ...

[12] Much more direct experimental evidence and theoretical studies are available currently; we will report on this elsewhere.

Remark 3.2 *The role of the Boltzmann constant and Avogadro Number, as possible Pi-group elements, and their role in connecting natural units with our "bulky", ad-hoc selected measurement units, will be considered elsewhere.*

3.4.2 Equipartition Hypothesis

Note that the number of states for each principal quantum number n, is $2n^2 = \sum_{0\leq l<n}\sum_{-l\leq m\leq m}$. This corresponds to the irreducible representations of $SU(2)$, with spin $S = n/2$ h (s is the electron spin $\pm 1/2$).

Now let us interpret the above energy formula. Each n-orbital has an energy $E_n = (1/2n^2)\alpha E_{Int}$, hence with the *same* energy αE_{Int}, partitioned among its states.

Remark 3.3 *Note of course the expected role of Alpha:*

$$E_{Ext} = \alpha^2\, E_{Int},$$

which suggests a laminar flow *of the A-vector field auto-parallels constituting the orbital (connection interpretation), and a* turbulent flow *of the* gluonic flow *of the A_R, A_G, A_B vector potentials of the R, G, B quark fields of the proton [51].*

Recall that Alpha is the BPT analog of Reynolds Number, in the mechanic-EM analogy (Quantum Reynold Number).

3.4.3 The Electric "External" Energy

Hence the *electric energy / laminar flow* of all the orbitals, corresponding to all the quantum numbers n is:

$$E_{Ext} = \sum_{n\in N^+} E_{Ext}(n) = \sum_n \frac{1}{2n^2}\,\alpha E_{Int} = \frac{1}{2}\zeta(2)\,\alpha\, E_{Int}.$$

The attribute "electric" is used because the distribution of these orbitals is *radial*, corresponding to the radial part of Schrodinger equation; also because it is associated with the electron as a source of electric field, capable of mechanic work[13].

We will refer to "magnetic" as the key-term for the internal energy, associated with the curvature of the connection, closed lines of the A-gauge field(s) (R,G,B quarks as a 3D-frame for the emerging Space-Time / $SU(2)$-eigenvector basis "hypothesis").

Remark 3.4 *The decomposition "electric-magnetic" is of course Helmholdtz decomposition, work / hyperbolic transformations vs. geometry/ elliptic transformations etc.*

3.4.4 The Magnetic "Internal" Energy

We should mention that the Sommerfeld quantization condition $\oint_C p_i dq_i = n\hbar$[14], in a symplectic mechanics / algebra formalism.

Remark 3.5 *This is related to the quantum phase $\Delta\theta = \oint_C A dr$, holonomy and the "uniqueness of the quantum phase" for the "wave function" (projective representations etc.). We mention this to see that the mechanical interpretations around H-atom can be expressed geometrically, and that its quantum physics is* Periods Physics, *in* natural units, *the theme of this article.*

Indeed, relativistic time *emerges from the quantum phase [36]*[15].

[13] The "electron" has a magnetic field / moment also, it is not pointwise, and it is a toroidal structure, part of the H-atom: closing part of the quark fields; the unified model is still missing, yet it is associated with the Hopf fibration / principal bundle etc.

[14] $\theta = pdr - Edt$ is the symplectic potential (when no ramification is involved), with $\omega = d\theta$ the symplectic form.

[15] ... and 3D-space connection / parallelism, from quark fields [37]; when is the connection representable as a Levi-Civita connection? (locally we can find a metric in the conformal class, but what about global obstructions? What do they mean? Need "Gauge Theory with Singularities", on Ramified Principal Bundles; relate with Galois Theory and [35, 56].

3.4.5 2D vs. 3D finite subgroups

Returning to *reverse engineering* alpha, the PQN n counts the number of nodes of a "big circle" of an orbit (think Hopf fibration and Kepler's problem on S^3, $SO(4)$-symmetry of H-atom).

It is the multiplier of the "frequency of the orbit": $T = 1/n\ T_0$ (some fundamental time period / quantum phase), or can be interpreted as $Ab = Z/nZ$ cyclotomic units ("Feynman time" $e^{i\omega t}$ interpretation of the quantum phase associated with a particle; carries an Einstein's Clock).

Then, for the "internal energy", we should look for a corresponding *partition function decomposition*, associated to a class of finite subgroups of, perhaps, $SU(2)$, or more general as quotients of finite index congruence subgroups of the modular group $\Gamma \to SL_2(Z) \to G$.

Remark 3.6 *Level $N = p$ prime ($\Gamma(N) = SL_2(Z\ mod\ p)$ should play a* fundamental string mode, *associated with a "spectral decomposition" of the H-atom frequencies of the various n-spectral series*[16].

The fact *that Riemann zeros (R-spectrum) is Fourier dual to the $Z - Spec$, the primes, with their Polya-interpretation, is a hint that these (the above) might be related ...*

3.4.6 A crude attempt ...

Let us take the TOI subgroups (Γ for $N = p = 3, 5, 7?$), and count their conjugacy classes: QPN $n <-> N$, l the number of symmetry axes (faces plus nodes), m ... ? maybe l, m should be viewed as two directions, corresponding to the *interaction* between a particle with spin and the "ambient" M-field; hence l, m is a pair of directions, and of course their inner product, a quantum amplitude related: $< \psi_l, \psi_m >$ (connection geometry; Hyper-Kehler framework, icosians etc.).

The irreducible reps of Γ correspond to conjugacy classes. This should give another invariant / parameter.

With this "partial DATA", we coin the following definition, containing a hypothesis / program of study too:

Definition 3.1 *The* internal energy, *E_{Int}, interpreted as of "magnetic origin" (curvature/connection geometry), defines the* inertial mass, *according to Einstein's relativistic formula:*

$$E_{Int} = mc^2, \quad 1/c^2 = \epsilon \cdot \mu.$$

The corresponding decomposition of the canonical momentum $p = m\,v$ resulting from soldering the Gauge Theory Huper-Kahler principal bundle with a Space-Time (symplectic) Tangent Bundle defines the fine structure constant*:*

$$Symplectic\ Mechanics : \alpha = \frac{v_e}{c}, \quad Gauge\ Theory :\ E_{Ext} = \alpha^2\ E_{Int}.$$

A crude estimate of the internal portion of the partition function of the H-atom, results from:

$$G = \Gamma(N),\ N = 3, 5, 7; \quad n_{|G|} = |G/conj.|, \quad d_l = dim(V(G; l),\ l \in Conj.Cl.of G.$$

So, the PQN $n <-> Z/nZ$ is now (for internal symmetry group) $G \in \{\Gamma | modular\ quotiesnt\ N = 3, 5, 7\},$; For exach group $l <-> $ conj. cl./irred. rep. of such a G. For each such irred. rep $V(G; l)$, d_l is its dimension.

Then, for the internal energy, a similar sum may be expected:

$$Zint = \sum_G 1/[\sum_l \sum_m 1]^2.$$

Since

$$|G| = \sum_{irr} d_i^2, \quad d_i = dim(V_i),$$

the probabilities involved, analog to the $1/2n^2$ factor for E_{Ext}, would be $\sum_G 1/|G|$. For TOI such a sum is $1/12 + 1/24 + 1/120 = 2/15$.

[16] Is the structure of n into primes, of any significance for a finite *Chinese Remainder Theorem* application to Schrodinger's equation, in the finite / discrete case!?

3.4.7 ... and State Sum Invariants

A connection with the familiar framework of state-sum invariants should be investigated [17].

McKay-Thomson and String Theory. As a prospective plan, such a sum of dimensions of representations for a family of quotients (deck transformations of Belyi maps, of modular curves with dessins d'enfant as signatures of the vibration modes), is reminiscent with the Mckay-Thomson series in the case of infinite dimensional Lie algebras (Kac-Moody), more specifically associated with the Virasoro algebra (central extension of Witt algebra, the Lie algebra of the vector fields on the 1D-sphere).

Elliptic Curves and Jehle's Alpha. The loopforms model of H. Jehle (a Wilson loops analog), claims that a count of $N = 207$ toroidal fibrations of the Hopf fibration type is needed, to yield the correct electric Coulomb field and Bohr magnetic moment of the toroidal model of the electron (and quarks, as modeled by similar $SU(2)$-connections of EM type), but also the value of Alpha.

below a tentative connection between alpha and the periods of EC $\tau = \omega_1/\omega_2$, will be attempted, to involve j-invariant of EC, with its possible role in explaining fermions mass (electron, muon, tau: [9, 35]).

Energy Levels and Entropy The energy-probability relation in the partition function relates the above $p_i = E_i/kT$ and Shannon entropy, via Gibbs entropy:

$$Gibbs\ entropy:\ S_G = k_B Sh(p),\ Sh(p) = -p_i \log p_i\ Shannon\ entropy.$$

This allows to replace the thermodynamic temperature, by a direct counting of states framework.

Mass as a quantity of information Recall the entropy is the probabilistic average of *quantity of information*:

$$I(p) = -\sum log(p_i),$$

which is a measure of the "weight of generators" per mode / eigenspace of states for a certain (degenerate) energy level. In the author's opinion, this is a direct approach to understand the meaning of mass.

Boltzmann and Avogadro Numbers The role of these constants, in this context, needs a better understanding, in relation with the classical variables (metric, mass etc.).

3.4.8 The TOI Model of Internal Energy

If only TOI groups G are considered in the above state sum / partition function Z_{Int}, then [50] the weight factor for E_{Int} in $E_{Ext} = \alpha E_{Int}$, as an example, can be computed as follows.

Recall that the dimensions d_i of irreducible reps (conjugacy classes/ characters) and the size of the group are related by:

$$|G| = \sum d_i^2.$$

We will provide this relation for the corresponding *binary Platonic groups*, under the $2:1$ correspondence $SU(2) \to SO(3)$ [58].

Tetrahedron group.

$$G = A_4,\ d: 3\times 1D, 3\times 2D, 1\times 3D, \quad 2\times 12 = 3\cdot 1^2 + 3\cdot 2^2 + 1\cdot 3^2.$$

Octahedron Group. $O = S_4$. The *binary tetrahedral group* has size 48:

$$O = S_4:\ 48 = 2\times 4! = 2\times 1^2 + 3\times 2^2 + 4^2.$$

Icosahedron group. Binary icosahedral group $2I$, of size 120 (but not isomorphic to S_5 [59]) has dimensions of irreps $1, 2, 2, 3, 3, 4, 4, 5, 5, 6$ [58]:

$$G = S_5:\ 120 = 1^2 + 2\times 2^2 + 2\times 3^2 + 2\times 4^2 + 5^2 + 6^2.$$

Then the analog of the $\zeta(2)$ factor for E_{Ext}, for the family of groups $G = Z/n$, is:

$$\sum_{G=T,O,I}\ \sum_{V_i\in Irr(G)} \frac{1}{dim(V_i)^2} = (3\frac{1}{1^2} + 3\frac{1}{2^2} + \frac{1}{3^2}) + (2\frac{1}{1^2} + 3\frac{1}{2^2} + \frac{1}{4^2}) + (\frac{1}{1^2} + 2\frac{1}{2^2} + 2\frac{1}{3^2} + 2\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2})$$
$$= 31/9 + 9/4 + 383/200 = 13697/1800 = 7.609(4). \quad (2)$$

[17] Author's MS thesis supervised by Dr. Louis Crane ...

This is of course just an example of the direction suggested above, to compute a distribution of probabilities for the states yielding the internal energy (electron mass, of magnetic origin, i.e. 3D-finite groups / reps in the H. Jehle toroidal model, associated with the Hopf fibration), similar to $\sum_n 1/2n^2$ for the external energy. Here $\Gamma = G$ and the above $N(\Gamma) = \sum_{V_i \in Irr(\Gamma)} 1/dim(V_i)^2$ (see §3.4.6, Eq. 1).

3.5 Comparing with the Period Interpretation

Returning to the other loose end, the interpretation of α as the inverse of a period P [52]:

$$P = \pi(4\pi^2 + \pi + 1) \approx 137.036303776, \quad P - \alpha^{-1} = 0.00030469187, \tag{3}$$

Then the ratio $E_{Ext}/E_{Int} = \frac{1}{2}\,\alpha^2\zeta(2)$ becomes, under the assumption $P <-> 1/\alpha$:

$$\frac{1}{2}\cdot\zeta(2)\alpha^2 = \pi^2/12\cdot\frac{1}{\pi^2(4\pi^2+\pi+1)^2} = \frac{1}{12}\cdot\frac{1}{4\pi^2+\pi+1}.$$

This reduces the Ext/Int ratio of energies in the partition function, to the period-algebraic factor $4\pi^2 + \pi + 1$:

$$N(z) = 4z^2 + z + 1,$$

As explained in [52] §6.1-6.3, it can be related to the two periods of an EC, and viewed as an algebraic extension of π (loc. cit. 6.3).

Further investigations of the two sides, cohomologic / periods and symmetry / representation theory, are needed.

4 Conclusions and Further Developments

The progress in Unit Standards, e.g. Natural Units, together with Buckingham Pi-Theorem lead to a deep link with cohomological physics and de Rham Periods: as suggested by Post, physics laws may turn out to be "just" Period Laws (finally clarifying the "Unreasonable effectiveness of Mathematics").

There are several partial works that addressed deep questions, but unfortunately not well used yet: H. Jehle, E. Kiehn, E. J. Post, J. Schwinger etc. They attempt to refine pointwise physics and provide a singularity-free, gauge and topological framework for unifying electric and magnetic charges.

These contributions provide the base-camp for climbing the "Everest": what alpha is, or rather on what framework (mountain) is defined.

4.1 Cohomological Constants vs. Period Constants

Natural constants are of two types: A) cohomological "central charges" (central extensions; capturing a duality), e.g. h, c and B) periods in associated algebraic-geometric models: electric, magnetic and spin. While the general framework defining them, e.g. h and c, is rather known (Heisenberg group, quaternion algebra, Hopf fibration), and the "factorization" into periods, also known, e.g. $h = e \cdot g, 1/c^2 = \epsilon\mu$, the actual discrete structure which dictates the factorization (Galois group, covering spaces etc.) and how it yields alpha, still needs work: a study of modular curves as Belyi maps and counting elliptic curves (Jacobians?).

4.2 Natural Laws as Periods Laws

In a previous article [2], alpha was presented as a period, and the Natural Period Laws Program announced. In this article, the use of Buckingham's Pi-Theorem (Raynold Number interpretation of alpha, as a ratio between "inertial and elastic forces": electric and magnetic), was proposed, to better understand how the above Hopf fibration 3-periods (CST GT / quantum computing / spinor framework) are related to the the emergent relation $1/c^2 = \epsilon\mu$ (the quaternionic / Space-Time central charge), via Hodge duality.

The actual value needs a count of number of modes (see [2]) with some "Hodge data" structure coming from modular curves and j-invariant.

4.3 Further developments

4.3.1 On The Toroidal Model in SU(2)-EM

By now it is apparent that the work of H. Jehle needs a thorough study from the presented point of view: it relates spin, electric charge and magnetic charge, in the context of a unified EM field, based on the Hopf fibration (QM: $CP^1 \cong S^3$: *qubits*/*QC*). It notably relates the Physics with an unfinished mathematical model, including a computation of Alpha, based on an estimate of the number of subfibrations of the Hopf bundle, which seams associated to finite subgroups, as for quark flavors and generations.

4.3.2 Quantized Theory of Spin

The above leads towards a unification of *spin, e-charge and m-charge*.

Now the Theory of Spin needs "quantized", as announced by the author in previous works: everything is (locally) finite ("Quantize everything!"); hence axes of symmetry, e.g. of Platonic geometries, should correspond to the available directions for interactions (implying the quantization of angular momentum). Also finite groups implies finite spins and finite number of irreducible representations; but when having a class of such finite groups, there is a "refinement" of the theory, akin with the one for H-atom: fine structure, hyperfine, Zeeman effect etc.. It will clarify quantitative aspects of the gravitational split of energy levels.

A comparison with quatum groups at roots of unity is mutually beneficial.

Regarding the "big picture" (Weil and the role of analogies), we included a general description of how the Network Model is a universal framework, unifying not only many specific aspects and solving "paradoxes" in Quantum Physics, but also providing a unifying approach to Sciences in general.

A The Network Model as a Paradigm in Sciences

In several prior articles the role of the Network Model was explained [57], and shown how it solves many of the "weird" experimental facts of Quantum Physics in a natural way: 2-slit experiment, delayed choices, the problem of "time" and its "arrow", entanglement etc.

A.1 The Hierarchy

The main point of the Net-Model is that a complex system has a hierarchy of subsystems, with irreducible parts etc. like a human organism for instance, and various sciences study specific level of description, agregation, interaction etc. The NM provides a unified framework for this.

A.2 One Interaction, several phases / regimes

In Physics, Classical and Quantum, best understood from a Gauge Theory fibration perspective, it also unifies particles and fields / fermionic sources and gauge bosons, as interaction channels between the elementary nodes of the Network, baryons. The channels are mesonic-leptonic, depending on the regime of the interaction (laminar or turbulent, "near field" or "far field"), with decays for instance "just" phase transitions, in addition to the tree-like or loops, created in the turbulent stage, or just as steady-state bound systems (e.g. orbitals).

A.3 An Expert System vs. TOE

The neutron is the fundamental baryon, quasi-stable with broken symmetry the hydrogen atom, an irreducible system where the baryonic field closes in an orbital, reminiscent of a capacitor (yet magnetic moments are present: it is a Quantum LC-circuit).

The NM paradigm can be seen scaled from Nuclear Physics to Atomic, Chemistry, Biology etc.

The application of GT concepts to Low Energy Physics and HEP alike, shows the universality of the gauge field (EM-type), and emergence of Space, Time, metric; the matter is composed of the nodes of

the Network, in interactions via channels or bonds, quite similar to VLSI and "QI busses" connecting various components (mesoni bonds in nuclei, electronic bonds in modecules, together with valence orbitals for further connections etc.)

Gravity is by now well understood, as a spin dependent component of the baryonic field, with its three eigenvectors of $U(1)$-symmetry, called quarks, with fields of EM type each. The nuclear spin polarization is quite similar with the electronic one, which yields macroscopic magnetism, and can be controlled, affecting the inertial mass ("resitance" to momentum change), similar to what happens in electronics or LEP (supraconductivity).

A.4 QCD and EWT

What remains is for Gravity to be "recognised" as part of the description of the Nuclear Force, once QCD and EWT are unified by understanding that $SU(3)$ is a group of symmetry of the 3D-frame of these quark directions, of $SU(2)$ type.

The duplication *W's* (WF) and *pions* (NF) can be resolved, once mesons are viewed as "valence" quark bonds (duplex channel), through which gluons transfer E/p etc, since the $U(1)$-regime (3D "Space Frames" aligned interaction) is just the electron (or lepton) - photon interaction (quark-quark channel, no torsion etc. and structure group reduction to $U(1)$-gauge boson).

A.5 Period Laws

The basic elements of a quantum circuit will be subject to *period Laws*, once we use math models including: Hopf fibration, RS / Belyi maps (and their unitary tangent line bundles / quantum phase), together with discrete "Hodge structure Data" (tesselations coming from the modular curves approach).

B Mechanical vs. Electrical Laws

The *Mechanical-EM Analogy* is often used, e.g. in the interpretation of Alpha as v/c of Bohr's model of the hydrogen atom. The underlying justification is the generalized momentum $P = p + eA$, which also relats Space-Time base manifold an Gauge Theory fiber ($U(1)$ or $SU(2)$ versions of EM).

B.1 Emergence of Space and Time

In previous articles [37, 36] it was explained how Space and Time emerge in the GT Paradigm of the SM.

This should also allow to relate the cohomological, quantum parameters h, c and the constitutive, emergent parameters ϵ_0 and μ_0.

B.2 Permitivity and Permeability

ϵ_0: relates the units for electric charge to mechanical quantities such as length and force:

$$Coulomb\ constant:\ k_C = \frac{1}{2\pi\epsilon_0},\ F_E = \frac{1}{4\pi}\frac{1}{\epsilon_0}\frac{Q_1 Q_2}{r^2},\quad Hodge\ duality:\ D = \epsilon_0\, E.$$

Mu: "scalar constant linking the electromagnetic quantities and the mechanical quantities; "Magnetic Coulomb Law" [47]:

$$"Ampere's\ constant":\ k_A = \frac{1}{2\pi}\mu_0,\ F_M/L = k_M\,\frac{I_1 I_2}{d},\quad Hodge\ duality:\ B = \mu_0\, H.$$

Note that the *linear density of force* used here is "not natural", and a closed loop version needs to be used instead: radial / electric / work vs. rotational / magnetic / geometric ("space curvature").

Moreover, the natural *conjugate momenta* are the electric charge Q and magnetic flux Φ [42, 43, 44] (What is the analog of, or relation with $p + eA$?)[18].

A linking-number (Gauss [49] form would allow the interpretation in the context of Chern-Simins Theory, relating the three periods: e, g, h.

[18] For more insight into the role of the vector potential, see [45]

B.3 Unification and Quantum-Relativistic Split

A unifying framework is provided by Hodge Theory in the context of Gauge Theory $F = dA$ etc. (Seiberg-Witten / Hitchin eq. [48][19]); how this corroborates with the polar decomposition of conformal transformations (electric and magnetic components), or alternatively with "Minkowski Space-Time structure" / Lorentz metric (QC/Symplectic vs. central extension: 2+2=3+1) needs analysed.

What controls the factorization also relates to what Alpha is.

B.4 ... and Natural Units

The corresponding Natural Laws should be expressed using DFT of an algebraic-geometric model, with EC part of Hopf fibration formalism. This is similar to DFT for Z/nZ [46] applied to congruence subgroups of the *modular group* $SL_2(Z)$, with Belyi maps as such an AG-model [35]. The TOI Platonic symmetries are just a basic level of groups $p = 3, 5, 7$ (?), for modular curves plethora and rich theory.

B.5 Periodic Time and Quantum Space

The local time[20] and space for a quantum object, in the sense of Einstein-Cartan (3+1 tetrad local frame) are now "quantized" and bound states and transitions occur in the context of finite groups of symmetry.

The natural units should be de Rham / cohomological of origin; e.g. for a version of Q-Bohr atom using $SO(4)$, a.k.a. Hopf fibration.

ϵ_0 and μ_0 should emerge from such a GT/Machanical/EM analogy.

B.6 ... and Buckingham Pi Theorem

This is formulated for "free, independent physical units", that correspond to *commuting variables* (classical).

Including relations, like $hc = e'g'$, deformations / quantization / central extensions / Hodge-Hopf duality h, c (Dirac-Heisenberg, Hodge) needs a "cohomological Pi-Theorem", which should be inspired from Hilbert's syzygy theorem (short exact sequences). The role of Hopf fibration and its associated homotopy sequence remains to be considered and used.

References

[1] E. J. Post, The Gaussian version of Ampere's law, JMP, Volume 19, Issue 1, January 1978, https://pubs.aip.org/aip/jmp/article-abstract/19/1/347/445455

[2] L. M. Ionescu, The fine structure constant: revisited, https://vixra.org/abs/2308.0180

[3] L. M. Ionescu, Our Universe: to Model and to Simulate, https://vixra.org/abs/2305.0162

[4] L. M. Ionescu, Algebraic-Geometry Tools for Particle Physics, https://vixra.org/abs/2305.0001

[5] Jeff Yee, The relationship of the Fine Structure Constant and Pi, 2019, https://www.researchgate.net/publication/333224718

[6] Roskies R., et. al, 1971. A new pastime–calculating alpha to one part in a million. Phys. Today, 24 9.

[7] L/ M/. Ionescu, related to Superconductivity and Quantum Computing / Quantum LC-Circuits perspective: https://vixra.org/abs/2305.0162, https://vixra.org/abs/2307.0084

[8] M. E. Tobar, An Alternative View of the Fine Structure Constant and its Drift: Bringing the Flux Quanta into the Definition, https://arxiv.org/vc/hep-ph/papers/0306/0306230v2.pdf

[19] Higgs bundles etc.

[20] Local periodic time is the quantum phase; spin is related to a relative change of symmetry axes, relative to another quantum object, with its own local 3+1D frame.

[9] F. Potter, http://www.sciencegems.com/Icosahedron.html

[10] Herbert Jehle, Quantization of magnetic flux: articles 1971-1977.

[11] M. H. Mac Gregor, The power of alpha: Electron Elementary Particle Generation with α-Quantized Lifetimes and Masses, https://www.worldscientific.com/worldscibooks/10.1142/6213#t=aboutBook

[12] L. M. Ionescu, The Beauty and The Beast.

[13] L. M. Ionescu, On elementary particles: rethinking the Standard Model, TBA.

[14] NIST, Inverse fine-structure constant, https://physics.nist.gov/cgi-bin/cuu/Value?alphinv

[15] D. Griffith, Introduction to Elementary Particle Physics, 2nd ed., 2008

[16] R. M. Kiehn, Periods on manifolds, quantization and gauge, JMP, Vol. 18, No.4, p.614-624, 1976.

[17] Sabine Hossenfelder, Lost in Math: How Beauty Leads Physics Astray Hardcover – Illustrated, June 12, 2018.

[18] L. M. Ionescu, A modular group approach to completion of the rationals, work in progress.

[19] Wiki: Legendre relation, https://en.wikipedia.org/wiki/Legendre%27s_relation

[20] L. M. Ionescu, related to *periods*: 1) On Hodge structure and Periods, https://arxiv.org/abs/2105.13115; 2) On Periods: From Local to Global, https://arxiv.org/abs/1806.08726; 3) https://arxiv.org/abs/1708.09277 4) From Periods to Anabelian Geometry and Quantum Amplitudes; 5) Several articles at: https://about.illinoisstate.edu/lmiones/research/

[21] L. M. Ionescu, on the arrow of time, https://arxiv.org/abs/0708.4180

[22] L. M. Ionescu, Variations sur une theme de Kontsevich, IHES presentation Feb. 6, 2018, https://drive.google.com/file/d/1uU_40ZB3-uvxIl5RsGC2UheXqQguYaTE/view

[23] Masatsugu MINAMI, Dirac's Monopole and the Hopf Map, Progress of Theoretical Physics, Vol. 62, No. 4, October 1979.

[24] Wikipedia: Buckingham Pi Theorem, https://en.wikipedia.org/wiki/Buckingham_%CF%80_theorem

[25] Wikipedia: Legendre's relation.

[26] Wikipedia: Chern-Simons Theory.

[27] B. Mazur and W. Stein, Prime numbers and the Riemann hypothesis, 2016.

[28] S. Gukov and E. Witten, Gauge Theory, Ramification and Langlands program, IP Current Developments in Mathematics, 2006, https://www.intlpress.com/site/pub/files/_fulltext/journals/cdm/2006/2006/0001/CDM-2006-2006-0001-a002.pdf

[29] E. Frankel, Gauge Theory and Langlands duality, http://www.numdam.org/item/AST_2010__332__369_0.pdf

[30] J. Stasheff, Cohomological Physics, Cohomological physics, 2006; see also "Cohomological Physics in the XX-th c: a survey, https://ncatlab.org/nlab/files/StasheffCohomPhysicsSurvey.pdf

[31] C. Duston, Modeling Spacetime as a Branched Covering Space over 2-Knots.

[32] J. Schwinger, A magnetic model of matter, 1969.

[33] Google 1,2 and 3D cymatics: images.

[34] Wikipedia: Natural Units.

[35] L. M. Ionescu, The Beauty and The Beast, https://vixra.org/abs/2304.0051

[36] L. M. Ionescu, On Emergence of Relativistic Time from Quantum Phase, https://vixra.org/abs/2303.0149; see also

[37] L. M. Ionescu, A Quark Fields Emergent Space-Time and Gravity, https://vixra.org/abs/2303.0030

[38] Charles Day, Pseudovortices Aid in Modeling the Synchronization Behavior of Neurons, https://physics.aps.org/articles/v16/s109

[39] L. M. Ionescu, "A Quark Fields Emergent Space-Time and Gravity", https://vixra.org/abs/2303.0030; see also [36] and https://vixra.org/abs/2305.0162

[40] L. M. Ionescu, On Hodge Structures and Periods – Part I: Algebraic Aspects, https://arxiv.org/abs/2105.13115

[41] L. M. Ionescu, "Hodge Structures and Motives" (Google Slides), IHES online presentation, Nov.11, pdf; https://about.illinoisstate.edu/lmiones/research/

[42] M. H. Devoret, Quantum fluctuations in electrical circuits

[43] M. H. Devoret and M. Martinis, Superconducting Integrated Circuits, Quantum Information Processing, Vol. 3 Nos. 1-5, Oct. 2004, https://link.springer.com/article/10.1007/s11128-004-3101-5

[44] Bernard Yurke and John S. Denker, Quantum network theory, Phys. Rev. A 29, 1419 – Published 1 March 1984

[45] L. M. Ionescu, To understand the Universe: Follow the Qi, https://vixra.org/abs/2307.0084

[46] A. Terras, Fourier Analysis on finite groups and applications, Cambridge University Press Online publication date:July 2010.

[47] Electropedia: "permitivity of vacuum", https://www.electropedia.org/iev/iev.nsf/display?openform&ievref=121-11-03; "permeability of vacuum", https://www.electropedia.org/iev/iev.nsf/display?openform&ievref=121-11-14

[48] N. Hitchin, The Self-Duality Equations on a Riemann Surface N. J. Hitchin, Proc. LMS, Volumes 3-55, Issue1 July 1987, Pages 59-126

[49] Wikipedia: "Linking Number", https://en.wikipedia.org/wiki/Linking_number#Gauss's_integral_definition

[50] L. M. Ionescu, From Platonic Solids to Gravity via Representation Theory, ISU Algebra Seminar, Oct. 28, 2021, https://about.illinoisstate.edu/lmiones/research/

[51] L. M. Ionescu, On quark fields and nuclear physics, https://vixra.org/abs/2212.0217

[52] L. M. Ionescu, Fine structure constant: revisited, https://vixra.org/abs/2308.0180

[53] L. M. Ionescu, The Qubit Model: A Platonic and Exceptional Quantum Theory, https://vixra.org/abs/1912.0547; see also: The Platonic Universe (and the ADE-Classification), https://about.illinoisstate.edu/lmiones/research/

[54] L. Hecht, The geometric basis for the periodicity of elements, pp. 18-25, https://21sci-ech.com/Articles%202004/Spring2004/Periodicity.pdf; see also R. J. Moon, "Space must be quantized", pp.26-3; http://21sci-tech.com/articles/moon_nuc.html; Dr. R. J. Moon: life and work, http://21sci-tech.com/Articles%202005/moon_F04.pdf

[55] by Hal Wm. Vaughan, "Archimedean Polyhedra And the Boundary: The Missing Link", https://21sci-tech.com/Articles%202005/Archimedean_Polyhedra.pdf

[56] L. M. Ionescu, Algebraic-Geometric tools for particle physics, https://vixra.org/abs/2305.0001

[57] L. M. Ionescu, A Note on Manifolds VS Networks as Mathematics Models in Modern Physics, https://vixra.org/abs/1912.0095; see also "On The Current Physics Crises and The Hierarchy Problem", https://vixra.org/abs/2301.0041; https://vixra.org/abs/2301.0012

[58] J. Bakland, Notes: Character tables and McKay quivers; see link in https://ncatlab.org/nlab/show/icosahedral+group#character_table

[59] Wikipedia: Binary icosahedral group.