On non-principal arithmetical numberings and families

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Abstract

The paper studies $\boldsymbol{\Sigma^0_n}$-computable families ($\boldsymbol{n \geqslant 2}$) and their numberings. It is proved that any non-trivial $\boldsymbol{\Sigma^0_n}$-computable family has a complete with respect to any of its elements $\boldsymbol{\Sigma^0_n}$-computable non-principal numbering. It is established that if a $\boldsymbol{\Sigma^0_n}$-computable family is not principal, then any of its $\boldsymbol{\Sigma^0_n}$-computable numberings has a minimal cover and, if the family is infinite, is incomparable with one of its minimal $\boldsymbol{\Sigma^0_n}$-computable numberings. It is also shown that for any $\boldsymbol{\Sigma^0_n}$-computable numbering $\boldsymbol{\nu}$ of a $\boldsymbol{\Sigma^0_n}$-computable non-principal family there exists its $\boldsymbol{\Sigma^0_n}$-computable numbering that is incomparable with $\boldsymbol{\nu}$. If a non-trivial $\boldsymbol{\Sigma^0_n}$-computable family contains the least and greatest elements under inclusion, then for any of its $\boldsymbol{\Sigma^0_n}$-computable non-principal non-least numberings $\boldsymbol{\nu}$ there exists a $\boldsymbol{\Sigma^0_n}$-computable numbering of the family incomparable with $\boldsymbol{\nu}$. In particular, this is true for the family of all $\boldsymbol{\Sigma^0_n}$-sets and for the families consisting of two inclusion-comparable $\boldsymbol{\Sigma^0_n}$-sets (semilattices of the $\boldsymbol{\Sigma^0_n}$-computable numberings of such families are isomorphic to the semilattice of $\boldsymbol{m}$-degrees of $\boldsymbol{\Sigma^0_n}$-sets).

Keywords: non-principal numbering, complete numbering, minimal cover, minimal numbering.

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1 Introduction

One of the basic properties of the Gödel numbering $x \mapsto W_x$ is its *principality*, i.e. for every computable numbering ν of a family of c.e. sets there exists a computable function f such that $\nu(x) = W_{f(x)}$ for each x. This property is intensively studied in the literature (cf., e.g., [1–4]), since the principal numberings contain information about all computable numberings of the numbered family. Another key property of the Gödel numbering is that for any partially computable function ψ there exists a computable function f such that, for every x, $W_{f(x)} = W_{\psi(x)}$ if $\psi(x)$ converges, and $W_{f(x)} = \emptyset$ otherwise. This property called by Mal'tsev [5, 6] the *completeness* (with respect to $\emptyset$) is also actively studied in the theory of numberings (cf., e.g., [1, 7–13]) and was used by Ershov [14] to prove Kleene's recursion theorems in arbitrary (not necessarily computable) numberings (i.e. surjective mappings from $\mathbb{N}$ onto nonempty countable sets).

In this paper, we consider generalized computable numberings of families of arithmetical sets which were first introduced and studied in Goncharov and Sorbi's paper [15]. Let us fix, until the end of the paper, $n \geqslant 2$. By [15], a numbering ν of a nonempty family of arithmetical sets is said to be Σ^0_n*-computable* if

$$G_\nu = \{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : y \in \nu(x)\} \in \Sigma^0_n.$$

Families with Σ^0_n-computable numberings are themselves called Σ^0_n*-computable*. If G_ν is c.e., then the numbering ν is called *computable*.

Even if a Σ^0_n-computable family is not principal (i.e. has no principal numberings), it always has a Σ^0_n-computable numbering complete with respect to any preselected element (cf., e.g., [8]). In Section 3, we discuss how the algorithmic expressiveness of such a complete numbering can be improved and prove that it can be chosen to be complete simultaneously with respect to all elements of the numbered family.

Another motivation for studying non-principal Σ^0_n-families is that some structural properties of their numberings are proved for principal and non-principal families separately (cf., e.g., [16–18]). In Sections 4 and 5, we prove that for every Σ^0_n-computable numbering, say ν, of a non-principal Σ^0_n-computable family there exists its minimal cover (for arbitrary Σ^0_n-computable families this question was raised by Badaev and Podzorov in their paper [19]) and, if ν is not the least, a Σ^0_n-computable numbering that is incomparable with ν.

Our notation from computability theory is mostly standard. In the following, φ_e denotes the partially computable function with the Gödel number e. We write $\varphi_e(x)\downarrow$ if this computation converges, and $\varphi_e(x)\uparrow$ otherwise. For a partial function ψ we denote its domain and range by $\operatorname{dom}\psi$ and $\operatorname{ran}\psi$ respectively. We let $c(x, y)$ denote the computable pairing function $2^x(2y + 1) - 1$. For unexplained notions we refer to Soare [20, 21].

2 Preliminaries on the theory of numberings

For the main concepts and notions of the theory of numberings we refer to the book by Ershov [14] and his paper [22].

Definition 1. *A numbering ν of a set S is said to be complete with respect to a special element $a \in S$ if for every partially computable function ψ there exists a computable function f such that, for each x, $\nu(f(x)) = \nu(\psi(x))$ if $\psi(x)$ converges, and $\nu(f(x)) = a$ otherwise.*

We say that a numbering ν is *complete* if it is complete with respect to some special element.

Given numberings μ and ν, we say that μ is *reducible* to ν (denoted by $\mu \leqslant \nu$) if there exists a computable function f such that $\mu(x) = \nu(f(x))$ for each x (in this case, we say that μ is reducible to ν via f). We note that if $\mu \leqslant \nu$, then $\operatorname{ran}\mu \subseteq \operatorname{ran}\nu$. We write $\mu < \nu$ if $\mu \leqslant \nu$ and $\nu \nleqslant \mu$. The numbering μ is called *minimal* if $\mu \leqslant \alpha$ for every numbering $\alpha \leqslant \mu$ with $\operatorname{ran}\alpha = \operatorname{ran}\mu$. The numberings μ and ν are said to be *incomparable* if $\mu \nleqslant \nu$ and $\nu \nleqslant \mu$. If in the definition of reducibility of numberings we replace the computable function f by an X-computable one ($X \subseteq \mathbb{N}$), then we obtain the notion of *X-reducibility* $\leqslant^X$. For numberings ν_0 and ν_1, their *direct sum* is defined by $(\nu_0 \oplus \nu_1)(2x + i) = \nu_i(x)$, $i = 0, 1$, $x \in \mathbb{N}$.

A Σ^0_n-computable numbering ν of a family $\mathcal{A}$ is said to be a *minimal cover* of its Σ^0_n-computable numbering μ if $\mu < \nu$ and there is no numbering α such that $\mu < \alpha < \nu$. It is said to be the *least numbering* if $\nu \leqslant \alpha$ for each Σ^0_n-computable numbering α of $\mathcal{A}$. We say that ν is *principal* if $\alpha \leqslant \nu$ for each Σ^0_n-computable numbering α of $\mathcal{A}$. Families with Σ^0_n-computable principal numberings are called *principal* as well. By replacing the reducibility $\leqslant$ with $\leqslant^X$, we obtain the definitions of the *X-principality.*

3 Complete non-principal numberings

The study of the sets of special elements of complete numberings was initiated by Denisov and Lavrov in their paper [9]. From results by Khisamiev [11], it follows that every Σ^0_n-computable family containing the least element under inclusion has a Σ^0_n-computable numbering complete simultaneously with respect to all of its elements. It was proved by Badaev, Goncharov, and Sorbi [12] that there exists a Σ^0_n-computable principal family with a Σ^0_n-computable non-principal numbering complete simultaneously with respect to all elements of the family. The following theorem shows that any non-trivial (i.e. containing more than one element) Σ^0_n-computable family has such a numbering.

Theorem 1. *Every non-trivial Σ^0_n-computable family $\mathcal{A}$ has a complete with respect to each of its elements Σ^0_n-computable non-principal numbering.*

Proof. Let ν be a Σ^0_n-computable numbering of the family $\mathcal{A}$ such that $\nu(0) \neq \nu(1)$. Without loss of generality we assume that if $\mathcal{A}$ is finite and $|\mathcal{A}| = k > 1$, then $\nu(i) \neq \nu(j)$ for all $i < j \leqslant k - 1$ and $\nu(i) = \nu(k - 1)$ for each $i \geqslant k$.

Now we are going to define by induction sequences $\{\mu_s\}_{s\in\mathbb{N}}$ and $\{\alpha_s\}_{s\in\mathbb{N}}$ of partial mappings form $\mathbb{N}$ to $\mathcal{A}$ such that

- $\mu_s \subseteq \mu_{s+1}$ and $\alpha_s \subseteq \alpha_{s+1}$ for each s;
- $\mu = \bigcup_s \mu_s$ is a Σ^0_n-computable numbering of $\mathcal{A}$ complete with respect to each of its elements;
- $\alpha = \bigcup_s \alpha_s$ is a Σ^0_n-computable numbering of a subfamily of $\mathcal{A}$ such that $\alpha \nleqslant \mu$.

At the same time, for every s, we will define an equivalence relation η_s on $\mathbb{N}$ and a strictly increasing computable sequence of integers $\{z^s_i\}_{i\in\mathbb{N}}$.

Let $\mu_0(x)$ and $\alpha_0(x)$ be undefined for each x. We define η_0 to be the equality relation on $\mathbb{N}$ and $z^0_i = i$ for each i.

Assume by induction that the partial mappings $\mu_s : \mathbb{N} \to \mathcal{A}$ and $\alpha_s : \mathbb{N} \to \mathcal{A}$, the equivalence relation η_s, and the strictly increasing computable sequence $\{z_i^s\}_{i\in\mathbb{N}}$ have already been defined and satisfy the following conditions:

1. $\mu_t \subseteq \mu_s$, $\alpha_t \subseteq \alpha_s$, and $\eta_t \subseteq \eta_s$ for each $t \leqslant s$;
2. the sequence $\{z_i^s\}_{i\in\mathbb{N}}$ strictly increasing and computable;
3. $\langle z_i^s, z_j^s\rangle \notin \eta_s$ for any distinct i and j;
4. $\operatorname{dom}\mu_s = \mathbb{N} \setminus (\bigcup_i [z_i^s]_{\eta_s})$, where $[z]_{\eta_s}$ is used to denote the η_s-equivalence class of an integer z;
5. for all $x, y \in \operatorname{dom}\mu_s$, if $\langle x, y\rangle \in \eta_s$, then $\mu_s(x) = \mu_s(y)$.

It is not hard to see that conditions 1–5 hold for $s = 0$. To define the partial mappings μ_{s+1} and α_{s+1}, the equivalence relation η_{s+1}, and the sequence $\{z_i^{s+1}\}_{i\in\mathbb{N}}$, we consider the following several cases.

i. $s = 3t$ for some t.

We choose the least $y \notin \operatorname{dom}\mu_s$ and fix an integer l such that $y \in [z_l^s]_{\eta_s}$. In this case, we provide that $\mu(y) = \nu(t)$. So we will obtain that $\operatorname{ran}\mu = \mathcal{A}$. For every z, we define

$$\mu_{s+1}(z) = \begin{cases} \nu(t), & \text{if } z \in [z_0^s] \cup \cdots \cup [z_l^s], \\ \mu_s(z), & \text{if } z \in \operatorname{dom}\mu_s, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We set $\alpha_{s+1} = \alpha_s$ and define η_{s+1} to be the equivalence relation generated by the binary relation

$$\eta_s \cup \{\langle z_i^s, z_j^s\rangle : i, j \leqslant l\}.$$

For every i, we let

$$z_i^{s+1} = z_{l+i+1}^s.$$

It is not hard to see that induction assumptions 1–5 hold and $\mu(y) = \nu(t)$.

ii. $s = 3t + 1$ for some t.

In this case, we provide the completeness of μ with respect to $\nu(t)$. We define η_{s+1} to be the equivalence relation generated by the binary relation

$$\eta_s \cup \left\{\left\langle z_{c(e,x)}^s, \varphi_e(x)\right\rangle \in \mathbb{N}\times\mathbb{N} : \varphi_e(x)\downarrow\right\}.$$

Using the Recursion Theorem we choose a strictly increasing computable sequence $\{c(e_i, 0)\}_{i\in\mathbb{N}}$ such that

$$\varphi_{e_i}(0) = z_{c(e_i,0)}^s \qquad (1)$$

for each i. For every i, we set

$$z_i^{s+1} = z_{c(e_i,0)}^s.$$

Thus, induction assumptions 2 holds. From the definition of the equivalence η_{s+1} and equalities (1), it follows that induction assumption 3 also holds.

Next, we let $\alpha_{s+1} = \alpha_s$ and $\mu_{s+1}(x) = \mu_s(x)$ for each $x \in \operatorname{dom}\mu_s$. Thus, induction assumption 1 also holds. Now, for every $x \notin \operatorname{dom}\mu_s$, we define

$$\mu_{s+1}(x) = \begin{cases} \mu_s(\varphi_e(y)), & \text{if } \left\langle x, z^s_{c(e,y)} \right\rangle \in \eta_{s+1} \,\&\, \varphi_e(y) \downarrow \in \operatorname{dom}\mu_s, \\ \text{undefined}, & \text{if } \exists i \, [\langle x, z^{s+1}_i \rangle \in \eta_{s+1}], \\ \nu(t), & \text{otherwise.} \end{cases}$$

Therefore, induction assumptions 4 and 5 hold. By the definition of μ_{s+1} we have that, for all e, y,

$$\mu_{s+1}\left(z^s_{c(e,y)}\right) = \begin{cases} \mu(\varphi_e(y)), & \text{if } \varphi_e(y)\downarrow, \\ \nu(t), & \text{if } \varphi_e(y)\uparrow, \end{cases}$$

whenever $z^s_{c(e,y)} \in \operatorname{dom}\mu_{s+1}$. We also have that $\varphi_{e_i}(0) = z^s_{c(e_i,0)}$ for each i. Hence, the numbering μ will be complete with respect to $\nu(t)$.

iii. $s = 3t + 2$ for some t.

In this case, we provide that α is not reducible to μ via φ_t. Let us first assume that $\mathcal{A}$ is infinite. If there exist integers $x \notin \operatorname{dom}\alpha_s$ and l such that $\langle \varphi_t(x)\downarrow, z^s_l \rangle \in \eta_s$, then, for every $y \leqslant x$, we define

$$\alpha_{s+1}(y) = \begin{cases} \nu(0), & \text{if } y \notin \operatorname{dom}\alpha_s, \\ \alpha_s(y), & \text{if } y \in \operatorname{dom}\alpha_s. \end{cases}$$

For every z, we also define

$$\mu_{s+1}(z) = \begin{cases} \nu(1), & \text{if } z \in [z^s_0]_{\eta_s} \cup \cdots \cup [z^s_l]_{\eta_s}, \\ \mu_s(z), & \text{if } z \in \operatorname{dom}\mu_s, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We let η_{s+1} to be the equivalence relation generated by the binary relation

$$\eta_s \cup \{\langle z^s_i, z^s_j \rangle : i, j \leqslant l\}.$$

For every i, we set

$$z^{s+1}_i = z^s_{l+i+1}.$$

Since $\nu(0) \neq \nu(1)$, we have

$$\alpha(x) \neq \mu(\varphi_t(x)).$$

If the required x and l do not exist, then we let $\mu_{s+1} = \mu_s$, $\eta_{s+1} = \eta_s$, and $z^{s+1}_i = z^s_i$ for each i. Next we choose the least $y \notin \operatorname{dom}\alpha_s$ and define

$$\alpha_{s+1}(y + u) = \nu(u)$$

for each $u \leqslant s$. For every $z \in \operatorname{dom}\alpha_s$, we set $\alpha_{s+1}(z) = \alpha_s(z)$. Thus, $\operatorname{ran}\alpha$ will be infinite. Since $\mathcal{A}$ is infinite and $\operatorname{ran}\mu_s$ is finite, we will have that α cannot be reduced to μ via φ_t.

Now suppose that $\mathcal{A}$ is finite. If there exist integers $x \notin \operatorname{dom} \alpha_s$ and l such that $\langle \varphi_t(x) \downarrow, z_l^s \rangle \in \eta_s$, then we define μ_{s+1}, α_{s+1}, η_{s+1}, and $\{z_i^{s+1}\}_{i \in \mathbb{N}}$ in the same way as in the case of infinite $\mathcal{A}$. Otherwise, if $\varphi_t(y) \downarrow$ for the least $y \notin \operatorname{dom} \alpha_s$ (note that $\varphi_t(y) \in \operatorname{dom} \mu_s$), then we take an $i < k$ with

$$\nu(i) \neq \mu_s(\varphi_t(y))$$

(such i can be chosen effectively by the finiteness of $\mathcal{A}$, the choice of ν, and the definition of μ_s) and define $\alpha_{s+1}(y) = \nu(i)$. For every $z \in \operatorname{dom} \alpha_s$, we set $\alpha_{s+1}(z) = \alpha_s(z)$. Let $\mu_{s+1} = \mu_s$, $\eta_{s+1} = \eta_s$, and $z_i^{s+1} = z_i^s$ for each i. Therefore, we will have again that α is not reducible to μ via φ_t.

It is not hard to see that in this case inductive assumptions 1–5 hold as well.

Thus, by the definition of the numbering μ, we have that it is Σ_n^0-computable and complete with respect to $\nu(t)$ for each t. Since $\alpha \oplus \mu \not\leqslant \mu$, it is also not principal. This completes the proof of the theorem. □

4 Minimal covers

The minimal covers of Σ_n^0-computable numberings were first studied by Badaev and Podzorov in their paper [19]. In that paper, a series of sufficient conditions for the existence of minimal covers was proved, among which there is the non-$\emptyset'$-principality of a numbering being covered. The following theorem shows that instead of $\emptyset'$ one can take any non-computable c.e. set. Using this theorem, we then prove that any Σ_n^0-computable numbering of a Σ_n^0-computable non-principal family has a minimal cover.

Theorem 2. *Let C be a non-computable c.e. set. If a Σ_n^0-computable numbering ν of a family $\mathcal{A}$ is not C-principal, then it has a minimal cover.*

Proof. By [23, Lemma 3.3], for every non-computable c.e. set B there exists a c.e. equivalence $\eta \leqslant_T B$ such that

a) the class $[y]_\eta$ is finite for each y;
b) for every e, if $\operatorname{ran} \varphi_e$ is infinite, then

$$\mathbb{N}_{/\eta} =^* \{[\varphi_e(y)]_\eta : \varphi_e(y) \downarrow\},$$

where for arbitrary sets X and Y the notation $X =^* Y$ means that their symmetric difference is finite.

Let $\eta \leqslant_T C$ be a c.e. equivalence relation satisfying conditions a) and b). Fix a Σ_n^0-computable numberings α of the family $\mathcal{A}$ such that $\alpha \not\leqslant^C \nu$ and choose a C-computable sequence $\{a_i\}_{i \in \mathbb{N}}$ of pairwise non-η-equivalent integers with

$$\mathbb{N}_{/\eta} = \{[a_i]_\eta : i \in \mathbb{N}\}.$$

Now we define a Σ^0_n-computable numbering μ of $\mathcal{A}$ by letting

$$\mu(x) = \alpha(i)$$

whenever $\langle x, a_i \rangle \in \eta$. Since $\alpha \not\leqslant^C \nu$, we have $\mu \not\leqslant \nu$.

Let β be an arbitrary Σ^0_n-numbering of $\mathcal{A}$ such that $\nu < \beta \leqslant \mu \oplus \nu$. To prove that $\mu \oplus \nu$ is a minimal cover of ν, it remains to show that $\mu \leqslant \beta$. Fix an index n such that $\beta \leqslant \mu \oplus \nu$ via φ_n. Since $\beta \not\leqslant \nu$, $\operatorname{ran} \varphi_n$ contains infinitely many even integers. Now, by condition b), the c.e.-ness of η, and the equalities $\mu(x) = \mu(y)$ for all x, y with $\langle x, y \rangle \in \eta$, we have $\mu \leqslant \beta$. □

Corollary 3. *Every Σ^0_n-computable numbering of a Σ^0_n-computable non-principal family has a minimal cover.*

Proof. Let ν be a Σ^0_n-computable numbering of a Σ^0_n-computable non-principal family $\mathcal{A}$ and let C be a low_2 non-computable c.e. set. Since

$$C <_T \emptyset' <_T \emptyset'' \equiv_T C'',$$

$\emptyset'$ is high over C. It follows that there exists an $\emptyset'$-computable sequence $\{f_n\}_{n\in\mathbb{N}}$ consisting of all C-computable functions (cf., e.g., [20, 24]). Since the family $\mathcal{A}$ is not principal, its Σ^0_n-computable numbering

$$\beta : c(n, x) \mapsto \nu(f_n(x))$$

is not principal as well. Therefore, the numbering ν is not C-principal (because otherwise β would be principal). By Theorem 2, ν has a minimal cover. □

The question of the existence of minimal covers of numberings of principal families remains open.

Now, using the technique from the proof of Theorem 2 we prove that for any Σ^0_n-computable numbering ν of a Σ^0_n-computable non-principal family there exists its minimal Σ^0_n-computable numbering that is not reducible to ν.

Proposition 4. *For every Σ^0_n-computable numbering ν of an infinite Σ^0_n-computable non-principal family $\mathcal{A}$ there exists its minimal Σ^0_n-computable numbering μ such that $\mu \oplus \nu$ is a minimal cover of ν and, hence, $\mu \not\leqslant \nu$.*

Proof. Let C be a low_2 non-computable c.e. set. In the same way as in the proof of Corollary 3, it is proved that ν is not C-principal. Let us define a numbering μ in the same way as in the proof of Theorem 2. Thus, $\mu \oplus \nu$ is a minimal cover of ν. Since the family $\mathcal{A}$ is infinite, for every index e, if $\mu \circ \varphi_e$ is a numbering of $\mathcal{A}$, then $\operatorname{ran} \varphi_e$ is infinite. Now, by condition b) in the proof of Theorem 2, the c.e.-ness of η, and the equalities $\mu(x) = \mu(y)$ for all x, y with $\langle x, y \rangle \in \eta$, we have that μ is minimal. □

The corresponding question (on the existence of such minimal numberings for arbitrary not necessarily principal families) was posed by Badaev and Goncharov in [25]. In [25, 26], some other partial answers to this question were obtained, but in the general case it remains open.

5 Incomparable numberings

One of the classical theorems of the theory of numberings proved by Badaev [27] states that for any non-principal non-least computable numbering ν of a family of c.e. sets there exists its computable numbering that is incomparable with ν. It is unknown whether Badaev's theorem holds for Σ^0_n-computable families, however, if the family is not principal, then the following theorem holds.

Theorem 5. *Let $\mathcal{A}$ be a Σ^0_n-computable non-principal family. Then for every Σ^0_n-computable non-least numbering ν of $\mathcal{A}$ there exists its Σ^0_n-computable numbering μ that is incomparable with ν.*

Proof. If the family $\mathcal{A}$ is infinite, then the conclusion of the theorem follows immediately from Corollary 4. So, we will assume that $\mathcal{A}$ is finite. Let

$$\mathcal{A} = \{P_0, \dots, P_m\}.$$

In [3, 19], it was proved that a finite Σ^0_n-computable family is principal if and only if it has the least element under inclusion. Let $R_0, \dots, R_k$ be all the pairwise distinct and minimal under inclusion elements of $\mathcal{A}$. Since $\mathcal{A}$ is not principal, $k > 0$. Just as in the proof of [14, I § 2, Proposition 4], we choose finite sets $F_0, \dots, F_k$ that are pairwise incomparable under inclusion and

$$F_i \subseteq R_i \,\&\, F_i \nsubseteq R_j$$

for any distinct $i, j \leqslant k$. Fix a strongly $\emptyset^{(n-1)}$-computable double sequence of finite sets $\{\nu_t(x)\}_{t\in\mathbb{N}}$ such that

$$\nu_t(x) \subseteq \nu_{t+1}(x) \,\&\, \nu(x) = \bigcup_s \nu_s(x)$$

for all t, x.

Now we proceed to defining a Σ^0_n-computable numbering μ of the family $\mathcal{A}$ that is incomparable with ν. Let $\mu_0(x) = P_x$ for each $x \leqslant m$. Assume by induction that the partial mapping $\mu_s : \mathbb{N} \to \mathcal{A}$ has already been defined. Let us define the partial mapping μ_{s+1}. For every $y \in \operatorname{dom}\mu_s$, we define $\mu_{s+1}(y) = \mu_s(y)$. Take the least $x \notin \operatorname{dom}\mu_s$. If $\varphi_s(x)\downarrow$, then we fix the least t such that there exists an $i \leqslant k$ with $F_i \subseteq \nu_t(\varphi_s(x))$ and define

$$\mu_{s+1}(x) = \begin{cases} R_1, & \text{if } i = 0, \\ R_0, & \text{if } i > 0. \end{cases}$$

If $\varphi_s(x)\uparrow$, then we set $\mu_{s+1}(x) = P_0$. Therefore, the numbering $\mu = \bigcup_s \mu_s$ is not reducible to ν via φ_s. If there exists a y with $\varphi_s(y)\downarrow > x$, then we fix the least t for which there exists an $i \leqslant k$ such that $F_i \subseteq \nu_t(y)$. For every z with $x < z \leqslant \varphi_s(y)$, we define

$$\mu_{s+1}(z) = \begin{cases} R_1, & \text{if } i = 0, \\ R_0, & \text{if } i > 0. \end{cases}$$

Thus, ν is not reducible to μ via φ_s. If the required y does not exists, then φ_s is bounded above. Hence, if $\nu \leqslant \mu$ via φ_s, then ν is the least numbering of $\mathcal{A}$. This contradicts the condition of the theorem.

It follows directly from the definition of partial mappings $\mu_s : \mathbb{N} \to \mathcal{A}$, $s \in \mathbb{N}$, that $\mu = \bigcup_s \mu_s$ is a Σ^0_n-computable numbering of $\mathcal{A}$, not comparable to ν. □

Denisov [28] and Khutoretskii [29] proved, respectively, that for any non-principal non-least computable numbering ν of a family consisting of two c.e. sets comparable by inclusion (recall that the semilattice of its computable numberings is isomorphic to the semilattice of c.e. m-degrees [14]) or of the family of all c.e. sets there exist its computable numberings that are incomparable with ν. The following theorem generalizes these results to the case of Σ^0_n-computable families.

Theorem 6. *Let $\mathcal{A}$ be a non-trivial Σ^0_n-computable family with the least and the greatest elements under inclusion. Then for every Σ^0_n-computable non-principal and non-least numbering ν of $\mathcal{A}$ there eixsts its Σ^0_n-computable numbering μ that is incomparable with ν.*

Proof. If the family $\mathcal{A}$ is not principal, then the conclusion of the theorem follows immediately from Theorem 5. Suppose $\mathcal{A}$ is principal. Let α be a Σ^0_n-computable principal numbering of $\mathcal{A}$ such that $\alpha(0)$ is the least element of $\mathcal{A}$ under inclusion and $\alpha(1)$ is its inclusion-greatest element. Fix strongly $\emptyset^{(n-1)}$-computable double sequences of finite sets $\{\alpha_t(x)\}_{t\in\mathbb{N}}$ and $\{\nu_t(x)\}_{t\in\mathbb{N}}$ such that

$$\alpha_t(x) \subseteq \alpha_{t+1}(x) \,\&\, \alpha(x) = \bigcup_s \alpha_s(x),$$

$$\nu_t(x) \subseteq \nu_{t+1}(x) \,\&\, \nu(x) = \bigcup_s \nu_s(x)$$

for all t, x.

Now we proceed to defining a binary function $f \leqslant_T \emptyset^{(n-1)}$ such that

$$f(x,t) \neq f(x,t+1) \Rightarrow f(x,t) = 0 \,\vee\, f(x,t+1) = 1,$$

$$\exists y \,[\lim_s f(y,s) = x]$$

for all x, t. Hence, the numbering

$$\mu : x \mapsto \alpha(\lim_s f(x,s))$$

will be a Σ^0_n-computable numbering of $\mathcal{A}$. We will also provide that the numberings μ and ν are incomparable. In what follows, we denote

$$\mu_t(x) = \alpha_t(f(x,t))$$

for all x, t.

For every x, we let $f(x, 0) = 0$. Next, we need binary functions l and m defined as follows:

$$l(e,t) = \max\{r \leqslant t : \forall x \leqslant r\,[\varphi_{e,s}(x)\downarrow \;\&\; \mu_s(x) \restriction r = \nu_s(\varphi_e(x)) \restriction r]\},$$

$$m(e,t) = \max\{r \leqslant t : \forall x \leqslant r\,[\varphi_{e,s}(x)\downarrow \;\&\; \nu_s(x) \restriction r = \mu_s(\varphi_e(x)) \restriction r]\}$$

for all e, t. It is not hard to see that for every e there exist limits $\lim_t l(e,t)$, $\lim_t m(e,t)$ and

$$\lim_t l(e,t) = \infty \Leftrightarrow \mu = \nu \circ \varphi_e,$$
$$\lim_t m(e,t) = \infty \Leftrightarrow \nu = \mu \circ \varphi_e.$$

We assume by induction on s that all the values $f(x, s)$, $x \in \mathbb{N}$, have already been defined. To define the values $f(x, s+1)$, we consider the following several cases.

i. $s = 3t$ for some t.
In these cases, we provide that $\operatorname{ran}\mu = \mathcal{A}$. Fix the least e such that $f(c(2e, x), s) \neq e$ for each x and define

$$f(c(2e, z), s+1) = e$$

for the least z with $f(c(2e, z), s) = 0$. For every $y \neq c(2e, z)$, we set $f(y, s+1) = f(y, s)$.

ii. $s = 3t+1$ for some t.
In these cases, we provide that $\mu \nleqslant \nu$. If there exists an $e \leqslant s$ such that

$$l(e, t+1) > l(e, t),$$

then we take the least such e and define

$$f(c(2e+1, x), s+1) = x$$

for the least $x > 0$ with $f(c(2e+1, x), s) = 0$. Thus,

$$\forall k < e\,[\lim_u l(k,u) < \infty \;\&\; \lim_u m(k,u) < \infty] \Rightarrow \lim_u l(e,u) < \infty.$$

Indeed, otherwise we would have that

$$\mu(c(2e+1, x)) = \alpha(x)$$

for all (except for a finite number) integers x. Therefore, $\mu \leqslant \nu$ via φ_e and hence $\alpha \leqslant \nu$. This contradicts the non-principality of ν. For each y (not equal to $c(2e+1, x)$ if the required e and x exist), we define $f(y, s+1) = f(y, s)$.

iii. $s = 3t+2$ for some t.
In these cases, we provide tha $\nu \nleqslant \mu$. If there exists an $e \leqslant s$ such that

$$m(e, t+1) > m(e, t),$$

then we take the least such e and define

$$f(z, s+1) = 1$$

for the least $z = c(i, v)$ with $i > 2e+1$ and $f(z, s) \neq 1$. Thus,

$$\forall k < e\,[\lim_u l(k, u) < \infty \,\&\, \lim_u m(k, u) < \infty] \Rightarrow \lim_u m(e, u) < \infty.$$

Indeed, otherwise we would have that

$$\mu(c(i, x)) = \alpha(1)$$

for all $i > 2e+1$ and x. For all $j \leqslant 2e+1$ and for all (except for a finite number) integers x, we would have that $\mu(c(j, x)) = \alpha(0)$. Hence, $\nu \leqslant \mu$ via φ_e and the numbering μ is the least. This contradicts the fact that the numbering ν is not the least. For each y (not equal to z if it exists), we define $f(y, s+1) = f(y, s)$.

Now it follows directly from the definition of the function f that the numbering μ is Σ^0_n-computable and incomparable with ν. □

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