A Novel Derivation of Black Hole Entropy in all Dimensions from truly Point Mass Sources

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Abstract

It is explicitly shown how the Schwarzschild Black Hole Entropy (in all dimensions) emerges from truly point mass sources at $r = 0$ due to a non-vanishing scalar curvature involving the Dirac delta distribution. It is the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ which furnish the Schwarzschild black hole entropy in all dimensions $D \geq 4$ after evaluating the Euclidean Einstein-Hilbert action. As usual, it is required to take the inverse Hawking temperature β_H as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time. The appealing and salient result is that there is *no* need to introduce the Gibbons-Hawking-York boundary term in order to arrive at the black hole entropy because in our case one has that $\mathcal{R} \neq 0$. Furthermore, there is no need to introduce a complex integration contour to *avoid* the singularity as shown by Gibbons and Hawking. On the contrary, the source of the black hole entropy stems entirely from the scalar curvature *singularity* at the origin $r = 0$. We conclude by explaining how to generalize our construction to the Kerr-Newman metric by exploiting the Newman-Janis algorithm. The physical implications of this finding warrants further investigation since it suggests a profound connection between the notion of gravitational entropy and spacetime singularities.

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The static spherically symmetric (SSS) *vacuum* solution of Einstein's field equations [1] (in Lorentzian signature) was originally found by Schwarzschild

[3], but is historically more widely known in terms of the solution provided by Hilbert [2] as

$$ds^2 \ = \ - \ (1 - \frac{2GM}{r}) \ (dt)^2 \ + \ (1 - \frac{2GM}{r})^{-1} \ (dr)^2 \ + \ r^2 \ (d\Omega)^2 \qquad (1)$$

where the solid angle infinitesimal element is $(d\Omega)^2 = (d\theta)^2 + sin^2(\theta)(d\phi)^2$. We shall use throughout this work the units of $\hbar = c = k_B = 1$.

The higher-dimensional extension of the metric (1) was found by Tangherlini [4] and can be obtained by simply replacing $(d\Omega)^2 \rightarrow (d\Omega_{D-2})^2$ (the $D-2$-dim solid angle) and $1 - \frac{2GM}{r} \rightarrow 1 - (\frac{r_h}{r})^{D-3}$ where r_h is the horizon radius expressed in terms of M and the gravitational coupling G_D in D dimensions whose units are $(length)^{D-2}$. The higher dimensional metric is given by

$$ds^2 \ = \ - \ f(r) \ (dt)^2 + \frac{(dr)^2}{f(r)} + r^2 \ (d\Omega_{D-2})^2, \ \ f(r) \ = \ 1 \ - \ \frac{16\pi G_D M}{(D-2)\Omega_{D-2} r^{D-3}} \qquad (2a)$$

where G_D is the D-dim Newton's constant, M the black hole mass. The solid angle of a $D-2$-dim hypersphere is $\Omega_{D-2} = 2\pi^{\frac{D-1}{2}} / \Gamma(\frac{D-1}{2})$. The horizon radius is determined from the condition $f(r_h) = 0$ giving

$$r_h \ = \ \left(\frac{16\pi G_D M}{(D-2) \ \Omega_{D-2}} \right)^{\frac{1}{D-3}} \qquad (2b)$$

such that the metric (2a) can be rewritten as

$$ds^2 \ = \ - \ [\ 1 - (\frac{r_h}{r})^{D-3} \ ] \ (dt)^2 \ + \ [\ 1 - (\frac{r_h}{r})^{D-3} \ ]^{-1} \ (dr)^2 \ + \ r^2 \ (d\Omega_{D-2})^2 \quad (3)$$

The Schwarzschild metric leads to a vanishing Ricci tensor and scalar curvature $\mathcal{R} = 0$, hence in order to arrive at a key delta function singularity at the origin one has to replace r for $|r|$ in the metric (1). More precisely, one needs to make the replacement

$$1 - \frac{2GM}{r} \ \rightarrow 1 - \frac{2GM}{|r|} \ = \ 1 - \frac{2GM}{r} \ \frac{r}{|r|} \ = \ 1 - \frac{2GM sgn(r)}{r}, \ \ r = |r| sgn(r) \quad (4)$$

where $sgn(r)$ is the sign function. The sign function is defined by $sgn(r) = 1$, for $r > 0$; $sgn(r) = -1$, for $r < 0$; and $sgn(r = 0) = 0$, the arithmetic mean of $1, -1$, and it will be instrumental in deriving the non-zero scalar curvature. The derivative of the sign function is $\frac{d}{dr} sgn(r) = 2\delta(r)$ [1]. It is the derivatives of the sign unction appearing in eq-(4) which will generate the key $\delta(r)$ terms in the scalar curvature. If one wishes to be mathematically rigorous in using

[1]The factor of 2 is due to the jump of 2 from -1 to $+1$

distributions in nonlinear theories like general relativity one needs to recur to the Colombeau's theory of distributions [7] instead of the Dirac delta distributions.

In doing so one finds that the scalar curvature is no longer zero $\mathcal{R} \neq 0$ but has a delta function singularity at $r = 0$ [2]

$$\mathcal{R} \;=\; 4GM \;\left(\frac{\delta'(r)}{r} \;+\; 2\;\frac{\delta(r)}{r^2} \right) \;=\; 4GM\;\frac{\delta(r)}{r^2}, \tag{5}$$

where the identities involving the derivatives of the delta functions have been used

$$\delta'(r) = -\frac{\delta(r)}{r}, \quad \delta^{(n)} \;=\; (-1)^n\; n!\;\frac{\delta(r)}{r^n} \tag{6}$$

Because now one has that $\mathcal{R} \neq 0$, the Euclidean Einstein-Hilbert action is no longer zero. The inverse Hawking temperature $\beta_H = 8\pi GM$ is the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory and resulting after a Wick rotation, $it = \tau$, to imaginary time. The non-trivial Euclidean Einstein-Hilbert action is given by the integral

$$I \;=\; -\;\frac{i}{16\pi G}\;\int_0^{\beta_H}\;d\tau\;\int_0^{\infty}\;\mathcal{R}\;4\pi r^2\;dr \tag{7}$$

Note the presence of an $-i$ factor in the Euclidean action I which results from the measure $\sqrt{-g}$ piece since the determinant $g = det(g_{\mu\nu}) > 0$ is now positive due to the Euclidean signature. The minus sign $-i$ is chosen so that $exp(iS_g) = exp(-I)$ in the gravitational path integral ($I = -iS_g$).[3] Furthermore, because the radial integral (7) is symmetric in r due to $\delta(-r) = \delta(r)$, one has to extend the radial domain of integration as follows

$$\int_0^{\infty}\;\delta(r)\;dr \;=\; \frac{1}{2}\;\int_{-\infty}^{\infty}\;\delta(r)\;dr \;=\; \frac{1}{2} \tag{8}$$

in order to fully integrate the delta function. Given $\beta_H = 8\pi GM$, $\mathcal{R} = 4GM\delta(r)/r^2$, and eq-(8), the magnitude of the integral (7) becomes

$$|I| = \;\frac{1}{2}\;M\;\beta_H \;=\; 4\pi GM^2 \;=\; \frac{4\pi(2GM)^2}{4G} \;=\; \frac{4\pi r_h^2}{4G} \;=\; \frac{Area}{4L_P^2} \tag{9}$$

Therefore, the (magnitude of the) Euclidean Einstein-Hilbert action S_E associated with the delta function point mass source yields precisely the Schwarzschild black hole entropy and given by one quarter of the horizon's area in Planck units. Note that it was *not* necessary to introduce the Gibbons-Hawking-York *boundary* term [5], [6] in order to evaluate the entropy and involving the extrinsic curvature K

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[2]The Kretschmann invariant $\mathcal{R}_{\mu\nu\rho\tau}\mathcal{R}^{\mu\nu\rho\tau} \sim (\frac{2GM}{r^3})^2$ is singular at $r = 0$ for the Schwarzschild metric

[3]The scalar curvature $\mathcal{R}$ remains unaffected due to the fact that the change of sign in g_{tt} and $\mathcal{R}_{tt}$ cancels out when one evaluates the trace of the Ricci tensor component $g^{tt}\mathcal{R}_{tt}$

$$S = \frac{1}{16\pi G} \int_M \sqrt{|g|} \; \mathcal{R} \; d^4x \; + \; \frac{1}{8\pi G} \int_{\partial M} \sqrt{|h|} \; K \; d^3x \tag{10}$$

h is the determinant of the induced metric on the boundary ∂M. The bulk Einstein-Hilbert action for the metric (1) vanishes (due to the vanishing of $\mathcal{R} = 0$), consequently, the contribution to the entropy stems entirely from the extrinsic curvature K of the boundary term. Gibbons and Hawking argued that in order to obtain an action which depends on the first derivatives of the metric, as is required by the composition property of the path-integral approach, the second derivatives appearing in the curvature scalar $\mathcal{R}$ had to be removed by an integration by parts resulting in the need to introduce the boundary term. In the case of asymptotically flat metrics the boundary region can be chosen to be the product of the Euclidean time axis (a circle of size β_H) with a sphere S^2 of large radius. Gibbons and Hawking evaluated the action for the gravitational field on a section of the *complexified* spacetime which avoids the singularity. The boundary integral in the limit that the sphere's radius goes to infinity yielded an action I given by $i4\pi GM^2$, and which agrees with the black hole entropy (up to an i factor).

Let's proceed with the evaluation of the higher dimensional Schwarzschild black hole entropy. Once more, by replacing $r \rightarrow |r|$ in the metric (2a, 3) it gives

$$ds^2 \; = \; - \; f(|r|) \; (dt)^2 \; + \; \frac{(dr)^2}{f(|r|)} \; + \; |r|^2 \; (d\Omega_{D-2})^2 \; =$$

$$- \; (1 - (\frac{r_h}{|r|})^{D-3}) \; (dt)^2 \; + \; (1 - (\frac{r_h}{|r|})^{D-3})^{-1} \; (dr)^2 \; + \; |r|^2 \; (d\Omega_{D-2})^2 \tag{11}$$

After a very lengthy and laborious calculation one learns that the scalar curvature associated with the metric (11) is

$$\mathcal{R} \; = \; \frac{d^2 f}{dr^2} \; + \; \frac{2(D-2)}{r} \; \frac{df}{dr} \; - \; \frac{(D-2)(D-3)}{r^2} \; (1 - f) \tag{12}$$

Taking into account that $\frac{d|r|}{dr} = sgn(r)$ [4] where $sgn(r)$ is the sign function it leads to the following results

$$\frac{d}{dr} sgn(r) \; = \; 2 \; \delta(r), \quad \frac{df}{dr} \; = \; (D-3) \; r_h^{D-3} \; \frac{sgn(r)}{|r|^{D-2}},$$

$$\frac{d^2 f}{dr^2} \; = \; - \; (D-2) \; (D-3) \; r_h^{D-3} \; \frac{1}{|r|^{D-1}} \; + \; 2(D-3) \; r_h^{D-3} \; \frac{\delta(r)}{|r|^{D-2}} \tag{13}$$

Inserting the results of eq-(13) into eq-(12) and taking into account the *identity* $r = |r| sgn(r)$ which leads to key exact *cancellations*, the scalar curvature in eq-(12) turns out to be

[4]The derivative of $|r|$ is discontinuous at $r = 0$, but because it jumps from -1 to $+1$, one may take their arithmetic mean which is 0 and which agrees with the value of $sgn(r = 0) = 0$

$$\mathcal{R}_D \;=\; 2\;\frac{16\pi G_D M}{(D-2)\Omega_{D-2}}\;(D-3)\;\frac{\delta(r)}{|r|^{D-2}} \;=\; 2\;r_h^{D-3}\;(D-3)\;\frac{\delta(r)}{|r|^{D-2}} \tag{14}$$

The use of $|r|$ in $f(|r|)$ in eq-(11) was instrumental in generating the delta function in (14). Had one used $f(r)$ one would have obtained $\mathcal{R} = 0$. In the case when $D = 4$ one recovers the same result as in eq-(5) for $\mathcal{R}$.

The Hawking temperature of the D-dim Schwarzschild black hole is $T_D = (D-3)/4\pi r_h \Rightarrow \beta_D = 4\pi r_h/(D-3)$. The non-trivial Euclidean Einstein-Hilbert action in D-dim is given by the integral

$$I \;=\; -\;\frac{i}{16\pi G_D}\;\int_0^{\beta_D}\;d\tau\;\int_0^{\infty}\;\mathcal{R}_D\;\Omega_{D-2}\;r^{D-2}\;dr \tag{15}$$

In the region where $r \geq 0$ one can replace $|r|^{D-2}$ for r^{D-2}, hence after taking into account eq-(8), setting $\beta_D = 4\pi r_h/(D-3)$, and inserting the expression (14) for $\mathcal{R}_D$ into (15), one arrives finally at

$$|I| \;=\; \frac{\Omega_{D-2}\;r_h^{D-2}}{4G_D} \;=\; \frac{\Omega_{D-2}}{4G_D}\;\left(\frac{16\pi G_D M}{(D-2)\;\Omega_{D-2}}\right)^{\frac{D-2}{D-3}} \tag{16}$$

which is the Schwarzschild black hole entropy in D-dimensions.

Next we shall find the expressions for the density and pressure of the point-matter source leading to a non-vanishing scalar curvature and which furnishes the higher dimensional black hole entropy. Given the trace of the stress energy tensor $\mathcal{T}_D = T^\mu_\mu$, the *trace* of the Einstein tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}$ obeys the following relation stemming from the field equations

$$-\;\mathcal{R}_D\;\frac{(D-2)}{2} \;=\; 8\pi G_D\;\mathcal{T}_D \;=\; -\;(8\pi G_D)\;\left(2\;(D-3)\;\frac{M}{\Omega_{D-2}}\;\frac{\delta(r)}{|r|^{D-2}}\right) \tag{17}$$

since the spherically symmetric energy-mass density ρ in D-dim for a point mass source is given by [5]

$$\rho \;=\; \frac{2M}{\Omega_{D-2}\;|r|^{D-2}}\;\delta(r) \;\Rightarrow\; \int_0^{\infty}\;\rho\;\Omega_{D-2}\;r^{D-2}\;dr \;=\; 2M\int_0^{\infty}\;\delta(r)\;dr \;=\; M \tag{18}$$

one finds that the trace of the stress energy tensor is

$$\mathcal{T}_D \;=\; -\;(D-3)\;[\;\frac{2M}{\Omega_{D-2}\;|r|^{D-2}}\;\delta(r)\;] \;=\; -\;(D-3)\;\rho \tag{19}$$

Due to the (hyper) spherical symmetry, the $D-2$ transverse pressure components $p_\perp$ to the radial direction are all equal, then the expression in (19) leads to

[5] Note the key extra factor of 2 in eq-(18) that is required to evaluate the integral of $\delta(r)$

$$\mathcal{T}_D \;=\; -\,\rho \;+\; p_r \;+\; (D-2)\; p_\perp \;=\; -\;(D-3)\;\rho \tag{20}$$

One must supplement eq-(20) with the Einstein field equations in order to determine ρ, p_r and the $D-2$ transverse pressure components $p_\perp = p_{\theta_i}, i = 1, 2, \cdots, D-2$,

$$\mathcal{R}^t_t - \frac{1}{2}\,\delta^t_t\,\mathcal{R} \;=\; 8\pi G_D T^t_t \;=\; -\,8\pi G_D\,\rho,\;\; \mathcal{R}^r_r - \frac{1}{2}\,\delta^r_r\,\mathcal{R} \;=\; 8\pi G_D T^r_r \;=\; 8\pi G_D\,p_r \tag{21}$$

$$\mathcal{R}^\perp_\perp \;-\; \frac{1}{2}\,\delta^\perp_\perp\,\mathcal{R} \;=\; 8\pi G_D T^\perp_\perp \;=\; 8\pi G_D\;p_\perp \tag{22}$$

After a lengthy but straightforward algebra one finds that the density and pressure components are

$$\rho \;=\; \frac{2M}{\Omega_{D-2}\;|r|^{D-2}}\;\delta(r), \quad p_r \;=\; -\;\frac{2(D-3)}{(D-2)}\;\rho,$$

$$p_\perp \;=\; \left(\frac{(4-D)(D-2)+2(D-3)}{(D-2)^2}\right)\;\rho \;\Rightarrow\; -\rho + p_r + (D-2)p_\perp \;=\; -(D-3)\rho \tag{23}$$

The solutions (23) satisfy the *strong* energy conditions $\rho + \sum p_i \geq 0$ but not the weak energy conditions $\rho + p_i \geq 0$ for all $i = 1, 2, \cdots, D-1$.

One may object to the above expressions (23) because the angular coordinates are not well defined at $r = 0$. This is not a problem because one can simply perform a coordinate change of the stress energy tensor $T_{\mu\nu}$ to Cartesian coordinates which are well defined at $r = 0$ [6]. The solutions (23) are consistent with the conservation equation of the stress energy tensor $\nabla_\mu T^{\mu\nu} = 0$. It can be more easily verified in $D = 4$ where one arrives at

$$\rho \;=\; -\;p_r \;=\; \frac{2M}{4\pi r^2}\;\delta(r), \quad p_\perp \;=\; \frac{1}{2}\;\rho \;=\; \frac{M}{4\pi r^2}\;\delta(r) \tag{24}$$

One can check that the expressions (24) are consistent with the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \;\Rightarrow\; p_\perp \;+\; \rho \;+\; \frac{r}{2}\,\frac{d\rho}{dr} \;=\; 0 \tag{25}$$

and which can be verified explicitly after using the identities $r\frac{d}{dr}(\delta(r)) = -\delta(r)$; $r^n \frac{d^n}{dr^n}(\delta(r)) = (-1)^n n!\delta(r)$. Similar results as those found in eq-(24) were obtained in [11] by choosing a mass density given by a Gaussian $M(\sigma)^{-3/2} exp(-r^2/\sigma)$ where the Gaussian width $\sqrt{\sigma}$ was related to the noncommutativity parameter associated with the noncommutative spacetime coordinates $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}\mathbf{1}$ after equating the norm to σ : $\sqrt{\Theta_{\mu\nu}\Theta^{\mu\nu}} = \sigma$. As the width of the Gaussian goes to zero one recovers the product of three delta functions

[6] In Cartesian coordinates the stress energy tensor will have off-diagonal components

$$lim_{\sigma\to 0} \left(M\ (\sigma)^{-3/2} exp(-r^2/\sigma)\right) \ \to\ \rho\ =\ M\ \delta(x)\ \delta(y)\ \delta(z) \tag{26}$$

Our mass density does not involve the product of three delta functions but involves the term $\frac{\delta(r)}{r^2}$ instead. Because the authors [11] used a Gaussian mass density to *smear* the point mass source and introduce "fuzziness" of the spacetime points into the picture, their value of $\mathcal{R}$ was finite at $r = 0$. Their physical model could be viewed as a self-gravitating *anisotropic* fluid droplet. Our effective mass function in eq-(4) is $\mathcal{M}(r) = M sgn(r)$, whereas the mass function $\mathcal{M}(r)$ in [11] was given by an incomplete gamma function as a result of integrating the Gaussian mass density across a spherical region of radius r.

The importance of using the modulus $|r|$ can also be seen when one evaluates the Laplacian : $\nabla^2(1/r) = 0$ but $\nabla^2(1/|r|) \sim \delta(r)/r^2$. In $3D$ the radius is defined as $r = \pm\sqrt{x^2+y^2+z^2}$. In general, one must include both $\pm$ signs so an analytical extension from $r \to -r$ is possible by using $|r|$ in the metric solutions and without having to switch the signs $M \to -M$.

After this discussion one concludes that the expressions (23) are the density and *anisotropic* pressure components associated with the point mass delta function source at the origin $r = 0$ and which furnish the Schwarzschild black hole entropy (up to a factor of $-i$) in all dimensions $D \geq 4$ by a direct evaluation of the Euclidean Einstein-Hilbert action. As usual, it was required to take the inverse Hawking temperature β_H as the length of the circle S^1_β obtained from a compactification of the Euclidean time in thermal field theory which results after a Wick rotation, $it = \tau$, to imaginary time. The appealing result is that there was *no* need to introduce the Gibbons-Hawking-York boundary term in order to arrive at the black hole entropy because in our case one has that $\mathcal{R} \neq 0$. And, furthermore, there was no need to introduce a complex integration contour to *avoid* the singularity as done in [6]. On the contrary, we found that the source of the black hole entropy stems entirely from the scalar curvature *singularity* at the origin $r = 0$. The physical implications of this finding warrants further investigation since it suggests a profound connection between the notion of gravitational entropy and spacetime singularities.

A plausible explanation why the interior region of the black hole inside the horizon played no role in our derivations is because there is a procedure to remove the interior. In [8] we argued how the action of *active* diffeomorphisms (diffs) $r \to f(r)$ on the Schwarzschild metric leads to metrics which are also static spherically symmetric solutions of the Einstein vacuum field equations. It was shown how in a limiting case it allows to introduce a deformation of the manifold such that $f(r = 0) = 0$, and $f(r = 0_+) = 2GM$ corresponding, respectively, to the spacelike singularity and horizon of the Schwarzschild metric. In doing so, one ends up with a spherical *void* surrounding the singularity at $r = 0$. In order to explore the "interior" region of this void, we introduced complex radial coordinates whose imaginary components have a direct link to the inverse Hawking temperature, and which furnish a path that provides access to interior region. The black hole entropy admitted a geometrical interpretation as the area of a rectangular strip in the complex radial-coordinate plane associated to this path.

Note that this proposal was very different form the one undertaken by some authors who have proposed in the past to "avoid", "remove", the black hole interior via an antipodal identification of the points on a sphere of radius 2GM, and associated with the topology of a RP^3 projective space [9]. Closely related to the aforementioned picture of a spherical *void* is that 't Hooft has referred to the empty region in the "interior" of a black hole as a vacuole [10]. He proposed a procedure for a better understanding of the evolution laws of black holes in terms of pure quantum states. In his most recent work [19], 't Hooft came up with a simpler proposal such that all the states defined in the external region II are exact quantum clones of those in the other external region I. He argued that not only does this make all physical states observable, but, most surprisingly, this suffices to restore unitarity as well.

It remains to explore if the procedure proposed in this work also works for the Reissner-Nordstrom and the more general Kerr-Newman metric solutions. In the latter case, when $M \neq 0, a = \frac{J}{M} \neq 0$, one has a true ring singularity of radius a located in the $z = 0$ plane, so one would expect a scalar curvature delta function singularity with support on the ring [12]. The clue how to proceed is to recall how Newman and Janis [13] showed that the Kerr metric could be obtained from the Schwarzschild metric by means of a coordinate transformation and allowing the radial coordinate to take on *complex* values. The Newman-Janis shift was $r \to r - ia$. Originally, no clear reason for why the algorithm works was known and many physicists considered it to be an ad hoc procedure or a "fluke" not worthy of further investigation until Drake and Szekeres [14] gave a detailed explanation of the success of the algorithm and proved the uniqueness of certain solutions. In particular, the Kerr–Newman metric associated to a charged-rotating black hole can be obtained from the Reissner-Nordstrom metric by means of a coordinate transformation and allowing the radial coordinate to take on *complex* values. Therefore, all we have to do is to modify the Newman-Janis shift by writing $|r - ia|$. In this way one would generate $\delta(r - ia)$ terms corresponding to a ring singularity of imaginary radius $r = ia$.

Acknowledgments

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