Discrete models of the Universe

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Abstract

In paper [16], we developed the local and global structure of a discrete physical space, and constructed and investigated discrete black and white holes using methods developed in the field of digital topology. In this paper, we develop various discrete models of the universe, as well as explore the discrete structure of wormholes.

Key words: Discrete physical space; Discrete models of the universe; Discrete wormholes; Digital Space

1. Introduction

There are currently various models of the continuous universe in physics, but the Friedman–Lemaitre–Robertson-Walker (FLRW) models are mainly used.
In these models, the Universe is either a three-dimensional Euclidean space, or a three-dimensional hypersphere, or a three-dimensional hyperbolic space.
In recent years, many works have appeared in physics in which a physical space is considered not as continuous but as a discrete space (see e.g. [1-14]).
In many of these approaches, physical space is a lattice consisting of individual points. However, these papers do not give the mathematical structure of lattices, including the topology and geometry of these lattices, the relationship between different points, and so on.
Meanwhile, back in 1916, Einstein declared the need to use discrete models of physical space and regretted the lack of such models in mathematics [15].
In this article, we use digital topology to describe discrete models of the spatial universe and wormholes.
Digital topology as an independent branch of mathematics arose in connection with the widespread use of computers.
In particular, these methods make it possible to create discrete models of continuous spaces with the same mathematical properties as the continuous spaces themselves.
Methods of replacing continuous two-dimensional surfaces with a finite set of two-dimensional cells were developed in works [17-21]. These methods preserve the basic mathematical characteristics of continuous two-dimensional surfaces. An algebraic approach to constructing discrete models of continuous spaces using coverings was studied and developed in papers [22-38]. In this approach, a discrete model of continuous space is considered as a simple undirected graph with a certain structure. An important feature of this approach is the similarity of the properties of a

discrete model with the properties of its continuous analog in terms of algebraic topology.

2 Preliminaries. Discrete spaces in digital topology

This section is an information section and is a copy of a similar section set out in article [16]. In this section, for ease of understanding, we present the basic definitions and properties of discrete spaces obtained within the framework of digital topology. Our approach for constructing discrete spaces was introduced and studied in works [22-38].

2.1 General definitions and properties

Definition 2.1

A 'digital space G=(V,W)' is a pair of sets V and W. $V=\{v_1,v_2,...v_n,\dots\}$ is a finite or countable set of points. W is a set of edges. Each edge (v_pv_q) connects two different points, $v_p \neq v_q$. Two edges (v_pv_q) and (v_qv_p) are the same. Two points can only be connected by one edge: $W=\{ (v_pv_q),| v_p, v_q \in V, v_p \neq v_q, (v_pv_q)=(v_qv_p)\}$.

Note that digital spaces are called simple undirected graphs in graph theory. We will use the terminology of graph theory whenever it is convenient. For an edge (uv) of G, the points u and v are called its endpoints and u and v are incident with (uv). Points u and v are called adjacent or neighbors if they are the endpoints of an edge (uv). Such notions as the connectedness, the adjacency, the dimensionality and the distance on G are completely defined by sets V and W (e.g. [25, 28, 31, 35]).

Definition 2.2

The digital space H=(P,S) is called the subspace of the digital space G=(V,W), if $P \subseteq V$ and H is induced by the set of points P.

Let G and v be a digital space and a point of G. The subspace O(v) induced by the set of points of G that are adjacent to v (without v) is called the nearest neighborhood or the rim of point v in G, (fig. 1). The subspace $O(v) \cup v=U(v)$ is called the ball of v. The joint rim $O(uv)=O(u) \cap O(v)$ of points u and v is a subspace, each point of which is adjacent to both the point u and the point v.

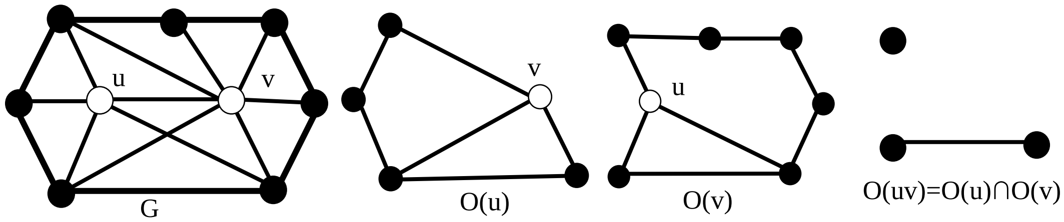

Figure 1. O(u) and O(v) are the rims of points u and v. O(uv) is the joint of points u and v.

According to this definition, H is obtained by removing from G points that are not contained in P together with their incident edges. The space G and subspaces O(u), O(v) and O(uv) are shown in fig. 1.

Contractible digital spaces were defined and studied in [25-27, 32]. To define contractible digital spaces, we use the inductive definition.

Definition 2.3

The one-point digital space K(1)=v is the contractible space.

Let G be a contractible space containing n-points, |G|=n and H be a contractible subspace of G, H⊆G. Then the space P=G∪x, that is obtained by gluing a point x to G in such a way that the rim O(x)=H, is a contractible space.

Contractible digital spaces with a number of points less than or equal to 4 are depicted in fig. 2.

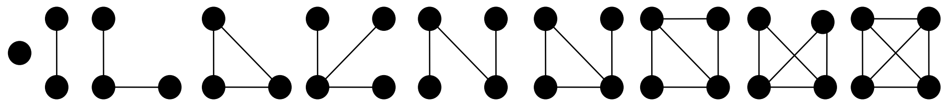

Figure 2. Contractible spaces with the number of points n<5.

Definition 2.4

A point v of the digital space G is called simple if the rim O(v) is a contractible space.

Let points v and u of G are adjacent and the joint rim O(vu)=O(v)∩O(u) is a contractible space. Then the edge (vu) is called simple.

Proposition 2.1

A contractible digital space G can be transformed into a point of G by sequentially removing simple points and edges.

Definition 2.5

Digital spaces G and H are called homotopy equivalent if G can be obtained from H by gluing and (or) removing simple points and edges. Gluing and removing simple points or edges are called contractible transformations.

Properties of the Euler characteristic and the homology groups of digital spaces were studied in [22-27, 31]. It was shown that the Euler characteristics and the homology groups of homotopy equivalent digital spaces G and F are equal.

2.2 Digital n-dimensional Manifolds. Basic definitions and properties

This part contains definitions of n-dimensional digital spaces and transformations of these spaces. There is abundant literature devoted to the study of different approaches to digital lines, surfaces, and spaces used by researchers. Just mention some of them [17-21]. A digital n-manifold is a special case of a digital n-surface defined and investigated in [26, 28, 30, 34].

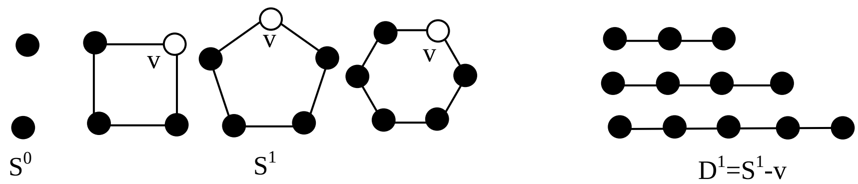

Figure 3. Zero- and one-dimensional spheres S^0 and S^1 and one-dimensional disks D^1.

Definition 2.6.

A 0-dimensional sphere, S^0, is a disconnected digital space with just two points: a and b (fig. 3).

A connected digital space, S^1, with a finite number of points is called a one-dimensional sphere, if the rim of any point v is a zero-dimensional sphere $O(v)=S^0$.

A contractible space, $D^1=S^1$-v, is called a digital one-disk (fig. 3) with the (spherical) boundary ∂D^1=O(v) and the interior $IntD^1=D^1-\partial D^1$.

A connected digital space S^n with a finite number of points is called n-dimensional sphere, if the rim of any point v is (n-1)-dimensional sphere $O(v)= S^{n-1}$ and $D^n=S^n-v$ is a contractible space. $D^n=S^n-v$ is called a digital n-disk (fig. 3) with the (spherical) boundary $\partial D^n=O(v)$ and the interior $IntD^n=D^n-\partial D^n$ (fig. 4).

Definition 2.7.

A connected digital space M with a finite number of points is called a closed n-dimensional manifold, n>0, if the rim O(v) of any point v is an (n-1)-dimensional sphere.

Obviously, if a closed n-manifold is not an n- sphere, then the rim O(v) of any point v is not a contractible space.

Digital models of continuous spaces can be obtained using LCL coverings of these

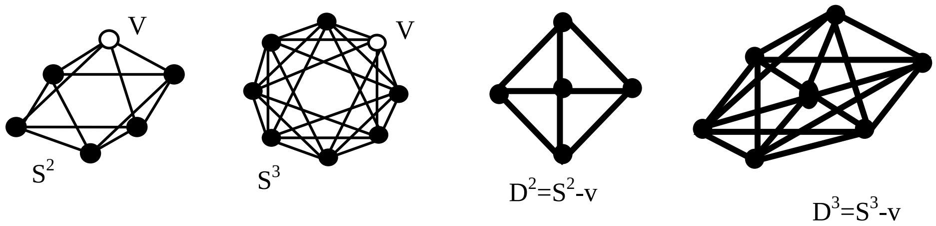

Figure 4. Minimal 2- and 3-dimensional spheres and disks.

spaces [31, 33, 35]. Intersection graphs of LCL coverings are digital models of these spaces with the same mathematical characteristics as the continuous spaces themselves including the dimension, the Euler characteristic, the homology groups and so on.

Digital spaces can be transformed from one space to another by various types of transformations. One type of transformation models the connection between homotopy equivalent continuous spaces in classical topology, the other type of transformation models the homeomorphism between spaces of classical topology. Consider transformations that translate one digital space into another digital space

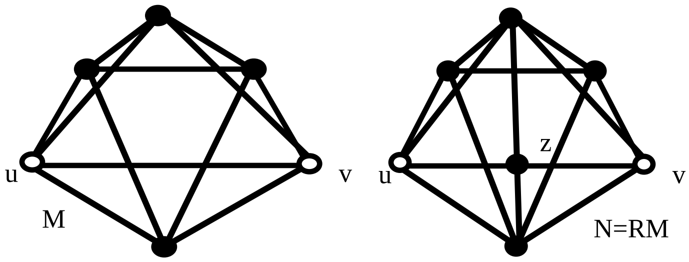

Figure 5. The replacement of the edge (uv) by the point z or R-transformation N=RM.

with the same mathematical characteristics and properties. At the same time, the number of points and edges can vary arbitrarily [32].

Definition 2.8

Let M be a digital space and (vu) be the edge in M. Remove the edge (vu) from M and glue the point z to M, where z adjoins points u, v and all points in the subspace $O(uv)=O(u)\cap O(v)$, $O(z)=v\cup u\cup O(vu)$. This pair of contractible transformations is called the replacement of an edge by a point or R-transformation, R: M→N. The obtained space N is denoted by $N=RM=(M-(vu))\cup z$ (fig. 5).

Obviously, the R-transformation increases the number of points in the digital space M.

Definition 2.9

Let M be a digital space, (vu) be the edge in M and any point x belonging to O(u)-O(v) is not adjacent to any point y belonging to O(v)-O(u).
Remove the points u and v from M and glue the point z to M, such that z is adjacent to all points in the subspace O(u)∪O(v), O(z)= O(u)∪O(v).
This pair of contractible transformations is called the contraction of points u and v or C-transformation. CM=(M∪z)-{u,v} and the points u and v are called a simple pair {u,v} of points (fig. 6).

Obviously, the C-transformations reduce the number of points in the digital space M.

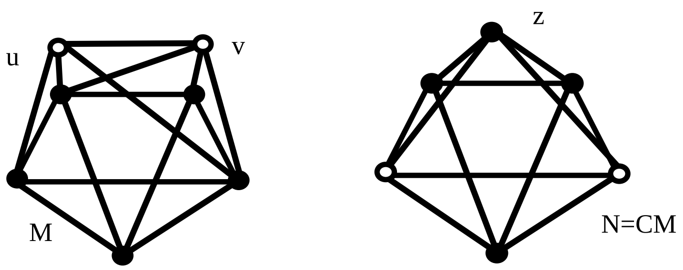

Figure 6. The contraction of points u and v or C-transformation. N=CM.

The properties of R- and C-transformations have been studied in a number of papers. In paper [34], the contraction of simple pairs of points was applied for classification of digital n-manifolds. The following result can be used to study properties of closed digital n-manifolds.

Proposition 2.2.

Let M be a closed digital n- manifold, n>0 and N=CM (N=RM) be the space obtained from M by C-transformations (R-transformations). Then N and M are homeomorphic (N is closed digital n- manifold with the same mathematical properties as M).

The join of two discrete spaces makes it possible to obtain a new space with new properties . This is especially important for discrete closed manifolds.

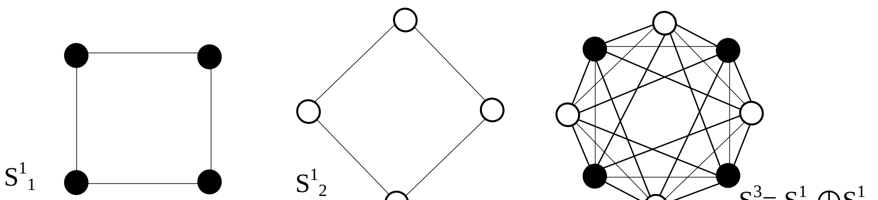

Figure 7. The join of two one-dimensional spheres S^1_1 and S^1_2 is a three-dimensional sphere S^3.

Definition 2.10

The join G⊕H of two discrete spaces G=(X,U) and H=(Y,W) is the space that contains G, H and all edges joining every point in G with every point in H [29, 30, 31].

The following result was proven in [30].

Proposition 2.3

For example, at the initial moment, the discrete models under consideration contain 8 Let G^n and H^m be discrete closed n- and m-dimensional spaces. Then their join $G^n \oplus H^m$ is a discrete closed (n+m+1)-dimensional manifold.

In fig. 7, the join of two one-dimensional spheres S^1_1 and S^1_2 is a three-dimensional sphere S^3.
This property can be used to construct various models of a discrete physical universe. Consider a method for obtaining closed discrete spaces from discrete spaces with boundaries by gluing two discrete spaces with boundaries along their boundaries [29, 30, 31].
Proposition 2.4

Let G=IntG∪∂G and H=IntH∪∂H be discrete n-dimensional spaces with the same boundaries ∂G=∂H. Boundaries ∂G and ∂H are closed (n-1)-dimensional spaces. Glue G and H along their boundaries in such a way that each point on the boundary ∂G is the same as the corresponding point on the boundary ∂H. Then the obtained space W=IntG∪∂G∪IntH is a closed discrete n-dimensional space without boundary.

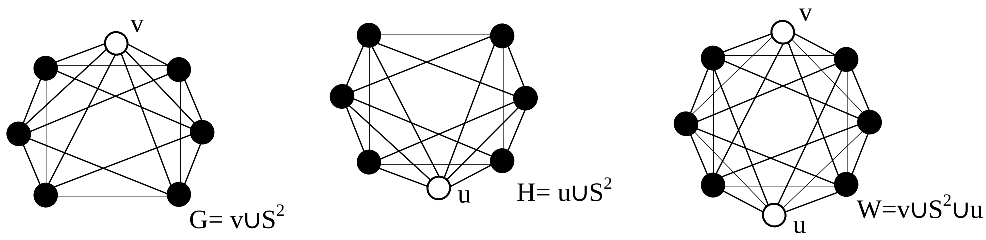

Figure 8. G and H are minimal 2-d disks with boundary S^2. W is the minimal 3-d sphere.

In fig. 8, G and H are minimal three-dimensional disks with a boundary that is a two-dimensional sphere S^2 . Gluing G and H along the boundary we get a minimal three-dimensional sphere W, which contains eight points. This is a closed three-dimensional space.

3 Continuous models of the universe with wormholes

3.1 Mathematical constructions and of spaces

Let's consider at the beginning several mathematical constructions and spaces.

3.1.1. A g-holed torus. In topology, a genus *g* surface is the connected sum of *g* distinct tori.

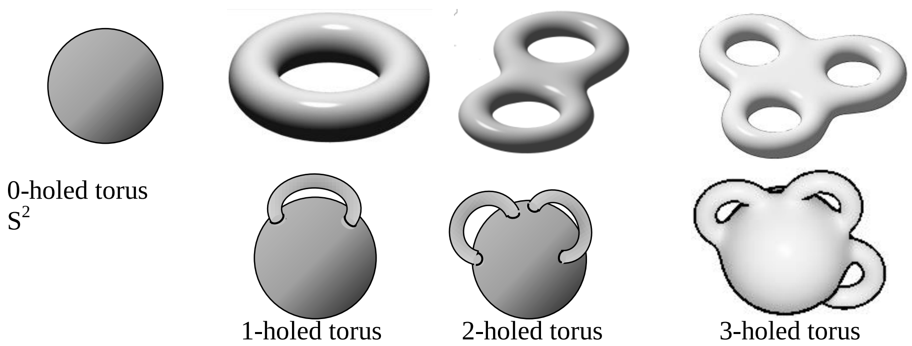

Figure 9. g-holed tori, g=0, 1, 2, 3.

Obviously, a genus *g* surface is a closed 2-dimensional manifold (without boundary). A genus zero surface is the 2-sphere S^2, a genus one surface is the torus itself, a genus two surface is the connected sum of two tori and so on. A **genus *g* surface** is also called a **g-holed torus** or a **2-sphere with g handles** (fig. 9). Note that in mathematics, a g-holed torus and a 2-sphere with g-handles are homeomorphic spaces.

3.1.2 A solid g-holed torus. A solid torus is a torus plus the volume inside the torus.

A solid torus (toroid) is the three-dimensional manifold with a boundary. The boundary of a toroid is the torus, i.e. is the two-dimensional closed space. Therefore, a solid g-holed torus (**g-holed toroid**) is a 3-dimensional manifold with the boundary. The boundary is the solid g-holed torus is the 2-dimensional closed manifold. Topologically, a solid g-holed torus can also be represented as **a 3-ball (3-dimensional disk) with g solid handles.**

3.1.3 The double of a manifold. In topology, if G is a connected n-manifold with boundary ∂G, its double D(G) is obtained by gluing two copies of G together along their common boundary ∂G.

That is, the double of G is $D(G)=G \times \{1,2\}$ where $(x,1) \sim (x,2)$ for all $x \in \partial G$. In other words, $D(G)=G\cup_f G$ where $f:\partial G\rightarrow\partial G$ is an identity map. By construction, D(G) is a closed 3-manifold (without boundary).

In figure 10, G is a 2-disc with boundary ∂G which is a 1-sphere, G×1 and G×2 are two copies of G and D(G) is the double of G. D(G) is a 2-sphere. Similarly, B and H are a 2- manifolds with boundary, D(B) and D(H) are double of B and H. By construction, D(B) and D(H) are closed 2-manifolds. D(B) is a 1-holed torus, D(H) is a 2-holed torus.

Let G be a solid n-holed torus, that is, G is a 3-ball with g solid handles.

The double D(G) is a closed 3-manifold (without boundary), $D(G)=G\cup_f G$ where $f:\partial G\rightarrow\partial G$ is an identity map. Since the double of a 3-ball is the 3-sphere S^3, then D(G) is a closed three-dimensional manifold containing a three-dimensional sphere S^3 and g wormholes attached to S^3. D(G) is a model of a continuous universe with g wormholes.

Let's now consider the application of these mathematical structures to the description of the model of a continuous universe with wormholes.

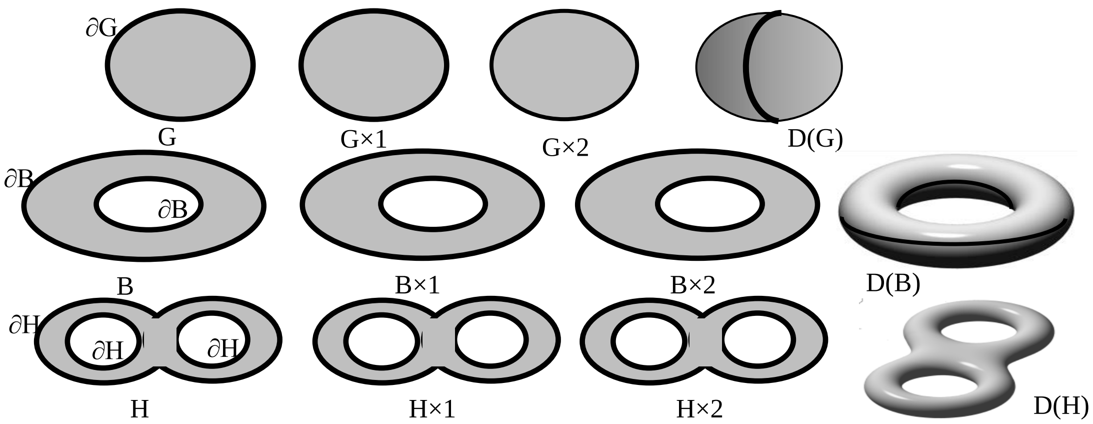

Figure 10. . G is a 2-disc. A one-sphere ∂G is the boundary of G. G×1 and G×2 are copies of G. D(G) is the double of G. D(G) is a two- sphere. B is a 2- manifold with boundary. D(B) is the double of B. D(B) is a torus. H a 2- manifold with boundary. D(H) is the double of H. D(H) is a 2-holed torus.

3.2 A continuous universe as a hypersphere S^3.

One of the continuous models the universe is a closed expanding model of the universe, which is mathematically described by an expanding three-dimensional sphere.
Let G be a solid 0-holed torus, i.e. is a 3- dimensional ball (cube, disk) without

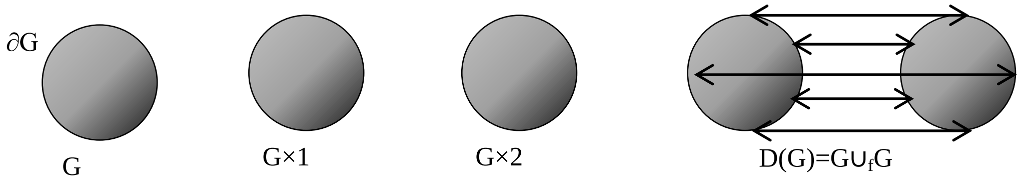

Figure 11. G is a solid 0-holed torus (a ball). The boundary ∂G is a 2-sphere S^2.
G×1 and G×2 are copies of G. D(G) is the double of G i.e. a 3-sphere S^3.

handles.
Therefore, the double of G is the 3-dimensional sphere S^3, $D(G)=S^3$.
In fig. 11, G is a solid 0-holed torus (a ball). G=IntG∪∂G, G×1=IntG∪∂G and G×2=IntG ∪∂G are balls. Glue G×1 and G×2 along their boundaries ∂G in such a way that each point on ∂G in G×1 is the same as the corresponding point on ∂G in G×2. The obtained space D(G)= IntG∪∂G∪IntG a three-dimensional sphere S^3. D(G) is a model of a continuous closed universe without wormholes.

3.3 A continuous universe as a three-dimensional torus.

A torus model of the universe was proposed in 1984. According to this model, topologically the universe is a three -dimensional torus.
Let G is a solid 1-holed torus G (fig. 12). The boundary ∂G is a 1-holed torus, i.e. a simple torus. ∂G is a closed 2-manifold (without boundary). G×1 and G×2 are copies

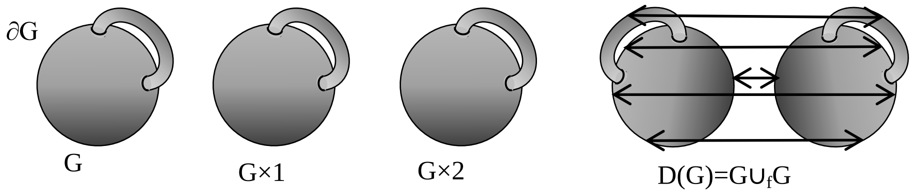

Figure 12. G is a solid 1-holed torus (toroid). The boundary ∂G is a torus.
G×1 and G×2 are copies of G. D(G) is the double of G i.e. a three -dimensional torus.

of G. D(G) is the double of G. D(G) is obtained by gluing two copies of G together along their common boundary ∂G. Thus, D(G) is a closed three-dimensional manifold without boundary containing a three-dimensional sphere S^3 and one wormhole attached to S^3. D(G) is a model of a continuous closed universe without boundary with one wormhole. Topologically D(G) is a three -dimensional torus.

3.4 A continuous universe as a hypersphere S^3 with n wormholes

Consider a mathematical model of a continuous universe, which is a three-dimensional sphere with n wormholes.
Let G be a solid g-holed torus, which can be represented as a 3-ball (3-dimensional disk) with g solid handles. The boundary ∂G of G is the g-holed torus, i.e. is the 2-dimensional closed manifold. The double of G, $D(G)=G\cup_f G$, is obtained by gluing two copies of G together along their common boundary ∂G. D(G) is a closed three-

dimensional manifold containing a three-dimensional sphere S^3 and g wormholes attached to S^3.
In figure 13, G is a solid 3-holed torus G. The boundary ∂G is a 3-holed torus, i.e. a closed 2-manifold (without boundary). D(G) is the double of G. D(G) is obtained by

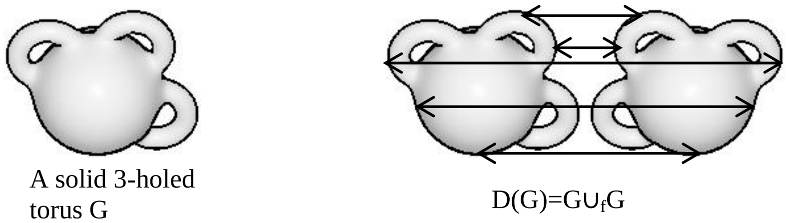

Figure 13. G is a solid 3-holed torus G. The boundary ∂G is a 3-holed torus. G×1 and G×2 are copies of G. D(G) is the double of G.

gluing two copies of G together along their common boundary ∂G.
Thus, D(G) is a closed three-dimensional manifold without boundary containing a three-dimensional sphere S^3 and three wormholes attached to S^3. D(G) is a model of a continuous closed universe without boundary with three wormholes.
Unfortunately, in physics, such models of the universe have never been proposed or studied from the point of view of observations and physical processes occurring in the universe.

4 Discrete models of the universe with wormholes

4.1 Discrete 3-dimensional dicks (balls)

In papers [25, 30, 31, 33, 35, 38] it is shown how to construct discrete models of n-dimensional continuous objects and manifolds. These models retain topological properties of their continuous counterparts.
In order to build a discrete model of the universe with wormholes, we can first build the discrete model of a solid n-holed torus and then glue two discrete models along their boundaries.
As in the continuous case, at the beginning we construct a discrete model of a solid n-holed torus G with the boundary ∂G. Then we construct the double of G by gluing

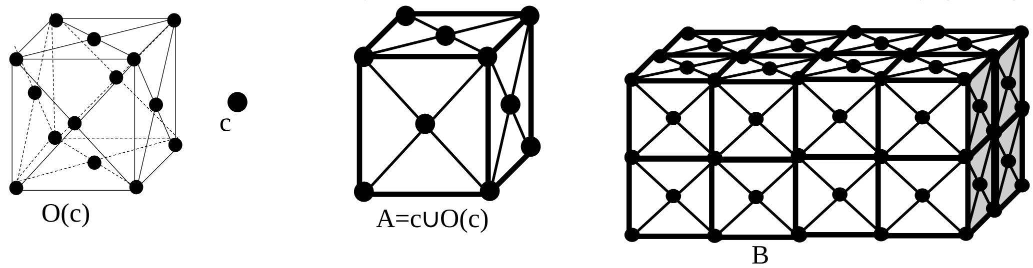

Figure 14. Discrete 3-d disks A and B. O(c) is the rim of point c.

two copies of G together along their common boundary ∂G. $D(G)=G\cup_f G$, where $f:\partial G\rightarrow\partial G$ is an identity map. By construction, D(G) is a discrete closed 3-manifold (without boundary). D(G) is a discrete model of a closed 3-dimensional universe with n wormholes.
We use the same approach that we used in work [16].
The discrete three-dimensional disks A and B are shown in Figure 14. All of them are homotopy equivalent to one another.

The surface of A is a discrete two-dimensional sphere O(c) with 14 points. Point c is located inside O(c) and is adjacent to any point in O(c). Topologically, A is a discrete 3-d ball (cube).
The discrete 3-dimensional disk B is the union of some number of A-cubes obtained by gluing their faces together in pairs. It is easy to verify that the surface of B is a discrete two-dimensional sphere and B is a three-dimensional discrete disk (ball).

4.2 A discrete model of a continuous spherical three-dimensional universe S^3

As in the continuous case, consider discrete solid 0-holed torus G (disk) (fig. 15). The boundaries ∂G is the discrete 0-holed torus, i.e. is the discrete two-dimensional sphere. G×1 and G×2 are copies of G containing ∂G. Therefore, as in the continuous case, the double of G is the discrete 3-dimensional sphere S^3, D(G)=S^3 (fig. 15).
In fig. 15, G is a discrete solid 0-holed torus (a ball). G=IntG∪∂G, G×1=IntG∪∂G and G×2=IntG ∪∂G are discrete balls. Glue G×1 and G×2 along their boundaries ∂G in such a way that each point on ∂G in G×1 is the same as the corresponding point on

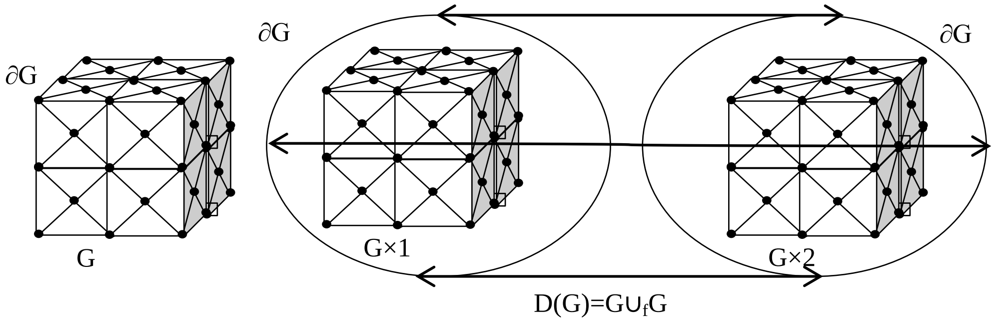

Figure 15. G is a discrete solid 0-holed torus (a 3- disk). The boundary ∂G is a discrete 2-sphere S^2. G×1 and G×2 are copies of G. D(G) is the double of G i.e. a discrete 3-sphere S^3.

∂G in G×2. The obtained space D(G)= IntG∪∂G∪IntG a discrete three-dimensional sphere S^3. D(G) is a discrete model of a continuous closed universe without wormholes. Note that the minimum number of points in a discrete three-dimensional sphere is eight. The minimal discrete 3-d sphere is shown in fig. 4.

4/3 Discrete model of the Universe which is a three-dimensional torus

As mentioned above, the torus model of the universe was proposed in 1984. As in

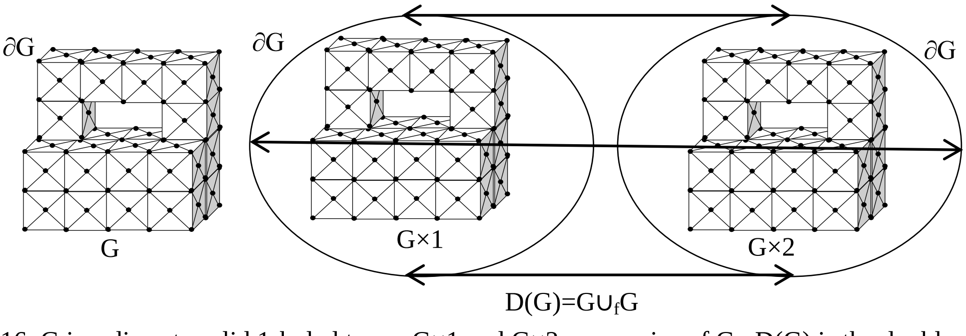

Figure 16. G is a discrete solid 1-holed torus. G×1 and G×2 are copies of G. D(G) is the double of G. D(G) is a discrete three-dimensional torus.

the continuous case, consider a discrete solid 1-holed torus G (fig. 16). The boundary

∂G is a discrete 1-holed torus, i.e. is the discrete two-dimensional closed space. As in the continuous case, the double of G is the discrete 3-dimensional closed space. Geometrically, this space is a discrete three-dimensional sphere S^3 with one handle attached to S^3. In other words, it is a discrete model of a closed three-dimensional spherical universe with a single wormhole.
In figure 16, G is a discrete solid 1-holed torus. D(G) is a discrete closed universe with one wormhole in this universe. The minimum number of points in a discrete three-dimensional torus is twenty.

4.4 Discrete model of the Universe which is a three-dimensional sphere with n wormhole

Let's build a discrete model of a continuous universe, which is a three-dimensional sphere with n wormholes. For example n=4 In figure 17, G is a discrete solid 4-holed torus G. The boundary ∂G is a discrete 4-holed torus, i.e. a closed 2-manifold

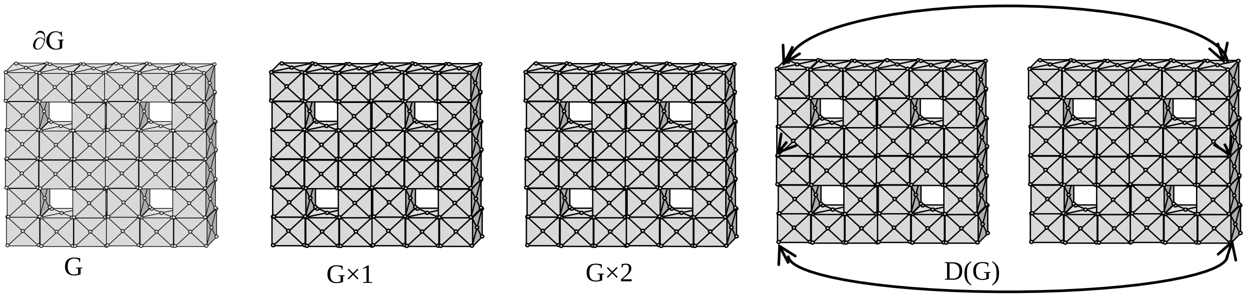

Figure 17. G is a discrete solid 4-holed torus. The boundary ∂G is a discrete 4-holed torus. G×1 and G×2 are copies of G. D(G) is the double of G i.e. a discrete 3-sphere S^3 with 4 wormholes.

(without boundary). G×1 and G×2 are copies of G. D(G) is the double of G. D(G) is obtained by gluing two copies of G together along their common boundary ∂G. Thus, D(G) is a discrete closed three-dimensional manifold without boundary containing a three-dimensional sphere S^3 and four wormholes attached to S^3. D(G) is a model of a discrete closed universe without boundary with four wormholes.

Hypothesis

Dark matter in the universe is a hypothetical form of matter that does not emit electromagnetic radiation and does not interact with it.
There are many hypotheses explaining the nature of dark matter.
In article [16], we hypothesized that the source of dark matter in the universe are the points forming the structure of the discrete universe.

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