A DARK MATTER THEORY BY GRAVITATION FOR GALAXIES AND CLUSTERS-V2

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1. ABSTRACT

This paper develops carefully an original theory of dark matter, whose main hypothesis is that DM is generated by the own gravitational field. This work introduces the best version of the theory physical and mathematically, which has been developed and published since 2013.

The hypothesis of DM by gravitation has two main consequences: the first one is that the law of DM generation has to be the same, in the halo region, for all the galaxies and the second one is that the haloes are unlimited so the total DM goes up without limit. Both properties are crucial for the success of this theory.

This work has two newness: it is demonstrate that Direct mass comes from Bernoulli mass formula when it is considered an initial point belonging to the Buckingham halo curve, (ideal curve related to dynamical equilibrium), to calculate the parameter C. The second one is two astonishing calculus of masses in LG. This work begins studying the Rotation curves, from M31 and MW, published by Sofue,Y.2015 and 2020 . The regression curves at halo region are fitted with power regression functions whose exponents are the same for both galaxies.

By the fitted function is possible to calculate a dark matter density function depending on radius which is transformed into a DM density depending on gravitational field. This change is the core of the theory because at such moment it is possible to study the formula of dark matter density by the Buckingham theorem in order to change the statistical calculus by physical formulas which depend on the Universal constants G, h and c.

From now on the statistical theory becomes a perfect physical theory that despite it is based on the Newtonian framework allows to get new formulas for DM density and for total mass, including DM.In chapters 11, 12 and 13 are calculated the parameters $\mathbf{a^2}$ for MW, M33 and LCM galaxies. In chapter 14 is calculated the mass of the Local Group, which match with results got by prestigious researchers. In chapter 15 is got the solution of Poisson equation. In chapter 16, is shown a method to extend the theory to cluster of galaxies and chapter 17 is dedicated to show how DM is counter balanced by dark energy with negative mass.

2. INTRODUCTION

Since 2013 up to 2019 I have published several papers studying DM in galactic halos, especially in M31 and Milky Way although also I have published some papers studying DM in Coma cluster [3] Abarca,M.2019.

As reader knows M31 is the twin galaxy of Milky Way in the Local Group of galaxies. According [5] Sofue, Y. 2015. Its baryonic masses are $M_{BARYONIC-M31} = 1,61\cdot10^{11}\ M_{SUN}$ and $M_{BARYONIC-MILKY\ WAY} = 1,4\cdot10^{11}\ M_{SUN}$. Where Msun represents the mass of Sun. Msun = 1.99E+30 kg that will be used frequently throughout the paper.

The DM by gravitation theory was introduced in [1] Abarca, M.2014. *Dark matter model by quantum vacuum* . It considers that DM is generated by the own gravitational field. In order to study purely the phenomenon it is needed to consider a radius dominion where it is supposed that baryonic matter is negligible. i.e. radius bigger than 30 kpc for MW and 40 kpc for M31, as it will be shown in chapter 6.

This hypothesis has two main consequences: the first one is that the law of dark matter generation, in the halo region, has to be the same for all the galaxies and the second one is that the haloes are unlimited so the total dark matter goes up without limit.

In the chapter 9, is demonstrated mathematically that total mass by Direct mass formula goes up proportionally to the root square of distance, so this property may explain how the ratio of dark matter versus baryonic matter at cluster scale is bigger than such ratio at galactic scale.

By other side the growing of the total mass is so slow that Dark energy phenomenon may counter balance the DM when it is considered radius measured in mega parsecs. Precisely, this fact may explain the size of galactic clusters. This issue will be studied in chapter 16.

The first consequence before mentioned, dark matter generated by a Universal law, has been studied by all my papers, especially inside M31 and Milky Way thanks the remarkable data of rotation curves published in papers [5] Sofue, Y.2015 and [6] Sofue, Y.2020.

In fact I could develop rigorously the theory because the rotation curve of M31 at halo region decreased with a power regression fitted curve whose exponent is -1/4. However with data published for Milky Way at the same paper (2015) it was not possible to fit rigorously the rotation curve with such exponent.

Fortunately, in a new paper [6] Sofue,Y.2020, the author gives a new rotation curve data for Milky Way at halo region whose fitted curve has an exponent -1/4. Such result is good news for DM by gravitation theory, because the theory states a universal law of DM generation in the halo region of galaxies or clusters.

In this paper it is firstly developed all the theory carefully with M31 rotation curve data up to chapter 10 and the chapter 11 is dedicated to apply the theory to Milky Way with magnificent results.

In the chapters 12 and 13 is estimated the total mass for M33 and the Large Magellanic Cloud by the Direct mass.

The chapter 14 is dedicated to estimate the total mass of Local Group. The total mass calculated for MW, LMC, M31 and M33 is $4,97*10^{12}$ Msun. The mass at 770 kpc for MW and M31 is accepted to be $5*10^{12}$ Msun. See [18] Azadeh Fattahi, Julio F. Navarro.2020. So it is possible to state that both results match perfectly.

In addition it is estimated the total mass for LG at 3 MPc being equal to 9,8E+12 which match perfectly with 1E+13 Msun, the estimation published by the well known researchers in [17] Azadeh Fattahi, Julio F. Navarro.2020.The importance of these findings is high because there is not any other theory able to explain such amount of mass offering a physic nature of dark matter. The chapter 15 shows two important results: The divergence of Bernoulli field verify the Gauss law if it is changed baryonic density by Bernoulli density in addition is got the solution of Poisson equation using the direct D.M. density. The chapter 16 introduces a method to extend the Direct mass formula to clusters, knowing its virial mass and radius. Finally in chapter 17 are developed some calculus showing how the Dark Energy is able to counter balance the DM.

As I have mentioned before, this theory has been developed assuming the hypothesis that DM is a quantum gravitational effect. However, it is possible to remain into the Newtonian framework to develop the theory. In my opinion there are two factors to manage the DM conundrum with a quite simple theory.
The first one, that it is developed into the halo region, where baryonic matter is negligible. The second one, that the mechanics movements of celestial bodies are very slow regarding velocity of light, which is supposed to be the speed of gravitational bosons. It is known that community of physics is researching a quantum gravitation theory since many years ago, but does not exist yet, however I think that my works in this area support strongly that DM is a quantum gravitation phenomenon.

Use a more simple theory instead the general theory is a typical procedure in physics.
For example the Kirchhoff ´s laws are the consequence of Maxwell theory for direct current and remain valid for alternating current, introducing complex impedances, on condition that signals must have low frequency. However these laws do not work for electromagnetic microwaves because of its high frequency.

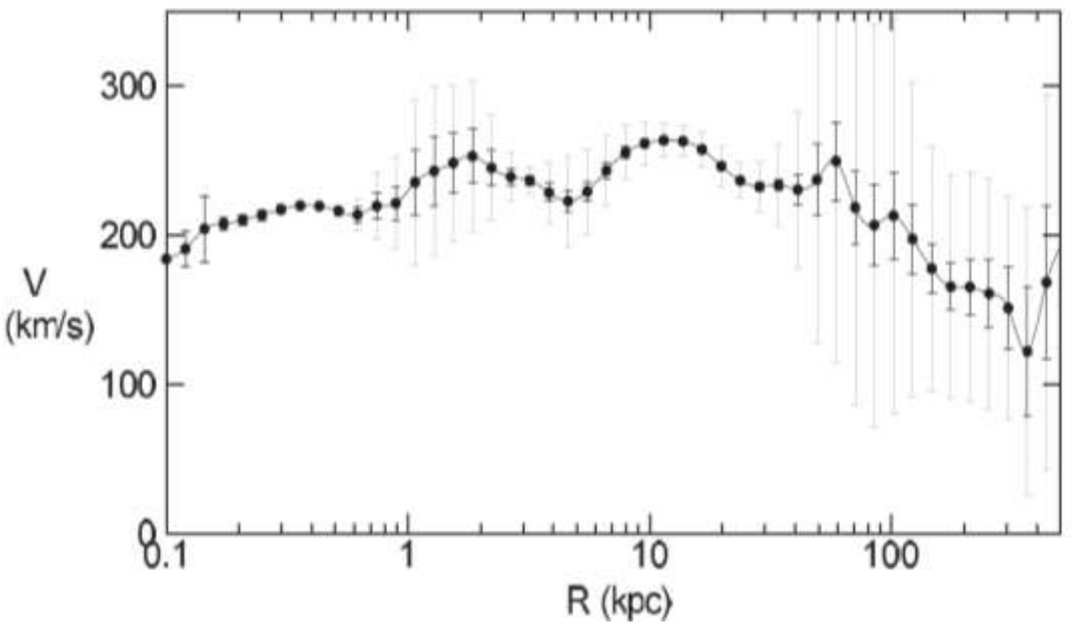
Thanks the possibility to study the gravitational effect of DM pure, in halo regions of M31 and MW, it have been possible to develop a theory mathematically simple. When baryonic mass is mixed with dark matter as it happens inside the galactic disc the mathematical treatment is by far more complex.

Taking into account that the only ones giant galaxies quite close to be able to study with accuracy the rotation curve at halo region are Milky Way and M31, the coincidence of the same exponent to the fitted function for the rotation curves for both galaxies is crucial in order to state that dark matter is generated according an Universal law.

3. OBSERVATIONAL DATA FOR M31 GALAXY FROM SOFUE. 2015 DATA

Graphic come from [5] Sofue,Y. 2015. The axis for radius has logarithmic scale. Although Sofue rotation curve ranges from 0,1 kpc up to 352 kpc the range of dominion considered for this work is only the halo region where ratio baryonic matter is negligible. In chapter 6, will be shown that this happens for radius bigger than 40 kpc, despite the fact that disc radius for M31 is accepted to be 35 kpc.

kpc	km/s
40,5	229,9
49,1	237,4
58,4	250,5
70,1	219,2
84,2	206,9
101,1	213,5
121,4	197,8
145,7	178,8
175	165,6
210,1	165,6
252,3	160,7
302,9	150,8



The measure at 352 kpc has been rejected because has a velocity too high, so does not match with the other measures. May be an celestial object captivated by the gravitational field of M31 afterwards to M31 formation and it is right to think that it is not in dynamical equilibrium with M31.So it is a good criteria to consider the Virial mass associated to M31 as the dynamical mass up to 302,9 kpc.

3.1 POWER REGRESSION TO ROTATION CURVE

The measures of rotation curve have a very good fitted curve by power regression.

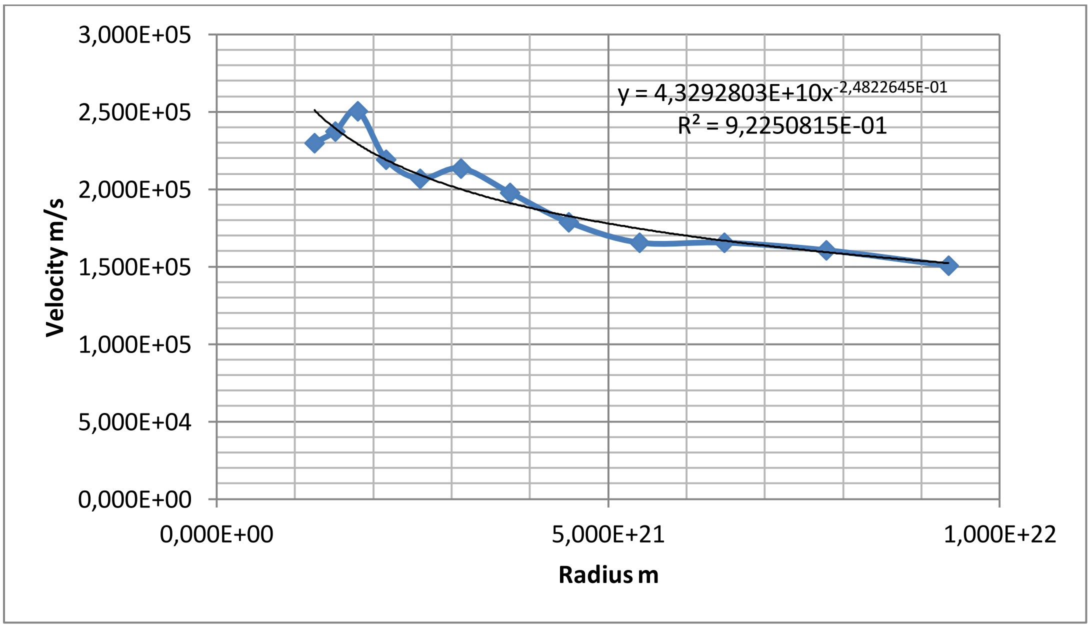
Power regression for M31 rot. curve	
V=a*r^b	
a	$4{,}32928 \cdot 10^{10}$
b	-0.24822645
Correlation coeff.	0,96

In particular coefficients of $v = a \cdot r^{b}$ are in table below. Units are into I.S.

Radius kpc	Vel. km/s	Radius m	Vel. m/s	Vel. fitted	Relative Diff.
40,5	229,9	1,250E+21	2,299E+05	2,510E+05	8,397E-02
49,1	237,4	1,515E+21	2,374E+05	2,393E+05	7,777E-03
58,4	250,5	1,802E+21	2,505E+05	2,292E+05	-9,304E-02
70,1	219,2	2,163E+21	2,192E+05	2,190E+05	-8,154E-04
84,2	206,9	2,598E+21	2,069E+05	2,093E+05	1,138E-02
101,1	213,5	3,120E+21	2,135E+05	2,000E+05	-6,755E-02
121,4	197,8	3,746E+21	1,978E+05	1,911E+05	-3,500E-02
145,7	178,8	4,496E+21	1,788E+05	1,826E+05	2,107E-02
175	165,6	5,400E+21	1,656E+05	1,745E+05	5,115E-02
210,1	165,6	6,483E+21	1,656E+05	1,668E+05	7,100E-03
252,3	160,7	7,785E+21	1,607E+05	1,594E+05	-8,307E-03
302,9	150,8	9,347E+21	1,508E+05	1,523E+05	9,891E-03

Data fitted are in grey columns below. In fifth column is shown results of fitted velocity and sixth column shows relative difference between measures and fitted results.

Below is shown a graphic with measures data and power regression function.



Correlation coefficient equal to 0,96 which is a superb result especially when dominion measures is up to 303 kpc. There is not any other galaxy to measure a rotation curve so magnificent. According theory of DM generated by field, galaxy haloes are unlimited although up to a half of distance i.e. 375 kpc toward Milky Way direction is dominated by M31 field whereas the other half distance is dominated by Milky Way.

4. DIRECT FORMULA FOR DM DENSITY ON M31 HALO GOT FROM ROTATION CURVE

4.1 GETTING DIRECT DM DENSITY FROM NEWTONIAN DYNAMICS

Outside disk region, rotation curve it is fitted by power regression with a high correlation coefficient according formula $v = a \cdot r^b$. As $M_{DYNAMIC}(< r) = \dfrac{v^2 \cdot R}{G}$ represents total mass enclosed by a sphere with radius r, by substitution of velocity results $M = \dfrac{v^2 \cdot R}{G} = \dfrac{a^2 \cdot r^{2b+1}}{G}$. Hereafter this formula will be called Direct Mass $M_{DIRECT}(< r) = \dfrac{a^2 \cdot r^{2b+1}}{G}$ because it has been got rightly from rotation curve.

If it is considered outside region of disk where baryonic matter is negligible regarding dark matter it is possible to calculate DM density by a simple derivative. In next chapter will be show that for r > 40 kpc baryonic matter is negligible.

As density of D.M. is $D_{DM} = \dfrac{dm}{dV}$ where $dm = \dfrac{a^2 \cdot (2b+1) \cdot r^{2b} dr}{G}$ and $dV = 4\pi r^2 dr$ results

$$D_{DM} = \frac{a^2 \cdot (2b+1)}{4\pi G} \cdot r^{2b-2}$$

Writing $L = \dfrac{a^2 \cdot (2b+1)}{4\pi G}$ results $D_{DM}(r) = L \cdot r^{2b-2}$. In case b = -1/2 DM density is cero which is Keplerian rotation.

4.2 DIRECT DM DENSITY FOR M31 HALO

Parameters a & b from power regression of M31 rotation curve allow calculate easily direct DM density

Direct DM density for M31 halo 40 < r < 300 kpc
$D_{DM}(r) = L \cdot r^{2b-2}$ kg/m^3 being L= 1.1255E+30 and 2b-2= -2.4964529

It is important to highlight that at this moment this formula is only a statistical approximation of DM density able to explain the rotation curve, without any physic meaning.

5. DARK MATTER DENSITY AS POWER OF GRAVITATIONAL FIELD

As independent variable for this function is E, gravitational field, previously will be studied formula for E in the following paragraph.

5.1 GRAVITATIONAL FIELD E BY VIRIAL THEOREM

As it is known total gravitational field may be calculated through Virial theorem, formula $E = v^2/R$ whose I.S. unit is m/s^2 is well known. Hereafter, Virial gravitational field, E, got through this formula will be called E.

The key to state the Virial theorem is the dynamical equilibrium. It is supposed that celestial bodies are quite close to dynamical equilibrium, because the most of celestial bodies belong to a specific galactic system from its formation times.

By substitution of $v = a \cdot r^b$ in formula $E = \frac{v^2}{r}$ it is right to get $E = \frac{a^2 \cdot r^{2b}}{r} = a^2 \cdot r^{2b-1}$ briefly $E = a^2 \cdot r^{2b-1}$

5.2 DARK MATTER DENSITY AS POWER OF GRAVITATIONAL FIELD

According hypothesis dark matter by quantum vacuum $D_{DM} = A \cdot E^B$.Where A & B are parameters to be calculated. This hypothesis has been widely studied by the author in previous papers. [1] Abarca,M. [2] Abarca,M. [8] Abarca,M. y [10] Abarca,M. This hypothesis fulfils the physic meaning of D.M. Density formula in the halo region because it is supposed that such D.M. density is generated as a consequence of gravitational field propagation in the framework of a quantum gravitation theory.

As it is known direct DM density $D_{DM} = \frac{a^2 \cdot (2b+1)}{4\pi G} \cdot r^{2b-2}$ depend on a & b parameters which come from power regression formula for velocity. In previous paragraph has been shown formula for gravitational field $E = \frac{a^2 \cdot r^{2b}}{r} = a^2 \cdot r^{2b-1}$ which depend on a & b as well. Through a simple mathematical treatment it is possible to get A & B to find function of DM density depending on E. Specifically formulas are $A = \frac{a^{\frac{2}{2b-1}} \cdot (2b+1)}{4\pi G}$ & $B = \frac{2b-2}{2b-1}$.

M31 galaxy	$D_{DM} = A \cdot E^B$
A	$3{,}6559956 \cdot 10^{-6}$
B	1,6682469

According parameters a & b got in previous chapter, A & B parameters are written beside.

Conversely $b = \frac{B-2}{2B-2}$ and $a = \left[\frac{4\pi G A(B-1)}{2B-3}\right]^{\frac{2b-1}{2}}$ being $B \neq 1$ *and* $B \neq 3/2$.

As conclusion, in this chapter has been demonstrated that a power law for velocity $v = a \cdot r^b$ *is mathematically equivalent to a power law for DM density depending on E.* $D_{DM} = A \cdot E^B$ *if it is considered as physic hypothesis that D.M. is generated by the gravitational field.*

5.3 THE IMPORTANCE OF $D_{DM} = A . E^B$

This formula is vital for theory of dark matter generated by gravitational field because it is supposed that DM is generated locally according an unknown quantum gravity mechanism. In other words, the propagation of gravitational field has this additional effect on the space as the gravitational wave goes by.

The formulas $D_{DM} = \frac{a^2 \cdot (2b+1)}{4\pi G} \cdot r^{2b-2}$ and $E = a^2 \cdot r^{2b-1}$ have been got rightly from rotation curve. Therefore it can be considered more specific for each galaxy. However the formula $D_{DM} = A \cdot E^B$ is much more essential.

The basis of this theory is that such formula is right for different gravitational systems. Therefore A & B parameters have to be similar for different galaxies on condition that the galaxies are similar. In further chapters will be got that power B is exactly the same for M31 and Milky Way although parameter A will be a bit different.

However, there is an important fact to highlight. It is clear that A depend on a and b, both parameters are global parameters. As the gravitational interaction time between masses is proportional to distance, it is right to think that DM generated by a gravitational field has a bigger proportion as the system increase its size. For example inside the Solar system it is clear that Newton and General Relativity Theory is able to explain with total accuracy every gravitational phenomenon without DM hypothesis. Therefore it is right to conclude that DM arises when gravitational interaction takes a longer time to link the matter. Namely, at galaxy scale or cluster of galaxies.

6. RATIO BARYONIC MASS VERSUS DARK MATTER MASS DEPENDING ON RADIUS FOR M31

In this paragraph will be estimated radius which is needed to consider negligible baryonic density regarding DM density in M31 galaxy. [5] According Sofue, Y. data for M31 disk are

M31 Galaxy	Baryonic Mass at disk	a_d	Σ_0
	$M_d = 2\pi \cdot \Sigma_0 \cdot a^2{}_d$		
	$M_d = 1{,}26 \cdot 10^{11}$ Msun	5,28 kpc	1,5 kg/m^2

Where $\Sigma(r) = \Sigma_0 \exp(-r/a_d)$ represents superficial density at disk. Total mass disk is given by integration of superficial density from cero to infinite. $M_d = \int_0^\infty 2\pi \cdot r \Sigma(r) \cdot dr = 2\pi \cdot \Sigma_0 \cdot a^2{}_d$

To convert superficial baryonic density to volume density it is right to get the formula $D^{VOLUME}_{BARYONIC} = \dfrac{\Sigma(r)}{2r}$ so $D^{VOLUME}_{BARYONIC}(40kpc) = 3.1 \cdot 10^{-25}\ kg/m^3$.

The formula of Direct Dark matter density is got afterwards, in page 17. $D_{DM}(r) = L \cdot r^{\frac{-5}{2}}$ being L = 1,33E+30. For example D_{DM} (40 kpc) = 2,5E-23 kg/m^3 . So the ratio of both volume density is 0,0124 .In conclusion it is right to consider negligible the baryonic density at 40 kpc, therefore it is possible to estate that halo dominion begins at 40 kpc in M31.

7. A BERNOULLI DIFFERENTIAL EQUATION FOR THE GRAVITATIONAL FIELD

7.1 INTRODUCTION

This formula $D_{DM} = \dfrac{a^2 \cdot (2b+1)}{4\pi G} \cdot r^{2b-2}$ is a local formula because it has been got by differentiation. However E, which represents a local magnitude $E = \dfrac{G \cdot M(<r)}{r^2} = \dfrac{a^2 \cdot r^{2b}}{r} = a^2 \cdot r^{2b-1}$ has been got through $v = a \cdot r^b$ whose parameters a & b were got by a regression process on the whole dominion of rotation speed curve. Therefore, D_{DM} formula has a character more local than E formula because the former was got by a differentiation process whereas the later involves M(<r) which is the mass enclosed by the sphere of radius r.

In other words, the process of getting D_{DM} involves a derivative whereas the process to get E(r) involves M(r) which is a global magnitude. This is a not suitable situation because the formula $D_{DM} = A \cdot E^B$ involves two local magnitudes. Therefore it is needed to develop a new process with a more local nature or character.

It is clear that a differential equation for E is the best method to study locally such magnitude.

7.2 A BERNOULLI DIFFERENTIAL EQUATION FOR GRAVITATIONAL FIELD IN THE GALACTIC HALO

As it is known in this formula $\vec{E} = -G\frac{M(r)}{r^2}\hat{r}$, M(r) represents mass enclosed by a sphere with radius r. If it is considered a region where does not exit any baryonic matter, such as any galactic halo, then the derivative of M(r) depend on dark matter density essentially and therefore $M'(r) = 4\pi r^2 \varphi_{DM}(r)$.

If $E = G\frac{M(r)}{r^2}$, vector modulus, is differentiated then it is got $E'(r) = G\frac{M'(r)\cdot r^2 - 2rM(r)}{r^4}$

If $M'(r) = 4\pi r^2 \varphi_{DM}(r)$ is replaced above then it is got $E'(r) = 4\pi G\varphi_{DM}(r) - 2G\frac{M(r)}{r^3}$ As $\varphi_{DM}(r) = A\cdot E^B(r)$ it is right to get $E'(r) = 4\pi\cdot G\cdot A\cdot E^B(r) - 2\frac{E(r)}{r}$ which is a Bernoulli differential equation.

$E'(r) = K\cdot E^B(r) - 2\frac{E(r)}{r}$ being $K = 4\pi\cdot G\cdot A$ Calling y to E, the differential equation is written in this simple way $y` = K\cdot y^B - \frac{2\cdot y}{r}$ Bernoulli family equations $y` = K\cdot y^B - \frac{2\cdot y}{r}$ may be converted into a differential linear equation with this variable change $u = y^{1-B}$. This is $\frac{u`}{1-B} + \frac{2u}{r} = K$

The homogenous equation is $\frac{u`}{1-B} + \frac{2u}{r} = 0$ whose general solution is $u = C\cdot r^{2B-2}$ being C the integration constant.
If it is searched a particular solution for the complete differential equation with a simple linear function u = z*r then it is got that $z = \frac{K\cdot(1-B)}{3-2B}$. Therefore the general solution for u- equation is $u = C\cdot r^{2B-2} + z\cdot r$
When it is inverted the variable change it is got the general solution for field E.

General solution is $E(r) = \left(Cr^{2B-2} + \frac{K(1-B)\cdot r}{3-2B}\right)^{\frac{1}{1-B}}$ $with\ B \neq 1\ and\ B \neq 3/2$ where C is the parameter of initial condition of gravitational field at a specific radius.

Calling $\alpha = 2B-2$ $\beta = \frac{1}{1-B}$ and parameter $D = \left(\frac{K(1-B)}{3-2B}\right)$ then $E(r) = \left(Cr^{\alpha} + Dr\right)^{\beta}$

Calculus of parameter C through initial conditions Ro and Eo

Suppose R_0 and E_0 are the specific initial conditions for radius and gravitational field, then $C = \frac{E_0^{1/\beta} - D\cdot R_0}{R_0^{\alpha}}$

Final comment

It is clear that the Bernoulli solution contains implicitly the fact that it is supposed there is not any baryonic matter inside the radius dominion and the only DM matter is added by $\varphi_{DM}(r) = A\cdot E^B(r)$. Therefore this solution for field works only in the halo region and R_0 *and* E_0 could be the border radius of galactic disk where it is supposed begins the halo region and the baryonic density is negligible, also it is possible to select another point belonging to halo.

8. DIMENSIONAL ANALYSIS FOR D.M. DENSITY AS POWER OF E FORMULA

8.1 POWER OF E BY BUCKINGHAM THEOREM

As it is supposed that DM density as power of E come from a quantum gravity theory, it is right to think that constant Plank h should be considered and universal constant of gravitation G as well.

So the elements for dimensional analysis are D, density of DM whose units are Kg/m^3, E gravitational field whose units are m/s^2, G and finally h.

In table below are developed dimensional expression for these four elements D, E, G and h.

	G	h	E	D
M	-1	1	0	1
L	3	2	1	-3
T	-2	-1	-2	0

According Buckingham theorem it is got the following formula for Density

$D = \frac{K}{\sqrt[7]{G^9 \cdot h^2}} \cdot E^{\frac{10}{7}}$ Being K a dimensionless number which may be understood as a coupling constant between field E and DM density.

As it is shown in previous epigraph, parameters for M31 is B = 1, 6682469

In this case relative difference between B = 1, 6682469 and 10/7 is 16, 7 %. A 17% of error in cosmology could be acceptable. However by the end of the chapter it will be found the right value for B.

8.2 POWER E FORMULA FOR DM DENSITY WITH TWO PI MONOMIALS

As this formula come from a quantum gravitation theory it is right to consider that Universal constants involved are G, h and c. So elements to make dimensional analysis are D, E, G, h and c =2.99792458$\cdot 10^8$ m/s.

	G	h	E	D	c
M	-1	1	0	1	0
L	3	2	1	-3	1
T	-2	-1	-2	0	-1

According Buckingham theorem, as matrix rank is three, there are two pi monomials. The first one was calculated in previous paragraph and the second one involves G, h, E and c.

These are both pi monomials $\pi_1 = D \cdot \sqrt[7]{G^9 \cdot h^2} \cdot E^{-\frac{10}{7}}$ and $\pi_2 = \frac{c}{\sqrt[7]{G \cdot h}} E^{-\frac{2}{7}}$. So formula for DM density through two pi monomials will be $D = \frac{J}{\sqrt[7]{G^9 \cdot h^2}} \cdot E^{\frac{10}{7}} \cdot f(\pi_2)$ being J a dimensionless number and $f(\pi_2)$ an unknown function, which cannot be calculated by dimensional analysis theory.

8.3 MATHEMATICAL ANALYSIS TO DISCARD FORMULA WITH ONLY ONE PI MONOMIAL

As it was shown in paragraph **5.2** $A = \frac{a^{\frac{2}{2b-1}} \cdot (2b+1)}{4\pi G}$ and $B = \frac{2b-2}{2b-1}$. Being a, b parameters got to fit rotation curve of velocities $v = a \cdot r^b$

Conversely, it is right to clear up parameters a and b from above formulas.

Therefore $b = \frac{B-2}{2B-2}$ and $a = \left[\frac{4\pi GA(B-1)}{2B-3}\right]^{\frac{2b-1}{2}}$ being $B \neq 1$ *and* $B \neq 3/2$.

As A is a positive quantity then $2b+1>0$. As $2b+1 = \frac{2B-3}{B-1} > 0$ therefore $B \in (-\infty, 1) \cup (3/2, \infty)$.

If B=3/2 then 2b+1=0 and A=0 so dark matter density is cero which is Keplerian rotation curve.

In every galactic rotation curve studied, B parameter has been bigger than 3/2. See Abarca papers quoted in Bibliographic references. Therefore experimental data got in several galaxies fit perfectly with mathematical findings made in this paragraph, namely for $B \in (3/2, \infty)$.

The main consequence this mathematical analysis is that formula $D = \frac{K}{\sqrt[7]{G^9 \cdot h^2}} \cdot E^{\frac{10}{7}}$ got with only one pi monomial is wrong because B=10/7 < 3/2. Therefore formula $D = \frac{J}{\sqrt[7]{G^9 \cdot h^2}} \cdot E^{\frac{10}{7}} \cdot f(\pi_2)$, being $\pi_2 = \frac{c}{\sqrt[7]{G \cdot h}} E^{-\frac{2}{7}}$, got by two pi monomials it is the right physic formula of D.M. density depending on gravitational field E.

This formula is physically more acceptable because it is got considering G, h and c as universal constant involved in formula of density. As the hypothesis of the theory estates that DM is generated through a quantum gravitation mechanism it is right to consider not only **G** and **h** but also **c** as well.

8.4 LOOKING FOR A D.M. DENSITY FUNCTION COHERENT WITH DIMENSIONAL ANALYSIS

It is right to think that $f(\pi_2)$ should be a power of π_2, because it is supposed that density of D.M. is a power of E.

M31 galaxy	$D_{DM} = A \cdot E^B$
A	$3{,}6559956 \cdot 10^{-6}$
B	1,6682469

Taking in consideration A &B parameters on the left, power for π_2 must be -5/6. This way, power of E in formula $D_{DM} = A \cdot E^B$ will be 5/3, which is the best approximation to B= 1.6682469.

Finally $D = \frac{J}{\sqrt[7]{G^9 \cdot h^2}} \cdot E^{\frac{10}{7}} \cdot f(\pi_2)$ becomes $D = \frac{M}{\sqrt[6]{G^7 \cdot c^5 \cdot h \cdot}} E^{\frac{5}{3}}$ being M a dimensionless number.

9. RECALCULATING FORMULAS IN M31 HALO WITH B = 5/3

Findings got through Buckingham theorem are crucial. It is clear that a physic formula has to be dimensionally coherent .Therefore it is a magnificent support to the theory of DM generated by gravitational field that statistical value got by regression analysis in M31, differs less than 2 thousandth regarding value got by Buckingham theorem.

Now it is needed to rewrite all the formulas considering B=5/3. Furthermore, with B= 5/3, a lot of parameters of the theory become simple fraction numbers. In other words, theory gains simplicity and credibility.

In chapter 5 was shown the relation between a & b parameters and A&B parameters. Now considering B= 5/3

as $A = \frac{a^{\frac{2}{2b-1}} \cdot (2b+1)}{4\pi G}$ & $B = \frac{2b-2}{2b-1}$. It is right to get $b = \frac{B-2}{2B-2} = -\frac{1}{4}$ and $A = \frac{a^{\frac{-4}{3}}}{8\pi G}$

Therefore, the central formula of theory becomes $D_{DM} = A \cdot E^{\frac{5}{3}} = \frac{a^{\frac{-4}{3}}}{8 \cdot \pi \cdot G} \cdot E^{\frac{5}{3}}$. In chapter 11 will be studied the rotation curve of Milky Way according the data published by Sofue in 2020, and it will be shown that in the halo region the rotation curve decreases with the same exponent b = -1/4. This fact is crucial for DM by gravitation theory because both giant galaxies are the only ones with rotation curves measures in halo region.

9.1 RECALCULATING THE PARAMETER a IN M31 HALO

Table below comes from chapter 3 and represents regression curve of velocity depending on radius.

Regression for M31 dominion 40-303 kpc	
$V=a*r^b$	
a	$4{,}32928*10^{10}$
b	-0.24822645
Correlation coeff.	0,96

Due to Buckingham theorem it is needed that b= -1/4. Therefore it is needed to recalculate parameter **a** in order to find a new couple of values a &b that fit perfectly to experimental measures of rotation curve in M31 halo.

RECALCULATING a WITH MINUMUN SQUARE METHOD

When it is searched the parameter a, a method widely used is called the minimum squared method. So it is searched a new parameter **a** for the formula $V= a*r^{-0.25}$ on condition that $\sum_e (v - v_e)^2$ has a minimum value. Where v represents the value fitted for velocity formula and v_e represents each measure of velocity. It is right to calculate the formula for a.

New parameters a &b and A&B	
B	5/3
$b = \frac{B-2}{2B-2}$	b = -1/4
a new	$4{,}727513*10^{10}$
A new	$3.488152*10^{-6}$

Where r_e represents each radius measure and V_e represents its velocity associated. See table page 5 data in columns in grey.

$$a = \frac{\sum_e Ve \cdot r_e^{-0.25}}{\sum_e r_e^{-0.5}} = 4.727513 \cdot 10^{10} \frac{m^{5/4}}{s}$$

9.2 FORMULAS OF DIRECT D.M. FOR DENSITY FOR MASS AND FIELD

With these new parameters recalculated it is going to get the direct formulas got at the beginning of paper.

Function of Density DM depending on radius.

$D_{DM}(r) = L \cdot r^{2b-2} = L \cdot r^{\frac{-5}{2}}$ being $L = \frac{a^2 \cdot (2b+1)}{4\pi G} = \frac{a^2}{8 \cdot \pi \cdot G}$ =1,3326*10 ^{30} $\frac{Kg}{m^{1/2}}$

Function of E depending on radius $E = a^2 \cdot r^{2b-1} = a^2 \cdot r^{\frac{-3}{2}}$ being a^2 = 2,235*10 ^{21} $\frac{m^{5/2}}{s^2}$

Mass enclosed by a sphere of radius r, known as dynamical mass because it is calculated with velocity.

$M_{DYN}(<r) = \frac{v^2 \cdot R}{G}$ When velocity is replaced by its fitted function it is got $M_{DIRECT}(<r) = \frac{a^2 \cdot r^{2b+1}}{G} = \frac{a^2 \cdot \sqrt{r}}{G}$

9.3 BERNOULLI SOLUTION FOR E AND DENSITY IN M31 HALO

In chapter 7 was got the solution for field in the halo region, now thanks dimensional analysis it is possible to get the formulas because some parameters and algebraic expression, now are simple fractions. Namely:

$E(r) = \left(Cr^{\alpha} + Dr\right)^{\beta}$ being $\alpha = 2B - 2 = \frac{4}{3}$ and $\beta = \frac{1}{1-B} = \frac{-3}{2}$ By other side, the initial condition C

$C = \frac{E_0^{1/\beta} - D \cdot R_0}{R_0^{\alpha}}$ becomes $C = \frac{E_0^{\frac{-2}{3}} - D \cdot R_0}{R_0^{\frac{4}{3}}}$ and $D = \left(\frac{4 \cdot \pi \cdot G \cdot A(1-B)}{3-2B}\right) = 8 \cdot \pi \cdot G \cdot A$ As $A = \frac{a^{\frac{-4}{3}}}{8\pi G}$ (see chap. 9 at the beginning) then $D = a^{\frac{-4}{3}}$ = 5,85*10^{-15}

Therefore $E(r) = \left(Cr^{\frac{4}{3}} + Dr\right)^{\frac{-3}{2}}$ being C the initial condition of differential equation solution for E and $D = a^{\frac{-4}{3}}$ is a parameter closely related to the global rotation curve at halo region, being parameter **a** =4,7275*10^{10}

AND THE BERNOULLI SOLUTION FOR DENSITY IN HALO REGION

$$D_{BERNI} = A \cdot E^{\frac{5}{3}} = A\left(Cr^{\frac{4}{3}} + Dr\right)^{\frac{-5}{2}} = \frac{D}{8\pi G}\left(Cr^{\frac{4}{3}} + Dr\right)^{\frac{-5}{2}}$$

9.4 DARK MATTER AT A SPHERICAL CORONA BY BERNOULLI SOLUTION IN HALO REGION

Formula below express the dark matter contained inside a spherical corona defined by R_1 and R_2 belonging at halo.

$$M_{BERNI} = \int_{R1}^{R2} 4\pi \cdot r^2 \cdot D_{BERNI} dr = \int_{R1}^{R2} 4\pi \cdot \mathrm{r}^2 A E^B dr = 4\pi A \int_{R1}^{R2} \mathrm{r}^2 \left[C \cdot r^{4/3} + D \cdot r\right]^{\frac{-5}{2}} \cdot dr$$

The indefinite integral $I = 4\pi A \cdot \int \frac{r^2}{\left(C \cdot r^{4/3} + D \cdot r\right)^{2.5}} = \frac{8\pi A \sqrt{r}}{D \cdot \left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}} = \frac{\sqrt{r}}{G \cdot \left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}}$

As $\dfrac{8\pi A}{D} = \dfrac{1}{G}$. Calling $M_{BER}(r) = \dfrac{\sqrt{r}}{G \cdot \left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}}$ and **by** the Barrow`s rule, it is got

$M_{BERNI}\Big|_{R1}^{R2}$ = M(R2)-M(R1) that provided the DM contained inside the spherical corona defined by R2 and R1.

9.5 NEWTON`S THEOREM WITH BERNOULLI MASS FORMULA

The name for this theorem has been chosen because the relation between field E and total mass M(<r) is the same that in Newton's theory.

From Bernoulli field $E(r) = \left(Cr^{\frac{4}{3}} + Dr\right)^{\frac{-3}{2}} = \dfrac{1}{r^{3/2} \cdot \left(C \cdot r^{1/3} + D\right)^{3/2}}$

Bernoulli mass formula $M_{BERNI}(r) = \dfrac{\sqrt{r}}{G \cdot \left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}}$ so $G \cdot M_{BER}(r) = \dfrac{\sqrt{r}}{\left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}}$ and

$$\frac{G \cdot M_{BER}(r)}{r^2} = \frac{\sqrt{r}}{r^2\left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}} = \frac{1}{r^{3/2} \cdot \left(C \cdot r^{1/3} + D\right)^{3/2}} = E(r)$$

Therefore $E(r) = \dfrac{G \cdot M_{BER}(r)}{r^2}$ this is the intensity of field in Newton´s theory. This identity shows how the ***DM by gravitation theory,*** adding an extra of mass depending on radius, being the halo region unlimited, is able to explain the DM measures in galaxies and cluster in the Newtonian framework.

COROLLARY

According the Newtonian framework, the function of mass included in the formula of field E, means the total mass included inside the sphere with radius r. Therefore M_{BER} (R) must be renamed as $M_{BER}(< r) = M_{TOTAL}(< r)$ where r ranges in the halo region. In conclusion, the total mass enclosed by radius r is given by the formula:

$M_{TOTAL}(< r) = \dfrac{\sqrt{r}}{G \cdot \left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}}$ where r belong to halo region. i.e. r > 40 kpc for M31 galaxy.

9.6 CALCULUS OF PARAMETER C

$$C_O = \frac{E_0^{\frac{-2}{3}} - D \cdot R_0}{R_0^{\frac{4}{3}}}$$

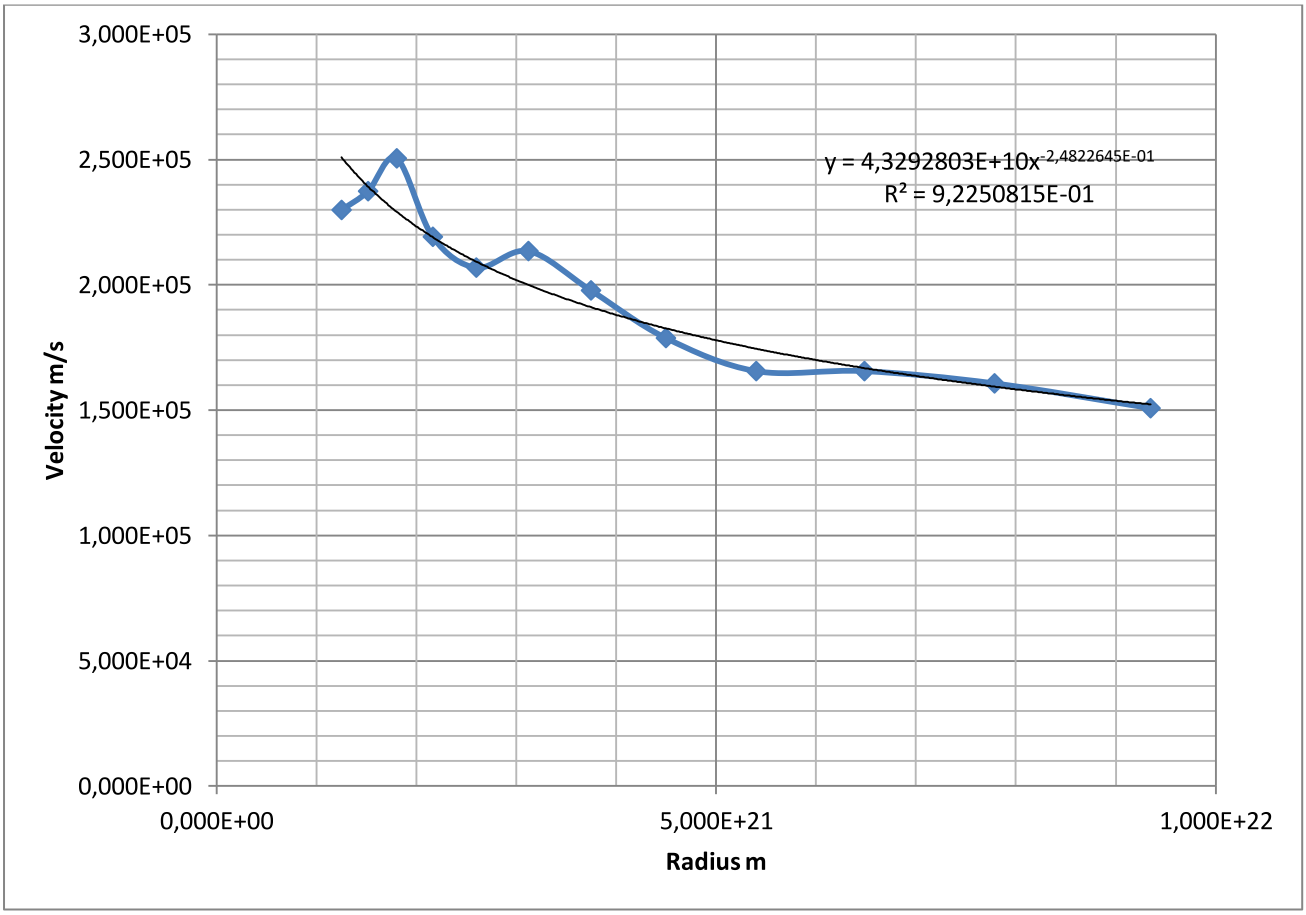
This parameter is calculated by a data belonging to rotation curve in halo region, (R_0, V_0). E_0 is the gravitational field at R_O radius. Considering that data measures are in dynamical equilibrium, it is possible to estimate $E_O = V_0^2 / R_0$ In the following epigraph it will see studied more in deep the hypothesis about the dynamical equilibrium of data.

It is only an approximation because the celestial bodies are not in perfect dynamical equilibrium, but data selected in the halo region are quite close to the dynamical equilibrium. In addition experimental measures have errors. By these two reasons, it will be calculated Co for every data in order to calculate Bernoulli mass formula for all of them.

Points	Radius	Radius	Velocity	Field Eo	Eo^(-2/3)	Eo^(-2/3)-D*R_0	Parameters C_0
	kpc	m	m/s	m/s^2			
1	40,5	1,250E+21	2,299E+05	4,23E-11	8239256,95	9,27E+05	6,8830E-23
2	49,1	1,515E+21	2,374E+05	3,72E-11	8973615,749	1,11E+05	6,3753E-24
3	58,4	1,802E+21	2,505E+05	3,48E-11	9377625,482	-1,16E+06	-5,3082E-23
4	70,1	2,163E+21	2,192E+05	2,22E-11	12654654,13	1,20E+03	4,2804E-26
5	84,2	2,598E+21	2,069E+05	1,65E-11	15443489,73	2,45E+05	6,8683E-24
6	101,1	3,120E+21	2,135E+05	1,46E-11	16732951,68	-1,52E+06	-3,3317E-23
7	121,4	3,746E+21	1,978E+05	1,04E-11	20928758,45	-9,85E+05	-1,6934E-23
8	145,7	4,496E+21	1,788E+05	7,11E-12	27043372,43	7,42E+05	9,9988E-24
9	175	5,400E+21	1,656E+05	5,08E-12	33846627,1	2,26E+06	2,3822E-23
10	210,1	6,483E+21	1,656E+05	4,23E-12	38232870,38	3,08E+05	2,5446E-24
11	252,3	7,785E+21	1,607E+05	3,32E-12	44959131,06	-5,83E+05	-3,7771E-24
12	302,9	9,347E+21	1,508E+05	2,43E-12	55281485,72	6,02E+05	3,0572E-24

Values in yellow are negatives because these points are above the fitted curve . See graph below. In addition, the more close to curve the point is, the smaller, in absolute value, the parameter C is. The cyan value is the smaller.

Afterwards will be shown that this fact explain that data measures are close to dynamical equilibrium. For example points 1 and 3 are quite far away from dynamical equilibrium despite to be placed in halo region.



The close oscillation around the fitted curve suggests strongly that this curve might be the ideal curve of perfect ideal dynamical equilibrium for the celestial bodies belonging to M31 galaxy.

In the following epigraph it will be demonstrated that if it is considered as initial condition a point belonging to the ideal fitted curve at halo region, this point will have parameter C = 0. In addition in the epigraph 9.8 it will be shown that Bernoulli mass formula becomes direct mass if parameter C = 0.

9. 7 PARAMETER C EQUAL ZERO THEOREMS

Definition. Hereafter, it will be named Buckingham halo curve to the points **(r, v)** **r** belonging to the halo region and the velocity $v = a \cdot r^{-1/4}$ being **a** the parameter associated to galactic halo. It is an ideal curve because its points are in perfect dynamical equilibrium.
As the exponent -1/4 was got by the Buckingham theorem, it has been select such name for that curve.

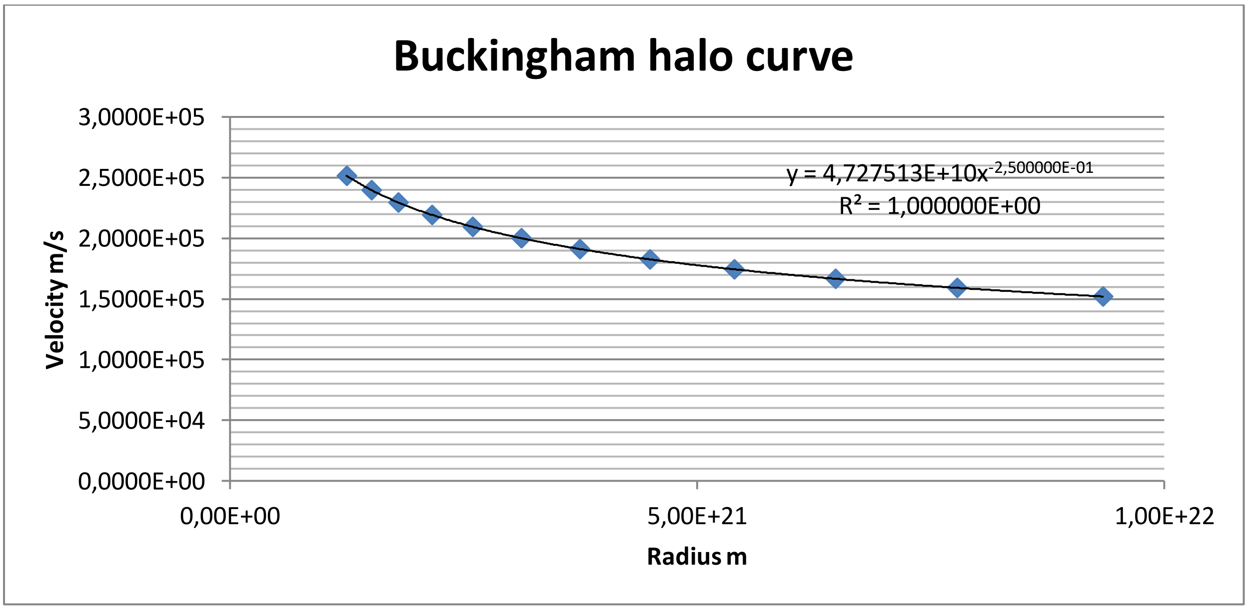
Direct Theorem: If it is supposed that a point belonging to Buckingham halo curve is in dynamical equilibrium and if it is selected such point as initial point to calculate C, then such parameter is zero.
Proof : Suppose a point (Ro, Vo) belonging to Buckingham halo curve, then $V_0 = a \cdot R_0^{-1/4}$ As dynamical equilibrium leads to $E_0 = \frac{GM(<r)}{r^2} = \frac{V_0^2}{r}$ then $E_0 = a^2 \cdot R_0^{-3/2}$ and $E_0^{\frac{-2}{3}} = a^{-4/3} \cdot R_0 = D \cdot R_0$ because $D = a^{\frac{-4}{3}}$ when B=5/3 as it was shown at epigraph 9.3. Therefore C= 0 because its numerator is zero. $C = \frac{E_0^{\frac{-2}{3}} - D \cdot R_0}{R_0^{\frac{4}{3}}}$

It is important to highlight that such points are in perfect dynamical equilibrium, whereas the data measures are **close** to dynamical equilibrium. In addition, the real gravitational field never has a perfect spherical symmetry. See in table below the relative difference between data and Buckingham points.

Reverse Theorem
If it is selected a point (Ro,Vo) which is supposed to be in dynamical equilibrium and its parameter C=0 then such point belong to Buckingham halo curve. Suppose that $V_0 = a \cdot R_0^b$ being exponent b unknown.

Proof: If C = 0 then $E_0^{-2/3} = D \cdot R_0$ and as there is dynamical equilibrium $E_0 = \frac{V_0^2}{r}$ then $E = \frac{a^2 \cdot r^{2b}}{r} = a^2 \cdot r^{2b-1}$

and $E_0^{-2/3} = a^{-4/3} \cdot R_0^{\frac{2-4b}{3}}$ so $a^{-4/3} \cdot R_0^{\frac{2-4b}{3}} = D \cdot R_0$ or $D \cdot R_0^{\frac{2-4b}{3}} = D \cdot R_0$ which leads to $R_0 = R_0^{\frac{2-4b}{3}}$ so b= - 1/4



Radius kpc	Radius m	Buckingham Velocity m/s	Data velocity m/s	Relative Diff. %
40,5	1,25E+21	2,5144E+05	2,2990E+05	8,57E+00
49,1	1,52E+21	2,3962E+05	2,3740E+05	9,27E-01
58,4	1,80E+21	2,2945E+05	2,5050E+05	-9,17E+00
70,1	2,16E+21	2,1921E+05	2,1920E+05	5,58E-03
84,2	2,60E+21	2,0939E+05	2,0690E+05	1,19E+00
101,1	3,12E+21	2,0004E+05	2,1350E+05	-6,73E+00
121,4	3,75E+21	1,9109E+05	1,9780E+05	-3,51E+00
145,7	4,50E+21	1,8257E+05	1,7880E+05	2,06E+00
175	5,40E+21	1,7440E+05	1,6560E+05	5,04E+00
210,1	6,48E+21	1,6660E+05	1,6560E+05	6,03E-01
252,3	7,79E+21	1,5915E+05	1,6070E+05	-9,72E-01
302,9	9,35E+21	1,5204E+05	1,5080E+05	8,18E-01

The third column got with the formula $V_0 = a \cdot R_0^{-1/4}$ is called Buckingham velocity because such formula is calculated by the Buckingham theorem. See the graph above.

Parameter a_{M31} = 4,727513E+10 $m^{5/4}$ /s

The fourth column are the measures of velocity.

Data at radius 302,9 kpc has the lowest relative difference. Although data radius 252, 3 kpc has almost the same relative difference but negative.

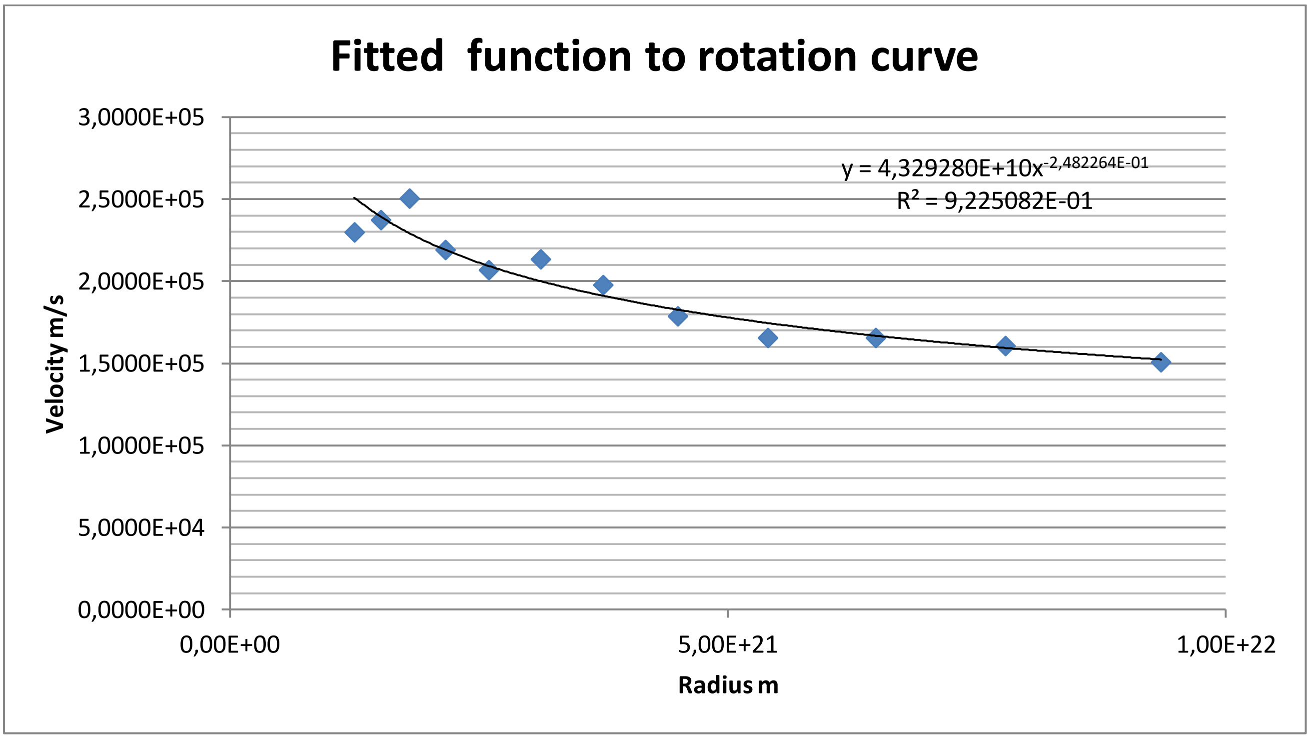
Final comments

Data measures do not belong to Buckingham halo curve by two reasons:

The first one it is simple: measures have experimental errors. The second one is more subtle, the celestial bodies are not in perfect dynamical equilibrium. It is right to think that celestial bodies which belong to M31 gravitational system from its formation times, more than ten billions years ago, will be closer to dynamical equilibrium regarding other ones that were captivated by the gravitational field of M31 afterwards.

Watching the graph, it is clear that point 1 and point 3 at 40,5 kpc and 58,4 kpc are the points more distant regarding Buckingham halo curve. This important difference regarding dynamical equilibrium curve may be explained by the asymmetries of gravitational field during the history of dynamic evolution.

Anyway it is undeniable that in general data are very close to dynamical equilibrium, just as Sofue data measures are magnificent.



As it is shown in the graph the exponent of fitted function differs 18 thousands regarding -0.25

9.8 BERNOULLI FORMULAS BECOME DIRECT FORMULAS WHEN PARAMETER C = 0

Thanks demonstration made above, it is clear why data measures close to Buckingham halo curve give values for C very close to zero. The more close point measure to Buckingham halo curve is, the more close to zero parameter C is. Now parameter C will be zero, because at C = 0 are got the formulas with initial point belonging to Buckingham halo curve. i. e. a initial point which is in perfect dynamical equilibrium.

FOR FIELD E

When in formula $E(r)=\left(Cr^{\frac{4}{3}}+Dr\right)^{\frac{-3}{2}}$ C= 0 then it is got $E=a^2\cdot r^{\frac{-3}{2}}$ because $D=a^{\frac{-4}{3}}$, being $a^2 = 2.235*10^{21}$

which is precisely direct formula for E. Which is the field calculated by perfect dynamical equilibrium.

FOR D.M. DENSITY

As $D_{DM}= A*E^{5/3}$ Using field got by Bernoulli solution it is right to get

$D_{DM}(r)=A\left(Cr^{\frac{4}{3}}+Dr\right)^{\frac{-5}{2}}$ Being $A=\frac{D}{8\pi G}$ and $D=a^{\frac{-4}{3}}$ if **C** = 0 then formula becomes

$D_{DM}(r)=A\cdot D^{\frac{-5}{2}}\cdot r^{\frac{-5}{2}}=L\cdot r^{\frac{-5}{2}}$ being $L=\frac{a^2}{8\cdot\pi\cdot G}$ = $1.3326*10^{30}$ which is direct DM density formula.

FOR DIRECT MASS FORMULA

If C=0 then $M_{BERNI}(<r)=\frac{\sqrt{r}}{G\cdot\left(C\cdot\sqrt[3]{r}+D\right)^{\frac{3}{2}}}$ becomes $M_{DIRECT}(<r)=\frac{a^2\cdot\sqrt{r}}{G}$ being $\frac{a^2}{G}=3.349\cdot 10^{31}$

Final comment

As it is clear Direct formulas are a particular case of Bernoulli formulas when parameter C =0.

At the beginning of this paper, the direct formulas were only right up to 300 kpc, and they were statistical formulas to calculate the extra additional amount of matter needed to explain the rotation curves. However, thanks to findings got in this chapter, Direct formulas are the Bernoulli formulas when it is considered as initial point a theoretical point of rotation curve in halo region, which are in perfect dynamical equilibrium, therefore according DM theory by gravitation its dominion is unbounded. Also direct formulas depend on parameter **a** solely, instead two parameters C and D associated to Bernoulli formulas, which is a magnificent simplification of the theory.

In the last chapter will be shown how the dark energy is able to counter balance dark matter at cluster scale, so the total mass of dark matter generated by a galaxy or cluster of galaxies do not diverge at infinitum.

Conclusion Hereafter it will be use Direct formulas with unbounded dominion instead Bernoulli formulas.

10. MASSES IN M31

In this chapter, It will be calculated and compared three different types of masses related to M31.

10.1 DYNAMICAL MASS VERSUS DIRECT MASS

As it is known, dynamical mass represents the total mass enclosed by a sphere with a radius r in order to produce a balanced rotation with a specific velocity at such radius, so it is right to consider dynamical mass as the total mass, baryonic and DM mass, enclosed at radius R. Ranging radius in the interval of radius measured.

The formula of dynamical mass is $M_{DYN}(<r) = \frac{V^2 \cdot r}{G}$. and $M_{DIRECT}(<r) = \frac{a^2 \cdot \sqrt{r}}{G}$ being $a^2/G = 3{,}35.10^{31}$

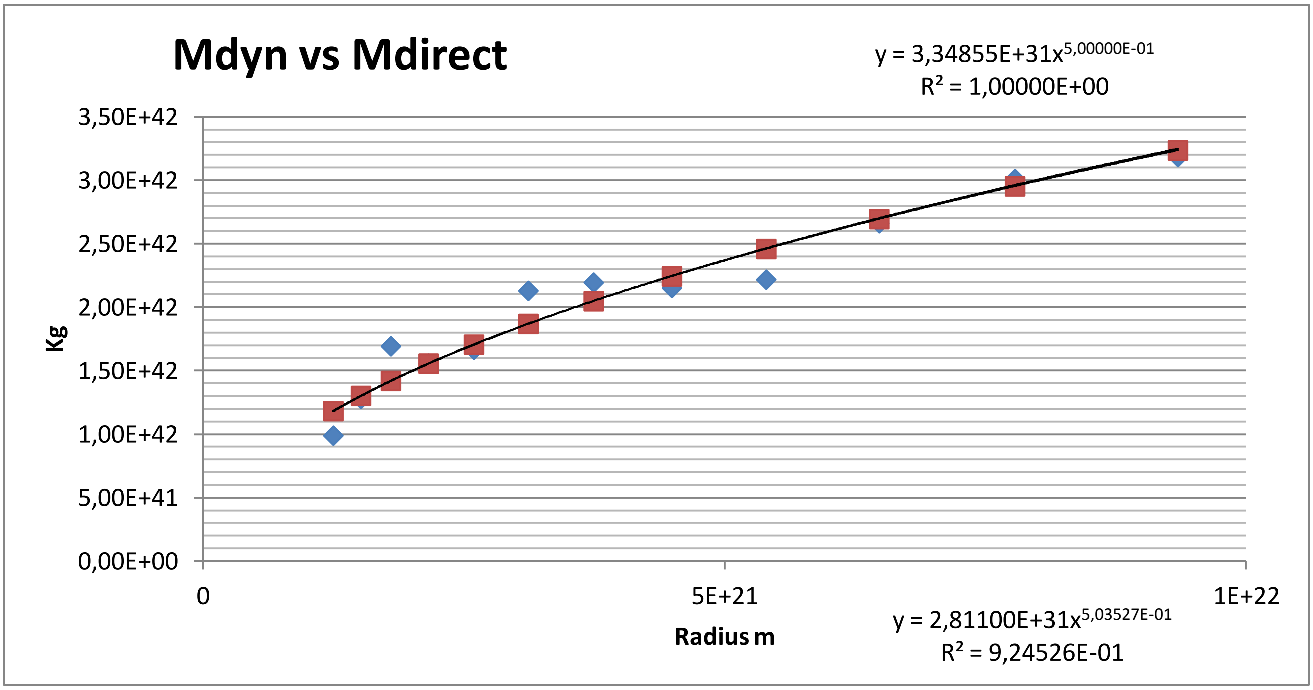
			Dyn Mass	Direct mass	Rel diff
kpc	m	m/s	Msun	Msun	%
40,5	1,250E+21	2,299E+05	4,974E+11	5,95E+11	1,639E+01
49,1	1,515E+21	2,374E+05	6,429E+11	6,55E+11	1,849E+00
58,4	1,802E+21	2,505E+05	8,514E+11	7,14E+11	-1,919E+01
70,1	2,163E+21	2,192E+05	7,825E+11	7,83E+11	1,419E-02
84,2	2,598E+21	2,069E+05	8,373E+11	8,58E+11	2,373E+00
101,1	3,120E+21	2,135E+05	1,071E+12	9,40E+11	-1,392E+01
121,4	3,746E+21	1,978E+05	1,103E+12	1,03E+12	-7,143E+00
145,7	4,496E+21	1,788E+05	1,082E+12	1,13E+12	4,087E+00
175	5,400E+21	1,656E+05	1,115E+12	1,24E+12	9,833E+00
210,1	6,483E+21	1,656E+05	1,339E+12	1,35E+12	1,204E+00
252,3	7,785E+21	1,607E+05	1,514E+12	1,48E+12	-1,951E+00
302,9	9,347E+21	1,508E+05	1,600E+12	1,63E+12	1,629E+00

In the fifth column is tabulated the direct masses in order to be compared with dynamical masses.

Below in the graph are plotted both functions, blue points are dynamical masses and brown point are direct masses.

The first and third points have the maximum difference regarding fitted curve whereas relative differences decreased as radius increased. Namely, relative differences are below 10% for radius bigger than 120 kpc and are below 2 % for radius bigger than 210 kpc. So direct mass is a very good approximation for total mass (baryonic and DM) enclosed at radius R. Ranging radius in the interval of radius measured.

As it was shown in chapter 3, the data set was selected up to 303 kpc because the following data was too different to value associated to Buckingham halo curve, which means that such data is not in dynamical equilibrium.
In conclusion 303 kpc may be considerate as the Virial radius of M31.



10.2 BERNOULLI MASS VERSUS DIRECT MASS

In this epigraph will be shown that relative difference between both kinds of formulas is negligible.

Below are both function formulas.

$M_{DIRECT}(<r) = \dfrac{a^2 \cdot \sqrt{r}}{G}$ being $a^2/G = 3{,}35.10^{31}$, as was pointed in previous paragraph, will be used to approximate total mass at radius R.

In the corollary of Newton´s theorem, epigraph 9.5, was demonstrated that the Bernoulli mass is the total mass enclosed by a sphere with radius r. Now it will be shown that relative differences between Bernoulli and direct mass are quite small, even for extended haloes.

In the paper [20] Abarca,M.2023, epigraph 9.6 it is developed a method to calculate the optimal parameter C, taking into account that every data have errors not only as a result of measures but celestial bodies are not in perfect dynamical equilibrium, so such value for parameter C is the average associated to different data inside the M31 halo.

M31 PARAMETERS C & D	This selected value C_{M31} belong to the initial point radius equal to 252,3 kpc. See epigraph 9.6
$C_{M31} = -3{,}777 * 10^{-24}$	
$D_{M31} = a^{-\frac{4}{3}} = 5{,}85 * 10^{-15}$	

$$M(<r) = \frac{\sqrt{r}}{G \cdot \left(C \cdot \sqrt[3]{r} + D\right)^{\frac{3}{2}}}$$

Below are tabulated both function and its relative difference. It is remarkable that even at 3 Mpc its difference is only 4,35 %, despite the fact that its dominion has been extended 10 times.

		Direct mass C=0	Bernoulli mass $C \neq 0$	Rel diff
kpc	m	Msun	Msun	%
40,5	1,250E+21	5,949E+11	6,011E+11	1,04E+00
60	1,851E+21	7,240E+11	7,327E+11	1,19E+00

80	2,469E+21	8,361E+11	8,471E+11	1,31E+00
100	3,086E+21	9,347E+11	9,481E+11	1,41E+00
200	6,171E+21	1,322E+12	1,346E+12	1,77E+00
385	1,188E+22	1,834E+12	1,875E+12	2,20E+00
500	1,543E+22	2,090E+12	2,142E+12	2,40E+00
770	2,376E+22	2,594E+12	2,668E+12	2,77E+00
1000	3,086E+22	2,956E+12	3,048E+12	3,02E+00
1500	4,629E+22	3,620E+12	3,750E+12	3,46E+00
2000	6,171E+22	4,180E+12	4,346E+12	3,80E+00
3000	9,257E+22	5,120E+12	5,353E+12	4,35E+00

Bernoulli mass have been got with the DM by gravitation theory so its dominion is the unlimited haloes.

It is obvious that direct mass is easier to calculate than Bernoulli mass because it has only one parameter. Hereafter it will be used Direct mass instead Bernoulli mass.

11. DARK MATTER BY GRAVITATION THEORY IN MILKY WAY

In the new rotation curve published by [6] Sofue.2020, the radius of data range from 0,1 kpc up to 95,5 kpc whereas in the previous rotation curve [5] Sofue.2015 the radius range up to 300 kpc. Afterwards will be discussed the importance to reduce the dominion up to 95 kpc. However firstly it is needed to calculate the lowest radius for the halo region, where the baryonic density is negligible versus DM density.

11.1 AN ESTIMATION FOR THE HALO RADIUS

As it is known, Bernoulli solution is only right into the halo region, where the baryonic mass is negligible.
To calculate baryonic volume density has been used model provided by Sofue for baryonic disc.

This Table comes from Sofue [6],see table 3, page 12. Parameters for baryonic matter at disc in Milky Way.

Component	Parameter	Fitted Value	χ^2
Expo. disk	a_d	4.38 ± 0.35 kpc	
	Σ_0	$(1.28 \pm 0.09) \times 10^3 M_\odot \mathrm{pc}^{-2}$	

Where $\Sigma(r) = \Sigma_0 \exp(-r/a_d)$ represents superficial density at disc region. To convert superficial baryonic density to volume density it is right to get the formula $D^{VOLUME}_{BARYONIC} = \dfrac{\Sigma(r)}{2r}$ so $D^{VOLUME}_{BARYONIC}(30{,}5kpc) = 1.34 \cdot 10^{-24}\ kg/m^3$.

The formula of Direct Dark matter density was got in page 17. $D_{DM}(r) = L \cdot r^{\frac{-5}{2}}$ being $L_{MILKY\text{-}WAY}$ = 9,1E+29, according with parameter **a** got in epigraph 11.3. For example D_{DM} (30,5 kpc) = 3,35E-23 kg/m^3 . So the ratio of both volume density at 30,5 kpc is 0,04 .In conclusion it is right to consider negligible the baryonic density for radius bigger than 30,5 kpc, therefore it is possible to estate that halo dominion begins at 30,5 kpc for Milky Way. According with DM by gravitation theory, the halo region is unbounded as it is the gravitational field. This property will be discussed newly in chapter 13, where is calculated the total mass of Local Group of galaxies.

11. 2 ROTATION CURVE OF MILKY WAY BY SOFUE 2020 DATA

kpc	km/s	radius m	vel m/s
30,448	229,6	9,40E+20	229600
33,493	222,5	1,03E+21	222500
36,842	215	1,14E+21	215000
40,527	207,1	1,25E+21	207100
44,579	200,3	1,38E+21	200300
49,037	194,7	1,51E+21	194700
53,941	189,8	1,66E+21	189800
59,335	186,2	1,83E+21	186200
65,268	184,7	2,01E+21	184700
71,795	183,9	2,22E+21	183900
78,975	181,4	2,44E+21	181400
86,872	175,5	2,68E+21	175500
95,56	167,7	2,95E+21	167700

This table of rotation curve of Milky Way comes from [6] Sofue.2020, and there have been selected data with radius bigger than 30 kpc.

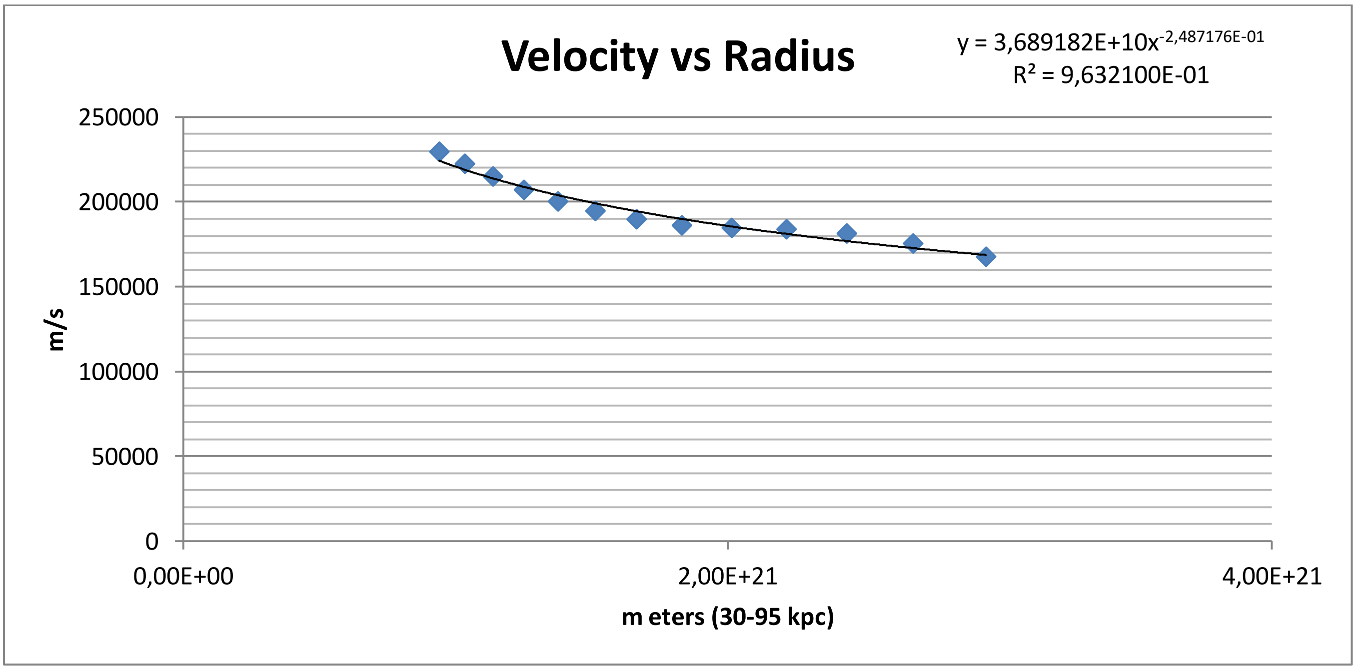
This new set of Sofue data is very important for the theory of DM by gravitation theory because gives a rotation curve at halo region with a power for radius very close to -1/4 which is the same for M31. This fact backs strongly the hypothesis of this theory.

In the previous paper [5] Sofue, Y.2015, the author gave an extended dominion up to 300 kpc, However data with radius bigger than 100 kpc have too high velocity and fitted power function did not fit properly with exponent -1/4.

The logical explanation about the "bad" behaviour of these data is to consider that such celestial bodies are not in dynamical equilibrium. Perhaps they came from the outskirts of MW and were captivated by MW gravitational field afterwards so it is right to consider that these data are far away to dynamical equilibrium, whereas celestial bodies below 100 kpc of radius are properly in dynamical equilibrium with Milky Way.

Anyway, the important data are those closer, because it is right to think that celestial objects with lower radius belong to MW from times of MW formation so these objects may have a better dynamic equilibrium.

11.3 FITTED FUNCTION VELOCITY VERSUS RADIUS AT HALO REGION



According the statistical procedure $v=a*R^b$ Being a = 3.68918E+10 and b = - 0.248717

$$a_{OPTIMAL} = \frac{\sum_e Ve \cdot r_e^{-0.25}}{\sum_e r_e^{-0.5}} = 3.90787373 \cdot 10^{10}$$

Using Buckingham theorem it has been stated b= - 1/4 so it is needed recalculated parameter **a** through the formula as it was made with M31 rotation curve, using the formula for **a** optimal, where Ve is the experimental velocity and r_e is its associated radius.

New parameters a&b - A&B for Milky Way	
B	5/3
$b = \dfrac{B-2}{2B-2}$	b = -1/4
a optimal	3,90787373*10^{10}
$A = \dfrac{a^{\frac{-4}{3}}}{8\pi G}$	New parameter A 4,496262*10^{-6}

A good approximation for parameter $\mathbf{a}_{\mathbf{M\text{-}W}}$ = 3,9E+10

Which is lightly bigger compared with the which one associated to b= - 0.248717

Parameter **a** is similar for similar galaxies, for example $\mathbf{a}_{\mathbf{M31}}$ = 4,7275E+10

Dark matter by gravitation theory stated that B has to be the same for all galaxies. However parameter **a** depend on each galaxy because it depend on baryonic matter enclosed by the galaxy.

Table beside shows the most important parameters for M-W for the halo region.

11.4 MASSES SSOCIATED TO MILKY WAY UP TO 3 MPC

Radius	Radius	Masses
kpc	m	Msun
30,448	9,3953E+20	3,524E+11
33,493	1,033E+21	3,696E+11
36,842	1,137E+21	3,877E+11
40,527	1,251E+21	4,066E+11
44,579	1,376E+21	4,264E+11
49,037	1,513E+21	4,473E+11
53,941	1,664E+21	4,691E+11
59,335	1,831E+21	4,920E+11
65,268	2,014E+21	5,160E+11
71,795	2,215E+21	5,412E+11
78,975	2,437E+21	5,676E+11
86,872	2,681E+21	5,953E+11
95,56	2,949E+21	6,244E+11
770	2,38E+22	1,772E+12
1000	3,09E+22	2,020E+12
2000	6,17E+22	2,856E+12
3000	9,26E+22	3,498E+12

As it has been demonstrated previously, the direct mass is the Bernoulli mass when parameter C is zero.

$$M_{DIRECT}(< r) = \frac{a^2 \cdot \sqrt{r}}{G}$$ being $a^2/G = 2{,}2885.10^{31}$

It is remarkable data of mass at 95,5 kpc equal to 6.2E+11Msun, such quantity may be considerate the virial mass because that radius is the biggest value where data are close to Buckingham halo curve, which represents data in dynamical equilibrium.

11.5 COMPARING DIRECT MASS WITH RESULTS FROM GAIA DR2 PUBLISHED IN JCAP 2020

In this section will be compared result got by direct mass formula with result published in the prestigious Journal of Cosmology and Astroparticle Physics by [19] E.V. Karukes et al. 2020 in the paper *A robust estimate of the Milky Way mass from rotation curve data.*

These results come from GAIA DR2 and others remarkable sources.

In table below is made the comparison only with the four radiuses bigger than 30 kpc, as DM by gravitation theory only works in the halo region.

In the last column is shown the relative difference between both kind of mass, being quite small indeed.

Radius kpc	Radius m	Direct Mass Msun	Karukes et al. Msun	Relative difference %
45,79	1,4129E+21	4,305E+11	4,27E+11	8,23E-01
74	2,2834E+21	5,473E+11	5,68E+11	-3,78E+00
119,57	3,6896E+21	6,957E+11	7,26E+11	-4,35E+00
193,24	5,9628E+21	8,845E+11	8,95E+11	-1,19E+00

It is awesome how a simple theory which associates only one parameter to galactic halo is able to give results so close with results got by GAIA DR2 which have been got with the highest current technology and processed through sophisticated software.

These tables comes from **[**19**]** E.V. Karukes et al. In page 25

Total mass in MW at some radius
E.V. Karukes et al. 2020 in page 25

Radius kpc	Total mass x 10^{11} Msun
45.79	$4.27^{+0.22\,(0.43)}_{-0.19\,(0.37)}$
74.0	$5.68^{+0.40\,(0.83)}_{-0.37\,(0.65)}$
119.57	$7.26^{+0.66\,(1.40)}_{-0.58\,(1.03)}$
193.24	$8.95^{+0.98\,(2.07)}_{-0.84\,(1.48)}$

r [kpc]	$M_{\rm tot}$ [10^{11} M$_\odot$]	$M_{\rm DM}$ [10^{11} M$_\odot$]
2.57	$0.23^{+0.006\,(0.011)}_{-0.005\,(0.011)}$	$0.10^{+0.03\,(0.05)}_{-0.02\,(0.05)}$
4.15	$0.45^{+0.008\,(0.014)}_{-0.007\,(0.012)}$	$0.22^{+0.04\,(0.07)}_{-0.03\,(0.08)}$
6.71	$0.80^{+0.01\,(0.02)}_{-0.01\,(0.03)}$	$0.45^{+0.05\,(0.09)}_{-0.05\,(0.11)}$
10.85	$1.33^{+0.02\,(0.04)}_{-0.03\,(0.05)}$	$0.87^{+0.06\,(0.12)}_{-0.06\,(0.12)}$
17.53	$2.07^{+0.04\,(0.09)}_{-0.04\,(0.08)}$	$1.51^{+0.07\,(0.14)}_{-0.07\,(0.13)}$
28.33	$3.04^{+0.10\,(0.19)}_{-0.08\,(0.17)}$	$2.46^{+0.08\,(0.18)}_{-0.12\,(0.21)}$
45.79	$4.27^{+0.22\,(0.43)}_{-0.19\,(0.37)}$	$3.63^{+0.23\,(0.42)}_{-0.18\,(0.36)}$
74.0	$5.68^{+0.40\,(0.83)}_{-0.37\,(0.65)}$	$5.06^{+0.38\,(0.79)}_{-0.38\,(0.67)}$
119.57	$7.26^{+0.66\,(1.40)}_{-0.58\,(1.03)}$	$6.59^{+0.69\,(1.43)}_{-0.55\,(0.97)}$
193.24	$8.95^{+0.98\,(2.07)}_{-0.84\,(1.48)}$	$8.26^{+1.21\,(2.09)}_{-0.92\,(1.43)}$

It is important to notice that results by the direct mass formula at 216 kpc is calculated through Dark Matter by gravitation theory using a formula which was got with a data set whose dominion ranges between 30 kpc and 100 kpc and there is a perfect concordance if it is considered the interval of errors.

11.6 RESULTS GOT BY JEFF SHEN ET AL.ApJ.2022 VERSUS DIRECT MASS AT MW HALO

In this epigraph will be compared result published in The astrophysical journal 2022. See [21] Jeff Shen, with results calculated by the Direct mass formula in Milky Way.

Below is placed rightly the abstract of the paper where it is possible to see two masses results at different

Abstract

The mass of the Milky Way is a critical quantity that, despite decades of research, remains uncertain within a factor of two. Until recently, most studies have used dynamical tracers in the inner regions of the halo, relying on extrapolations to estimate the mass of the Milky Way. In this paper, we extend the hierarchical Bayesian model applied in Eadie & Juri to study the mass distribution of the Milky Way halo; the new model allows for the use of all available 6D phase-space measurements. We use kinematic data of halo stars out to 142 kpc, obtained from the H3 survey and Gaia EDR3, to infer the mass of the Galaxy. Inference is carried out with the No-U-Turn sampler, a fast and scalable extension of Hamiltonian Monte Carlo. We report a median mass enclosed within 100 kpc of $M(<100\text{ kpc}) = 0.69^{+0.05}_{-0.04} \times 10^{12}\,M_\odot$ (68% Bayesian credible interval), or a virial mass of $M_{200} = M(<216.2^{+7.5}_{-7.5}\text{ kpc}) = 1.08^{+0.12}_{-0.11} \times 10^{12}\,M_\odot$, in good agreement with other recent estimates. We analyze our results using posterior predictive checks and find limitations in the model's ability to describe the data. In particular, we find sensitivity with respect to substructure in the halo, which limits the precision of our mass estimates to ∼15%.

radiuses.Now it will be calculated the total mass at the same radius with direct mass formula $M_{DIRECT}(<r) = \frac{a^2 \cdot \sqrt{r}}{G}$

being $\frac{a^2}{G} = 2.2885 \cdot 10^{31}$, (units in I.S.). Msun =1.99E+30 kg. In the last column is shown the relative difference between both results.

Radius kpc	Total mass x 10^{12} Msun Direct formula	Data total mass x 10^{12} Msun Jeff Shen et al.	Relative difference %
100	0.64	0.69 ± 0.04	7
216 ± 7.5			
216+7.5	0.96	1.08 ± 0.11	11

As it is shown the relative difference is small especially at 100 kpc. In adittion both results match if it is considered the range of errors of measures.

It is important to notice that results by the direct mass formula at 216 kpc is calculated through Dark Matter by gravitation theory using a formula which was got with a data set whose dominion ranges between 30 kpc and 100 kpc and there is a perfect concordance if it is considered the interval of errors.

12. DIRECT MASS FORMULA FOR M33 GALAXY

radius kpc	Vel km/s	Mdyn Msun
8	116,3	2,514E+10
9	118,7	2,947E+10
10	118,7	3,274E+10
11	118,7	3,601E+10
12	119,3	3,969E+10
13	118,7	4,256E+10
14	119,3	4,630E+10
15	119,88	5,009E+10
16	119,88	5,343E+10
17	119,88	5,677E+10
18	119,88	6,011E+10
19	119,88	6,345E+10
20	119,88	6,679E+10
21	119,88	7,013E+10
22	119,3	7,276E+10
23	119	7,568E+10

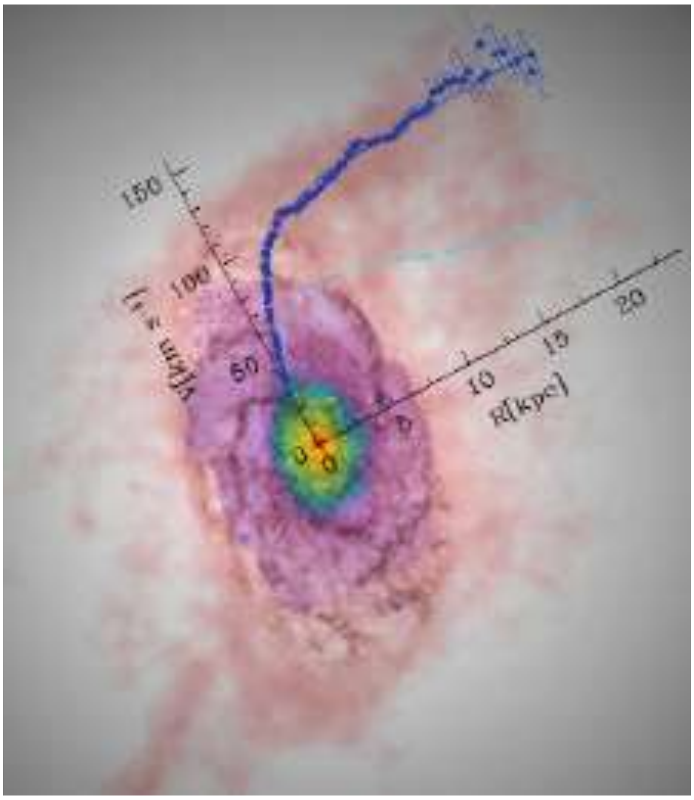
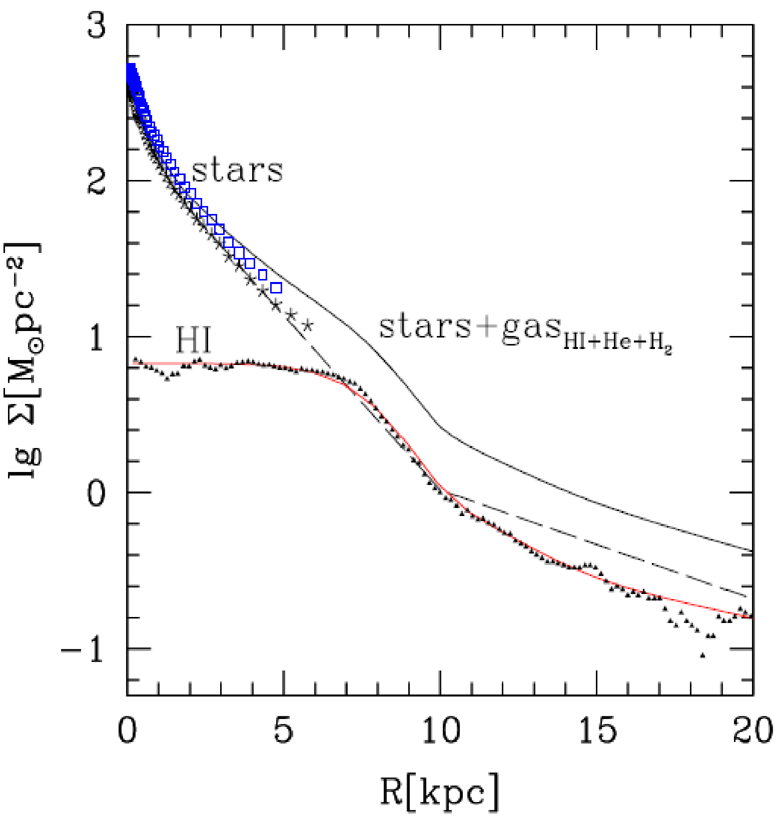


Table and graphic has been taken from [4] Corbelli,E.2014 . The rotation curve from 10 kpc to 23 kpc is quite horizontal. The stellar surface density ranges from red to magenta colours, pink cloud represents the HI gas.



The above Figure and text below has been taken from [4] Corbelli,E.2014 .

According the figure at 20 kpc Superficial density is 0,4 $Msun/pc^2$ and by linear extrapolation it is got superficial density at 23 kpc equal to Dsup = 0,28 $Msun/pc^2$

The formula D_{VOLUME} = Dsup/(2R) is a good approximation to calculate the volume density. So at 23 kpc volume density is aprox. 6,1E-6 $Msun/pc^3$ which is $D_{BARYONIC}$ = 4,2E-25 kg/m^3.

Afterwards will see that such density is 2,3 % regarding DM density at 23 kpc. So it is acceptable to consider negligible the baryonic matter at such distance.

Fig. 10. The HI surface density perpendicular to the galactic plane of M33 (small filled triangles) and the function which fits the data (continuous line, **red in the on-line version**) after the 21-cm line intensity has been deconvolved according to tilted ring model-shape. Asterix symbols indicate the stellar mass surface density using the *BVIgi* stellar surface density map. The dashed line is the fit to the stellar surface density and the extrapolation to larger radii. Open squares (**in blue in the on-line version**) show for comparison the surface density using the BVI mass map. The heavy weighted line is the total baryonic surface density, the sum of atomic and molecular hydrogen, helium and stellar mass surface density.

Text beside from [4] Corbelli,E.2014 explains carefully the figure, although the most important line is the black continuous one. i.e. Total baryonic superficial density.

According a new paper, [7] Carignan.2017 Mass dynamic (< 23 kpc) is 8E+10 Msun, which is a bit large mass than value of table above, but as this paper is more recent it is right to consider such date.

As $M_{DIRECT}(<r) = \dfrac{a^2 \cdot \sqrt{r}}{G}$ taking the Carignan data it is got $\mathbf{a^2}$ = 3,987E20 which is rounded to $\mathbf{a^2}$ = 4E20 (I.S.)

With such parameter direct density DM $D_{DM}(r) = L \cdot r^{\frac{-5}{2}}$ being $L = \dfrac{a^2}{8 \cdot \pi \cdot G}$ =2,385E+29

Then D_{DM}(23 kpc) = 1,78E-23 kg/m^3. So comparing such data with baryonic density, calculated in previous page, it is right to get that baryonic density is 2,3% regarding DM density. Therefore it is right to consider negligible baryonic matter and so to consider acceptable the value for parameter **a**, calculated with data at 23 kpc.

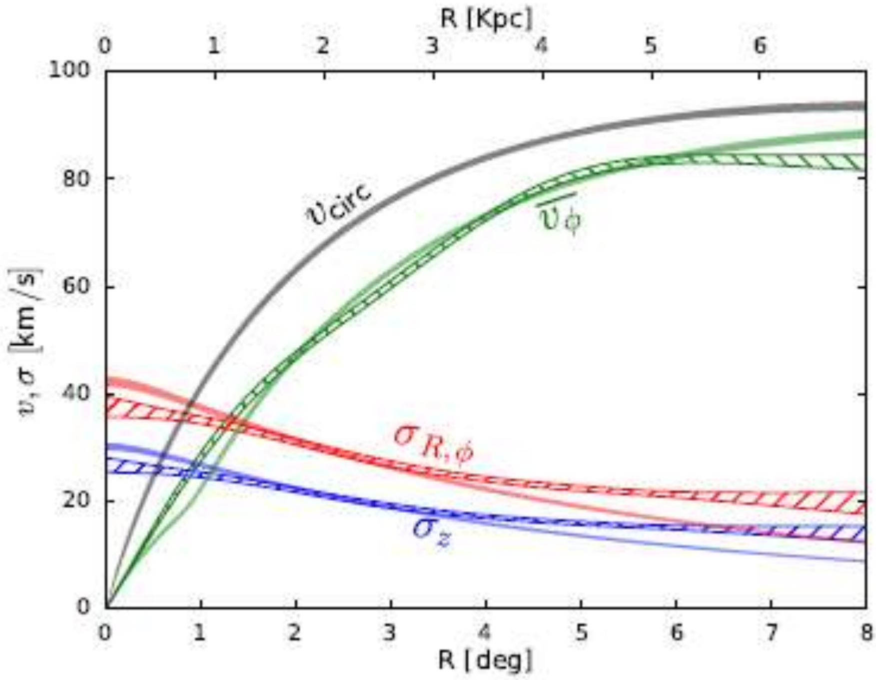
Beside are calculated some masses at different radii, using Direct mass.

In chapter 14 it will be added the direct mass function of M31,MW, LMC and M33, and it will be shown that M33 add 10 % aprox. of total mass associated to the four galaxies, the two giants and its satellite galaxies.

Direct Mass M33 function at some radii		
Radius kpc	radius m	Mass Msun
23,00	7,10E+20	8,025E+10
50,00	1,54E+21	1,183E+11
200,00	6,17E+21	2,366E+11
500,00	1,54E+22	3,742E+11
1000,00	3,09E+22	5,291E+11
1500,00	4,63E+22	6,480E+11
3000,00	9,26E+22	9,165E+11

13. DIRECT MASS FOR THE LARGE MAGELLANIC CLOUD GALAXY

The graph comes from [16] Eugene Vasiliev, and shows the rotation curve of LMC galaxy up to 7 kpc. According the author, the circular velocity reaches approximately 90 km/s at 5 kpc, and it is not lower at 7 kpc.



The upper axe ranges in kpc, whereas the lower axe ranges in degrees. So the LMC has a diametric angular extension in the sky equivalent to 16 degrees at least.

As the reader knows the LMC is a satellite galaxy of MW which is 50 kpc far away from MW.

Figure 5. Internal dynamics of LMC recovered by JAM (shaded) and thin-disc (hatched) models. Red and blue are the radial and vertical velocity dispersions; green is the mean azimuthal velocity; gray is the rotation curve (circular velocity in the $z = 0$ plane) in JAM.

The graph below comes from [15] Knut A.G. Olsen et al. 2011 and according the author, the amplitude of the rotation curve is V = 87±5 km/s beyond a radius R= 2.4 ± 0.1 kpc.

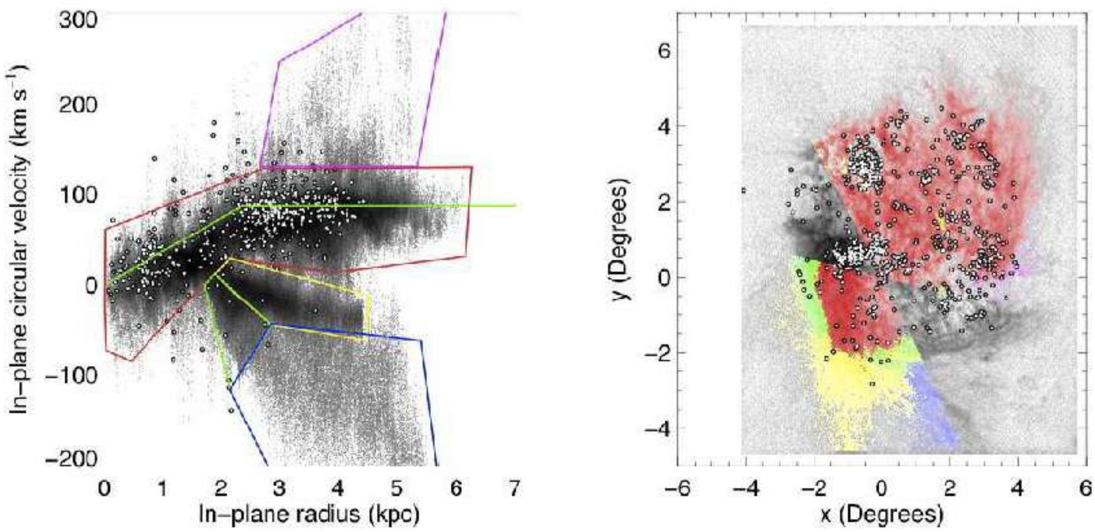


Figure 3. The rotation curve of the LMC as fit to red supergiants. *Left:* The line-of-sight velocities of red supergiants (circles) and of H I gas (grayscale; Kim et al. 1998) have been deprojected to show their in-plane circular velocities, as described in the text. The green line is our parameterized fit of the rotation curve to the RSGs. The colored polygons identify the distinct kinematic components discussed in paper I, most of which were identified by Staveley-Smith et al. (2003): the flat rotation curve (red), arm S (purple), arm E and its E2 extension (yellow and green), and arm B (blue). *Right:* The H I gas and red supergiants are shown as projected on the sky, with color-coding corresponding to the kinematic components identified at left.

As it is shown above the text, the green line is the rotation curve, the white dots are the supergiant stars, which range up to 4 kpc, and the grey scale represents the HI gas which begins to fade at 6 kpc. According this picture it is right to consider negligible the baryonic density at 7 kpc, so it is a good approximation to consider that the Buckingham halo curve begins at 7 kpc, and using its formula, it is right to calculate the parameter **a** by the formula $a = V \cdot R^{1/4}$ considering V = 90 km/s and R = 7 kpc, then $a = 1.09 \cdot 10^{10}$, $a^2 = 1.19 \cdot 10^{20}$ and $a^2 / G = 1.78 \cdot 10^{30}$ all of them into the international system of units.

14. THE MASS CALCULUS FOR THE LOCAL GROUP OF GALAXIES

According [18] Azadeh Fattahi, Julio F. Navarro.2020. The pair MW, M31 considering its distance and velocity have a mass around 5E+12 Msun. By other side, according [5] Sofue, the relative velocity between M31 and Milky Way is 170 km/s. Assuming that both galaxies are linked gravitationally, it is possible to calculate the total mass of the Local group by a simple formula because of the Virial theorem.

To suppose that there is dynamical equilibrium in the Local Group of galaxies is a plausible hypothesis supported by the fact that only one third of dwarf galaxies that belong to LG are satellite associated to M31 or MW, being the other two thirds linked to global gravitational field of LG. See [17] Azadeh Fattahi, Julio F. Navarro, and Carlos S. Frenk.2020, for more details about this statement. Therefore, in dynamical equilibrium, $M = \frac{v^2 \cdot r}{G}$ As r = 770 kpc and v= 170 km/s then $M_{LOCAL\ GROUP} = 5.17 * 10^{12}$ Msun

According [5] Sofue, using the current models of DM, the total mass of M31 and Milky Way is approximately $3*10^{12}$ Msun, so there is a mass lack of $2*10^{12}$ Msun which is a considerable amount of matter. Namely read epigraph 4.6 of [5] Sofue paper.

Up to now, in order to do calculus with data of rotation curve, the border of M31 is right to be placed at a half the distance to Milky Way because it is supposed that up to such distance its gravitational field dominates whereas for bigger distances is Milky Way field which dominates.

This hypothesis is right when it is considered rotation curves of different systems bounded to each galaxy i.e. stars or dwarf galaxies satellites. However when it is considered the gravitational interaction between both giant galaxies it is needed to extend their haloes up to 770 kpc, because according DM by gravitation theory the phenomenon of Dark matter is linked to gravitational field, which is unlimited.

Therefore the M31 halo is extend up to 770 kpc and reciprocally the Milky Way halo is extend up to 770 kpc, when it is calculated the gravitational interaction between both galaxies.

In the following paragraph it will considered MW and its main satellite galaxy and M31 and its main satellite M33 in order to do calculus at different radii.

Galaxies	Parameter a^2
M31	2,235E+21
MW	1,527E+21
M33	4E+20
LMC	1,1881E+20
Parameter a^2 Local Group	4,28E+21

14.1 STIMATING TOTAL MASS FOR THE L.G. AT DIFFERENT RADII

In order to estimate the total mass of Local Group will be considered only M31, M33, MW and LMC. The rest of galaxies have a mass negligible to estimate the total mass of Local Group. In fact M33 add only a 9% of total mass approx. and LMC add only a 2.8 % . It will be used Direct mass formulas to do calculus.

The values of parameters $\mathbf{a^2}$ have been got in previous chapters.

By the formula of Direct mass it is right to get the table of masses at different distances using parameters a^2 associated to galaxies.

Calculus written in table below are only an estimation, as the gravitational interaction between the four galaxies is quite complex. The masses calculated below are in Msun units.

kpc	m	MW Msun	LMC Msun	M31 Msun	M33 Msun	Total Mass LG
40	1,234E+21	4,040E+11	3,143E+10	5,912E+11	1,05E+11	1,132E+12
50	1,543E+21	4,516E+11	3,514E+10	6,609E+11	1,18E+11	1,266E+12
100	3,086E+21	6,387E+11	4,969E+10	9,347E+11	1,67E+11	1,790E+12
200	6,171E+21	9,033E+11	7,027E+10	1,322E+12	2,36E+11	2,531E+12
375	1,157E+22	1,237E+12	9,622E+10	1,810E+12	3,23E+11	3,466E+12
500	1,543E+22	1,428E+12	1,111E+11	2,090E+12	3,73E+11	4,002E+12
770	2,376E+22	1,772E+12	1,379E+11	2,594E+12	4,63E+11	4,967E+12
1000	3,086E+22	2,020E+12	1,571E+11	2,956E+12	5,27E+11	5,660E+12
1500	4,629E+22	2,474E+12	1,924E+11	3,620E+12	6,46E+11	6,932E+12
2000	6,171E+22	2,856E+12	2,222E+11	4,180E+12	7,46E+11	8,004E+12
2500	7,714E+22	3,194E+12	2,485E+11	4,674E+12	8,34E+11	8,949E+12
3000	9,257E+22	3,498E+12	2,722E+11	5,120E+12	9,13E+11	9,803E+12

So a^2 for Local Group is 4,28E+21 which is got adding the a^2 parameter associated to the four galaxies.

In conclusion adding MW+LMC+M31+M33 at 770 kpc the mass calculated is 4,97E+12 Msun, which is equivalent to dynamical mass commonly associated to Local Group at 770 kpc equal to 5E+12 Msun.

Another astonishing result is the mass of LG at 3 Mpc. At last row in table above is shown the mass associated to the four galaxies equal to 9,8E+12 Msun.

In the paper [17] Azadeh Fattahi, Julio F. Navarro, C. Frenk.2020.These three well known researchers have estimated the total mass of LG at 3 Mpc equal to 1E+13 Msun.

These two results are a magnificent success of Dark Matter by Gravitation theory.

When it is considered the universal expansion with a Hubble constant H_o = 70 km/s/Mpc, the local expansion between M31 and MW should be 54 Km/s, therefore to justify an approach velocity equal to 170 km/s it is needed even more mass than 5E+12 Msun. Fortunately, DM by gravitation theory stated that the more distance between galaxies the more mass have both galaxies, as it is shown in table above. For example the total mass if both galaxies would have at 1.5 Mpc distance, would be 6,9E+12Msun.

According [17] Azadeh Fattahi, Julio F. Navarro, and Carlos S. Frenk.2020, there is 42 dwarf galaxies in LG with masses bigger than 1E+7 Msun. However this amount of mass is negligible when it is estimated the total mass of LG, which is totally dominated by the four galaxies studied in this chapter.

15. DIVERGENCE FOR FIELD AND POISSON EQUATION FOR POTENTIAL IN THE D.M. THEORY

15.1 DIFFERENTIAL OPERATORS IN SPHERICAL COORDINATES

In the DM by gravitation theory, field has spherical symmetry so differential operators have quite simple formulas, namely: Gradient $\nabla V = \frac{\partial V}{\partial r}\hat{r}$ Divergence $\nabla \bar{E} = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 E_r]$ and the Laplacian $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right]$

In this paper, the formulas for field, as E_{BERNI} or as E_{DIRECT} always represent the vector modulus, therefore $\bar{E} = -E_{BERNI}(r)\hat{r}$ and $E_{DIRECT} = a^2 \cdot r^{\frac{-3}{2}}$ is the vector modulus so $\bar{E} = -E_{DIRECT}(r)\hat{r}$.

15.2 THE GAUSS LAW FOR THE GENERAL BERNOULLI FIELD THEOREM

It is well known the Gauss law for Newtonian field $\boldsymbol{\nabla \bar{E} = -4\pi G\rho}$ where $\boldsymbol{\rho}$ letter represents baryonic density, which is zero outside of the celestial body.

In the following theorem it will be shown that Bernoulli field verify the same differential law, changing baryonic density by Dark matter density, which is non zero in the whole space. Namely in the halo region, which is the dominion of the theory of DM by gravitation.

THE GAUSS LAW FOR THE GENERAL BERNOULLI FIELD THEOREM

In the previous version of this paper, it was shown that Bernoulli field, in case B= 5/3, verify the gauss law changing the baryonic density by DM density. In this version it will be demonstrated the same theorem for any value of B. i.e. $\boldsymbol{\nabla \overline{E_B} = -4\pi G D_{DM}}$ being $\boldsymbol{E_B(r) = (C \cdot r^{\alpha} + D \cdot r)^{\beta}}$ the vector modulus of field and $D_{DM} = A{\cdot}E^B$.

In the chapter 7 it was demonstrated that a general law for DM density $D_{DM} = A{\cdot}E^B$ for any value of B leads to a Bernoulli differential equation for field E, whose solution is $\boldsymbol{E(r) = (C \cdot r^{\alpha} + D \cdot r)^{\beta}}$ being $\boldsymbol{D = \frac{4\pi GA(1-B)}{3-2B}}$ or equivalently $\boldsymbol{4\pi GA = \frac{(3-2B)\cdot D}{1-B}}$ and $\boldsymbol{C = \frac{E_0^{1/\beta} - D\cdot R_0}{R_0^{\alpha}}}$ represents the initial constant of differential equation, E_0 and R_0 are the initial conditions to calculate C, in addition $\boldsymbol{\alpha = 2B - 2}$ and $\boldsymbol{\beta = \frac{1}{1-B}}$. Notice that $\boldsymbol{\alpha.\beta = -2}$.

Doing the divergence in spherical coordinates to $-E_B(r) = -(C \cdot r^{\alpha} + D \cdot r)^{\beta}$**,** which is the radial component of vector field, it is got $\nabla\overline{E_B} = -(\boldsymbol{C \cdot r^{\alpha} + Dr})^{\boldsymbol{\beta - 1}} \cdot \boldsymbol{\frac{D\cdot(3-2B)}{1-B}}$

Is not easy to get this formula. For example it is used the equivalence $\alpha.\beta = -2$ or $D\beta = \frac{D}{1-B}$. The last step to get the previous formula is $\nabla\overline{E_B} = -(\boldsymbol{C \cdot r^{\alpha} + Dr})^{\boldsymbol{\beta-1}} \cdot \left[\boldsymbol{2C \cdot r^{\alpha-1} - 2C \cdot r^{\alpha-1} + 2D + \frac{D}{1-B}}\right]$

By other side $-4\pi G D_{DM} = -4\pi G A{\cdot}E^B = -4\pi GA(Cr^{\alpha} + Dr)^{B\cdot\beta} = -\frac{D(3-2B)}{1-B}{\cdot}(C{\cdot}r^{\alpha} + Dr)^{B\beta}$ **.But** $\boldsymbol{B\beta = \frac{B}{1-B} = \frac{B-1+1}{1-B} = -1 + \frac{1}{1-B} = \beta - 1}$ So $\boldsymbol{-4\pi G D_{DM} = -\frac{D(3-2B)}{1-B}(C \cdot r^{\alpha} + Dr)^{\beta-1}}$ Therefore it is demonstrated the Gauss law for the general Bernoulli field for any value of B i.e. $\boldsymbol{\nabla\overline{E_B} = -4\pi G D_{DM}}$ **.** Notice that D_{DM} is not zero for any point of the dominion, the whole halo region.

Obviously the theorem is right for B=5/3 and for Direct field and Direct DM density as these formulas come from Bernoulli field when B=5/3 and C=0. Being $E_{DIRECT} = a^2 \cdot r^{\frac{-3}{2}}$ and $D_{DIRECT} = \frac{a^2}{8\pi G} \cdot r^{\frac{-5}{2}}$ where a= $D^{-3/4}$.

15.3 POISSON EQUATION SOLUTION FOR DM DIRECT DENSITY AND DIRECT FIELD IN MILKY WAY

As Bernoulli field has only radial component, mathematically verify $\nabla x\bar{E} = 0$, therefore it exist a scalar potential such as $\overline{E_{BERNI}} = -\nabla V$ and by application the Gauss law it is right to get to Poisson equation for Potential $\nabla^2 V = 4\pi G D_{DM}$

In spherical coordinates Laplacian operator is quite simple $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right]$ as V depend on radius only.

POISSON EQUATION SOLUTION FOR DM DIRECT DENSITY AND DIRECT FIELD IN MILKY WAY

As $4\pi G D_{DIRECT} = \frac{a^2}{2} r^{-5/2}$ then Poisson equation becomes $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = \frac{a^2}{2} r^{-5/2}$ so $\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = \frac{a^2}{2} r^{-1/2}$ then $r^2\frac{\partial V}{\partial r} = a^2 r^{1/2} + K$ being K a constant to calculate by an initial condition, so $\frac{\partial V}{\partial r} = a^2 r^{-3/2} + \frac{K}{r^2}$ finally $V = -2a^2 r^{-1/2} - \frac{K}{r}$+ N

If V=0 at infinitum then N=0 so $V_{DIRECT} = -2a^2 r^{-1/2} - \frac{K}{r}$ and taking the field by gradient formula

$E_{POISSON} = -\frac{\partial V}{\partial r} = -a^2 r^{-3/2} - \frac{K}{r^2}$ being $E_{POISSON}$ the radial component of field, so $\bar{E} = E_{POISSON}\hat{r}$

However $E_{DIRECT} = a^2 \cdot r^{\frac{-3}{2}}$ is the vector modulus got in the halo region, therefore it is right to deduce that in this case K = 0.

In other words, the field E_{DIRECT} is calculated completely by the parameter **a**, and by the exponent of radius = -3/2 = 2b-1 which were got by a regression method to rotation curve in the halo region, so in these parameters **a** and **b** is enclosed gravitational field generated by the baryonic matter of the galaxy and its dark matter. For example, the value a^2_{MW}=1.52*10^21 got in page 24 is the parameter for $E_{DIRECT} = a^2 \cdot r^{\frac{-3}{2}}$ which gives the whole field E in the halo region of Milky Way.

So with these physics disquisitions it is proved that the Poisson constant k has to be zero and in the following paragraph will be proved a theorem demonstrating mathematically the same result.

15.4 POISSON CONSTANT EQUAL ZERO THEOREM

In the previous epigraph it was shown that the Poisson solution for direct field is $E_{POISSON} = -\frac{\partial V}{\partial r} = -a^2 r^{-\frac{3}{2}} - \frac{K}{r^2}$

In this epigraph will be shown that the constant **k** of Poisson solution has to be zero if it is required as solution of Bernoulli differential equation of field. As preliminaries before the demonstration it will be checked the general solution of Bernoulli differential equation. In this epigraph it will be used the value B=5/3 to check the solution as this value is right for M31 and Milky Way haloes, although it could be made for any value of B.

As it is known in this formula $\vec{E} = -E\,\hat{r}$, E is the vector modulus $E = G\frac{M(r)}{r^2}$

If $E = G\frac{M(r)}{r^2}$, the vector modulus, is differentiated then it is got $E'(r) = 4\pi GA \cdot E^{5/3}(r) - 2\frac{E(r)}{r}$ which is the Bernoulli differential equation for field at the halo region. See epigraph 7.2, and whose solution is

$E(r) = \left(Cr^{\frac{4}{3}} + Dr\right)^{\frac{-3}{2}}$ (1) being $D = 8\cdot\pi\cdot G\cdot A$ and C the value got by an initial condition. In the epigraph 9.8 it was shown that for C =0 the formula for field becomes $E = a^2 \cdot r^{\frac{-3}{2}}$ as $D = a^{\frac{-4}{3}}$.

By differentiation of the solution (1) it is easy to get that $E` = -2C \cdot r^{1/3} \cdot E^{5/3} - 12\pi GA \cdot E^{5/3}$ where $E^{5/3} = \left[C \cdot r^{4/3} + D \cdot r\right]^{-5/2}$. Writing this result as $E` = -2C \cdot r^{1/3} \cdot E^{5/3} - 16\pi GA \cdot E^{5/3} + 4\pi GA \cdot E^{5/3}$ and as

$D = 8 \cdot \pi \cdot G \cdot A$ then $E^{`} = -2 \cdot E^{5/3} \cdot \left(C \cdot r^{\frac{1}{3}} + D\right) + 4\pi GA \cdot E^{5/3}$ and finally $E^{`} = -2 \cdot \frac{E}{r} + 4\pi GA \cdot E^{5/3}$ because $\frac{E}{r} = \frac{\left(C \cdot r^{4/3} + Dr\right)^{-3/2}}{r} = \left(C \cdot r^{4/3} + D \cdot r\right)^{-5/2} \cdot \frac{\left(C \cdot r^{4/3} + Dr\right)}{r} = E^{5/3} \cdot \left(C \cdot r^{1/3} + D\right)$

So it is checked the solution of the Bernoulli differentia equation for field in case B= 5/3.

POISSON CONSTANT EQUAL ZERO THEOREM

If it is required a solution of Poisson equation that has to verify the Bernoulli differential equation then ***k*** *must be zero.*

In the epigraph 15.3 it was solved the Poisson equation for E direct $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = \frac{a^2}{2} r^{-5/2}$ getting the solution

$E_{POISSON} = -a^2 r^{-\frac{3}{2}} - \frac{K}{r^2}$ as radial component. Hereafter $E_{POISSON} = a^2 r^{-\frac{3}{2}} + \frac{K}{r^2}$ will be the vector modulus, and **k** will be named Poisson constant. Obviously when k= 0 then $E_{POISSON}$ becomes E_{DIRECT}
Now it will be shown that this solution does not verify the Bernoulli differential equation for the vector modulus of field $E'(r) = 4\pi GA \cdot E^{5/3}(r) - 2\frac{E(r)}{r}$ if **k** is not zero.

As it was shown in epigraph 9.8 $E_{DIRECT} = a^2 \cdot r^{\frac{-3}{2}}$ comes from Bernoulli solution when C= 0 therefore this solution verify the Bernoulli differential equation. $E'_{DIRECT}(r) = 4\pi GA \cdot E^{5/3}_{DIRECT}(r) - 2\frac{E_{DIRECT}}{r}$ (1) . Now the Poisson solution $E_{POISSON} = a^2 r^{\frac{-3}{2}} + \frac{K}{r^2}$ written as $E_{POISSON} = E_{DIRECT} + \frac{K}{r^2}$ will be differentiated so $E'_{POISSON} = E'_{DIRECT} - \frac{2K}{r^3}$ and using the previous expression (1) $E'_{POISSON} = 4\pi GAE^{5/3}_{DIRECT} - 2 \cdot \frac{E_{DIRECT}}{r} - \frac{2K}{r^3} = 4\pi GAE^{5/3}_{DIRECT} - 2 \cdot \frac{E_{POISSON}}{r}$ as $\frac{E_{POISSON}}{r} = \frac{a^2 r^{\frac{-3}{2}}}{r} + \frac{K}{r^3} = \frac{E_{DIRECT}}{r} + \frac{2K}{r^3}$
So it is demonstrated that the Poisson solution does not verify the Bernoulli differential equation unless **k** is zero.

16. VIRIAL THEOREM AS A METHOD TO CALCULATE THE DIRECT MASS IN CLUSTERS

Virial theorem states that $M_{DYNAMICAL}(< r) = \frac{V^2 \cdot r}{G}$ is a formula right for a cluster of galaxy on condition that velocity and radius are calculated for galaxies in dynamical equilibrium.
If it is considered that the virial radius is the border of halo cluster where galaxies are in dynamical equilibrium and at the same time is negligible the amount of Baryonic matter outside the sphere with such radius, then it is possible to apply the formula of $M_{DIRECT}(< R_{VIRIAL})$. Then by equation of both formulas will be possible to clear up a^2.

$M_{VIRIAL}(< r) = \frac{a^2 \cdot \sqrt{R_{VIRIAL}}}{G}$ Getting the value for $a^2 = \frac{G \cdot M_{VIRIAL}}{\sqrt{R_{VIRIAL}}}$ This formula is only a way to estimate parameter a^2 because outside the virial radius always there will be a fraction of the galaxies belonging to cluster. Anyway, this method may estimate a lower bound of parameter a^2 associated to cluster.

For example, in the Local Group of galaxies, the dynamical data according [5] Sofue, Y.2015, are 770 kpc for distance between M31 and MW and 170 km/s for it relative velocity, getting $M_{LOCAL\text{-}GROUP} = 5.17 \cdot 10^{12}$ Msun. = 1.03E+43 kg, supposing that there is dynamical equilibrium between MW and M31, by equation with

$M_{DIRECT}(<r) = \dfrac{a^2 \cdot \sqrt{r}}{G} = 1.03 \cdot 10^{43}$ kg with Radius= 770 kpc. It is got a^2 = 4,45E+21, which is very close to parameter a^2 = 4,28E+21 got in previous chapter adding parameters a^2 associated to M31 plus M33 and MW plus LMC and even these values would be closer if it is considered another galaxies such as the Small Cloud of Magellan and the others dwarfs satellite galaxies of MW and M31. Anyway its relative difference is lower than 4%.

It is remarkable the fact that the parameter a^2 of M31 and MW were calculated with data in halo whose radius range from 40 kpc up to 300 kpc in M31 case and range from 35 kpc up to 100 kpc in MW case, whereas calculus for parameter a^2 for Local Group has been made with only one data radius 770 kpc and velocity 170 km/s. However, both methods have given similar results.

With the virial data for some important clusters such as Virgo or Coma cluster will be calculated its parameter $\mathbf{a^2}$ with formula $a^2 = \dfrac{G \cdot M_{VIRIAL}}{\sqrt{R_{VIRIAL}}}$. As it has been comment, it is only estimation for parameter a^2, whose precision also depend on the precision of measures for virial mass and radius. According [12] Karachentsev I.D. the Virgo cluster, which is 17 Mpc far away from MW, has M_{VIRIAL} = 7E+14 Msun and R_{VIRIAL} = 1.8 Mpc which leads to an approximate parameter $\mathbf{a^2}$ = 3.94E+23

Table beside comes from [13] Joo Heon Yoon,
In order to compare with one of the biggest known cluster of galaxies, the Coma cluster has a Virial radius R=2,9 Mpc with a Virial Mass 1,4E+15 Msun, that gives a parameter $\mathbf{a^2}$ = 6,215E+23

Table 1. The properties of the Coma and Virgo Clusters

	Coma	Virgo	ref.
M_{vir} ($M_\odot$)	1.4×10^{15}	2.19×10^{14}	a,b
R_{vir} (Mpc)	2.9	1.57	a,b
D (Mpc)	105	16.5	c,d
v (km s^{-1})	6930	1138	e,d
σ_v (km s^{-1})	1008	544	e,d

NOTE—a: Lokas & Mamon (2003) b: Mamon et al. (2004) c: Ferrarese et al. (2000) d: Mei et al. (2007) e: Struble & Rood (1999)

Using these data for Virgo cluster it is got parameter $\mathbf{a^2}$ = 1.3E+23 that give a value for a^2 three times lower than value got from paper [12] Karachentsev I.D. published in 2014. It is supposed that more recent papers give trustworthy data.

17. DARK MATTER IS COUNTER BALANCED BY DARK ENERGY

The basic concepts about DE on the current cosmology can be studied in [9] Chernin,A.D.

According [11] Biswajit Deb. Plank satellite data (2018) give a new updated, Hubble constant, H = 67.4 ± 0.5 km/s/Mpc and a new $\Omega_{DE} = 0.6889 \pm 0.0056$. In this paper it will be used $\Omega_{DE} = 0.69$ as the fraction of Universal density of DE.

In the current cosmologic model ΛCDM, dark energy has an effect equivalent to antigravity i.e. the mass of dark energy is negative and the dark energy have a constant density for all the Universe equal to

$\varphi_{DE} = \varphi_C \bullet \Omega_{DE} = -5.865 \cdot 10^{-27}\, kg/m^3$ being $\Omega_{DE} = 0.69$ and $\rho_C = \dfrac{3H^2}{8\pi G}$ = 8.5E-27 kg/m^3 the critic density of the Universe , updated with Plank satellite data 2018.

As DE density is constant, the total DE mass is proportional to Radius with power 3, whereas DM mass grows with radius power 0.5 so it is right to get a radius where DM is counter balanced by DE.

According [9] Chernin,A.D. $M_{DE}(<R) = -\frac{\rho_{DE} 8\pi R^3}{3}$, and using the values of data Plank 2018, for H = 67.4 Km/s/Mpc and $\Omega_{DE} = 0.69$ a good approximation for mass of DE is given by the formula $M_{DE}(<R) = -\varphi_{DE}\frac{8\pi R^3}{3} = -4.91 \cdot 10^{-26} \cdot R^3\ kg$. It is important to highlight that this formula is proposed by [9] Chernin,A.D. Notice that this author multiply by two the volume of a sphere i.e. he considers that the effective density of dark energy is two times the $\varphi_{DE} = \varphi_C \bullet \Omega_{DE}$. Anyway, notice that DE theory is being developed currently.

[9] Chernin defines gravitating mass $M_G = M_{DE} + M_{TOTAL}$, where M_{TOTAL} is baryonic plus DM mass, and defines R_{ZG} , Radius at zero Gravity as the radius where $M_{DE} + M_{TOTAL} = 0$. When the gravitating mass is zero, this leads to equation $M_{TOTAL} = \varphi_{DE}\frac{8\pi R_{ZG}^3}{3}$. As Direct mass $M_{DIRECT}(<r) = \frac{a^2 \cdot \sqrt{R_{ZG}}}{G} = M_{TOTAL}(<r)$, in the framework of ***DM by gravitation theory*** it is possible to clear up rightly $R_{ZG} = \left[\frac{3a^2}{8\pi G \rho_{DE}}\right]^{2/5}$ where the only local parameter is $\mathbf{a^2}$.

Below it is calculated R_{ZG} for some clusters.

For the Local group of galaxies, if $a_{L-G}^2 = 4{,}28 \bullet 10^{21}$ then R_{ZG} =2.27 Mpc. So at that radius the gravitating mass is zero, in other words, for radius under 2,27 Mpc dark matter dominates and for bigger radius dark energy dominates and it is not possible to link by gravitation a galaxy more than 2,27 Mpc far away to centre of mass of Local Group, because there is more dark energy than dark matter. Notice that if were used the current $\varphi_{DE} = \varphi_C \bullet \Omega_{DE}$ instead the its twice then R_{ZG} would be 3 Mpc. Anyway 2.3 Mpc or 3 Mpc is only an estimation of R_{ZG}.

For Coma cluster with its previously calculated parameter a^2 = 6,215E+23 , it is possible to get rightly R_{ZG} = 16.7 Mpc .In other words 16,7 Mpc is the radius of region where the DM of Coma Cluster dominates versus dark energy.

Similarly for Virgo cluster with parameter $\mathbf{a^2}$ = 3.944E+23 leads to R_{ZG} = 13,9 Mpc. With this simple calculus it is possible to state that the Local Group is placed in the outskirts of gravitational influence of Virgo cluster, as distance to Virgo cluster from MW is estimated to be 17 Mpc.

With these three important clusters of galaxies, it has been illustrated how the total mass, approximated by $M_{DIRECT}(<r) = \frac{a^2 \cdot \sqrt{r}}{G}$, is counter balanced by dark energy at mega parsecs scale, and precisely this Radius at zero gravity determines the region size where the cluster has gravitational influence.

18. CONCLUSION

As it has been outlined at the introduction, this work is the consequence of the new set of data for rotation curve in Milky Way halo. With these new data, it is possible to state that the rotation curve of MW at halo is governed by the ideal curve named Buckingham halo curve, which has the same exponent for M31 and Milky Way galaxies in the framework of DM by gravitation theory.

This fact back strongly the main hypothesis of Dark gravitation theory i.e. Dark matter is generated according an unknown quantum gravitational mechanism, which depend on the gravitational field, so it is a Universal law.

Through the first ten chapters is developed the theory using M31 rotation curve. This chapters are identical to the previous paper [2] Abarca,M.2019, excepting the process of getting the parameter C , the initial condition of solution for Bernoulli differential equation associated to field E. See chapter 9. Also as a newness in this work, it has been found that the ideal curve called *Buckingham halo curve* linked to dynamical equilibrium of systems leads to Direct mass formula as parameter C is zero in Bernoulli mass formula.

In the chapter 10 are calculated and compared three types of masses linked to M31, dynamical, direct and Bernoulli mass, finding that they are quite close, being Direct mass preferable as it contains only one parameter $\mathbf{a^2}$.
In the chapter 11 has been calculated parameter $\mathbf{a^2}$ associated to Milky Way halo, also it is calculated Direct mass at different radii up to 3 Mpc. In addition results got by the Direct mass are compared with data published by two prestigious astrophysics teams in a dominion radius which ranges from 45 kpc up to 220 kpc. The relative differences are below 4 % regarding a team and below 11% regarding the other team.

In the chapters 12 and 13 are calculated the parameter $\mathbf{a^2}$ associated to M33 and LMC, and with such parameter is possible to estimate the total mass of M33 and LMC at different radii.

In the 14 chapter is calculated the mass of Local Group, being $4{,}97 \cdot 10^{12}$ Msun at 770 kpc, which is equivalent to current accepted dynamical mass $5 \cdot 10^{12}$ Msun associated to MW and M31 . Also it is calculated the total mass at 3 Mpc being equal to 9.8E+12 Msun, which is equivalent to calculated by [17] Azadeh Fattahi, Julio F. Navarro. These calculus are a big success of ***DM by gravitation theory*** because this theory link DM to the quantum gravity nature and is able to calculate the data of masses with quite simple formulas offering accuracy results.

The chapter 15 shows two important results: The divergence of Bernoulli field verify the Gauss law if it is changed baryonic density by Bernoulli density, which is non zero in the whole halo region. The second one is that it is solved the Poisson equation using the direct D.M. density.

In the chapter 16 it is shown a method to estimate the Direct mass formula for a cluster of galaxies, only with its Virial Mass and radius. In the chapter 17, it is shown how at 2.27 Mpc the total dark energy is able to counter balance the dark matter contained in the Local group, or in the Virgo cluster the gravitating mass is zero at 13.9 Mpc whereas in Coma cluster the radius zero gravity is 16.7 Mpc. Put in short, DM is counter balanced by dark energy at some mega parsecs of distance according data of current ΛCDM cosmology.

This theory introduces a powerful method to study DM in the halo region of galaxies and cluster of galaxies and conversely measures in galaxies and clusters offer the possibility to check the theory.
In my opinion, it is not possible to develop anymore the theory into the Newtonian framework. However a natural way to develop more in depth the ***DM by gravitation theory*** would be to consider General Relativity.
Namely, it is right to get the density of energy associated to DM, multiplying density of dark matter by c^2 i.e.
$D_{DM}^{ENERGY} = \frac{a^2 \cdot c^2}{8 \cdot \pi \cdot G} \cdot r^{-5/2}$. So this density of energy would be a new term to consider into the tensor of energy of Einstein's gravitational field equations.

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